Feedforward control of robotic arms with noisy actuators

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Feedforward control of robotic arms with noisy actuators

MASTER OF SCIENCE THESIS

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The variability of feedforward controlled robotic arms is influenced by motor noise. This motor noise originates from multiple sources including the motors, transmission and bearings. It is modeled as additive and multiplicative noise. Human arms also suffer from motor noise in feedforward controlled movements, but humans are believed to perform movements that minimize the end-point variability due to their multiplicative input noise. Minimizing the end-point variability for human arms leads to smooth and energy-efficient movements. For robotic arms performing a pick-and-place task the end-point variability should also be as small as possible. The goal of this research is to minimize the end-point variability of feedforward controlled movements in robotic arms.

The variability of a movement is determined in simulation using a technique based on the Extended Kalman Filter. The minimum-variability movements are found by performing an optimization of the end-point variability. In simulation it was shown that the minimum-variability movements can make use of the friction in the system, low input torques, and the state of the system. It depends on the exact noise supplied to the system, which aspects are used to minimize the end-point variability. These movements are different from energy-optimized or time-optimized movements. The minimum-variability movements are also applied to an experimental robotic arm set-up. It was found that the variability in the set-up is input-dependent. The minimum-variability movements had lower end-point variability than the energy-optimized or time-optimized movements. This research shows that minimizing the end-point variability for a robotic arm in simulation, leads to movements that have low end-point variability in an experimental set-up too.
# Table of Contents

1 Introduction  
1-1 Human motor control .................................. 1  
1-2 Robot control ............................................ 2  
1-3 Research goal ............................................ 4  
1-4 Structure of the report ................................. 4  

2 Motor Noise  
2-1 Motor noise in humans .................................. 5  
2-2 Motor noise in robotic systems ......................... 8  
  2-2-1 Electrical noise sources .......................... 8  
  2-2-2 Mechanical noise sources ......................... 8  
2-3 Noise model for robotic systems ....................... 9  
  2-3-1 Multiplicative noise .............................. 9  
  2-3-2 Additive noise .................................... 9  
  2-3-3 Total noise model .................................. 9  
  2-3-4 Variance for different sample times .......... 10  
2-4 Discussion ................................................ 11  
2-5 Conclusion ............................................... 11  

3 Method  
3-1 Systems .................................................. 13  
  3-1-1 Simulation .......................................... 14  
  3-1-2 Experiments ....................................... 15  
3-2 Task ........................................................ 15  
3-3 Analysis ................................................. 17  
  3-3-1 Non-linear optimization problem .............. 17  
  3-3-2 Comparison ....................................... 18  
3-4 Discussion ............................................... 19  
3-5 Conclusion ............................................... 19
4 Determining Variability 21
   4-1 Monte Carlo variability determination . ........................................ 21
   4-2 Algebraic variability determination for linear systems . .................. 23
      4-2-1 Gaussian calculus .............................................................. 23
      4-2-2 Extended Kalman Filter variability determination .................... 24
      4-2-3 Algebraic variability determination for non-linear systems ........... 25
   4-3 Comparison between different methods ......................................... 26
   4-4 Discussion ................................................................. 28
   4-5 Conclusion .............................................................. 29

5 One DOF Robotic Arm 31
   5-1 One DOF robotic arm with viscous friction .................................... 31
      5-1-1 Additive noise .............................................................. 31
      5-1-2 Multiplicative noise ........................................................ 33
      5-1-3 Both noise types ............................................................ 35
   5-2 One DOF robotic arm with the full friction model ............................. 38
      5-2-1 Additive noise .............................................................. 38
      5-2-2 Multiplicative noise ....................................................... 40
      5-2-3 Both noise types ............................................................ 41
   5-3 Input parameterization .......................................................... 44
   5-4 Discussion ................................................................. 45
   5-5 Conclusion .............................................................. 46

6 Two DOF Robotic Arm 49
   6-1 Two DOF robotic arm with viscous friction .................................... 49
      6-1-1 Additive noise .............................................................. 49
      6-1-2 Multiplicative noise ........................................................ 54
      6-1-3 Both noise types ............................................................ 54
   6-2 Two DOF robotic arm with the full friction model ............................. 58
      6-2-1 Additive noise .............................................................. 58
      6-2-2 Multiplicative noise ....................................................... 60
      6-2-3 Both noise types ............................................................ 63
   6-3 Discussion ................................................................. 66
   6-4 Conclusion .............................................................. 66

7 Experiments 67
   7-1 Model versus set-up ............................................................. 67
   7-2 Motor noise identification ....................................................... 68
   7-3 Experimental results ............................................................ 70
      7-3-1 One DOF robotic arm ....................................................... 70
      7-3-2 Two DOF robotic arm ....................................................... 72
   7-4 Discussion ................................................................. 75
   7-5 Conclusion .............................................................. 76
# Table of Contents

8 Discussion 77
  8-1 Simulations .................................................. 77
  8-2 Experiments .................................................. 78

9 Conclusion 79

10 Recommendations 81

A System Models 83
  A-1 One DOF Robotic Arm Model .................................. 83
  A-2 Two DOF Robotic Arm ....................................... 84

B EKF variability algorithm 87

C Minimum variability in human-like robotic arm 89

D Influence of model parameters 93

  Bibliography 95
### List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>Movement variability of human saccadic eye movements</td>
<td>2</td>
</tr>
<tr>
<td>1-2</td>
<td>Graphical indication of the path-dependency of variability in human movements</td>
<td>3</td>
</tr>
<tr>
<td>2-1</td>
<td>Graphical indication of the influence of motor noise on the execution of a movement</td>
<td>6</td>
</tr>
<tr>
<td>2-2</td>
<td>Signal-dependent noise</td>
<td>7</td>
</tr>
<tr>
<td>2-3</td>
<td>Constant noise</td>
<td>7</td>
</tr>
<tr>
<td>3-1</td>
<td>The experimental robotic arm set-up on which the models are based</td>
<td>13</td>
</tr>
<tr>
<td>3-2</td>
<td>A pick-and-place task for the one DOF robotic arm model and the two DOF robotic arm model</td>
<td>16</td>
</tr>
<tr>
<td>3-3</td>
<td>A typical piecewise constant input with 10 steps</td>
<td>17</td>
</tr>
<tr>
<td>3-4</td>
<td>A minimum-energy and a minimum-time input for a one DOF system</td>
<td>18</td>
</tr>
<tr>
<td>4-1</td>
<td>Input and angle of different movements for Monte Carlo variability determination</td>
<td>22</td>
</tr>
<tr>
<td>4-2</td>
<td>Dice throw example of Monte Carlo variability determination</td>
<td>23</td>
</tr>
<tr>
<td>4-3</td>
<td>Probability density function of a Gaussian distribution</td>
<td>24</td>
</tr>
<tr>
<td>4-4</td>
<td>End-point variability for the linear model of the one DOF robotic arm</td>
<td>27</td>
</tr>
<tr>
<td>4-5</td>
<td>Velocity variability distribution for the one DOF robotic arm with the full friction model when the mean velocity approaches zero</td>
<td>27</td>
</tr>
<tr>
<td>4-6</td>
<td>End-point variability distribution for the one DOF robotic arm with the full friction model</td>
<td>28</td>
</tr>
<tr>
<td>5-1</td>
<td>Schematic drawing of the one DOF robotic arm</td>
<td>32</td>
</tr>
<tr>
<td>5-2</td>
<td>Minimum-energy and minimum-time movement for the one DOF robotic arm with viscous friction</td>
<td>32</td>
</tr>
<tr>
<td>5-3</td>
<td>Minimum-variability movement for the one DOF robotic arm with viscous friction and additive noise</td>
<td>33</td>
</tr>
<tr>
<td>5-4</td>
<td>Variability for the one DOF robotic arm with viscous friction and additive noise</td>
<td>34</td>
</tr>
</tbody>
</table>
5-5 Minimum-variability movement for the one DOF robotic arm with viscous friction and multiplicative noise .............................................. 34
5-6 Variability for the one DOF robotic arm with viscous friction and multiplicative noise ................................................................. 36
5-7 Minimum-variability movement for the one DOF robotic arm with viscous friction and full noise ....................................................... 36
5-8 End-point variability for different movements for the one DOF robotic arm with viscous friction and all noise combinations ...................... 37
5-9 Minimum-energy and minimum-time movement for the one DOF robotic arm with the full friction model .......................................... 38
5-10 Minimum-variability movement for the one DOF robotic arm with the full friction model and additive noise ................................. 39
5-11 Variability for the one DOF robotic arm with the full friction model and additive noise ................................................................. 39
5-12 Minimum-variability movement for the one DOF robotic arm with the full friction model and multiplicative noise ......................... 41
5-13 Variability for the one DOF robotic arm with viscous friction and multiplicative noise ............................................................... 42
5-14 Minimum-variability movement for the one DOF robotic arm with the full friction model and full noise ........................................... 43
5-15 End-point variability for different movements for the one DOF robotic arm with the full friction model and full noise ....................... 43

6-1 Schematic drawing of the two DOF robotic arm .................................................. 50
6-2 Minimum-energy movement for the two DOF robotic arm with viscous friction 50
6-3 Minimum-time movement for the two DOF robotic arm with viscous friction ... 51
6-4 Minimum-variability movement for the two DOF robotic arm with viscous friction and additive noise .................................................. 51
6-5 Variability for different movements of the two DOF robotic arm with viscous friction and additive motor noise of level 0.1 ....................... 52
6-6 End-point variability for different movements of the two DOF robotic arm with viscous friction and additive motor noise ...................... 53
6-7 Minimum-variability movement for the two DOF robotic arm with viscous friction and multiplicative noise ........................................ 53
6-8 Variability for different movements of the two DOF robotic arm with viscous friction and multiplicative noise ....................................... 55
6-9 End-point variability for different movements of the two DOF robotic arm with viscous friction and multiplicative noise ......................... 55
6-10 Minimum-variability movement for the two DOF robotic arm with viscous friction and full noise ...................................................... 56
6-11 End-point variability for different movements for the two DOF robotic arm with viscous friction and all noise level combinations ............... 57
6-12 Minimum-energy movement for the two DOF robotic arm with the full friction model ................................................................. 58
6-13 Minimum-time movement for the two DOF robotic arm with the full friction model ................................................................. 59
6-14 Minimum-variability movement for the two DOF robotic arm with the full friction model and additive noise ........................................ 59
6-15 Variability for different movements of the two DOF robotic arm with the full friction model and additive input noise of level 0.1 ..................... 61
6-16 End-point variability for different movements of the two DOF robotic arm with the full friction model and additive input noise .................. 61
6-17 Minimum-variability movement for a two DOF robotic arm with a full friction model and multiplicative noise ........................................... 62
6-18 Variability for different movements of the two DOF robotic arm with the full friction model and multiplicative motor noise of level 0.1 ........................................... 63
6-19 End-point variability for different movements of the two DOF robotic arm with the full friction model and multiplicative noise ........................................... 64
6-20 Minimum-variability movement for the two DOF robotic arm with the full friction model and full noise ........................................... 64
6-21 End-point variability for different movements of the two DOF robotic arm with the full friction model and all noise level combinations ........................................... 65
7-1 The robotic arm set-up ........................................... 67
7-2 Fit of the model on the experimental data ........................................... 68
7-3 Fit of the variability data for two different movements ........................................... 69
7-4 Mean angle and variability of minimum-variability movements for different noise types ........................................... 70
7-5 Mean angle and variability of four different movements ........................................... 71
7-6 Probability density data of the end-point distribution for four different movements ........................................... 72
7-7 Mean angle and variability of minimum-variability movements for different noise assumptions ........................................... 73
7-8 Mean angle and variability of three different movements ........................................... 74
7-9 Probability density data of the end-point distribution for three different movements ........................................... 75
A-1 Schematic overview of the one DOF robotic arm ........................................... 84
A-2 Schematic overview of the two DOF robotic arm ........................................... 84
C-1 Minimum-velocity-variability and minimum-energy movement for a human-like robotic arm ........................................... 90
D-1 Minimum-variability movement for half the inertia ........................................... 94
## List of Tables

4-1  Table of minimum and maximum difference between the EKF and the MC variability determination for different models  29

5-1  Table of mean minimum end-point variability for different numbers of input steps  44

A-1  Table of variables of the one DOF robotic arm models  83

A-2  Table of variables of the two DOF robotic arm models  85

D-1  Table of end-point variability for different values of the variables  94
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Delft, University of Technology
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“We can change the future. We can’t change the past. Feedforward helps people envision and focus on a positive future, not a failed past.”

— Marshall Goldsmith
The field of biorobotics studies humans and other biological systems in order to improve robot technology. However, robots are not yet our mechanical counterparts as they are sometimes envisioned in science fiction. Creating a humanoid robot that truly matches ourselves is still a distant goal. On the other hand, simpler systems like robotic arms have surpassed our abilities on some fronts like speed, endurance, and accuracy. In other cases, humans still outperform robots, like versatility. The difference between humans and robots lies in the control strategy. Most robotic factory arms control their motions by feedback control (Spong & Vidyasagar, 2004). This control method uses state information from the system to update the control action. In contrast, human motor control consist of a combination of feedback and feedforward control methods (Desmurget & Grafton, 2000). Feedforward control uses only a model of the system to find the desired control action.

**1-1 Human motor control**

If we look at humans we can see that particular human movements are only controlled by feedforward control (Harris & Wolpert, 1998). Feedforward control is the only control method available when the movements are faster than the human feedback loop.

The speed of the feedback loop, depends on the task and the complexity of this task (Flanders & Cordo, 1989). Reflexive movements are governed by very fast feedback loops, from 25 ms up to 110 ms (Van der Kooij et al., 2008). In voluntary human motor control higher brain regions govern the movements, these feedback loops range from 110 ms upwards. The simplest voluntary proprioceptive control tasks have feedback loops that take 110 ms, however voluntary visuomotor control tasks have feedback loops from 190 ms up (Flanders & Cordo, 1989). Movements that are faster than the feedback loop are not very rare. For example, a typical dart throw (a voluntary visuomotor control task) takes less than 150 ms (Smeets et al., 2002; Tamei et al., 2011).

A feedforward controlled movement is error-prone. There are three problems that can result in an error in feedforward controlled systems. First of all, an error in the modeling of the system...
due to uncertain parameters can result in a different movement of the system compared to the simulated movement. Secondly, an external disturbance (an interference with the world) can result in a deviation from the desired movement. Finally, motor noise can result in variability between movements that are intended to be the same.

Motor noise is one of the main causes of variability in feedforward controlled saccadic eye movements (van Beers, 2007) (the other cause is perceptual noise). The variability due to motor noise in human movements changes with the direction of the movement. Van Beers showed that in saccadic eye movements, the variability changes with the direction of the saccades (van Beers, 2007). His results are depicted in Figure 1-1. For a movement in a different direction, other muscles and muscle combinations are used. All muscles in the human body have different noise properties, leading to the change in variability with the direction of the movement.

Van Beers also suggested that different paths could result in different end-point variability. An indication of the path-dependency of variability is shown in Figure 1-2. Harris and Wolpert suggested that humans optimize their movements towards end-point variability (Harris & Wolpert, 1998). They argue that this optimization method results not only in accurate movements, but also in energy-efficient and smooth movements.

1-2 Robot control

For humans feedforward control is useful, and the human body has smart ways to deal with the problems of feedforward control. Although robotic arms are usually controlled with state feedback (Spong & Vidyasagar, 2004), feedforward control is useful for robotics too.

In very different circumstances feedforward control (with or without feedback) has been implemented. Kuo 2002 showed that in rhythmic limb movements the combination of feedback and feedforward control performs best for disturbance rejection. Schaal & Atkeson 1993 showed
Figure 1-2: Graphical indication of the path-dependency of variability in human movements. Moving your hand from one point to another can be done via different paths. Movements A and B are two of these possible paths. The blue ellipse shows the variability at the end-point. It is clear that the variability of movement A is less than the variability of movement B. Since humans optimize their movements to end-point variability, movement A is the movement that people prefer. Taken from Wolpert & Ghahramani 2000.

that robotic juggling can be performed with an open-loop (read: feedforward) controller. This controller has no information on the position of the ball. The stability of running with a humanoid robot is improved by using a particular feedforward control scheme for swing leg retraction (Seyfarth et al., 2003). Stable walking and running is also possible by using feedforward stable periodic movements (Mombaur et al. 2005a, 2005b).

In some situations feedback is too slow, too imprecise, too expensive or even infeasible for the control of a system. In that case feedforward control is the only option. An example of a system that is too slow for classical state feedback is a system where the state information is supplied by a 30 Hz camera. The bandwidth of this total set-up is limited to 30 Hz. If the dynamics of the controlled system are faster than the 30 Hz, the total set-up can become unstable under state feedback control (Gu et al., 2003).

This report focuses only on feedforward control. The result of different control inputs will be clearer in a system that is only governed by feedforward control, than a system with added feedback.

Two of the three problems of feedforward control (i.e. parameter uncertainty and external disturbances) have already been addressed by others. First of all, Plooij et al. have focused on parameter uncertainty in feedforward controlled systems (Plooij, Wolfslag & Wisse Plooij et al.a). They have shown that the error in the final position due to an inaccurate modeling of the Coulomb, viscous, or torque-dependent friction can be reduced to almost zero if the right feedforward controller is used. Secondly, external disturbances in feedforward controlled systems are studied by Plooij and Wolfslag et al. (Plooij, Vries, Wolfslag & Wisse Plooij et al.b). For a two degree of freedom robotic arm they have shown that feedforward stable
cycles can be found. Since these feedforward cycles are stable, errors will diminish over time without the need for feedback.

1-3 Research goal

The last problem of feedforward control, the variability due to motor noise, has not yet been studied. Research in human motor control shows that humans minimize the variability due to motor noise by choosing specific movement paths. An extra advantage is the smooth and energy efficiency of these movements. Motor noise is also present in most robotic systems (Goh et al., 1993), therefore a similar strategy could be applied for feedforward control in robots.

