LIMITATIONS TO NON-ISOTHERMAL FLOW IN LUBRICANT FILMS DUE TO VISCOS-SHEAR HEATING

Proefschrift

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The thermal effects due to viscous-shear-heating have been studied for the severe steady-state regime of band-shaped thermo-elastohydrodynamic lubricant films where these effects may well become critical in various respects. This regime is characterized by velocities of the two rubbing surfaces so high that the convection of the heat inside the film becomes negligible with respect to the withdrawal through these surfaces. Then, as first indicated by W.R.D. Wilson, the classical momentum equation of Reynolds may be combined with both the energy equation and the viscosity-temperature-pressure relationship of the lubricating oil into one single equation. The latter, the "thermal Reynolds equation", amounts to a relationship between the local pressure gradient in the film cross-section considered individually and the two viscosities at the local surface temperatures and at the equally local pressure in that same cross-section. Numerical integration of this equation yields the distribution of pressure along the film, and therefore that of the shear distortion of pressure along the film, and as a result, the frictional losses, provided that a realistic set of boundary conditions be used. That is, in the present regime W.R.D. Wilson's thermal boundary condition of uniformity of the two distributions of the surface temperatures can no longer be upheld and has to be replaced by the really representative one that follows from the flash temperature theory.

Although the present procedure in its full generality is valid even for complete analyses, those covering thermo-elastohydrodynamic films over their entire length, in the present work it has been applied only to the so-called "thermal inlet zone analyses". The latter analyses yield a fair estimate of the pumping, or rather "boosting", characteristic of the inlet zone which determines the constant rate of flow in all the successive film cross-sections. Therefrom a thermal correction factor may be derived for evaluating the most critical, smallest film thicknesses, those occurring in the conjunctive zone of the film. In fact, this factor may be applied to the thicknesses that are accurately known from literature on the limiting, isothermal elastohydrodynamic regime.

For the present purpose W.R.D. Wilson et al.'s thermal Reynolds equation has been generalized by introducing Slotté's triparametric isobaric viscosity-temperature relationship instead of their exponential biparametric one. The former relationship has been chosen for two reasons, namely: first, in being triparametric it covers a much wider range of temperatures, whilst like its limiting case of the exponential relation, it is still referentially invariant in Pawlowski's sense; second, one of its three viscometric parameters has been shown to be conceivable as an index of viscometric homology, which is closely related to the well-known Slope Index of Roelands.

Reynolds equation various basic problems have been worked out in the general problem. Therefore the existence of both limiting shear stresses and limiting pressure gradients could be established, which cannot be exceeded anywhere in thermo-elastohydrodynamic or thermo-plastohydrodynamic oil films. In the same light the question of various kinds of instabilities of the flow in such films has also been discussed.

SUMMARY

The thermal effects due to viscous-shear-heating have been studied for the severe steady-state regime of band-shaped thermo-elastohydrodynamic lubricant films where these effects may well become critical in various respects. This regime is characterized by velocities of the two rubbing surfaces so high that the convection of the heat inside the film becomes negligible with respect to the withdrawal through these surfaces. Then, as first indicated by W.R.D. Wilson, the classical momentum equation of Reynolds may be combined with both the energy equation and the viscosity-temperature-pressure relationship of the lubricating oil into one single equation. The latter, the "thermal Reynolds equation", amounts to a relationship between the local pressure gradient in the film cross-section considered individually and the two viscosities at the local surface temperatures and at the equally local pressure in that same cross-section. Numerical integration of this equation yields the distribution of pressure along the film, and as a result, the frictional losses, provided that a realistic set of boundary conditions be used. That is, in the present regime W.R.D. Wilson's thermal boundary condition of uniformity of the two distributions of the surface temperatures can no longer be upheld and has to be replaced by the really representative one that follows from the flash temperature theory.

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Symbols

a Hertzian semi-width for band contacts [m]
A viscometric parameter in Slotte's viscosity-temperature relationship [l/°C]
b = \frac{W_d c}{N_{T_1}} , thermal contact coefficient [N/(ms°C)]
\beta_n laser beam diameter [m]
c Specific heat per unit mass \([J/(kg°C)]\)
c_p Specific heat of vapour \([J/(kg°C)]\)
d_f fringe spacing in the control volume [m]
d diameter of the laser beam leaving the lens [m]
d_m thickness of the measuring volume [m]
d_p line spacing of the grating, [m]
E' = \frac{2l}{L_1} L_2 , reduced modulus of elasticity \([N/m^2]\)
f = \frac{f_f}{f_v} , coefficient of rolling friction
f_v rotating frequency \([1/s]\)
f_s frequency difference \([1/s]\)
F Frictional force per unit length of the rubbing members \([N/m]\)
g = \frac{\partial p}{\partial x} , pressure gradient in the direction opposite to that of flow \([N/m^2]\)
G Roelands empirical constant in viscosity-temperature relationship
h lubricant flow film thickness \([m]\)
h_c central film thickness of elastohydrodynamic films \([m]\)
h_r latent heat of vapourization \([J/m^2] \text{kg}\)
k thermal conductivity \([W/(m°C)]\)

NOMENCLATURE

- Length of the measuring volume \([m]\)
- Condenser length \([m]\)
- Exponent in Slotte's viscosity-temperature relationship
- Mass rate of flow of condensate \([kg/s]\)
- Pressure \([N/m^2]\)
- Reduced pressure \([N/m^2]\)
- Maximum Hertzian pressure \([N/m^2]\)
- Volumetric rate of flow \([m^3/s]\)
- Heat partition ratio
- Radial coordinate in axisymmetric flow
- Non-dimensional quantity
- Line spacing of pipes
- Radius of pipes
- Radius of conformity
- Material function in Pawlowski's standard representation
- Distance between collimated laser beams \([m]\)
- Roelands Slope Index in viscosity-temperature relationship
- Temperature \([°C]\)
- Velocity of flow \([m/s]\)
- Entering velocity \([m/s]\)
- Sliding velocity \([m/s]\)
- Characteristic velocity \([m/s]\)
- Load per unit length of the rubbing members \([N/m]\)
- Coordinate across the film \([m]\)
- Roelands Slope Index in viscosity-pressure relationship
- Coefficient in viscosity-pressure relationship \([N/m]\)
- Taper angle \([°]\)
- Coefficient in viscosity-temperature relationship for water \([N/m]\)
- Coefficient in viscosity-temperature relationship \([N/m]\)
- Coefficient in viscosity-temperature relationship for water \([N/m]\)
- Film thickness of condensate in the heat pipe \([m]\)
- Saturation temperature difference \([°C]\)
- Dynamic viscosity of liquids \([cP]\)
- Density per unit volume \([kg/m^3]\)
- Yield limit of the plastically deforming work piece \([N/m^2]\)
- Extrusion pressure \([N/m^2]\)
- Shear stress \([N/m^2]\)
- Reynolds stress \([N/m^2]\)
- Heat generation due to viscous dissipation per unit volume per unit time \([W/m^3]\)
- Total heat generation across the film \([W/m^2]\)
- Function in Pawlowski's standard representation
- Rotational speed \([rad./s]\)

- Frictional drag number
- Thermal back flow parameter
- \(G = \frac{h_{ben}}{k} \) for elastohydrodynamic films
- Coefficient related to \(f\)
- Non-dimensional quantity with exponential viscosity-temperature relationship
- Thermal sliding parameter with exponential viscosity-temperature relationship
- Non-dimensional quantity with Slotte's exponent \(m=1\)
- Thermal sliding parameter with Slotte's viscosity-temperature relationship
- \(H = \frac{h_{ben}}{k} \) , Russel number
- \(F = \frac{h_{ben}}{k} \) for plan-parallel film flows
- \(F^* = \frac{p'}{\mu'}, \text{ non-dimensional lubricant pressure}\)
- \(Q = q/q_is \) , correction factor to isothermal flow
- \(R = \frac{1}{L} \frac{dL}{dx} \) (for isothermal flow)
As most essential structural element in lubricated rubbing systems, full fluid oil film shows a performance in terms of strength, stiffness, and stability that may be influenced considerably by the viscosity characteristics of the oil. Especially under severe operating conditions such as occurring in elastohydrodynamic films between counterformal rubbing surfaces, or in plastohydrodynamic films in metal-working, the local pressures and temperatures will vary widely from point to point in these films. Accordingly, the same is true of viscosity and of the internal heating by viscous shear. Therefore, the performance of the films concerned may be affected quite appreciably by this internal heating and, of course, also by the concurrent withdrawal of the heat concerned by the two rubbing surfaces bounding the film.

The dissipative property which characterizes the generation of heat due to viscous shear is intrinsic to a flowing liquid as much as its physical property, i.e. viscosity, which characterizes the flow resistance. The higher the viscosity, the greater is the dissipation.

To demonstrate the quantity of heat involved while subjecting a lubricant to viscous shear heating the following illustration proves to be interesting.

The heat \( \phi_v \) generated by viscous shear per unit time and per unit volume of the lubricating oil inside the film may be represented by,

\[
\phi_v = \mu \left( \frac{du}{dy} \right)_v = \mu \left( \frac{\partial u}{\partial y} \right)_v
\]

In high speed operations involving surface speeds, \( u \), of say 10 m/s, lubricant films 2 x 10^{-6} m (2 thousandth of a millimeter) thick of oils with their viscosity in the range of \( \mu = 0.1 \text{ Ns/m}^2 \) (100 centipoise), will be subjected to a rate of shear:

\[
\frac{du}{dy} = 3 \times 10^6 \text{s}^{-1}
\]

so that

\[ \phi_v = 0.25 \times 10^6 \text{ W/litre} \]

In this connection consider that the electric power generated by all the powerhouses in The Netherlands during peak hours exceeds 10^9 MW (ten thousand megawatt) only slightly. So, the viscous shear heating in these lubricant films subjected to even moderate sliding speeds is tantamount to exposing only four litres of oil continuously to all the heat that may be obtained in this country from total conversion of the maximum electric power available.
For a thermally steady state to prevail there should locally exist a complete balance between the heat that is generated due to effects of viscosity and the heat that is responsible to maintain the latter viscosity through the thus involved temperatures. There is then a need to establish accurately the relationship between the variation of viscosity with that of temperature.

In attempts to establish the flow behaviour in lubricating films so as to generate a safe full fluid film, and thereby to protect the lubricants, direct experimental methods have been applied by various investigators. The results thus obtained have been used by these and other investigators for a comparison with the results available from theories of thermo-elastohydrodynamic lubrication for oils hypothesised to be truly Newtonian behaviour under the operating conditions concerned. Fairly generally they felt that deviations found between the experimental and theoretical results might be ascribed to non-Newtonian behaviour. However, most of the models underlying the theories applied were far from being sufficiently representative in one or even more respects. Shortcomings in these theories may be exemplified by inaccuracy of viscosity-temperature-pressure relationships, introduced for supposedly Newtonian oils, and by inadequacy of the thermal boundary conditions imposed on the films.

It is meanwhile fairly well established that most lubricating oils follow a linear relationship between shear stress and rate of shear, i.e. a Newtonian behaviour, up to rather high shear stresses of about 10 Pa. But for already at lower shear stresses the thermal effects may already play a dominant role in the performance of the films concerned, including the generation of shear heat, or say that of frictional losses. In fact, in the regime of operations yielding comparatively low shear stresses the thermal effects already pose certain limitations to the flow. The role played by these effects may be judged from comparing the theoretical results obtained with due consideration of these thermal effects for film performance with those obtained by ignoring them. The latter results characterize isothermal flows in which the viscosity remains unchanged throughout film cross-sections individually and varies only with pressure from one cross-section to the other.

