Nonlinear Inversion in Electrode Logging in a Highly Deviated Formation with Invasion Using an Oblique Coordinate System

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Abstract—Electrode logging as known in the oil industry is a method for determining the electrical conductivity distribution around a borehole or between two boreholes from the static-field (dc) measurements in the borehole. In this paper, we discuss the reconstruction of the three-dimensional (3-D) conductivity around a borehole in a highly deviated formation with invasion. At this moment, we have not included the borehole effect. To solve this problem, the full vector analysis is required. In most available algorithms, for the forward and inverse modeling of the resistivity data, the dipping bed environment is approximated using the staircase-discretization grid. In contrast, we have modeled the dipping-bed environment by introducing an oblique (nonorthogonal) coordinate system. By using the oblique coordinate system, we have gained some advantages over the usual approach. First, the use of the staircasing approximation for the dipping-bed environment can be avoided. This means that we reduce the discretization error, and we can suffice with less discretization points to obtain the results with the same degree of accuracy as the problem formulated in the Cartesian coordinate system. Secondly, the horizontally-symmetry constraints of the conductivity distribution can be included easily in the inversion procedure. Several numerical results are presented to demonstrate the performance of the inversion method using the synthetic “measured” data, which are generated by solving a forward-scattering problem numerically with the help of the conjugate gradient fast fourier transform (CGFFT) method.

Index Terms—Electrode logging, nonlinear inversion, oblique coordinate system, three-dimensional method.

I. INTRODUCTION

ELECTRODE-TYPE resistivity tools as known in the oil industry are often used in electromagnetic well logging to probe the electrical-conductivity distribution around a borehole. These electrode-type tools usually operate at very low frequency. Such a measurement system allows us to increase the investigation range while sacrificing the resolution.

In the last decade, the nonlinear-inversion method became an efficient tool to invert the data from various measurement systems. A class of iterative reconstruction methods undertakes repeated modifications of the background configuration (green function), and both the internal field and the conductivity are updated after each iteration. It has been used by Liu [13] to invert low-frequency, electrode-type resistivity measurements in two-dimensional (2-D), axisymmetric, inhomogeneous media.

Another method that updates the fields and the contrast simultaneously by a nonlinear conjugate-gradient algorithm that minimizes the error both in the object equation and the data equation has been used by Abubakar [2] to solve the full-vector electrode-logging inverse problem in a simple three-dimensional (3-D) configuration.

When logging tools are used in highly deviated wells with invasion (see Fig. 1) a more complicated treatment of the pertaining problem is required. For the forward modeling of the induction logging, there are already several efficient numerical schemes available, in which the effect of dip is modeled by taking a linear combination of vertical and horizontal magnetic dipole. Eigenfunction solutions to dipping-bed induction response are given by Hardman and Shen [10], Anderson et al. [4], and Gianzero and Su [9]. The 3-D forward modeling of the induction response is given by Howard and Chew [11] and Anderson et al. [5]. A one-dimensional (1-D) inversion method with dip is given by Minerbo [14]. A full 3-D inversion method using the finite-difference approach is given by Koelman et al. [12]. Here, the dipping-bed environment is modeled using the staircasing-discretization grid.

In contrast to the available methods, we approach the effect of dip differently. The approach uses the electric-current, density-integral equation developed by Abubakar and Van den Berg [2], [3]. This integral equation is transformed to its equivalent in an oblique coordinate system. The new vertical
axis (the borehole axis) is defined to make an angle with the vertical Cartesian axis. The dip angle is assumed to be known and constant. By using this approach, we have some advantages above the common approach. First of all, the discretization errors made by the staircasing approximation for the dipping-bed environment can be avoided. Secondly, the a priori information, such as the formation, is horizontally symmetric around the borehole and can be included easily in the inversion procedure.

The integral equation that has been formulated in this oblique basis is our point of departure to develop the forward and inversion procedures. First, we note that this integral equation is a singular integral equation. In order to discretize such an integral equation, the weak form of the Green function is used, and the gradient-divergence operator occurring in the integral equation is computed with the finite-difference rule [1]. After the discretization procedure, the forward scattering problem is solved with the conjugate gradient fast fourier transform (CGFFT) method. The contrast source inversion (CSI) method in [3] is applied to solve the present inverse problem. Like the modified-gradient method [7], one of the advantages of the CSI method over the other optimization methods is that the CSI method does not require some artificial regularization techniques to deal with the problems of the nonuniqueness in the inversion of data, but it uses the object equation itself as a regularizer. The CSI method considers the inverse-scattering problem as an inverse-source problem. Similar to the modified-gradient method, in each iteration there is no full inversion of the object equations involved. A cost functional consists of errors in the data and object equation. Updates in the sources are found as a conjugate-gradient step, after which the conductivity contrast is updated by minimizing only the error in the object equation. The source updates are similar to those used in the modified gradient method, while the conductivity contrast updates are found in a simple fashion, in which a priori information about the conductivity distribution such as the positivity of the conductivity and the horizontally-symmetric formation around the borehole can be included in a simple fashion. In view of the efficiency of the CSI method, 3-D inversion problems can now be handled with moderate computer power. In all of our numerical examples, the borehole effect has not been taken into account such that we can study the formation-related effect without further complication. In a later study, the borehole effect can be included in the algorithm by replacing the homogeneous Green function with the inhomogeneous Green function.

II. BASIC EQUATIONS AND INTEGRAL REPRESENTATIONS

We define an inhomogeneous domain as a contrasting domain \( D \) with conductivity \( \sigma (\mathbf{x}) \) in a homogeneous background medium with a constant conductivity \( \sigma_0 \). The excitation source is a point source located in the domain of \( S(\mathbf{x}^S \in S) \). The measurement is also made in the domain of \( S(\mathbf{x}^D \in S) \). In the present electrode-logging problem, we want to determine the 3-D conductivity distribution around the borehole (the object domain \( D \)) from the measurements made in the borehole (the data domain \( S \)) (see Fig. 1).

Under most situations, the low-frequency resistivity measurements are essentially electrostatic. To this end, we introduce the electric-potential field \( V \) as follows:

\[ \mathbf{E} = -\nabla V \]  \hspace{1cm} (1)

where \( \mathbf{E} \) is the electric field. Here, \( \nabla = (\partial_1, \partial_2, \partial_3) \) denotes the spatial differentiation with respect to the position vector \( \mathbf{x} = (x_1, x_2, x_3) \). By using (1) and neglecting the nonzero-frequency effects in Maxwell equations, we arrive at the so-called “stationary electric current flow equations” [2], in which the electric-current density field \( \mathbf{J} \) and the electric-potential field \( V \) occur as the basic state quantities. These basic equations are given by

\[ \nabla V + \sigma^{-1} \mathbf{J} = 0 \]  \hspace{1cm} (2)

\[ \nabla \cdot \mathbf{J} = \rho^{\text{ext}} \]  \hspace{1cm} (3)

where \( \rho^{\text{ext}} \) is the volume density of charge due to the sources.

In (2) and (3), all of the spatial variations in medium properties are contained in the conductivity \( \sigma \).