The goal of this research is to minimize the variability of a feedforward controlled robotic arm with noisy actuators. This is done by choosing the movements that minimize the end-point variability. This will be studied in simulation and in hardware experiments.

1-4 Structure of the report

In order to minimize the variability of a feedforward controlled robotic arm, a more thorough understanding of motor noise is needed. Chapter 2 investigates the noise in human motor system and in robotic systems and introduces the motor noise model used in the rest of this research. Chapter 3 explains the method of this research, including the task used, the robotic arm configurations studied, and the optimization methods applied in this research. Chapter 4 explains how the variability is determined. The determination techniques introduced in this Chapter are used to find the minimum variability movements. These techniques are applied in simulation on a one linked robotic arm first. The results of these simulations are given in Chapter 5. Secondly, Chapter 6 studies the results of the simulations on a two linked robotic arm. Thirdly, Chapter 7 shows the results of the experiments on a robotic arm set-up. The results are discussed in Chapter 8. Finally, conclusions are drawn in Chapter 9 and recommendations for future research are given in Chapter 10.
Humans are believed to perform movements that minimize the variability due to motor noise. Section 2-1 investigates where this motor noise comes from and how it can be modeled. The sources of motor noise in robotic systems and their characteristics are described in Section 2-2. Section 2-3 presents the model that is used throughout this thesis to model motor noise in robotic arms. The limitations of the proposed noise model are discussed in Section 2-4.

### 2-1 Motor noise in humans

Human movement execution consist of two phases: the planning phase and the execution phase. The planning phase is situated in the higher brain regions, here the movement is determined. After the movement is determined the execution phase starts; the motor command is send to the muscles and the muscles perform the movement.

Motor noise is the combination of the noise from the planning phase and from the execution phase of a human movement. It is often impossible to indicate whether the main contribution of motor noise lies in the planning or execution phase (van Beers, 2007).

A graphical indication of the influence of motor noise on a movement is shown in Figure 2-1. The actual control action differs from the desired control action due to the motor noise. The level of the off-set from the desired control action varies, since the motor noise has a stochastic distribution.

Motor noise is often modeled with three components: signal-dependent noise (SDN), constant noise (CN), and temporal noise. Signal-dependent noise was proposed by Harris and Wolpert (Harris & Wolpert, 1998). This signal-dependent noise is white noise corrupting the magnitude of the signal with zero mean and the standard deviation is proportional to the absolute value of the signal. This means that more powerful signals contain more noise, which is also graphically shown in Figure 2-2. This signal dependent noise is observed in firing of motor neurons and in force production during isometric contraction (van Beers et al., 2004). The standard deviation of this noise type, $\sigma_{SDN}$, is modeled by van Beers as:
Figure 2-1: Graphical indication of the influence of motor noise on the execution of a movement. The desired motor command (yellow surface) is specified as function of position and velocity of the hand. Although the motor command should lie on the yellow surface, the motor noise creates a different motor command (black dots). Taken from Orban & Wolpert 2011.

\[
\sigma_{SDN}^2 = k_{SDN}^2 u^2
\]  \hspace{1cm} (2-1)

Where \( u \) is the motor command and \( k_{SDN} \) is the level of the noise, dependent of the muscle properties.

The second component of motor noise (i.e. constant noise) is white noise with a standard deviation independent of the magnitude of the signal. Figure 2-3 shows an example of this noise type. This noise type could result from background activity of the motor neurons (van Beers et al., 2004). Secondly, constant noise is added to account for the effects of co-activation of muscles with signal-dependent noise. Due to co-activation the torques generated by the two muscles will partially cancel, but the variances in this torque will add up. Therefore, the net torque will not have pure signal-dependent noise (van Beers, 2007). The standard deviation of constant noise, \( \sigma_{CN} \), is modeled as:

\[
\sigma_{CN}^2 = k_{CN}^2
\]  \hspace{1cm} (2-2)

Where \( k_{CN} \) is the noise level of the constant noise.

The combined model of the standard deviation of motor noise with signal-dependent noise and constant noise becomes:

J.H. Warnaars  
Master of Science Thesis
Figure 2-2: Signal-dependent noise. The desired motor command (red dashed line) shows a step-like profile through time. The actual motor output is plotted in blue. It is clear from this figure that the larger the motor command gets the noisier the signal becomes. Created using the model presented in this chapter.

Figure 2-3: Constant noise. The desired motor command (red dashed line) shows a step-like profile through time. The actual motor output is plotted in blue. It is clear from this figure that the noise level remains the same independent of the motor command. Created using the model presented in this chapter.
\[ \sigma_u = \sqrt{\sigma_{SDN}^2 + \sigma_{CN}^2} = \sqrt{k_{SDN}^2 u^2 + k_{CN}^2} \] (2-3)

Human movements also deviate in movement time. With longer movement time, the peak velocity decreases. Van Beers models this as temporal noise, although it has no influence on the standard deviation of the motor noise (van Beers et al., 2004; van Beers, 2007). To model this noise the movement time was scaled by a factor \( c \) and the motor command \( u \) by a factor \( 1/c^2 \).

2-2 Motor noise in robotic systems

The motor noise in robotic systems is defined as all the noise that is intrinsic to the system. Motor noise in robotic systems can arise from electrical or mechanical sources. This section gives an overview of the possible noise sources. First, electrical noise sources are discussed. Additionally, mechanical noise sources are discussed.

2-2-1 Electrical noise sources

Motor noise arises from three different electrical sources: the electronic circuits, the electric motors, and the motor control.

Noise is always present in all electronic circuits. Thermal noise, shot noise and flicker noise are the main noise sources in electronic circuits (Demir & Sangiovanni-Vincentelli, 1998). The thermal noise and shot noise are modeled as additive Gaussian white noise. The flicker noise is a frequency dependent noise, but it is only present in low frequency regions.

Electric motors display a certain torque ripple. This ripple is the effect of phase shift between the current in the different coils of the motor. This torque ripple increases with the load (Raju et al., 2012). Therefore, this torque ripple could be seen as a multiplicative noise source.

The torque of an electric DC motor can be controlled by controlling the current through the motor. This is done in a feedback loop for which the output current is measured using a current sensor. The measurement noise from the sensor is propagated to the torque through the feedback loop, resulting in motor noise (Moberg & Öhr, 2005).

2-2-2 Mechanical noise sources

Noise from mechanical components, can arise from multiple sources and include friction, stick-slip, backlash in the transmissions, cogging in motors and reflected inertia through a transmission (Robinson, 2000). This motor noise is often modeled as additive Gaussian white noise (Swevers et al., 1997). Sometimes it is also modeled as multiplicative noise and additive noise. Shell & Mataric 2006 do this to model uncertainty in the actuation of small scale robots for multi-robot systems.

A source of torque-dependent mechanical noise is the friction in the bearings of timing belts. The friction in these bearings depends on the load. The load is again dependent on the torque, however this relationship depends on the angle of income and outcome of the timing
2-3 Noise model for robotic systems

Noise in robotic systems is usually assumed to be Gaussian white noise (Swevers et al., 1997). This means that the noise in every sample is unrelated to any sample and that the magnitude of the noise is taken from a Gaussian distribution with a certain mean $\mu$ and a certain standard deviation $\sigma$. For the simulations the full noise model composes of multiplicative noise and additive noise.

For means of notation we use the variance, $\Sigma$, instead of the standard deviation, $\sigma$, unless noted otherwise. The relation between the both is very simple.

$$\Sigma = \sigma^2 \tag{2-4}$$

2-3-1 Multiplicative noise

The multiplicative noise is modeled equivalent to the signal-dependent noise in the human noise model. The mean of the multiplicative noise, $\mu_m$, is taken as 1, so the mean of the desired input signal does not change, but only the standard deviation of the input does due to the multiplicative noise. Equivalent to Equation 2-1, we get:

$$\Sigma_m = k_m \cdot u^2 \tag{2-5}$$

Where $\Sigma_m$ is the variance of the multiplicative noise and $k_m$ is the multiplicative noise level.

2-3-2 Additive noise

The additive noise is also modeled equivalent to the constant noise of the human noise model. The mean of the additive noise, $\mu_a$, is taken as zero, so the mean of the desired input signal does not change due to the additive noise. Equivalent to Equation 2-2, we get:

$$\Sigma_a = \sigma_a^2 \tag{2-6}$$

Where $\Sigma_a$ is the variance of the additive noise and $k_a$ is the additive noise level.

2-3-3 Total noise model

The total noise model consists of the combination of the multiplicative noise and the additive noise. The variability is added to the desired input signal, this is done in the following way. Where $u^*$ is the desired input, $u$ is the disturbed input, $m$ is the multiplicative noise, $a$ is the additive noise.

$$u = u^* \cdot m + a \tag{2-7}$$
The variance of the input then becomes:

\[ \Sigma_u = \Sigma_m + \Sigma_a, \quad (2-8) \]

Or equivalently,

\[ \Sigma_u = k_m^2 u^2 + k_a^2 \quad (2-9) \]

### 2-3-4 Variance for different sample times

Gaussian white noise is defined by its mean, \( \mu \), and variance, \( \Sigma \) \( (G \sim \mathcal{N}(\mu, \Sigma)) \). Gaussian white noise is the generalized derivative of the Wiener process. For Gaussian white noise with zero mean this relation is defined by the following equations (Wellner, 2003):

\[ G \sim \mathcal{N}(0, \Sigma) = S \cdot dW_t, \quad \text{where} \ S^2 = \Sigma \quad (2-10) \]

So the Gaussian white noise is the same as some scaling factor \( S \) times the derivative of the Wiener process \( dW_t \). The variance \( \Sigma \) of the Gaussian white noise is equivalent to the square of this scaling factor.

The derivative of the Wiener process is a Gaussian distribution \( \epsilon_t \) with a standard normal distribution \( (\epsilon_t \sim \mathcal{N}(0, 1)) \) times the square root of the sample time \( \triangle t \).

\[ dW_t = \epsilon_t \sqrt{\triangle t}, \quad \text{where} \ \epsilon_t \sim \mathcal{N}(0, 1) \quad (2-11) \]

To make this noise independent of the sample time, \( S \) needs to scale with:

\[ S = \frac{a}{\sqrt{\triangle t}} \quad \text{and thus}, \quad (2-12) \]

\[ \Sigma = \frac{a^2}{\triangle t} \quad (2-13) \]

Where \( a \) is a constant scaling factor. The variance of the Gaussian white noise is thus scaled opposite to the sample time. The mean of this noise can be altered by a simple addition of a constant value with the size of the desired mean.

This means that if the sample time is taken ten times smaller, the variance of the noise \( \Sigma \) needs to be ten times larger. This may seem surprising, but this is a consequence of the independence between the samples for Gaussian white noise. The average over one hundred samples is typically closer to the expectancy of the Gaussian white noise distribution than the average over ten samples. This can also be seen from the dice example in Figure 4-2. If we would not apply this scaling the noise has less influence on the system when the sample time is decreased. If the sample time would go to zero the influence of the noise would then become zero.
2-4 Discussion

The noise model presented in this chapter captures two important noise types: additive and multiplicative Gaussian white noise. This assumes that these noise types are the dominant noise sources in robotic arms. The frequency-dependent noise of the electrical circuit (i.e. flicker noise) is thus not included in the model.

Frequency-dependent noise has also been used to model the behavior of humans. The variability of human stance could for instance be modeled as 1/f noise (Duarte & Zatsiorsky, 2001). Additionally, human cognition is believed to suffer from 1/f noise (Gilden et al., 1995).

The noise model is limited in the noise sources it includes. Another limitation becomes clear from the scaling of the variance with the sample time. A real system is continuous and displays a certain variability for feedforward control. This variability should also stay constant in a discrete time simulation regardless of the sample time. For a sample time approaching 0 the variance of the noise would become infinite.

2-5 Conclusion

The motor noise for humans and robotic arms is modeled in the same way. The noise model incorporates two noise types: additive and multiplicative noise. The noise is assumed to be Gaussian white noise.

Motor noise in robotic arms is defined as all the noise that is intrinsic to the system. This motor noise arises from multiple sources, including the electric motors, the transmission and the bearings. The noise has an additive nature and also a torque-dependent nature. The torque-dependent noise is captured by the multiplicative noise.
This chapter explains the method followed in the rest of this research. In order to study the influence of motor noise effectively, choices have to be made regarding the systems studied, the task performed, and the method of analysis. First, Section 3-1 introduces the systems that are studied in this research. Second, the tasks that these systems have to perform is described in Section 3-2. Third, Section 3-3 introduces the method of analysis performed in this research. The chapter ends with a brief discussion of the chosen method in Section 3-4 and conclusions are drawn in Section 3-5.

### 3-1 Systems

The goal of this research is to minimize the variability of a feedforward controlled robotic arm with noisy actuators. The robotic arm used is the arm presented in the paper of Plooij & Wisse 2012 without the spring mechanism. Figure 3-1 shows this arm for the one DOF and two DOF set-up. This robotic arm can be modified to a one DOF arm or a two DOF arm.

![Figure 3-1: The experimental robotic arm set-up on which the models are based. On the left, the one DOF set-up of the robotic arm is shown. On the right, the two DOF set-up is shown. The hand was replaced by a mass in this research.](image-url)
both degrees of freedom allow only movements in the horizontal plane. First, this robotic arm is studied in simulation to understand the influence of motor noise in simplified systems. Section 3-1-1 describes what systems and models are used in the simulations. Finally, the movements that minimize the variability in simulation are then tested on the experimental set-up. How these experiments are conducted is explained in Section 3-1-2.

3-1-1 Simulation

The movements that minimize variability are investigated using simulations of robotic arm systems. In Appendix A the exact models for these simulations are given.

The motor noise in an experimental set-up depends on many factors, as was explained in Chapter 2. Therefore, it is hard to predict what the exact noise model is. In simulation the noise model can be changed at will. In this way the influence of different noise sources and different sizes of these disturbances is examined.

In a brushless DC motor the torque ripples are reported to have a size of around ten percent of the commanded torque, when no efforts where taken to reduce the torque ripple (Salisbury et al., 1988). The torque ripple was eventually decreased to two percent of the commanded torque. The torque ripple from the motor is a multiplicative noise source, other multiplicative noise sources are not investigated by Salisbury et al. 1988.

The multiplicative noise levels investigated in this study are chosen such that they include the highest torque ripple reported by Salisbury et al. 1988. Nevertheless, the range of multiplicative noise levels is larger to include also other multiplicative noise sources. The highest multiplicative noise level has a standard deviation of ten percent of the commanded torque.

The range of the noise levels of the additive motor noise is taken to be the same as for the multiplicative noise. The additive noise has more influence at smaller torque inputs than at larger torque inputs. The highest additive noise level makes that the input torque has a standard deviation of 5 percent of the maximum torque.

The influence of motor noise is investigated in three different situations: additive noise, multiplicative noise, and combinations of both noises types. This is done for the full range of noise levels. The noise level of the multiplicative noise is investigated between 0 and 0.1. The noise level of the additive noise is investigated between 0 and 0.1. This noise level is taken for a time step of one thousandth of a second, the same time step as was taken for the experiments. The noise levels for both noise types are defined in Chapter 2.

One DOF

One DOF robotic arms are very simple, therefore they give a clear insight in the influence of minimizing towards end-point variability. For one DOF systems two cases will be studied. First, a one DOF robotic arm model with only viscous friction in the joints (a linear model) is studied. A linear model is insightful and gives basic information on how motor noise affects a system and what movements minimize the end-point variability due to motor noise. However, a model with only viscous friction is far from reality. So secondly, a one DOF robotic arm
with a full friction model (i.e. viscous, Coulomb, and torque-dependent friction) is studied.\textsuperscript{1} The addition of the Coulomb friction and the torque-dependent friction make this a non-linear one DOF model. The results of these simulations are shown in Chapter 5.

**Two DOF**

For two DOF robotic arms, two models are studied. First, a two DOF robotic arm with only viscous friction is studied. This gives a good comparison with the one DOF model with viscous friction. Secondly, a two DOF robotic arm with a full friction model is studied. This model comes closest to the experimental set-up and incorporates multiple degrees of freedom as are normal in a robotic arm. This model gives best insight in how motor noise affects robotic arms. The results of these simulations are shown in Chapter 6.

**3-1-2 Experiments**

Experiments are conducted, to see if the influence of motor noise is the same in reality as in the simulations. Moreover, the experiments show if the minimum-variability movements found in this research also work in practice.

The motor noise of the real robotic arm is determined, via fitting of the motor noise model to the variability data of the actual set-up. This is done by performing more than 200 feedforward exactly similar movements on the set-up. This noise model is used to find the minimum-variability movement in simulation. This movement is performed multiple times by the experimental robotic arm, and compared to the outcome of the simulations. The results of these experiments are shown in Chapter 7.

**3-2 Task**

In order to analyze the influence of motor noise on robotic arms, we have chosen to look at a pick-and-place task. Robotic arms are widely used in factories around the world. One of the most basic tasks of these robotic arms is the pick-and-place task. This task is simplified into the objective to move from an initial position to a final position.

In a pick-and-place task it is important to arrive at the goal position and to do this in time. For this research we assume that the movement time is only a constraint and not an objective. In other words, the goal position has to be reached within time only. The main objective for the robot arm is repeatably reaching the goal position with high precision. In other words, the objective is to reach the goal position with an end-point variability that is as small as possible. In Section 3-3 we will explain how this can be achieved.

For the task the robot arm has to move from point A to point B in 1 second. The initial position, final positions and the final time are chosen such that the task is suboptimal. This means that the task can be performed by various movements.

---

\textsuperscript{1}Note that the viscous friction coefficient is positive when the model only incorporates viscous friction. The viscous friction coefficient for the full friction model is negative in order to fit to the experimentally found friction data.
The initial conditions are assumed to be fully known, so there is no uncertainty initially. In this way, we study only the effects of motor noise on the feedforward movement.