Among other characteristic behaviours that are associated with the isothermal flows, a monotonic increase of shear stress with the faster sliding of the boundaries is the most important one. This behaviour has been depicted for both isothermal (also isoviscous, since the present illustration concerns pure shear flows in the absence of any variation in pressure) flows with uniform viscosity and the aforementioned thermally affected ones in Figure 1.1. Shear stresses for isothermal flows linearly increase to indefinitely high values at high sliding speeds. On the contrary, thermally affected, non-isothermal flows are realizable in their steady state only when they are generated under a shear stress not exceeding a limiting finite value, which falls well below that at which the fluid concerned is still Newtonian. Many analyses of such fluids have been published, especially as to certain limitations, remain theoretical in nature. They have not so far been experimentally verified. Their predictions well deserve further study since they are deemed important for designing lubricating films for safe operation. Surprisingly, most of the theoretical information seems so far to have escaped notice of machine designers.

The aforementioned maximum conceivable finite value of shear stress, or of pressure gradient in the case of flows generated by pressure differences imposed, varies from one kind of liquid to another, and also with the kind of thermal boundary conditions imposed. Steady-state non-isothermal flows are possible for a given liquid, specified thermal boundary conditions, and imposed speeds of the boundaries. But they are, of course, conceivable only in the regime of shear stresses lower than the aforementioned maximum. Results including the transition point and the regime of operation following it, have been achieved through mathematical models including an energy equation that accounts for heat generation due to viscous-shear heating, and...
Further only for heat transfer due to cross-conduction, i.e., in a direction perpendicular to the direction of flow. The results thus obtained by various investigators have led them, if only on theoretical grounds, to proclaim existence of certain limitations to the steady-state solutions at the transition point and in the second regime of operation. These results also indicate excessive temperatures and rates of flow in the second regime. There are indications of certain even detrimental effects to the flow in the same operating regime.

1.3 SCOPE OF THE PRESENT INVESTIGATION

The aforementioned behaviour of non-isothermal flows of liquids such as lubricating oils is strongly related to their dissipative property. This dissipative property is, in turn, dependent on the variation of viscosity of these liquids with the local changes in temperatures and pressures. Thus it is of primary importance to relate accurately the viscosity variation with temperature and pressure, and to devise relationships that are better suited to the theoretical analysis of the flow problems concerned. Now, the viscosity-temperature-pressure relationships so far used in literature about non-isothermal flows may be deemed too inaccurate for reliably predicting the various thermally affected processes and phenomena. Therefore the author was induced to refine the theory on non-isothermal flows by introducing more accurate such relationships.

Various investigators have, in the past, carried out elaborate studies to establish very accurate empirical viscosity-temperature relationships on the basis of precise viscometric data. However, most of those empirical relationships, as well as the theoretical ones based on thermodynamic considerations, although prove useful to accurately predicting viscosities even at the very high pressures and temperatures prevailing in certain lubricating films, they are definitely far too involved for inclusion in models underlying the theory of non-isothermal flow. Therefore, there is a need for simpler viscosity-temperature relationships.

The aforementioned limitations to the previous published analyses of steady-state non-isothermal flows are here distinguished, namely: one caused by inaccurate description of the variation of viscosity: the other caused by inadequacy of the kind of thermal boundary conditions imposed. There is a growing need for analyses of non-isothermal flows that are refined in a more realistic manner.

Remarkably enough, governing energy balance equations for non-isothermal flow are characteristically similar for various viscous such kinds of flow: shear flows induced by the motion and drag of their moving boundaries; pressure-induced flows with thermal as well as kinematic asymmetry such as in pipes and capillaries, and those with plane symmetry in plan-parallel flows, and finally, flows of a more general nature involving both thermal and kinematic asymmetries caused by the motion and drag of their moving boundaries. In the first and the last type of flows, thermal asymmetry may be caused by the two distributions of the boundary temperatures being unequal.

It has been pointed out in literature that the aforementioned types of governing equations are even more or less similar to those for other physical systems than those constituted by lubricating films, namely those that are bound up with processes involved in: two-dimensional vortex motion of incompressible fluids; space charge of electricity of a glowing wire; distribution of mass of gaseous interstellar material under the influence of its own gravitational field; nuclear rods in operation; the breakdown of solid insulating materials used in the presence of alternating electric fields; etc. Solutions to such governing equations provide the domain in which steady-state operation of the above physical systems can be maintained and outside which this proves impossible.

To machine designers the safe operation of lubricating films is becoming increasingly important in view of the still increasing sliding and rolling speeds and loads. The analysis of those physical systems constituted by thermo-elasto/plastohydrodynamic oil film and its two bounding and moving solid surfaces should be refined enough to realistically account for the thermal processes involved. Such an improved analysis is crucial in delimiting safe operating regimes of this system where lubricating films are thick enough for avoiding any undue damage by contact of the bounding solid surfaces.

1.4 CONSIDERATIONS BASIC TO THE PRESENT ANALYSIS

1.4.1 Viscosity-temperature-pressure relationship

Although the viscosity characteristics of lubricating oils are in general to be represented by complete viscosity-temperature-pressure relationships, for the present purpose it is convenient to use isobaric viscosity-temperature relationships for various pressures that cover the range inherent in the application to be considered. For instance, in the thermo-elasto/hydrodynamic films, owing to their thinness, in contrast to temperature, will in every cross-section be constant whilst it will vary only along such films, that is from one cross-section to another.

Now, certain kind of functions, which is called "referentially invariant" has proved advantageous for describing the variation of material properties, such as viscosity, with some such state variable as temperature or pressure. Indeed, the mathematical structure of the results of analyses of physical processes, including thermo-elasto/ hydrodynamic lubrication, in which such variations have to be accounted for, may be simplified in that the number of dimensionless groups required may be reduced to its minimum. Among isobaric viscosity-temperature relationships the classical triparametric one developed empirically by Slotte is the most general "referentially invariant" one. One of its algebraically equivalent forms is,

$$\mu(T) = \mu_r \left[ 1 + 2\frac{T - T_r}{\Delta T} \right]^m$$

where \( \mu \) and \( \mu_r \) are the three viscometric parameters, the dynamic viscosity \( \mu \) being expressed as a function of temperature \( T \), \( \mu_r \) being the reference viscosity defined at some suitable reference temperature \( T_r \). In its form,

$$\frac{\mu(T)}{\mu_r} = \left[ 1 + 2\frac{T - T_r}{\Delta T} \right]^m$$

the viscosity-temperature coefficient \( B \) of the given oil at the reference temperature \( T_r \) is brought out. Slotte's relationship yields two
well-known limiting biparametric cases, the hyperbolic and the exponential relationship, i.e. when m is assigned the value unity and infinity, respectively. The latter relationship is given by,

\[ u = e^{RT_{R}} \]

where the temperature coefficient \( R \) is independent of the reference temperature \( T_{R} \).

### 1.4.2 Energy equation for non-isothermal flow

The study to follow deals with flow analyses general enough to represent steady flows of liquids in elasto/plastohydrodynamic films, like those schematically represented in Figure 1.2. It is based on the assumptions herebelow.

**Assumptions**

1. The flow is assumed steady and two-dimensional. Transient effects resulting from occasional disturbances have been excluded from consideration.

2. The fluid is considered incompressible, i.e. density is independent of temperature and pressure. Also the thermal conductivity \( k \) of the oil has such an independence been assumed. These two assumptions may, however, be relaxed whilst evaluating solutions along the lines employed in the analysis.

3. The fluid is assumed to follow Newtonian behaviour, i.e. shear stress independent of rate of shear. Then the local shear stress \( \tau \) is proportional to the local velocity gradient \( du/dy \), viz.

\[ \tau = \mu \frac{du}{dy} \]

4. The extent of cross-conduction of heat across the film is far more dominant so as to yield negligible effects of conduction of heat along the flow direction. In view of the extreme thinness of lubricant films the conduction of the viscous-shear heat along the film may be ignored with respect to the cross conduction. The local heat flux by cross-conduction may be represented by,

\[ -k \frac{dT}{dy} \]

5. The dominant mechanism of heat generation in the film is that by viscous shear, this being per unit volume and per unit time represented by,

\[ \phi_v = \mu \left( \frac{du}{dy} \right) \]

where Newton's definition of shear stress has been implied. The present form of the dissipation term requires the additional use of the momentum balance given in its basic form by,

\[ \frac{d\rho}{dx} = \frac{dp}{dx} \]

where \( g = dp/dx \) represents the opposite to the pressure gradient in the direction \( x \) of flow, and which, like pressure itself, is constant across any film cross-section and varies only along the film, that is with \( x \). The resulting equation for energy balance considering the above assumptions, can then be given by,

\[ \rho c u \frac{dT}{dy} = k \frac{dT}{dy} + \phi_v \]

where the first term may be described as follows.

6. The first term represents the heat transfer due to liquid convection along the film. For very thin lubricating films, such as occur in elasto/plastohydrodynamic lubrication, this term may be considered negligible as compared with either term on the right-hand side. As has been pointed out by Blok\(^*\), even at very high sliding speeds of the bounding solid surfaces, its relative importance is so small that it may be ignored.

With the consequent exclusion of the convection term the governing energy

\[ \boxed{-} \]

For mineral oils this linear relationship is in general applicable only up to moderate shear stresses, not exceeding 10\(^5\) N/m\(^2\). At higher shear stresses a non-Newtonian relationship may be expected to set in.

\[ \phi_v = \mu \left( \frac{du}{dy} \right) \]

\[ \frac{d\rho}{dx} = \frac{dp}{dx} \]

\[ \rho c u \frac{dT}{dy} = k \frac{dT}{dy} + \phi_v \]

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\[ -k \frac{dT}{dy} \]

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walls can then be dealt with to a good approximation by the "rod-model" move at relatively high speeds (Péclet numbers of the surfaces involved used in the analysis of non-isothermal fluid flows in lubricating films. Then being high). Accordingly the heat transfer from the film to the simple especially for the particular case in which the bounding walls films operating in a thermally steady state. The relationship is quite

dients normal to the boundary involved, these, in conjunction with the being unknown a priori. When expressed in terms of temperature gra­

temperature. It is hardly possible to achieve uniformity of temperature over wide surface areas of the two boundaries delimiting and guiding the flow. In the lubricant films concerned, it is however possible to conceive that the successive cross-sections in the direction of flow are coupled internally by the flow, and externally by elements, as finite as desired, of the boundaries concerned. Thus the analysis yet to be carried out is readily applicable to successive elements, each being at some uniform temperature.

The most general thermal boundary conditions are to cover unequal and non-uniform distributions of boundary temperature, say $T_1$ and $T_2$, both being unknown a priori. When expressed in terms of temperature gradients normal to the boundary involved, these, in conjunction with the heat conductivity of the oil, yield the proportional flow of heat.

Both these two kinds of general thermal boundary conditions will be used in the analysis of non-isothermal fluid flows in lubricating films. They are in fact, related to each other in successive cross-sections of films operating in a thermally steady state. The relationship is quite simple especially for the particular case in which the bounding walls move at relatively high speeds (Péclet numbers of the surfaces involved then being high). Accordingly the heat transfer from the film to the walls can then be dealt with to a good approximation by the "rod-model" of the flash temperature theory that is characterized by one dimensional transient heat conduction. The aforementioned two kinds of thermal boundary conditions, one referring to the temperature, and the other to its gradient, can at any point $x$ along the flow and at the boundary 1 or 2 be given by the so-called "flash temperature" boundary conditions,

$$T_1(x) = \int_{0}^{x} \frac{\phi_{1}(s)}{k} dy$$

where $u$ is the velocity with which the boundary moves, and $b$ is the thermal contact coefficient of the solid rubbing material concerned. The quantity $\phi_{1}$ represents the distribution of heat flux along the boundary, and it is given by:

$$\phi_{1}(x) = - \frac{dT(x)}{dy}$$

These two "flash temperature" boundary conditions reflect the approximation typical of the aforementioned "rod-model", namely, that in any given film cross-section there would be no feed-back of the heat from the portion of either rubbing surface that has not yet passed over that cross-section. The present boundary conditions have proved to be the most general and realistic ones in that they account for the non-uniformity of distributions of temperature and heat flux along the two rubbing surfaces.