Equations (2) and (3) can be formulated in terms of an integral equation in which either the vector electric-current density or the scalar electric potential occurs as the main unknown. Abubakar [2] has shown that inversion based on the vector-integral equation for the electric-current density leads to substantially better results than inversion based on the scalar-integral equation for the electric potential. This is due to the presence of a spatial differentiation (gradient operator) inside the integral over \( D \) in the integral equation for the electric potential, which yields larger numerical errors than spatial differentiations (gradient-divergence operator) operating outside the integral over \( D \) in the integral equation for the electric current density. Thus, in the present work, we will take the integral equation for the electric-current density as our point of departure, and this vectorial-integral equation is given by

\[ \mathbf{J}'(\mathbf{x}) = \left[ 1 - \frac{\sigma(\mathbf{x}) - \sigma_0}{\sigma(\mathbf{x})} \right] \mathbf{J}(\mathbf{x}) - \nabla V \cdot \int_{\mathbf{x} \in D} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \sigma(\mathbf{x}') - \sigma_0 \mathbf{J}(\mathbf{x}') \, d\mathbf{x}' , \quad \mathbf{x} \in D \]  \hspace{1cm} (4)

where \( \mathbf{J}' \) denotes the primary electric-current density field in the absence of the contrasting domain \( D(\sigma(\mathbf{x}) = \sigma_0) \). In order to obtain an object equation in the CSI method, it is required to multiply (4) with the contrast \((\sigma - \sigma_0)/\sigma\). Therefore, to avoid the \(((\sigma - \sigma_0)/\sigma)\mathbf{J} \) term, which is highly nonlinear, we will reformulate the integral equation by using the constitutive relations

\[ \mathbf{J} = \sigma \mathbf{E} \quad \text{and} \quad \mathbf{J}' = \sigma_0 \mathbf{E}' . \]  \hspace{1cm} (5)

Substituting (5) in (4), we arrive at the integral equation for the electric field for \( \mathbf{x} \in D \)

\[ \mathbf{E}'(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - \nabla V \cdot \int_{\mathbf{x} \in D} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \chi(\mathbf{x}') \mathbf{E}(\mathbf{x}') \, d\mathbf{x}' \]  \hspace{1cm} (6)
where
\[ \chi(x) = \frac{\sigma(x) - \sigma_0}{\sigma_0} \]  
and \( E^p \) is denoted as the primary electric field and is given by
\[ E^p(x) = -\sigma_0^{-1} \nabla \int_{x' \in D} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \chi(x') \mathbf{E}(x') \, dx' \]  
in which \( \chi(x') \) is a point source
\[ \chi(x') = \delta(x - x^s). \]

In the electrode-logging problem, we are interested in the secondary electric-potential field \( V^s \) at \( x = x^R \). This is given as an integral representation over the contrasting domain \( D \), namely
\[
V^s(x^R) = -\nabla^R \cdot \left( \int_{x' \in D} \frac{1}{4\pi|x^R - x'|} \chi(x') \mathbf{E}(x') \, dx' \right)
+ \int_{x' \in D} \chi(x') \nabla^R \left[ \frac{1}{4\pi|x^R - x'|} \right] \cdot \mathbf{E}(x') \, dx'.
\]
where \( \nabla^R = (\partial_1^R, \partial_2^R, \partial_3^R) \) denotes the spatial differentiation with respect to \( x^R = (x_1^R, x_2^R, x_3^R) \), and \( \nabla^R' = (\partial_1^R, \partial_2^R, \partial_3^R) \) denotes the spatial differentiation with respect to \( x' = (x_1', x_2', x_3') \).

As mentioned before, we will take the effect of the dipping-bed environment by using a nonorthogonal coordinate system. To this end, we will formulate the integral equation in (6) in the oblique (nonorthogonal) basis. Before discussing the transformation of the integral equation, we first write it as
\[ E_\kappa - B_\kappa = E^p_\kappa, \quad (x_1, x_2, x_3) \in D, \quad \kappa \in \{1, 2, 3\} \]
where the vector \( B_\kappa \) is given by
\[ B_\kappa = \partial_\kappa \left[ \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 \right] \]
in which the normalized vector potential \( A_\kappa \) is given by
\[
A_\kappa(x_1, x_2, x_3) = \int_{x' \in D} \chi(x_1' - x_1, x_2' - x_2, x_3' - x_3) \cdot (x_1', x_2', x_3') E_\kappa(x_1', x_2', x_3') \, dx'.
\]
The scalar Green function is given by
\[ G(x_1, x_2, x_3) = \frac{1}{4\pi(x_1^2 + x_2^2 + x_3^2)^{1/2}}. \]

III. TRANSFORMATION PROCEDURE

As shown in Fig. 2, the borehole axis is defined to make an angle of \( \theta \) with respect to the vertical Cartesian axis \( (\hat{z}_3-\text{axis}) \), and the rotation is about the \( \hat{z}_1 \)-axis. The dip angle \( \theta \) is assumed to be known and constant. We therefore introduce a new coordinate system such that the deviated vertical axis \( (\hat{z}_3-\text{axis}) \) is defined to coincide with the borehole axis. By formulating and discretizing the integral equation in this oblique coordinate system, we are able to avoid the use of the staircasing approximation as shown in Fig. 3(a). Note that the use of the staircasing approximation will always need more discretization points (in the logging direction) than the model formulated in the oblique coordinate system. This aspect plays a crucial roll in the real application of the logging methods, because the size of the log interval can be very large.

A. Oblique Coordinate System

In this oblique coordinate system, the volume or the surface is defined by independent coordinate curves \( \hat{x}_1, \hat{x}_2, \) and \( \hat{x}_3 \). The relation between this coordinate and the Cartesian coordinate is given by
\[ \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \tan(\theta) \\ 0 & 0 & \cos^{-1}(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \]
The vectors associated with these curves form a nonorthogonal system, so that the covariant and the contravariant components of a vector must be introduced. Following the same procedure as in Stratton [16], the unitary vectors are given by

$$\mathbf{\hat{e}_1} = \mathbf{i}_1, \quad \mathbf{\hat{e}_2} = \mathbf{i}_2, \quad \mathbf{\hat{e}_3} = -\sin(\theta)\mathbf{i}_2 + \cos(\theta)\mathbf{i}_3.$$  

(16)

Defining the volume of the parallelepiped formed by these three vectors as \(V = \mathbf{\hat{e}_1} \cdot (\mathbf{\hat{e}_2} \times \mathbf{\hat{e}_3})\), we can form the reciprocal unitary vectors as

$$\mathbf{\check{e}_1} = \mathbf{i}_1, \quad \mathbf{\check{e}_2} = \mathbf{i}_2 + \tan(\theta)\mathbf{i}_3, \quad \mathbf{\check{e}_3} = \cos^{-1}(\theta)\mathbf{i}_3.$$  

(17)

The unitary vectors satisfy

$$\mathbf{\hat{e}_i} \cdot \mathbf{\hat{e}_j} = \delta_{ij}$$  

(18)

where \(\delta_{ij} = 1\) when \(i = j\) and \(\delta_{ij} = 0\) when \(i \neq j\) and any arbitrary vector \(\mathbf{A}\) can be expanded in the contravariant or the covariant projections on the unitary vectors or the reciprocal unitary vectors

$$\mathbf{A} = \sum_{\kappa = 1}^{3} \mathbf{\hat{A}}^\kappa \mathbf{\hat{e}}^\kappa = \sum_{\kappa = 1}^{3} \mathbf{\check{A}}^\kappa \mathbf{\check{e}}^\kappa.$$  

(19)

where \(\mathbf{\hat{A}}^\kappa\) and \(\mathbf{\check{A}}^\kappa\) are the contravariant and the covariant components and need not have the same units. The covariant components \(\mathbf{\hat{A}}^\kappa\) are related to the covariant components \(\mathbf{\check{A}}^\kappa\) as follows:

$$\begin{bmatrix} \mathbf{\hat{A}}^1 \\ \mathbf{\hat{A}}^2 \\ \mathbf{\hat{A}}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\cos(\theta)}{\cos^2(\theta)} & \frac{\sin(\theta)}{\cos^2(\theta)} \\ 0 & \frac{\sin(\theta)}{\cos^2(\theta)} & \frac{1}{\cos^2(\theta)} \end{bmatrix} \begin{bmatrix} \mathbf{\check{A}}^1 \\ \mathbf{\check{A}}^2 \\ \mathbf{\check{A}}^3 \end{bmatrix}.$$  

(20)

We can perform the volume integral with the differential elements of volume, given by

$$dv = \cos(\theta) \, dx^1 \, dx^2 \, dx^3.$$  

(21)

The gradient and divergence operators in this oblique coordinate system are given by

$$\nabla v = \partial_1 V \mathbf{\hat{e}}^1 + \partial_2 V \mathbf{\hat{e}}^2 + \partial_3 V \mathbf{\hat{e}}^3$$  

(22)

and

$$\nabla \cdot \mathbf{A} = \partial_1 \mathbf{\hat{A}}^1 + \partial_2 \mathbf{\hat{A}}^2 + \partial_3 \mathbf{\hat{A}}^3$$  

(23)

in which \(\partial_\kappa\) denotes the spatial differentiation with respect to \(\hat{x}_\kappa\) with \(\kappa \in \{1, 2, 3\}\).