For the one DOF systems the task is depicted on the left of Figure 3-2. The task is to go from the initial state $s_0$ to the final state $s_{\text{end}}^*$ in 1 second:

\[
s_0 = [\theta_0 \dot{\theta}_0]' = [0 0]'
\]
\[
s_{\text{end}}^* = [\theta_{\text{end}} \dot{\theta}_{\text{end}}]' = [1 0]'
\]  

(3-1)

where $\theta_0$ is the initial angle in radians and $\dot{\theta}_0$ is the initial angular velocity in radians per second. $\theta_{\text{end}}$ and $\dot{\theta}_{\text{end}}$ are the final angle in radians and angular velocity in radians per second.

For the two DOF systems the task is depicted on the right of Figure 3-2. The task is to go from the initial state $s_0$ to the final state $s_{\text{end}}$ in 1 second:

\[
s_0 = [\theta_{1_0} \theta_{2_0} \dot{\theta}_{1_0} \dot{\theta}_{2_0}]' = [-0.25 0 0 0]'
\]
\[
s_{\text{end}}^* = [\theta_{1_{\text{end}}} \theta_{2_{\text{end}}} \dot{\theta}_{1_{\text{end}}} \dot{\theta}_{2_{\text{end}}}]' = [0.25 0 0 0]'
\]  

(3-2)

where $\theta_{1_0}$ and $\theta_{2_0}$ are the initial angles in radians, and $\dot{\theta}_{1_0}$ and $\dot{\theta}_{2_0}$ are the initial angular velocities in radians per second. $\theta_{1_{\text{end}}}$ and $\theta_{2_{\text{end}}}$ are the final angles in radians, and $\dot{\theta}_{1_{\text{end}}}$ and $\dot{\theta}_{2_{\text{end}}}$ are the final angular velocities in radians per second.
3-3 Analysis

Chapter 4 explains how to determine the variability of a movement. In this section, we will show how the optimization is done to find the movement with the least variability at the end-point. Furthermore, this section will introduce two other movements for the pick-and-place task. These movements are a minimum-energy movement, and a minimum-time movement. These movements are used as a comparison for the minimum-variability movement.

3-3-1 Non-linear optimization problem

The minimization of the variability is for most cases a non-linear optimization problem, the total problem is shown in Equation 3-3. Only for linear systems it becomes a linear optimization problem. The optimization problem is constrained by an equality constraint on the desired final state. This constraint makes that only movements that reach and stay at the final state within time are feasible movements.

Since this is a non-linear equality constrained optimization problem, a suitable optimization algorithm is the SQP algorithm. In MATLAB this algorithm can be used within \texttt{fmincon}.

Since, this is a non-linear optimization problem, there are local minima. The multistart procedure (i.e. running the optimization multiple times from random initial conditions) is used to find the global optimum, however there is no guarantee that it is found.\(^2\)

The optimization can change the input torque in order to minimize the variability. The input is chosen to be a piecewise constant signal with 10 steps. Figure 3-3 shows a typical input signal. This choice, although still arbitrary, is made out of practical reasons. The optimization has only 10 free parameters per input, limiting the optimization time. A 10 step input signal has enough freedom to give information on how the optimization of a fully continuous signal would look like. This is checked by performing simulations with other numbers of steps.

\(^2\)The optimization was sometimes started with the minimum-variability input of another noise level as an initial guess.

Figure 3-3: A typical piecewise constant input with 10 steps. In this figure a torque input is plotted through time. The torque level makes 10 steps in 1 second and stays constant in between these steps.
Figure 3-4: A minimum-energy and a minimum-time input for a one DOF system. The minimum-energy movement is shown on the left: the input torques are gradually decreasing from a positive torque at the beginning of the movement, to negative at the end of the movement. The minimum-time movement is shown on the right: the input torques are near maximum and alternating for the first two torque steps, this is followed by a settling phase. The input torque is zero for the rest of the movement. These movements where required for a similar robotic arm model as is used in this research.

The minimum variability optimization problem is the following:

$$\min_{T_{in}} \Sigma_{x_{end}}(T_{in}), \quad \text{subject to } s_{end} - s^{*}_{end} = 0 \quad (3-3)$$

Where $\Sigma_{x_{end}}$ is the square of the end-point variability of the position, which is a function of the input torque $T_{in}$. $s_{end}$ is the end state and $s^{*}_{end}$ is the desired end state at time $t = t_{end}$. In the two DOF case we multiplied the end-point variance of the first link by a factor four. This was done to include the non-linear propagation of end-point variability of the first link to the second link.

### 3-3-2 Comparison

The end-point variability of the minimum variability movement (i.e. the solution to Equation 3-3) is compared to two other optimal solutions of the pick-and-place task: the minimum-energy solution and the minimum-time solution (i.e. the solution to Equation 3-4 and to Equation 3-5). Both these optimization problems are solved with a multistart of the SQP algorithm in fmincon.

The minimum-energy movement is found by solving the following optimization problem:

$$\min_{T_{in}} \sum_{t=0}^{t_{end}} T_{in} \cdot R \cdot T_{in}^T, \quad \text{subject to } s_{end} - s^{*}_{end} = 0 \quad (3-4)$$

Where $T_{in}$ is again the input torque. $R$ is a weighting matrix, we set this as an identity matrix, so in the case with two motors both input signals are equally weighted. $s_{end}$ is the end state and $s^{*}_{end}$ is the desired end state at time $t = t_{end}$.

Minimizing this cost function leads to a good approximation of the minimum-energy movement. The main energy losses for a DC motor are losses in the windings of the motor, this
is minimized in the cost function above. However, losses due to friction are not taken into account. For reasons of simplicity we will call the minimization of this cost function the minimum-energy movement throughout this research. A typical input signal for a minimum-energy movement of a one DOF robotic arm can be seen in Figure 3-4.

Since the task defines an end-time of one second, it is difficult to optimize the movement time directly. Therefore, we optimize the difference between the movement and the desired end state for the entire movement. Going to the desired end-state quicker, also gives a lower error. Therefore this metric can be used as an approximation of the minimum-time movement. A typical input signal for a minimum-time movement of a one DOF robotic arm can be seen in Figure 3-4.

The minimum-time movement is found by solving the following optimization problem:

$$\min_{T_{in}} \sum_{t=0}^{t_{end}} e \cdot Q \cdot e^T, \text{ where } e = s_t - s_{end}^*, \text{ subject to } s_{end} - s_{end}^* = 0$$  \hspace{1cm} (3-5)$$

Where $s_t$ is the state at time $t$ (this is of course a function of the $T_{in}$), $s_{end}$ is the end state, and $s_{end}^*$ is the desired end state. $e$ is the difference between the actual state and the desired end state. $Q$ is a weighting matrix, we set this as a diagonal matrix with weight one on the position and weight zero on the velocity. Only positional errors are penalized, because the initial and final velocities are zero in our tasks. Penalizing also deviations in the velocity would mean that moving is penalized, but velocities should be high since we try to get to the end state as fast as possible.

### 3-4 Discussion

Having all the different movements for different motor noise properties and different systems, gives us a good idea of the influence of motor noise and the possibilities of optimizing towards end-point variability in these cases. However, the question that remains is if the optimized movements are robust for changes in the motor noise properties. This is investigated in the following chapters.

### 3-5 Conclusion

The task used in this research is a pick-and-place task executed by a robotic arm. The objective of this task is to reach the end-point in one second and with minimal end-point variability. The input is a piecewise constant input with 10 steps. In simulation, the influence of additive noise, multiplicative noise, and the combination of both noise sources is examined. These simulations use different robotic arm models, ranging from a one DOF robotic arm model with only viscous friction to a two DOF robotic arm model with a full friction model. The optimization problem of minimizing the end-point variability is solved with a multistart of the SQP algorithm in `fmincon`. The minimum-variability movement is compared to the variability of the minimum-energy movement and the minimum-time movement. The decision variable of the optimization is the torque level of the piecewise constant input and the final state is used as a constraint to the optimization problem.
Determining Variability

The variability needs to be determined, in order to find movements that have minimal endpoint variability and to evaluate the variability of other movements. There are various methods to determine the variability of a movement. This chapter explains two methods. Firstly, Section 4-1 introduces the most basic approach for determining the variability: the Monte Carlo variability determination. Secondly, the variability can also be determined in an algebraic way: the Extended Kalman Filter variability determination. This method is explained in Section 4-2. Section 4-3 discusses the performance of both methods by comparing them. The chapter ends with a discussion in Section 4-4 and a conclusion in Section 4-5.

4-1 Monte Carlo variability determination

The most basic variability determination is the Monte Carlo (MC) variability determination. Every input is randomly disturbed by the motor noise, which leads to a different outcome for every trial. Repeating the trial over and over again, gives a distribution in the movement data. The variability at every point in time is then determined by looking at that distribution. Figure 4-1 shows this principle.

Since the input signal is disturbed by Gaussian white noise, the output is expected to be Gaussian distributed too. The variability distribution is fitted with a Gaussian distribution, defined by its mean and standard deviation. Below, it will be shown that for a linear system the variability stays Gaussian for Gaussian motor noise. For non-linear system a Gaussian distribution is often a good estimate of the real distribution, if the non-linearities are not "too large."

The problem with Monte Carlo variability determination, is that it takes many trials before a good estimate of the real distribution can be made. An analogy can be made with rolling a dice. This example is also depicted in Figure 4-2. The chance that you will get any of the numbers on the dice is in theory the same. Therefore, the distribution is a flat one, with a chance of rolling any number of one sixth. However, if we roll the dice five times the distribution doesn’t look anything like the statistical distribution. Rolling for 1000 times,
Figure 4-1: Input and angle of different movements for Monte Carlo variability determination. In this figure we see on the left the disturbed input signal and on the right the corresponding movement. One movement is shown in (a), another movement is added in (b), and a third in (c) all differ from each other due to the statistical properties of the motor noise. In (d) we see the movements plotted for 50 disturbed input signals. At each point in time the position of all movements has a distribution with a certain mean and standard deviation.
4-2 Algebraic variability determination for linear systems

For linear systems with Gaussian motor noise, it is also possible to determine the variability by algebraic calculations. In Section 4-2-1 the mathematical rules of adding and transforming Gaussian distribution are described. In Section 4-2-2 these rules are used to determine the variability in a way similar to the Extended Kalman Filter.

4-2-1 Gaussian calculus

If two Gaussian distributions are independent, as is the case with Gaussian white noise, these distributions can be linearly transformed and added. A variable $X$ has a Gaussian distribution $\mathcal{N}$ which is defined by its mean, $\mu_x$, and variance, $\Sigma_x$:

$$X \sim \mathcal{N}(\mu_x, \Sigma_x)$$ (4-1)

The probability density function of a Gaussian distribution is depicted in Figure 4-3. Or in an equation the probability density function of a Gaussian distribution is:

$$\mathcal{N}(\mu_x, \Sigma_x) = \frac{1}{\sqrt{2\pi \Sigma_x}} \exp\left(-\frac{(x - \mu_x)^2}{2\Sigma_x}\right)$$ (4-2)

Linear transformation and addition of variables

Such Gaussian distribution can be linearly transformed and added. This is a useful operation as we will see later. Equation 4-3 shows how transformation and addition works and Equation...
Determined Variability

Figure 4-3: Probability density function of a Gaussian distribution. The distribution is symmetric around the mean, \( \mu \). 68.3 percent of the data falls between the mean minus one standard deviation, \( \sigma \) and the mean plus one standard deviation. 95.4 percent of the data falls between plus and minus two standard deviations from the mean. 99.74 percent of the data falls between plus and minus three standard deviations from the mean.

4-4 shows the special case where Gaussian distributed variables are just added without any transformation.

Suppose we have two independent variables which are Gaussian distributed: \( X \sim \mathcal{N}(\mu_x, \Sigma_x) \) and \( Y \sim \mathcal{N}(\mu_y, \Sigma_y) \). These variables are transformed by matrices \( A, B \in \mathbb{M}_{c \times d} \) giving \( Z \). Then

\[
Z = AX + BY
\]

\[
Z \sim \mathcal{N}(A\mu_x + B\mu_y, A\Sigma_x A^T + B\Sigma_y B^T)
\]

So \( Z \) is also Gaussian distributed.

Addition

Let \( X_i \sim \mathcal{N}(\mu_i, \Sigma_i), i \in 1, \ldots, N \) be independent variables. Then the sum of these distributions is again a Gaussian distribution, with the mean the sum of all means and the variance the sum of all variances.

\[
\sum_{i}^{N} X_i \sim \mathcal{N}(\sum_{i}^{N} \mu_i, \sum_{i}^{N} \Sigma_i)
\]

Note: This is a direct implication of equation 4-3 if \( A \) and \( B \) are identity matrices.

4-2-2 Extended Kalman Filter variability determination

The Extended Kalman Filter is used to estimate the states of a non-linear system with process and observation noise. The Extended Kalman Filter predicts the next state and its variability. These state estimates are updated using measurements. In the Extended Kalman Filter variability determination presented here the update step is not used, since in feedforward control no measurements are made.

A linear system can be written in state space form:

\[
\dot{x} = Ax + Bu
\]

J.H. Warnaars
Master of Science Thesis
where $\dot{x}$ is the derivative of the state vector $x$. $A$ is the system matrix, $B$ is the input matrix and $u$ is the input vector.

The desired input $u^*$ is disturbed by multiplicative input noise and additive input noise as was defined in Section 2-3. Let $m$ be the multiplicative noise, with a Gaussian distribution: $m \sim \mathcal{N}(\mu_m, \Sigma_m)$. Let $a$ be the additive noise, with a Gaussian distribution: $a \sim \mathcal{N}(\mu_a, \Sigma_a)$. Let $u$ be the actual input with noise.

Thus, the input is the desired input times the multiplicative noise plus the additive noise,

$$ u = u^* \cdot m + a \quad (4-6) $$

Since both $m$ and $a$ have a Gaussian distribution, following Equation 4-3 and Equation 4-4 we get,

$$ u \sim \mathcal{N}(u^* \mu_m + \mu_a, u^* \Sigma_m u^* + \Sigma_a) \quad (4-7) $$

Now we apply the Gaussian calculus to Equation 4-5 and we calculate the distribution of $\dot{x}$,

$$ \dot{x} = Ax + Bu \sim \mathcal{N}(A\mu_x + B\mu_u, A\Sigma_x A^T + B\Sigma_u B^T) \quad (4-8) $$

Or equivalently,

$$ \dot{x} = Ax + B(u^* m + a) \quad (4-9) $$

These equations of motion can than be solved using numerical integration. This leads to the new state at time $t + 1$. For the sake of simplicity only Euler’s method is shown here. For the simulations, the variability is calculated via the Euler method in discrete time, but the state is updated using a Runge-Kutta 4 integration step.

$$ \dot{x}_t = Ax_t + B(u^*_t \cdot m + a) \quad (4-10) $$

$$ x_{t+1} = x_t + \dot{x}_t $$

Since $\dot{x}_t$ has a certain distribution, at the following time instant the new state will also be distributed. Following Equation 4-4,

$$ x_{t+1} \sim \mathcal{N}(\mu_{x_{t+1}}, \Sigma_{x_{t+1}} + \Sigma_{\dot{x}_t}) \quad (4-11) $$

Solving the equations of motion in such a manner gives information about the statistical distribution of the state at any time instant during the movement. For multiplicative noise with a mean of 1 and additive noise with a mean of 0 on the input, the equations of motion lead to a state with the exact same mean as with an undisturbed input.

### 4-2-3 Algebraic variability determination for non-linear systems

In order to calculate the variability of non-linear systems, the above equations are not directly applicable. For non-linear systems we assume that the distribution has a mean that is the
same as the undisturbed movement. Additionally, we assume that the variability distribution remains Gaussian throughout the movement.

For non-linear systems the system matrix and the input matrix depend on the state and/or the input: \( A(x, u) \) and \( B(x, u) \). In order to use the EKF variability determination technique, the system has to be linearized at every time instant.

\[
A_{lin} = \left. \frac{\partial \dot{x}}{\partial x} \right|_{x, u} \\
B_{lin} = \left. \frac{\partial \dot{x}}{\partial u} \right|_{x, u}
\] (4-12)

Where \( A_{lin} \) is the linearized system matrix around the current state, and \( B_{lin} \) is the linearized input matrix around the current state.

The linearized system matrices can then be used for the calculation of the standard deviation of the state. Equation 4-9 is in this case transformed to Equation 4-14. The non-linear equation of motion are shown in Equation 4-13, these are used to update the mean of the state.

\[
\dot{x} = A(x, u)x + B(x, u) \cdot (u^* m + a),
\] (4-13)

Where \( A(x, u) \) is the non-linear state and input-dependent system matrix and \( B(x, u) \) is the non-linear state and input-dependent input matrix.

\[
\dot{x} \sim N(A(x)\mu_x + Bu^*, A_{lin} \Sigma_x A_{lin}^T + B_{lin} \cdot u^* \Sigma_m u^* B_{lin}^T)
\] (4-14)

### 4-3 Comparison between different methods

For a linear model of a one DOF robotic arm the MC variability determination and the EKF variability determination are similar. Small changes between both determination methods arise only from the fact that a finite number of trials is used for the MC variability determination. Figure 4-4 shows the probability density function of the raw data, the MC determination, and the EKF determination. It can be seen that there are small differences between the three. The difference between the MC determination and the EKF determination is maximally ten percent (see Table 4-1).

For a non-linear system with a full friction model, when the velocity goes to zero the distribution of the velocity is not Gaussian anymore. Due to the friction all small velocities will die out (if the input is lower than the friction force). Therefore the velocity distribution becomes skewed towards zero velocity with less values below zero velocity.

