The transition to turbulence in flows where the laminar profile is linearly stable requires perturbations of finite amplitude. “Optimal” perturbations are distinguished as extrema of certain functionals, and different functionals give different optima. We here discuss the phase space structure of a 2D simplified model of the transition to turbulence and discuss optimal perturbations with respect to three criteria: energy of the initial condition, energy dissipation of the initial condition, and amplitude of noise in a stochastic transition. We find that the states triggering the transition are different in the three cases, but show the same scaling with Reynolds number.

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1. Introduction

In parallel shear flows like pipe flow, plane Couette flow or Poiseuille flow and in boundary layers like the asymptotic suction boundary layer or the Blasius profile, turbulence appears when the laminar profile is linearly stable against perturbations [1]. Accordingly, finite amplitude perturbations are required to trigger turbulence, a scenario referred to as by-pass transition [2]. Many studies in the above flows have shown that the transition to turbulence is associated with the presence of 3D exact coherent states [3]. They appear in saddle-node bifurcations which in the state space of the system create regions of initial conditions that do not decay to the laminar profile, but instead are attracted towards the node-state [4]. As the Reynolds number increases, the region widens, the node state undergoes further bifurcations and chaotic attractors or saddles are formed [5–7]. Initial conditions can only trigger turbulence when they reach into that interior region, i.e., cross the stable manifold of the saddle state on the boundary of the region [8]. An “optimal” perturbation is one that can trigger turbulence and at the same time is a minimum of a prescribed functional. Popular is an optimization based on amplification or energy gain over a given interval in time [9–17] or on the total time-averaged dissipation [18,19]. Because they take the time-evolution into account, they connect to optimization problems in control theory [20,21].

We simplify matters here and focus on a geometric optimization by identifying initial conditions that will eventually become turbulent, without regard of the time it takes for them to become turbulent. The states are optimized so that a certain quadratic functional, such as energy content or dissipation, is extremal: it is a maximum in the sense that all initial conditions with a lower value of the quadratic functional will not become turbulent, and it is minimal in that the first initial conditions that become turbulent have values larger than this optimum. At the optimal value there will then be at least one trajectory which neither becomes turbulent nor returns to the laminar profile: it lies on the stable manifold of the edge state [8], so that the optimum is reached when the isocontours of the optimization functional touch the stable manifold of the edge state (similar descriptions of the state space structure can be found in Refs. [17,19,22,23]).

2. The Model

To fix ideas and to keep the mathematics as simple as possible, we take the 2D model introduced by Baggett and Trefethen [24]. The model we use is one of a set of many low-dimensional models of various levels of complexity [20,22,25–31]. It has a non-normal linear part and an energy conserving nonlinearity, and, this being the most important feature for the present application, it is 2D so that the entire phase space can be visualized (a property it shares with the illustrative model of Ref. [20]). Despite its simplicity, the model can be used to illustrate several features of the transition mechanisms in shear flows.
The model has two variables, which may be thought of as measuring the amplitudes of streaks $x$ and vortices $y$ (see also Ref. [32]), and one parameter $R$ that plays the role of the Reynolds number.

\[
\dot{x} = -x/R + y - y\sqrt{x^2 + y^2}, \\
\dot{y} = -2y/R + x\sqrt{x^2 + y^2}.
\]

In order to highlight more clearly what happens near the origin, we magnify by rescaling the variables with the Reynolds number $R$ (see Ref. [33]), i.e., we redefine the amplitudes $x = x'/R^2, y = y'/R^2$ and the time $t = R\tau$ such that (with the primes dropped)

\[
\dot{x} = -x + Ry - y\sqrt{x^2 + y^2}/R, \\
\dot{y} = -2y + x\sqrt{x^2 + y^2}/R.
\]