**B. Integral Equation in Oblique Coordinate System**

The integral equation for the electric field in (11)–(14) can now be formulated in the oblique coordinate system. For our integral-equation modeling, the covariant components are chosen to be the field quantities. Using the transformation rules (15)–(23) in (11)–(14), we arrive at

$$\hat{E}_\kappa - \hat{B}_\kappa = \hat{E}_\kappa^\kappa, \quad (\hat{x}^1, \hat{x}^2, \hat{x}^3) \in D, \quad \kappa \in \{1, 2, 3\}$$  

(24)

where the vector \(\hat{B}_\kappa\) is given by

$$\hat{B}_\kappa = \partial_\kappa (\partial_1 \mathbf{\hat{A}}^1 + \partial_2 \mathbf{\hat{A}}^2 + \partial_3 \mathbf{\hat{A}}^3)$$  

(25)

in which \(\hat{A}_\kappa^\kappa\) for \(\kappa \in \{1, 2, 3\}\) are the contravariant components of the normalized-vector potential \(\mathbf{A}\) and can be expressed in its covariant components \(\mathbf{\check{A}}_\kappa\) [see (20)]. For this particular theory of our oblique coordinate system, we may replace the expression of the normalized-vector potential (13) in Cartesian coordinates by the covariant components \(\mathbf{\check{A}}_\kappa\) given by

$$\mathbf{\check{A}}_\kappa(\hat{x}^1, \hat{x}^2, \hat{x}^3) = \int_{C=\kappa} G(\hat{x}^1 - \hat{x}^1', \hat{x}^2 - \hat{x}^2', \hat{x}^3 - \hat{x}^3') \cdot \chi(\hat{x}^1', \hat{x}^2', \hat{x}^3') \hat{E}_\kappa'(\hat{x}^1', \hat{x}^2', \hat{x}^3') \, d\Gamma'.$$  

(26)

The Green function becomes

$$G(\hat{x}^1, \hat{x}^2, \hat{x}^3) = \frac{1}{4\pi[(\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 - 2\sin(\theta)\hat{x}^2\hat{x}^3]^{1/2}}.$$  

(27)

Once (24) is solved, the secondary electric potential \(V^s\) at \(\mathbf{x}^R = (\hat{x}^R_1, \hat{x}^R_2, \hat{x}^R_3)\) is obtained from (10) as

$$V^s(\mathbf{x}^R) = \int_{C=\kappa} \sum_{\kappa = 1}^{3} \mathbf{\hat{G}}_\kappa(\hat{x}^1 - \hat{x}^1', \hat{x}^2 - \hat{x}^2', \hat{x}^3 - \hat{x}^3') \cdot \chi(\hat{x}^1', \hat{x}^2', \hat{x}^3') \hat{E}_\kappa'(\hat{x}^1', \hat{x}^2', \hat{x}^3') \, d\Gamma'.$$  

(28)

where

$$\begin{bmatrix} \mathbf{\hat{E}}^1 \\ \mathbf{\hat{E}}^2 \\ \mathbf{\hat{E}}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\cos(\theta)}{\cos^2(\theta)} & \frac{\sin(\theta)}{\cos^2(\theta)} \\ 0 & \frac{\sin(\theta)}{\cos^2(\theta)} & \frac{1}{\cos^2(\theta)} \end{bmatrix} \begin{bmatrix} \mathbf{\hat{G}}_1 \\ \mathbf{\hat{G}}_2 \\ \mathbf{\hat{G}}_3 \end{bmatrix}.$$  

(29)

and \(\mathbf{\hat{G}}_\kappa\) is the gradient of the scalar Green function and is given by

$$\mathbf{\hat{G}}_\kappa(\hat{x}^1, \hat{x}^2, \hat{x}^3) = \frac{\hat{x}^\kappa}{4\pi[(\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 - 2\sin(\theta)\hat{x}^2\hat{x}^3]^{3/2}}.$$  

(30)

$$\mathbf{\hat{G}}_1(\hat{x}^1, \hat{x}^2, \hat{x}^3) = \frac{\hat{x}^3}{4\pi[(\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 - 2\sin(\theta)\hat{x}^2\hat{x}^3]^{3/2}}$$  

(31)

and

$$\mathbf{\hat{G}}_1(\hat{x}^1, \hat{x}^2, \hat{x}^3) = \frac{\sin(\theta)\hat{x}^2 + \hat{x}^3}{4\pi[(\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 - 2\sin(\theta)\hat{x}^2\hat{x}^3]^{3/2}}.$$  

(32)

We observe that the integral equation (24) is a singular integral equation, in which the gradient-divergence operator acts on a normalized-vector potential. Numerical implementation of such an integral equation must be carried out carefully. This discretization procedure is discussed in Appendix I.
IV. FORWARD-SCATTERING PROBLEM

From the results of Appendix I, when we substitute (76)–(77) in (70)–(72), and using the results in (69), we obtain a linear system of equations for the electric field $\mathbf{E}$ when the conductivity contrast $\chi$ is known. This linear system of equations can be written compactly in an operator notation. The operator expression $\mathbf{LE}$ is directly obtained from the left-hand side of (69) of the Appendix.

Since the matrix operator consists of spatial convolutions, we can use an FFT routine advantageously (see Zwamborn and Van den Berg [17]). However, we then need an iterative solution, and a conjugate-gradient method seems to be one of the most efficient methods. With this so-called CGFFT technique, we are able to solve complex 3-D problems efficiently. Furthermore, it also gives the fundamentals of our solution to the inverse problem. We observe that the matrix describing this linear system of equation is nonsymmetric. Therefore, we need the adjoint operator to set up the conjugate-gradient scheme. The matrix form of this adjoint operator is given in the Appendix. In such a scheme, we need the definition of the norm and inner product. The norm on $D$ is defined as

$$||\mathbf{E}||_D^2 = \sum_{s=1}^{3} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=1}^{P} \hat{E}_{k,m,n,p} \hat{E}^*_{m,n,p}$$

where $\hat{E}_{k,m,n,p}$ and $\hat{E}^*_{k,m,n,p}$ are the discretized contravariant and covariant components of the vector $\mathbf{E}$ (see (66) of the Appendix).

With these definitions, we are now able to apply a conjugate-gradient iterative scheme to solve the operator equations

$$\mathbf{(LE)}|_{k,m,n,p} = \hat{E}^p_{k,m,n,p}$$

for $m = 1, \cdots, M$, $n = 1, \cdots, N$, and $p = 1, \cdots, P$, where $\mathbf{(LE)}|_k$ denotes the covariant components of the vector $\mathbf{LE}$. Once the normalized error

$$\text{ERR}^{1/2} = \frac{||\mathbf{r}||_D}{||\mathbf{E}^*||_D}$$

where

$$\mathbf{r} = \mathbf{E}^* - \mathbf{LE}$$

is small enough, the approximate solution of $\mathbf{E}$ is substituted in (28) to arrive at secondary potential field

$$V^s(\mathbf{x}^R) = \cos(\theta) \Delta^1 \Delta^2 \Delta^3 \sum_{k=1}^{3} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=1}^{P} \hat{G}_k \hat{E}^p_{m,n,p} \cdot \chi_{m,n,p} \mathbf{L}_{m,n,p}$$

V. INVERSE PROBLEM

We now assume that the inhomogeneous object $D$ is irradiated successively by a number ($i = 1, \cdots, I$) of known primary electric fields. For each excitation, the forward-scattering problem may be reformulated as the domain-integral equation [cf. (34)]

$$\mathbf{LE}^{(i)} = \mathbf{E}^{p(i)}$$

where the operator $\mathbf{L}$ depends on the conductivity contrast $\chi$. To show this explicitly, we rewrite this operator equation as

$$\mathbf{E}^{(i)} - \mathbf{K}_D \chi \mathbf{E}^{(i)} = \mathbf{E}^{p(i)}$$

where $\mathbf{K}_D \chi \mathbf{E}^{(i)}$ follows from (70)–(72) by replacing $\hat{A}_m(\mathbf{r})$ with $\hat{A}_m(\mathbf{r})$. Equation (39) is denoted as the object equation that holds in the object domain $D$.