The Gaussian approximation of the Monte Carlo data does not behave well if the mean velocity approaches zero. Since the distribution of the velocity becomes more and more skewed to zero, the fitting of a Gaussian function on to this distribution has a poor result. The EKF variability determination tends to find a mean that approximates zero and fits nicely to the positive velocity values. However, it thus also determines negative velocity values, that are
Gaussian approximation of end-point variability of the position

![Histogram of end-point variability distribution](image1)

**Figure 4-4**: End-point variability for the linear model of the one DOF robotic arm. The probability density of 500 trials is plotted in the bars. The higher the bar the more likely the movement ended in this bar. The data does not represent a true Gaussian distribution, due to the finite number of trials. This is visible, for example, in the lower probability density of the bar at 1, compared to the bars next to it. The red dashed line is the outcome of the MC variability determination, in other words it is the Gaussian distribution fitted to the data. The fitted Gaussian distribution is an approximation of the real distribution. The green solid line shows the Gaussian distribution of the end-point variability found by the EKF variability determination. In this linear case the difference between the EKF variability determination and the MC variability determination is only due to the finite number of trials of the MC variability determination.

![Histogram of velocity variability distribution](image2)

**Figure 4-5**: Velocity variability distribution for the one DOF robotic arm with the full friction model when the mean velocity approaches zero. This figure shows the variability in velocity for a one DOF robotic arm with multiplicative and additive noise of level 0.1. At this point in the movement the mean velocity approaches zero. It is clear that both the EKF variability determination (green solid line) and the MC variability determination (red dashed line) do not represent the probability density function of the data correctly.
not present. An example of the velocity variability when the velocity becomes zero can be seen in Figure 4-5. It is clear that both the MC and EKF variability determination do not capture the variability distribution. This effect is dependent on the inertia of the arm, the Coulomb friction, and the noise type and level.

This effect is most visible in the movement of the second link of the two DOF robotic arm with the full friction model and high noise levels and additive noise. The torques and velocities are low in these movements. The high noise levels let some of the movements stop, therefore the variability in velocity is lower. This velocity variability is propagated to the position variability of the second link and to the first link, leading to lower end-point variability. This effect happens both at the beginning of the movement and at the end. Therefore, the movement still ends up in the desired end-state.

The EKF variability determination does not capture this effect and overestimates the variability in these movements. However, the overall variability in these movements is still larger than the other movements, so the overestimation is not viewed as a large problem in this research.

The difference in end-point variability between the determination methods for the different models can be seen in Table 4-1. The large difference for the two DOF robotic arm with the full friction model are only from the minimum-energy movements, the minimum-variability and minimum-time movements did not show larger differences than 14 percent.

4-4 Discussion

The EKF variability determination is based on the Extended Kalman Filter. Instead of the EKF also the Unscented Kalman Filter (UKF) could be used. This approach is based on the intuition that it is easier to approximate a Gaussian distribution, than it is to approximate a non-linear function. The UKF leads to more accurate results than the EKF (van der Merwe et al., 2001). Although the order of function evaluations in the UKF approach should be the same as in the EKF approach, the computationally complexity of the UKF approach can be
Table 4-1: Table of minimum and maximum difference between the EKF and the MC variability determination for different models. The one DOF robotic arm with viscous friction shows the difference between the EKF and the MC variability determination in a linear model. The differences in this model are completely due to the finite number of trials for the MC variability determination. For the one DOF robotic arm model with the full friction model and the two DOF robotic arm with viscous friction the variability stays below 10 percent, so both methods perform well. For the two DOF robotic arm with the full friction larger deviations are seen. This is due to the overestimation of the variability of minimum-energy movements with high noise levels and additive noise.

<table>
<thead>
<tr>
<th>Model</th>
<th>Difference EKF and MC variability determination (minimum)</th>
<th>Difference EKF and MC variability determination (maximum)</th>
</tr>
</thead>
<tbody>
<tr>
<td>One DOF viscous friction</td>
<td>-10 %</td>
<td>+10 %</td>
</tr>
<tr>
<td>One DOF full friction</td>
<td>-10 %</td>
<td>+5 %</td>
</tr>
<tr>
<td>Two DOF viscous friction</td>
<td>-8 %</td>
<td>+10 %</td>
</tr>
<tr>
<td>Two DOF full friction</td>
<td>-14 %</td>
<td>+40 %</td>
</tr>
</tbody>
</table>

bigger (Farina et al., 2002). In a optimization this added computationally complexity can lead to far longer optimization times. Furthermore, due to the linearization of the model the EKF variability determination gives more insight into why certain movements have minimum variability. Therefore, the EKF variability determination is used in this research.

4-5 Conclusion

The MC variability determination and the EKF variability determination methods can be used to investigate the variability of a certain movement. The Monte Carlo variability determination is computationally expensive. When the variability is optimized using this method, a long computation time is imminent. The EKF variability determination can compute the variability algebraically. Therefore, the EKF variability determination is used for the minimization of the end-point variability.

The EKF variability determination, determines the variability perfectly for linear systems. In non-linear systems the determination is not always correct. This happens, for a system with Coulomb friction, when the velocity goes to zero. However, the determination of the end-point variability stays close to the Monte Carlo determination of the end-point variability.
This chapter shows the results for minimizing the end-point variability for the one DOF robotic arm in simulation. The chapter consists of two main sections. First, Section 5-1 discusses the results for the one DOF robotic arm with only linear viscous friction. This model is linear, leading to easily interpretable results. Secondly, Section 5-2 discusses the results for the one DOF robotic arm with a non-linear friction model including viscous, Coulomb and torque-dependent friction. This model is a better approximation of the real robotic arm. This model gives more insight in what the minimum-variability movement for the real robotic arm is. A schematic drawing of the one DOF robotic arm is depicted in Figure 5-1. The models used for the robotic arms can be found in Appendix A.

5-1 One DOF robotic arm with viscous friction

First, we will look at the one DOF robotic arm with only viscous friction. Three different noise cases will be discussed: additive motor noise, multiplicative motor noise, and the combinations of both noise types.

The minimum-energy and minimum-time movements are the same for all noise conditions. The minimum-energy movement has relatively low torques and goes smoothly to the goal state; the minimum-time movement has relatively large torques and goes quickly to the goal state. These movements are depicted in Figure 5-2.

5-1-1 Additive noise

For additive noise the influence of input noise is not dependent on the movement. This can be seen from equation 5-1.

\[ \dot{x} = Ax + B(u^* + a) \sim \mathcal{N}(A\mu_x + Bu^*, A\Sigma_x A^T + B\Sigma_a B^T) \]  

So the variance is the final element in Equation 5-1

\[ \Sigma_\dot{x} = A\Sigma_x A^T + B\Sigma_a B^T \]  

Master of Science Thesis

J.H. Warnaars
Figure 5-1: Schematic drawing of the one DOF robotic arm. The robotic arm is controlled by the input torque $T_{in}$. The task is to go from A to B in 1 second. $\theta_0$ is the initial angle and $\theta_{end}$ is the desired final angle. The friction is applied on the joint: either viscous friction or the full friction model is applied.

Figure 5-2: Minimum-energy and minimum-time movement for the one DOF robotic arm with viscous friction. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-energy (red dashed lines) and the minimum-time movement (green dashed dotted lines). For the minimum-energy movement: the torques decrease steadily from almost 1 Nm to almost -1 Nm in one second, the accompanying momenta-profile is therefore smooth and parabolic, the angle increases also smoothly to the desired final angle in exactly one second. For the minimum-time movement: the torques are maximal (2 Nm and -2 Nm) at first followed by a phase with lower torques after 0.5 seconds, the accompanying momenta-profile shows a large peak followed by a settling phase, the angle goes rapidly to the desired final angle with a small overshoot and a settling phase after 0.6 seconds.
5-1 One DOF robotic arm with viscous friction

Figure 5-3: Minimum-variability movement for the one DOF robotic arm with viscous friction and additive noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-variability movement for additive noise. Since the variability is independent of the movement, the minimum-variability movement is just a movement that satisfies the constraints. The torque input is erratic and so is the momenta, but the final angle is 1 radian and the final momenta is zero.

Both the system matrix $A$ and the input matrix $B$ are constant for a linear system. Therefore, the variability does not depend on the input (no $u^*$ in Equation 5-2). The standard deviation will thus always increase with the same rate independent of the movement. Figure 5-4b shows this for the minimum-energy, minimum-time, and minimum-variability movement for an additive noise level of 0.1. The rate of increase of the variability depends on the size of the elements in matrices $A$ and $B$. Other system models can have a larger or smaller variability with additive input noise than the robotic arm model used in this research.

Figure 5-3 shows the minimum-variability movement for an additive noise level of 0.1. Since the variability cannot be influenced by the input signal, the optimization gives a result that only has to satisfy the constraints: the angle will end at the desired final state of 1 radian, the velocity ends at 0 radians per second.

Figure 5-4a shows the influence of the additive noise level on the variability. This figure shows that the size of the end-point variability scales linearly with the additive noise level.

Since the time-optimal movement stays at the final state after 0.9 seconds, one could argue that this is the point where the end-point variability should be measured. This would lead to the fact that the fastest movement is also the least variable movement. In this research the task is specified as going to the final state in one second, therefore the variability is measured after 1 second.

5-1-2 Multiplicative noise

For multiplicative noise the influence of motor noise does depend on the input and thus also on the movement. This can be seen from Equation 5-3.

$$\dot{x} = Ax + B(u^* \cdot m) \sim \mathcal{N}(A\mu_x + Bu^*\mu_m, A\Sigma_xA^T + Bu^*\Sigma_mu^{*T}B^T)$$ (5-3)
(a) End-point variability for different movements in a one DOF robotic arm with viscous friction and additive noise.

(b) Variability for different movements for the one DOF robotic arm with viscous friction and additive noise of level 0.1.

**Figure 5-4:** Variability for the one DOF robotic arm with viscous friction and additive noise. (a) shows the end-point variability for different additive noise levels for three different movements (minimum-energy, minimum-time, and minimum-variability). The end-point variability scales linearly with the additive noise level. Additionally, it is independent of the movement. (b) shows the variability through time for the three different movements, this also shows that the variability is independent of the movement. The minimum-energy and minimum-time movements are shown in Figure 5-2. The minimum-variability movement is shown in Figure 5-3.

**Figure 5-5:** Minimum-variability movement for the one DOF robotic arm with viscous friction and multiplicative noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-variability movement for multiplicative noise. The torque increases slowly from 0.40 Nm to 0.54 Nm in 0.5 seconds, after 0.6 seconds it decrease rapidly to -2 Nm at 0.9 seconds. This results in a increasing momenta up to 0.8 seconds and a rapid decreases to zero in the last 0.2 seconds. The angle is accelerating for the first 0.8 seconds, in the last 0.2 seconds it rapidly comes to a hold at the desired final angle.
So the variance is the final element in Equation 5-3:

\[ \Sigma_{\dot{x}} = A \Sigma_x A^T + B u^* \Sigma_m u^{*T} B^T \]  

(5-4)

Smaller input torques \( u^* \) will have less noise in absolute sense. Therefore, the variability of these movements increases slower compared to movements with larger input torques. The rate of this increase does also depend on the size of the elements in matrices \( A \) and \( B \) and the multiplicative noise level. The end-point variability scales linearly with the multiplicative noise level. This can be seen in Figure 5-6a. The end-point variability of the minimum-time movement is 4 times larger than the minimum end-point variability. The end-point variability of the minimum-energy movement is only 1.4 times larger than the minimum end-point variability.

The variability through time for an additive noise level of 0.1 can be seen in Figure 5-6b. The variability increases throughout the movement, even for the minimum-time movement that has zero torque after 0.9 seconds. This increase is due to the size of the elements in the system matrix \( A \) (squearing \( A \) will give a matrix with larger values than in \( A \). Even when the input is zero, due to matrix \( A \) any variability of the state at time \( t \) will lead to an increase in variability for time \( t+1 \).

The end-point variability of the minimum-time movement is the largest, since the input torques for this movement are larger than for the other movements. The input torques of the minimum-energy movement are lower compared to the minimum-time movement. Thus the end-point variability for the minimum-energy movement is lower. The minimum-variability movement uses the lowest torques to come to the end-state, however the end-point variability of the minimum-energy movement is still higher than the minimum-variability movement.

The torques of the minimum variability movement are lower in the first part of the movement compared to the minimum-time movement (see Figure 5-5 and compare with Figure 5-2). This leads to a lower increase in variability for the minimum-variability movement for the first part of the movement. Since the existing variability is always increased through \( A \Sigma_x A^T \) no matter what the torque is, the lower variability at the beginning of the movement will also lead to a lower variability for the rest of the movement.

5-1-3 Both noise types

For a system with full noise on the input signal the influence of motor noise is dependent on the movement. However, this dependency is only due to the multiplicative component of the motor noise. As was explained before, it is not possible to do anything about additive noise in the one DOF robotic arm with viscous friction. Therefore, minimizing the end-point variability of systems with both motor noise types will result in the same input as was found for minimization of the end-point variability in a system with only multiplicative noise. The minimum-variability movement for multiplicative and additive noise is shown in Figure 5-7. This movement is the same as the minimum-variability movement for only multiplicative noise (see Figure 5-5).

The size of the end-point variability does not increase linearly with the noise level of both noise types combined. This can be seen in Figure 5-8. The end-point variability scales as follows:
Figure 5-6: Variability for a one DOF robotic arm with viscous friction and multiplicative noise. (a) shows the end-point variability for different noise levels for three different movements (minimum-energy, minimum-time, and minimum-variability). The end-point variability scales linearly with the multiplicative noise level. The end-point variability for the minimum-energy movement is 1.4 times larger than the minimum end-point variability. The end-point variability of the minimum-time movement is 4 times larger than the minimum end-point variability. (b) shows the variability through time for the three different movements. The variability of the minimum-time movement increases rapidly at the beginning and keeps increasing. The variability of the minimum-energy movement is lower than of the minimum-time movement, but it stays above the variability of the minimum-variability movement. The variability of the minimum-variability movement increases slowly at the beginning of the movement. The variability increases more rapidly at the end of the movement due to the large torques, however the variability remains lower than for the other two movements. The minimum-energy and minimum-time movements are shown in Figure 5-2. The minimum-variability movement is shown in Figure 5-5.

Figure 5-7: Minimum-energy and minimum-time movement for the one DOF robotic arm with viscous friction and full noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-variability movement for multiplicative and additive noise of level 0.1. This movement is exactly the same as the minimum-variability movement for multiplicative noise only (see Figure 5-5).
Figure 5-8: End-point variability for different movements for the one DOF robotic arm with viscous friction and all noise combinations. The minimum-variability movement and the minimum-energy movement have almost similar end-point variability for all noise cases, but the minimum-variability movement does have a smaller variability for all cases. The influence of the multiplicative noise on the end-point variability for these two movements is smaller for increasing levels of additive noise. The variability of the minimum-time movement increases more with the level of multiplicative noise, but is also largely influenced by an increase in additive noise. NOTE: the scale of the z-axis in the right graph is different from the other graphs.

\[ \sigma_{\text{end,combined}} = \sqrt{\sigma_{\text{end,a}}^2 + \sigma_{\text{end,m}}^2}, \]  

(5-5)

where \( \sigma_{\text{end,a}} \) is the end-point variability with a specific level of additive noise only, \( \sigma_{\text{end,m}} \) is the end-point variability with with a specific level of multiplicative noise only, and \( \sigma_{\text{end,combined}} \) is the end-point variability when both noises are present at their respective level.

The minimum-variability movement and the minimum-energy movement suffer the most from additive noise. The end-point variability of the minimum-time movement is mainly influenced by the multiplicative noise.

**Robustness of the minimum-variable movements**

The movements found perform well in simulation, but the assumed noise levels could be different from the real noise levels. The minimum-variable movement for only multiplicative noise is the movement that has the minimum end-point variability for all combinations of additive and multiplicative noise. This movement is thus robust against differences in the assumed noise model and the actual noise model.
This section studies the influence of motor noise on the one DOF robotic arm with the full friction model. This model is similar to the one DOF robotic arm with viscous friction, but the friction model now includes the non-linear Coulomb friction and torque-dependent friction. The model is described in Appendix A. Three different noise cases will be discussed: additive input noise, multiplicative input noise, and the combinations of both noise types.

The minimum-energy and minimum-time movements are the same for all noise conditions. The minimum-energy movement has relatively low torques and goes smoothly to the goal state; the minimum-time movement has relatively large torques and goes quickly to the goal state. These movements are depicted in Figure 5-2.

Compared to the one DOF robotic arm with only viscous friction, the torques for both the minimum-time and minimum-energy movement are higher in an absolute sense. This is due to the Coulomb and torque-dependent friction. The Coulomb friction needs to be overcome when the velocity is zero (at the beginning of the movement), therefore higher torques are needed. The torque-dependent friction counters the input torque of course, also leading to higher torques to arrive at the goal state.

### 5-2-1 Additive noise

The variability due to additive motor noise depends on the movement in the one DOF robotic arm with full friction in contrast to the one DOF robotic arm with viscous friction only. The
Figure 5-10: Minimum-variability movement for the one DOF robotic arm with the full friction model and additive noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-variability movement for additive noise. The torque shows two positive and negative phases. The accompanying momenta profile shows two positive peaks and one negative peak, these resemble the three staged movement. The angle also shows this three staged movement, first the angle is increasing, then decreasing, and increasing again afterwards. This multistage behavior leads to minimum-variability for additive noise.

(a) End-point variability for different movements in (b) Variability for different movements for the one DOF robotic arm with full friction and additive input noise.