Time evolution under the nonlinear terms alone preserves $x^2 + y^2$, which may be thought of as a kind of energy, so that the nonlinear terms are “energy” conserving. For $R < R_c = \sqrt{8}$ the only fixed point is $x = y = 0$, henceforth referred to as the “laminar” fixed point. At $R = R_c$, symmetry related fixed points appear at $(x_c, y_c)$ and $(-x_c, -y_c)$, with

\[
x_c = R(2R^2 \pm 2\sqrt{R^2 - 8})/D_{\pm} , \\
y_c = R(2R^2 - 4 \pm R\sqrt{R^2 - 8})/D_{\pm}.
\]

where $D_{\pm} = \sqrt{8 + 2R^2 \pm 2R\sqrt{R^2 - 8}}$. The two fixed points closest to the origin are unstable, hence are saddle states, and the two further out are stable and hence nodes. The saddle states are the “edge states” [8] and the node states are in the regions where turbulence would form, if more degrees of freedom were available. Nevertheless, we will refer to them as the “turbulent” states.

For $R \to \infty$, the saddles are to leading order in $1/R$ located at $\pm(2, 2/R)$, which in the original coordinates represents an approach to the origin like $\pm(2/R^2, 2/R^2)$. The stable manifolds rotate so as to become parallel to the $x$-axis, as we will see in the following.

3. Optimal Initial Conditions of Minimal Energy

The Euclidean distance to the origin can be obtained from a quadratic form

\[
E = (1/2)(x^2 + y^2),
\]

which has the form of kinetic energy. This assignment is further supported by the observation that $E$ is preserved under time evolution by the nonlinear terms alone. In the sense described in the introduction, optimality with respect to this energy functional thus means the largest value up to which all trajectories return to the laminar state, and the smallest one where the first trajectories that evolve towards the turbulent state become possible. On the boundary between these two cases are states that neither return to laminar nor become turbulent, that lie on the stable manifold of the edge state. Geometrically, we are thus looking for the circle with the largest radius that we can draw around the origin that just touches the stable manifold. Algorithmically, we find this point by a modified edge tracking which minimizes the energy (7) as described in the Appendix.

An example of such an optimal circle is given in Fig. 1, and its variation with $R$ is shown in Fig. 2. As the Reynolds number increases, the fixed point moves towards $(2, 0)$ on the abscissa, and the stable manifold rotates to being parallel to the abscissa. The point of contact between circles of equal energy and the stable manifold moves away from the edge state, approaches the $y$-axis and moves inwards to the origin like $1/R$.

In an insightful discussion of the energy functional, Cossu [23] notes that in the time-derivative of the energy functional only the linear parts of the equations of motion remain and that the nonlinear ones drop out because energy is preserved. This observation allows to define a necessary condition for the location...
of the extremum, which for the 2D example studied here implies that the optimum lies along the line connecting the laminar and the turbulent fixed points. One could then find the optimum by a one-dimensional search along this line. However, we did not pursue this further, as we also want to find optima with respect to other functionals that are not preserved by the nonlinear terms.

4. Optimal Initial Conditions of Minimal Energy Dissipation

The diagonal terms in the linear part of the equations of motion correspond to the dissipation in the original Navier–Stokes equation. Accordingly, we can define a dissipation functional \[ \epsilon = (1/2)(x^2 + 2y^2) \] and study initial conditions that are minimal or optimal with respect to this functional. As in the previous example, the geometrical condition is that we now have to find the point where an ellipse touches the stable manifolds. This gives the ellipses shown for different \( R \) in Fig. 3. Note that the points where the ellipses touch the stable manifolds are different from the ones of the energy functional, but their asymptotic behavior for large \( R \) seems to be similar (see below).

