50. A number of relations characteristic for the energy exchange between the components of turbulence and, when present, between the main motion and those components, can be illustrated by means of the solutions of a differential equation, much simpler than the hydrodynamical equations but retaining certain important features of the latter. It has been found that these solutions also lead to statistical problems, in many ways analogous to statistical problems occurring in turbulence. It is worthwhile therefore to give attention to this equation and its solutions, which may be considered as giving an introduction into some of the deeper turbulence problems.

The particular equation chosen for this purpose has the form:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2}$$

(1)

The variable \(v\) can be taken as the analogue of the turbulent velocity; it is dependent on the time and a coordinate \(y\) (something like the transverse dimensions of the field of flow in a tube). The equation has a non-linear term of the first order, and a term of the second order multiplied by a coefficient \(\nu\) which is assumed to be very small (analogue of the kinematic viscosity). The interplay of these terms is of primary importance in the dynamics of turbulence. Since the equation refers to a single variable and a single coordinate, it does not picture those geometric relations of hydrodynamic turbulence, which are dependent on the three dimensional nature of the field, the properties of vortices and the phenomena of shearing motion. There is no pressure term in the equation and there is no equation of continuity; hence there is nothing which reflects the condition of incompressibility of ordinary hydrodynamics. In a way the equation is more illustrative of certain phenomena peculiar to compressible media, in particular to shock waves. In the results which can be obtained from eq. (1) there is no approach
to the special geometric features (tensor relations etc.) which are considered in the theory of isotropic turbulence.

The equation can be completed on the right hand side by introducing either a term representing an exterior force, or a term describing a coupling between \( v \) and another parameter, to be considered as the analogue of the velocity of the main motion. It is necessary to specify the domain of the variable \( y \) to which the equation shall refer. This can be either the complete \( y \)-axis, from \(-\infty \), to \( +\infty \); or it can be a limited domain, for instance \( 0 \leq y \leq b \), in which case we shall require that \( v \) vanishes at both limits.

Equation (1) can be reduced to a linear equation by substituting a new variable for \( v \). It is useful, however, first to consider the equation as it stands. We look for a particular solution of the form:

\[
v = \beta \left( y - \delta \right),
\]

(2)

where \( \beta \) and \( \delta \) are functions of \( t \). When this expression is substituted into (1), the equation is satisfied identically if

\[
\beta = 1 / (t - t_0); \quad \delta = \text{constant}
\]

(3)

\( t_0 \) being a constant. Hence a solution represented by a straight line will turn to the right, while its point of intersection with the axis \( \delta \) remains unchanged. The angle \( \alpha \) between the segment and the vertical direction increases according to the relation:

\[
\tan \alpha = t - t_0.
\]

The result can be extended to the case where the course of \( v \) is represented by a series of straight segments. To prevent difficulties with quantities becoming infinite, we assume that the angles between consecutive segments are slightly rounded off. Since \( \partial^2 v / \partial y^2 \) is not zero in the neighborhood of the angles, the term \( v \left( \partial^2 v / \partial y^2 \right) \) will have influence on the form of the solution. Keeping in mind the effect of a similar term in the equation of diffusion, we may expect
that the rounding off will gradually spread to a more extended region. In the first phases of the motion this detail will not have much influence and we can find the approximate history of the solution simply by applying the rule that every segment turns about its "hinge point", \( \delta \), according to form \((2)\) and \((3)\). The point of intersection of two consecutive segments retains a constant value of \( v \) and moves with a velocity equal to \( v \). If the point is below the axis, \( v \) is negative and the movement is directed to the left.

51. When the initial slope of the segment was positive, it will remain so with the slope gradually decreasing to zero. If the initial slope is negative, the slope will increase in absolute measure and after a finite lapse of time the segment will approach a vertical position. Our simple result can then no longer be used.

To see what will happen in such a case, we consider a particular solution of \((1)\), obtained by assuming that \( v \) is determined by an expression of the form:

\[
v = V(\eta) (t - t_0)^{-1/2}
\]

where \( V(\eta) \) is a function of the variable \( \eta = (y - \delta)(t - t_0)^{-1/2} \)

Equation \((1)\) is then transformed into:

\[
v V'' - vV' + \frac{1}{2} \eta V' + \frac{1}{2} v = 0
\]

which can be integrated and gives:

\[
v V' - \frac{1}{2} v^2 + \frac{1}{2} \eta V = 0,
\]

(the integration constant has been adjusted so that \( V \) may vanish at infinity). We now put:

\[
V = -2v \frac{d \ln u}{d \eta},
\]

and obtain a linear equation for \( u \):

\[
u'' = -\left(\frac{\eta}{2v}\right) u'.
\]
Integration of the latter equation finally leads to the following expression for $V$:

$$V = \frac{2\nu}{h} \exp \left\{ \frac{h^2 - \eta^2}{4\nu} \right\} \int_0^h d\eta \exp \left\{ \frac{(h^2 - \eta^2)}{4\nu} \right\}$$

$h$ being another integration constant. Since this expression looks rather involved, it is useful to note the following approximations, valid when $h^2/\nu$ is large:

(a) when $\eta = h + \delta$ where $\delta \ll h$,

$$V \approx \frac{1}{2} h \left[ 1 - \tanh \left( \frac{h\delta}{4\nu} \right) \right]$$

(b) when $0 < \eta < h$ and $\eta$ not too near to one of the end points:

$$V \leq \eta$$

(c) when $\eta$ is near zero or is negative, the value of $V$ becomes practically zero.

With the aid of the approximations we can obtain a general picture of the course of $V$. This can be readily translated into a picture for $v$ as a function of $y$ and $t$, leading to the result given in the diagram:

![Diagram](image-url)
It will be seen that the course of \( v \) is represented approximately by a triangular figure with constant area \( \frac{1}{2} h^2 \). The slope of the hypotenuse is \( 1/(t - t_o) \) as before. The velocity of advance of the right hand side (the "front velocity") is \( \frac{1}{2} h (t - t_o)^{-1/2} \), which is equal to one half the height of the front.

An approximation to the course of \( v \) at the front is given by:

\[
v = \frac{h}{2 \sqrt{t - t_o}} \left[ \frac{1 - \tanh \left( \frac{h(y - \xi)}{4\sqrt{t - t_o}} \right)}{\tanh \left( \frac{h(y - \xi)}{4\sqrt{t - t_o}} \right)} \right] \tag{4}
\]

with \( \xi = \delta + h(t - t_o)^{1/2} \)

It should be noted that there is also a solution in which the signs of \( V \) and \( \gamma \) both are changed; the triangle points downward and its front moves to the left.

52. The result that the area of triangle is constant expresses the "conservation of momentum" for the solution. By integrating eq. (1) with respect to \( y \) we obtain

\[
\frac{d}{dt} \int v \, dy = 0, \tag{5}
\]

for any domain at the limits of which \( v \) vanishes and \( v \, (\partial v/\partial y) \) is either rigorously zero or sufficiently small to be negligible.

If we multiply eq. (1) by \( v \) and integrate, we obtain an "equation of energy"

\[
\frac{d}{dt} \int \frac{v^2}{2} \, dy = -v \int \left( \frac{\partial v}{\partial y} \right)^2 \, dy, \tag{6}
\]

for any domain at the limits of which \( v \) vanishes. In the case of the solution considered above the energy integral amounts to:

\[
\frac{1}{6} h^3 (t - t_o)^{-1/2};
\]
Making use of (4) the "dissipation integral" is found to be:

\[ \frac{1}{12} h^3 (t - t_0) - 3/2 \]

It is easily verified that these expressions satisfy (6).

The result that a nearly vertical front in the course of \( v \) can be described approximately by a hyperbolic tangent function and that it has a certain velocity of advance, can be generalized. We return to eq. (1) and introduce a coordinate system which shall move with the front. To this end we put:

\[ y' = y - \xi; \quad t' = t, \]

\( \xi \) being a function of \( t \). From these formulas we deduce:

\[ \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - c \frac{\partial}{\partial y'} \]

where \( c = \frac{d \xi}{dt} \). Substitution of these expressions into (1) gives:

\[ \frac{\partial v}{\partial t'} + (v - c) \frac{\partial v}{\partial y'} = \nu \frac{\partial^2 v}{\partial y'^2}. \]

We expect that the derivative \( \frac{\partial v}{\partial y'} \) will be of the order \( \nu \) in the region with a steep gradient, and that \( \frac{\partial^2 v}{\partial y'^2} \) will be of order \( \nu^{-2} \). If \( c \) is properly adjusted, \( \frac{\partial v}{\partial t'} \) will be of normal order of magnitude both at the steep front and elsewhere. This term can then be discarded and there remains:

\[ (v - c) \frac{\partial v}{\partial y'} = \nu \frac{\partial^2 v}{\partial y'^2}. \]

Integration gives:

\[ \frac{1}{2\nu} (v - c)^2 - \nu \frac{\partial v}{\partial y'} = \text{constant}. \]

Since this expression must be valid through the region of a rapid change of \( v \) and also on both sides of it, where \( \frac{\partial v}{\partial y'} \) returns to
to a normal order of magnitude and $v \left( \frac{\partial v}{\partial y} \right)$ can be neglected, we find:

$$\frac{1}{2} (v_I - c)^2 = \frac{1}{2} (v_{II} - c)^2 = \text{constant},$$

where the subscripts I and II denote the values of $v$ on the two sides of rapid change. From this:

$$c = \frac{1}{2} \left( v_I + v_{II} \right) \quad (7)$$

Hence the velocity of advance of a steep front is given by half the sum of the values of $v$ at both ends. A further integration gives:

$$v = \frac{1}{2} (v_I + v_{II}) - \frac{1}{2} \left( \frac{v_{II} - v_{I} }{v_{I} - v_{II}} \right) \left( \frac{v_{I} - v_{II} }{v_{I} - v_{II}} \right) \tanh \frac{(v_{I} - v_{II} ) (y - c)}{4v} \quad (8)$$

By making use of this result we can now construct the development of $v$ from any initial state given by a chain of straight segments, at least so long as the rounding off of the angles in consequence of the viscosity has not proceeded too far. Every segment turns about its "hinge point" according to the form, (2) and (3); any time when a downward sloping segment reaches the vertical position, it does not turn any further but remains vertical, obtaining a velocity of advance given by form, (7). This velocity of advance will not be constant, since the values of $v_I$ and $v_{II}$ usually will change in course of time.

It is possible that consecutive vertical segments overtake each other. When this occurs we combine them to form a single segment, moving from now on with a velocity again given by (7) provided $v_I$ and $v_{II}$ refer to the velocities at the ends of the new segment.

It is found that every vertical segment is the seat of dissipation of energy, to the amount $(v_I - v_{II})^3/12$ (It should be observed that $v_I - v_{II}$ is positive for every vertical segment, as will be evident from the process by which these segments are generated).

All these features can be checked by making use of the exact solution of eq. (1), which can be obtained, as was indicated by J. D. Cole and V. Bargmann, by making the substitution:

$$v = -2v \left( \ln u \right)/\partial y,$$
which is similar to the one used to transform the ordinary (not partial) differential equation for $V$.

53. One may ask what these considerations have to do with the turbulence problem.

The first point, already noted, is that eq. (1) has the two important features of the hydrodynamic equations: a typical non-linear term of the first order and a linear term of the second order multiplied by a small coefficient. It is possible to apply the similarity theory to eq. (1) and it is found that its solutions are characterized by a Reynolds number formed out of the product of a typical velocity and a typical length, divided by the kinematic viscosity $v$. Moreover, the solutions of eq. (1) which vanish at infinity satisfy the condition of conservation of momentum, and an energy equation can be formed, expressing the loss of energy through dissipation. (The non-linear term of eq. (1) vanishes both from the momentum and from the energy equation).

The presence of the non-linear term has the consequence that features of the solutions are propagated with a velocity given by the magnitude of $v$ itself, which has the further consequence that steep fronts can be generated which, once formed, keep their individuality so long as they do not merge with a neighboring front. This property can be compared with the tendency found in fluid motion that masses, having acquired a certain velocity, displace themselves with this velocity and push aside elements having a smaller velocity, in which process usually surfaces of intensive shearing motion are generated. Shearing motion is not represented by the solutions of eq. (1), but the analogy is retained when we do not look at the shear itself, but at the dissipation accompanying it: a similar dissipation is to be found in each steep front developing in the course of $v$.

Surfaces of shear flow, separating masses of liquid with different velocities, can merge together, in a similar way as the steep fronts in the curve describing our function $v$; both in the case of fluid motion and in that case of eq. (1) this process has the importance that it leads to an increase of the scale of the pattern.

A system intermediate in its character between the complete hydrodynamic equations and eq. (1) is formed by equations (13) considered
shortly in section 45 of the preceding chapter. The solutions of the system (13) actually exhibit the formation of regions of shear flow, which already are quite comparable to those of actual fluid flow.

It is possible to make a Fourier analysis of the solutions of eq. (1), in a similar way as this can be done in hydrodynamics. It is then found, in both cases, that there is a coupling between the various components, depending upon the non-linear terms of the equations, of such nature, that the interaction between any two components leads to the appearance of the influencing, of other components, having the sum or the difference of the frequencies. Hence, when we describe the system by means of a spectrum, there is a transfer of energy both upward and downward along the frequency scale. Viscosity in both cases operates most effectively on the components of small wavelength (high frequency); hence in both cases there must be an overall flow of energy towards the high frequency end of the spectrum. In all these aspects and in a number of statistical features, the solutions of eq. (1) can help to understand relations appearing in actual turbulence. This will be shown in greater detail in the next sections.

Attention may be drawn to one further point: How convenient the resolution of a pattern of fluid motion into Fourier components may be from many points of view, there is also something unnatural in it and the strong coupling between the various components due to the non-linear terms of the equations present a mathematical problem which thus far cannot be solved. It is difficult, therefore, to predict the history of any single Fourier component. The same applies to any other method of resolution, based on the characteristic solutions of some linear partial differential equation (for instance, the equation used in investigations on the stability of laminar motion). On the other hand the type of solutions of eq. (1) considered in the preceding section, with their steep fronts, are directly connected with the non-linear term and present, so to say, an "individualized" character. Steep fronts have a certain life time and it is possible to describe a certain class of solutions of (1) by giving a list of the positions and strengths of the front. The history of the field then becomes a description of the motion of these fronts and their merging together, leading to a gradual decrease of their number.
and increase of their mean distance. Such a description brings into evidence those properties of the field which are dependent on the presence of the non-linear term in the equation, whereas the Fourier resolution is based primarily on the linear terms and is only poorly adapted to the treatment of non-linear effects.

In the next sections we shall consider various particular cases of fields described by eq. (1).
### Solutions of Eq. (1) Representing the Propagation of Impulses in One Direction along the Y-Axis

54. In section 51 we have become acquainted with a particular solution of eq. (1) corresponding to an impulse of magnitude \( h^2/2 \) introduced at the instant \( t_0 \) at the point \( y = 0 \). The course of \( v \) is represented by a triangle, having the area \( h^2/2 \), while the velocity of advance of the front is given by one half the height of the front. These two rules completely determine the solution.

We now imagine that a series of impulses of magnitudes \( \omega_1, \omega_2, \omega_3, \ldots \) is introduced at the point \( y = 0 \) at instants \( t_1, t_2, t_3, \ldots \), all impulses being positive. Each impulse leads to the appearance of a triangle and the various triangles are superposed on each other in the way as indicated in the accompanying figure. For any instant this figure can be constructed by drawing a set of straight lines starting from \( y = 0 \) and having slopes given by \( 1/(t - t_i) \); between these lines we draw vertical segments in such a way that a triangle with the area \( \omega_i \) is formed between the lines with slopes \( 1/(t - t_i) \) and \( 1/(t - t_{i-1}) \).

We denote the position of the corresponding vertical segment by \( \xi_i \) and provisionally assume that all \( \xi_i \) satisfy the conditions \( \xi_i < \xi_{i-1} \).

We then have a representation of the course of \( v \) at the instant \( t \) (heavy broken line in the diagram). The development in time of the
curve is determined by two rules: all inclined lines turn to the
right in conformity with the expression \(1/(t - t_i)\) for their slope;
every triangle must retain a constant area \(\omega_i\). One finds:

\[
\xi_i = \sqrt{\frac{2 \omega_i (t - t_i) (t - t_{i-1})}{t_i - t_{i-1}}} \tag{9}
\]

We assume that new impulses are introduced at \(y = 0\) as time advances,
which will give rise to new triangles to be superposed at the left end
of the curve. - The corrections connected with viscosity are neglected
in this picture, but we can assume that they are very small.

The curve obtained gives a crude analogy to turbulence produced
by a screen in a windtunnel and carried along by the general flow. In
the present case there is no special "carrying flow": the propagation
along the y-axis is determined by the magnitude of the impulses them­
selves. We suppose that at \(y = 0\) impulses are continuously introduced
in a random manner, but so that it is possible to define a mean value
\(\bar{\omega}\) of the \(\omega_i\) and a mean value \(\bar{\theta}\) of the time intervals \(T_i = t_i - t_{i-1}\),
and also a mean value of the ratio \(\omega_i/T_i\). The picture then obtained
has a stationary statistical character for every given point of the
y-axis; it changes gradually as we go along the y-axis.

The supposition that all \(\xi_i\) satisfy \(\xi_i < \xi_{i-1}\) cannot be
upheld in general. The velocities of advance of the various vertical
fronts will be unequal and whenever we find \(d \xi_i/dt > d \xi_{i-1}/dt\),
there will be a decrease of the distance \(\xi_{i-1} - \xi_i\). When this has
become zero, the vertical front \(\xi_i\) has overtaken the segment \(\xi_{i-1}\);
from then onward we must count these fronts as a single one. This
requires that from now onward we leave out the inclined line with the
slope \(1/(t - t_{i-1})\) and work as if the combined impulse \(\omega_i + \omega_{i-1}\)
had been introduced at the instant \(t_i\), so that from now on it follows
the impulse \(\omega_{i-2}\) introduced at \(t_{i-2}\).

This process leads to a gradual decrease of the number of fronts.
Since new fronts are continually introduced at \(y = 0\), the statistical
state of the system can be stationary.
We now come back to the rule for constructing the state of the system at a given instant \( t \). Whenever in carrying out the rules given before, we find a case where \( \xi_{i-1} > \xi_i \), we take out the instant \( t_{i-1} \) and combine \( \omega_i \) and \( \omega_{i-1} \) into a single impulse corresponding to the instant \( t_i \). If we should find that the new value of \( \xi_i \) would surpass \( \xi_{i-2} \), or that it would be smaller than \( \xi_{i+1} \), the process has to be repeated, until all cases which did not satisfy the condition \( \xi_i < \xi_{i-1} \) have been eliminated. The result of this elimination is not dependent on the order in which it is carried out.