Figure 5-11: Variability for the one DOF robotic arm with the full friction model and additive noise. (a) shows the end-point variability for different additive noise levels for three different movements (minimum-energy, minimum-time, and minimum-variability). The end-point variability scales linearly with the additive noise level. (b) shows the variability through time for the three different movements, this also shows that the variability is dependent on the movement. The minimum-time movement shows a buckle at 0.65 seconds and at 0.85 seconds. The minimum-variability movement shows buckles at 0.3 seconds and at 0.4 seconds. The minimum-energy and minimum-time movements are shown in Figure 5-9. The minimum-variability movement is shown in Figure 5-10.
variability due to the additive noise is for all three optimal movements smaller than in the case with only viscous friction.

Due to the Coulomb friction, the variability stays constant if the velocity is zero. The Coulomb friction needs to be overcome before a movement is made. Therefore any noise that is not high enough to overcome this friction, will have no influence on the system. Therefore, the variability is also not increased in this situation. This can be seen for the minimum-time movement in Figure 5-11. When the velocity is zero the variability does not increase.

The minimum-variability movement is very different than the minimum-variability movement for the one DOF robotic arm with viscous friction. The movement has three major stages for which the velocity increases first and then decreases again until the system comes to a hold. The increase of the variability in the moving phases is not linear and increases more rapidly the longer the arm is moving. This can be seen in the variability for the minimum-variability movement in Figure 5-11. If it makes a small stop in between, the influence of the motor noise is set to zero again, and only builds up when the system starts moving again. Therefore, the minimum-time movement with two movement phase has a larger end-point variability than the minimum-variability movement with three movement phases. Since, the energy-optimal movement is moving for the largest part of the movement, the end-point variability of this movement is the largest.

The variability scales linearly with the additive noise level (see Figure 5-11a). This also holds for the minimum-variability movement, while the minimum-variability movement is the same for all additive noise levels. The minimum-energy movement has the highest end-point variability: 1.9 times higher than the minimum end-point variability. The minimum-time movement has a lower end-point variability: 1.3 times higher than the minimum end-point variability.

5-2-2 Multiplicative noise

For multiplicative noise the end-point variability scales linearly with the multiplicative noise level. This is also true for the minimum-variability movement, while the minimum-variability movement is independent of the noise level. The minimum-time movement has the highest end-point variability, with an end-point variability that is 1.9 times higher than the minimum end-point variability, the minimum-energy movement has an end-point variability that is 1.3 times higher than the minimum end-point variability. This is shown in Figure 5-13b.

Figure 5-12 shows that the minimum-variability movement for multiplicative noise does not show a multistaged movement, as we have seen for the case with only additive input noise (see Section 5-2-1). Large inputs are more noisy for multiplicative input noise, thus smaller inputs are favorable. A multistaged movement needs higher inputs to overcome the friction and then to stop again. Despite the slower increase in variability after a moving stage, the end-point variability is still larger due to the higher torques. The minimum-variability movement is thus very similar to the minimum-variability movement for the one DOF robotic arm with viscous friction. The torques are slightly higher, due to the Coulomb and the torque-dependent friction.
5-2 One DOF robotic arm with the full friction model

Figure 5-12: Minimum-variability movement for the one DOF robotic arm with the full friction model and multiplicative noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-variability movement for multiplicative noise. The movement is very similar to the minimum-variability movement for the one DOF robotic arm with viscous friction and multiplicative noise. The torque slowly increases from 0.55 Nm to 0.98 Nm in 0.7 seconds, after 0.7 seconds it decrease rapidly to -2 Nm at 0.9 seconds. This results in a increasing momenta up to 0.8 seconds and a rapid decreases to zero in the last 0.2 seconds. The angle accelerates for the first 0.8 seconds, in the last 0.2 seconds it rapidly comes to a hold at the desired final angle.

5-2-3 Both noise types

The minimum-variability movement is a two staged movement for the multiplicative and additive noise level of 0.1 (see Figure 5-14). The second movement stage shows a movement that is close to the minimum-variability movement for only multiplicative noise. The multistaged movement was already seen in the case with only additive noise. The movement is a compromise between minimizing the variability due to the additive and the multiplicative noise.

The minimum-variability movement is dependent on the levels of additive and multiplicative noise. This is part of the reason for the wobbly plane for the minimum end-point variability shown in Figure 5-15.\(^1\) A large influence of one of the noise types, makes that the movement resembles more the case for when only this noise type is present.

The minimum-energy and minimum-time movement show a end-point variability that scales with the same function as for the one DOF robotic arm with only viscous friction (see Equation 5-5).

Robustness of the minimum-variable movements

The minimum-variability movements perform well in simulation, but in real life the estimated noise levels could be different from the real noise levels. Different movements have been shown for the case with only additive noise and the case with only multiplicative noise. The more

\(^1\)The other reason is local minima, due to the limited amount of multistarts the global minimum was not always found.
(a) End-point variability for different movements in the one DOF robotic arm with the full friction model and multiplicative noise. (b) Variability for different movements for the one DOF robotic arm with full friction and multiplicative input noise.

**Figure 5-13:** Variability for the one DOF robotic arm with viscous friction and multiplicative noise. (a) shows the end-point variability for different noise levels for three different movements (minimum-energy, minimum-time, and minimum-variability). The end-point variability scales linearly with the multiplicative noise level. The end-point variability for the minimum-energy movement is 1.3 times larger than the minimum end-point variability. The end-point variability of the minimum-time movement is 1.9 times larger than the minimum end-point variability (b) shows the variability through time for the three different movements with a multiplicative noise level of 0.1. The variability of the minimum-time movement increases rapidly at the beginning, it increases slower after the overshoot and stays constant when the angle stays constant. The variability of the minimum-energy movement is lower than of the minimum-time movement, since the torques are also lower than the minimum-time movement. The variability of the minimum-variability movement increases the slowest of all three movements, since the torques are also the lowest. The variability increases more rapidly at the end of the movement due to the large torques, however the variability remains lower than for the other two movement. The minimum-energy and minimum-time movements are shown in Figure 5-9. The minimum-variability movement is shown in Figure 5-12.
Figure 5-14: Minimum-variability movement for the one DOF robotic arm with the full friction model and full noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-variability movement for multiplicative and additive noise of level 0.1. The momenta profile shows two peaks and thus two moving phases. This can also be seen from the angle; the angle stays constant at 0.1 radians between 0.2 and 0.3 seconds. The torque levels are slightly lower than we have seen for the minimum-variability movement with only additive noise. The second movement phase shows a torque and momenta profile that looks similar to the profiles seen for the minimum-variability movement with only multiplicative noise (longer acceleration phase, than the deceleration phase).

Figure 5-15: End-point variability for different movements for the one DOF robotic arm with the full friction model and full noise. The minimum-variability movement has a lower end-point variability than the minimum-energy movement when the multiplicative noise levels are higher than the additive noise levels. The minimum-variability movement shows a non-smooth plane, because the movement is a compromise of minimizing the variability due to additive or multiplicative noise.
<table>
<thead>
<tr>
<th>$T_{nr}$</th>
<th>$\sigma_{end}$</th>
</tr>
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<tbody>
<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>0.009709</td>
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<tr>
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Table 5-1: Table of mean minimum end-point variability for different numbers of input steps. The end-point variability $\sigma_{end}$ is estimated for additive and multiplicative input noise of level 0.1. The end-point variability is determined for five to twenty torque steps ($T_{nr}$). A minimum end-point variability is found for 12 torque steps. The increase in torque steps leads to larger freedom in the optimization. Therefore movements with slightly lower end-point variability can be found. After 12 torque steps the variability increases again, but not gradually. This is the result of the increase in local minima for more torque steps.

5-3 Input parameterization

Taking a different input signal changes the variability, but also a different parameterization of the input signal results in different variability. The input in the simulations discussed above is parameterized as a piecewise constant torque signal with 10 steps. Taking more steps in the same time, will result in a lower minimum end-point variability. The increase of steps allows for more movement stages, thus lowering the variability. However, a minimum is found for 12 steps (see Table 5-1).

The optimum, depends on the constraints of the task and on the system. A longer task time would result in more multistaged behavior. Every movement phase the non-linear friction has to be overcome. Due to the piecewise constant input, for the rest of the step the system will be moving with constant acceleration.
The smallest possible end-point variability would be reached, where the friction force would just be overcome and then the velocity becomes zero again, for such a small time that the motor noise would have no effect anymore on the system. This would be repeated until the final state is reached.

The increasing freedom in the optimization with an increasing number of input steps gives problems with local minima. For more input steps, more local minima exist. In the simulation the number of multistarts remained the same, therefore the chance of ending up in a local minimum becomes bigger with more input steps. Increasing the number of multistarts was not an option, due to the increase in computation time.

The choice is made for input signals with 10 steps for computational reasons. The movements for a 10 step input are similar to the movements of a 12 step input. Therefore, the analysis on one of these input signals will not give different conclusions.

5-4 Discussion

In this section, the results of the one DOF robotic arm with viscous friction will be discussed first. Secondly, the results for the one DOF robotic arm with the full noise model are discussed. The minimum-variability movements found in this research are different than the results found by Harris & Wolpert 1998 for human arm movements. These differences will be discussed last.

In the one DOF robotic arm with viscous friction with additive noise, the end-point variability does not depend on the movement. For inputs disturbed by multiplicative noise, the minimum-variability movement is a movement with small increasing inputs and a final larger negative input. This is also the minimum-variable movement for all cases with both additive and multiplicative noise.

The minimum-energy movement, performs second best, with end-point variabilities that are 1.4 times higher than the minimum-variability in the case of only multiplicative noise. When both noise sources are combined it has a variability that is close to the minimum-variability. The minimum-time movement performs the worst, with end-point variability that can be 4 times larger than the minimum-variability.

For the one DOF robotic arm with viscous, Coulomb and torque-dependent friction, the variability always depends on the movement. The minimum-variability movement is dependent on the level of additive and multiplicative noise. For the additive noise the minimum-variability movement is multistaged (the velocity becomes zero multiple times throughout the movement). For multiplicative noise the minimum-variability movement has only one movement stage. This is due to the inertia of the arm: large torques are needed to accelerate the arm and decelerate it. These large torques contribute to large variability in a system with multiplicative noise. Therefore the minimum-variability movement does not stop.

The minimum-energy movement performs better than the minimum-time movement in the case of multiplicative noise. In the case of additive noise it is the other way around: the minimum-time movement has a lower end-point variability than the minimum-energy movement. This is not surprising, while the higher torques in the minimum-time movement lead to larger noise with multiplicative noise. With additive noise the noise is independent of the
torque, but due to the Coulomb friction the variability does not increase when the velocity becomes zero. In Appendix D, it is shown that for a robotic arm with lower inertia the minimum-variability movement for multiplicative noise is also multistaged.

The difference between end-point variability for the model with only viscous friction and the full noise model is large. In both cases one thing is clear, that the minimum-variable movement is different from both the minimum-energy movement and the minimum-time movement.

The results shown above are different from the results shown for human arm movements by Harris & Wolpert 1998. Harris & Wolpert 1998 stated that for human arm movements with multiplicative noise the minimal-variable movement is a smooth (bell shaped velocity) and energy-efficient movement. This was not the case for the models used in this research.

The difference lies mostly in the friction of a human arm. The human arm has very low viscous friction coefficients and Coulomb friction coefficients, respectively 0.003 and 0.01 (Serway et al., 2003). The Coulomb friction is thus easily overcome and has less of a dampening effect on the variability. Additionally, Harris minimized the end-point variability for the velocity and not for the position as is done in this research.

The method of Harris & Wolpert 1998 is used on the robotic arm model with human like friction coefficients (see Appendix C). The minimum-variability movement lies very close to the minimum-energy movement, however the friction does have a small impact. The minimum-variability movement has a slightly longer acceleration phase and smaller deceleration phase, compared to the minimum-energy movement. This leads to lower end-point variability for the velocity, but the end-point variability for the position becomes larger.

Opposite to human arms, robotic arms have far larger Coulomb friction coefficients. Minimizing the end-point variability for robotic systems, does not lead to smooth and energy efficient movements. It leads to multistage movements, that use the Coulomb friction to keep the variability to a minimum.

5-5 Conclusion

For one DOF robotic arms with viscous friction, the end-point variability only depends on the movement for systems with multiplicative noise. Additive noise will always cause variability in the end-point with the magnitude depending on the time of the movement and the system properties. For all noise combinations, there is one and the same minimum-variability movement.

For one DOF robotic arms with the full friction model, the end-point variability always depends on the movement. For additive input noise the minimum-time movement has a lower end-point variability than the minimum-energy movement. For multiplicative noise it is the other way around. The minimum-variability movement with additive noise shows a multistage movement with a momenta profile with three spikes. In the case with multiplicative noise, the minimum-variability movement had only one stage with a longer acceleration phase than a deceleration phase.

When both noise types are combined, the minimum-variability movement is a combination of the minimum-variability movements found for the different noise types alone. The exact
movement depends on the level of both noise types. Therefore, the movements found are not completely robust against deviations from the noise model.
Chapter 6

Two DOF Robotic Arm

This chapter shows the results for minimizing the end-point variability for a two DOF robotic arm in simulation. The two DOF robotic arm is interesting since it adds more complexity to the system. Additionally, it has a closer resemblance to robotic arms that are used in factories. The chapter consists of two main sections. First, Section 6-1 discusses the results for a two DOF robotic arm with only viscous friction. This model is the extension to the one DOF robotic arm model with viscous friction and it gives insight in how the added degree of freedom influences the variability. Secondly, Section 6-2 discusses the results for a two DOF robotic arm with a non-linear friction model including viscous, Coulomb and torque-dependent friction. This model is a better approximation of a real robotic arm and the results give insight in what the minimum-variability movements for the real robotic arm are. A schematic drawing of the two DOF robotic arm is depicted in Figure 6-1. The models used for the robotic arms can be found in Appendix A.

6-1 Two DOF robotic arm with viscous friction

First, we will look at the two DOF robotic arm with only viscous friction. Three different noise cases will be discussed: additive input noise, multiplicative input noise, and the combination of both noise types.

The minimum-energy and minimum-time movements are the same for all noise conditions. The minimum-energy movement has relatively low torques and goes smoothly to the goal state; the minimum-time movement has relatively large torques and goes quickly to the goal state. These movements are depicted in Figure 6-2 and Figure 6-3.

6-1-1 Additive noise

This section shows the end-point variability of the two DOF robotic arm with viscous friction that is disturbed by additive motor noise. First, a comparison is made with the one DOF
Figure 6-1: Schematic drawing of the two DOF robotic arm. The robotic arm is controlled by the input torques for both joints, $T_{in1}$ and $T_{in2}$. The task is to go from A to B in 1 second. $\theta_{10}$ and $\theta_{20}$ are the initial joint angles and $\theta_{1_{end}}$ and $\theta_{2_{end}}$ are the desired final joint angles. The friction is applied on the joint; either viscous friction or the full friction model is applied.

Figure 6-2: Minimum-energy movement for the two DOF robotic arm with viscous friction. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (red solid lines) and for the second link (dark red dashed lines). The input torques for both links show an evenly decreasing trend, with the accompanying parabolic momenta profiles. The angle of the first link goes smoothly to the desired final state, the angle of the second link stays close to zero and shows a small overshoot in the first 0.5 seconds and a small undershoot for the last 0.5 seconds.
Figure 6-3: Minimum-time movement for the two DOF robotic arm with viscous friction. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (green solid lines) and for the second link (dark green dashed dotted lines). The input torques are positive and near maximal for the first 0.3 seconds, than negative and near minimal for the next 0.3 seconds, the movement ends with a settling phase. The momenta profile for both links is similar, increasing for the first 0.3 seconds, than decreasing for the next 0.3 seconds, and settling around the desired final momenta for the last part of the movement. The angle of the first link goes quickly to the desired final state with a small overshoot. The angle of the second link deviates from zero for the first part of the movement and than goes back to zero again with a small overshoot.

Figure 6-4: Minimum-variability movement for the two DOF robotic arm with viscous friction and additive noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (blue solid lines) and for the second link (dark blue dotted lines). The torque inputs are negative for the first 0.2 seconds, followed by positive and near maximum torques between 0.4 and 0.7 seconds, the input torques are than reversed and stay near minimum for the rest of the movement. The angle of the second joint becomes first very negative, despite the negative torques on the first joint this joint moves in the positive direction. Than the angle of the first joint decreases again after 0.25 seconds. It increases rapidly for the rest of the time and comes to a quick halt at the desired final state. In the same time the angle of the second joint goes back to zero. The momenta are decreasing for the negative torques and increasing for the positive torques. The momenta profiles of both links lie close together and have a zig-zag shape.
robotic arm with viscous friction. Secondly, the differences in end-point variability between the different movements are presented. Lastly, these differences are explained.

For the two DOF robotic arm with viscous friction the end-point variability does depend on the movement. Contrastingly, the end-point variability is independent of the movement in the one DOF robotic arm with viscous friction with additive noise. In the one DOF robotic arm the variability only depends on the movement-time and the system. Since the two DOF system is a non-linear system, the variability depends also on the state of the system. Figure 6-5 shows the variability for the minimum-energy, minimum-time and minimum-variability movement with an additive noise level of 0.1.

The end-point variability scales linearly with the additive noise level (see Figure 6-6). The end-point variability for the minimum-energy movement and the minimum-time movement are almost the same. The minimum-energy movement has a slightly larger end-point variability for the first link and a slightly lower end-point variability for the second link compared to the minimum-time movement. The differences is only 0.3 percent for the first link and 0.6 percent for the second link. The minimum-variability movement is independent of the noise level. The minimum-variability movement has an end-point variability that is 20 percent smaller for the first link and 23 percent smaller for the second link.