1.5 ANALYSIS AND CONCLUDING REMARKS

The foregoing considerations as to viscosity-temperature relationships, the governing equations for energy and momentum balance, as well as to the thermal boundary conditions, in conjunction represent a realistic model of thermally affected flows of Newtonian liquids, particularly in thermo-elasto/plastohydrodynamic lubricant films. They can be culmi­

tated to formulate the analysis on somewhat realistic basis that proves to be accurately representative of the aforementioned lubricant films.

They also permit a wide choice of operating conditions that can be ana­

lysed theoretically in order to thereafter verify the results experimentally, for instance as to the delimitation of Newtonian from non-Newtonian behaviour of the oil concerned.

The analysis will provide solutions to various sets of operating condi­
tions that are physically realizable. It also yields conditions under which certain limitations to the present flow model may occur. Some result sets for checking the numerical representativity of those of previous investigations, whilst others may extend previous solutions to much more severe sets of operating conditions.

Further, by virtue of the aforementioned simple, nevertheless accurate, viscosity-temperature relationship of Slotte the present analysis is expected to provide a reliable guide for a selection of lubricating oils well adapted to the operating conditions to be considered. It is also expected to yield criteria for a sufficiently strict control over the thermal state of the boundaries so as to achieve safe operation. The results obtained will serve as basic data for an analysis, in gen­

eral, of the previously specified lubricant film functioning in elasto/plastohydrodynamic lubrication under rolling/sliding conditions.

Early predictions on potentialities for increasing the load-carrying capacity of elastohydrodynamically operating rubbing elements have now
been surpassed, and film pressures on such rubbing surfaces have already exceeded $10^9 \text{ N/m}^2$; similarly the surface speeds in such bearings have already reached, or perhaps even surpassed, 100 m/s. Under such extreme operating conditions designers can no longer rely upon the conventional methods of calculation, these being based on solutions originally developed for oversimplified flow models, mainly isothermal, or even isoviscous ones.

In literature there are, of course, indications about the refinements required in conventional methods of design calculation, that is, by accounting for the non-isothermal characters of the flows concerned. But these refinements have so far been confined mainly to oils that hypothetically follow an exponential viscosity-temperature relationship, and the solutions that have been obtained have been shown both accurate and easy to use in such analyses. It has been shown that the heat pipe has been shown to operate excellently especially up to heat loads of 350 watt and circumferential speeds of about 8.5 m/s. Chapter 7 reports the experimentally obtained variation of shear stress with circumferential speed in the aforementioned viscometer. Also included are the distributions of velocity of flow which have been recorded by means of the laser-Doppler method, using a powerful argon-ion laser.

The subject of Chapter 8 is recognized by the author not directly relevant to lubricating films. Yet, it has been included in the present thesis because of its importance in chemical industry. It deals with axisymmetric flows of liquids in capillaries, pipes and other conduits. Characteristics of the solution for an imposed pressure gradient have been analysed also in this chapter.

Chapter 9 is an extension to Chapter 8 in that it covers both thermally and kinematically symmetric flows in plan-parallel channels. Certain major characteristics of these flows, the analysis of which is amenable only to numerical evaluation, are similar to those of axisymmetric flows in, for instance, pipes. Chapter 10 extends the analysis of Chapter 9 to thermally and kinematically asymmetric flows which involve unequal temperatures and relative sliding of the surfaces. Solutions obtained through this analysis prove to be sufficiently complete; they cover various sets of operating conditions that are conceivable in practical physical processes, involving lubricating films in elasto/plastohydrodynamic lubrication. Indeed, the present solutions, apart from substantiating the specialized ones of the previous chapters, cover all possible combinations of thermal and kinematic asymmetry that may be expected in the successive cross-sections of the aforementioned lubricating films. For the particular case of pure rolling and equal boundary temperatures the "thermal Reynolds equation" relating the pressure gradient and the rate of flow, and covering a wide range of operating conditions has been provided.

Chapter 11 points out the importance of the effects of viscous shear heating in the inlet, or boosting zone of elastohydrodynamic lubricating films. It deals qualitatively with the influence of the "boosting" effect on the rate of flow generated, and thus on the possibly critical small film thickness in the following "conjunctional" zone, the one of closest approach between the two rubbing surfaces. The traction of such films has been demonstrated to vary with the operating conditions. Finally, Chapter 12 gives an overall picture of the effects of viscous shear heating, and of the possible limitations to flow and temperature in the light of the present investigation.
Chapter II

VISCOSITY-TEMPERATURE-PRESSURE RELATIONSHIPS FOR LUBRICATING OILS

2.1 INTRODUCTION

The physical property that characterizes the flow resistance of fluids, i.e. viscosity, has long been of concern mainly to chemical engineers, and to tribologists later; to the latter especially to incorporate this physical property in controlling the behaviour of a lubricant film as a load-carrying machine element. The sensitivity of viscosity towards temperature, and also pressure, has called for constantly refined models relating viscosity to either temperature or pressure, or to both temperature and pressure.

Early attempts by several investigators in the beginning of this century have yielded viscosity-temperature relationships at atmospheric pressure, which are even now being used by tribologists because of their simplicity, notwithstanding their somewhat limited accuracy. These relationships have mainly been established on empirical grounds and are based on experimentally obtained viscosity data at various temperatures. Elaborate theoretical treatments, such as Eyring's [2.1], based on a model for the movement of an activated molecule jumping over a potential energy barrier, have proved to be neither accurate in a wide temperature range nor applicable to complex liquids such as lubricating oils. Such treatments have constantly been extended to cover lubricating oils, not only base oils but also Viscosity Index (VI) improved oils, operating under various conditions. The pertinent models cover non-Newtonian behaviour, which is a subdivision of the discipline of rheology, i.e. the science of deformation and flow of materials. Empirical relationships for the variation of viscosity with temperature and pressure as fitted to experimental viscometric results in general are sufficiently suitable for tribological studies. Their simplicity is enhanced when one considers the Newtonian model, the shear stress then being just proportional to the local velocity gradient, du/dy, viz.:

\[ \tau = \mu \frac{du}{dy} \]  
(2.1)

where \( \mu \) is the dynamic viscosity. The present model is accurately representative of many lubricant oils, especially those which do not contain highly polymeric additives such as VI improvers. It is also valid whenever the latter oils, even including mineral oil fractions with mean molecular weights of 1000, are not subjected to excessively high shear stresses exceeding about 10^6 N/m². For hydrocarbons such as alkanes up to n-C₃₅H₇₂ and derivatives of cyclohexane these limiting shear stresses are appreciably lower, namely about 5x10^5 N/m². Thus up to about such rather high shear stresses may analyses of thermal effects in lubricant films as these are induced by the viscous-shear heat be carried out to a reasonable accuracy by means of the Newtonian model.

For Newtonian liquids quite a variety of empirical relationships is available in literature. Many of these relationships, however, show certain limitations as to their correlational accuracy and usable simplicity. Relationships containing three parameters (tri-parametric relationships), although generally more accurate than some of their counterparts, are rather cumbersome to use. Some of three-parameter relationships are readily reducible to nearly equally accurate two-parameter ones, at least for certain oils and for certain operating regimes.

Viscosity-temperature relationship due to Slotte has long proved to be one of the more convenient three-parameter relationships. This relationship belongs to the particular group known as "referentially invariant" material functions. The use of a referentially invariant material function simplifies the theory for the present physical process, namely non-isothermal elasto/plasto/hydrodynamic lubrication, yet to be studied in the chapters to follow. In what follows Slotte's relationship has been studied for its temperature range of accuracy, and as to how effective it will be in terms of the viscosity-temperature coefficient \( B_\vartheta \), which proves suitable for analysing thermally affected flows of oils. Slotte's relationship has been reduced from its original tri-parametric form to a corresponding biparametric one, and has also been compared for its accuracy with the well-known Roelands two-parameter viscosity-temperature relationship.

2.2 VISCOSITY-TEMPERATURE RELATIONSHIPS

The approximate theory of the viscosity of liquids developed by Eyring and his co-workers [2.1], [2.2] embodies the mechanism conceived by them for the formation of viscous forces in liquids. In a liquid at rest the molecules continually undergo a rearrangement their motion being ascribable to their thermally induced vibrations. When the liquid is set flowing the frequency of this rearrangement is increased. Correspondingly, due to the work done on the molecules, there appear viscous forces, which are larger than those required when the liquid is stationary. Further, with an increase in temperature the vibrational frequency of molecules increases, and thus the work required for the rearrangement decreases so that viscosity will then be reduced. Whilst isobaric viscosity-temperature relationship is governed by the activation energy for viscous flow, i.e. the energy required by a molecule to perform a jump over a potential-energy barrier into a "hole", the isothermal viscosity-pressure relationship is governed by the size of such a "hole". With increased pressures the free volume constituted out of "holes" decreases so that the viscosity will then be increased. The simplest equation on Eyring's model is that of Arrhenius,

\[ \mu = B_\vartheta \exp \left( \frac{A_1}{T} \right) \]  
(2.2)

This equation is known to be accurate especially at high temperatures. In the entire temperature range of concern to tribologists, the viscosity-temperature relationship of the highest choice would be Vogel's given by:

\[ \mu = B_\vartheta \exp \left( \frac{A_2}{(T-C_0)} \right) \]  
(2.3)

In expressions (2.2) and (2.3), \( B_\vartheta, A_1, A_2 \) and \( A_2, C_0 \) are viscometric param-
eters. In Vogel’s equation \( T_g \) would represent the temperature at which the free volume is hypothetically zero and \( u \) infinite, whilst \( A \) would be inversely proportional to the free-volume expansion coefficient. With a due consideration of molecular properties, improved viscosity-temperature (also viscosity-pressure) relationships have been proposed by Witt [2.3]. But the complexity of these relationships handicaps their use in the present investigation of non-isothermal flows of liquids.

2.2.1 Slottle’s isobaric viscosity-temperature relationship

One of the viscosity-temperature relationships that offers analytical simplicity in the mathematical analysis of viscous flow problems whilst being reasonably accurate in a wide temperature range, the justification of which is postponed till a later section, is Slottle’s relationship given by,

\[
\mu = B \left( \frac{T + A}{T_g} \right)^m
\]  

(2.4)

where \( B \) can be conceived to be a function of any reference viscosity yet to be defined, and \( m \) and \( A \) are viscometric parameters characterizing the type of oil. When the reference viscosity is chosen at the zero level of the temperature scale, viz. \( \mu_0 \), viscosity-temperature relationship (2.4) reduces to,

\[
\mu = \mu_0 \left( 1 + \frac{T + A}{T_g} \right)^m
\]  

(2.5a)

where the reference viscosity \( \mu_0 \) replaces \( B \) through its definition,

\[
\mu_0 = B A^m
\]

Alternatively, at any other temperature level, the corresponding viscosity takes the place of \( \mu_0 \), whilst relationship (2.4) can be shown to reduce to:

\[
\mu = \mu_g \left( \frac{T + A}{T_g} \right)^m
\]  

(2.5b)

or,

\[
\mu = \mu_g \left[ 1 + \frac{T - T_g}{T - T_g} \right]^m
\]  

(2.5c)

2.2.2 Roelands isobaric viscosity-temperature relationship

While the aforementioned three-parameter viscosity-temperature relationships given by equation (2.3) through (2.5c) are sufficiently accurate, the simplicity offered by the two-parameter relationship of equation (2.2) is far too decisive for a convenience of analytical and numerical manipulation. Roelands [2.4] has carried out a thorough investigation to propose more accurate two-parameter viscosity-temperature relationships than the Arrhenius equation which is known to be accurate only in a high temperature regime. He has also proposed ‘rectifying’ scales for constructing viscosity-temperature charts both for temperature and pressure effects on viscosity by making use of his own two-parameter relationships. These ‘rectifying’ charts have been proved to be more suitable while analyzing the problems involved in lubrication practice than the widely used standardized ASTM kinematic viscosity-temperature chart for viscosity variation with temperature and Sanderson’s [2.5a, 2.5b] kinematic viscosity-pressure chart which is known to yield erroneous results especially in the range of low pressures.