A particular case is obtained when all impulses are exactly of the same magnitude \( \omega \) and are spaced at exactly equal intervals of time \( \theta \). In that case merging of fronts will occur only at the foremost end of the sequence. If the process has been going on for a long time, this end will have moved far out to the right, and in certain cases can be considered as having disappeared from the field under observation. The velocity of advance of all other vertical fronts will then be very nearly equal to \( \omega / \theta \). It should be observed that when the initial impulses were not exactly equal, or when there should have been slight inequalities of their spacing in time, these inequalities would not disappear in the process, but will lead to the merging of fronts somewhere inside the series. This is a typical feature characteristic for systems which are governed by a non-linear equation, belonging to the hyperbolic type (having real characteristics), where there is no smoothing out of irregularities introduced by the boundary conditions or by the initial conditions.

In the beginning of this chapter it had been remarked that eq. (1) is also illustrative of phenomena peculiar to compressible media. The solution we have been considering can be taken as a simplified picture of the behavior of a series of plane shock waves, introduced one after the other into a gas. Real shock waves will show the same feature of overtaking one another and of merging together, as do the steep fronts found in our solution of eq. (1).
55. **Statistical Problems Connected with the Propagation of Consecutive Impulses.** If $t$ is chosen so that for a group of consecutive impulses $t - t_i$ is large compared with the time intervals $T_i$ within this group, we can replace (9) by the approximation:

$$
\phi_i \approx u_i(t - t_i + \frac{1}{2} T_i)
$$

where $u_i = \sqrt{2 \omega_i / T_i}$.

In this case the condition $\phi_i < \phi_{i-1}$ is not compatible with a large positive difference $u_i - u_{i-1}$. Whenever $u_i > u_{i-1}$ we can find a value of $t$ for which the condition $\phi_i < \phi_{i-1}$ will be violated, so that we must combine impulses in the way as described before. Since we have assumed that the system has a statistically stationary character, we can expect that in the course of time all appreciable differences between consecutive $u_i$ will be eliminated in this way. There might remain large negative differences $u_i - u_{i-1}$ or series with consecutive negative differences. But in view of the randomness of the values of the $u_i$, a large value of $u_{i-k}$ would have great chance to be followed by a smaller one, and then elimination again would be necessary.

To make these considerations more precise, we observe that when two vertical fronts merge together into a single one, the values of the $\omega_i$ are added, so as also are the values of the intervals $T_i$. The new value of $u_i$ resulting from the process is given by:

$$
u_i^* = \sqrt{\frac{u_i^2 T_i + u_{i-1}^2 T_{i-1}}{T_i + T_{i-1}}}$$

If now we define a mean square value of the $u_i$ by means of the formula:

$$U^2 = \frac{\Sigma u_i^2 T_i}{\Sigma T_i},$$

where the summation is extended over a series of consecutive $i$-values, it follows that this mean square value is not affected by the process of merging together of fronts and that it is a constant. It is easily seen that this constant has the value $2 \omega / \theta$. 
On the other hand, since:

\[ u_{i}^{*}T_{i} = \sqrt{(u_{i}T_{i} + u_{i-1}T_{i-1})^2 + T_{i}T_{i-1}(u_{i} - u_{i-1})^2}, \]  

it is seen that \( \Sigma u_{i}T_{i} \) increases each time two fronts merge. Hence the (linear) mean value \( \bar{u} \) of the \( u_{i} \) increases. It thus follows that the fluctuations of the \( u_{i} \) tend to become smaller and smaller as time goes on.

This can be put into a slightly different form if we write:

\[ u_{i} = U(1 + \delta_{i}). \]  

We then have for the mean value of the square of \( 1 + \delta_{i} \):

\[ (1 + \delta_{i})^2 = 1, \]

from which:

\[ 2\bar{\delta} = -\sigma^2. \]

Hence \( \bar{\delta} \leq 0 \) and \( \bar{u} < U \). Since \( \bar{u} \) increases, \( \bar{\delta} \) increases towards zero and \( \sigma^2 \) decreases. This implies that the absolute values of the \( \delta_{i} \) gradually become smaller.

A consequence of these considerations is that, if we have started with two independent distributions for the values of the \( \omega_{i} \) and the \( T_{i} \), these will not remain independent of each other. If they are represented simultaneously by means of points in an \( \omega_{i}, T_{i} \) - diagram, the combinations occurring every time when two fronts merge, will lead to the appearance of both larger \( \omega_{i} \) - values and larger \( T_{i} \) - values, in such a way that the ratios \( \omega_{i}/T_{i} \) will cluster more and more closely round the mean value \( \bar{\omega}/\bar{T} \).

56. Application of Similarity Considerations. We will follow the history of a group, originally formed from \( N_{o} \) impulses of total amount \( S_{i} \), introduced at \( y = 0 \) in a period of duration \( G \). When we consider the group at an instant \( t \), the number of fronts will have
decreased to say $N$, and the mean values $\bar{\omega}$ and $\Theta$ are given by:
\[ \bar{\omega} = \frac{\sum \omega}{N}; \quad \Theta = \frac{\sum \Theta}{N}. \]

When the merging process has already gone so far that the $\delta_i$ are small compared with unity, it follows from (12) that when two fronts merge (which makes $N$ decrease with $1$) the sum $\sum u_i T_i$ increases by approximately:
\[ \frac{T_i T_{i-1} (u_i - u_{i-1})^2}{2(u_i T_i + u_{i-1} T_{i-1})}. \]

It seems fairly safe to assume that the mean value of this quantity is given by:
\[ \frac{1}{4} \bar{\omega} \Theta (\delta_i - \delta_{i-1})^2 = \frac{1}{2} \bar{\Theta} \delta_i^2. \]
Hence the mean value $\bar{\omega}$ increases on the average by $\frac{1}{2} \bar{\omega} \delta_i^2/N$, from which there further follows:
- average increase of $\bar{\omega} = \frac{1}{2} \delta_i^2/N$
- average decrease of $\bar{\Theta} = \delta_i^2/N$

If we assume that a certain statistical similarity is conserved in the process, we must write this in the form:
\[ \frac{\text{decrease of } \delta^2}{\delta^2} = \frac{\text{decrease of } N}{N}, \]
and find:
\[ \delta^2 \propto N. \quad (14) \]

Now the instant at which $\xi_i$ and $\xi_{i-1}$ merge, (this was the typical combination considered in the formulas, assuming that $u_i - u_{i-1} > 0$) is given by:
\[ t - t_1 + \frac{1}{2} T_i = \frac{u_i - 1 (T_i + \delta_{i-1})}{2(u_i - u_{i-1})} = \frac{(1 + \delta_i)(T_i + \delta_i - 1)}{2(\delta_i - \delta_{i-1})} \]

We can take this expression as an estimate of the "age" of the group under consideration. Since the average value of \(|\delta_i - \delta_{i-1}|\) is \(\sqrt{2\delta^2}\), we obtain:

\[
\text{average age} \approx \sqrt{2\delta^2} \approx \sqrt{2\delta^2} \approx \frac{3}{2}.
\]

In this way we arrive at the results:

\[
N \approx (\text{age}) - 2/3 \tag{15}
\]

\[
\delta^2 \approx (\text{age}) - 2/3 \tag{16}
\]

It also follows that:

\[
\delta \text{ and } T \approx N^{1/2} \approx (\text{age})^{2/3} \tag{17}
\]

The dependence on the age can be transformed into a dependence on the mean position of the group along the y-axis, if we observe that according to (10):

\[
\xi_i \approx U_i (\text{age}) \tag{18}
\]

57. An observer located at a fixed value of y will see the broken curve representing the course of v pass over him with a velocity approximately equal to U. His observations will show a slow increase of v during periods \(T_i\), alternating with a sudden decrease. The magnitude of the decrease, observed at the instant when the front \(\xi_i\) passes the observer, is given by:

\[
\left[ v_i \right] = \sqrt{\frac{2\omega_i T_i}{(t - t_i) (t - t_{i-1})}} = \frac{\sqrt{2\omega_i T_i}}{t - t_i + \frac{1}{2} T_i} \tag{19}
\]
From this expression it appears that so long as merging of fronts does not take place, the values of \( \left[ v_1 \right] \) (which are quantities \( v_x - v_{II} \) considered in section 52) decrease inversely proportionally with the age, and consequently also inversely proportionally with the distance of the observer from the origin on the y-axis. However, every merging of two successive fronts introduces an increase of \( \omega_1 \) and \( T_1 \); the decrease of the average value of the \( \left[ v_1 \right] \) consequently will be slower.

From (17) we see that the numerator increases proportionally with \( \text{age}^{2/3} \); hence we find:

average value of \( \left[ v_1 \right] \propto (\text{age})^{-1/3} \).  \( \text{(20)} \)

Certain other problems can be taken in view connected with the \( v \)-curve passing over a stationary observer. Since the process should be stationary in the statistical sense, we can consider the Eulerian time-correlation \( v(t) v(t + \tau) \) for a fixed value of \( y \). It can be estimated that this correlation will become zero when \( \tau \) exceeds a few times \( \Theta \).

We can also fix attention to a particular point of the curve for \( v \). Such a point displaces itself without change of height with its velocity \( v \), until the instant where the two fronts between which it is situated, happen to merge. The interval of time from the instant at which the corresponding impulses were introduced until the instant at which the fronts merge, is of the order of the age of the group to which the fronts belong. Since this age is much larger than \( \Theta \), an individual point of the curve, followed in its motion, will keep its velocity usually over a long period. Hence a Lagrangian correlation calculated for the motion of individual points will be different from zero over a much longer interval than the Eulerian time correlation.

When the age of a group is large, and the group still contains many members, it is also possible to calculate an Eulerian space correlation \( v(y) v(y + \gamma) \) for the group, which quantity will be a function of time. These Eulerian space correlations, however, can be better investigated if we turn to a different type of solution of eq. (1), in which the broken curve giving the course of \( v \) is formed of parallel upward sloping segments (instead of segments all starting from the point
y = 0). Such a solution can be obtained from the one considered here by means of a limiting process, but it is more convenient to define it in an independent way.

It is useful to notice the quantity $\omega_1 = \frac{1}{2} U^2 T_1$ for the solution thus far considered. This quantity represents the difference in area between the two triangles OAB and OCD connected with the interval $T_1 = t_i - t_{i-1}$. The value can be positive or negative and the mean value is zero. Since

$$\omega_1 = \frac{1}{2} U^2 T_1 = U^2 T_1 \left( \delta_1 + \frac{1}{2} \delta_1^2 \right) = U^2 T_1 \delta_1,$$

we must expect that the average absolute value will increase proportionally with $(\text{age})^{1/3}$.
Solutions of Eq. (1) Illustrating Spatially Homogeneous Turbulence Decaying with Time

58. We consider a solution of eq. (1) in which the initial form of the curve for \( v \) is given by a series of straight parallel segments. In consequence of formulas (2) and (3) of section 50, the parallelism will be retained during the whole history, though the magnitude of the slope will decrease. It is convenient to take the slope equal to \( 1/t \), omitting the unimportant constant \( t_o \). The upward sloping segments are again separated by vertical fronts. If account must be taken of the influence of viscosity, a more correct expression for the course of \( v \) in these fronts can be given by making use of eq. (3) of section 52. If we introduce the notation which is illustrated in the accompanying diagram, it will be seen that the height of a vertical front is measured by \( v_I - v_{II} = \tau_i/t \). We change the first term of (8) in such a way that account is taken of the slope of the curve to the left and to the right of the front and obtain:

\[
v = \frac{y - \xi_{i-1}^1}{t} - \frac{\tau_i}{2t} \tanh \frac{\tau_i (y - \xi_{i-1})}{2y t}
\]  

(21)

\[
\frac{1}{2} (\sigma_{i-1} + \sigma_i) \quad \quad \quad \tau_i = \xi_i - \xi_{i-1}
\]
At the same time:

\[ \frac{d \xi_1}{dt} = -\frac{\xi_1 - \sigma_1 + \frac{1}{2} \tau_1}{t} \]  

(22)

The distribution of the lengths \( \tau_1 \) and \( \lambda_1 \) to a large extent can be chosen arbitrarily at the initial instant, provided the distribution is sufficiently homogeneous in order that mean values \( \overline{\tau} \) and \( \overline{\lambda} \) shall exist. These mean values are obtained by taking a certain number \( N \) of consecutive \( \tau_1 \) or \( \lambda_1 \) and dividing their sum by \( N \); the result must approach to a definite limit when \( N \) increases more and more, which limit should be independent of the choice of the starting point. The mean values \( \overline{\tau} \) and \( \overline{\lambda} \) moreover shall be equal. In consequence of the circumstance that the development of the system in the course of time leads to the merging of fronts, there can arise a certain interdependence between the distributions of the \( \tau_1 \) and \( \lambda_1 \), and we shall see later (section 65.) that such an interdependence may be of importance.

In the calculations we shall also have to do with mean values obtained by integrating a quantity with respect to \( y \) over a certain length \( S \) of the \( y \)-axis and dividing by this length; such mean values will be denoted by means of a simple bar, as for instance \( \overline{y} \). It is often convenient to make \( S = N \ell \).

There is still a freedom in the system: the relative situation of the two series of points \( \sigma_i \) and \( \xi_i \). We assume that this is determined in such a way that the statistical properties of the system do not change when the direction of \( y \) and the sign of \( v \) are simultaneously changed. The mean value \( \overline{v} \) of \( v \) with respect to \( y \) will then be zero.

The properties of homogeneity and of statistical invariance with respect to simultaneous change of order and of the sign of \( v \) just mentioned, remain valid throughout the development of the system if they have existed at some initial instant. This is a consequence of the invariance of the differential equation, both with respect to a shift along the \( y \)-axis and with respect to a simultaneous change of signs of \( v \) and \( y \).

Concerning the development of the system in the course of time we observe that the \( \sigma_i \) and the lengths \( \tau_1 \) are constants. The points
\( \varepsilon_{n+1} \) move according to eq. (22); it follows that the lengths \( \lambda_i \) are functions of time and

\[
\frac{d \lambda_i}{dt} = \lambda_i - \frac{1}{2} \left( \frac{\tau_i + \tau_{i-1}}{t} \right)
\]

It is convenient to introduce quantities \( \zeta_i \) defined by:

\[
\zeta_i = \varepsilon_{n+1} - \frac{1}{2} \left( \sigma_i + \tau_{i-1} \right) = \varepsilon_{n+1} - \sigma_i + \frac{1}{2} \tau_i;
\]

then:

\[
\frac{d \zeta_i}{dt} = \frac{d \varepsilon_{n+1}}{dt} = \frac{\zeta_i}{t}; \quad \frac{d \lambda_i}{dt} = \frac{\zeta_i - \zeta_{i-1}}{t}
\]

The \( \zeta_i \) can be positive as well as negative, and the mean value \( \overline{\zeta_i} \) must be zero.

The property of statistical invariance with respect to a simultaneous change of sign of \( \nu \) and \( \nu' \), referred to before, can also be expressed by the rule that any statistical quantity formed out of the \( \tau_i, \lambda_i \) and \( \zeta_i \) will remain unchanged when the signs of all the \( \zeta_i \) are changed simultaneously with a change of the direction in which \( i \) is counted, no change of sign being made in \( \tau_i \) and \( \lambda_i \). For instance:

\[
\overline{\lambda_i \tau_i} = \lambda_i \tau_i; \quad \overline{\lambda_i \zeta_i} = \lambda_i \zeta_i; \quad \overline{\tau_i \zeta_i} = 0.
\]

When two consecutive \( \varepsilon_{n+1} \) (say \( \varepsilon_{n+1}^{i-1} \) and \( \varepsilon_{n+1}^i \)) become equal to each other, the point \( \varepsilon_{n+1}^{i-1} \) and the segment \( \lambda_i \) disappear from the arrangement. The values of \( \tau_{i-1} \) and \( \tau_i \) are added and the law of motion of the resulting segment is again determined by (22), provided we replace \( \tau_i \) by \( \tau_i + \tau_{i-1} \).

By this process both the number of segments \( \tau_i \) and that of segments \( \lambda_i \) are gradually reduced in the course of time.

59. Simple Mean Values. We write:

\[
\frac{\tau}{\lambda} = \frac{\tau}{\lambda} = \ell
\]

and further:

\[
\tau^2 = \ell^2 (1 + \omega); \quad \tau^3 = \ell^3 (1 + \omega) \sqrt{\tau_i \zeta_i^2} = \ell^3 \omega
\]

To calculate the mean value of \( \nu \) (which is zero, as mentioned before), we observe that \( \nu \) changes linearly with \( \nu' \) over a segment \( \lambda_i \); hence for a single segment:

\[
\int v \, dv = t \int v \, dv = \frac{1}{2t} \left\{ \left( \zeta_i + \frac{1}{2} \tau_i \right)^2 - \left( \zeta_{i-1} - \frac{1}{2} \tau_{i-1} \right)^2 \right\}.
\]
We sum this expression with respect to $i$ over all segments contained in a part of the $y$-axis of great length $S$; the sum must be divided by $S = N \ell$, where $N$ is the number of segments $\lambda_1$ contained in $S$ (and also the number of segments $\tau_1$ contained in $S$). Having regard to the relation $\frac{1}{S} \sum_{i} \tau_i = 0$, it immediately follows that $\mathbf{v} = 0$.

To find the mean value of $v^2$ a similar procedure is applied. Integration over the length of the segment $\lambda_1$ gives:

$$\int v^2 \, dy = t \int v^2 \, dv = \frac{1}{2t^2} \left\{ (\xi_1 + \frac{1}{2} \tau_1)^3 - (\xi_{i-1} - \frac{1}{2} \tau_{i-1})^3 \right\}.$$ 

Hence the mean value becomes:

$$\bar{v}^2 = \frac{1}{\ell t^2} \left( \frac{\xi_1^2}{\ell} + \frac{1}{12} \tau_1^3 \right) = \frac{\ell^2}{t^2} \left\{ \frac{\xi_1^2}{\ell} + \frac{1}{12} \left( 1 + \omega^* \right) \right\}.$$ 

It has been mentioned in section 52 that the dissipation is to be found only in the vertical segments, each segment giving a contribution which in the present notation is equal to $\frac{1}{12} \tau_1^3 / t^3$. It follows that the mean dissipation of energy in unit time per unit length of the $y$-axis is given by:

$$\epsilon = \epsilon = \frac{1}{12} \ell \frac{t^3}{\ell} = \frac{\ell^2}{12 t^3} \left( 1 + \omega^* \right).$$ 

In consequence of the merging of fronts the number $N$ of segments contained in a constant great length $S$ decreases and $\ell$ increases with time. It is also possible that the values of the dimensionless parameters $\omega$, $\omega^*$, $\bar{\omega}$, will be functions of $t$. If the arrangement retains a statistical similarity, these parameters will be constants.

So long as we do not know how $\ell$ changes with time, very little can be said about the behavior of $\bar{v}^2$ and $\epsilon$.