Further, in the inverse problem the secondary potential field is known or measured at the measurement points $\mathbf{x}^R$. We assume that all the measurement points are located in the data domain $S$, outside $D$, see Fig. 1. We can also write (37) in the shorthand notation,

$$\mathbf{K}_S \cdot \chi \mathbf{E}^{(i)} = V^s(\mathbf{i})$$

where $V^s(\mathbf{i})$ follows from (37) after replacing $\hat{E}^p_{m,n,p}$ with $\hat{E}^p_{m,n,p}$. Equation (40) is denoted as the data equation that holds in the data domain $S$.

The data equation contains both the unknown field and the unknown conductivity contrast, but they occur as a product that can be considered as a contrast source that produces the secondary electric-potential field at the measurement points. Hence, there is no unique solution to the problem of inverting the data equation itself. The contrast source inversion (CSI) method attempts to overcome this difficulty by recasting the problem as an optimization, in which we seek not only the contrast sources but also the conductivity contrast itself to minimize a cost functional consisting of two terms, the $L_2$ errors in the data equation and in the object equation, and rewritten in terms of the contrast and the contrast sources rather than the fields. An alternating method of iteratively solving this optimization problem is proposed, in which first the contrast sources are updated in the conjugate-gradient step weighted so as to minimize the cost functional, and then the conductivity contrast is updated to minimize the error in the object equation using the updated sources. This latter minimization can be carried out analytically.

To this end, we introduce the contrast source $\mathbf{W}$ as follows:

$$\mathbf{W}^{(i)} = \chi \mathbf{E}^{(i)}$$

The data equation becomes

$$\mathbf{K}_S \cdot \mathbf{W}^{(i)} = V^s(\mathbf{i})$$

In order to arrive at a suitable object equation for the CSI method, we multiply (39) with the conductivity contrast $\chi$, and by using (41), we arrive at

$$\mathbf{W}^{(i)} - \chi \mathbf{K}_D \mathbf{W}^{(i)} = \chi \mathbf{E}^{p(i)}$$

These latter two equations are the basic equations to develop the CSI method.
A. Inversion Method

In the present method, we alternatingly construct sequences of contrast sources $W_k^{(i)}$, fields $E_k^{(i)}$, and conductivity contrast $\chi_k$ for $k \geq 1$ to minimize a cost functional iteratively. The cost functional in the $k$th iteration is defined as

$$ F_k = n_S \sum_i \left| \rho_k^{(i)} \right|^2_S + n_{D,k-1} \sum_i \left| r_k^{(i)} \right|^2_D $$  \hspace{1cm} (44)

where

$$ n_S = \left( \sum_i \left| V_s^{(i)} \right|^2_S \right)^{-1} $$

$$ n_{D,k-1} = \left( \sum_i \left| \chi_{k-1} E_k^{(i)} \right|^2_D \right)^{-1} $$ \hspace{1cm} (45)

in which the norm on $S$ is given by

$$ \left| V^{(i)} \right|^2_S = \sum_{\mathbf{x} \in S} \left| V^{(i)} (\mathbf{x}) \right|^2 $$ \hspace{1cm} (46)

and the norm on $D$ is given in (33). This cost functional $F_k$ consists of two terms. The first term measures the error in the data equation and is defined to be

$$ \rho_k^{(i)} = V_s^{(i)} - \mathbf{K}_S \cdot W_k^{(i)}. $$ \hspace{1cm} (47)

The second term measures the error in the object equation and is defined to be

$$ r_k^{(i)} = \chi_k E_k^{(i)} - W_k^{(i)} $$ \hspace{1cm} (48)

where

$$ E_k^{(i)} = E_s^{(i)} + \mathbf{K}_D W_k^{(i)} $$ \hspace{1cm} (49)

The normalization in the cost functional $F_k$ in (44) is chosen so that both terms are equal to one if the contrast source $W_{k-1}^{(i)}$ vanishes. Note that the object equation acts as a regularization for the data equation, and we have not employed other regularization techniques.

To solve this inverse problem, we propose an iterative minimization of the cost functional $F_k$ using an alternating method that first updates the contrast sources $W_k$, and then updates the conductivity contrast $\chi_k$. Thus, we construct sequences $\{W_k^{(i)}\}$ and $\{\chi_k\}$ for $k \geq 1$. Now suppose $W_{k-1}^{(i)}$ and $\chi_{k-1}$ are known. We update $W^{(i)}$ as follows:

$$ W_k^{(i)} = W_{k-1}^{(i)} + \alpha_k w_k^{(i)} $$ \hspace{1cm} (50)

where $\alpha_k$ is a constant that is determined by minimizing the cost functional $F_k$, and $w_k^{(i)}$ are the update directions of the contrast sources. In order to determine the constant $\alpha_k$, we observe that the errors satisfy the following recursive relations for the data error

$$ \rho_k^{(i)} = \rho_{k-1}^{(i)} - \alpha_k \mathbf{K}_S \cdot w_k^{(i)} $$ \hspace{1cm} (51)

and for the object error

$$ r_k^{(i)} = r_{k-1}^{(i)} - \alpha_k \left( w_k^{(i)} - \chi_{k-1} \mathbf{K}_D w_k^{(i)} \right). $$ \hspace{1cm} (52)

Substituting these residual errors in (44) and minimizing this cost functional, the constant $\alpha_k$ is determined explicitly to be

$$ \alpha_k = \left( n_S \sum_i \left( \rho_k^{(i)} \cdot \mathbf{K}_S \cdot w_k^{(i)} \right)_S + n_{D,k-1} \sum_i \left( r_k^{(i)} \cdot w_k^{(i)} - \chi_{k-1} \mathbf{K}_D w_k^{(i)} \right)_D \right) \times \left( n_S \left| \mathbf{K}_S \cdot w_k^{(i)} \right|^2_S + n_{D,k-1} \left| w_k^{(i)} - \chi_{k-1} \mathbf{K}_D w_k^{(i)} \right|^2_D \right)^{-1} \hspace{1cm} (53)

Once the contrast source $W_k^{(i)}$ is determined, the field $E_k^{(i)}$ is obtained via (49). With $E_k^{(i)}$ and $W_k^{(i)}$, the approximate conductivity contrast $\chi_k$ follows by minimizing the following cost functional $F_{D,k}$

$$ F_{D,k} = \sum_i \left| \chi_k E_k^{(i)} - W_k^{(i)} \right|^2_D $$ \hspace{1cm} (54)

leading to

$$ \chi_k = -\frac{\sum_i \left| E_k^{(i)} \cdot E_k^{(i)} \right|^2}{\sum_i \left| E_k^{(i)} \right|^2} $$ \hspace{1cm} (55)

