OPTIMUM SHAPES OF BODIES IN FREE SURFACE FLOWS

Hydrodynamics
free surface
optimalisation
theory

by

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ABSTRACT

The general problem of optimum shapes arising in a wide variety of free-surface flows can be characterized mathematically by a new class of variational problems in which the Euler equation is a set of dual integral equations which are generally nonlinear, and singular, of the Cauchy type. Several approximate methods are discussed, including linearization of the integral equations, the Rayleigh-Ritz method, and the thin-wing type theory. These methods are applied here to consider the following physical problems:

(i) The optimum shape of a two-dimensional plate planing on the water surface, producing the maximum hydrodynamic lift;

(ii) the two-dimensional body profile of minimum pressure drag in symmetric cavity flows;

(iii) the cavitating hydrofoil having the minimum drag for prescribed lift.

Approximate solutions of these problems are discussed under a set of additional isoperimetric constraints and some physically desirable end conditions.

1. Introduction

The general problem of optimum shapes of bodies in free-surface flows is of practical as well as theoretical interest. In applications of naval hydrodynamics these problems often arise when attempts are made to improve the hydromechanical efficiency and performance of lifting and propulsive devices, or to achieve higher speeds of operation of certain vehicles. Some examples of problems that fall under this general class are illustrated in Fig. 1. The first example is to evaluate the optimum profile of a two-dimensional plate planing on a water surface without spray formation, and producing the maximum hydrodynamic lift under the isoperimetric constraints of fixed chord length ℓ and fixed wetted arc-length S of the plate. The second example depicts the problem of determining the shape of a symmetric two-dimensional plate so that the pressure drag of this plate in an infinite cavity flow is a minimum, again with fixed base-chord ℓ and wetted arc-length S. The third is an example concerning the general lifting cavity flow past an optimum hydrofoil having the minimum drag for prescribed lift, incidence angle α, chord length ℓ and the wetted arc-length S. In these problems the gravitational and viscous effects may be neglected as a first approximation for operations at high Froude numbers. Physically, there is no definite rule for choosing the side constraints and isoperimetric conditions, but the existence and the characteristic behavior of the solution can depend decisively on what constraints and conditions are chosen. Mathematically, it has been observed in a series of recent studies that the determination of the optimum hydromechanical shape of a body in these free-surface flows invariably results in a new class of variational problems. Only a very few special cases from this general class of problems have been solved, the optimum lifting-line solution of Prandtl being an outstanding example.

There are several essential differences between the classical theory and this new class of variational problems. First of all, the unknown argument functions of the functional under extremization are related, not by differential equations as in the classical calculus of variations, but by a singular integral equation of the Cauchy type. Consequently, the "Euler equation" which results from the consideration of the first variation of the functional in this new class is also a singular integral equation which is, in general, nonlinear. This is in sharp contrast to the Euler differential equation in classical theory. Another characteristic feature of these new problems is that while regular behavior of solution at the limits of the integral equation may be required on physical grounds, the mathematical conditions which insure such behavior generally involve functional equations which are difficult, and sometimes just impossible, to satisfy.

Because of these difficulties and the fact that no general techniques are known for solving nonlinear singular integral equations, development of this new class of variational problems seems to require a strong effort. Attempts are made here to present some general results of the current study. Some necessary conditions for the existence of an optimum solution are derived from a consideration of the first and second variations of the functional in question. To solve the resulting nonlinear, singular integral equation several approximate methods are discussed. One method is by linearization of the integral equation, giving a final set of dual singular integral equations of the Cauchy type. When the variable coefficients of this system of integral equations satisfy a certain relationship, this set of dual integral equations can be solved analytically in a closed form; the results of this special case provide analytical expressions which can be extensively investigated to determine the behavior of a solution near the end points. Another approximate method is the Rayleigh-Ritz expansion; it has the advantages of retaining the nonlinear effects to a certain extent, of incorporating the required behavior of the solution near the end points into the discretized expansion of the solution, but the method is generally not convergent. A third approach depends on a thin wing type theory to describe the flow at the very beginning, a variational calculation is then made on an approximate expression of the physical quantities of interest. These mathematical methods will be discussed and then applied to three problems described earlier. While the results to be presented should be considered as still preliminary, since exact solutions to these problems have not yet been found, it is hoped that this paper will succeed in stimulating further interest in the development of the general theory, and, in turn, aid in the resolution of many hydromechanic problems of great importance.