60. Quantities Connected with the Momentum Integral. We write:

$$\mu_1 = \tau_1 \xi_1 / t.$$ 

Every quantity $\mu_1$ corresponds to a segment $\tau_1$ and to a vertical front with position $\xi_{\tau_1}$. With the aid of the relations indicated in the diagram of section 58, it can easily be proved that $\mu_1$ is the algebraic
value of the area ABCDE

\[ \text{area ABCDE} \]

It will be seen that the integral

\[ \int_{y}^{y+S} v \, dy \]

extended over a certain length \( S \) of the \( y \)-axis will be equal to

\[ \sum_{k=0}^{N-1} \mu_{i+k} + \Delta, \]

where the sum refers to the values of \( \mu_i \) for all segments \( \tau_i \) contained in \( S \), while \( \Delta \) stands for the additional parts appearing at the ends of the integration interval.
Since \( \frac{d \xi}{dt} = \frac{\xi}{t} \), from which it follows that the \( \xi \)'s are proportional with \( t \), the \( \mu_i \) are independent of the time. Whenever two \( \xi_i \) come to coincidence and the corresponding vertical fronts merge into a single front, the \( \mu_i \) corresponding to these fronts are added. Hence \( \Sigma \mu_i \) does not change. This result is evidently connected with the property of the momentum integral, mentioned in section 52.

We now consider the mean value \( \bar{M} \) of the quantity:

\[
\frac{1}{S} \left( \sum_{k=0}^{N-1} \mu_i + k \right)^2
\]

which evidently is independent of the time. If the \( \mu_i \) would be completely independent of each other, the mean value would reduce to:

\[
\bar{M} = \frac{\mu^2}{\ell}
\]

It is possible, however, that relations exist between successive \( \mu_i \), so that the mean values \( \frac{\mu_i \mu_{i+k}}{\mu_{i+k}} \) may be different from zero. We can expect nevertheless that these mean values rapidly will approach zero when \( k \) increases. Hence, when \( N \) is large, we may write:

\[
\bar{M} = \frac{1}{\ell} \left\{ \frac{\mu^2}{\ell} + 2 \sum_{k=1}^{\infty} \mu_i \mu_{i+k} \right\}
\]

which formula contains the preceding one as a special case.

If the arrangement should retain a statistical similarity (which in itself is not quite certain, we shall see afterwards), an important argument could be deduced from the fact that \( \bar{M} \) is independent of the time. If there would be statistical similarity, mean values like \( \mu^2 \) and \( \mu_i \mu_{i+k} \) would be proportional to the fourth power of the mean length \( \ell \) of the segments \( \tau_i \), divided by the square of the time \( t \). Hence \( \bar{M} \) would become proportional to the third power of \( \ell \) divided by \( t^2 \), and since this must be a constant, it follows that \( \ell \) must increase according to the formula:

\[
\ell \sim t^{2/3}
\]

As \( \omega, \omega^*, \bar{\omega} \) would be constants, it further follows from eqs. (25) and (26) that:

\[
v^2 \sim t^{-2/3} ; \bar{v} \sim t^{-5/3}.
\]
These results would be in accordance with those obtained in sections 56 and 57 for the preceding type of solutions (see eqs. 17 and 20).

Even if there should be similarity, the argument would break down if \( \mu \) would be zero. This can occur where there exist certain relations between the mean values \( \mu_{i+k} \) and \( \mu_{i} \), of such nature that the mean value of \( ( \sum \mu_{i+k} )^2 \) would not increase proportionally with \( N \), but at a slower rate or would approach to a constant value.

With reference to this point it is of importance to mention that the solutions we have been considering can be obtained from an initial state in which a series of concentrated impulses, each of a finite integrated magnitude \( A_m \), is introduced at an infinite series of points of the \( y \)-axis, arbitrarily spaced, but so that the distribution is statistically homogeneous. The \( A_m \) can be positive or negative and follow each other at random, while the mean value of the \( A_m \) over any large domain of the \( y \)-axis shall be zero. It can then be proved that the \( \mu_{i+k} \) are either equal to certain \( A_m \) or are equal to sums of consecutive \( A_m \) (in consequence of the merging together of consecutive impulses), so that \( \sum \mu_{i+k} \) over any length \( S \) of the \( y \)-axis is equal to \( \sum A_m \) for the same length. Hence if the random distribution of the \( A_m \) is subjected to the condition that the mean value of \( ( \sum A_m )^2 \) will not increase proportionally with the length \( S \), but increases at a slower rate or approaches to a constant value, the same property will apply to \( ( \sum \mu_{i+k} )^2 \). Such arrangements can be obtained by choosing a particular rule for determining the magnitudes of the \( A_m \).

In the next section we shall see that \( M \) is connected with an important invariant referring to correlation functions.

61. Correlations. For the type of solutions we are considering now, the Eulerian space correlation \( \bar{v}(y) \bar{v}(y+\eta) \) can be obtained, for which we shall write \( \bar{v}_1 \bar{v}_2 \). Another important Eulerian correlation function is \( \frac{[v(y)]^2 v(y+\eta)}{\bar{v}_1 \bar{v}_2} \), to be written \( \bar{v}_1 v_2^2 \). It will be evident that \( \bar{v}_1 \bar{v}_2 \) is a symmetric function of \( \eta \), having its maximum at \( \eta = 0 \), whereas \( \bar{v}_1 v_2^2 \) is an odd function of \( \eta \), which is zero for \( \eta = 0 \). Both functions decrease to zero when \( \eta \) becomes large. It is understood that the mean values refer to a particular instant. Both quantities therefore are functions of the time.

Starting from eq. (1) the following equation can be formed...
\[ \frac{\partial}{\partial t} \left( v_1 v_2 \right) = v_1 v_2 \left( \frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial y} \right) + v \left( v_2 \frac{\partial^2 v_1}{\partial y^2} + v_1 \frac{\partial^2 v_2}{\partial y^2} \right). \]

When mean values are taken from both sides, we obtain (according to a well known procedure):

\[ \frac{\partial}{\partial t} \overline{(v_1 v_2)} = \frac{\partial}{\partial y} \overline{(v_1^2 v_2)} + 2v \frac{\partial^2}{\partial y^2} \overline{(v_1 v_2)}. \quad (31) \]

This equation is the analogue of the important equation of von Kármán and Howarth for hydrodynamic turbulence. It has a central place in investigations on the statistical behavior of the solutions we are considering. If we consider not too small values of \( \gamma \), the term with \( v \) can be discarded.

Integrating (31) with respect to \( \gamma \) from 0 to \( \infty \) we find:

\[ \frac{d\bar{J}}{dt} = 0 \quad (32) \]

where

\[ \bar{J} = \int_{0}^{\infty} \overline{v_1 v_2} \ d\gamma \quad (32a) \]

The quantity \( J \) is the analogue of Loitsiansky's invariant in hydrodynamic turbulence.

To obtain the connection between this invariant and the quantity \( M \) of the preceding section, we write:

\[ M = \frac{1}{S} \left( \int_{y}^{y+S} v \ dy \right)^2. \]

The error committed by neglecting the quantity denoted by \( \Delta \) decreases to zero when \( S \) is made larger and larger. The mean value indicated by the bar must be understood in the following way: we calculate the integral with various starting points \( y \), keeping \( S \) constant; then we
determine the mean value with respect to the starting point. The expression for \( M \) can also be written:

\[
M = \frac{1}{S} \int_0^S \int_0^S dy_1 dy_2 v(y + y_1) v(y + y_2)
\]

It will be seen that \( v(y + y_1) v(y + y_2) \) reduces to the function \( v_1 v_2 \) with \( \gamma = y_1 - y_2 \). It is thus easily found that the double integration gives:

\[
M = \frac{1}{S} \cdot 2S \int_0^\infty v_1 v_2 d\gamma = 2J.
\]

62. **Scales Connected with the Solutions under Consideration.** In all investigations concerning turbulence the concept of certain scales, referring either to the macroscopic aspect or to features of detail, plays an important part. Similar quantities can be formed for our solutions of eq. (1).

A macroscopic scale of length is given by \( \ell \), representing the mean values \( \bar{\tau} = \bar{\lambda} \).

An average amplitude for \( v \) is given by \( \ell / t \). Making use of both quantities we can define a Reynolds number:

\[
Re = \ell^2 / v t.
\]

This should be a large number; otherwise the approximations, involving either the neglect of the viscosity or, if greater accuracy is needed, the use of the expression (21), would not be valid. If the arrangement in the system would remain statistically similar and the results obtained in section 60 could be applied, it is found that \( Re \) would increase proportionally with \( t^{1/3} \).

A microscale \( m \) can be defined by means of the development of \( v_1 v_2 \) according to powers of \( \gamma \), when this is written in the
\[
\overline{v_1v_2} = \overline{v^2} \left(1 - \frac{\overline{\gamma^2}}{2m^2} + \ldots \right) \tag{34}
\]

According to a well known formula we then have:

\[
\left(\frac{\partial \overline{v}}{\partial y}\right)^2 = \frac{\overline{v^2}}{m^2}.
\]

We introduce the mean kinetic energy \( E = \frac{1}{2} \overline{v^2} \) and the mean dissipation \( \varepsilon = \nu \left( \frac{\partial v}{\partial y}\right)^2 \); we then have the equation:

\[
\frac{dE}{dt} = -\varepsilon, \text{ or } \frac{dv^2}{dt} = -\frac{2\nu v^2}{m^2}. \tag{35}
\]

Making use of eqs. (25) and (26) we obtain:

\[
m^2 = \frac{\tau^3 + 12\tau_1\tau_4^2}{\tau_3^3} \nu t. \tag{36}
\]

It follows that for a system for which the arrangement in the large would remain statistically similar, we should have:

\[
m \sim \sqrt{\nu t}
\]

This would mean that there could be no similarity over the whole range, since \( \frac{\ell}{m} \) would increase with time.

One can define a Reynolds number connected with the microscale and obtain:

\[
Re_m \sim Re^{1/2}.
\]

63. When we develop the correlation function \( v_1^2v_2 \) with respect to \( \gamma \), it is found that the term of the first degree disappears, in consequence of the circumstance that the mean value of \( v^2 \left( \frac{\partial v}{\partial y}\right) \)

*In the existing literature this microscale is denoted by \( \lambda \). We used \( m \) to avoid confusion with the segments \( \lambda_1 \).
is zero. Since $v_1^2 v_2$ is an odd function of $\eta$, its development begins with a term in $\eta^3$.

The series for the two functions $v_1 v_2$ and $v_1^2 v_2$ can be written:

$$
\frac{v_1 v_2}{v_1^2 v_2} = v^2 - \frac{\eta^2}{2} \left( \frac{\partial v}{\partial y} \right)^2 + \frac{\eta^4}{24} \left( \frac{\partial^2 v}{\partial y^2} \right)^2 - \ldots
$$

$$
\frac{v_1^2 v_2}{v_1^2 v_2} = - \frac{\eta^3}{6} \left( \frac{\partial v}{\partial y} \right)^3 + \ldots
$$

Approximate expressions for the mean values in the right hand members can be obtained by making use of eq. (21). The contributions from the upward sloping segments and from the nearly vertical fronts can be calculated separately; the values obtained are:

<table>
<thead>
<tr>
<th>upward sloping segments</th>
<th>vertical fronts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial v}{\partial y}$</td>
<td>$\frac{1}{t}$</td>
</tr>
<tr>
<td>$(\frac{\partial v}{\partial y})^2$</td>
<td>$\frac{1}{t^2}$</td>
</tr>
<tr>
<td>$(\frac{\partial v}{\partial y})^3$</td>
<td>$\frac{1}{t^3}$</td>
</tr>
<tr>
<td>$(\frac{\partial^2 v}{\partial y^2})^2$</td>
<td>0</td>
</tr>
</tbody>
</table>

It must be observed, however, that these formulas give the most important parts only, and a more refined calculation would bring terms with smaller powers of $v$ in the denominators.

It will be seen that the overall mean value of $\partial v / \partial y$ is zero, as is necessary. The mean values of the other quantities are practically given by the contributions from the fronts. The fact that the mean value of $(\partial v / \partial y)$ is different from zero, is an indication of a
certain skewness in the distribution of $v$. A similar skewness appears in hydrodynamic turbulence. A "skewness factor" can be defined by:

$$\frac{(\overline{\partial v/\partial x})^3}{[\overline{\partial v/\partial x}]^{3/2}} = \frac{\sqrt{12}}{10} \frac{\rho^2}{\nu} \frac{\tau^5}{[\tau^3]}^{3/2} \frac{\eta_3}{\nu^2 L^4 t^5},$$

which is proportional to $\text{Re}^{1/2}$. The latter result is different from what is obtained in hydrodynamic turbulence, where values of order unity are found (Batchelor gives: $-0.395$; compare Chapter IV, section 21). The concentration of the gradient of $v$ in the model system, which is not subjected to any equation of continuity, is much larger than the concentration of vorticity in the vortex sheets between eddies in actual turbulence. The series for the correlation functions now take the form:

$$\overline{v v v} = \overline{v^2} - \frac{\eta^2}{2} \frac{\tau^3}{12 \nu \frac{\rho^2}{\nu} L^3} + \frac{\eta^4}{24} \frac{\tau^5}{240 \nu^2 L^5 t^5} - \ldots$$

$$\overline{v_1 v_2} = - \frac{\eta^3}{6} \frac{\tau^5}{120 \nu^2 L^5 t^5} + \ldots$$

When these results, which are applicable only for very small values of $\eta$, are substituted into (31) and terms independent of $\eta$ are compared, we obtain:

$$\frac{d \varepsilon^2}{dt} = - \frac{\tau^3}{6 \frac{\rho^2}{\nu} L^3 t^5} = - 2 \varepsilon,$$

which is the same as eq. (34) (compare eq. (26) for $\varepsilon$). Comparison of the terms in $\eta^2$ does not lead to a useful result, since the terms with $\tau^5$ should be completed with other terms, for which no expression has been obtained. In current turbulence theory the comparison of the terms in $\eta^2$ is used for discussing the decay of vorticity, but also here in a qualitative way only, since detailed expressions for the quantities appearing in the equations are unknown.
Expressions for the Correlation Functions $v_1 v_2$ and $v_2^2$

64. For further discussion of the statistical properties of the solutions it is necessary to obtain expressions for the correlation functions $v_1 v_2$ and $v_2^2$ and their first derivatives with respect to $\gamma$. This will also bring us to certain conditions to be satisfied by the arrangement of the segments $\tau_i$ and $\lambda_i$. Since in constructing the formulas it is necessary to distinguish between various cases, the full mathematical deduction is rather involved. The most simple case is presented by $\frac{\partial (v_1 v_2)}{\partial \gamma}$ and if we content ourselves with a less rigorous deduction, the expression for this quantity can be obtained in the following way.

We start from the relation:

$$\frac{\partial (v_1 v_2)}{\partial \gamma} = \frac{v(y) \frac{\partial}{\partial y} v(y+\gamma)}{v(y+\gamma)} = \frac{v(y-\gamma) \frac{\partial v}{\partial y}}{v(y-\gamma)}$$

Along the upward sloping segments of the curve for $v$ the derivative $\frac{\partial v}{\partial y}$ has the constant value $1/t$. It follows that the contribution of these segments to the desired mean value is given by $\frac{v}{t}$, which is zero. In the nearly vertical fronts $\frac{\partial v}{\partial y}$ assumes large negative values, the integrated amount for any particular front being equal to $-\frac{\tau_i}{t}$. The value of $v$ to be multiplied with this quantity is given by

\[ v \]

\[ \frac{v}{t} \]

\[ \frac{v(y+\gamma)}{v(y+\gamma)} \]

\[ \frac{v(y-\gamma) \frac{\partial v}{\partial y}}{v(y-\gamma)} \]
\[ (\zeta_i + \frac{1}{2} \tau_i - \eta) / t \text{ if } \eta < \lambda_i; \text{ or more generally by} \\
(\zeta_i + \frac{1}{2} \tau_i + \tau_{i-1} + \ldots + \tau_{i-k} - \eta) / t \text{ if } \eta \text{ should exceed the combined length of the } k \text{ segments } \lambda_i + \lambda_{i-1} + \ldots + \lambda_{i-k+1}, \text{ where } k \text{ can be any number from 1 upward. Since all these cases can occur, the mean value of } v(y - \eta) \cdot (\partial v / \partial y) \text{ is obtained as the mean with respect to } y \text{ of the expression:} \\
-\frac{\tau_i}{t^2} \left( \zeta_i + \frac{1}{2} \tau_i - \eta + \varepsilon_1 \tau_{i-1} + \varepsilon_2 \tau_{i-2} + \ldots \right), \\
\text{where the coefficients } \varepsilon_k \text{ (which are functions of } \eta) \text{ specify the probability that } \eta \text{ will exceed the sum of } k \text{ consecutive segments. We first take the mean value with respect to } i \text{ and afterwards divide by } \mathcal{L} \text{ in order to obtain the mean with respect to } y. \text{ The mean value of } \tau_i \text{ is } L; \text{ that of } \tau_i \varepsilon_i \text{ is zero; and we find:} \\
\frac{\partial \langle v_1 v_2 \rangle}{\partial \eta} = -\frac{\tau_i^2}{2L^2 t^2} + \frac{\eta}{t^2} - \frac{1}{\mathcal{L} t^2} \sum_{k=1}^{\infty} \varepsilon_k \tau_i \tau_{i+k} \quad (37) \\
\text{To obtain the } \varepsilon_k \text{ we introduce a set of distribution functions } f_k. \text{ The first one, } f_1, \text{ of these functions gives the distribution of the possible values of the length } \lambda_i \text{ of a single segment. The second one, } f_2, \text{ gives the distribution of the combined length of two consecutive segments, say } \lambda_{i+1} + \lambda_{i+2}; \text{ and } f_k \text{ gives the distribution of the combined length of } k \text{ consecutive segments. The functions } \varepsilon_k \text{ are the integrals of the } f_k: \\
\varepsilon_k(\eta) = \int_{\lambda} f_k(\lambda) \, d\lambda. \\
\text{A more rigorous deduction, paying attention to details, is given in Proc. Netherl. Acad. Sciences Amsterdam 53, p. 393, 1950. There the expression for } v_1 v_2 \text{ is deduced first, in order to eliminate difficulties concerning derivatives; the expressions obtained have been written in such a form that it is possible to take account of correlations which}
might exist between the values of \( \tau_i \tau_{i+k} \) and the distances \( \xi_{i+k} - \xi_i = \lambda_i + \lambda_{i+1} + \cdots + \lambda_i + k \).