B. Conductivity Constraints

The form of the cost functional $F_{D,k}$ is a particularly simple one and allows for easy implementation of a priori information or constraint on the conductivity contrast $\chi_k$. In the electrode-logging problem, we have the a priori information that the conductivity $\sigma$ is a positive quantity. Abubakar [2] has demonstrated that by explicitly incorporating this a priori information into the modified-gradient algorithm, the reconstruction of high contrast can be achieved. To incorporate this information into our inversion scheme, we note that $\alpha / \sigma_0$ is a positive quantity. Following a similar procedure (as used by Van den Berg and Kleinman [6]) to minimize the cost functional $F_{D,k}$ in (54), we obtain the following approximation:

$$ \chi_k = \frac{\sigma_k}{\sigma_0} - 1 = \left\{ \frac{\sum_i \left[ \frac{E_k^{(i)} \cdot W_k^{(i)}}{E_k^{(i)} \cdot E_k^{(i)}} + \left| E_k^{(i)} \right|^2 \right]^{1/2}}{\sum_i \left| E_k^{(i)} \right|^2} \right\}^{1/2} - 1. $$ \hspace{1cm} (56)

In the practical applications, we have a priori information about the conductivity distribution such as that around the borehole, the conductivity distribution cannot change abruptly. This means that the formation is horizontally symmetrical. When we have such a priori information, we can enforce this constraint by assigning, in each iteration, the mean value of the conductivity.
contrast $\chi_k$, in (55) or (56), to all constant points that have the same horizontal distance to the borehole axis ($\xi^3$-axis).

C. Update Directions

The iterative algorithm will be completely specified when the updated directions $\mathbf{w}_k^{(i)}$ are specified. The update directions $\mathbf{w}_k^{(i)}$ are chosen as the Polak-Ribiére conjugate-gradient directions

$$\mathbf{u}_0^{(i)} = 0, \quad \mathbf{u}_k^{(i)} = \partial \mathbf{w}_k^{(i)} + \gamma_k^{(i)} \mathbf{w}_k^{(i-1)}, \quad k \geq 1$$

(57)

where

$$\gamma_k^{(i)} = \frac{\sum_i \langle \partial \mathbf{u}_k^{(i)}, \partial \mathbf{u}_k^{(i)} \rangle_D}{\sum_i \| \partial \mathbf{u}_k^{(i)} \|^2_D}$$

(58)

in which $\partial \mathbf{u}_k^{(i)}$ is the gradient (Frechet derivative) of the cost functional $F_k$ in (44) with respect to $\mathbf{W}_k^{(i)}$ evaluated at $\mathbf{W}_k^{(i)}$ and $\chi_{k-1}$. Explicitly, the gradient is found to be

$$\partial \mathbf{u}_k^{(i)} = -n_\xi \mathbf{K}^e \mathbf{S}_k^{(i)} \cdot n_D \left( \mathbf{r}_k^{(i)} - \mathbf{K}^e \chi_{k-1} \mathbf{r}_k^{(i-1)} \right).$$

(59)

The various operators in covariant form are calculated as (omitting the iteration index $k = 1$)

$$\left( \mathbf{K}^e \mathbf{r}_k^{(i)} \right)_{\xi \eta \gamma, r, s} = \mathbf{C}^e \mathbf{r}_k^{(i)}_{\xi \eta \gamma, r, s},$$

$$\left( \mathbf{S}_k^{(i)} \right)_{\xi \eta \gamma, r, s} = \cos(\theta) \Delta x_1 \Delta x_2 \Delta x_3 \sum_{\xi' \eta' \gamma'} \rho^{(i)}(\xi' \eta' \gamma')$$

$$\cdot \mathbf{C}^{(i)}_{\xi' \eta' \gamma', \xi \eta \gamma} - \mathbf{C}^{(i)}_{\xi' \eta' \gamma, \xi \eta \gamma} - \mathbf{C}^{(i)}_{\xi' \eta \gamma, \xi \eta \gamma} + \mathbf{C}^{(i)}_{\xi \eta \gamma, \xi \eta \gamma},$$

(60)

where $\mathbf{C}^{(i)}_{\xi' \eta' \gamma, \xi \eta \gamma}$ is given in (80) by replacing $\xi' \eta' \gamma$, $r$, and $s$ in (81)–(83) with $\chi_{r, s} \mathbf{r}^{(i)}_{\xi \eta \gamma}$.

D. Starting Values

This completes the description of the algorithm except for designating the starting values for the contrast sources $\mathbf{W}_0^{(i)}$. Observe that we cannot start with $\mathbf{W}_0^{(i)} = 0$, since then $\chi_0 = 0$ and the cost functional $F_0$ in (44) is undefined for $k = 1$. Therefore, for starting values, we choose the values obtained by the backpropagation

$$\mathbf{W}_0^{(i)} = \frac{\| \mathbf{S}_1 \mathbf{V}_s^{(i)} \|_2^2}{\| \mathbf{K} \mathbf{S}_1 \mathbf{V}_s^{(i)} \|_2^2} \mathbf{S}_1 \mathbf{V}_s^{(i)}.$$ (62)

This completes the description of the inversion algorithm.

VI. Numerical Examples

In order to investigate the performance of the present inversion method, we will demonstrate two examples of inversion of synthetic “measured” data. The synthetic data were generated by solving the forward-scattering problem numerically with the help of the conjugate-gradient FFT method as described in Section IV. Note that in all of our numerical examples, we have not taken the borehole effect into account. Thus, we assume that the borehole is always filled with the conductivity of the background medium $\sigma_0$.

As first example, in order to investigate the possibility of whether or not we have inadvertently committed the “inverse crime,” we will invert the synthetic data generated by the forward solver using the Cartesian coordinate system. As a second example, we will consider a more complicated configuration.

A. Example 1

In this example, the forward and inversion procedures using the oblique coordinate system will be compared to the ones using the Cartesian coordinate system. The exact model (either in a Cartesian or in an oblique system) is given in Fig. 3. A single layer (the ‘black’ layer) with conductivity 0.9 S/m is embedded in an unbounded homogeneous background with conductivity $\sigma_0 = 0.3$ S/m. In this configuration, the borehole of radius 0.15 m is filled with the conductivity of the background medium $\sigma_0 = 0.3$ S/m. The borehole is deviated with a dip angle of 45° ($\theta = 0.7854$ rad).

1) Forward Solution: The configuration using the Cartesian coordinate system is given in Fig. 3(a). Here, the configuration is located entirely within an object (test) domain of 1.51.50.7 m. Then, the test domain is described by $x_1$, $x_2$, and $x_3$. This test domain was partitioned into equal-sized subdomains with side lengths $\Delta x_1 = \Delta x_2 = \Delta x_3 = 0.05$ m. Hence, $M = N = 30$ and $P = 14$. Thus, we have 12,600 discretization points. Note that the dipping-bed environment is approximated by using the staircase-discretization grid. While the configuration using the oblique coordinate system is located entirely within a test domain of 1.1×1.1×0.99 m (in the oblique coordinate system) as shown in Fig. 3(b). This test domain is discretized into equal-sized subdomains with side lengths $\Delta x_1 = \Delta x_2 = \Delta x_3 = 0.05$ m. Hence, $M = N = 22$ and $P = 14$. Then we have 6776 discretization points.

In Fig. 4, for a given source position at $\delta^{33}$, we present the secondary electric-potential field for different receiver positions $\delta^{R3}(-1 \leq \delta^{R3} \leq 1)$. The results using the oblique coordinate system with $22 \times 22 \times 14$ discretization points are given by the solid line, while the one using the Cartesian coordinate system with $30 \times 30 \times 14$ discretization points is given by the dashed line. We observe that the difference between the two forward solutions is large in the region nearby the source. Further away from the source, the agreements are very good.