2. General mathematical theory

To present a unified discussion of the general class of optimum hydromechanical shapes of bodies in plane free-surface flows, including the three examples (i) - (iii) depicted in Fig. I, we assume the flow to be inviscid, irrotational, and incompressible, taking as known that the physical plane $z = x + iy$ and the potential plane $f = \varphi + i\psi$ correspond conformally to the upper half of the parametric $\zeta = \xi + i\eta$ plane by the mapping that can be signified symbolically as

$$f = \varphi + i\psi = \nu(\zeta; c_1, \ldots c_n) \quad , \tag{1}$$

where ν is an analytic function of ζ and may involve geometric parameters $c_1, \ldots c_n$, so that the wetted body surface corresponds to $\eta = 0^+$, $|\xi| < 1$, and the free surface, to $\eta = 0^+$, $|\xi| > 1$. Specific forms of the function $\nu(\zeta)$ will be given later, but our purpose at this time is merely to illustrate the type of nonlinear variational problem that arises.

Description of the flow is effected by giving the parametric expressions $f = f(\zeta)$ and $\omega = \omega(\zeta)$,

$$\omega(\zeta) = -\log(df/dz) = \tau + i\theta \tag{2}$$

being the logarithmic hodograph. The boundary conditions for ω may be specified either as a Dirichlet problem, by giving

$$\tau^+ = \mathrm{Re}\,\omega(\xi + i0) = \begin{cases} \Gamma(\xi) & (\text{given for } |\xi| < 1) \quad , \\ 0 & (|\xi| > 1) \quad , \end{cases} \tag{3}$$

or as a Riemann-Hilbert problem,

$$\theta^+ = \mathrm{Im}\,\omega(\xi + i0) = \beta(\xi) \qquad (|\xi| < 1) \quad , \tag{4a}$$

$$\tau^+ = \mathrm{Re}\,\omega(\xi + i0) = 0 \qquad (|\xi| > 1) \quad . \tag{4b}$$

The formulation of the ω problem is completed by specifying a condition at the point of infinity, say

$$\omega \to 0 \quad , \qquad (|z| \to \infty) \quad , \tag{5}$$

and by prescribing a set of end conditions, which are generally on $\Gamma(\xi)$, as

$$\Gamma(\pm 1) = 0 \quad , \tag{6}$$

or similar ones. The end conditions are usually required on physical grounds in order that the fluid pressure is well behaved at the end points $\xi = \pm 1$, at which the free boundary meets the wetted body surface.

The solution to the Dirichlet problem (3), (5), (6), i.e.

$$\omega(\zeta) = \frac{1}{i\pi} \int_{-1}^{1} \frac{\Gamma(t)\,dt}{t - \zeta} \quad , \qquad \Gamma(\pm 1) = 0 \quad , \tag{7}$$

and the solution to the Riemann-Hilbert problem (4), (5), (6), given by

$$\omega(\zeta) = \frac{1}{i\pi} (\zeta^2-1)^{\frac{1}{2}} \int_{-1}^{1} \frac{\beta(t)dt}{(1-t^2)^{\frac{1}{2}}(t-\zeta)} , \tag{8a}$$

with

$$\int_{-1}^{1} \frac{\beta(t)dt}{(1-t^2)^{\frac{1}{2}}} = 0 , \tag{8b}$$

are equivalent to each other, as can be readily shown. Here, the function $(\zeta^2-1)^{\frac{1}{2}}$ is one-valued in the ζ-plane cut from $\zeta = -1$ to $\zeta = 1$. On the body surface, we deduce from (7), by applying the Plemelj formulas, that

$$\beta(\xi) = -\frac{1}{\pi} \oint_{-1}^{1} \frac{\Gamma(t)dt}{t-\xi} \equiv -H_{\xi}[\Gamma] \qquad (|\xi| < 1) , \tag{9}$$

where the integral with symbol C signifies its Cauchy principal value, and also defines the finite Hilbert transform of Γ (t), as denoted by $H_{\xi}[\Gamma]$.

From this parametric description of the flow we derive the physical plane by quadrature

$$z(\zeta) = \int_{-1}^{\zeta} e^{-\omega(\zeta)} \frac{df}{d\zeta} d\zeta . \tag{10}$$

With the solution (7) - (10) in hand, we see that the chord ℓ, wetted arc-length S, angle of attack α, as well as the drag D, lift L, etc. can all be expressed as integral functionals with argument functions $\Gamma(\xi)$ and $\beta(\xi)$, which are further related by (9).