65. We must expect that all correlation functions and thus also (37) will become zero for infinite values of \( \tau_i \). A verification of this property reveals that certain conditions must be fulfilled by the arrangement of the \( \tau_i \) and \( \lambda_i \). We write:

\[
\tau_i \tau_{i+k} = \ell^2 (1 + \omega_k); \quad \tau_i \tau_{i+k} = \ell^2 (1 + \omega_k).
\]

The \( \omega_k \) (with \( k \) different from zero) will be zero if there are no correlations between consecutive \( \tau_i \); they may be different from zero if there exist such relations, but nevertheless we can expect that the \( \omega_k \) will vanish for large \( k \). Since for a large value of \( \tau_i \) all \( \xi_k \) with \( k \) small compared with \( \tau_i / \ell \) will be practically equal to unity, we find:

\[
\sum \xi_k \frac{\tau_i \tau_{i+k}}{\ell^2} = \ell^2 \sum \xi_k + \ell^2 \sum \omega_k.
\]

The second sum is independent of \( \tau_i \). Concerning the first sum, we observe that we can assume:

\[
\sum f_k(\lambda) = 1/\ell \text{ for large } \lambda.
\]

Indeed, \( \sum f_k(\lambda) d\lambda \) is the probability to find a vertical front at a point \( \xi_{i+k} \) with arbitrary \( k \), satisfying \( \xi_{i+k} + \lambda < \xi_{i+k} < \xi_{i+k} + \lambda + d\lambda \). This probability must become independent of \( \lambda \) when \( \lambda \) is large enough, provided there is sufficient randomness in the arrangement of the \( \lambda_i \). It will then be equal to \( d\lambda / \ell \). It follows that:

\[
\sum \xi_k = \eta / \ell + \text{constant}.
\]

This proves that (37) becomes independent of \( \tau_i \) for large \( \tau_i \). It does not prove that (37) will become zero; for this it is necessary that:

\[
\sum \xi_k(\tau_i) = \frac{\eta}{\ell} - \frac{1}{2} - \frac{\omega}{2} - \sum \omega_k.
\]
The condition (38) is not fulfilled when we take all \( \tau_i \) and \( \lambda_i \) to be completely independent of each other and to be subjected to no other condition than that the mean values \( \bar{\tau} \) and \( \bar{\lambda} \) shall be equal to \( \ell \).

For instance if we take the simple distribution function:

\[
f_1(\lambda) = e^{-\lambda/\ell} \frac{\lambda}{\ell},
\]

which, with complete independence of the \( \lambda_i \), leads to:

\[
f_k(\lambda) = \frac{1}{(k-1)!} \left( \frac{\lambda}{\ell} \right)^{k-1} e^{-\lambda/\ell} \frac{d\lambda}{\ell},
\]

we find:

\[
\sum f_k = 1/\ell \quad \text{for all } \gamma;
\]

and:

\[
\sum s_k = \gamma/\ell \quad \text{for all } \gamma.
\]

Furthermore, complete mutual independence of the \( \tau_i \) gives \( \omega_k = 0 \) for all \( k \) different from zero. Equation (37) in this case becomes:

\[
\frac{\partial}{\partial \gamma} \left( \frac{v_1 - v_2}{\ell t^2} \right) = -\frac{\tau^2}{2\ell t^2} \quad \text{for all } \gamma.
\]

The result arrived at is understandable if we observe that with completely independent \( \tau_i \) and \( \lambda_i \) the curve for \( v \) becomes of a type as is found in problems of the "random walk", with arbitrary positive steps \( (\tau_i/\ell) \) alternating with arbitrary negative steps \( (\lambda_i/\ell) \). It is known that in such a case the mean square difference \( (v_1 - v_2)^2 \) between the values of \( v \) at two points a large distance \( \gamma \) apart, increases linearly with \( \gamma \). Indeed, we roughly have:

\[
v_1 - v_2 \approx \frac{1}{t} \sum_{k=0}^{N} (\tau_{i+k} - \lambda_{i+k})
\]

where \( N \) = largest integer in \( \gamma/\ell \); and with complete independence:

\[
(v_1 - v_2)^2 \approx \frac{1}{t^2} \left[ \sum (\tau_{i+k} - \lambda_{i+k}) \right]^2 \approx \frac{N}{t^2} \left( \tau_1 - \lambda_1 \right)^2 = \frac{\gamma}{\ell t^2} \left( \tau^2 + \lambda^2 - 2 \ell^2 \right).
\]
Since \((v_1 - v_2)^2 = 2v_1^2 - 2v_1v_2\), it follows that:

\[
\frac{\partial (v_1v_2)}{\partial \gamma} = -\frac{\tau^2 + \lambda^2 - 2\ell^2}{2\ell^2} .
\]

The result obtained above is in accordance with this formula, since we find \(\ell^2 = 2\ell^2\) with the expression taken for \(f_1\).

It follows that we must require that the arrangement of the \(\tau_i\) and \(\lambda_i\) shall satisfy the condition:

\[
\frac{1}{N} \left[ \sum_{k=1}^{N} (\tau_{i+k} - \lambda_{i+k}) \right]^2 \rightarrow 0 . \tag{38a}
\]

It can be expected that this condition will be connected in some way with (38). A general analysis of the connection has not been made, but a particular way to satisfy (38) is to require that the \(\tau_i\) and \(\lambda_i\) separately satisfy the conditions:

\[
\frac{1}{N} \left[ \sum \tau_{i+k} - N \ell \right]^2 \rightarrow 0 ; \quad \frac{1}{N} \left[ \sum \lambda_{i+k} - N \ell \right]^2 \rightarrow 0 .
\]

The first one leads to:

\[
(\omega + 2 \sum \omega_k) = 0 .
\]

The second one leads to a similar relation for the \(\lambda_i\). However, when an additional supposition is made, it can also lead to:

\[
\sum \xi_k = \frac{\gamma}{\ell} - \frac{1}{2} .
\]

In that way (38) would be fulfilled.

A further investigation is desirable. There may be also a connection with the problem whether \(J\) will be zero under certain circumstances.

66. The complete expressions for \(v_1v_2\) and \(\partial (v_1^2v_2^2) / \partial \gamma\), valid for values of \(\gamma\) large compared with \(v\ell / \ell\), are:
where the functions \( \varphi_k, x_k, \Phi_k \) are defined by:

\[
\varphi_k(\gamma) = \mathcal{L}^{-2} \int_0^{\eta} d\lambda \ f_k(\lambda) \frac{x_i \tau_{i+k}}{\tau_{i+k}}
\]

\[
x_k(\gamma) = \mathcal{L}^{-3} \int_0^{\eta} d\lambda \ f_k(\lambda) \lambda \frac{x_i \tau_{i+k}}{\tau_{i+k}}
\]

\[
\Phi_k(\gamma) = \mathcal{L}^{-3} \int_0^{\eta} d\lambda \ f_k(\lambda) \left( \frac{1}{2} \tau_{i+1} + \tau_{i+2} + \cdots + \tau_{i+k-1} + \frac{1}{2} \tau_{i+k} \right) \frac{x_i \tau_{i+k}}{\tau_{i+k}}
\]

The asterisk at the mean value signs indicates that, if necessary, the mean values should be considered as functions of the distance \( \lambda = \xi_{i+k} - \xi_i = \lambda_i + \lambda_{i+1} + \cdots + \lambda_{i+k} \).

It should always be kept in mind that all mean values introduced in these formulas are functions of the time.

Since the lower limit set to the value of \( \gamma \) is very small, it is possible to define a domain in which the formulas can be developed into series according to powers of \( \gamma \). The first few terms of these series can be written:

\[
\frac{v_1 v_2}{v} = v^2 - \frac{\eta}{\mathcal{L}} \frac{v^2}{2t^2} + \frac{\eta^2}{2\mathcal{L}t^2} \left( \frac{1}{2} - \frac{\xi_0}{\mathcal{L}t^2} \frac{x_i \tau_{i+1}}{\tau_{i+1}} \right)
\]

\[
\frac{\partial}{\partial \eta} \left( \frac{v_1^2 v_2}{v^2} \right) = -\frac{\tau}{6\mathcal{L}t^3} + \frac{\eta^2}{\mathcal{L}} \left\{ \frac{v^2}{t^3} - \frac{\xi_0}{2\mathcal{L}t^3} \frac{(x_i + \tau_{i+1}) \tau_{i+1}}{\tau_{i+1}} \right\}
\]
where \( f_0 \) is the limiting value of \( f_1(\lambda_1) \) for \( \lambda_1 \to 0 \), while the asterisk now means that, properly speaking, the mean values should be calculated for \( \lambda_1 = 0 \). (The fact that these formulas are not even functions of \( \gamma \) is connected with the circumstance that they are not valid for very small values of \( \gamma \); they cannot be continued analytically to 0 or through 0).

The expressions can be substituted into (31), from which the term multiplied by the viscosity \( \nu \) must be omitted. Comparison of the terms independent of \( \gamma \) on both sides brings us back to the ordinary dissipation equation; this can be considered as a first check on our results.

Comparison of the terms of the first degree in \( \gamma \) leads to:

\[
\frac{d}{dt} \left( \frac{2 \gamma^2}{\ell t^2} \right) = - \frac{2 \gamma^2}{\ell t^2} + \frac{f_0}{\ell t^2} \left( \frac{\gamma + \gamma t}{\ell} \right)^{\gamma - 1} + \frac{1}{t} \left( \frac{\gamma + \gamma t}{\ell} \right)^{\gamma - 1} \cdot \ldots \tag{43}
\]

This equation can also be obtained by means of a direct calculation. It will be evident that the first term on the right hand side is immediately connected with the presence of \( t^2 \) in the denominator on the left hand side.

To obtain the second term it is convenient to multiply by the constant length \( S = \lambda \ell \); we then have:

\[
S \frac{\gamma^2}{\ell} = \frac{\lambda \gamma^2}{\ell} = \Sigma \gamma^2.
\]

Now \( \Sigma \gamma^2 \) increases whenever two consecutive fronts merge, with an amount \( 2\gamma \). The frequency of this occurrence is given by the number of segments \( \lambda_i \) which become zero in unit time; this is given by

\[
-N f_0 (d \frac{\lambda_i}{dt}), \text{ where } d \frac{\lambda_i}{dt}, \text{ for } \lambda_i \text{ going to zero, has the value}
\]

\[
-\frac{1}{2} (\gamma + \gamma t + 1) / 2t \text{ (compare the expression for } d \frac{\lambda_i}{dt} \text{ given in section 53, p. 111).}
\]

Hence we obtain:

\[
\frac{d}{dt} \left( \Sigma \gamma_i^2 \right) = + N f_0 \cdot \frac{1}{t} \left( \frac{\gamma + \gamma t}{\ell} \right)^{\gamma - 1} \]

which, after division by \( S = \lambda \ell \) explains the second term of the equation. It is gratifying that this comes out so well, for it provides...
another check on the calculations leading to the expressions for \( v_1 v_2 \) and \( v_1^2 v_2 \) and on the rules for the merging together of vertical fronts.

67. An Objection Against the Applicability of the Similarity Hypothesis to the Solutions Considered Here. The expression obtained for the frequency of merging of vertical fronts can be applied to obtain some further results. The decrease of the average number of segments contained in a large length \( S \) and the corresponding increase of the average length \( \bar{\ell} \) of the segments are given by:

\[
\frac{1}{N} \frac{dN}{dt} = \frac{d\ell}{dt} = f_o \frac{\tau_i + \tau_{i+1}}{2t} \tag{44}
\]

It is also possible to check the expression for \( \frac{d\tau}{dt} \); to check the invariance of \( J \); and to find the value of \( d\varepsilon/\varepsilon \).

More important is the following formula, deduced from (43) and (44):

\[
\frac{d}{dt} \left( \frac{\tau^2}{\ell^2} \right) = \frac{f_o}{\ell^2} \left\{ \frac{\tau_i + \tau_{i+1}}{\tau_i \tau_{i+1}} \right\} - \frac{1}{2} \left( \frac{\tau_i + \tau_{i+1}}{\tau_i} \right)^2
\]

The quantity between brackets on the left hand side is non-dimensional; its value has been denoted by \( 1 + \omega \). It should be a constant if the arrangement would retain statistical similarity. However, it is difficult to see how the right hand member of the equation can become zero unless the \( \tau_i \) have to satisfy rather improbable conditions. If we suppose that the mean values to be used here are not different from ordinary mean values without the asterisk (i.e., for \( \lambda_i \) different from zero) and if further we suppose that consecutive \( \tau_i \) are approximately independent, we find:

\[
\frac{d}{dt} (1 + \omega) = f_o \frac{\ell}{t} (1 + \omega) \tag{46}
\]

This would point to a continuous increase with time of \( 1 + \omega \), which is completely different from any approach to an arrangement which will
remain statistically similar to itself.

The result is due to the circumstance that the equation for \( d\lambda_i/dt \), obtained in section 58, gives a big chance for shrinking to zero to those segments \( \lambda_i \), for which either \( \tau_i \) or \( \tau_{i-1} \) is large. On the whole, therefore, wherever we find large \( \tau_i \), there is a good chance that these will quickly combine with other \( \tau \)'s to form larger and larger \( \tau \)'s. Since small \( \tau_i \) will not combine at the same rate, there is tendency towards a greater spreading of values. Of course, such a qualitative argument is not sufficient to settle the problem, since there is a complicated interplay between the \( \tau_i \) and the \( \lambda_i \), and it is also possible that the value of \( f_0 \) may decrease. Attempts have been made to formulate the problem in terms of a distribution function and a Boltzmann equation, but these attempts so far have not been successful.

It may be of interest to mention that the laws of motion of the vertical fronts of these solutions can be illustrated with the aid of a molecular analogue, in which it is assumed that molecules move along a line of infinite extent and that they combine whenever there is a collision. We denote the coordinates of the molecules by \( \xi_i \);

the velocities by \( d\xi_i/dt = d\zeta_i/dt = \zeta_i/t \);

the masses by \( \tau_i \);

and the momenta by \( \tau_i \zeta_i/t \).

At every collision masses and momenta are added; kinetic energy is lost. The laws of motion for the molecules then conform to those of the fronts. The number of molecules in every finite domain of the line on which they are moving, decreases continually; however, when the domain is infinite, statistical considerations still can be applied, and it would be possible to ask whether there might be an approach to a definite pattern of mass distribution.

For such a molecular system our result would be that no definite pattern would be approached, but that on the contrary also there would be found a tendency towards increased spreading.

68. Correction of Formula (39) to Make it Valid for Very Small Values of \( \eta \). In the deduction of form. (37) the fronts had been treated as if the vertical slope did not occupy a finite length of the y-axis, but was of infinite steepness. The same supposition was used in the deduction
of form. (39) and (40), to which was referred. More accurate expressions can be obtained when use is made of form. (8) for the course of $v$ at a steep front, or of its equivalent, form. (21), p. 109. The integral of $v_1 v_2$ (in which $\gamma$ is supposed to be of order $v t/\tau_1$) is calculated with the aid of this expression over an interval of the $y$-axis extending to both sides of the front at $\xi$, and having a length of normal order of magnitude but not including any neighboring front. The difference of the result obtained with a finite (though small) value of $v$ and the limit for $v = 0$ gives the correction connected with the particular front considered. This correction appears to be:

$$\frac{\gamma \tau_1^2}{2 t^2} \left( 1 - \text{ctnh} \frac{\tau_1 \gamma}{4 \nu t} \right).$$

The correction to be applied to (39) is the mean value of this expression:

$$\frac{\gamma}{2 l^2} \left( \frac{\tau_1^2}{\tau_1} - \frac{\tau_1^2 \text{ctnh} \frac{\tau_1 \gamma}{4 \nu t}}{\nu} \right).$$

For values of $|\gamma|$ small in comparison with $4 \nu t/l$ we may even develop the hyperbolic cotangent function, which gives:

$$\frac{\gamma}{2 l^2} \left( \frac{\tau_1^2}{\tau_1} - \frac{4 \nu t \tau_1}{\gamma} \right) - \frac{\gamma}{12 \nu t} \left( \tau_1^3 \ldots \right) =$$

$$= - \frac{2 \nu}{t} + \frac{\eta l}{2 t^2} (1 + \omega) - \frac{\eta^2 l^2}{24 \nu t^3} (1 + \omega^*) \ldots \ldots .$$

In this way it is seen that there is also a small correction to the value of $\bar{v}^2$ given in (25). The expression obtained for $\bar{v}_1 \bar{v}_2$, in the domain $|\eta| \ll 4 \nu t/l$, is:

$$\bar{v}_1 \bar{v}_2 = \bar{v}^2 \text{uncorrected} - \frac{2 \nu}{t} - \frac{\eta^2 l^2}{24 \nu t^3} (1 + \omega^*) \ldots \ldots (45)$$

The term with $\gamma^2$ in (39) and the terms depending on the functions $\psi_k$ and $X_k$ have been omitted, since, for the values of $\gamma$ considered here, they are insignificant in comparison with the term appearing
in (45). It will be seen that the linear term has disappeared.

We can now apply Eq. (34) and obtain:

\[ \varepsilon = \nu \left( \frac{\partial v}{\partial y} \right)^2 \quad = \quad \frac{L^2}{12t^8} \left( 1 + \omega^* \right) \]

in conformity with (26).

69. Kolmogoroff's Similarity Theory for the Correlation Function.

Kolmogoroff has enunciated the hypothesis that for distances \( r \) small compared with a length, which is equivalent to our \( \mathcal{L} \), the correlation function should be determined only by the magnitude of the energy dissipation \( \varepsilon \) and the kinematic viscosity \( \nu \). The quantity \( \mathcal{L} \), which has played a prominent part in the formulas referring to our solutions, for such values of \( r \) should be unimportant.

Kolmogoroff defines the correlation by means of the quantities:

\[ \frac{(v_1 - v_2)^2}{(v_1 - v_2)^3} \]

where again \( v_1, v_2 \) represent the values of \( v \) in two points, \( y \) and \( y + \eta \) respectively. The mean value is taken with respect to \( y \) for a fixed value of \( \eta \). These quantities are connected with those used in the preceding sections by means of the equations:

\[ \frac{(v_1 - v_2)^2}{(v_1 - v_2)^3} = 2 \frac{(v_2 - v_1 v_2)}{v_1^2 v_2} \]

Application of dimensional reasoning on the hypothesis that no other physical quantities are relevant than \( \varepsilon \) and \( \nu \), leads to the formulas:

\[ (v_1 - v_2)^2 = \nu^{1/2} \varepsilon^{1/2} F_1(\eta \varepsilon^{1/4} \nu^{-3/4}) \]

\[ (v_1 - v_2)^3 = \nu^{3/4} \varepsilon^{3/4} F_2(\eta \varepsilon^{1/4} \nu^{-3/4}) \]

where \( F_1 \) and \( F_2 \) are unknown functions.
Kolmogoroff further supposes that if the Reynolds number for the field \( \text{Re} = \frac{L^2}{\nu \tau} \) is sufficiently high, there should be a domain in which \( \gamma \gg \nu^{-3/4} \) is a large number and in which these expressions would take forms independent of the viscosity. This prescribes a certain condition for the functions \( F_1 \) and \( F_2 \) and leads to the results:

\[
\frac{(v_1 - v_2)^2}{(v_1 - v_2)^3} = c_1 (\epsilon \gamma)^{2/3}
\]

\[
\frac{(v_1 - v_2)^3}{(v_1 - v_2)^2} = c_2 \epsilon \gamma,
\]

\( c_1 \) and \( c_2 \) being constants.