Next, we use a finer discretization grid in the $x_3$ direction for the forward solution using the Cartesian coordinate system. Now, the test domain is discretized into equal-sized subdomains with side lengths $\Delta x_1 = \Delta x_2 = 0.05$ m and $\Delta x_3 = 0.025$ m. Hence, $M = N = 30$ and $P = 28$. Thus, we have 25,200 discretization points. The results using $30 \times 30 \times 28$ discretization points are given by the plus sign in Fig. 4. We observe that the forward solution using the Cartesian coordinate system with finer discretization grid tends to be the one using the oblique system. In Fig. 4, we also give the forward solution using the oblique coordinate system with $22 \times 22 \times 28$ discretization
Fig. 4. Secondary electric-potential field \( V^S(\mathbf{r}^R) \) simulated for the input configuration given in Fig. 3.

points \((\Delta\delta^3 = 0.035)\). The results are given by the stars. Here we observe that the forward solution using the oblique coordinate system remains stable. Thus, it is clear that by using the oblique coordinate system, we can suffice with less discretization points to obtain the same results with the same degree of accuracy. This aspect plays an important role in the real applications of the logging methods, because the logging interval to be investigated can be very large.

2) Inverse Solution: To test our inversion method, we use the same numerical example as in the forward solution with a measurement setup consisting of five stations at the borehole axis \((\delta^3, \delta^3)\) with spacing of 0.2 m as shown in Fig. 3. Each station is denoted with an “×” in Fig. 3. The sources/receivers position are given by \(\delta^S; 3 = \delta^R; 3 = 0.4, -0.2, 0, 0.2, 0.4\) m, respectively. For a given source position, the secondary electric-potential field \( V^S(\mathbf{r}) \) is measured at all receiver positions. Thus, in this numerical example, we use only 25 data points.

In order to verify the validity of our approach, we have used our inversion method to invert the synthetic data generated by the forward solver using the Cartesian coordinate system with 30 \(\times\) 30 \(\times\) 28 discretization points. The input configuration that has been used to generate the synthetic data is given in Fig. 5(a).

The reconstruction, it was assumed that the test domain was located entirely within a test domain of \(1.1 \times 1.1 \times 0.99\) m\(^3\) (in the oblique coordinate system) as shown in Fig. 3(b). The test domain is described by \(-0.55 < \delta^1 < 0.55\) m, \(-0.55 < \delta^2 < 0.55\) m, and \(-0.495 < \delta^3 < 0.495\) m. This test domain was partitioned into equal-sized subdomains with side lengths \(\Delta\delta^1 = \Delta\delta^2 = 0.05\) m and \(\Delta\delta^3 = 0.354\) m. Hence, \(M = N = 22\) and \(P = 14\). Then we have 6776 discretization points.

The reconstruction after 256 iterations using the Cartesian data is given in Fig. 5(b). The figure gives the cross-sections of the 3-D conductivity distributions \((\sigma)\) in the \((\delta^2, \delta^3)\)-plane at \(x_1 = 0\). We note that after 256 iterations, the square root of the cost functional \(F_k^{1/2}\) has decreased to 0.18%. By including some constraints in the inversion procedure, we have reduced the solution space. This does not imply that the cost functional of the inversion procedure with constraint will be smaller than the cost functional of the inversion procedure without constraint. Therefore, in order to give a more indicative measure of the reconstructed profile, we introduce the error in the conductivity at the \(k\)th iteration as follows:

\[
\text{ERR}_k = \frac{\|\sigma_k - \sigma_0\|_D^2}{\|\sigma_0\|_D^2}.
\]
The square root of the error in the conductivity $ERR_k^{1/2}$ of the reconstructed profile without constraint amounts to 0.36265, while with constraint it amounts to 0.30171. Furthermore, in order to have more detailed information about the reconstructed profiles, we present in Fig. 6 six slices of the conductivity distribution in the $\hat{x}^3$ direction of the actual profiles (solid lines) and the reconstructed profiles without (dashed lines) and with constraint (dashed-dotted lines) for a fixed position of $\hat{x}^1$ and $\hat{x}^2$. The positions of $\hat{x}^1$ and $\hat{x}^2$ are given by $\hat{x}^1 = -0.3$, $-0.2$, $-0.1$, $0.1$, $0.2$, $0.3$ m.

In Fig. 7, we also give the reconstruction using the synthetic data generated by the forward code using the oblique-coordinate system. The input configuration is given in Fig. 7(a). The reconstruction result after 256 iterations without horizontally symmetrical constraints is given in Fig. 7(b), and with constrains, after 256 iterations the square root of the error functional $J_k^{1/2}$ has decreased to 0.20% (without constraint) and 0.22% (with constraint). Further iterations do not yield substantial improvement. The square root of the error in the conductivity $ERR_k^{1/2}$ of the reconstructed profile without constraint amount to 0.36832, while with constraint, they amount to 0.29878. We present again in Fig. 8 six slices of the conductivity distribution in the $\hat{x}^3$ direction of the actual profiles (solid lines) and the reconstructed profiles without (dashed lines) and with constraint (dashed-dotted lines) for a fixed position of $\hat{x}^1$ and $\hat{x}^2$. By comparing Figs. 5 and 6 and Figs. 6 and 8, we observe that the agreement between the reconstruction results from the two data sets are remarkable.

B. Example 2

We now consider a formation consisting of four beds (two with invasion), as shown in Fig. 9. The conductivity from top to bottom is 0.9, 0.45, 0.1, and 0.45 S/m. The vertical ($\hat{x}^3$) dimension of each layer is 0.3 m. The conductivity of the invasion zone of the first (top) layer is 0.6 S/m and of the third layer is 0.2 S/m. This formation is embedded in an unbounded homogeneous domain with conductivity $\sigma_0 = 0.3$ S/m. The radius of the borehole is 0.1 m, and the borehole is deviated with a dip angle of 45° ($\theta = 0.7854$ rad).
Fig. 9. Geometry of the four dipping-beds problem (two with invasion zones). This figure gives the cross-section of the 3-D conductivity distribution ($\sigma$) in the ($x_2$, $x_3$)-plane at $x_1 = 0$. The cross (×) denotes the source/receiver position.

Fig. 10. Conductivity distribution ($\sigma$) of the exact profile (a) and of the reconstruction results (b) in the ($x_2$, $x_3$)-plane at $x_1 = 0$, with the background conductivity $\sigma_0 = 0.3$ S/m.

In this numerical example, we consider 15 different source and receiver positions at the borehole axis ($\delta^3$-axis), with spacing of 0.1 m. The source/receiver positions are given by $\delta^3 = \delta_{S_i}^3 = -1.4$, $-1.2$, $\ldots$, $-0.2$, $0$, $0.2$, $\ldots$, $1.2$, $1.4$ m, respectively. Each station (source/receiver) is denoted with an “×” in Fig. 9. For a given source position, the secondary electric-potential field $V_{\text{sec}}(x)$ is measured at all receiver positions. Thus, we have 225 data points. In the reconstruction, it was assumed that the unknown configuration was located entirely within an object (test) domain of $1.4 \times 1.4 \times 1.4$ m$^3$. Then, the test domain is described by $-0.7 < \delta^1 < 0.7$ m, $-0.7 < \delta^2 < 0.7$ m, and $-0.7 < \delta^3 < 0.7$ m. This test domain was partitioned into equal-sized subdomains with side lengths $\Delta \delta^1 = \Delta \delta^2 = \Delta \delta^3 = 0.1$ m. Hence, $M = N = P = 14$. Thus, the total discretization points amount to 2744.

Fig. 11. Same as Fig. 10, but now in the ($\delta^2$, $\delta^3$)-plane at different levels of $\delta^1$.

We present in Fig. 10 the conductivity ($\sigma$) of the exact profile, along with that from the reconstruction, after 1024 iterations in the ($x_2$, $x_3$)-plane at $x_1 = 0$. Note that in this example, the horizontally symmetrical constraint has been used. We note that after 1024 iterations, the square root of the cost functional $f^{1/2}_{\text{cost}}$ has decreased to 0.27%, and further iterations do not yield substantial improvement. The square root of the error in the conductivity $\text{ERR}^{1/2}_{\text{cost}}$ of the reconstructed profile amounts to 0.1690. In Fig. 11, we also give the conductivity ($\sigma$) distribution in the ($\delta^1$, $\delta^2$)-plane at different
Fig. 13. Conductivity distribution ($\sigma$) of the exact profile (a) and of the reconstruction results (b) in the $(x_2, x_3)$-plane at $x_1 = 0$, with the background conductivity $\sigma_0 = 0.6$ S/m.