3. The variational calculation

The general optimum problem considered here is the minimization of a physical quantity which may be expressed as a functional of the form

$$I_o[\Gamma, \beta; c_1, \ldots c_n] = \int_{-1}^{1} F_o(\Gamma(\xi), \beta(\xi), \xi; c_1 \ldots c_n) d\xi \tag{11}$$

under M isoperimetric constraints

$$I_\ell[\Gamma, \beta; c_1, \ldots c_n] = \int_{-1}^{1} F_\ell(\Gamma, \beta, \xi; c_1, \ldots c_n) d\xi = A_\ell \tag{12}$$

where A_ℓ's are constants, $\ell = 1, 2, \ldots M$.

The original problem is equivalent to the minimization of a new functional

$$I[\Gamma,\beta;c_1, \ldots c_n] = I_o - \sum_{\ell=1}^{M} \lambda_\ell (I_\ell - A_\ell) \quad , \tag{13}$$

where λ_ℓ's are undetermined Lagrange multipliers.

We next seek the necessary conditions of optimality. Let $\Gamma(\xi)$ denote the required optimal function which, together with its conjugate function $\beta(\xi)$ given by (9), minimizes $I[\Gamma,\beta]$. We further let $\delta\Gamma(\xi)$ denote an admissible variation of $\Gamma(\xi)$, which is Hölder continuous, satisfies the isoperimetric constraints (12) and the end conditions (6). The corresponding variation in $\beta(\xi)$ is found from (9) as

$$\delta\beta(\xi) = -H_\xi[\delta\Gamma] \qquad (|\xi| < 1) \quad . \tag{14}$$

The variation of the functional I due to the variations $\delta\Gamma$ and $\delta\beta$ is

$$\Delta I = I[\Gamma+\delta\Gamma, \beta+\delta\beta; c_n+\delta c_n] - I[\Gamma,\beta;c_n] \quad , \tag{15}$$

where δc_n's are variations of parameters c_n. For sufficiently small $|\delta\Gamma|$, $|\delta\beta|$ and $|\delta c_n|$, expansion of the above integrand in Taylor's series yields

$$\Delta I = \delta I + \frac{1}{2!}\delta^2 I + \frac{1}{3!}\delta^3 I + \ldots \quad , \tag{16}$$

where the first variation δI and the second variation $\delta^2 I$ are

$$\delta I = \int_{-1}^{1} [F_\Gamma \delta\Gamma + F_\beta \delta\beta] d\xi + \delta c_n \int_{-1}^{1} (\partial F/\partial c_n) d\xi \quad , \tag{17}$$

$$\delta^2 I = \int_{-1}^{1} [F_{\Gamma\Gamma}(\delta\Gamma)^2 + 2F_{\Gamma\beta}\delta\Gamma\delta\beta + F_{\beta\beta}(\delta\beta)^2] d\xi + \delta c_n \delta c_m \int_{-1}^{1} \frac{\partial^2 F}{\partial c_n \partial c_m} d\xi$$

$$+ \text{ cross product term between } \delta c_n \text{ and } \delta\Gamma \text{ or } \delta\beta \quad , \tag{18}$$

in which the subindices denote partial differentiation. The variations δI, $\delta^2 I \ldots$ depend on $\delta\Gamma$ as well as on Γ. For I to be minimum, we must have

$$\delta I[\Gamma,\delta\Gamma] = 0 \quad , \tag{19}$$

$$\delta^2 I[\Gamma,\delta\Gamma] \geq 0 \quad , \tag{20}$$

in which β and $\delta\beta$ are understood to be related to Γ and $\delta\Gamma$ by (9) and (14). Relation (19) assures I to be extremal, and with the inequality (20), I is therefore a minimum.