From form (40), taken in connection with the expression for \( \epsilon \) given in (26), it will be seen that the second result is quite acceptable; the value of \( c_2 \) appears to be \(-12\).

The first result, however, is so much different from what we obtained in (39), that a judgment concerning its value is difficult. It may be a useful approximation over a certain domain, but sufficient evidence for this cannot be obtained. Nevertheless it is of importance to observe that Kolmogoroff's expression for \((v_1 - v_2)^2\) definitely increases less rapidly with \( \gamma \) than the result which is obtained on the assumption that the \( \tau_1 \) and \( \lambda_1 \) should be completely independent of each other. (Compare the considerations developed in section 65.) Hence in order that Kolmogoroff's formula may be valid, it is necessary that there exist certain relations between consecutive \( \tau_1 \) and \( \lambda_1 \), of such nature that the condition (38a) shall be satisfied. This seems to be an important point, which is not restricted to the solutions considered here, but has its bearing on the hydrodynamical problem.

It is perhaps possible to give the following turn to Kolmogoroff's reasoning: Consider a great number of cases of our type of solution, starting from various initial stages, so that there will be a great variety of values of \( \ell \) and of "ages" \( t \). Now take together all those cases for which \( \epsilon \) has the same value. This will require a certain relation between the value of \( \ell \) and the age; if we should return for a moment to the supposition of statistical similarity, proportionality between \( \ell^2 \) and \( t^3 \) would be required, so that "age" \( \sim \ell^{2/3} \).
We then obtain: \( \overline{v^2} \sim \frac{\sigma^2}{t^2} \sim \frac{\eta}{l^{2/3}} \).

If we now assume that for values of \( \eta \), small compared with \( l \) and large compared with \( vt/l \), the curve for \( (v_1 - v_2)^2 \) should be the same for all cases having the same \( \varepsilon \), it would be necessary that in the range specified we should have \( (v_1 - v_2)^2 \sim \eta^{2/3} \). The reasoning looks rather artificial.

It is to be observed that with Kolmogoroff's hypotheses the value of \( v^2 \) is not independent of \( v \), but is given by:
\[ \text{constant} \cdot \frac{\sigma}{l^{1/2}} \varepsilon^{1/2}. \]
Energy Transfer Through the Spectrum

70. Spectral Analysis of the Solutions. The general idea of the application of Fourier analysis to functions of a single variable $y$ and the connection with the correlation function has been considered in Chapter III, sections 16 and 17. The results given there can be applied to the present case. According to form (4) of that section, we write:

$$\bar{v}_1 v_2 = \int_0^\infty \Gamma(k) \cos k \eta \, dk \quad (46)$$

For $\eta = 0$ this gives the energy spectrum:

$$E = \frac{1}{2} \bar{v}^2 = \frac{1}{2} \int_0^\infty \Gamma(k) \, dk$$

Making use of form (6) of section 17 we have:

$$\Gamma(k) = \frac{2}{\pi} \int_0^\infty \bar{v}_1 v_2 \cos k \eta \, dk \quad (47)$$

Hence we see that

$$\Gamma(0) = \frac{2}{\pi} J$$

and if we develop the cosine-function we obtain:

$$\Gamma(k) = \frac{2}{\pi} J - \frac{k^2}{2} \Gamma_2 + \cdots \quad (47a)$$

where:

$$\Gamma_2 = \frac{2}{\pi} \int_0^\infty \bar{v}_1 v_2 \cdot \eta^2 \, d\eta$$

It does not seem easy to obtain expressions for the function $\Gamma(k)$ or quantities like $\Gamma_2$ connecting them with the statistical characteristics of the "saw tooth" profile for $v$. However, when $k$ is large compared with $1/\ell$ and still small in comparison with $\ell/\nu t$, we can obtain the approximation:

$$\Gamma(k) = \frac{2}{\pi k^2 t^2 \ell} \quad (48)$$
where the error is of the order $k^{-3}$ or $k^{-4}$. One would be tempted to combine (47a) and (48) into an expression of the type:

$$\Gamma \sim \frac{A}{1 + B k^2 \mathcal{L}^2} ,$$

but the data available do not permit to prove this expression.

If $k$ becomes of the order $\mathcal{L}/\nu t$ it appears that (48) must be replaced by:

$$\Gamma (k) \sim \frac{4 \pi \nu^2}{\mathcal{L}} \left( \frac{\sinh \frac{2 \pi k \nu t}{\tau_1}}{\tau_1} \right)^{-2} , \quad (48a)$$

although further mathematical investigations are necessary. From (48a) it would follow that at the end of the spectrum $\Gamma (k)$ decreases exponentially with $k$. Existing theories concerning the course of $\Gamma (k)$ mostly give a decrease according to some inverse power law. On general mathematical grounds it is likely, however, that ultimately an exponential decrease must be obtained, since otherwise the course of $\nu$ - and in the hydrodynamical case: the course of the flow - in dimensions of the order $\nu t / \mathcal{L}$ could not be smooth. (This point is stressed by J. von Neumann in a report on "Recent Theories of Turbulence").

From Kolmogoroff's similarity hypotheses it has been deduced that in a certain domain of $k$-values one should have:

$$\Gamma (k) \sim \varepsilon^{2/3} k^{-5/3} .$$

This of course differs notably from (48). The two results can, perhaps, be brought into a relation if we use the artifice mentioned in the preceding section. We consider a set of cases for which $\varepsilon$ has the same value. As observed, for the fields selected in this way we must then have "age" $(t) \sim \mathcal{L}^{2/3}$. From (48) we deduce:

$$\Gamma \sim \frac{\mathcal{L}}{k^2 \nu^2} \sim \frac{1}{k^2 \mathcal{L}^{1/3}} ,$$

If we assume that, within the domain of $k$-values mentioned above, (I) $\Gamma (k)$ is a function of $k \mathcal{L}$, and (II) that for all fields with the same $\varepsilon$ the curves for $\Gamma (k)$ should coincide, it would be necessary that $\Gamma$
should be proportional to $k^{-5/3}$. The reasoning, however, is just as artificial as the one given before.

A result obtained by Heisenberg for hydrodynamical turbulence, viz., that $\Gamma$ must become proportional to $k^d$ when $k$ decreases to zero, has as its analogue in our solutions that $\Gamma$ must approach a constant value for $k \to 0$. This is in accordance with what has been found above.

71. Energy Transfer Through the Spectrum. We introduce the Fourier transform of the correlation function $v_1^2 v_2$:

$$\widetilde{v_1^2 v_2} = \int_0^\infty \psi(k) \sin k \eta \, dk$$

with:

$$\psi(k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} v_1^2 v_2 \sin k \eta \, d\eta.$$

It is then possible to transform von Karman and Howarth’s equation (31) into:

$$\frac{\partial \Gamma}{\partial t} = k \psi - 2 \nu k^2 \Gamma$$

(49)

For $k = 0$ this gives $\nabla \Gamma(0) / \partial t = 0$, which is nothing else than the constancy of Loitsiansky’s invariant.

The equation for the energy transfer through the spectrum is obtained from (49) by integration with respect to $k$:

$$\frac{d}{dt} \int_0^k \Gamma \, dk = \int_0^k k \psi \, dk - 2 \nu \int_0^k k^2 \Gamma \, dk.$$  (50)

The term on the left hand side gives the change with time of (twice the) energy in the part of the spectrum which extends from 0 to $k$. The last term on the right hand side determines the amount that is lost by viscous dissipation. The other integral, when taken with the sign reversed, gives the energy that is transmitted from the part 0 .. $k$ to the part beyond $k$. 
Since we have

\[
\frac{\partial^2}{\partial \gamma^2} \frac{v_1^2}{v_2} = \int_0^\infty k \psi \cos k \gamma \, dk,
\]

and since the derivative of \( v_1^2/v_2 \) with respect to \( \gamma \) is zero for \( \gamma = 0 \), it follows that:

\[
\int_0^\infty k \psi \, dk = 0.
\]

Hence the term with \( \psi \) does not bring over-all gain or loss of energy; it represents pure interchange.

To find \( \psi \) we follow a similar method as was used in section 17 and calculate

\[
\frac{v_1^2}{v_2} = \frac{1}{2M} \int_{-\infty}^{+M} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(k_1) \varphi(k_2) \varphi(k_3) e^{i(k_1 + k_2 + k_3) \gamma + i\kappa \gamma} \, dk_1 \, dk_2 \, dk_3.
\]

When \( M \) is made infinite this gives:

\[
\frac{v_1^2}{v_2} = \frac{\pi}{M} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(k_1) \varphi(k_2) \varphi(-k_1 - k_2) e^{-i(k_1 + k_2) \gamma} \, dk_1 \, dk_2.
\]

In the double integral we shall write: \( k_1 + k_2 = k \); \( k_1 = k' \); it then transforms into:

\[
\frac{v_1^2}{v_2} = \frac{\pi}{M} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(k') \varphi(k - k') \varphi(-k) e^{-ik \gamma}.
\]
On the other hand, we bring (49) in the form:

\[
\frac{v_1^2}{v_2} = -\frac{1}{2i} \int_{-\infty}^{+\infty} \psi(k) e^{-ik\gamma} dk,
\]

assuming \( \psi(-k) = -\psi(k) \). Comparison now leads to the following expression for \( \psi \):

\[
\psi(k) = -\frac{2\pi i}{M} \int_{-\infty}^{+\infty} dk' \varphi(k') \varphi(k-k') \varphi(-k).
\]

72. It will be seen that the value of \( \psi \) depends on an integral of a product of three factors \( \varphi \) the arguments of which have an algebraic sum equal to zero. Every frequency \( k \) comes into connection with all other frequencies, both smaller and larger. From the expression for \( \psi \) alone it is not to be seen whether energy transport will be mainly from small \( k \) to large \( k \), or inversely, or both ways simultaneously.

It was observed in section 17 that the absolute value of \( \varphi \) is proportional to \( M^{-1/2} \). In order that \( \psi \) shall be independent of \( M \) (as it should be) it is therefore necessary that the outcome of the integration brings a factor \( M^{-1/2} \), since there is only the factor \( M^{-1} \) before the integral. Looking at the dimensions, each function \( \varphi \) has the dimensions (velocity \cdot length), while the function \( \psi \) must have the dimensions (velocity)\(^3\) \cdot (length). Hence an extra factor \( M^{-1/2} \) must be accompanied by a factor of dimension (length)\(^{1/2} \). The most probable supposition is that the full factor should be \( (k.M)^{-1/2} \).

How could the appearance of such a factor be explained? The explanation can be sought in the effect of phase relations. The circumstance that the algebraic sum of the arguments of the three functions \( \varphi \) is equal to zero, might in itself not guarantee a stationary phase for the product. The phase factor of the product could rapidly change with \( k' \), so that the contribution to the integral for most values of \( k' \) would be negligible. A stationary phase, however, might appear with certain simple rational relations between the three frequencies involved, for instance:
We may suppose that the "resonance breadth" on the k-scale would be proportional to \( k \cdot (kM)^{-1/2} \) (with different numerical factors for the various cases). The result of the integration would then be of the form:

\[
\psi = \frac{k^{3/2}}{M^{3/2}} \sum a_m \left| \varphi_1 \right| \left| \varphi_2 \right| \left| \varphi_3 \right|
\]

where the arguments of \( \varphi_1, \varphi_2, \varphi_3 \) would be in some simple rational relation, while the \( a_m \) would be numerical constants. Expressing the absolute values of the \( \varphi \) by means of the \( \Gamma \) according to Eq. (5) of section 17 and considering \( k \psi \), which is the actual term occurring in Eq. (50), we would find:

\[
k \psi = \left( \frac{k}{2\pi} \right)^{3/2} \sum a_m \left( \Gamma_1 \Gamma_2 \Gamma_3 \right)^{1/2}
\]

The most important term might be:

\[
\left( \frac{k}{2\pi} \right)^{3/2} a_1 \Gamma \left( \frac{k}{2} \right) \sqrt{\Gamma(k)}
\]

This reasoning is of course highly speculative, but the supposition that transfer of energy would mainly take place when the three frequencies involved make some "harmonic chord," would have the consequence that the number of steps involved in transporting energy over a certain region of the spectrum would be roughly proportional to the logarithm of the ratio of the frequencies limiting this region. Such a supposition has been made by L. Onsager in a note on turbulence, where he speaks of the transfer of energy as a "cascade process."
Solutions Representing Turbulence Coupled with a Primary Motion

73. The solutions of Eq. (1) considered thus far were subjected to decay. In order to be able to form solutions deriving energy from an outside source, in such a way that a stationary state of turbulence is maintained of a type analogous to that obtained in pipe flow, we replace Eq. (1) by an extended equation:

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2} + \frac{uv}{b}.
\]

In the term added on the right hand side \(U\) stands for a given quantity, an analogue of the average velocity of flow over the cross section in the case of pipe flow. The term has been given the form of a product of \(U\) and \(v\), with a factor \(b\) in the denominator to give it the proper physical dimensions. In this way Eq. (51) can be satisfied by taking \(v = 0\) and turbulence can derive energy from the main motion only when it is already in existence. The same is true of hydrodynamic turbulence, although in that case the coupling with the main motion depends on terms containing derivatives. (Compare Eq. (9) of Chapter VI, p. 72.)

We limit the domain of \(y\) to \(0 < y < b\) and subject \(v\) to the condition:

\[v = 0 \text{ at } y = 0 \text{ and } y = b.\]

It is possible to consider \(U\) itself as a variable quantity, depending on the time. We then subject \(U\) to the equation:

\[
\frac{dU}{dt} = \frac{P}{b} - \frac{\nu U}{b^2} - \frac{1}{b^2} \int_0^b v^2 dy.
\]

Now the parameter \(P\) is assumed to be a given quantity; it can be taken as the analogue of the pressure in pipe flow. The equation expresses that a motion with the velocity \(U\) experiences both "laminar" friction, given by the term \(\nu U/b^2\), and a resistance derived from the turbulence, given by the integral. Since we had assumed that \(U\) does not depend on \(y\), it has been inevitable to choose a rather artificial form for the resistance terms; nevertheless they serve their purpose and they allow to form an equation of energy for the "main motion" \(U\) and the "turbulence"
v together, as follows:

\[ \frac{d}{dt} \left[ \frac{1}{2} b U^2 + \frac{1}{2} \int_0^b \nu^2 \, dv \right] = PU - \frac{\nu U^2}{b} - \nu \int_0^b \left( \frac{\partial v}{\partial y} \right)^2 \, dy. \] (53)

The quantity between \( \left[ \right] \) on the left hand side represents the kinetic energy in the field. On the right hand side we have the energy derived from the pressure acting on the main motion, and losses due to the laminar friction of the main motion and to the dissipation in the turbulence. The terms of the second degree in Eqs. (51) and (52) have disappeared; they represented a coupling without loss or gain.

74. Stationary Solutions of the System (51) - (52). The system admits two types of stationary solutions. In one type \( v = 0 \); we shall say that they represent "laminar" flow without turbulence. In the other type \( v \) is different from zero, and although it is stretching the meaning of the word "turbulence", we shall use it for any solution in which \( v \) is not permanently and everywhere zero.

The "laminar" solution is given by:

\[ U = Pb/v ; \quad v = 0 \] (54)

It is seen that \( U \) is proportional to \( P \) in this solution.

The stability of the solution can be investigated by means of the method of small disturbances. The necessary formulas will come automatically in section 78, below; we mention that it is found that the laminar solution is stable so long as \( U < \pi^2 v/b \). When \( U \) exceeds this value, the solution becomes unstable and gives way to one in which \( v \) is different from zero. It can be observed that the dimensionless parameter \( U b/v \) plays the part of a Reynolds number for the system (51) - (52), with the critical value \( \pi^2 \).

To obtain stationary "turbulent" solutions, we start from (51) without the time derivative:

\[ \nu \frac{d^2 v}{dy^2} - \nu \frac{dv}{dy} + \frac{U v}{b} = 0, \] (55)

where \( U \) is a constant. We introduce an auxiliary variable

\[ \eta = - \frac{b}{U} \frac{dv}{dy} \]
(the \( \gamma \) used here has nothing to do with the \( \gamma \) occurring in correlation functions.) The following equation is obtained for \( \gamma \):

\[
\frac{vU}{b} \frac{d\gamma}{dv} + \nu(1 + \gamma) = 0.
\]

This equation can be solved. The complete solution of (55) can be expressed in parametric form, as follows:

\[
v = \sqrt{\frac{2vU}{b}} \ln \left( \frac{C - \gamma + \ln (1 + \gamma)}{1 + \frac{\gamma}{C - \gamma + \ln (1 + \gamma)}} \right);
\]

with \( C \) as the integration constant. The variable \( \gamma \) must be situated between two limits \( \gamma_1, \gamma_2 \), given by the roots of

\[
C - \gamma + \ln (1 + \gamma) = 0.
\]

When \( C \) is large, these roots are approximately equal to:

\[
\gamma_1 \approx -1 + e^{-C} - \frac{1}{C}, \quad \gamma_2 \approx C + \ln(C + 1).
\]

We take \( \gamma_1 \) as the lower limit of the integral; since \( \gamma = \gamma_1 \) gives \( v = 0 \), this can be made to correspond to \( y = 0 \). As \( v \) must be zero also at \( y = b \), a condition must be satisfied which fixes the value of \( C \).

Since there may be intermediate zeros the condition takes the form:

\[
\sqrt{\frac{vb}{2U}} \int_{\gamma_1}^{\gamma_2} \frac{d\gamma}{1 + \gamma} \frac{1}{\sqrt{C - \gamma + \ln (1 + \gamma)}} = \frac{b}{m},
\]

in which \( m \) must be a positive integer. The simplest solution is obtained with \( m = 1 \); see the accompanying diagram, curve \( a \), for the case of a large value of \( C \). A solution with \( m = 2 \) is represented by curve \( b \).

The value of the definite integral decreases when \( C \) decreases. Its minimum value, obtained for \( C \to 0 \), is \( \pi \sqrt{2} \). Hence there is a maximum for \( m \), determined by the largest integer contained in \( (Ub/n^2)^{1/2} = (Re/n^2)^{1/2} \).
There are other series of solutions, for instance the series obtained from the one just considered by means of the formula:

\[ v(y) = -v(b - y) + U \]

(See curve \( a \) for \( m \ = \) 1).