5. Optimal Noisy Transitions

As a third example we consider noise-driven transitions. To this end, the equations of motion are expanded to include a stochastic forcing of the individual terms,

\[
\begin{align*}
\dot{x} &= -x + Ry - y\sqrt{x^2 + y^2}/R + \xi_x, \\
\dot{y} &= -2y + x\sqrt{x^2 + y^2}/R + \xi_y,
\end{align*}
\]

where the noise is characterized by \( \langle \xi_x \rangle = 0 \) and \( \langle \xi_x(t)\xi_y(t') \rangle = D_0 \delta(t - t') \). We consider the case \( D_1 = D_2 = D \), so that both components are driven with equal noise amplitude. In a linear approximation around the laminar fixed point, the non-normal coupling between the two components results in a probability density function (pdf) for the two components that is Gaussian with a covariance matrix \[ \Sigma \] given by

\[
p(x, y) = \frac{1}{\pi D |\det(Q)|^{1/2}} \exp\left[-Q(x, y)/D\right]
\]

with

\[
Q = \frac{1}{2(R^2 + 9)} \begin{pmatrix} 9 & -3R \\ -3R & 3R^2 + 18 \end{pmatrix}
\]

and

\[
Q(x, y) = x^T Q x
\]

\[
= \frac{3}{2(R^2 + 9)}[3x^2 - 2Rxy + (R^2 + 6)y^2].
\]

Asymptotically, for \( R \to \infty \), the quadratic form becomes \( Q(x, y) \to 3y^2/2 \), so that the Gaussian stretches out along the \( x \)-direction for increasing \( R \).

In the noisy case, transition is induced when a fluctuation carries the system across the stable manifold. A good estimate of the likelihood of transition can be obtained by considering the probability density at the transition point. Given the functional form of the pdf, the biggest contribution to its variations comes from the quadratic form in the exponent. The equation shows that the iso-contours \( p = \text{const.} \) are ellipsoids determined by \( Q_0 = \text{const.} \); that decrease or increase with the noise amplitude \( D \). Therefore, if we want to describe where a noisy trajectory
crosses over to the turbulent state, we again have to study iso-
contours of a quadratic form, \(Q_N = \text{const.}\), and determine where
they touch the stable manifold of the saddle state. In contrast
to the energy functional (7) and the dissipation functional (8),
the fluctuation functional \(Q_N\) depends on the Reynolds number.
The point of contact between the probability iso-contours and the
stable manifold then corresponds to the point where trajectories
are most likely to cross over the stable manifold and to become
turbulent. Alternatively, if one wants to push the system to become
turbulent, small perturbations in that region are most effective
because the border is so close.

Figures 4 and 5 show the relative probability density to be at
\((x, y)\) in the region where the transition is expected to occur. It
is obtained by integrating 20000 initial conditions in time for 20
time units starting at the laminar state with a step size of \(dt = 10^{-3}\). As the phase space is symmetric with respect to the origin,
trajectories from the third quadrant are mirrored into the first
quadrant. Figure 4, obtained without the nonlinear part, shows the
Gaussian shape of the iso-contours. Out of the \(400 \times 10^6\) calculated
points of the trajectories, more than \(108 \times 10^6\) lay in the plotted
region of phase space. In Fig. 5 the nonlinear part is added and
the pdf stretches out along the path to the turbulent state. The
figure shows clearly that this happens close to the point where
the ellipsoid \(Q_N(x, y) = \text{const.}\) touches the stable manifold. Here more
than \(75 \times 10^6\) points lay in the interesting region of phase space. We
note that the shape of the iso-contours of the pdf is independent
of the noise amplitude \(D\) (within the linear approximation), so
that changes in \(D\) will predominantly influence the likelihood of
a transition, but not the path it takes.

More examples of such iso-contours are shown in Figs. 1
and 6 for different values of \(R\). With increasing \(R\) the ellipsoids
become more elongated in \(x\)-direction, as a result of the asymptotic
behavior of the quadratic form noticed above (14). As they are
rotated in the direction opposite to the rotation of the stable
manifolds, the point of contact stays close to the edge state. Within
the hydrodynamic interpretation, the transition is dominated by
the streaks (\(x\)-component) that form as a result of the vortices (\(y\-
component), not by the vortices themselves.