Now, substituting (14) in (17) reduces it to

$$\delta I = \int_{-1}^{1} \{F_{\Gamma} + H_{\xi}[F_{\beta}]\} \, \delta\Gamma(\xi) d\xi + \delta c_n \int_{-1}^{1} \partial F / \partial c_n d\xi \tag{17)'$$

after inter-changing the order of integration, which is permissible under certain integrability conditions (see Tricomi 1957, § 4. 3) which will be tacitly assumed to hold. Since the variations $\delta\Gamma(\xi)$ and δc_n are independent and arbitrary, the last integral in (17)' and the factor in the parenthesis of the first integrand must all vanish, hence

$$\int_{-1}^{1} \partial F(\Gamma(\xi), \beta(\xi), \xi; c_1 \; . \; . \; . \; c_n) / \partial c_j d\xi = 0 \quad (j=1, \; . \; . \; . \; n) \, , \tag{21}$$

$$F_{\Gamma}(\Gamma(\xi), \beta(\xi); \xi) = -H_{\xi}[F_{\beta}] = -\frac{1}{\pi} \int_{-1}^{1} \frac{F_{\beta}(\Gamma(t), \beta(t), t)}{t - \xi} dt \quad . \tag{22}$$

The nonlinear integral equation (22) combines with (9) to give a pair of singular integral equations for the extremal solutions. This is one necessary condition for $I[\Gamma]$ to be extremal; it is analogous to the Euler differential equation in the classical theory. Presumably, calculation of the extremal solution $\Gamma(\xi)$ from (22) and (9) can be carried out with $\lambda_1, \; . \; . \; . \; \lambda_M$ regarded as parameters, which are determined in turn by applying the M constraint equations (12). While we recognize the lack of a general technique for solving the system of nonlinear integral equations (9) and (22), we also notice the difficulty of satisfying the end conditions (6), as has been experienced in many different problems investigated recently. The last difficulty may be attributed to the known behavior of a Cauchy integral near its end points which severely limits the type of analytic properties that can be possessed by an admissible function $\Gamma(\xi)$ and its conjugate function $\beta(\xi)$.

Supposing that these equations can be solved for $\Gamma(\xi; c_1, c_2, \; . \; . \; . \; c_n)$, we proceed to ascertain the condition under which this extremal solution actually provides a minimum of $I[\Gamma]$. From the second variation $\delta^2 I$ we find it is necessary to have

$$\int_{-1}^{1} (\partial^2 F / \partial c_j^2) d\xi \geqslant 0 \quad , \tag{23}$$

$$\int_{-1}^{1} [F_{\Gamma\Gamma}(\delta\Gamma)^2 + 2F_{\Gamma\beta}\delta\Gamma\delta\beta + F_{\beta\beta}(\delta\beta)^2] d\xi \geqslant 0 \quad . \tag{24a}$$

By substituting (14) in (24a), interchanging the order of integration according to the Poincaré-Bertrand formula (Muskhelishvili, 1953) wherever applicable, it can be shown that (24a) can also be written as

$$\int_{-1}^{1} g(\xi)(\delta\Gamma)^2 d\xi + \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{h(t)-h(\xi)}{t-\xi}\,\delta\Gamma(t)\delta\Gamma(\xi)dt\,d\xi \geq 0 \tag{24b}$$

where

$$g(\xi) = F_{\Gamma\Gamma} + F_{\beta\beta} \quad , \quad h(\xi) = F_{\Gamma\beta} + H_\xi[F_{\Gamma\beta}] \quad .$$

If we suppose that $F_{\Gamma\Gamma}$, $F_{\Gamma\beta}$, $F_{\beta\beta}$ are Hölder continuous, and consider a special choice of $\delta\Gamma$ which vanishes for $|\xi - \xi_o| > \epsilon$, bounded $(|\delta\Gamma| \leq \beta)$ and is of one sign for $|\xi - \xi_o| < \epsilon$, where ξ_o is any interior point of $(-1, 1)$, then it can be shown that the first term on the left side of (24b) predominates, hence a necessary condition for (24b) to hold true is the inequality $g(\xi) \geq 0$, or

$$F_{\Gamma\Gamma} + F_{\beta\beta} \geq 0 \qquad (|\xi| < 1) \quad . \tag{24c}$$

This condition is analogous to the Legendre condition in the classical theory.

The preceding illustrates the method of solution of the extremum problem by singular integral equations. We should reiterate that the integral equations are nonlinear unless F is quadratic in Γ and β. No general methods have been developed for the exact solution of nonlinear singular integral equations. Further, it may not always be possible to satisfy the condition $\Gamma(\pm 1) = 0$, which are required on physical grounds. With these difficulties in mind, we proceed to discuss some approximate methods of solution.