Fig. 14. Same as Fig. 13, but now in the $(x^3, x^2)$-plane at different levels of $x^3$.

Fig. 15. Conductivity distribution ($\sigma$) of the exact profiles (solid lines), and those of the reconstructed profiles (dashed-dotted lines) as a function of $x^3$ for a fixed position of $x^1$ and $x^2$, with the background conductivity $\sigma_0 = 0.6$ S/m.

levels of $x^3$. The $x^3$-levels from top to bottom are $-0.55, -0.35, -0.25, 0.15, 0.35,$ and $0.55$ m. The exact profile is given in Fig. 11(a), and the inversion result after 1024 iterations is given in Fig. 11(b). In order to have more detailed information about the reconstructed profiles, we present in Fig. 12 six slices of the conductivity distribution in the $x^3$-direction of the actual profiles (solid lines) and the reconstructed profiles (dashed-dotted lines), for a fixed position of $x^1$ and $x^2$. The positions of $x^1$ and $x^2$ are given by $x^1 = 0$ and $x^2 = -0.55, -0.35, -0.15, 0.15, 0.35,$ and $0.55$ m.

Next, we also must consider the same configuration as given in Fig. 10, but now the unknown configuration is embedded in the unbounded homogeneous medium with conductivity $\sigma_0 = 0.6$ S/m. In Fig. 13(a), we again give the exact conductivity profile in the $(x_2, x_3)$-plane at $x_1 = 0$. The reconstruction after 1024 iterations is given in Fig. 13(b). In Fig. 14, we give the exact profile and the reconstruction after 1024 iterations in the $(x^1, x^2)$-plane at different levels of $x^3$. We note that after 1024 iterations, the square root of the cost functional $F_h^{1/2}$ has decreased to 1.27%, and further iterations do not yield substantial improvement. The square root of the error in the conductivity ERR$^{1/2}$ of the reconstructed profile amounts to 0.17904. In order to have more detailed information about the reconstructed profiles, we present in Fig. 15 six slices of the conductivity distribution in the $x^3$-direction of the actual profiles (solid lines) and the reconstructed profiles (dashed-dotted lines) for a fixed position of $x^1$ and $x^2$. We note that the inversion results with $\sigma_0 = 0.3$ S/m yield better results than the inversion results with $\sigma_0 = 0.6$ S/m. This can be understood because the conductivity contrast of the formation with $\sigma_0 = 0.6$ S/m is higher than the contrast with $\sigma_0 = 0.3$ S/m.

We have also tested our forward-inversion procedure with other configurations. We have observed that if we use a larger deviated angle (close to 90°), more iterations in the forward-inversion procedure will be needed in order to get numerical results with the same degree of accuracy as the example present in this paper. The reason is that with a larger deviated angle, the problem becomes more 3-D in nature. For horizontal, we use the forward-inversion procedure with the Cartesian coordinate system should be used.

All computations were carried out on a 200 MHz Pentium Pro Workstation running Windows NT 4.0. The Fortran code was compiled using a digital visual fortran 5.0 compiler, and the
computation time for one iteration in the inversion of the last example was 20 s. The use of parallel computers to compute the 3-D FFT routines will decrease this computation time significantly. If, in practice, the computer time used to invert large log intervals is too much, the use of the nonregular grid becomes necessary. This is a point for future research.

VII. CONCLUSIONS

In the present paper, a new approach to reconstruct a 3-D conductivity distribution in a highly deviated well formation with invasion from the electrode-logging data has been described. The integral equation in an oblique coordinate system is taken as our point of departure to develop the forward scheme using the conjugate-gradient method and inversion scheme using the contrast-source inversion method. By using the oblique (nonorthogonal) coordinate system, we have gained some advantages above the usual approach. Firstly, the use of the staircasing approximation for the dipping bed environment can be avoided. This means that we reduce the discretization error, and we can suffice with less discretization points to obtain the results with the same degree of accuracy of the problem formulated in the Cartesian coordinate system. Secondly, the horizontally symmetrical constraints of the conductivity distribution can be included easily in the inversion procedure. Note that if the horizontally symmetrical constraint cannot be used in the inversion procedure, the smaller-number-of-discretization-points requirement is still a great advantage in the use of the oblique coordinate system. The results of the numerical examples have shown that the present inversion method may be used as a new interpretation technique for the electrode-type resistivity measurements. Furthermore, the oblique coordinate system is a prerequisite to handle the highly deviated logging problem.

APPENDIX I

A. Discretization Procedure

We assume that the domain $D$ is a domain with boundaries along the $x^1$, $x^2$, and $x^3$-directions. We discretize the domain in a parallelogram mesh. The mesh is uniformly spaced in the $x^1$ and $x^2$-directions. The subdomains with a width of $\Delta x^1$ in the $x^1$-direction, $\Delta x^2$ in the $x^2$-direction, and $\Delta x^3$ in the $x^3$-direction are given by

$$D_{m,n,p} = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid x^1_m = \frac{1}{2} \Delta x^1, x^2_n = \frac{1}{2} \Delta x^2, x^3_p = \frac{1}{2} \Delta x^3 \}$$

for $m = 1, \ldots, M$, $n = 1, \ldots, N$, $p = 1, \ldots, P$, where

$$x^1_m = \frac{1}{2} \Delta x^1 + (m - \frac{1}{2}) \Delta x^1, \quad x^2_n = \frac{1}{2} \Delta x^2 + (n - \frac{1}{2}) \Delta x^2, \quad x^3_p = \frac{1}{2} \Delta x^3 + (p - \frac{1}{2}) \Delta x^3,$$

in which $x^1_{1/2}$ is the lower $x^1$-bound of the contrasting domain $D$, $x^2_{1/2}$ is its lower $x^2$-bound, and $x^3_{1/2}$ is its lower $x^3$. In each subdomain $D_{m,n,p}$ with center points $(x^1_m, x^2_n, x^3_p)$, we assume the conductivity contrast $\chi$ to be constant with value the same as the value at the center point $x^1_m, x^2_n, x^3_p$. We assume that the boundary of the domain $D$ lies completely outside the contrasting object. Using this spatial discretization grid, the covariant field quantities are defined as follows

$$\hat{E}_{k;m,n,p} = \hat{E}_{k}(x^1_m, x^2_n, x^3_p), \quad k \in \{1, 2, 3\}$$

(66)

(67)

(68)

Then (24) is discretized as

$$\hat{B}_{m,n,p} = \hat{B}_{k;m,n,p} = \hat{E}_{k}(x^1_m, x^2_n, x^3_p) \in D$$

(69)

for $m = 1, \ldots, M$, $n = 1, \ldots, N$, $p = 1, \ldots, P$, where $\hat{B}_{m,n,p}$ given in (25) is computed with the finite-difference rule [1], and the results are given as

$$\hat{B}_{1;m,n,p} = (\Delta x^1)^{-2}(A_{m-1,n,p} - 2A_{m,n,p} + A_{m+1,n,p}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m-1,n-1,p} - A_{m+1,n-1,p} - A_{m-1,n+1,p} + A_{m+1,n+1,p}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m-1,n,p-1} - A_{m-1,n,p+1} - A_{m,n-1,p+1} + A_{m,n+1,p+1}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m+n-1,p} - A_{m+n+1,p}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m-1,n-1,p} - A_{m+l_1,n-1,p} - A_{m-1,n+1,p} + A_{m+l_1,n+1,p}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m-1,n,p-1} - A_{m-1,n,p+1} - A_{m,n-1,p+1} + A_{m,n+1,p+1}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m,n-1,p} - A_{m,n+1,p}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m-1,n-1,p} - A_{m-1,n+1,p} - A_{m,n-1,p+1} + A_{m,n+1,p+1}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m,n-1,p} - A_{m,n+1,p}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m-1,n-1,p} - A_{m-1,n+1,p} - A_{m,n-1,p+1} + A_{m,n+1,p+1}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m,n-1,p} - A_{m,n+1,p}) + \frac{1}{2}(\Delta x^1)^{-2}(A_{m-1,n-1,p} - A_{m-1,n+1,p} - A_{m,n-1,p+1} + A_{m,n+1,p+1})$$