75. **Approximate Formulas for the Stationary Solutions.** We assume \( U_b/v \) to be large. In this case the solutions with small \( m \) (for which \( C \) is large) can be represented by means of approximate expressions. A not rigorous but convenient method to obtain these approximations is to divide the domain \( 0 \leq y \leq b \) into regions where the term \( v(d^2v/dy^2) \) can be neglected and where we find:

\[ v \approx \text{constant} + U y/b \]

and regions where the term \( Uv/b \) is relatively unimportant in comparison with the other two terms and where we find:

\[ v \approx -\text{Atanh} \left(A(y - B)/2v\right) \]

\( A \) and \( B \) being constants. The constants must be adjusted in such a way that the boundary conditions at \( y = 0 \) and \( y = b \) are satisfied and that no discontinuities are obtained in the course of \( y \). We mention the following examples, (the expressions combine the two partial solutions):*

---


(Continued next page)
The regions where the course of \( v \) is determined by the \( \tanh \)-function present a steep gradient and are the only regions which materially contribute to the dissipation. In the case of the first one of the solutions given above we find:

\[
\mathcal{V} = \sqrt{\left( \frac{\partial v}{\partial y} \right)^2} = \frac{U^3}{3}.
\]

When this result is substituted into the energy equation (53), in which the left hand member is zero for a stationary solution, while in the right hand member we can neglect \( vU^2/\beta \), we obtain:

\[
P = \frac{U^2}{3}.
\]

The same expression can be deduced from Eq. (52), in which likewise the left hand member is zero, while the term \( vU/\beta \) can be neglected. We thus arrive at the result that with stationary turbulence the relation between \( U \) and the pressure leads to a quadratic resistance law.

76. Non-stationary Solutions. If \( U \) is considered as a constant, non-stationary solutions of Eq. (51) can be obtained without great difficulty. In order to obtain an approximate picture, we can follow a similar way as in the preceding section. We divide the domain for \( y \) into regions where we can neglect the viscosity and simplify Eq. (51) to:

In that paper the parameter \( \beta \) had been replaced by unity, while on the other hand a factor 2 had been inserted before the term \( v(\partial v/\partial y) \), which causes certain numerical differences in the expressions. Part of this paper has been taken over, with some new material, in "A Mathematical Model Illustrating the Theory of Turbulence," Advances in Appl. Mechanics Vol. 1, pp. 171 - 199, (1966). In the latter paper (which does not reproduce the deduction of the approximations), the parameter \( \beta \) had been used and the factor 2 before \( v(\partial v/\partial y) \) was retained. The approximations are straightforward and can easily be reconstructed.
and other regions where the approximation

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} = \frac{v^2}{\partial y^2} \]

will be appropriate. The latter equation is identical with Eq. (1) and the method to be followed is the one indicated in section 52 (pp. 95/96), with the result given already in Eq. (8) of that section.

Equation (58) can be solved by means of its characteristics. We can, however, also start in a similar way as was followed in section 50 and look for a solution of the form \((2)\): 

\[ v = \beta (y - \sigma) \]

We then obtain:

\[ \text{either } v = \frac{U}{b} (y - \sigma) \frac{\epsilon}{1 + \epsilon}; \text{ or } v = -\frac{U}{b} (y - \sigma) \frac{\epsilon}{1 - \epsilon}, \]

where \( \epsilon = e^{U(t-t_0)/b} \), \( t_0 \) being an integration constant. The first form is appropriate when the initial slope \( \beta \) is positive; it is seen that for \( t \to \infty \) the slope ultimately approaches to \( U/b \). The second form must be used when the initial slope is negative; the adjustment of \( t_0 \) will then require that initially \( \epsilon \) is smaller than unity. With increase of \( t \) there comes an instant when the slope threatens to become infinite; we must then correct the result by making use of the appropriate form of Eq. (8) mentioned before.

Attention must be given to the conditions at the limits \( y = 0 \) and \( y = b \). We shall not go into details and mention only that as far as can be seen by means of the approximate method, solutions starting from arbitrary initial conditions approach asymptotically to one of the stationary solutions found before.

77. Application of Fourier Analysis. Having regard to the boundary conditions we can develop \( v \) into a Fourier series:

\[ v = -\Sigma \xi_n \sin \frac{muy}{b} \]

where, in general, the coefficients or amplitudes \( \xi_n \) will be functions of the time. The minus sign before the sum is of no importance in itself,
but has been introduced to obtain positive values for the amplitudes in
the case of the solution $c)$ of section 75. The stationary solutions con-
sidered in that section have Fourier series with amplitudes which are
independent of the time. The amplitudes can be calculated with suf-
ficient accuracy from the approximate formulas given in that section; for
the three cases $c), a), b)$ the following results are obtained:

$$
\xi_n = \frac{2\pi \nu}{b \sinh (\pi^2 \nu/Ub)} \quad (\text{for } c) \\
\xi_n = \frac{(-1)^n 2\pi \nu}{b \sinh (\pi^2 \nu/Ub)} \quad (\text{for } a) \\
\xi_n = \begin{cases} 
\frac{4\pi \nu}{b \sinh (2\pi^2 \nu/Ub)} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}
\end{cases} \quad (\text{for } b)
$$

It will be seen that these amplitudes ultimately decrease as ex-
ponential functions of the index $n$. Taking the first case we have:

$$
\xi_n \sim \frac{4\pi \nu}{b} e^{-\pi^2 \nu/Ub} \quad \text{for large } n . \quad (60a)
$$

In the case $a)$ the amplitudes have alternating signs; in case $b)$ the
amplitudes of odd order are zero, but the general principle of expo-
nential decrease is retained. It will be remembered that an exponential
decline was also surmised in the case of the Fourier development of de-
caying homogeneous turbulence, considered in section 70.

It is possible to translate the partial differential equation (51)
into an infinite system of simultaneous ordinary differential equations
for the amplitudes $\xi_n$. For this purpose we substitute (59) into (51)
and apply a simple transformation to the terms obtained from $v(\nu/v)$.It is then possible to separate the sine functions of various order and
the following result is obtained:

$$
\frac{d\xi_n}{dt} = \left(\frac{U}{b} - \frac{n^2 \nu^2}{b^2}\right)\xi_n + \frac{m\nu}{2b} \left(\frac{1}{2} \Sigma_{h=1}^{n-1} \xi_{n-h} - \Sigma_{h=1}^{\infty} \xi_{n+h} \xi_{n+h}\right) . \quad (61)
$$
In principle the development of any turbulent solution from a given initial state is determined just as well by the system of equations (61), as it is by the partial differential equation (51). No direct method for attacking the system (61) has been found thus far, but some important properties of it are easily recognizable.

78. In the first place we note that the system (61) is satisfied when all amplitudes $\xi_n$ are zero. This means that there is no "turbulence"; we have "laminar" flow and the auxiliary equation gives a linear relation between the value of $U$ and the pressure gradient $P$.

Looking at the terms on the right hand side of (61), we note that the first term is linear in $\xi_n$ and contains $U/b$ as factor. This term represents the coupling between the "mean motion" $\bar{U}$ and the turbulent amplitude $\xi_n$. If this term were the only one on the right hand side, an exponential increase of $\xi_n$ with time would result. The circumstance that all amplitudes $\xi_n$ have the same factor $U/b$ is accidental for the model; this does not represent a feature of general importance.

Next follows a term also linear in $\xi_n$, which represents the damping effect of viscosity. The factor $n^2$ occurring in this term exhibits a general feature which is always obtained with Fourier components. Since the sign of the term is negative, it causes an exponential damping which increases rapidly with increasing order.

If, on the supposition of small amplitudes, we provisionally neglect the non-linear terms in (61), the system reduces to a set of separate equations for the various amplitudes, without coupling between them. We immediately see that so long as $U < n^2 v/b$ (Reynolds number $Ub/v < n^2$) all components $\xi_n$, if stimulated, will be damped. When $U$ exceeds the critical value, one or more of these components, when stimulated, will increase exponentially, while the rest would show exponential decrease. This explains the transition from stability to instability of the laminar solution, to which reference had been made in section 74. We also see that as soon as the increasing components have obtained a certain amplitude, the non-linear terms can no longer be neglected and coupling between all components sets in. This coupling makes it possible that energy is detracted from the components which otherwise increase exponentially, so that their increase is reduced or ceases, while components...
which otherwise would disappear through viscous damping can gain energy and may remain in existence.

Two features of (61) now are of great importance. One has been mentioned already, viz., that the viscous damping has the factor $n^2$. The other feature is the presence of the sum $\sum_{n=1}^{\infty} \xi_n \xi_{n-h}$, which represents a "convolution" of the series of amplitudes $\xi_n$. If we ask for a term which can balance the viscous damping for very large $n$, that term must have the factor $n^2$. Now such a term can be obtained if we assume that the "convolution" $\sum \xi_n \xi_{n-h}$ becomes a linear function of $n$ for large $n$; since the convolution is already multiplied by a factor $n$, we then obtain the required term with $n^2$. It is easily seen that this result can be secured by supposing that for large $n$ the amplitudes $\xi_n$ depend exponentially on $n$. In that case the product $\xi_n \xi_{n-h}$ will become independent of $h$ and the convolution will contain a number of such terms, which number increases linearly with $n$.

To make this reasoning more precise, we assume that:

$$\xi_n = \beta e^{-\alpha n} \text{ for } n > N,$$

where $N$ is some definite number. We suppose that there are no terms in the series of the $\xi_n$ which drop out (as in the case of the solution b) - see section 77). We then write out Eq. (61) for a value of $n$ exceeding $2N$ (preferably much exceeding $2N$). In the two sums occurring in (61) we take apart the terms which contain amplitudes $\xi_n$ for which $h < N$ and amplitudes $\xi_{n-h}$ for which $n - h < N$. We then obtain:

$$\frac{d \xi_n}{dt} = \beta e^{-\alpha n} \left[ b - \frac{n^2 \pi^2}{b^2} + \frac{n^2 m^2}{b} - \frac{m}{2b} A \right]$$

(62)

where $A$ is short for

$$A = \beta (N+\frac{1}{2}) - \sum_{h=1}^{N} \xi_n (e^{-n\alpha} - e^{-n\alpha}) - \frac{\beta e^{-2(N+1)\alpha}}{1 - e^{-2\alpha}}$$

(62a)

It is not difficult to check that the expression for $A$ does not change, when the value of $N$ is increased by one or more units.
The result obtained shows that the effect of viscosity can be balanced if we take:

\[ \beta = \frac{\mu v}{b} \]  

(63)

which is in accordance with the formula mentioned before (60a).

This reasoning does not give us the value of \( \alpha \). If we look for a physical interpretation of \( \beta \), the only feature which suggests itself is that:

\[ \xi_n h \xi_{n-h} = \beta \xi_n, \]  

(64)

independently of \( n \) and \( h \), provided both \( h \) and \( n - h \) exceed \( N \). This is a non-homogeneous relation between amplitudes of a kind not found in linear problems.

If the condition (63) is satisfied, the term with \( n^2 \) disappears from the expression for \( \ddot{\xi}_n / dt \). There remains a term with \( n \). It is possible to get rid of this term as well, if we make \( \Delta \) vanish. This puts a condition on the amplitudes \( \xi_n h \) for \( h < N \). It will be seen that this condition can be satisfied to a large extent if we assume:

\[ \eta \xi_n h = \frac{\beta}{e^{h\alpha} - e^{-h\alpha}} \]  

(65)

This formula passes into (63) when \( h > N \), provided we can neglect \( e^{-2N\alpha} \) in comparison with unity. It is seen that this new result brings us to a hyperbolic sine function of the type as occurred in (60).

To make \( \Delta \) vanish rigorously, a more accurate calculation would be necessary. But we can now make a start from the other side, if we accept the expression (65) for small \( h \) and suppose that \( \alpha \) is very small. In that case we have the approximation

\[ \xi_n h = \frac{\beta}{2h\alpha} \]  

(65a)

We can now substitute this approximation into (61), neglecting the viscous damping for small \( n \). We have:
\[ \sum_{1}^{n-1} \xi_{h} \xi_{h-n} = \frac{\beta}{2na^2} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right) \]

\[ \sum_{1}^{\infty} \xi_{h} \xi_{n+h} = \frac{\beta}{2na^2} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} \right) \]

and (61) reduces to:

\[ \frac{d\xi_n}{dt} = \frac{U}{b} \frac{\beta}{2na} - \frac{n\pi}{2b} \left( \frac{\beta}{2na} \right)^2. \]

The two terms on the right hand side compensate each other if:

\[ \frac{\beta}{2a} = \frac{2U}{n\pi}, \]

giving:

\[ \xi_n = \frac{2U}{n\pi} \text{ for small } n, \quad (66) \]

which again is in accordance with (60).

These considerations are far from a rigorous treatment of the system (61), and the last point in particular seems to be intimately connected with the peculiar structure of the model system.

Nevertheless it would appear that the reasoning leading to the asymptotic exponential law for the amplitudes \( \xi_n \) represents a general feature, not limited to the special form of the model system. It ensures the possibility of balancing viscous dissipation for very large \( n \), that is, in the tail of the spectrum, by inflow of energy from part of the spectrum preceding the component considered, in which the head of the spectrum (where the exponential formula does not apply) is of small importance. It does not appear necessary that \( a \) should be independent of the time; there might be, for instance, a slow increase with time. The actual value of \( a \) probably will depend on the character of the head of the spectrum, or, in other words, on the way in which energy is fed into the system.
The following additional remarks are of interest. If the Fourier series \((57)\) is introduced into Eq. \((52)\), we obtain:

\[
\frac{dU}{dt} = \frac{P}{b} - \frac{\nu U}{b^2} - \frac{1}{2b} \sum_1^\infty \xi_n^2
\]  

(67)

When \(U\) is independent of the time, the left hand side is zero. It will be seen that the sum \(\sum_1^\infty \xi_n^2\) can be calculated with sufficient accuracy from the approximation \((66)\), giving \(\sum_1^\infty \xi_n^2 = 202/3\). Neglecting the term \(\nu U/b^2\), we find:

\[
P = \frac{2U^2}{3},
\]

in conformity with a result already mentioned. We can interpret this result by saying that the reaction of the turbulence on the mean motion is dependent on the amplitudes of the components at the head of the spectrum only.

The energy equation \((53)\) takes the form:

\[
\frac{d}{dt} \left[ \frac{1}{2} bU^2 + \frac{1}{4} b \sum_1^\infty \xi_n^2 \right] = \frac{\nu U^2}{b} - \frac{\nu^2}{b^2} \sum_1^\infty \frac{n^2}{2} \xi_n^2.
\]

(68)

In this equation the approximation \((66)\) is of no use, since it would make the sum \(\sum_1^\infty \xi_n^2\) divergent. Indeed, the approximation \((66)\) has the consequence that the dissipation has the same value for every component of the turbulence. The more accurate expression \((60)\) is needed to make the sum convergent — in other words, to set a limit to the part of the spectrum where there is equipartition of dissipation.

For certain purposes it is useful to multiply Eq. \((67)\) by \(b\xi_n^2/2\), which gives:

\[
\frac{b}{4} \frac{d}{dt} \xi_n^2 = \frac{1}{2} \left( \frac{U^2}{b} \right) \xi_n^2 + \frac{\nu n}{4} \xi_n \left( \frac{n-1}{2} \xi_n \xi_{n-h} \xi_{n+h} \right).
\]

The equation can now be considered as an analogue of Eq. \((1.9)\). If we take the sum with respect to \(n\), up to a certain value \(k\), we can write:

\[
\frac{b}{4} \frac{d}{dt} \left( \sum_1^k \xi_n^2 \right) = \frac{U}{2} \sum_1^k \xi_n^2 - \frac{\nu^2}{2b} \sum_1^k \frac{n^2}{2} \xi_n^2 - W_k
\]

(69)
where:

\[
W_k = \frac{\pi}{h} \sum_{n=1}^{\infty} \sum_{h=n}^{k} \left( \sum_{h=1}^{\infty} \xi_{n+h}^2 - \frac{1}{2} \sum_{h=1}^{n-1} \xi_{n-h}^2 \right).
\]

The sums appearing here can be rearranged, as follows:

\[
\sum_{n=1}^{k} \sum_{h=1}^{k-n} \xi_n^2 \xi_{n+h}^2 = \sum_{n=1}^{k} \sum_{h=1}^{k-n} \xi_n^2 \xi_{n-h}^2 - \sum_{n=1}^{k} \sum_{h=1}^{n} \xi_n^2 \xi_{n-h}^2.
\]

So that finally:

\[
W_k = \frac{\pi}{h} \sum_{n=1}^{k} \sum_{h=k+1}^{\infty} \xi_n^2 \xi_{n}^2.
\]

If we look at Eq. (69) and take the case represented by our first stationary solution, for which the left hand member of the equation naturally is zero, we can observe that the term

\[
\frac{1}{2} U \xi_n^2
\]

represents the energy which the turbulence derives from the mean motion \( U \).

Since with \( \xi_n = 2U/n \pi \) according to the approximation (66) the series is rapidly convergent, we can say that already with a moderate value of \( k \) this sum gives all energy which is fed into the system.

We can always assume that \( \nu \) is so small that for such a value of \( k \) the second sum on the right hand side of (69), which has the value \( kn^2 \nu (2U/n \pi)^2 / 2b \), is insignificant. Hence there must be a domain of values of \( k \) for which the value of \( W_k \) is practically independent of \( k \). (This was mentioned in the paper in Adv. in Appl. Mechanics, quoted in the footnote to section 75; see p. 180, Eq. (22d), where \( S_n \) has been used instead of \( W_k \)).
This result can be compared with an argument applied in Heisenberg's theory of the turbulent spectrum.

The result can be verified by calculating the value of \( W_k \) on the basis of the approximation (66). It is more convenient to calculate \( W_{k-1} - W_{k'} \) for which, after some simple transformations of series, one finds the value \( 1/(k+1)^2 \). This proves that \( W_k \) can be considered as a constant already for moderate values of \( k \), to the same degree of approximation to which the sum \( \sum_k \frac{1}{k^2} \) can be treated as a constant.

81. The question arises if a similar exponential law might be found for the amplitude function \( \varphi(k) \) occurring in the Fourier integral considered in connection with the solutions representing homogeneous decaying turbulence over an infinite extent of the \( y \)-axis, not influenced by exterior forces or by coupling with a mean motion.

We observe that the introduction of the Fourier integral as given by form. (1) of section 16 (p. 25) makes it possible to transform Eq. (1) into:

\[
\frac{\partial \varphi}{\partial t} = -k^2 \varphi - \frac{ik}{2} \int_{-\infty}^{+\infty} dk' \varphi(k') \varphi(k-k') .
\]

In comparing this equation with Eq. (61) the following circumstances require attention:

(a) the function \( \varphi(k) \) contains a phase factor which was not determined by the relation between \( \varphi(k) \) and \( \Gamma(k) \) and which may behave in some complicated or irregular manner;

(b) the absolute magnitude of \( \varphi(k) \) must be proportional to \( \tilde{M}^{1/2} \), where \( 2\tilde{M} \) is the length of the domain to which the Fourier integral applies;

(c) the dimensions of \( \varphi(k) \) are \( \text{(length) \cdot (velocity)} \), whereas the dimensions of the amplitudes \( \varphi_n \) were those of a velocity.