The variability for the minimum-variability only deviates largely from the other two movement in the last 0.3 seconds (see Figure 6-5). At 0.7 seconds there is a small period of time that both links have the same speed and essentially move as a rigid arm. In the minimum-energy and minimum-time movement there is no point when both joints have the same velocity. It is thus hypothesized that when the links are moving with the same velocity, the variability increases less.
Figure 6-6: End-point variability for different movements of the two DOF robotic arm with viscous friction and additive motor noise. On the left side, the end-point variability for the first link is shown for different additive noise levels, on the right side the same is shown for the second link. The three movements depicted here are the minimum-energy movement (red and dark red dashed lines with the round markers), the minimum-time movement (green and dark green dashed dotted lines with the star markers), and the minimum-variability movement (blue and dark blue solid lines with the plus sign markers). The end-point variability scales linearly with the additive noise level for both links and all movements. The end-point variability for the second link is larger than for the first link for all movements.

Figure 6-7: Minimum-variability movement for the two DOF robotic arm with viscous friction and multiplicative noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (blue solid lines) and for the second link (dark blue dotted lines). For both links the torques are positive and below 0.5 Nm up till 0.8 seconds, then the torque decreases by steps of more than 0.5 Nm. The accompanying momenta profile shows a long increasing phase and a rapid decreasing phase for the last 0.2 seconds. The angle of the first link accelerates for the first 0.8 seconds, in the last 0.2 seconds it rapidly comes to a hold at the desired final angle. The angle of the second link decreases only slightly during the first half of the movement, in the last half it goes back to zero radians.
6-1-2 Multiplicative noise

This section shows the end-point variability of the two DOF robotic arm disturbed by multiplicative input noise. First, a comparison is made with the one DOF robotic arm with viscous friction. Secondly, the differences between the different movements are presented. Lastly, these differences are explained.

For the two DOF robotic arm with viscous friction the minimum-variability movement is very similar to the minimum-variability movement for the one DOF robotic arm with multiplicative noise. The torques are positive for the first part of the movement, the last part of the movement the torques are negative and larger than in the first part. Figure 6-7 shows this minimum-variability movement.

Figure 6-8 shows the variability for the minimum-energy, minimum-time and minimum-variability movement with a multiplicative noise level of 0.1. The same trend is seen here as in the one DOF robotic arm with viscous friction and multiplicative noise. High input torques have high noise, leading to large variability. The inputs of the minimum-energy movement are mostly larger than of the minimum-time movement, therefore the variability is also larger. The torques for the minimum-variability movement are larger in absolute sense than the minimum-energy movement. In contrast with the one DOF robotic arm with viscous friction, there is no larger increase in variability for the last part of the movement. In these states the non-linear system and input matrices have a dampening effect on the variability, this was also seen for the minimum-variability movement with additive noise (see Section 6-1-1).

The end-point variability scales linearly with the multiplicative noise level (see Figure 6-9). The end-point variability for the minimum-time movement is the largest, due to the large input torques required for this movement: it is 4.8 times larger for the first link and 5.3 times larger for the second link compared to the minimum end-point variability. The minimum-energy movement has an end-point variability that is only 1.4 times larger for the first link and 1.5 times larger for the second link compared to the minimum end-point variability. The end-point variability scales also linearly for the minimum-variability movement, this indicates that the minimum-variability movement is independent of the level of multiplicative noise.

6-1-3 Both noise types

This section shows the end-point variability of the two DOF robotic arm with viscous friction disturbed by multiplicative and additive input noise. First, the minimum-variability movement is investigated for when both additive and multiplicative noise at a level of 0.1 are present. Secondly, the end-point variability for different combinations of the noise types is examined. Lastly, the minimum-variability movements found under one condition but applied to another condition are investigated for robustness.

Figure 6-10 shows the minimum-variability movement for the additive and multiplicative noise of level 0.1. This movement is similar to the minimum-variability movement with only multiplicative noise (see Figure 6-7). The difference is best seen in the angle of the second link: it does not only deviate from zero in the negative direction, but it also deviates from zero in the positive direction. This makes the influence of the additive noise on the variability smaller.
Figure 6-8: Variability for different movements of the two DOF robotic arm with viscous friction and multiplicative noise. On the left side, the variability for the first link is shown for multiplicative noise of level 0.1, on the right side the same is shown for the second link. The three movements depicted here are the minimum-energy movement (red and dark red dashed lines with the round markers), the minimum-time movement (green and dark green dashed dotted lines with the star markers), and the minimum-variability movement (blue and dark blue solid lines with the plus sign markers). The results for both links are the same. The variability of the minimum-energy movement is lower than of the minimum-time movement, but it stays above the variability of the minimum-variability movement. The variability of the minimum-variability movement increases less throughout the movement. The variability does not increase more at the end of the movement, despite the higher torques for the minimum-variability movement compared to the minimum-energy movement. The minimum-energy and minimum-time movements are shown respectively in Figure 6-2 and in Figure 6-3. The minimum-variability movement is shown in Figure 6-7.

Figure 6-9: End-point variability for different movements of the two DOF robotic arm with viscous friction and multiplicative noise. This figure shows the end-point variability for different noise levels for three different movements: minimum-energy (red and dark red dashed lines with round markers), minimum-time (green and dark green dashed dotted lines with star markers), and minimum-variability movement (blue and dark blue solid lines with plus sign markers). On the left side the end-point variability against the noise level for the first link is given and on the right side for the second link. The end-point variability scales linearly with the multiplicative noise level. The end-point variability for the minimum-energy movement is more than 1.4 times larger than the minimum end-point variability for both links. The end-point variability of the minimum-time movement is more than 4.5 times larger than the minimum end-point variability for both links.
Figure 6-10: Minimum-variability movement for the two DOF robotic arm with viscous friction and full noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (blue solid lines) and for the second link (dark blue dotted lines) of the minimum-variability movement for multiplicative and additive noise of level 0.1. This movement looks very similar to the minimum-variability movement for multiplicative noise only (see Figure 6-7). The difference is that for this movement the positive torques increase over a longer period of time and that the negative torques for the second link deviate more from the torques of the first link. The momenta profile of the second link is similar to the momenta profile of the first link. The difference is most clearly seen in the angle of the second link. The angle of the second link decreases first, then it increases to positive values, and finally it goes to zero.

The size of the end-point variability does not increase linearly with both noise types. This can be seen in Figure 6-11. The end-point variability for the minimum-time and minimum-energy movement for both links scales by the same relation as was shown for the one DOF robotic arm with viscous friction (see Equation 5-5). The combined end-point variability is the square root of the sum of the squares of the end-point variability for both noise sources alone. The minimum-energy movement has a lower end-point variability than the minimum-time movement for all cases with multiplicative noise.

Figure 5-8 shows that for the minimum-variability movement the end-point variability does not scale following Equation 5-5. This indicates that for different combinations of the noise types, a different minimum-variability movement exists. Already with small multiplicative noise, the minimum-variability movement will look like the minimum-variability movement with multiplicative noise.

Robustness of the minimum-variability movements

The minimum-variability movement is mostly determined by the multiplicative noise. This means that a minimum-variability movements for a combination with multiplicative noise, will show low end-point variability for all other combinations with multiplicative noise. In the case with only additive input noise (or where additive input noise is the main noise source) these movements will perform worse.
Figure 6-11: End-point variability for different movements for the two DOF robotic arm with viscous friction and all noise level combinations. This figure shows the end-point variability for different combinations of the level of multiplicative and additive noise. The top graphs show the end-point variability for the first link, the bottom graphs show this for the second link. From left to right the graphs show the end-point variability for the minimum-variability movement, the minimum-energy movement, and the minimum-time movement. The end-point variability for the minimum-energy and minimum-time movement scales as in Equation 5-5. The end-point variability of the minimum-energy movement is lower for all cases with multiplicative noise. The end-point variability of the minimum-variability movement does not scale with the same relation, since different minimum-variability movements are found for the different noise level combinations. This can best be seen from the increase of the end-point variability in the left corner of the lower right graph.

Figure 6-11: End-point variability for different movements for the two DOF robotic arm with viscous friction and all noise level combinations. This figure shows the end-point variability for different combinations of the level of multiplicative and additive noise. The top graphs show the end-point variability for the first link, the bottom graphs show this for the second link. From left to right the graphs show the end-point variability for the minimum-variability movement, the minimum-energy movement, and the minimum-time movement. The end-point variability for the minimum-energy and minimum-time movement scales as in Equation 5-5. The end-point variability of the minimum-energy movement is lower for all cases with multiplicative noise. The end-point variability of the minimum-variability movement does not scale with the same relation, since different minimum-variability movements are found for the different noise level combinations. This can best be seen from the increase of the end-point variability in the left corner of the lower right graph.
Two DOF Robotic Arm

6-2 Two DOF robotic arm with the full friction model

In this section the two DOF robotic arm with the full friction model is studied. This section is subdivided for the three different noise cases. First, Section 6-2-1 studies the variability due to additive input noise. Secondly, Section 6-2-2 studies the variability due to multiplicative input noise. Lastly, Section 6-2-3 studies the variability for the combination of both noise types.

The minimum-energy and minimum-time movements are the same for all noise conditions. The minimum-energy movement has relatively low torques and goes smoothly to the goal state; the minimum-time movement has relatively large torques and goes quickly to the goal state. The minimum-energy movement has larger torques for the first link compared to the two DOF robotic arm with only viscous friction. Additionally, the minimum-energy movement has a longer acceleration phase and a shorter deceleration phase compared to the case with only viscous friction. The minimum-time movement has less maximum-torques compared to the case with only viscous friction. The minimum-energy and minimum-time movements are depicted in Figure 6-12 and Figure 6-13.

6-2-1 Additive noise

This section shows the end-point variability of the two DOF robotic arm with a full friction model and additive motor noise. First, the characteristics of the minimum-variability movement are shown and an explanation is given for why the end-point variability is minimal in this movement. Secondly, the variability of the minimum-energy and minimum-time movements
**Figure 6-13:** Minimum-time movement for the two DOF robotic arm with the full friction model. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (green solid lines) and for the second link (dark green dashed dotted lines). The input torques for the first link are positive and maximal for the first 0.3 seconds, than negative for the next 0.4 seconds, followed by a settling phase with smaller torques with alternating signs. The same trend is seen for the torques of the second link, although it first starts with a near maximal negative torque. The accompanying momenta profiles show a large peak at the beginning of the movement, followed by a settling phase with smaller alternating momenta after 0.6 seconds. The angle of the first link goes quickly to the desired final state with a small overshoot. The angle of the second link deviates negatively from zero for the first part of the movement and then it goes back to zero again with a small overshoot.

**Figure 6-14:** Minimum-variability movement for the two DOF robotic arm with the full friction model and additive noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (blue solid lines) and for the second link (dark blue dotted lines). The torque inputs are alternating between positive and negative values for the first link, for the second link the torque steps are alternating too but less frequent. The accompanying momenta profiles show the same erratic behavior. For the first link the momenta shows a lot of spikes, the momenta of the second link follow more gradually. The angle of the first link shows five movement stages, where the arm moves than stops for a short period of time. The angle of the second link shows four movement stages.
are explained. Thirdly, the influence of different noise levels on the end-point variability is investigated.

The minimum-variability movement is a multistaged movement for the two DOF robotic arm with a full friction model and additive noise. This movement is shown in Figure 6-14. The multistaged movement was also the minimum-variability movement for the one DOF robotic arm with a full friction model. When the movement comes to a stop the variability does not increase anymore due to the Coulomb friction. Furthermore, the variability of the second link is also propagated to the first link. Therefore, if the second joint is not moving relative to the first joint, the variability increases slower.

The propagation of variability is best seen in the minimum-energy movement. Figure 6-15 shows the variability through time for the minimum-energy, minimum-time and minimum-variability movement with an additive noise level of 0.1. Once the second link stops moving relative to the first joint the variability of the first link shows a buckle. This buckle can also be seen for the other movements.

The minimum-energy movement has the largest end-point variability, since the first link is moving constantly and the second link has only two movement stages. The minimum-time movement has the second largest end-point variability. The second link has three movement phases and the it reaches the desired goal state the fastest, so the variability does not increase anymore when the both links have reached the desired goal state.

The end-point variability scales linearly with the additive noise level for all three movements. This indicates that the minimum-variability movement is independent of the noise level. The end-point variability for both links of the minimum-energy movement is approximately 2.5 times larger than the minimum end-point variability. For the minimum-time movement, the end-point variability is approximately 1.5 times larger than for the minimum-variability movement for both links.

6-2-2 Multiplicative noise

This section shows the end-point variability of the two DOF robotic arm with the full friction model and multiplicative motor noise. First, the characteristics of the minimum-variability movement are shown and an explanation is given for why the end-point variability is minimal in this movement. Secondly, the variability of the minimum-energy and minimum-time movements are explained. Thirdly, the influence of different noise levels on the end-point variability is investigated.

The minimum-variability movement shows a smooth movement of the first link and a multistaged movement of the second link (see Figure 6-17). The minimum-variability movement makes use of the decreasing noise with decreasing torque. Additionally, it uses the fact that when one link is not moving the variability is not propagated to the other link. Most of the time only one of the links moves, while the other keeps a constant angle. Therefore, the variability in both links increases more slowly, than when both links are moving together (see also Figure 6-18).

Figure 6-18 shows the variability through time for the three movements for multiplicative noise of level 0.1. The minimum-time movement has the highest variability. This is due to the large torques required for this movement. The variability is however closer to the variability of the
Figure 6-15: Variability for different movements of the two DOF robotic arm with the full friction model and additive input noise of level 0.1. On the left side, the variability for the first link is depicted, on the right side the same is shown for the second link. The three movements depicted here are the minimum-energy movement (red and dark red dashed lines), the minimum-time movement (green and dark green dashed dotted lines), and the minimum-variability movement (blue and dark blue solid lines). The variability of the links is propagated to each other, this can be seen for all movements: when the variability in one link stays constant, the variability of the other link shows a buckle. The variability of the minimum-variability movement shows most of these buckles and therefore the variability is the smallest. The variability for the minimum-time movement shows two buckles. The minimum-energy movement shows only one and has the largest variability.

Figure 6-16: End-point variability for different movements of the two DOF robotic arm with the full friction model and additive input noise. On the left side, the end-point variability for the first link is shown for different additive noise levels, on the right side the same is shown for the second link. The three movements depicted here are the minimum-energy movement (red and dark red dashed lines with the round markers), the minimum-time movement (green and dark green dashed dotted lines with the star markers), and the minimum-variability movement (blue and dark blue solid lines with the plus sign markers). The end-point variability scales linearly with the additive noise level for both links and all movements. The end-point variability for the first link is larger than for the second link for all movements. The end-point variability of the minimum-energy movement is the largest for both links (approximately 2.5 times larger than the minimum-end point variability). The minimum-time movement is approximately 1.5 times larger than the minimum-end point variability.
Figure 6-17: Minimum-variability movement for a two DOF robotic arm with a full friction model and multiplicative noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (blue solid lines) and for the second link (dark blue dotted lines). The torques for the first link are mostly increasing and positive up to 0.7 seconds and then become negative for the remainder of the movement. The torques for the second link follow a less clear pattern. The momenta profile for both links shows a longer acceleration phase and a shorter deceleration phase, for both links. The momenta of the second link shows some more bumps. For the first 0.1 seconds the angle of the first link stays constant, while the angle of the second link increases. For the next 0.6 seconds the first link is moving and the angle of the second link stays constant. Then there is a phase where both links move together. After this phase the first link stays at its desired final angle and the second link moves to its own desired final angle.
Figure 6-18: Variability for different movements of the two DOF robotic arm with the full friction model and multiplicative motor noise of level 0.1. On the left side, the variability for the first link is, on the right side the same is shown for the second link. The three movements depicted here are the minimum-energy movement (red and dark red dashed lines), the minimum-time movement (green and dark green dashed dotted lines), and the minimum-variability movement (blue and dark blue solid lines). The variability of the links is propagated to each other, this can be seen for all movements. When the variability in one link stays constant, the variability of the other link shows a buckle. The torque level has a higher impact on the variability, therefore we see that the minimum-time movement has the largest variability. The minimum-energy movement has a larger variability than the minimum-variability movement since the variability increase is larger at the beginning of the movement and this propagates for the rest of the movement.

minimum-energy movement, because the second-link does not move for some period of time during the movement. The minimum-energy movement performs second-best, due to the low torques compared to the minimum-time movement.

The end-point variability scales linearly with the level of multiplicative noise, as can be seen in Figure 6-19. The minimum-variability movement is thus not dependent on the noise level. The minimum-time movement has an end-point variability that is 2.6 times larger for the first link and 3.3 time larger for the second link compared to the minimum end-point variability. For the minimum-energy movement the end-point variability is 2 times larger for the first link and 1.5 times larger for the second link compared to the minimum-variability movement. The minimum-energy movement has a lower end-point variability for the second link. This is not surprising, since the link is not moving for most of the time.

6-2-3 Both noise types

This section shows the end-point variability of the two DOF robotic arm with the full friction model that is disturbed by multiplicative and additive motor noise. First, the minimum-variability movement is investigated for when both additive and multiplicative noise are present at a level of 0.1. Secondly, the end-point variability for different combinations of the noise types is examined. Lastly, the minimum-variability movements found under one condition but applied to another condition are investigated for robustness.

Figure 6-20 shows the minimum-variability movement for the additive and multiplicative noise of level 0.1. The movement shows a multistaged movement, with alternating movement for both links. The torques are also quite low compared to the minimum-time movement, leading to less variability due to the multiplicative noise.
Figure 6-19: End-point variability for different movements of the two DOF robotic arm with the full friction model and multiplicative noise. On the left side, the end-point variability for the first link is shown for different additive noise levels, on the right side the same is shown for the second link. The three movements depicted here are the minimum-energy movement (red and dark red dashed lines with the round markers), the minimum-time movement (green and dark green dashed dotted lines with the star markers), and the minimum-variability movement (blue and dark blue solid lines with the plus sign markers). The end-point variability scales linearly with the multiplicative noise level for both links and all movements. The end-point variability of the minimum-energy movement is larger for the first link (2 times the minimum end-point variability) than for the second link (1.5 times the minimum end-point variability). The end-point variability of the minimum-time movement is 2.6 for the first link and 3.3 times larger for the second link compared to the minimum end-point variability for the corresponding links.