The aforementioned rectifying charts have been prepared from his two-parameter equation in which a graph showing the viscosity-temperature relationship of a liquid would reduce, with a reasonably good approximation, to a straight line. For constructing a rectifying chart the condition that an n-parameter relationship should satisfy can be laid down as follows.

Let the basic, original relationship with n parameters, \( q_1, q_2, ..., q_n \), be represented by,

\[
\mu = \phi \left[ q_1, q_2, ..., q_n \right]
\]  

(2.6a)

Then the correct condition for rectifiability of this relationship is such that it proves reducible to an algebraically equivalent form as follows,

\[
E(\mu) = C(q_1, q_2, ..., q_n) + D(q_1, q_2, ..., q_n) F(T)
\]  

(2.6b)

where the functions \( E(\mu) \) and \( F(T) \) should be derived, if possible, from the basic relationship such that they are entirely free from the parameters \( q_1, q_2, ..., q_n \). These two rectifying functions (or any other two functions that are linearly related to \( E \) and \( F \), respectively) can then be used as the rectifying scales, i.e. as the coordinates of the rectifying chart to be constructed at any prescribed pressure. It may be realized that the smaller the number of parameters, such as only two, the better are the possibilities for rectification.

Such rectifying charts when prepared for given typical two-parameter equations would facilitate, in the first place, a laboratory use to directly read the viscosity at any required temperature; secondly, they would obviate the need to evaluate the parameters involved in the equations concerned. The charts devised by Roelands cover temperatures ranging from -80 °C to 400 °C. They have been based on his isobaric viscosity-temperature relationship at any prescribed pressure, namely:

\[
\log \mu = log \mu_0 + \left( \frac{G}{T + 100} \right)^3
\]  

(2.7a)
or, logarithmized:

$$\log(\log(u + 1.200)) = - S \log\left(1 + \frac{T}{135}\right) + \log C$$

(2.7b)

where \(C\) and \(S\) are two dimensionless parameters; this number of parameters proves to be absolutely minimum for characterizing the viscosity-temperature relationship of an arbitrary liquid. On these charts, the downward slope of viscosity-temperature lines when plotted is numerically equal to the parameter \(S\) in equations (2.7a) and (2.7b). The parameters \(G\) and \(S\) are replaced by \(G_0\) and \(S_0\), respectively, whenever they are prescribed at atmospheric pressure in the analysis to follow.

While Roelands' two-parameter viscosity-temperature relationship has been proved both easy to use than three-parameter relationships and accurate in a wide range of temperatures, it gives rise to rather involved evaluations of thermal effects on lubricant films. Although the numerical analyses would not be so much jeopardized, the analytical treatment involved would, however.

### 2.3 Referential Invariance of Viscosity-Temperature Relationships

Certain kind of functions describing the dependence of the material property, i.e. viscosity, on the state variable, i.e. temperature, have been proved preferable in the theory of the physical process constituted by thermo-elasto/plastohydrodynamic lubrication. The use of such functions simplifies the theory in that the number of dimensionless quantities required for a complete description and evaluation will be reduced.

Certain concepts concerning the abovementioned functions, also called "material functions" have been introduced by Pawlowski [2.6]. A study of these concepts has recently been carried out by Blok [2.7] to show that Slote's relationship indeed belongs to the group of aforementioned material functions. In his book [2.6] Pawlowski introduces the concepts of "the standard representation" of "material functions" and that of "the referential invariance" of such a representation. By definition a material function, \(\mu(q)\), describes the variation of some material property, \(\mu\), with a state variable, \(q\), such as temperature or pressure. Such a function may be postulated from theoretical physics, for instance from the theory of the viscosity of liquids, or it may be known empirically through fitting to the measured trend of the variation concerned. Here below Pawlowski's aforementioned concepts will be illustrated for the variation of viscosity with temperature \(T\) as it is reflected by the material function \(\mu(T)\) for a viscosity isobar at some given pressure such as the atmospheric one. It is left to the reader to apply similar considerations to viscosity isotherms, or say to material functions \(\mu(p)\) which reflect the variation of viscosity with pressure \(p\) when temperature is fixed at some prescribed level. The considerations to follow are meant to show that Pawlowski's aforementioned concepts may be used to advantage as a basis to the development of the theory of the physical process and system that is constituted by thermo-elasto/plastohydrodynamic lubrication and the thermo-elasto/plastohydrodynamic lubricant film, respectively.

#### 2.3.1 Standard Representation of Viscosity Isobar

Pawlowski's "standard representation" of a viscosity isobar \(\mu(T)\) is characterized by the following transform. First, choose some reference temperature, \(T_0\). Then introduce two new variables, \(w\) and \(u\), which both are dimensionless, viz.:

$$w = \frac{\mu(T)}{\mu(T_0)} = \frac{\mu}{\mu_0}$$

(2.8)

and

$$u = \frac{\mu(T)}{\mu(T_0)} - \frac{1}{\mu_0} \left[\frac{\mu(T_0)}{\mu_0}\right] = \frac{\mu(T)}{\mu_0} - \frac{1}{\mu_0}$$

(2.9)

where

$$-\frac{\partial \mu(T)}{\partial T} = \frac{1}{\mu_0} \left[\frac{\mu(T_0)}{\mu_0}\right] = \frac{\mu_0}{\mu(T_0)}$$

(2.10)

denotes the viscosity-temperature coefficient of the given oil at the reference temperature \(T_0\) and at the pressure considered, the minus sign having been introduced to make \(B_T\) positive. Accordingly, definition (2.9) may be rewritten as follows,

$$u = B_T(T-T_0)$$

(2.11)

The transform of the original material function, \(\mu(T)\), through the new variables \(u\) and \(w\) will result in a new function, \(\phi(w)\), viz.:

$$w = \phi(u)$$

(2.12)

which, in view of definitions (2.8) and (2.9), is tantamount to,

$$\frac{\mu(T)}{\mu_0} = \phi\left[B_T(T-T_0)\right]$$

(2.13)

Of course, the function \(\phi\) can be determined analytically when the material function \(\mu(T)\) has been postulated theoretically. But in case the latter function is known only empirically from viscometric measurements, \(\phi\) may also be determined graphically.

In either case, Pawlowski calls the function \(\phi\) the "standard representation" of the material function. Further, it is worthy of note that the new variables, \(u\) and \(w\), are not only dimensionless but also normalized in that they impose the following two conditions upon the standard representation, \(\phi\), at the point corresponding with the standard referen-
This characteristic feature may be employed usefully, for instance for checking whether some experimentally assessed viscosity-temperature curves that, although having the point $u=0$ in common, yield $w$-curves that, although having the point $w=0$ at $u=0$ and their slope at that point in common (see definitions (2.14a) and (2.14b)), show trends that differ with the $T_r$-value chosen.

But Pawlowski has been able to show that there is one class of material functions that is "referentially invariant", namely in that the aforementioned transform through the new variables $u$ and $w$ (see definition (2.3) and (2.8), respectively) yields one single "standard representation". So, this representation then is unique in that it is invariant towards the choice of the reference state variable, which here is the standard reference temperature $T_r$. In other words, then the corresponding $u$-$w$-curve is also unique for the material function concerned. This characteristic feature may be employed usefully, for instance for checking whether some experimentally assessed viscosity-temperature curve is referentially invariant or not. Indeed, one may then choose different reference temperatures so as to find out whether the successive curves depicting the corresponding standard representations coincide, at least to a reasonable approximation and in a sufficiently wide range of temperatures and viscosities.

Let us start with the class of referentially invariant standard representations $\sigma(u)$, so as later to deal with the corresponding class of referentially invariant material functions. The former class is specified by the following definition of Pawlowski,

$$w = \phi(u) = (1 - \frac{1}{n})^n$$

where $n$ may assume any finite value, either positive or negative. In the limiting case of $n$ attaining infinitely large values, in accordance with the classical definition of the base number $e=2.7183$ of the natural logarithms, the standard representation given by equation (2.15) reduces to the exponential relationship, viz.,

$$w = \phi(u) = \lim_{n \to \infty} (1 - \frac{1}{n})^n = e^u$$

The present quantity $u$ is the opposite of the one defined by Pawlowski in that the sign has been reversed so as to make $\beta$ positive. This may explain why the sign of the second term in expression (2.16) is opposite to that of Pawlowski's corresponding relationship Blok has described in his work. Indeed, Pawlowski was able to prove that there exist no standard representations other than those specified by definitions (2.15) and (2.16), and that in his sense would also be referentially invariant. Now, in the present case of viscosity-temperature relationships $\mu(T)$, the corresponding class of referentially invariant material functions may in general be defined, in accordance with additional findings of Pawlowski, by the following tri-parametric class,

$$\mu(T) = \lambda (c + c \cdot T)$$

which includes, as a limiting case for $n=0$, the biparametric exponential relationship,

$$\mu(T) = \lambda \exp (a \cdot T)$$

Further it may be seen that the following basic form of Slotte's relationship already given by equation (2.4), viz.,

$$\mu(T) = \beta (T + A)^{-n}$$

can be algebraically rearranged to Pawlowski's general form (2.17), or rather to his basic form (2.15). In fact, Slotte's relationship can be reduced by replacing $n$ with $-m$.

$$u = (1 + \frac{1}{n})^{-n}$$

which is valid for any reference temperature $T_r$ in conjunction with the corresponding reference viscosity $\mu_r$.
terms of the original variables for the viscosity isobar, viz.,

\[ \frac{v_r}{v_r'} = \left[1 + \frac{8}{m} (1-T_r') \right]^m \]  \hspace{1cm} (2.19b)

where from definition (2.10),

\[ \beta_r = - \frac{\partial \mu}{\partial T} \]  \hspace{1cm} (2.20)

It can be shown that two limiting cases, i.e. hyperbolic and exponential relationships, result when \( m \) is prescribed with unity and infinitely large value, respectively. Referential invariance of the same limiting cases is also self-evident.

The standard representation of Slotte's relationship given by equation (2.19a) has been shown in Figure 2.1. There the variation of the material function \( \mu(T) \) with the state variable \( T \) has been depicted making use of the variables \( w \) and \( u \) given by expressions (2.8) and (2.9). A family of curves has been depicted in the same figure for various values of Slotte's exponent \( m \) in the range \( 1 \leq m \leq 6 \), covering at least at the atmospheric pressure, most of the liquids used in lubrication practice. At increased pressures, however, the \( m \)-values for the same liquids are likely to be high, and the corresponding curves if plotted in Figure 2.1 would be less curved than that given by \( m=6 \). The limiting case of \( m=\infty \), representing the exponential relationship gives rise to a straight line in the same figure.

Further the set of the three parameters \( A, B \), and \( m \), which viscometrically specify a given oil at the pressure to be considered, can be evaluated from three or more viscosities measured at an equal number of different temperatures. Of course, if more than three viscosities should be available, regression analysis may be applied for determining the statistically most significant values of \( A, B \), and \( m \). So, Slotte's viscosity-temperature relationship may be considered a referentially invariant one in Pawlowski's sense. In passing it may be noted that Slotte's set of three parameters, \( A, B \), and \( m \) may be expressed in terms of that of Pawlowski, \( c_1, c_2, \) and \( m \).

The referential invariance of Slotte's viscosity-temperature relationship is the major reason why this relationship, including the exponential one as a limiting case, has been chosen as the material function in the theory for the present physical process, thermo-elastohydrodynamic lubrication. Indeed, as demonstrated also by Pawlowski, the use of referentially invariant material functions simplifies such theories in that the number of dimensionless groups required for a complete description and evaluation may be reduced as compared with that required when using other kinds of material functions.