It does not look easy to develop a formula for \( \varphi(k) \) which takes account of all these facts and gives a basis for the calculation of the integral occurring in (71). The only hypothesis which presents itself is the (very tentative) assumption that for \( k_1 \) and \( k_2 \) both exceeding a certain limit \( K \) we should have:
\[ \varphi(k_1) \cdot \varphi(k_2) = 2iv \varphi(k_1 + k_2) + \text{some irregularly fluctuating part}, \] (72)

where it is supposed that the "irregularly fluctuating" part does not give a contribution to the integral, increasing linearly with \( k \).

It is interesting to observe that there is a formal analogy between (72) and the relation satisfied by an exponential correlation function, as considered at the end of the Appendix to Chapter II. In both cases the only fact which matters is that an exponential relation exists, while the coefficient entering into the exponent is irrelevant. This idea would suggest that the amplitude function \( \varphi(k) \) would have in it something of the nature of a correlation function, which does not look unacceptable.

82. It is of interest to look back over the results obtained with the mathematical model.

It was possible in the first place to obtain a solution representing "stimulated turbulence", produced by the action of a time dependent force at a particular spot of the field (the origin of the \( y \)-axis). The force was supposed to be a random function of the time; we had taken a system of irregularly distributed positive impulses as the simplest case, but more general cases can be studied as well. It was found that the motion produced propagated itself in the direction of the \( y \)-axis, which made it possible to consider these solutions as a rough analogue of grid-produced windtunnel turbulence. The propagation was governed by such laws that every part of the curve where \( \partial^2 v/\partial y^2 \) is negative, developed into a steep front, which displaces itself with a velocity equal to the arithmetic mean of the velocities at the top and the bottom. Owing to the differences in velocities of the various fronts, fronts can overtake each other and then "merge" together. The consequence of this process is that, whereas originally the curve for \( v(y) \) reflects most details of the time behavior of the producing force, details gradually are eliminated. The resulting turbulence thus differs in two respects from the pattern, originally produced by the force: through the development of steep fronts, and through the elimination of detail. We, consequently,
may say that on the long run the pattern shown by the turbulence becomes independent of the force system; only features connected with long periods in the random behavior of the force are retained over an appreciable length of time in the propagation of the turbulence.

It must be observed that this example can also be considered as illustrating in a simplified way the propagation of a system of shock waves in a tube, when waves are being produced repeatedly at the origin.

Certain results could be obtained governing the decay of the mean amplitude of the curve in its propagation.

At great distances from the origin the statistical properties of the curve for \( v(y) \) change only very slowly with \( y \), so that there is an approach to homogeneous turbulence.

A solution representing homogeneous turbulence, however, can better be obtained by starting from a case in which the \( v \)-curve over the whole \( y \)-axis is produced at a single instant, by a system of impulsive forces acting simultaneously in a system of points, arbitrarily distributed, after which the system is left to itself. Initially the \( v \)-curve will picture the distribution of the original impulsive forces. But again there is the tendency to produce steep fronts wherever \( \frac{\partial v}{\partial y} \) happens to be negative, and following that more and more details will be gradually eliminated through the merging of fronts.

This case was appropriate for an investigation of spatial Eulerian correlations of the type \( v_{12} \) and \( v_{22} \). An equation could be formed, which can be considered as a very simplified analogue of von Karman and Howarth’s equation applied in the theory of homogeneous isotropic turbulence. It is also possible to obtain a Fourier integral representing the solutions, and the amplitude function appearing in the integral is related to the spectral function in the same way as in other correlation theories. A number of properties considered in the theory of homogeneous hydrodynamic turbulence can be illustrated by means of analogous properties of the solutions for the model.

The problem was considered whether in this case there is a tendency to assume a pattern developing according to a law of similarity.
Certain aspects were brought to light which seem to speak against this. It may be that these aspects are peculiar to the model and that in hydrodynamic turbulence, with its three-dimensional field, matters will be different.

The last case considered concerned turbulence coupled with a quantity representing something like the mean flow of the fluid in pipe flow. To obtain a satisfactory example, it was necessary to complete Eq. (1) with a coupling term, as was done in Eq. (51). The field had been limited to the domain $0 \leq y \leq b$ and $v$ had been subjected to the boundary condition $v = 0$ at both "walls." The turbulence could be analyzed into a Fourier series so that it was easy to speak of separate components of turbulence. The equation governing these components was studied, which revealed that depending on the Reynolds number, a "laminar" solution and a "turbulent" solution is possible. Expressions for the components could be obtained by deriving stationary turbulent solutions directly from the partial differential equation and calculating the Fourier coefficients for these solutions. The formula giving the values of the amplitudes $\mathcal{F}_n$ showed that at the head of the spectrum they decrease inversely proportional to the index $n$, in such a way that the dissipation of energy has the same value for each component; at the tail end of the spectrum an exponential law is obtained which makes the sum of the dissipation terms convergent.

The circumstance that the turbulent solutions for this system were found to assume a time-independent form, is exceptional and is due to the simple form which was given to Eqs. (1) and (51). It is possible to construct other systems, slightly more complicated, so that the turbulent motion has two components, $v$ and $w$, where the turbulent solution must be time-dependent. However, although it can be shown that the equations for such a system have many properties in common with the simpler equations considered here, it has not been possible to discuss them so fully.
Chapter VIII

Homogeneous Isotropic Turbulence

The preceding Chapter gave an extensive investigation of some problems connected with a non-linear equation characteristic for a dissipative system. The simplest type of equation had been chosen, referring to a single variable which was a function of one coordinate and of the time. It is impossible at present to develop a similar treatment for the hydrodynamic equations. However, a great amount of research has been carried out on the correlation problems connected with these equations and on the theory of their spectra. This research has had its greatest success in the domain of homogeneous isotropic turbulence where considerations of symmetry make it possible to express certain sets of correlation functions with the aid of a single function. Equations can then be obtained for the basic functions expressing relations pertaining to energy decay and to the transmission of energy through the spectrum. A short account of this theory will be given in this last chapter. The principal mathematical relations will be deduced in a somewhat condensed form, and the reader is referred to original publications for a number of proofs. The main object is to obtain relations for which the model system gave a simplified form. This will show for which purpose the model system had been introduced and a comparison with the treatment developed for the model system may give some idea of the problems still concealed behind the equations for hydrodynamic turbulence.

All mean values to be introduced in the following pages will be space mean value of the Eulerian type. They will be functions of the time if the field is decaying. Although the dependence on the time will not be indicated explicitly in the expressions, one of the objects is to obtain equations determining the time derivatives.

83. Ordinary (or double) Velocity Correlations. The first quantity to be considered is the correlation product:

$$Q_{ij} = \overline{u_i u_j}$$

(1)
formed out of two components of the velocity, $u_1$ observed at a point $P$ with coordinates $x_1$, $x_2$, $x_3$, and $u'_j$ observed at a point $P'$ with coordinates $x_1 + r_1$, $x_2 + r_2$, $x_3 + r_3$. Since the field is homogeneous, the mean value will depend only on the relative position of $P'$ with respect to $P$, as determined by the $r_i$. The absolute value of the distance follows from $r^2 = r_i^2$. To simplify notation, summation signs are omitted; it is understood that summation is carried out over any repeated index.

In principle there are nine correlation functions, each depending on the three relative coordinates $r_i$ in its own way. When the field is isotropic, meaning that its statistical properties are not dependent on a direction fixed in space, these nine functions can be expressed by means of two functions $Q_{\|}(r)$, $Q_{\perp}(r)$, as follows:

$$Q_{ij} = Q_{\|} r_i r_j + Q_{\perp} \delta_{ij},$$

where $\delta_{ij}$ denotes a tensor which is equal to unity when $i = j$ and equal to zero when $i \neq j$.

The condition of isotropy also has the consequence that

$$Q_{ij} = Q_{ji}.$$ 

This is implied in (2) and can be seen by observing that the correlation product must not change when we interchange $P$ and $P'$ and simultaneously change the signs of the coordinates, which changes the signs of the $r_i$ and also that of the $u_i$ and $u'_j$.

If we take $r_2 = r_3 = 0$, so that $PP'$ is parallel to the $x_1$-axis and $r = r_1$, we have:

$$\overline{u_1 u'_1} = Q_{\|} r^2 + Q_{\perp}$$

$$\overline{u_2 u'_2} = Q_{\perp}$$

$$\overline{u_1 u'_2} = 0; \quad \overline{u_2 u'_3} = 0.$$ 

For $r = 0$ ($P$ and $P'$ coinciding) we have:

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2} = Q_{\perp}(0),$$
for which one also writes \( \overline{u^2} \), so that \( \overline{u^2} \) is the mean value of the square of a component of the velocity taken in an arbitrary given direction. The mean value of the square of the absolute velocity is \( 3\overline{u^2} = 3Q_{\|}(0) \).

It is customary to introduce two normalised correlation functions, \( f(r) \) and \( g(r) \), defined by means of the following diagrams:

\[
\begin{align*}
\delta \rightarrow \ldots \rightarrow u^2 \cdot f(r) \\
\uparrow \quad \uparrow \\
0 \ldots 0 \quad u^2 \cdot g(r)
\end{align*}
\]

If we return to the case \( r_2 = r_3 = 0, r = r_1 \), we find:

\[
\begin{align*}
Q_{11} &= \overline{u_1 u_1'} = \overline{u^2} \cdot f(r) \\
Q_{22} &= \overline{u_2 u_2'} = \overline{u^2} \cdot g(r)
\end{align*}
\]

\( \delta 4. \) In the case of an incompressible fluid the equation of continuity enables us to obtain a relation between \( Q_{\perp} \) and \( Q_{\|} \). We have:

\[ \nabla u_j / \nabla x_j = 0, \]

and consequently:

\[ u_1 \cdot \nabla u_j / \nabla r_j = 0. \]

This immediately leads to:

\[ \nabla Q_{ij} / \nabla r_j = 0. \]

When this is worked out on the basis of (2) we find:

\[ rQ_{11}'' + 4Q_{\perp} + Q_{\|}'' / r = 0 \]

(accents here denoting derivatives with respect to \( r \)). Hence we can express \( Q_{\|} \) in \( Q_{\perp} \) or both \( Q_{\perp} \) and \( Q_{\|} \) in terms of an auxiliary function \( Q(r) \). The form ordinarily chosen is:

\[ Q_{\perp} = -Q'/2r; \quad Q_{\|} = \frac{1}{2}Q' r + Q, \]

so that \( Q_{\perp} r^2 + Q_{\|} = Q \).

It is then possible to write (2) in the form:

\[ Q_{ij} = -(Q'/2r) \cdot \delta_{ij} + (Q + \frac{1}{2}Q' r) \delta_{ij}. \]

With \( r_2 = r_3 = 0, r = r_1 \), we now have:
\[ Q_{11} = u^2 \cdot f(r) = Q \]
\[ Q_{22} = u^2 \cdot g(r) = Q + \frac{1}{2} Q' r . \]

From the \( Q_{ij} \) we derive the function \( R \):
\[ R = Q_{11} + Q_{22} + Q_{33} = u_1^2 u_1^r + u_2^2 u_2^r + u_3^2 u_3^r = 3 Q + Q' r. \] 

(5)

Whereas \( Q_{11}, Q_{22}, Q_{33} \) are dependent on the orientation of the vector \( r \) giving the distance between \( P \) and \( P' \), the function \( R \) is dependent only on the absolute value of \( r \).

For \( r = 0 \) we have:
\[ R(0) = 3 Q(0) = 3 u^2 . \]

It is also easily found that:
\[ \int_0^r R r^2 \, dr = Q r^3 . \]

If we may assume that for indefinitely increasing \( r \) the function \( Q(r) \) goes to zero sufficiently rapidly to make \( \lim (Q r^3) = 0 \), we find:
\[ \int_0^\infty R r^2 \, dr = 0 . \]

This shows that somewhere \( R \) must have negative values and the same must be true of the sum \( Q_{11} + Q_{22} + Q_{33} \).

85. Pressure-velocity Correlations

We take the value \( p \) of the pressure at the point \( P \) and a velocity component \( u_j' \) at \( P' \), and consider the mean value of the product of these two quantities. It is readily seen that in an isotropic field one can expect a correlation different from zero only between the pressure at \( P \) and the velocity component \( u_j' \) at \( P' \) in the direction of \( PP' \) (or in the inverse direction). Now if this correlation would be different from zero, it must have the same value for all directions of \( PP' \). But this would mean that there would be an average radial flow of fluid, either away from \( P \) if the correlation were positive, or towards \( P \) if the correlation were negative. This is impossible when there is no source or
sink at $P$. Consequently we must conclude that in the absence of sources and sinks the pressure-velocity correlation is always zero in isotropic turbulence.

Hence:

$$p u_j' = 0 \quad \text{for all } j.$$  \hfill (7)

86. **Triple Velocity Correlations**

Equations for the behavior of the double correlations as functions of the time can be deduced from the hydrodynamic equations:

$$\frac{\partial u_i}{\partial t} + u_h \frac{\partial u_i}{\partial x_h} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta u_i.$$  \hfill (8)

The equation referring to $u_i$ at $P$ is multiplied by $u_j'$ (at $P'$); a similar equation for $u_j'$ at $P'$ is multiplied by $u_i$; the results are added and the mean value is taken. This gives:

$$\frac{\partial}{\partial t} u_i u_j' + \left( u_h \frac{\partial u_i}{\partial x_h} u_j' + u_i u_h' \frac{\partial u_j}{\partial x_h}' \right) = 2 \nu \Delta \left( u_i u_j' \right).$$  \hfill (9)

Here a new type of mean values appears, referring to products of the third degree. This leads to the investigation of triple correlation products:

$$T_{ihj} = \overline{u_i u_h u_j'},$$  \hfill (10)

where $u_i$ and $u_h$ are taken at $P$ and $u_j'$ at $P'$.

In the case of isotropic turbulence the tensor $T_{ihj}$ can be expressed with the aid of three functions of $r$, as follows:

$$T_{ihj} = T_{hij} = T_{I} r_i r_j r_h + T_{II} (r_i \delta_{hj} + r_h \delta_{ij}) + T_{III} r_j r_i \delta_{ih}.$$  \hfill (11)

The functions $T_{I}$, $T_{II}$, $T_{III}$ are related to three basic types of triple correlation functions defined by means of the following diagrams. (Each arrow indicates a velocity component; a double arrow indicates a component squared):
If again we take \( r_2 = r_3 = 0, \ r = r_1 \), we have:

\[
\begin{align*}
T_{111} &= T_1 r^3 + 2 T_{11} r + T_{111} r = (u^2)^{3/2} \cdot k(r) \\
T_{221} &= T_{11} r = (u^2)^{3/2} \cdot h(r) \\
T_{122} &= T_{11} r = (u^2)^{3/2} \cdot g(r) . \tag{12}
\end{align*}
\]

From \( T_{ij} \) we can form:

\[
T_{ij} = \frac{1}{(u_1^2 + u_2^2 + u_3^2)} u_j^i .
\]

Since the first factor is a scalar, independent of direction, this mean value must be zero on the same grounds as made \( p u_j^i \) zero. This gives a relation between \( T_1, T_{11}, \) and \( T_{111} \). Two more relations are obtained from the equation of continuity, which gives:

\[
\frac{\partial T_{ihj}}{\partial r_j} = 0 .
\]

One relation appears to be a consequence of the two others and all three functions can be expressed by means of a single one. The following form can be chosen:

\[
\begin{align*}
T_1 &= 2 \theta + \theta' r \\
T_{11} &= -\frac{5}{2} \theta r^2 - \frac{1}{2} \theta' r^3 \\
T_{111} &= \theta r^2 , \tag{13}
\end{align*}
\]

where \( \theta \) is a function of \( r \). For certain purposes it is useful to introduce a function \( T = 10 T_{111} + 2 T_{111}' r \); then:

\[
T = 14 \theta r^2 + 2 \theta' r^3 . \tag{13a}
\]
The expressions for the functions $k(r)$, $h(r)$, $q(r)$ in terms of $\Theta$ are as follows:

$$
\begin{align*}
(u')^{3/2} & \cdot k(r) = -2 \Theta r^3 \\
(u')^{3/2} & \cdot h(r) = + \Theta r^3 \\
(u')^{3/2} & \cdot q(r) = - \frac{6}{2} \Theta r^3 - \frac{1}{2} \Theta^3 r^4.
\end{align*}
$$

(13b)

Triple correlation products in which two components refer to the point $P'$ and only one component to $P$, can be obtained from those considered thus far by interchanging the points $P$ and $P'$. In isotropic turbulence the value of any triple correlation product does not change by this process, provided we simultaneously change the signs of the $r_i$ and of the velocity components. It follows that one has the relation:

$$
\frac{u_i u_j u'}{u_i u_j u'} = - \frac{u_i u_j u'}{u_i u_j u'}.
$$

(14)

87. A new quantity is now derived from the $T_{ihj}$ by means of the formula:

$$
T_{ij} = T_{ji} = - \frac{\partial}{\partial r_h} (T_{ihj} + T_{jhi}).
$$

(15)

Since we have:

$$
\begin{align*}
\frac{\partial}{\partial r_h} T_{ihj} &= \frac{\partial}{\partial r_h} (u_{i1} u_{j1} u') = - \frac{\partial}{\partial r_h} (u_{i1} u_{j1} u') = - \frac{\partial}{\partial r_h} \frac{u_{i1} u_{j1}}{u_{i1} u_{j1}}.
\end{align*}
$$

$$
\begin{align*}
\frac{\partial}{\partial r_h} T_{jhi} &= \frac{\partial}{\partial r_h} (u_{j1} u_{i1} u') = - \frac{\partial}{\partial r_h} (u_{j1} u_{i1} u') = - \frac{\partial}{\partial r_h} \frac{u_{j1} u_{i1}}{u_{j1} u_{i1}}.
\end{align*}
$$

it follows that:

$$
T_{ij} = \frac{\partial}{\partial r_h} \frac{u_{i1} u_{j1}}{x_{i1}} + \frac{u_{i1} u_{j1}}{x_{i1}} \frac{\partial}{\partial r_h},
$$

(16)

which is just the term occurring in Eq. (9). This equation consequently can be written:

$$
\frac{\partial}{\partial t} \frac{Q_{ij}}{Q_{ij}} + T_{ij} = 2 v \triangle Q_{ij},
$$

(17)
It is convenient to make use of the following reductions. By substituting the expression (11) into (15) and working out the differentiations, the result is obtained:

\[
T_{ij} = -\left[10 T_{1} + 2 T_{1}r + \frac{2T_{II}}{r} + \frac{2T_{III}}{r} \right] r_{i}r_{j} - \left[8 T_{II} + 2 T_{II}r + 2 T_{III} \right] \delta_{ij}
\]

and by making use of (13)

\[
T_{ij} = -\left[11 \theta + 10 \theta'r + \theta''r^2 \right] r_{i}r_{j} + \left[28 \theta r^2 + 12 \theta'r^3 + \theta''r^4 \right] \delta_{ij}.
\]

Having regard to (13a) this can also be written:

\[
T_{ij} = -\left(\frac{T'}{2r} \right) r_{i}r_{j} + \left(T + \frac{1}{2} T'r \right) \delta_{ij} \quad (18)
\]

From form. (4) one obtains:

\[
\triangle Q_{ij} = \left[\frac{2Q'}{r^3} - \frac{2Q''}{r^2} - \frac{Q'''}{2r} \right] r_{i}r_{j} + \left[\frac{2Q'}{r} + 3Q'' + \frac{1}{2} Q'''r \right] \delta_{ij}.
\]

If we write:

\[
D = \frac{4Q'}{r} + Q''
\]

this reduces to:

\[
\triangle Q_{ij} = -\left(\frac{D'}{2r} \right) r_{i}r_{j} + \left(D + \frac{1}{2} D'r \right) \delta_{ij} \quad (19)
\]

88. The expressions (4), (18) and (19) all have the same structure. It follows that equation (17) will be satisfied if:

\[
\frac{\partial Q}{\partial t} + T = 2 \nu D \quad (20)
\]

Another equation can be deduced from (17) by taking \(i = j\), which requires summation with respect to \(i\). As follows from (5), the first term of (17) then becomes: \(\partial R/\partial t\). We further write:

\[
S = T_{11} + T_{22} + T_{33} = 3 T + T'r = 70 \theta r^2 + 26 \theta' r^3 + 2 \theta'' r^4
\]

Since:

\[
\triangle Q_{11} = \triangle \left(Q_{11} + Q_{22} + Q_{33} \right) = \triangle R,
\]

(21)
we obtain:

\[ \frac{\partial R}{\partial t} + S = 2 \nu \Delta R . \tag{22} \]

Equation (20) can still be transformed by returning to the normalized correlation functions \( f(r) \) and \( h(r) \). We have:

\[
Q = u^2 \cdot f(r) ; \quad T = 2 \left( \frac{u^2}{r} \right)^{3/2} \cdot \left( \frac{h}{r} + h' \right) ; \quad D = u^2 \cdot \left( \frac{hf'/r + f''}{r} \right),
\]

and find:

\[
\frac{\partial}{\partial t} \left( u^2 f \right) + 2 \left( \frac{u^2}{r} \right)^{3/2} \left( \frac{hf'}{r} + h' \right) = 2 \nu u^2 \left( \frac{hf'}{r} + f'' \right). \tag{23}
\]

This is the fundamental equation for the propagation of correlation, as given by von Karman and Howarth.