Figure 6-20: Minimum-variability movement for the two DOF robotic arm with the full friction model and full noise. This figure shows from left to right: the input torque, the angle, and the momenta through time for the first link (blue solid lines) and for the second link (dark blue dotted lines) of the minimum-variability movement for multiplicative and additive noise of level 0.1. The angle of the first link stays constant at first, while the second link moves. In the movement phase for the first link the angle of the second link stays constant. For the second movement stage of the first link, the links move also together. However, between 0.6 and 0.7 seconds the angle of the second link is constant, while the other link moves. The accompanying torque profile for the first link shows two phases of positive torques alternated by short periods of negative torque. The momenta profile shows two peaks for the first link and three peaks for the second link.
Figure 6-21: End-point variability for different movements of the two DOF robotic arm with the full friction model and all noise level combinations. This figure shows the end-point variability for different combinations of the level of multiplicative and additive noise. The top graphs show the end-point variability for the first link, the bottom graphs show this for the second link. From left to right the graphs show the end-point variability for the minimum-variability movement, the minimum-energy movement, and the minimum-time movement. The end-point variability for the minimum-energy and minimum-time movement scales as in Equation 5-5. The minimum-time movement has lower end-point variability than the minimum-energy movement for all combinations where the additive noise level is higher than the multiplicative noise level. The end-point variability for the second link of the minimum-variability movement shows an erratic plane, this indicates that local minima were found. The minimum-variability movement is dependent on the noise levels, indicating that the minimum-variability movement is different depending on the noise level.

The size of the end-point variability for both links scales by the same relation as was shown for the one DOF robotic arm with viscous friction (see Equation 5-5) for both the minimum-time and minimum-energy movement. The combined end-point variability is the square root of the sum of the squares of the end-point variability for both noise sources alone. The minimum-time movement has lower end-point variability than the minimum-energy movement for all combinations where the additive noise level is higher than the multiplicative noise level. This means that the minimum-time movement performs best in systems with a larger influence of additive motor noise; the minimum-energy movement has a lower end-point variability for systems with a large influence of multiplicative motor noise. Figure 6-21 shows the relation between noise level and end-point variability for all three movements.

Figure 6-21 shows that for the minimum-variability movement the end-point variability does not scale by Equation 5-5. This indicates that for different combinations of the noise types, a different minimum-variability movement exists. This is probably also one of the reasons, why so many local optima where found.
6-3 Discussion

The state of the system determines the influence of additive noise on the variability for the two DOF robotic arm with viscous friction. The system matrix is dependent on the state of the system contrary to the one DOF robotic arm with viscous friction. For multiplicative noise this dependency does also exist, however it is of minor influence. The variability is mostly influenced by the multiplicative noise level. Already a small amount of multiplicative noise compared to the level of additive noise, leads to a minimum-variability movement that closely resembles the minimum-variability movement for only multiplicative noise.

For the two DOF robotic arm with the full friction model the minimum-variability movements are similar to the ones for the one DOF robotic arm with the full friction model. The added link, leaves one extra option for less variability in a movement: if one of the links is moving and the other not, the variability of the other link is not propagated. This leads to a slower increase in variability. Due to the inertia of the links the torque applied to the links should counteract the influence of the inertia of the other link.

The minimum-variability movement for multiplicative noise shows multistage behavior for the movement of the second link. The first link does not show multistage behavior due to the large inertia at the end of the arm. If this inertia was smaller there would be more freedom in the task and multistage behavior could also be observed in the minimum-variability movement of the first link.

6-4 Conclusion

In a two DOF robotic arm with only viscous friction the minimum-variability movement depends on the noise level. Due to the state-dependent system and input matrices this is also true for additive noise. Still, the multiplicative noise has a larger influence on the end-point variability. Therefore, the minimum-variability movement with relatively small multiplicative noise already looks very similar to the minimum-variability movement with only multiplicative noise.

For a two DOF robotic arm with a full friction model, the variability is not propagated from one link to the other when one of the links has a constant angle. The minimum-variability movement, shows a multistage movement. The moving stages for both links are alternated in order to keep the influence of the noise on the variability to a minimum.
Chapter 7

Experiments

In simulation minimum-variability movements are found for robotic arms using the EKF variability determination. In this chapter, the minimum-variability movements are performed on an experimental robotic arm set-up. From these experiments it can be shown if the EKF variability determination is a good approximation of reality. First, the model used in the simulations is compared to the set-up. This is done in Section 7-1. After the evaluation of the model, Section 7-2 discusses the identification of the noise model for the set-up. Section 7-3 evaluates the performance of the minimum-variability movements on the experimental set-up by comparing them to minimum-energy movements and minimum-time movements.

7-1 Model versus set-up

The model of the robotic arm uses parameters taken form the actual set-up. The friction model is fitted to the actual friction data of the motors with the gearboxes attached. The

Figure 7-1: The robotic arm set-up. On the left, the one DOF set-up of the robotic arm is shown. On the right, the two DOF set-up is shown. For the experiments the hand was replaced by a mass. The DC motors are the black cylinders on the top of the housing. The driving-belts are the red belts.
mass and inertia parameters used in the models are taken from the Solidworks design of the robotic arm. The model uses torque as the input signal. In the set-up the arm is controlled by current control. The current is calculated using the motor parameters from the manufacturer.

The average movements for the same torque signal are different in simulation and practice. Figure 7-2 shows an example of the difference between the model and the set-up using the same input (i.e. commanded input torque is the same). The difference between the model and the set-up is also dependent on the specific movement.

The simplifications of the model lead to the differences. For instance, the model neglects the transmission between the motors and the joints. In the set-up the motors are not directly attached to the joints, the torque is transmitted using timing belts. The friction model used in the model does not incorporate the friction in the bearings of the timing belts and the friction in the joints. Furthermore, the timing belts themselves are not modeled.

Despite the imperfections of our model, this model is used for the experiments. Extensive system identification could lead to a better fit of the model to the behavior of the set-up. However, the risk remains that this only leads to a good fit for certain movements. Due to time limitations in this research, a further system identification is not performed.

## 7-2 Motor noise identification

In order to find the minimum-variability movement for the experimental robotic arm set-up, the noise in the set-up needs to be identified. This is done by acquiring variability data, fitting of the noise model to this data, comparing the end-point variability for minimum-variability movements for different noise types.

In order to get the variability data, a feedforward movement is performed for more than 250 times. After every feedforward movement, the set-up is moved back to its initial state using a feedback controller. From the data of the feedforward movements the variability is
determined. This is done by using the MC variability determination method introduced in Chapter 4.

The noise model is then fitted onto the variability data by minimizing the difference between the variability data and the variability of the model. The optimization is again solved using a multistart of the SQP algorithm in \textit{fmincon}. The noise levels of the multiplicative and additive noise are the decision variables of the optimization and the constraint is the non-negativity of these noise levels. The optimization minimizes the following cost function:

$$\min_{\sigma_{\text{noise}}} \sum_{t=0}^{t_{\text{end}}} \Sigma_{\text{data}} - \Sigma_{\text{model}}, \quad \text{subject to } \left| \sigma_{\text{noise}} \right| - \sigma_{\text{noise}} = 0 \quad (7-1)$$

Where $\sigma_{\text{noise}}$ is a vector containing the noise level of both noise types, $\Sigma_{\text{data}}$ is the variance acquired from the experiments and $\Sigma_{\text{model}}$ is the variance calculated with the noise model.

The noise identification is first performed on the one DOF set-up. The minimum-energy and minimum-time movement found in simulation are used for the noise identification. The minimum-energy movement showed variability data that matched with pure additive noise with a noise level of 0.11. The minimum-time movement showed variability data that had the closest fit with only multiplicative noise with a noise level of 0.11. The fits can also be seen in Figure 7-3.

For noise identification on the second link two approaches have been tried. First, the first link was held still by feedback control, so the second link would act as an one DOF system. Vibrations in the system led to large settling times before another feedforward movement could be made. Getting enough data, was impractical in this set-up. Secondly, the noise identification was tried for the two DOF set-up. For different movements, the optimization would find noise fits that were different. Furthermore, the noise model found for the first link, did not reflect the noise identification of the first link alone.
The noise identification could not find a single fit of the noise model for all movements and configurations of the robotic arm set-up. Therefore, the variability is investigated for a minimum-variable movement for pure additive noise, pure multiplicative noise, and the combination of both. The motor noise in the second link is assumed to be the same as in the first link.

7-3 Experimental results

Experiments are performed for the one DOF robotic arm and the two DOF robotic arm. The results of these experiments are given below.

7-3-1 One DOF robotic arm

The one DOF set-up is a one linked robotic arm with a mass at the end of the arm. This set-up is shown in Figure 7-1, for which the hand is replaced by a mass. In simulation minimum-variability movements are found for three different noise situations: additive noise of level 0.1, multiplicative noise of level 0.1, and the combination of both at level 0.1. All feedforward movements are performed at least 225 times to get the variability data.

The minimum-variability movements all have an end-point variability around 0.006 radians. The maximum difference in end-point variability between the movements is just $5 \cdot 10^{-4}$. The minimum-variability movement found is the movement for both noise types combined. From this it is concluded that the system suffers from a combination of additive and multiplicative motor noise. The mean movements and the variability can be seen in Figure 7-4.

To put the variability of the minimum-variability movement in perspective, a comparison is made with the minimum-energy, minimum-time, and a 'maximum-variability' movement.
Figure 7-5: Mean angle and variability of four different movements. The mean angle (left graph) and the variability (right graph) are shown for four different movements: the minimum-energy movement (red dashed line), the minimum-time movement (green dashed dotted line), the minimum-variability movement (blue solid line), and the "maximum-variability" movement (black dotted line). The end-point variability is the lowest for the minimum-variability movement at 0.004 radians; the minimum-energy movement has a 2 times higher end-point variability, the minimum-time movement a 2.5 times higher end-point variability, and the "maximum-variability" movement has a three times higher variability. The minimum-variability movement starts to differ from the other minimum-time and "maximum-variability" movement after 0.1 seconds. This is the moment where the input torque for the minimum-variability movement is lower than the other two movements, and the variability stays lower.

The "maximum-variability" movement is a movement found in simulation with additive noise of level 0.1.\(^1\)

The end-point variability of the minimum-energy movement is twice as high as the minimum-end point variability. The minimum-time movement has an end-point variability that is almost 2.5 times higher than the minimum end-point variability. The end-point variability of the "maximum-variability" movement is 3.2 times higher than the minimum end-point variability.

The difference in end-point variability of the different movements corresponds to the results found in simulation. In simulation, minimum-time movement has the largest end-point variability, followed by the minimum-energy movement for systems with multiplicative noise combined with additive noise at the same level. This also holds for all situations where the multiplicative noise level is higher than the additive noise level. The size of the end-point variability is in the same order as was found in simulation. The exact difference between the end-point variability of the movements is larger than in simulation.

End-point distribution

The end-point distribution is not always Gaussian as can be seen from Figure 7-6. Only the minimum-variability movement shows an end-point distribution that looks Gaussian. The

\(^1\)Maximum-variability movements found with multiplicative noise or a combination of both noise types did not show variability larger than the minimum-time movement. The reason for this is that the nonlinearities are exploited in the maximization, therefore the EKF variability determination does not work well for maximization.
Figure 7-6: Probability density data of the end-point distribution for four different movements. From left to right this figure shows the minimum-energy movement (a), the minimum-time movement (b), the minimum-variability movement (c), and the "maximum-variability" movement (d). For all four movements the MC variability determination is also plotted, besides the raw probability density data. Only the minimum-variability movement for which the Gaussian probability density function of the MC variability determination is a good fit. However, the MC variability determination gives a representation of the end-point variability for all four movements, since the mean and standard deviation can be determined for every distribution. The Gaussian approximation of the distribution, shows that almost all data falls between the 95 percent confidence interval of the Gaussian distribution.

standard deviation is just a mathematical measurement of the variability. Although the end-point distribution is not Gaussian, the standard deviation is still a good measurement of the variability. The EKF variability determination showed end-point variabilities of the same order as in the experiments. Therefore, this method is useful for finding minimum-variability movements for real-life systems.

7-3-2 Two DOF robotic arm

The two DOF set-up is a two linked robotic arm with a mass at the end of the arm. This set-up is shown on the right in Figure 7-1, for the experiments the hand is replaced by a mass. In simulation minimum-variability movements are found for three different noise situations: additive noise of level 0.1, multiplicative noise of level 0.1, and the combination of both at level 0.1. The noise level for both inputs is the same. All feedforward movements are performed at least 150 times to get the variability data.

The minimum-variability movements for the different noise assumptions show very different variability on the experimental set-up. The minimum-variability movement for only multiplicative noise performs best. The minimum-variability movement for only additive noise performs the worst. From this variability data it is concluded that the system is subjected to motor noise with a large multiplicative character. The movements and the variability for both links can be seen in Figure 7-7.

Comparing the minimum-variability movement to the minimum-time and minimum-energy movement, indicates that the system is subjected to large multiplicative motor noises. The minimum-energy movement has a more than ten times larger end-point variability than the minimum-variability movement. The minimum-energy movement shows a slightly larger end-point variability than the minimum-variability movement (1.4 times larger for the first link and 1 percent lower for the second link). The end-point variability is of the same order for the minimum-variability and the minimum-energy movements as was found in simulation with
Figure 7-7: Mean angle and variability of minimum-variability movements for different noise assumptions. The mean angle is depicted in the left graphs and the variability is depicted in the right graphs. The top graphs show this for the first link and the bottom graphs for the second link. The data is depicted for three minimum-variability movements for different noise conditions: additive noise (magenta dashed lines), multiplicative noise (cyan dashed dotted lines), and the combination of both noise types (blue solid lines). The three movements are different shown by the mean angle. The variability is also very different, the lowest variability is for the minimum-variability movement with multiplicative noise.
Figure 7-8: Mean angle and variability of three different movements. The mean angle is depicted in the left graphs and the variability is depicted in the right graphs. The top graphs show this for the first link and the bottom graphs for the second link. The data is depicted for three movements: minimum-energy movement (red dashed lines), minimum-time movement (green dashed dotted lines), and the minimum-variability movement (cyan solid lines). The fast changing angle for the minimum-time movement leads to the largest end-point variability. The minimum-energy movement and the minimum-variability movement show much lower variability compared to the minimum-time movement. The second link does only move a few hundreds of a radian, leading to low variability for the second link. The variability for the first link is higher in the minimum-energy movement compared to the minimum-variability movement. The end-point variability of the minimum-variability movement for both noise sources is the second lowest for the first link, but the highest for the second link. This is the other way around for the minimum-variability movement for additive noise.

the multiplicative noise level of 0.1. The exact difference in end-point variability between the movements is not of the same order as in the simulation.

End-point distribution

The end-point distribution is not always Gaussian. For the minimum-variability movement and the minimum-energy movement, a Gaussian function is a good approximation of the real end-point distribution. For the minimum-time movement the end point distribution is spread out over a large area, with a high peak at one side. The spread is even over the area, so these movements cannot be considered as outliers. A Gaussian end-point distribution is not a good fit for the minimum-time movement. The probability density function of the end-point distribution of the first link can be seen in Figure 7-9.
7-4 Discussion

The experimental results are different from the results of the simulations, but there are also similarities. First of all, the variability results in general will be discussed. Secondly, the distribution data is discussed.

The variability results tell something about the character of the input noise in general. The lowest end-point variability is shown by the minimum-variability movement for the combination of noise types for the one DOF set-up. In the two DOF set-up, the lowest end-point variability is shown by the minimum-variability movement for multiplicative noise only. The variability of the minimum-time movement is the highest for both the one DOF and the two DOF set-up. This was only seen in simulation for multiplicative input noise that is of the same or higher level compared to the additive input noise. Therefore, the noise in the robotic arm is believed to have a multiplicative nature.

Although the above results indicate a large multiplicative noise part, the results for the one DOF case and the two DOF case are still different. One of the differences between the one DOF set-up and the two DOF set-up is the extra drive belt to the second joint. During the experiments this drive belt showed a lot of backlash. The backlash is one of the noise sources of the robotic arm. Kaneko et al. 1992 showed that the backlash for a tendon-driven robot hand was torque-dependent. The extra backlash could thus result in the increase of the torque dependent nature of the variability for the two DOF set-up.

In the simulations Gaussian noise was applied to the model and it was assumed that the variability distribution stayed Gaussian throughout the movement. In the experiments we have seen non-Gaussian distributions. These are probably due to the non-linearities in the system. Non-Gaussian distributions are also seen in the non-linear simulation for some movements. Some movements, however, do show an approximately Gaussian end-point distribution. Some of the other movements show a bimodal distribution (a mixture of two Gaussian distributions with different means). These distributions could be caused by Gaussian input noise, but the evidence is not conclusive.
7-5 Conclusion

The use of the simulations in practice is tested by performing several experiments. These experiments show that the model used in simulation behaves differently from the experimental set-up. Furthermore, the variability for the set-up does not have a single fit to the noise model. Therefore, several noise assumptions have been tested.

For the one DOF robotic arm the movement that showed the lowest variability was the minimum-variability movement for a combination of additive and multiplicative noise of the same size. For the two DOF robotic arm the movement that showed the lowest variability was the minimum-variability movement for multiplicative noise only. In all cases the minimum-time movement performed the worst, followed by the minimum-energy movement. This leads to the belief that the noise in robotic arms is not pure additive but depends largely on the torque input.
In this Chapter the combined results of this research are discussed. The discussion is divided into two sections. First, the results of the simulations will be discussed in Section 8-1, regarding the characteristics of a minimum-variability movement and the EKF variability determination. Secondly, in Section 8-2 the results of the experiments are put in perspective, regarding standard system modeling, the minimum variability, and the improvements achieved.

8-1 Simulations

The end-point variability for systems can be minimized by making use of several aspects (of the noise itself, of the friction, and of the state of the system).