### 2.4 COEFFICIENT \( \beta \) AND SLOTTE'S PARAMETERS \( m \) AND \( A \)

Parameters \( m \) and \( A \) in Slotte's relationship (2.7) are very difficult to determine as they are contained implicitly in the equation. Among several methods of evaluation, the method explained below proves to be quite accurate. This consists of making use of the viscosity-temperature coefficient in its general form, i.e. \( \beta \), at the two extremes of the temperature range of interest for any given oil and at the pressure concerned. In material functions of referentially invariant nature as proposed by Pawlowski, it has been observed that the viscosity-temperature coefficient \( \beta \) has certain important significance. This coefficient can be defined in its general form by,

\[ \beta = - \frac{1}{\rho} \frac{\partial \mu}{\partial T} + \frac{S_0}{\rho} \]  \hspace{1cm} (2.20)

where the minus sign has been introduced to make \( \beta \) positive, and where \( S_0 \) stands for the natural logarithm. Since \( \beta \) denotes the fractional change of viscosity per unit of temperature rise, it has the dimension of reciprocal temperature.

![Figure 2.1](image-url)  
Standard representation of Slotte's relationship for the variation of the material function \( \mu(T) \) with the state variable \( T \) in terms of \( w \) and \( u \), respectively. The family of curves corresponds to Slotte's exponent \( 1 \leq m \leq 6 \) and \( m=\infty \), the last representing the exponential relationship. Each of the curves is referentially invariant for its corresponding \( m \)-value.
2.4.1 Relationship between the parameters \( \beta_g, \beta, \) and \( A \)

Now it follows from Slotte's relationship given by expression (2.4) that,

\[
\beta_g = \frac{m}{T_g A} \tag{2.21}
\]

On the other hand, temperature coefficient \( \beta_R \) as it follows from Roe-land's relationship (2.7a) can be assessed at any prescribed pressure as,

\[
\beta_R = 2.303 \times \frac{\log_10(1.200)}{T + 133} \tag{2.22}
\]

First of all the dependence of the coefficient \( \beta_g \) on viscosity \( \mu \) at any temperature \( T \) is apparent when expression (2.21) is rearranged. To start with, Slotte's relationship (2.5b) can be expressed in the form:

\[
\left[ \frac{\mu}{\mu_0} \right]^m = \frac{T/A}{T_g/A} \tag{2.23}
\]

which when substituted in equation (2.21) yields,

\[
\beta_g = \frac{m}{T_g A} \left[ \frac{\mu}{\mu_0} \right]^m \tag{2.24a}
\]

It is now seen that \( \beta_g \) varies with the viscosity \( \mu \) while it is a function of the viscosity grade \( \gamma_g \), the grade temperature \( T_g \) and Slotte's parameters \( m \) and \( A \).

One may further relate explicitly Slotte's exponent \( m \) with \( \beta_g \) when the latter is given at any two temperature levels say \( T_1 \) and \( T_2 \), covering a wide temperature range. Indeed expression (2.23) yields,

\[
\beta_{g1} = \frac{m}{T_g A} \left[ \frac{\mu}{\mu_0} \right]^{m_1} \tag{2.24a}
\]

and,

\[
\beta_{g2} = \frac{m}{T_g A} \left[ \frac{\mu}{\mu_0} \right]^{m_2} \tag{2.24b}
\]

from which it follows that,

\[
\frac{\beta_{g1}}{\beta_{g2}} = \left( \frac{\mu}{\mu_0} \right)^{m_1}
\]

so that,

\[
\ln \left( \frac{\mu}{\mu_0} \right) \frac{1}{m_1} = \ln \left( \frac{\mu}{\mu_0} \right) \frac{1}{m_2}
\]

On similar lines one may also obtain Slotte's coefficient \( A \) in terms of the coefficient \( \beta_g \) and the viscosity at the same given two temperatures \( T_1 \) and \( T_2 \), respectively. From equation (2.21), the parameter \( A \) can be given by,

\[
A = \frac{n}{\beta_g} - T
\]

at any temperature level \( T \) and at the pressure to be considered. Thus \( A \) can be approximated over the entire temperature range covered by \( T_1 \) and \( T_2 \) in terms of the coefficients \( \beta_{g1} \) and \( \beta_{g2} \); viz.,

\[
A = \frac{n}{\beta_{g1}} - T
\]

First of all, it will be shown that Slotte's relationship (2.4), or more conveniently, its variant given by expression (2.5c) can be expressed in terms of the viscosity-temperature coefficient \( \beta_{tg} \) while the latter is specified at the grade temperature \( T_g \), namely

\[
\beta_{tg} = \frac{m}{T_g A} \tag{2.27}
\]

With the introduction of the coefficient \( \beta_{tg} \), Slotte's relationship (2.5c) then reduces to a convenient form given by:

\[
\frac{\mu}{\mu_0} = \left[ 1 + \frac{\beta_{tg}}{n} (T - T_g) \right]^m \tag{2.28a}
\]

In the present expression, however, the suffix \( g \) refers to the "grade temperature" \( T_g \), which replaces the previously employed reference temperature \( T_r \) in equation (2.10). Similarly expression (2.28a) for the viscosity-temperature relationship while re-expressed in the form:

\[
\frac{\mu}{\mu_0} = \left[ 1 + \frac{\beta_{tg}}{n} (T - T_g) \right]^m \tag{2.28b}
\]

becomes identical to equation (2.19b). Therefore, equation (2.28b) can also be conceived to depict the "standard representation", i.e. equation (2.13) as suggested by Pawlowski. Correspondingly, the conditions specified on the material function, namely those given by expressions (2.14a) and (2.14b) can be made use of. In other words, the characteristic value of the coefficient \( \beta_{tg} \) at the "grade temperature" \( T_g \) can be employed for estimating with a fair accuracy the viscosity of any given oil at the required temperature within the range concerned.

It follows readily from equation (2.28b) and in accordance with the definition already provided by expression (2.16) concerning the classical definition of the base number \( e=2.7183 \) of the natural logarithms, that at extremely high values of \( m \):

\[
\frac{\mu}{\mu_0} = \frac{\beta_{tg}}{n} \left[ 1 + \frac{\beta_{tg}}{n} (T - T_g) \right]^m \tag{2.28c}
\]

where the viscosity-temperature coefficient \( \beta_g \), while satisfying its general definition given by expression (2.20), is free of the grade temperature \( T_g \).
2.4.2 Evaluation of the parameters $m$, $A$ and $\beta_a$

It has already been observed that the evaluation of the parameters $m$ and $A$ for any given oil is facilitated with the use of the viscosity-temperature coefficient $\beta$ instead of other well-known methods such as regression analysis. However, a direct evaluation of $\beta$ at any viscometrically obtained viscosity-temperature data proves to be cumbersome; the evaluation becomes straightforward while Roelands' empirically established analytical relationship is employed. Then one starts with two viscosities $\mu_1$ and $\mu_2$ known experimentally at two different temperature levels $T_1$ and $T_2$, preferably at the two extremes of the range of temperatures concerned.

Following the selection of the viscosity-temperature data at two temperatures and at the prescribed pressure, one can readily estimate Roelands' Slope Index, as well as his parameter $G$, for the oil concerned. One can further evaluate the ratio of the coefficients $B_R$ and $R_1$ at the aforementioned temperatures $T_1$ and $T_2$, as follows.

First of all, consider Roelands' isobaric viscosity-temperature relationship given by expression (2.7a), i.e.

$$\log \mu + 1.200 = \frac{S}{T + 135}$$

where the temperature $T$ is expressed in degrees centigrade and the viscosity $\mu$ in centipoise. By differentiating once and rearranging, the ratio of the coefficients $B_R$ and $B_2$ may be expressed in terms of the two viscosities $\mu_1$ and $\mu_2$ known at the two temperatures $T_1$ and $T_2$, viz.,

$$\log \frac{B_R}{B_2} = \left( \frac{H - H_j}{H - H_i} \right) \log \frac{T_1 + 135}{T_2 + 135}$$  \hspace{1cm} (2.29)

where, for convenience the function $H$, as originally introduced by Roelands, has been introduced, namely,

$$H = \log (1.200 + \log \mu)$$

Since the difference $H_i - H_j$ can be expressed solely in terms of $S$, $T_1$ and $T_2$, viz.,

$$H_i - H_j = S \log \frac{T_1 + 135}{T_2 + 135}$$

the same is true of the ratio $\frac{B_R}{B_2}$, viz.,

$$\log \frac{B_R}{B_2} = (S + 1) \log \frac{T_1 + 135}{T_2 + 135}$$  \hspace{1cm} (2.30)

Slotte's parameters $m$ and $A$ can then be evaluated at any prescribed pressure employing the ratio $\frac{B_R}{B_2}$ in the place of $\frac{\beta_1}{\beta_2}$ in expressions (2.25) and (2.26), in the form:

$$n = \frac{\log (H_i/H_j)}{\log \frac{T_1 + 135}{T_2 + 135}}$$  \hspace{1cm} (2.31)

$$A = \left( \frac{T_1 + 135}{T_2 + 135} \right)^r \left( \frac{1.200 + \log \mu_j}{1.200 + \log \mu_i} \right)$$

(2.32)

Now having established Slotte's parameters $m$ and $A$ employing the viscometrically obtained data at any two different temperature levels $T_1$ and $T_2$ and at the prescribed pressure, it is possible to provide with the coefficient $\beta_{sg}$ defined in expression (2.27) at any specified grade-temperature $T_g$. With the availability of the parameters $m$ and $\beta_{sg}$, therefore, it is possible to employ the "standard representation" of Slotte's relationship given by expression (2.28a) in the study of the physical process that is constituted by thermo-elasto/plastohydrodynamic lubrication.

Various kinds of oils used in lubrication practice, including mineral oils, fluorocarbons and other kinds of synthetic oils can be studied viscometrically through their parameters $m$, $A$ and $\beta_{sg}$, the last being defined at some specified grade temperature $T_g$.

Slope Index $S$, or more precisely its value $\beta_0$ at atmospheric pressure, is an index of "viscometric homology" which runs more or less parallel to "structural homology" in the sense of molecular structure. Indeed, for any truly structurally homologous series, $S$, shows one and the same value for all its members, thus being characteristic of the entire series considered. For instance, all the members of the series of $n$-paraffins ($n$-alkanes), at least in so far as they are liquid at atmospheric pressure, all show an $S$-value that happens to be very nearly equal to unity. More generally, for all mineral oils belonging to one single "naturally homologous" series, $S_j$ proves to have one and the same value which thus is characteristic of the entire series considered. Such groups have been defined by Roelands as those in which all the individual mineral oils of different viscosity grades have been produced from one and the same crude, or blend of crudes, and by means of similar refining procedures. For instance, the $L$-series and $H$-series of mineral oils, defined in the ASTM-standard for the kinematic viscosity index $VI$, show $S_j$-values of 1.50 and 1.23 or, in accordance with Roelands' following general definition, viscosity indices DVI of 0 and 100, respectively.

$$DVI = 220 - 7x10^{-5}$$  \hspace{1cm} (2.33)

While studying the atmospheric viscosity-temperature isobaric relationship, in accordance with the aforementioned definition, $S$ should be replaced by $S_j$ in expression (2.23). In Section 2.5 to follow, Formulae (2.31), (2.32) and (2.27) will be used for calculating Slotte's parameters $m$ and $A$, and the coefficient $\beta_{sg}$ from Roelands' parameter $S_j$. This will be done for quite a few sets of values of $S_j$ that is, for a wide variety of viscometrically homologous series of oils, and for a wide range of viscosity grades. Thus, when the variation of viscosity with temperature and pressure should have been specified through Roelands' relationship for the particular oil to be considered, this variation may then be expressed in terms of Slotte's relationship. The results thus obtained will prove useful in the later chapters where the thermal effects of the viscous-shear heat on the flow in lubricant films will be investigated in terms of the latter Slotte's relationship.
2.5 RESULTS AND DISCUSSION

2.5.1 Selection of lubricating oils for the analyses to follow

For investigating the variation of the parameters \( m \) and \( A \), and finally the coefficient \( \beta \) at a prescribed grade temperature \( T_{g} \), of various liquids used in lubrication practice, a systematic approach has been followed up. First of all, a range of the Slope Index \( S \), i.e., the coefficient \( \beta \) has been selected covering almost all liquids for their viscosity variation with temperature at any prescribed pressure. For each value of \( S \) selected in the above range at intervals of 0.25, a wide range of viscosity grades has been chosen giving rise to dynamic viscosities in the range \( 10^{-6} \text{ to } 10^{5} \) \( \text{cp} \), at intervals of decades of centipoise (\( \times 10^{2} \) \( \text{cp} \)). The latter range is representative of that covered by oils used in thermo-elastohydrodynamic lubrication, and at the temperatures and pressures there encountered in the oil films.