(70)

$$(71)$$

$$(72)$$

Next, we have to replace the continuous representation of the vector potential $A$ by a discrete one. In order to cope with the singularity of the Green function, we take first the spherical mean of the vector potential $A$. We integrate $A$ over a spherical domain with center at the center of subdomain $(x^1_m, x^2_n, x^3_p)$. The radius of these patches is taken to be $\frac{1}{2}\Delta x^3$. The results are divided by the volume of the spherical domain with radius $\frac{1}{2}\Delta x^3$. We then may write

$$\begin{bmatrix} A_{1,m,n,p} \\ A_{2,m,n,p} \\ A_{3,m,n,p} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2(\theta) & \sin(\theta) \\ 0 & \sin(\theta) & \cos^2(\theta) \end{bmatrix} \begin{bmatrix} \hat{A}_{1;m,n,p} \\ \hat{A}_{2;m,n,p} \\ \hat{A}_{3;m,n,p} \end{bmatrix}$$

(73)
where
\[
\hat{A}_{k,m,n,p} = \hat{A}(\hat{x}_m, \hat{x}_n, \hat{x}_p) = \int_{\mathbb{C}/D} G(\hat{x}_m, \hat{x}_n, \hat{x}_p) \cdot (\chi(\hat{x}_m')) \partial_x(\hat{x}_n') \partial_x(\hat{x}_p') \partial_x^{(3)} \hat{E}(\hat{x}_m', \hat{x}_n', \hat{x}_p') \mathrm{d}x' \quad (74)
\]
in which we have interchanged the order of integrations such that
\[
G(\hat{x}_m', \hat{x}_n', \hat{x}_p') = \frac{3}{4\pi \Delta x^3} \int_{|x'| < \frac{1}{2} \Delta x} G(\hat{x}_m', \hat{x}_n', \hat{x}_p') \mathrm{d}x' \quad (75)
\]
Note that this weakening of the singularity is different from the technique used by Van Bladel [8], where the spatial differentiations are acting on the Green function directly, while we compute the integral over the normalized vector potential \( \hat{A}_p \). Here, the Green function has been weakened by taking its spherical meridional and subsequenty, the differentiations are carried out numerically on the normalized vector potential \( \hat{A}_p \). This has proven to yield an efficient stable and accurate algorithm [17]. After this weakening procedure, we are now able to compute the integral in (74) numerically. In view of the functional properties of \( \hat{E}^n \), we approximate the integral in (74) using a midpoint rule. We then arrive at
\[
\hat{A}_{k,m,n,p} = \cos(\theta) \Delta^2 \hat{E}^n \Delta^2 \hat{E}^n \Delta^3 \hat{E}^n \sum_{m'=1}^{M} \sum_{n'=1}^{N} \sum_{p'=1}^{P} \cdot G(m', n', p', \theta') \hat{E}(m', n', p') \quad (76)
\]
for \( m = 0, \cdots, M + 1, n = 0, \cdots, N + 1, \) and \( p = 0, \cdots, P + 1 \), where
\[
G(m', n', p', \theta') = G(\hat{x}_m - \hat{x}_m', \hat{x}_n - \hat{x}_n', \hat{x}_p - \hat{x}_p'). \quad (77)
\]
Note that \( \hat{A}_{k,m,n,p} \) in (76) are discrete convolutions in \( m', n', \) and \( p' \) and can efficiently be computed by FFT routines [15].

B. Adjoint Operator

The adjoint operator is defined through the relation
\[
\langle \mathbf{r}, \hat{LE} D \rangle = \langle \hat{L}^* \mathbf{r}, \mathbf{E} \rangle D \quad (78)
\]
where \( \mathbf{r} \) and \( \mathbf{E} \) are both in the same vector space [see (36)]. Substituting the expression of the contravariant components \( (\hat{L}^*)_{q', r', s', q,s} \) of the operator \( \hat{LE} \) in the left hand side of (78) and interchanging the summation, the adjoint operator is recognized as
\[
(\hat{L}^*)_{k' ; q', r', s'} = \delta_{k' ; q', r', s'} \hat{C}_{k' ; q', r', s'} \quad (79)
\]
for \( q = 1, \cdots, M, r = 1, \cdots, N, \) and \( s = 1, \cdots, P \), where
\[
\hat{C}_{k,q,r,s} = \cos(\theta) \Delta^2 \hat{E}^n \Delta^2 \hat{E}^n \Delta^3 \hat{E}^n \sum_{q'=0}^{M+1} \sum_{r'=0}^{N+1} \sum_{s'=0}^{P+1} \cdot G(q-r-s') \hat{F}_{k,q', r', s'} \quad (80)
\]
in which
\[
\hat{F}_{k,q,r,s} = (\Delta^2 \hat{E}^n)^{-2} (\delta_{k+1,q,s} - 2\delta_{q,s} + \delta_{q+1,s} - \delta_{q-1,s} + 1) \hat{E}^n \Delta^2 \hat{E}^n \Delta^3 \hat{E}^n \hat{F}_{k,q,r,s} + \frac{1}{4} \Delta^3 \hat{E}^n \hat{F}_{k,q,r,s} + \frac{1}{4} \Delta^4 \hat{E}^n \hat{F}_{k,q,r,s} + \frac{1}{4} \Delta^5 \hat{E}^n \hat{F}_{k,q,r,s} \quad (81)
\]
and
\[
\hat{F}_{k,q,r,s} = (\Delta^2 \hat{E}^n)^{-2} (\delta_{k+1,q,s} - 2\delta_{q,s} + \delta_{q+1,s} - \delta_{q-1,s} + 1) \hat{E}^n \Delta^2 \hat{E}^n \Delta^3 \hat{E}^n \hat{F}_{k,q,r,s} + \frac{1}{4} \Delta^3 \hat{E}^n \hat{F}_{k,q,r,s} + \frac{1}{4} \Delta^4 \hat{E}^n \hat{F}_{k,q,r,s} + \frac{1}{4} \Delta^5 \hat{E}^n \hat{F}_{k,q,r,s} \quad (82)
\]
where
\[
\begin{bmatrix}
\delta_{k+1,q,s} \\
\delta_{q+1,s} \\
\delta_{q-1,s} \\
\delta_{k-1,q,s} \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\cos^2(\theta)} & \sin(\theta) & 0 \\
0 & \sin(\theta) & \cos^2(\theta) & 0 \\
0 & 0 & 0 & \cos^2(\theta) \\
\end{bmatrix}
\begin{bmatrix}
\delta_{k+1,q,s} \\
\delta_{q+1,s} \\
\delta_{q-1,s} \\
\delta_{k-1,q,s} \\
\end{bmatrix}
\quad (83)
\]
Since according to (80), \( q' \) runs from 0 to \( M + 1 \), \( r' \) runs from 0 to \( N + 1 \), and \( s' \) runs from 0 to \( P + 1 \), we set in (81)–(83)
\[
\hat{F}_{k,q,r,s} = 0 \quad \forall q \neq q' \quad (84)
\]
Note that \( \hat{C}_{k,q,r,s} \) is discrete convolution in \( q', r', \) and \( s' \) and that these convolutions can also be computed efficiently by FFT routines.

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REFERENCES