- First of all, for multiplicative noise lower input torques will lead to lower end-point variability. The variability is propagated throughout the movement, so torques at the beginning of the movement influence the end-point variability more than torques at the end of the movement.

- Secondly, the Coulomb friction makes that when the system is at standstill the variability stays constant, when the system starts moving again the variability due to the motor noise increases again. Additionally, in multi-DOF robotic arms the variability of one link influences the variability in the other link. When one of the links has a constant angle, the variability in the other link increases slower when it is moving.

- Thirdly, the state of the system is used to minimize the end-point variability. When the system matrix or the input matrix is dependent on the state, some states will have less influence of the motor noise. It was hypothesized that when both links move at the same speed, the variability increases slower.

As is indicated in Chapter 4 large motor noise combined with low torques results in an overestimation of the variability by the EKF variability determination. This happens especially with
minimum-energy movements with high additive noise levels (with or without multiplicative noise). In this research the minimum-variability movement and the minimum-time movement have significant lower variability in these cases, so this overestimation was not a problem. But for other systems or higher noise levels this could become a problem.

8-2 Experiments

The results of the experiments showed that the noise in robotic arms was torque-dependent and not constant as with additive noise. Contrastingly, in engineering problems noise is usually modeled as additive noise. Perhaps, when the input noise is modeled as input-dependent noise feedback controllers could be improved.

The set-up behaved quite differently than the simulations predicted. Therefore, the minimum-variability movements used in the experiments are probably not the real minimum-variability movements. One indication of this is that the minimum-variability movements in simulation are multistaged movements, but in the experiments the non-moving stages where much shorter.

The variability in experimental set-up for the one DOF robotic arm is only 0.0055 radians. If we assume a Gaussian distribution 95 percent of the movements has an end-point between plus and minus two times the variability. Then 95 percent of the minimum-variability movements are in a range of 0.022 radians. The robotic arm has a length of 0.41 m, so 95 percent of the movement are in a range of 9 mm. For the minimum-energy movement this is 18 mm and for the minimum-time movement this is 23 mm.\(^1\)

For the two DOF robotic arm the final angles are different for every movement. Due to the non-linear relationship between the angles and the end-effector position. Therefore, the variability of the end-effector has a non-linear relation with the angles. The analysis of this variability is to extensive to perform here.

The experiments showed that the variability in robotic arms is reduced by applying the minimum-variability movements for simulations. Due to model mismatches, the movement of the robotic arm is not the same as the movement in simulation. Although the end-point variability is decreased, the accuracy of feedforward control remains low and movement dependent.

\(^1\)The deviations are calculated perpendicular to the link, therefore the derivative of the sine function is one, leading to a linear calculation from angles to position.
Feedforward controlled systems suffer from motor noise, leading to variability in the movements. For robotic arms performing a pick-and-place task, this variability should be as small as possible. The goal of this research is to find movements that minimize the variability for a given task.

The motor noise is modeled as multiplicative and additive Gaussian white noise. In simulation the variability is determined via the Extended Kalman Filter variability determination method. This method is computationally less expensive than the Monte Carlo variability determination method.

The minimum-variability movement is found by minimizing the end-point variability using a multistart of the SQP algorithm constrained by the desired end-state. This minimum-variability movement is compared to the minimum-energy movement and the minimum-time movement.

For one DOF robotic arms with the full friction model, the minimum-variability movements found in simulation are movements that use several aspects to minimize the variability due to motor noise. For multiplicative noise small inputs, especially in the beginning lead to minimum end-point variability. For additive noise, a movement with multiple movement phases leads to minimum end-point variability. The exact minimum-variability movement depends on the additive and multiplicative noise levels. For an experimental robotic arm set-up the motor noise could not be captured in a single noise model fit. However, the movements found in simulation for additive and multiplicative noise combined at the same level lead to the the lowest end-point variability. This end-point variability was twice as low as for minimum-energy and minimum-time movements.

In two DOF robotic arms with the full friction model, other aspects can also be used to minimize the end-point variability. Minimum-variability movements for two DOF robotic arms show alternating movements in both links in order to minimize the end-point variability, furthermore when both links have the same speed the variability increases less. For an experimental robotic arm set-up minimal variability was found for a minimum-variability movement found in simulation with only multiplicative noise.
This research shows that despite modeling errors, the EKF variability determination can be used in simulation to find minimum-variability movements for real-life robotic arms.
Chapter 10

Recommendations

This research has its limits and therefore it inevitably leads to recommendations for future research. These recommendations are divided in three sections: the noise, the variability determination, and the simulations.

Recommendations regarding the noise:

- The noise was modeled as Gaussian white noise in this research, however it was indicated that if the time step used in simulation was smaller the noise level should be scaled up. In the limit case the noise would have an infinite variance. To avoid this scaling, the noise could be modeled as a Brownian motion. For a Brownian motion the influence of the noise is the same for any choice of the time step. Brownian motion as a noise model, assumes that the noise has a $1/f^2$ frequency spectrum. The question remains if this is a good approximation of the real noise.

- In order to get a better understanding of the characteristics of the noise in robotic systems, the noise of the electric motors should be identified. The electric motors are a small subsystem for which the noise is probably easier to be determined than for a whole system. With a good noise model for the electric motors, the noise from the addition of other components could also be identified.

- From the experiments on the robotic arm set-up it was hypothesized that the noise is torque-dependent. In this research the multiplicative noise was linearly related to the torque level. However, the experimental results indicate a non-linear relationship. This exact relationship can be identified using different torque inputs and measuring the variability.

Recommendations regarding the variability determination:

- The EKF variability determination, gives us information about the mean and covariance of the state distribution. In this research it was shown that the non-linearities in the system can make the variability distribution non-Gaussian. However, with the EKF
variability determination the actual distribution cannot be retrieved. The author would like to propose a hypothetical variability determination approach. A random distribution can be approximated by a mixture of Gaussian distributions with different weights. The Unscented Kalman Filter uses sigma-points to retrieve the mean and covariance of the state. The author suggest using combination of the EKF variability determination and the Unscented Kalman Filter approach in parallel with a UKF variability determination. The sigma points can be determined using the UKF approach, these sigma points are then replaced by Gaussian distributions of which the mixture makes up the total prior distribution. Propagating these Gaussian distributions through the system with the EKF variability determination and fitting them to the mean and covariance found by a normal UKF variability determination, could hypothetically result in a better approximation of the actual posterior distribution.\footnote{A downside to this hypothetical approach is that more function evaluations are needed. Thus leading to a computationally more complex variability determination method.}

Recommendations regarding the simulations:

- The simulations and the experiments on the two DOF robotic arm assume that the input noise for both links is the same. Although the motors used in the experimental set-up are of the same type, the noise could be different in both motors. Furthermore, the motor noise could also be different for the links, while the transmission from the motor to the joints is different for both links. In future research this can be investigated. A hypothetical outcome could be that it is best to actuate only the first link and use the coupling between the links to move the second link to the desired position.
This appendix shows the models used for the simulation. Section A-1 shows the Matlab code for the one degree of freedom robotic arm models. Section A-2 shows the Matlab code for the two degree of freedom robotic arm models.

A-1 One DOF Robotic Arm Model

Figure A-1 shows the schematic overview of the one DOF robotic arm, which is used for the models. The parameters of this one DOF robotic arm are described in Table A-1. Note that the viscous friction coefficient is positive for the one DOF robotic arm with only viscous friction and negative for the one DOF robotic arm with the full friction model. The full friction model is a fit to experimentally acquired friction data. Surprisingly, this incorporated a negative viscous friction.

<table>
<thead>
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<th>value</th>
<th>unit</th>
<th>remarks</th>
</tr>
</thead>
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<td>m</td>
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<td>0.41</td>
<td>m</td>
<td>length</td>
</tr>
<tr>
<td>$m$</td>
<td>0.809</td>
<td>kg</td>
<td>mass of the link</td>
</tr>
<tr>
<td>$I$</td>
<td>0.1719</td>
<td>kg·m²</td>
<td>Inertia around the joint with added mass</td>
</tr>
</tbody>
</table>

Table A-1: Table of variables of the one DOF robotic arm models.
A-2 Two DOF Robotic Arm

Figure A-2 shows the schematic overview of the two DOF robotic arm, which is used for the models. The parameters of this two DOF robotic arm are described in the Table A-2. Note that the viscous friction coefficient is positive for the two DOF robotic arm with only viscous friction and negative for the two DOF robotic arm with the full friction model. The full friction model is a fit to experimentally acquired friction data. Surprisingly, this incorporated a negative viscous friction.
<table>
<thead>
<tr>
<th>variable</th>
<th>value</th>
<th>unit</th>
<th>remarks</th>
</tr>
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<tbody>
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</tr>
<tr>
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<td>mass of link 1</td>
</tr>
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<td>kg</td>
<td>mass of link 2 + end mass</td>
</tr>
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</tr>
<tr>
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<td>0.0980</td>
<td>kg·m²</td>
<td>Inertia around the CoM link 2</td>
</tr>
</tbody>
</table>

**two DOF with viscous friction**

| $\mu_{v1}$ | 0.0521 | viscous friction coefficient joint 1 |
| $\mu_{v2}$ | 0.0432 | viscous friction coefficient joint 2 |

**two DOF with the full friction model**

| $\mu_{C1}$ | 0.1939 | Coulomb friction coefficient joint 1 |
| $\mu_{C2}$ | 0.3328 | Coulomb friction coefficient joint 2 |
| $\mu_{v1}$ | -0.0521| viscous friction coefficient joint 1 |
| $\mu_{v2}$ | -0.0432| viscous friction coefficient joint 2 |
| $\mu_t$    | 0.2195 | torque-dependent friction coefficient joint 1 |
| $\mu_t$    | 0.3195 | torque-dependent friction coefficient joint 2 |

*Table A-2*: Table of variables of the two DOF robotic arm models.
Appendix B

EKF variability algorithm

The EKF variability determination method uses a discrete time update of the variance together with a Runge-Kutta 4 continuous time update of the state. Algorithm 1 shows the EKF variability determination in pseudo code.

**Result:** EKF variability determination

Get initial state $s$;
Get desired input vector $u$;

for $t = 0 \to t_{end}$ do
  Get the new variance;
  $A_{lin}(s_t, u_t)$;
  $B_{lin}(s_t, u_t)$;
  $\Sigma_{\dot{s}_{t+1}} = A_{lin} \cdot \Sigma_{s_t} \cdot A_{lin}^T + B_{lin} \cdot (k_m^2 \cdot u^2 + k_a^2) \cdot B_{lin}^2$;
  $\Sigma_{s_{t+1}} = \Sigma_{\dot{s}_t}$;
  Get the new state;
  $k_1 = EOM(s_t, u_t) \cdot \Delta t$;
  $k_2 = EOM(s_t + 0.5 \cdot k_1, u_t) \cdot \Delta t$;
  $k_3 = EOM(s_t + 0.5 \cdot k_2, u_t) \cdot \Delta t$;
  $k_4 = EOM(s_t + k_3, u_t) \cdot \Delta t$;
  $\dot{s}_{t+1} = \frac{k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4}{6}$;
  $s_{t+1} = s_t + \dot{s}_{t+1}$;

end

$\sigma_s = \sqrt{\Sigma_{s_{t+1}}}$

**Algorithm 1:** EKF variability determination method. The EKF variability determination method uses a discrete time update of the variance $\Sigma_s$ together with a Runge Kutta-4 continuous time update of the state $s$ using the non-linear Equations of Motion (*EOM*). The variability $\sigma_s$ is the square root of the variance. $A_{lin}$ and $B_{lin}$ are the discrete system and input matrix.
In this chapter we will see if we can repeat the results of Harris & Wolpert for a human-like robotic arm.

Harris & Wolpert (1998) proposed that humans make movements that minimize the variability. They used a linear model of a human arm to show this. In this research we have seen that the addition of friction to a system results in different minimum-variability movement.

The main difference between this research and the study of Harris & Wolpert is the variability of what state is minimized. Harris & Wolpert choose to minimize the velocity variability some time after the movement. In this research the positional variability is minimized at the end of the movement.

In order to repeat the results of Harris & Wolpert the friction coefficients in the human arm are changed to the friction coefficient of a human synovial joint (Serway et al., 2003) human-like robotic arm. The human arm has low viscous friction coefficients and Coulomb friction coefficients, respectively 0.003 and 0.01. The other parameter of the robotic arm are kept the same.

The resulting minimum-velocity-variability movement is very similar to the minimum-energy movement (see Figure C-1). However, there are some very small differences in the torque levels. This results in a slightly \(7 \cdot 10^{-8}\) radians per second) smaller velocity-variability for the minimum-variability movement. This lower velocity-variability comes at the price of higher end-point positional variability of \(4 \cdot 10^{-5}\) radians. The velocity-variability of both movements is shown in Figure C-2a and the positional variability C-2b.

The differences are so small that in a MC variability determination as Harris & Wolpert used, they would not be visible.

The human-like robotic arm in our example has quite high inertia compared to a real human arm. Lowering this inertia would lead to larger difference between the minimum-variability
movement and the minimum-energy movement. These movements would still very much look like human-like movements.

The question remains if the minimization of the velocity variability is a logical one for humans. Harris & Wolpert argued that the data for minimization is readily available in the brain. However, positional information is probably easier to interpret, so one could also argue that humans optimize their movements on this data. Minimizing the positional variability, does lead to different movements as we have seen for the one DOF robotic arm with only viscous friction. For human-like robotic arms the same results are expected.

Harris & Wolpert (1998) has argued that humans minimize their movements to end-point variability. They prove this by minimizing the end-point velocity variability. This has been tested on a human-like robotic arm and the movements found are slightly different but very much alike minimum-energy movements. However, one could argue that humans optimize their movements on positional data, this would result in different movements if friction is included in the model. These movements are not energy-optimal, but they are energy-efficient. These movements are also smooth, but they have a longer acceleration phase than the deceleration phase.

Figure C-1: Minimum-velocity-variability and minimum-energy movement for a human-like robotic arm. The minimum-velocity-variability movement (blue solid lines) is very similar to the minimum-energy movement (red dashed lines). The only difference is visible in the torque levels. The first torque step of the minimum-variability movement is slightly larger and the last torque step is slightly smaller. The rest of the differences are not visible by the naked eye.
(a) Velocity-variability for a human-like robotic arm. The velocity-variability of the minimum-velocity-variability movement (blue solid lines) is very similar to the minimum-energy movement (red dashed lines). The difference in end-point velocity variability is only $7 \cdot 10^{-8}$ radians per second, where the minimum-velocity-variability movement has the lowest velocity-variability.

(b) Positional-variability for a human-like robotic arm. The positional-variability of the minimum-velocity-variability movement (blue solid lines) is very similar to the minimum-energy movement (red dashed lines). The difference in end-point positional variability is large than for the velocity variability but still small. The difference in end-point positional variability is $4 \cdot 10^{-5}$ radians, where the minimum-energy movement has the lowest end-point positional variability.
Appendix D

Influence of model parameters

The model used throughout this research is based on a real experimental robotic arm. Other robotic arms are different and have different parameters. This chapter shows the influence of different parameters on the minimum end-point variability for the one DOF robotic arm with the full friction model. The parameters used in this research can be found in Appendix A.

The main parameters that determine the movement of the one DOF robotic arm are: the inertia around the joint, the Coulomb friction, the torque-dependent friction, and the viscous friction. The values of the parameters was decreased and increased by 50 percent in the case with additive and multiplicative noise of level 0.1. Only for the viscous friction the value was set to zero and to a positive friction coefficient. Table D-1 shows the end-point variability of the minimum-variability movements for the different values of the parameters.

The inertia is of largest influence on the minimum-variability movement. If the inertia is halved, the freedom in the tasks becomes bigger. Therefore the possibilities of multistaged movements are larger. Figure D-1 shows the minimum-variability movement when the inertia is only half for the one DOF robotic arm with the full friction model. This movement shows a three staged movement, with one large skewed momenta peak to keep the influence of the multiplicative noise as low as possible.

Surprisingly, a higher Coulomb friction leads to larger movement variability. Due to the larger torques required to overcome the Coulomb friction also the variability due to multiplicative noise becomes larger.

Higher torque-dependent friction leads to lower end-point variability. Torque-dependent friction reduces the influence of multiplicative noise.

A positive viscous friction coefficient leads to lower end-point variability. A positive viscous friction coefficient will stop movement and thus lead to lower variability.
Table D-1: Table of end-point variability for different values of the variables. Larger inertia leads to a lower minimum end-point variability. Higher Coulomb friction leads to higher end-point variability. Higher torque-dependent friction leads to lower end-point variability. More positive viscous friction leads to lower variability. Note that -100 percent viscous friction is equivalent to a positive friction coefficient, while the normal viscous friction coefficient is negative.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>End-point variability for parameter at</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50 %</td>
</tr>
<tr>
<td>Inertia</td>
<td>0.0110</td>
</tr>
<tr>
<td>$\mu_C$</td>
<td>0.0083</td>
</tr>
<tr>
<td>$\mu_t$</td>
<td>0.0091</td>
</tr>
<tr>
<td></td>
<td>-100 %</td>
</tr>
<tr>
<td>$\mu_v$</td>
<td>0.0077</td>
</tr>
</tbody>
</table>

Figure D-1: Minimum-variability movement for half the inertia in the one DOF robotic arm with the full friction model. This figure shows from left to right: the input torque, the angle, and the momenta through time for the minimum-variability movement for multiplicative and additive noise of level 0.1. This figure shows from left to right, the torque, the angle, and the momenta through time. The movement has three movement stages, these are clearly seen in the momenta graph. The first two peaks are symmetric, the last and largest peak is skewed. This is the momenta peak that accompanies the last and largest movement stage. This peak, keeps the variability due to the multiplicative noise as low as possible.


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Master of Science Thesis