Following the above-mentioned analysis, a number of liquids have been selected for their parameters \( m \) and \( \beta \) at atmospheric pressure. These liquids have been selected from the ASTM-tables for Viscosity Index (VI) of mineral oils, and also from the tables of miscellaneous liquids such as paraffins, hydrocarbons, fluorocarbon oils etc. (see Roelands' [2.4] Tables VI-6, VI-7, VIII-2, VIII-4, VIII-7 and VIII-10). In addition, also water has been investigated because of its typical illustrative nature, i.e. quadratic dependence of its fluidity (the inverse of viscosity) on temperature.

The selection of Slotte's parameters \( m \) and \( A \) from equations (2.31) and (2.32), viscosities at the two temperatures have been employed. These temperatures cover a wide range that is generally encountered in lubrication practice: at the lower extreme a temperature level of 40 \( ^{\circ} \text{C} \) and at the higher extreme that of 150 \( ^{\circ} \text{C} \). The coefficient \( \beta \) has been finally evaluated from expression (2.27) at the temperature specified.

Finally the viscosity \( \mu \) at 100 \( ^{\circ} \text{C} \) has been evaluated using the previously mentioned Roelands' relationship and Slotte's relationship (2.5c), separately. The last evaluation yields a measure of the degree of fitting of Slotte's to Roelands' relationship for the given oil in the aforementioned temperature range.

2.5.2 Arbitrarily chosen Slope Indices and viscosity grades

In the present analysis numerical results for Slotte's parameters \( m \) and \( A \), as well as the coefficient \( \beta \) at some chosen definite grade temperature \( T_{g} \), have been tabulated for several viscometrically homologous series of oils, each series being characterized by its own value of Roelands' Slope Index. Further, in each series the viscosity grade as defined by the dynamic viscosity \( \mu_{40} \) at Roelands' standard reference temperature and at the prescribed pressure, has been varied, whilst the corresponding value of \( C \) has also been given in Table 2.1.

From Table 2.1 it may be observed that for specified group of oils, i.e., oils with a particular value of the Slope Index \( S \), the selected variation of viscosity grade through its dynamic viscosity gives rise to a variation of Roelands' parameter \( G \). There will correspondingly exist a variation in the parameter \( m \), and also in \( \beta \) and parameter \( A \) will remain unaffected by any change in the viscosity grade. In other words, the parameter \( A \) remains constant for any selected \( S \)-value, the specified increase of the viscosity grade being reflected through the increasing trend of the parameters \( m \), \( G \) and \( \beta \). The trend of \( m \) and \( G \) have been depicted in Figure 2.2 for the complete range of the Slope Index \( S \) and the dynamic viscosity \( \mu \) selected, the former Slope Index \( S \) being

<table>
<thead>
<tr>
<th>( S )</th>
<th>( r_{10^{6} \text{cp}} )</th>
<th>( r_{10^{5} \text{cp}} )</th>
<th>( r_{10^{4} \text{cp}} )</th>
<th>( r_{10^{3} \text{cp}} )</th>
<th>( r_{10^{2} \text{cp}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>10</td>
<td>2.8227</td>
<td>29.70</td>
<td>0.1220</td>
<td>0.0223</td>
</tr>
<tr>
<td>1.00</td>
<td>2.8227</td>
<td>29.70</td>
<td>0.1220</td>
<td>0.0223</td>
<td>1.8055</td>
</tr>
<tr>
<td>1.25</td>
<td>3.0403</td>
<td>30.66</td>
<td>0.1328</td>
<td>0.0217</td>
<td>1.8151</td>
</tr>
<tr>
<td>1.50</td>
<td>3.2470</td>
<td>31.62</td>
<td>0.1437</td>
<td>0.0211</td>
<td>1.8246</td>
</tr>
<tr>
<td>1.75</td>
<td>3.4466</td>
<td>32.63</td>
<td>0.1545</td>
<td>0.0205</td>
<td>1.8339</td>
</tr>
<tr>
<td>2.00</td>
<td>3.6420</td>
<td>33.64</td>
<td>0.1653</td>
<td>0.0199</td>
<td>1.8430</td>
</tr>
<tr>
<td>2.50</td>
<td>3.9200</td>
<td>34.64</td>
<td>0.1813</td>
<td>0.0183</td>
<td>1.8600</td>
</tr>
</tbody>
</table>

Table 2.1 Slotte's parameters \( m \) and \( A \) evaluated using viscosities at temperatures 40 \( ^{\circ} \text{C} \) and 150 \( ^{\circ} \text{C} \), and the coefficient \( \beta \) at the grade temperature \( T_{g} \). Also included are the fitted dynamic viscosities at 100 \( ^{\circ} \text{C} \), using Roelands' and Slotte's relationships. Slope Index \( S \) and the viscosity grade have been arbitrarily chosen to cover most of the liquids employed in lubrication practice.
Figure 2.2 Variation of Slotte's parameter $m$, Roelands' parameter $G$ and the coefficient $3_s$ at the grade temperature $T_g=40\,^\circ\text{C}$ for a few selected values of the Slope Index $S$. Viscosity grades have been arbitrarily chosen to cover most of the liquids employed in lubrication practice.

Straight lines relating $m$ to $\mu$ are only scarcely spaced at higher $S$-values, viz. oils having ASTM Viscosity Index VI close to zero or Roe-lands' Dynamic Viscosity Index DVI negative values. For the oils given by high $S$-values, therefore, the criterion concerning "naturally" homologous behaviour becomes unimportant; however, the criterion concerning the viscosity grade remains to play a dominant role. In other words, for such oils the effect of pressure on $m$-values remains important through the viscosity grade concerned, but the group nature becomes only of minor concern. Thus the above-mentioned trend of $m$-values might mean certain simplicity in the analysis of hydrodynamic lubrication, since the data obtained for any one of the oils in the range of high $S$-values can readily be used with a reasonably good accuracy for other oils in the same group. Then there remains only one important characteristic behaviour of the given oil to distinguish from the others, namely the viscosity grade, which is conceived to be solely representative of the effect of pressure on any given oil.

Finally the last coefficient $3_s$, which depends on $m$, $A$ and the grade temperature $T_g$, as given by expression (2.27) depicts a similar trend as that of $m$ and $G$. The former coefficient $3_s$ is also somewhat uniformly spread, in a way similar to $G$, over the range of $S$-values considered in obtaining Figure 2.2. It may then be concluded that the parameters $3_s$, $m$ and $A$ can as effectively be used as Roelands' Slope Index $S$ and parameter $G$ to represent the viscosity variation with temperature $T$ of any liquid at any prescribed pressure. It may also be noted that any two of the above-mentioned three parameters would be sufficient to evaluate the viscosity variation with temperature for the oil specified through the viscosity grade $\mu_g$ at the grade temperature $T_g$; $m$ and $A$ alone constitute Slotte's relationship, whereas $3_s$ together with $m$ provides a new relationship already given by expression (2.28a).

From its definition, Slotte's relationship is a three-parameter one, whilst Roelands' is a two-parameter one. The former Slotte's relationship, however, has been found, at least in a reasonably wide temperature range, to reduce to a two-parameter one. First of all, Slotte's parameter $A$ is found to correlate strictly with Roelands' Slope Index $S$ of the isobaric relationship, and even with $S$ at increased pressures. In fact, this involves that $A$, just like $S$, may be considered an index of "viscometric homology". Since this homology is similar to "structural homology" the index $A$ may be considered to represent the latter homology also. The fact that such an index, that is $A$, might be constructed for Slotte's relationship fairly easily seems to have escaped the recognition of the previous investigators. It has been found that the parameter $A$, like Roelands' Slope Index $S$, is also almost entirely free of the choice of the reference temperature. The correlation between Slotte's parameter $A$ and Roelands' Slope Index $S$ can be given with a fair degree of accuracy, by:

$$A = B_1 \log S + B_2,$$

where $B_1=114.21$ $\,{^\circ}\text{C}$ and $B_2=26.75$ $\,{^\circ}\text{C}$ for the complete range of $S$ given by $0.75 \leq S \leq 2.0$.

Further it has been observed that Slotte's exponent $m$ can be correlated reasonably well, at least at the prescribed pressure, with the viscosity grade of the oil chosen and, again, using Roelands' Slope Index $S$. The correlational-formula for $m$, however, turns out to be somewhat involved, viz.:

$$m = (1 - e^{7S}) (1.20 + \log \mu)$$

where $\gamma=0.146 S+0.740$.

With the introduction of the parameters $A$ and $m$, Slotte's viscosity-temperature relationship given by expression (2.5c) reduces, for an oil of viscosity grade $\mu_g$ at the grade temperature $T_g$, to the form:

$$\log \frac{\mu}{\mu_g} = (1 - e^{7S}) (1.20 + \log \mu_g) \log \left[ \frac{1.20+0.74 S}{1.20+0.74 A} \right]$$

The viscosity-temperature coefficient $3_s$, which is dependent from its definition (2.27) on $m$ and $A$, can also be expressed in terms solely of Slope Index $S$, the viscosity grade $\mu_g$ and the grade temperature $T_g$, viz.:

$$3_s = \frac{(1 - e^{7S}) (1.20+0.74 S) \log \mu_g}{1.20+0.74 S \log \mu_g}$$
With the formulation of a new approximately correlated form, that is through equation (2.36), Slotte's relationship has now reduced to a two-parameter one, and it also constitutes a referentially invariant relationship in the "standard representation" already discussed in Section 2.4. In such a representation the coefficient \( B_{SG} \) carries the significance as discussed by Pawlowski [2.6].

Several attempts have been reported also before concerning a simpler relationship having only two parameters and replacing Slotte's three parameters, at least for the most usual kind of lubricating oils, i.e., mineral ones. These attempts have yielded a common average value of the parameter \( A \), albeit to some approximation. When the temperature is expressed in degrees Fahrenheit \( T_f \), as provided by Hersey [2.8] in his book, the Slotte's relationship reduces to,

\[
u = -\frac{B_f}{T_f^{m-1.35}}
\]

where \( m \) usually assumes values from about 2.5 to 3.5 and where \( B_f \) is the viscosity at the temperature \( T =18^\circF \). If one expresses \( T \) in degrees centigrade \( T \), Hersey's approximate formula may be written in the form:

\[
u = -\frac{B_f}{(T+18)^m}
\]

where \( B_f = B_f(5/9)^m \), so that then \( A=18^\circC \). Cameron [2.9] suggests that \( A \) usually amounts to 20 to 30 \( ^\circC \), a value slightly exceeding Hersey's value of 18 \( ^\circC \) in the above expression.

From what precedes in this section, it is clear that \( A \) can vary in a wide range for various liquids, including mineral oils. For instance in the range of Roelands' Slope Index 1.75 > \( S_0 \) > 0.75, which corresponds with the range -174 < DVI < 181 for Roelands' Dynamic Viscosity Index, the \( A \)-value ranges from -1.05 \( ^\circC \) to 41.60 \( ^\circC \). So, both the constant value of 18 \( ^\circC \) and that of Cameron of about 20 to 30 \( ^\circC \) constitute only approximations that may be considered acceptable only in the DVI-range from about 100 to 150 (note that Roelands' DVI, at least in this range comes close to the ASTM kinematic Viscosity Index VI). It may thus be concluded that whilst Slotte's relationship, either in tri-parametric or in biparametric form, is employed it is preferable to employ the correct \( A \)-value which is indicative of the "viscometric homology" of the liquid concerned.