89. Additional Remarks

The function \( Q(r) \) is an even function of \( r \), which starts with a term independent of \( r \). The same applies to the functions \( Q_1, Q_2, R \), and also to \( f(r) \) and \( g(r) \).

The function \( T(r) \) likewise is an even function of \( r \), which, however, starts with a term in \( r^2 \) (see Eq. 13a). The function \( \Theta (r) \) is even and starts with a term independent of \( r \). It follows from (13) that \( T_1 \) starts with a term independent of \( r \), whereas \( T_{II} \) and \( T_{III} \) start with terms in \( r^2 \). The functions \( k(r), h(r), \) and \( q(r) \) are odd functions of \( r \), which start with terms in \( r^3 \).

It will be seen from (18) that \( T_{i,j}(0) = 0 \).

If we multiply Eq. (21) through with \( r^2 \), we obtain:

\[
\frac{\partial}{\partial t} \left( r^2 \right) + \frac{\partial}{\partial r} \left( r^2 \right) = 2 \nu \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right)
\]

if regard is given to the expression (21) for \( S \) and to the formula for \( \Delta R \) in spherical polar coordinates. If we assume that \( r^2 \) and \( r^2 (\partial R / \partial r) \) vanish for \( r = \infty \), integration gives:

\[
\frac{d}{dt} \int_0^\infty \frac{N}{2} r^2 dr = 0.
\]
This is in accordance with the result previously obtained, according to which
\[ \int_0^\infty R \, r^2 \, dr = 0. \]

If we multiply with \( r^4 \), we obtain
\[ \frac{\partial (Rr^4)}{\partial t} + \frac{\partial}{\partial r} \left( 10 \, r^7 + 2 \, r^5 \right) = 2 \, \nu \, r^2 \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right). \]

Integration now gives:
\[ \frac{d}{dt} \int_0^\infty R \, r^4 \, dr = 12 \, \nu \int_0^\infty R \, r^2 \, dr = 0. \]

Hence
\[ \int_0^\infty R \, r^4 \, dr \]
is independent of the time. If we express \( R \) by means of \( Q \) according to (5), we also find that
\[ J = \int_0^\infty Q \, r^4 \, dr \quad (24) \]
is independent of the time. This integral is called Loitsiansky's invariant for homogeneous isotropic turbulence. The result also can be deduced from Eq. (23) by multiplying this equation by \( r^4 \) and integrating it, since \( u^2 \cdot f(r) = Q \).
Our main interest now is the spectral analysis connected with the functions introduced in the preceding pages.

Since the field is three-dimensional, a triple Fourier integral is needed. In extension of form (1) of section 16 we use the following representation for the velocity component \( u_1(x_1, x_2, x_3) \), which shall be valid within a cubical region of space with sides equal to \( 2M \):

\[
u_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y dk_z \; \phi_1(k_x, k_y, k_z) e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)}.
\]

(25)

It is expected that no confusion will arise through the use of \( i = \sqrt{-1} \) along with \( i \) as an index. The amplitude function must satisfy the condition

\[
\phi_1(k_x, k_y, k_z) = \phi_1^*(-k_x, -k_y, -k_z),
\]

where the asterisk denotes the complex conjugate function. The amplitude function can depend on the time, but it is not necessary to state this explicitly. There is a function \( \varphi \) corresponding to each of the three components \( u_1 \). The equation of continuity leads to the following relation between these functions:

\[
k_1 \cdot \varphi_1 + k_2 \cdot \varphi_2 + k_3 \cdot \varphi_3 = 0.
\]

(26)

In order to reduce writing, we introduce the vectors \( \mathbf{x} \) and \( \mathbf{k} \); replace

\[
k_1 x_1 + k_2 x_2 + k_3 x_3
\]

by the scalar product \( \mathbf{k} \cdot \mathbf{x} \); and abbreviate

\[
dk_x dk_y dk_z
\]

by \( dk \), writing a single integral sign only. Further, in the same way as before, summation signs will be omitted; summation has to be carried out every time an index is repeated. Equations (25) and (26) then become:

\[
u_1 = \int dk \; \varphi_1(k) e^{i \mathbf{k} \cdot \mathbf{x}}
\]

(25a)

\[
k_1 \varphi_1 = 0
\]

(26a)

The inverse of (25a) is the formula:

\[
\varphi_1(k) = \frac{1}{8\pi^3} \int_{-M}^{M} u_1(x) e^{-i \mathbf{k} \cdot \mathbf{x}} dx.
\]

(27)
To obtain Fourier integrals for the correlation functions, we follow a similar procedure as was applied in section 17. Since the coordinates of \( P \) determine the vector \( \mathbf{x} \) and those of \( P' \) the vector \( \mathbf{x} + \mathbf{r} \), we have:

\[
\mathbf{u}_j \mathbf{u}_j' = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_i(k') \varphi_j(k') \, e^{i(k'_1 + k''_1) \cdot \mathbf{x} + i k''_2 \cdot \mathbf{r}}
\]

To obtain the mean value we integrate with respect to \( x_1, x_2, x_3 \) from \( -M_1 \) to \( +M_1 \), where \( M_1 > M \), and divide by \( 8 M^3 \). This gives:

\[
\overline{\mathbf{u}_j \mathbf{u}_j'} = \frac{1}{M^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_i(k') \varphi_j(k') \, e^{i \mathbf{k} \cdot \mathbf{r} \sin(k'_1 + k''_1)M_1 \sin(k'_2 + k''_2)M_2 \sin(k'_3 + k''_3)M_3}
\]

which for \( M_1 \rightarrow \infty \) transforms into (writing \( \mathbf{k} \) for \( \mathbf{k}' \)):

\[
\frac{\pi^3}{M^3} \int \mathbf{k} \varphi_i(\mathbf{k}) \varphi_j(-\mathbf{k}) \, e^{-i \mathbf{k} \cdot \mathbf{r}}
\]

Hence if we introduce:

\[
\Gamma_{i,j}(\mathbf{k}) = \frac{\pi^3}{M^3} \varphi_i(\mathbf{k}) \varphi_j(-\mathbf{k})
\]

we have:

\[
Q_{ij} = \overline{\mathbf{u}_j \mathbf{u}_j'} = \int \mathbf{k} \Gamma_{i,j}(\mathbf{k}) \, e^{-i \mathbf{k} \cdot \mathbf{r}}.
\]

In isotropic turbulence the \( \Gamma_{ij} \) form a tensor in the \( k \)-space of similar nature as the \( Q_{ij} \), so that, in analogy with (2):

\[
\Gamma_{ij} = A(\mathbf{k}) \mathbf{k}_i \mathbf{k}_j + B(\mathbf{k}) \hat{\varphi}_{ij},
\]

where \( A(\mathbf{k}) \) and \( B(\mathbf{k}) \) only depend on the absolute value of \( k \):

\[
= \sqrt{k_1^2 + k_2^2 + k_3^2}.
\]

Since the equation of continuity gives:

\[
k_j \cdot \Gamma_{ij} = 0,
\]

it is possible to express the function \( A(\mathbf{k}) \) by means of \( B(\mathbf{k}) \) and we can
write:

$$\Gamma_{ij} = B(k) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) .$$  \hspace{1cm} (29)

When we introduce:

$$\Gamma = \Gamma_{ii} = 2B$$

we find:

$$R(r) = \int \frac{dk}{2\pi} \Gamma(k) e^{-i k \cdot r} .$$

Since \(\Gamma(k)\), like \(B(k)\), depends on the absolute value of \(k\) only, this integral can be transformed by introducing polar coordinates into the \(k\)-space, using the direction of the vector \(r\) as polar axis. If \(\vartheta\) is the angle between the vector \(k\) and the vector \(r\), we have \(k \cdot r = kr \cos \vartheta\); for \(dk\) we substitute \(2\pi k^2 \sin \vartheta\, dk\), and obtain:

$$R(r) = \frac{i}{4\pi} \int \frac{dk}{\pi} \Gamma(k) \frac{k \sin kr}{r} .$$  \hspace{1cm} (30)

It is convenient to write:

$$\frac{i}{4\pi} k^2 \Gamma(k) = F(k) ,$$

so that:

$$R(r) = \int F(k) \frac{\sin kr}{kr} \, dk .$$  \hspace{1cm} (30a)

For \(r = 0\) this reduces to:

$$R(0) = \int F(k) \, dk .$$  \hspace{1cm} (30b)

Since \(R(0)\) is equal to the mean square value of the absolute velocity and thus - apart from a factor \(\frac{1}{2}\) - measures the mean kinetic energy per unit volume, this formula gives the spectral resolution of the kinetic energy, in the sense that \(F(k)\, dk\) represents the contribution to the kinetic energy, derived from harmonic components with an absolute spatial frequency \(k\) (wavelength \(2\pi/k\)).

From the theory of Fourier integrals it follows that the inverse of \((30a)\) is \(x\).
\[ F(k) = \frac{2k}{\pi} \int_{0}^{\infty} r \sin kr \, R \, dr \quad (30c) \]

To obtain the spectral resolution of the kinetic energy for a single component, we observe that according to (29):

\[ \overline{u_1^2} = Q_{11} = \int dk' \Gamma_{11}^i(k') = \int dk' B(k') \left( 1 - \frac{k_1^2}{k'^2} \right) = \]

\[ = \int_{-\infty}^{\infty} dk' \int_{0}^{\infty} 2\pi k'' B(k') \left( 1 - \frac{k_1^2}{k'^2} \right) dk'' , \]

where \( k'' = \sqrt{k_2^2 + k_3^2} \). Since \( k''^2 = k^2 - k_1^2 \), we have \( k'' \, dk'' = k \, dk \) for constant \( k_1 \). Hence if we write:

\[ F_1(k_1) = \int_{k_1}^{\infty} \frac{F(k)}{2k} \left( 1 - \frac{k_1^2}{k'^2} \right) \, dk' , \quad (31) \]

we find:

\[ \overline{u_1^2} = \int_{0}^{\infty} F_1(k_1) \, dk_1 . \quad (31a) \]

This is often denoted as the formula for the one-dimensional or Taylor-spectrum.

We finally observe:

\[ \Delta R = -\int_{0}^{\infty} k^2 F(k) \frac{\sin kr}{kr} \, dk \quad (32) \]

92. A calculation similar to the one applied for obtaining \( \Gamma_{ij} \) gives:

\[ T_{ihj} = \overline{u_i u_h u_j} = \frac{\pi^3}{\mathbf{k}^3} \int dk' \int dk'' \varphi_i(k') \varphi_h(k'') \varphi_j(-k'-k'') e^{-i(k' + k'')} \cdot \mathbf{r} \]

We write \( k' + k'' = \mathbf{k} \); hence \( k'' = \mathbf{k} - k' \); the integration with respect to \( dk' \) and \( dk'' \) can then be replaced by one with respect to \( dk \) and \( dk' \).
In this way we obtain:

\[ T_{ihj} = \int \frac{dk}{2\pi} \psi_{ihj}(k) e^{-i k \cdot r} ; \]  

(33)

where:

\[ \psi_{ihj}(k) = \frac{\pi^3}{H^3} \int \frac{dk'}{2\pi} \phi_i(k') \phi_h(k-k') \phi_j(-k) . \]  

(33a)

We shall also write:

\[ T_{ij} = \int \frac{dk}{2\pi} \psi_{ij}(k) e^{-i k \cdot r} ; \]  

(34)

then:

\[ \psi_{ij} = \psi_{ji} = \frac{1}{H} (\psi_{ihj} + \psi_{jhk}) = \]

\[ = \frac{\pi^3}{H} \int \frac{dk'}{2\pi} \left\{ k_1 \phi_1(k-k') + k_2 \phi_2(k-k') + k_3 \phi_3(k-k') \right\} \]

\[ \cdot \left\{ \phi_1(k') \phi_3(-k) + \phi_2(k') \phi_2(-k) + \phi_3(k') \phi_3(-k) \right\} . \]

In order to be able to express \( S = T_{ii} \) by means of a Fourier integral, we must form:

\[ \Psi = \psi_{ii} = \frac{2\pi^3}{H^3} \int \frac{dk'}{2\pi} \left\{ k_1 \phi_1(k-k') + k_2 \phi_2(k-k') + k_3 \phi_3(k-k') \right\} \]

\[ \cdot \left\{ \phi_1(k') \phi_3(-k) + \phi_2(k') \phi_2(-k) + \phi_3(k') \phi_3(-k) \right\} . \]  

(35)

This is a function of the absolute value of \( k \) only. Hence we obtain:

\[ S = T_{ii} = \int \frac{dk}{2\pi} \psi e^{-i k \cdot r} = \int_0^\infty w(k) \frac{\sin kr}{kr} \, dk , \]  

(36)

with

\[ 4\pi k^2 \psi(k) = w(k) . \]  

(36a)

The inverse formula of (36) is:

\[ w(k) = \frac{2k}{\pi} \int_0^\infty r \sin kr \, s \, dr . \]  

(36b)

For \( r = 0 \) we have \( S(0) = 0 \); hence it follows from (36) that

\[ \int_0^\infty w(k) \, dk = 0 . \]  

(36c)
93. We can now translate equation (22) into a relation between the quantities $F(k)$ and $w(k)$, as follows:

$$\frac{\partial F(k)}{\partial t} = -w(k) - 2 \nu k^2 F(k). \quad (37)$$

This equation is an analogue of Eq. (49) given in section 71 for the simple mathematical model, the function $-w(k)$ playing the part of $k \nu$ occurring in (49). It is often given in an integrated form, with limits 0 and $k$. We shall write:

$$\int_0^k w(k) \, dk = W(k);$$

then:

$$\frac{\partial}{\partial t} \int_0^k F(k) \, dk = -W(k) - 2 \nu \int_0^k k^2 F(k) \, dk. \quad (38)$$

This equation is an analogue of Eq. (50) of section 71 (p. 136). It is also an analogue of Eq. (69) of section 80 (p. 151), provided the term with the factor $U/2$ is omitted. The left hand side of Eq. (38) gives the change of energy (apart from the factor $1/2$) associated with harmonic components of wave numbers not exceeding $k$ (wavelength $\geq 2\pi/k$). The second term on the right hand side gives the loss of energy from this part of the spectrum through viscous decay; the first term gives the loss from this part of the spectrum through interaction with the rest of the spectrum. It is evident from its definition that $W(0) = 0$, but we have also proved that $W$ vanishes at infinity (see 36c).

Equation (38) forms the starting point for certain deductions by Heisenberg and by von Karman and others, who all make use of the hypothesis that $U(k)$ may be represented by an expression consisting of the product of two independent integrals, as follows:

$$\text{const.} \left( \int_0^\infty F^\alpha k^\beta \, dk \right) \cdot \left( \int_0^k F^{\alpha'} k^{\beta'} \, dk \right).$$

We shall not enter into these considerations since they have been
treated very fully in the literature. The corresponding expression for
the model system, that is, the integral

\[- \int_0^k k \varphi \, dk\]

occurring in Eq. (50) of page 136, can be brought into the
form of a double integral of the following type:

\[- \frac{\ln \pi}{M} \int_0^k \int_0^\infty \frac{dk_1 k_1 \varphi(k_1)}{\varphi(k_2) \varphi(-k_2)} \varphi(k_2 - k_1) \, dk_2 \, dk_1\]

It will be seen that this has a character differing from that assumed by
Heisenberg.

It follows from (26a) that \( w(k) \) vanishes whenever the vectors \( k' \)
and \( k \) have the same direction, which entails that also \( k - k' \) will have
that same direction. (Remark made by Dr. Ch. M. Tchen in Washington.)
The function \( w(k) \) vanishes likewise when the scalar product of the com-
plex quantities \( \varphi(k') \) and \( \varphi(-k) \) is zero.

It can be attempted to find a condition, analogous to Eq. (72) of
section 81 (p. 154), which might ensure that the terms of the highest
degree in \( k \) will cancel in (37). Owing to the three-dimensional struc-
ture of the formulas, the result is more complicated. A possible formul-
lation seems to be:

\[
\text{mean value for all directions of the vector } k', \text{ the abso-
late value of } k' \text{ remaining constant}
\]

\[
\left\{ \left[ \frac{k_j}{k} \varphi_j(k - k') \right] \cdot \varphi_1(k') \right\}
\]

should be equal to

\[
\frac{3i}{\ln \pi} \frac{\nu}{k^2} \varphi_1(k),
\]

for a given vector \( k \), of arbitrary direction and magnitude, provided
the absolute values \( |k'| \) and \( |k - k'| \) both exceed a fixed number \( K \).

The appearance of \( k^2 \) in the denominator of the right hand side is
connected with the circumstance that \( \varphi \) has the dimensions

\[(\text{velocity}) \cdot (\text{length})^3.\]