6. Summary and Conclusions

The calculations illustrate how different optimization criteria
select different optimal initial conditions for the transition to
turbulence. Geometrically, this is to be expected since different
quadratic forms give rise to different ellipsoids in their iso-
contours and hence also different points of contact with the stable
manifolds. We note that the results of Ref. [13] suggest that
for optimization with a time-integrated functional the difference
between energy and dissipation functionals is smaller and may
actually vanish. However, we have not pursued this question
further.

The variation of the optimal points of contact is summarized
in Fig. 7. The data indicates that while the optimal perturbations
are vortex like for the energy and the dissipation functional, they
are streak like for the noisy transition. The difference can be
rationalized by the different dynamics. In the deterministic cases,
with the energy and the dissipation functional, small vortex like
initial conditions can grow in time to develop the streaks which
then drive the transition. The noisy system is always exposed
to small perturbations which can grow to develop streaks, so
that the pdf is elongated in the streak direction by non-normal
amplification. Therefore, the transition happens on top of the
already existing streaks and noise driven flows [35–37] may show
different structures at the point of transition than flows driven by
judiciously chosen initial conditions.

A final quantity to study is the scaling of the functional with
Reynolds number, as shown in Fig. 8. Despite the differences in
dynamics, the functionals scale in all three cases like \(1/R^2\) for large \(R\).
The particular exponent is specific to the model studied here
and the type and form of the nonlinear interactions, as other nonlinearities
can require a rescaling near the origin [38]. However, the fact
that all three cases show the same scaling could also apply to the
full flow cases, as it is a consequence of the measure used and not
we can determine a point $x_0$ on the edge also when the point is very close to the edge and the time needed to pass the edge state becomes excessively large.

We then propagate this point along the time direction, $x_1 = x_0 + sf(x_0)$ by an amount $s$ that is chosen such that $Q(s)$ is minimized. Formally,

$$s = \frac{\int f^{-1}(x_0) Q x_0 + x_0^T Q f(x_0)}{2 \int f^{-1}(x_0) Q f(x_0)}$$

In numerical implementations, $\|sf\|$ is kept below a certain threshold to stay in a region where linear approximations are possible. Then a new edge tracking is started from $x_1$ and the process is repeated until the norm of the total shift $\|sf\|$ falls below a convergence threshold, here $10^{-5}$.

References


Appendix

In this appendix we discuss the modification of the edge tracking algorithm [8] used for the determination of the initial conditions on the edge that optimize a prescribed quadratic functional $Q_k(x, y)$. The functional may be the energy (7), the dissipation (8) or the argument in the pdf (12). To keep the notation compact, we denote the equations of motion in vectorial notation as $x = f$.

We begin with an arbitrary initial condition in the vicinity of the edge and we let it evolve in time towards the edge state. Unlike other edge tracking methods, where trajectories are integrated until they are sufficiently close to the laminar or the turbulent state, we here stop the integration at the time when the distance to the edge state is minimal. The trajectory's velocity at the turning point is then projected onto the normal of the stable eigenvector to decide if the tested initial condition moves upwards or downwards, towards the turbulent or the laminar state. With this criterion

the particular nonlinearity at play. What the model also shows is that deviations from the asymptotic behavior appear close to the point of bifurcation. It is tempting to speculate that such effects may be responsible for the different critical exponents that have been observed in pipe flow or plane Couette flow, but that clearly requires the transfer of the present analysis to realistic flow simulations and a careful analysis of the asymptotic properties.

The analysis of simple models has repeatedly helped to elucidate many features of the transition to turbulence in shear flows, and to develop tools to explore them [20,22,25–31]. It is in this spirit that we have used a forward integration technique to find the optimal points on the stable manifolds for different functionals and to explore the changes with Reynolds number. We expect that many of the features described here can also be found in the high-dimensional state spaces of realistic shear flows, perhaps after suitable modifications and adaptations of the methods used to explore the high-dimensional spaces.

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