An analysis of the parameters \( m \), \( A \) and \( B_{SG} \) has been carried out also for a number of miscellaneous liquids commonly used in lubrication practice, including a few series of synthetic oils which have recently achieved a considerable importance. The data required for such an analysis has been taken from Roelands' survey as well as his experimental results. A brief discussion of the results obtained in the present study has been provided in Appendices 2.A1 through 2.A4.

2.5.3 Referential invariance of Slotte's relationship for an ASTM oil

For the purpose of illustration of the previously explained Pawlowski's referentially invariant standard representation, an ASTM H-series oil with a dynamic viscosity of 10 \( \mu\text{P} \) at the reference temperature of 100 \( ^\circC \) and at atmospheric pressure has been investigated. This oil is represented through \( VI=100 \) from the definition of the ASTM H-series, \( S_0=1.23 \) by Roelands and the corresponding DVI of 101. The same oil is also represented, as shown in Table 2.A1, through Slotte's parameters \( m=3.02 \) and \( A=15.91 \) fitted to the viscosity data that was estimated using Roelands' isobaric relationship at 40 \( ^\circC \) and 150 \( ^\circC \) and employing the aforesaid value of \( S_0=1.23 \).

The standard representation of the material function \( \mu[T] \) with the state variable \( T \) has been depicted in Figure 2.3 in terms of the variables \( w \) and \( u \) given by expressions (2.8) and (2.9). In order to cover a wide range of temperatures, and thus \( u \)-values, the coefficient \( B_{SG} \) has been selected at two sufficiently widely displaced temperatures of 40 \( ^\circC \) and 150 \( ^\circC \) with the help of Slotte's and Roelands' relationships separately. The corresponding variation of \( w \) has been evaluated through es-
timating the viscosity at various levels of temperatures chosen and at the pressure concerned, and again using Slotte's and Roelands' relationships separately. The results thus displayed in Figure 2.3 cover temperature levels in the range of about -10 °C to 200 °C and the corresponding viscosities about 0.02 to 100 times the grades viscosities chosen.

It may be observed from Figure 2.3 that Roelands' isobaric relationship belonging to the group of two-parameter relationships and being fairly accurate in a wide range of temperatures gives rise to two different w-values at any chosen u-value different from zero: the farther the u-value from zero the higher is the difference in w-values. On the contrary, Slotte's relationship yields, in the complete range of u-values investigated, one and the same w-value for the two different coefficients $S_{0,2}$ chosen at 40 °C and 100 °C. This trend of w-values has been established previously to display the referentially invariant nature of the relationship concerned. It may thus be concluded that at least for the present oil belonging to the ASTM H-series, Slotte's relationship proves to be referentially invariant. A similar investigation using other oils belonging to ASTM L-series, or to any other group would yet depict the aforementioned referentially invariant nature of Slotte's relationship.

2.5.4 Mineral oils used in the present investigation

With a view of studying thermal effects on the flow of lubricating oils, mineral oils have been selected for a test on the Visco-Expert viscometer developed at the University of Technology at Delft. Requirements set forth in selecting the above oils consisted of a high viscosity grade accompanied by a high sensitivity of the viscosity to temperature. These requirements were met by Shell Vitrea series which are free of VI-improvers (polymeric compounds), but consisting only a limited amount of oxidation inhibitors. The Vitrea oils are also known to remain Newtonian for shear stresses up to at least $5 \times 10^3$ N/m².

The previously mentioned parameters $S_{0,2}$ and $m$ have been evaluated, in addition to Roelands' parameters $S_0$ and $G_0$, for three of the Shell Vitrea oils belonging to the highest viscosity grade. In order to do this, the viscosity was measured at three or four different temperatures in the range of about 17 °C to 75 °C at atmospheric pressure using "Ub-mode" type (capillary) conventional viscometer. The experimentally obtained viscosity values have been analysed on lines similar to those presented in the previous sections. The range of temperatures used in such an analysis has been confined to a rather narrow one of 20 °C to 75 °C. Nevertheless, this range is sufficiently wide to give an accurate estimation of the viscosity in the operating regime covered by the experiments on non-isothermal flow yet to be explained in Chapter 7. The results derived as to the required parameters have been listed in Table 2.2.

It may be noticed from Table 2.2 that the test oils, although belonging to the same group as to their type, covered viscosity grades which were widely spread. Oils denoted by IA and IB fell in the lowest, while 2A and 3B in the highest, viscosity grades. Correspondingly, the evaluated Roelands' parameter $G_0$ indicates a variation between 4.4 to about 5.2. The sensitivity of their viscosity with temperature can be evaluated on the basis of either the coefficient $G_0$ or Slotte's exponent $m$; it is far too evident that the same can be done on the basis of Roelands' Slope Index $S_0$ as well. The aforementioned oils prove to be quite sensitive as is evident from rather high values of $m$ and $S_0$, although only in a somewhat narrow range of temperatures as compared to that used in the analyses already presented in the preceding sections. The sensitivity of the oils is also evidenced by the high values of the Slope Index $S_0$ (cf. Appendix 2.4 for mineral oils where $S_0$-values barely exceed those of the present test oils). Although all the test oils belong to the Shell Vitrea series, their $m$ and $G_0$ values indicate an increased sensitivity of the viscosity variation with temperature for a higher viscosity grade; cf. oils IA and IB. Their $S_0$-values show only a small variation, and $A$-values a negligible variation. For the sake of comparison of the accuracy of the viscosity data obtained using Slotte's relationship with that using Roelands', the dynamic viscosity has been evaluated at the temperature of 40 °C, i.e. that falling within the range for which the parameters $m$ and $A$, the coefficient $G_0$, and the Slope Index $S_0$, were derived. There exists a reasonably good agreement between the results thus obtained as shown in Table 2.2.

### Table 2.2 Slotte's parameters $m$ and $A$, and the coefficient $S_{0,2}$ at the grade temperature $T_g=20$ °C in addition to Roelands' Slope Index $S_0$, and his parameter $G_0$, for the test oils belonging to Shell Vitrea mineral oil series. Also included are fitted viscosities at 40 °C using Roelands' and Slotte's relationships.

<table>
<thead>
<tr>
<th>Test oil</th>
<th>Viscosity, cp at 40°C</th>
<th>$S_{0,2}$</th>
<th>$G_0$</th>
<th>$m$</th>
<th>$A$</th>
<th>Viscosity, cp at 40°C using relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA</td>
<td>329.14</td>
<td>10.25</td>
<td>1.30</td>
<td>6.4474</td>
<td>0.0713</td>
<td>3.9060</td>
</tr>
<tr>
<td>1220.10</td>
<td>1250.26</td>
<td>1.30</td>
<td>5.1723</td>
<td>0.0851</td>
<td>4.6469</td>
<td>34.25</td>
</tr>
<tr>
<td>IA</td>
<td>1412.30</td>
<td>52.73</td>
<td>1.31</td>
<td>5.4195</td>
<td>0.0860</td>
<td>4.9406</td>
</tr>
<tr>
<td>IB</td>
<td>311.12</td>
<td>26.95</td>
<td>1.28</td>
<td>4.6105</td>
<td>0.0727</td>
<td>3.9060</td>
</tr>
<tr>
<td>1032.78</td>
<td>1054.69</td>
<td>1.30</td>
<td>5.0372</td>
<td>0.0816</td>
<td>4.5367</td>
<td>35.15</td>
</tr>
<tr>
<td>IB</td>
<td>1250.92</td>
<td>51.17</td>
<td>1.30</td>
<td>5.1318</td>
<td>0.0895</td>
<td>4.8507</td>
</tr>
</tbody>
</table>

2.6 VISCOSITY-PRESSURE RELATIONSHIPS

There exist a number of isothermal viscosity-pressure relationships, i.e. at a constant temperature. In essence, these relationships are quite similar to the previously discussed isobaric viscosity-temperature relationships, i.e. at a constant pressure. The former relationships generally consist of two or more parameters in addition to the viscosity at atmospheric pressure as one of the parameters. It can be shown that the number of parameters contained in these relationships is reduced by one as compared with the number of parameters in the analogous viscosity-temperature relationships. Such a reduction may be attributed to the characteristic behaviour of the former relationships, viz. they are aimed at correlating the viscosity at various pressures while the viscosity at the atmospheric pressure is already specified.
to be similar in its characteristic behaviour with Barus simple exponential function, namely:

$$\log \mu = \log \mu_0 + 1.200 \left(1 - \frac{T}{2000} \right)^2$$

(2.40)

where $\mu$ and $\mu_0$ denote the dynamic viscosity at the gauge pressure $p$ and the latter at atmospheric pressure, both being defined at the same temperature. $\alpha$ is the characteristic pressure-coefficient of viscosity which would depend solely on temperature, but not on pressure. By comparing the above-expression with expression (2.2), it may be noticed that the former Barus equation is identical with the latter Arrhenius equation relating the viscosity with temperature in that they do not require a reference variable: a reference pressure in equation (2.39) in a manner similar to a reference temperature in equation (2.2) can be dispensed with.

Barus equation (2.39) has proved to be reasonably accurate at low and moderate pressures. This seems to be evidenced by only a small curvature of log $\mu$-p isotherms. At high pressures the above-mentioned nearly straight lines become increasingly curved, and thus the accuracy of the viscosity-pressure relationship, although at any constant temperature, becomes somewhat limited. Being well aware of these limitations, especially at high pressures, Barus equation has been employed, in the mathematical analysis to follow, on non-isothermal, non-isoviscous flows of lubricating oils, mainly due to its mathematical simplicity.

The aforementioned Roelands' viscosity-pressure relationship at any given temperature is similar in its form with his viscosity-temperature relationship already given by expression (2.7a). The former Roelands' relationship involves a dimensionless quantity $S$ which is characteristically similar to his Slope Index $S$ in the latter relationship. His empirical relationship has already been proved to be far more accurate than the other known relationships for lubricating oils, and even for many other liquids. This relationship has been given by:

$$\log \mu = 1.200 \left(1 - \frac{T}{2000} \right)^2$$

(2.41)

where $\mu$ and $\mu_0$, denote, as before, dynamic viscosities at the gauge pressure $p$ and at atmospheric pressure, respectively, both being referred to some reference temperature $T_0$. Being well aware of the non-isothermal, non-isoviscous, flows of lubricating oils subjected to even moderately high shear stresses, the relationships concerned vary in their accuracy as to the estimation of viscosity with temperature, and their accuracy is not related to the simplicity with which they have been represented. Exponential relationship, although it offers simplicity, has been proved to be neither accurate in a wide temperature range nor applicable to liquids other than simple ones, viz. lubricating oils containing even very small amounts of additives to enhance their lubricating property. The well-known, especially for a high accuracy, Roelands' isobaric viscosity-temperature relationship proves to be rather inconvenient for using in the analytical treatment of problems involved in non-isothermal, and even in non-isoviscous, flows of lubricating oils.
Slotte's viscosity-temperature relationship, being a three-parameter one in its general form, belongs to the referentially invariant type, and it also possesses certain inherent advantages because of its type. This relationship is more general than many other relationships including the exponential one, and in fact the last relationship proves to be only a special case of Slotte's relationship. Further, the three-parameter Slotte's relationship is also reducible to a two-parameter one; then the latter offers certain additional mathematical simplicity in the analysis of lubricant flows.

Various liquids that are commonly employed in lubrication practice can accurately be represented using Slotte's relationship at any prescribed pressure. These liquids can be characterized for their viscosity variation with temperature by Slotte's exponent \( m \), which has been found to lie at atmospheric pressure in the range covered by \( 1 < m < 5 \). The same liquids can also be additionally characterized by Slotte's coefficient \( A \), which, just like Roelands' Slope Index \( S_0 \), may be considered an index of "viscometric homology"; this homology runs more or less parallel with "structural homology". For any structurally homologous series of oils, therefore, \( A \) shows one and the same value for all its members. This importance of the coefficient \( A \) in Slotte's relationship as an index for a series of oils seems to have escaped recognition in the past.