4. Linearized singular integral equation

The least difficult case of the extremal problems in this general class is when the fundamental function $F[\Gamma, \beta]$ is quadratic in Γ and β, that is

$$F(\Gamma, \beta, \xi; c_j) = a\Gamma^2 + 2b\,\Gamma\beta + c\beta^2 + 2b\Gamma + 2q\beta \quad , \tag{25}$$

in which the coefficients $a, b, \ldots q$ are known functions of ξ and may depend on the parameters $c_1, \ldots c_n$. It should be stressed that the above quadratic form of F can generally be used as a first approximation of an originally nonlinear problem in which F is transcendental or contains higher order terms than the quadratic. With this approximation the integral equation (22) is then linear in Γ and β, and reads

$$a\Gamma + b\beta + p = -H_\xi[b\Gamma + c\beta + q] \qquad (|\xi| < 1) \quad , \tag{26}$$

which combines with (9) to provide a set of two linear integral equations, both of the Cauchy type. The necessary condition (24c), obtained from the consideration of the second variation, now becomes

$$a(\xi) + c(\xi) \geqslant 0 \qquad (|\xi| < 1) \quad . \tag{27}$$

For the present linear problem (regarding the integral equations) two powerful analytical methods become immediately useful. First, the coupled linear integral equations (9) and (26) can always be reduced to a single Fredholm integral equation of the second kind. When the coefficients $a(\xi)$, $b(\xi)$ and $c(\xi)$ of the quadratic terms satisfy a certain relationship, the method of singular integral equations can be effected to yield an analytical solution in a closed form.

(4i) Fredholm integral equation

By substituting (9) in (26), we readily obtain

$$a(\xi)\Gamma(\xi)-b(\xi)H_\xi[\Gamma]+H_\xi[b\Gamma] - H_\xi[c(t)H_t[\Gamma]] = - H_\xi[q] -p(\xi) \quad .$$

Upon using the Poincaré-Bertrand formula (with appropriate assumptions) for the last term on the left side of the above equation, there results

$$\{a(\xi) +c(\xi)\} \Gamma(\xi) + \int_{-1}^{1} K(t,\xi)\Gamma(t)dt = - H_\xi[q] - p(\xi) \quad , \tag{28a}$$

where

$$K(t,\xi) = \frac{1}{\pi} \frac{b(t)-b(\xi)}{t-\xi} + \frac{1}{\pi^2} \oint_{-1}^{1} \frac{c(s)ds}{(s-t)(s-\xi)} \quad . \tag{28b}$$

This is a Fredholm integral equation of the second kind, with a regular symmetric kernel, for which a well-developed theory is available.

(4ii) Singular integral equation method

When the coefficients $a, b, c,$ satisfy the following relationship

$$a(\xi) + c(\xi) > 0 \quad , \quad b(\xi) = b_o \pm (ac)^{\frac{1}{2}} \quad , \quad b_o = \text{const} \quad , \tag{29}$$

the system of equations (26) and (9) can be reduced in succession to a single integral equation, each time for a single variable, and these equations are of the Carleman type, which can be solved by known methods (see Muskhelishvili 1953), yielding the final solution in a closed form.

In the first step we multiply (9) by b_o, and subtracting it from (26), giving

$$a^{\frac{1}{2}}\Phi_{\pm}(\xi) = H_{\xi}[\pm c^{\frac{1}{2}}\Phi_{\pm}] + \Psi(\xi) \qquad (|\xi| < 1) \quad , \tag{30a}$$

where

$$\Phi_{\pm}(\xi) = a^{\frac{1}{2}}\Gamma \pm c^{\frac{1}{2}}\beta \quad , \qquad \Psi(\xi) = -H_{\xi}[q] - p(\xi) \quad . \tag{30b}$$

After this Carleman equation for $\Phi_{\pm}$ is solved, a second Carleman equation results immediately upon elimination of β between the expression for $\Phi_{\pm}$ and (9). The details of this analysis are given by Wu and Whitney (1971). These analytical solutions are of great interest, since in their construction there are definite, but generally very limited degrees of freedom for choosing the strength of the singularity, or the order of zero, of the solution $\Gamma(\xi)$ and $\beta(\xi)$ at the end points $\xi = \pm 1$. It is in this manner that the analytical behavior of the solution $\Gamma(\xi)$ and $\beta(\xi)$ can be explicitly and thoroughly examined. This procedure will be demonstrated later by examples.

5. The Rayleigh-Ritz method

The central idea of this method, as in classical theory, consists in expansion of $\Gamma(\xi)$ and $\beta(\xi)$ in a finite Fourier series

$$\Gamma_m(\xi) = \sum_{k=1}^{m} \gamma_k \sin k\theta \quad , \qquad (\xi = \cos\theta \ , \ 0 \leqslant \theta \leqslant \pi) \tag{31a}$$

$$\beta_m(\xi) = \sum_{k=1}^{m} \gamma_k \cos k\theta \quad . \tag{31b}$$

This expansion is noted to satisfy (9) automatically. The functional $I[\Gamma, \beta]$ is now an ordinary function of the Fourier coefficients γ_k,

$$I[\Gamma, \beta; c_1, \ldots c_n] = \int_0^{\pi} F(\Gamma_m, \beta_m, \cos\theta; c_1 \ldots c_n)\sin\theta\, d\theta$$

$$= I(\gamma_1, \ldots \gamma_m; \ c_1, \ldots c_n) \quad . \tag{32}$$

For I to be extremum, we require that

$$\partial I/\partial\gamma_k = 0 \qquad (k = 1, \ldots m) \quad , \tag{33}$$

and

$$\partial I/\partial c_j = 0 \qquad (j = 1, \ldots n) \quad . \tag{34}$$

These $(m+n)$ equations together with M constraint equations (12) determine the m coefficients $\gamma_1, \ldots \gamma_m$, n parameters $c_1, \ldots c_n$, and M multipliers $\lambda_1, \ldots \lambda_M$. It should be pointed out, however, that the coefficients γ_k's and parameters c_j's generally appear in the expression for $I(\gamma_k, c_j)$ in a nonlinear or transcendental form, making their determination, by algebraic, numerical means or otherwise, extremely difficult even when their number is moderately small, such as three or more.

The preceding general theory will be further discussed and clarified with several specific examples in the presentation of this study.

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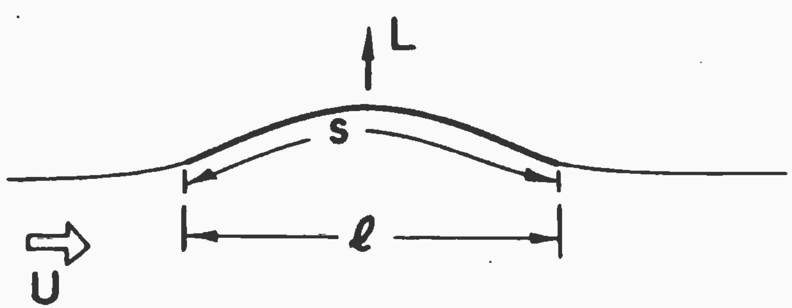
Examples of Physical Problems (2 — Dimensional)

Planing surface

($Fr = \infty$)

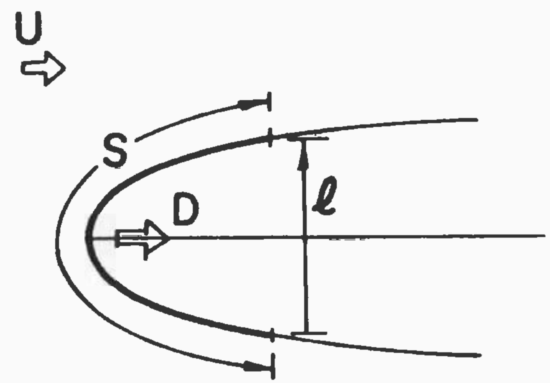


maximum lift L

(fixed ℓ, S)

Cavity flow

(Pure drag)

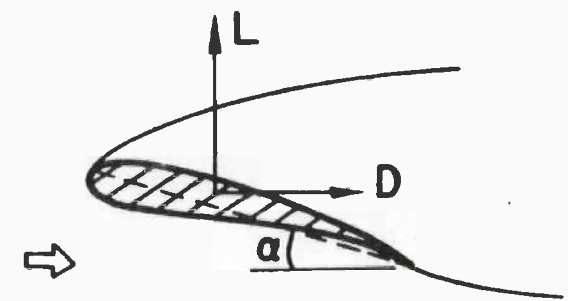


minimum drag D

(fixed ℓ, S)

Cavity flow

(Lifting)



minimum D

(fixed ℓ, S, α, L)