For analytical convenience, the viscosity-temperature coefficient \( g_s \) has been derived in terms of the aforementioned Slotte's parameters \( m \) and \( A \), and the prescribed grade temperature. Then Slotte's viscosity-temperature relationship can be formulated in terms of the coefficient \( g_s \) and the exponent \( m \), while dispensing with the coefficient \( A \). The resulting relationship has been proved to be a convenient one in the analysis intended for carrying out in the chapters to follow. It has also been shown that this new relationship reduces to the exponential one whenever \( m \) attains extremely high values; then \( g_s \) becomes equivalent to the well-known coefficient \( S \) in the exponential relationship.

Finally, the parameters \( g_s \) and \( m \) have been evaluated for various lubricating oils, such as hydrocarbons, fluorocarbon oils, mineral oils, etc., which belong to various viscosity grades. These parameters are believed to depict a useful information to tribologists, namely the sensitivity of various lubricating oils for their viscosity dependence on temperature.

2.8 REFERENCES


Chapter III

NON-ISOTHERMAL SHEAR FLOWS WITH THE EXPONENTIAL $u - T$ RELATIONSHIP

3.1 INTRODUCTION

Lubricant flow problems in thermo-elasto/plastohydrodynamic films can often be approximated in terms of the two basic patterns of flow involved, i.e. the one of shear flow and the other of pressure flow. These two basic patterns of flow in combination give rise to the most general type of flow. The basic as well as general patterns have all to be analysed through obtaining solutions to the coupled hydrodynamic and energy equations.

The analysis to follow, although confined to plan-parallel shear flow, is also applicable to shear flow patterns in individual cross-sections of the above-mentioned films. One may then have to conceive the successive cross-sections to be so represented separately, and accordingly to use local values of influential quantities such as velocity, viscosity and other material properties.

Certain important features of the thermally affected, i.e. non-isothermal, shear flows established in recent years with the help of refined mathematical analyses have hardly covered realistic representations of lubricant flow pattern. The following is a brief survey of such features which emphasises the importance of the accurate representation of influential quantities concerned with the flow: firstly, viscosity of the lubricant as thermally affected by viscous dissipation; secondly, realistic thermal boundary conditions at surfaces that enclose the lubricant film.

Early analyses of shear flows achieved solutions to momentum and energy equations uncoupled through assuming a constant viscosity for the liquid employed. For thermally steady states and in the absence of liquid convection, the energy equation could then readily be solved. With the introduction of the viscosity dependence on temperature the mathematical analysis became more involved. Nevertheless the solutions were obtained in a closed form by Nahm [3.1], Hagg [3.2] and Davis and Lawrence [3.3] for liquids whose viscosity varies with temperature according to the exponential relationship. Thermal boundary conditions assumed in their investigations consisted of two particular types: one, where both walls absorb equal amounts of heat at uniform and equal temperatures; the other, where one wall absorbs all the heat at some uniform temperature whilst the other is adiabatic. Results for a wide range of more general thermal boundary conditions which include the aforementioned particular ones have been obtained by Blok [3.4, 3.5]. An extension to his results [3.6] will, however, be given in the present chapter. It is known from the aforementioned published work [3.1, 3.2, 3.3] that at least in a certain range of thermal boundary conditions there exist two solutions for velocity and temperature distributions for any shear stress below a finite limiting value, and no solutions at all above the latter value. First of the two solutions oscillates correspondingly to that of the isoviscous flow under similar operating conditions whilst the other deviates considerably with any decreased shear stress.

The velocity and temperature distributions can also be obtained in terms of sliding velocity [3.3, 3.5, 3.6]. These solutions are single-valued in terms of an imposed sliding velocity; but they display the aforementioned double-valuedness in terms of an imposed shear stress. Further, the shear stress and the sliding velocity can also be related to each other as shown in Figure 3.1.

It may be observed from Figure 3.1 that beyond the transitional sliding velocity any further increase in sliding velocity will result in a monotonically decreasing shear stress for non-isothermal flows. The two branches of the shear stress behaviour, which join at the transition point, can be termed the first and the second branch. Whilst the rising trend of the first branch is similar to that for the isoviscous flow, the falling trend of the second branch can be attributed to the strong influence of viscous dissipation effects: an increase in sliding velocity results in an increase in shear rate which is more than offset by a reduction in viscosity, thereby decreasing the shear stress. There prevails a fairly general agreement on the existence of the second branch of the solution. But there exist only limited data towards the stability of flow along this branch while an extensive amount is available on the stability along the first branch: see Vanderborck and Platten [3.7] and Sukanek et al. [3.8].

All the aforementioned published analyses, excluding that originally worked out by Blok [3.4, 3.5], are confined to the previously mentioned two particular types of thermal boundary conditions. It is the object of the present investigation to delve into the shear stress behaviour with a variation of the sliding velocity for more general thermal boundary conditions specified at the bounding surfaces. The present chapter will deal also with the kind of thermal instability suggested by Blok [3.5] as well as that investigated by Vanderborck and Platten [3.7] and Sukanek et al. [3.8]. Although no new computations have been carried out to extend the latter kind of stability study for the aforementioned

Figure 3.1 Shear stress as a function of sliding velocity.
more general thermal boundary conditions, the results available for one particular type will be made use of to explore, at least in a qualitative sense, the regions of instability.

The present analysis also provides a basis to investigating plastohydrodynamic films in Chapter 4 and elastohydrodynamic films in Chapter 11, in particular as to the possibility of excessive generation of heat and, if so, under what specific operating conditions. The consequent friction and flow calculations aim at an insight into power loss, and at directives towards material selection and optimal cooling of lubricated surfaces such as in bearings and on gear teeth. These calculations are also believed to provide useful information for designing dies for increased life in wire drawing and metal extrusion processes.

3.2 MATHEMATICAL MODEL

The present study deals with a flow analysis general enough to represent the quasi-shear flow of elastohydrodynamic films in sliding contacts of bearings as well as plastohydrodynamic films in extrusion or wire drawing processes. We will first of all recall the basic equations for momentum and for heat transfer, and supplement these by some important considerations.

3.2.1 Assumptions

We first list the assumptions supplementary to those in Chapter 1. The assumptions that relate to the fluid flow are:

1. The flow is in the steady state.
2. The flow constitutes mainly of pure shear type, characterised by the constancy of the shear stress $\tau$ across the film, so that:

$$\frac{d\tau}{dy} = 0$$

The assumptions that correspond to the material and to the type of heat transfer are:

1. The viscosity can be represented by an exponential type relationship,

$$\mu = \mu_0 e^{-\alpha(T-T_0)}$$

where $\mu_0$ is the 'reference viscosity' taken at some standard reference temperature $T_0$ and at the pressure in the film cross-section concerned.

2. The dominant mechanism of heat development in the film is that by viscous shear which per unit volume and per unit time is represented by,

$$\dot{q}_v = \tau \left(\frac{du}{dy}\right) - \mu \left(\frac{du}{dy}\right)^2$$

Newtonian behaviour, i.e. linear shear stress dependence on shear strain, for all commonly used lubricants is not in general applicable at the excessive shear stresses that may be encountered in the film. This may constitute an important objection since the present analysis concentrates upon excessive sliding speeds and shear stresses leading to the thermal instability. Even though substantial data are provided in recent publications, e.g. Johnson and Tevaarwerk [3.9]. They do not yield any deterministic criterion for specifying the range of linear and non-linear dependence of shear stress to shear strain. Lacking concrete information I have chosen those lubricants that show reasonably Newtonian behaviour, at least in the range of shear stress that are covered in the present analysis. In any case the present analysis will yield a 'Newtonian' basis for comparison in later work.

3.2.2 Governing energy equation

Governing energy equation for a thermally steady state flow can, with due consideration of the afore-mentioned assumptions, be given by,

$$\kappa \frac{d^2T}{dy^2} + \dot{q}_v = 0$$

Boundary conditions can be expressed in a general form to account for a later consideration of boundary values for temperature and velocities which may lead to either a stable or an unstable mass and heat transfer, see Figure 3.2. They may be specified as:

$$y=0; u=u_1, T=T_1 \text{ and } -k \frac{dT}{dy} \mid _1 = \phi_1$$

$$y=h; u=u_2, T=T_2 \text{ and } -k \frac{dT}{dy} \mid _2 = \phi_2$$

where suffixes 1 and 2 refer to surface 1 and 2 respectively. The sign for the temperature gradient conforms with the usual convention, namely, that the positive direction of the heat flux is opposite to that of the temperature gradient. Of the above-mentioned two 'one-point' set of thermal boundary conditions, the first involving the temperature $T_1$ as well as the heat partition ratio $\phi_1$ at surface 1 suffices for obtaining the solutions for equation (3.4).
One may notice that the sum of the heat partition ratios at the slower and the faster surfaces, \( r_1 \) and \( r_2 \) (\( e^{-r_1} = e^{-r_2} \)) equals unity as long as the heat developed inside the film is totally transferred to the two bounding surfaces. Solutions can be developed, as shown by Blok [3.5], in terms of a set of boundary conditions relating to only one surface. Such a development of the solution is the well-known 'one-point' or marching type. The stability criterion of heat transfer can well be discussed wholly in terms of this particular set of boundary conditions.

### 3.2.3 Solutions

I will reproduce the solution obtained by Blok [3.5] in the first place, later to extend and express the results also in terms of dimensionless thermohydrodynamic quantities.

Under a constant shear stress which is characteristic of shear flows, the solution of the problem subject to the above-defined boundary conditions is specified as follows. Firstly the shear stress factor in dimensionless form, which can later be used to specify the limiting conditions of the stable operating regime of the lubricant film:

\[
\Gamma = \frac{\tau}{h} \left( \frac{9}{4} \right) \left[ \frac{1}{Z_1} + \frac{1}{Z_2} \right] \left( \text{arctanh} \ Z_1 + \text{arctanh} \ Z_2 \right)
\]

where \( N \) is a sliding parameter:

\[
N = \frac{\eta_1 u_0^2}{k}
\]

and:

\[
Z_1 = \frac{r_1}{(r_1^2 + 2/N)} ; \quad Z_2 = \frac{(1 - r_1)}{(r_1^2 + 2/N)}
\]

In the above expressions \( u_0 \) is the viscosity of the liquid at the wall temperature and at the pressure in the film cross-section considered.

Further, an implicit expression for the distribution, across the film, of the velocity of flow,

\[
Y = \frac{V}{h} = \frac{\text{arctanh} \ Z_1 + \text{arctanh} \ \left( \frac{r_1 - \varrho}{2/N} \right)}{\left( \text{arctanh} \ Z_1 + \text{arctanh} \ Z_2 \right)}
\]

In the above expressions \( u_0 = (u_0 - u) \) is the sliding speed given by the difference between the surface velocities of the two sliding surfaces involved. Expression (3.6a) can also be expressed in a form explicit in terms of the distribution of velocities of flow,

\[
V = \frac{u_0 - u}{h} \left[ \frac{(1 - r_1)}{(1 - r_1^2 + 2/N)} + \frac{r_1 - \varrho}{2/N} \right]
\]

Finally the viscosity and temperature distributions in the film cross-section can be expressed in the form,

\[
\frac{\varphi_T}{\eta} = 1 + \frac{1}{N} \left( \frac{r_1 - \varrho}{2/N} \right)
\]

The heat flux distribution in any cross-section of the film can be easily evaluated using the definition for the local heat generation per unit time and per unit volume by viscous shear, namely,

\[
\phi = \frac{T - T_w}{h} = \frac{u}{\eta_1 (0.5 - r_1) N}
\]

Equations (3.5) and (3.7) can be made use of to express \( \phi \) in a convenient dimensionless form,

\[
\Phi = \frac{(h/k)}{\varphi T} \left( \frac{r_1 - \varrho}{2/N} \right)
\]

Total heat \( \Phi_{\text{tot}} \) generated across the entire film thickness at any cross-section becomes,

\[
\Phi_{\text{tot}} = (h/k) \left( \frac{r_1 - \varrho}{2/N} \right)
\]

Blok has already shown that the specification of the thermally stable operating regime can be accomplished from various considerations of the above solutions, all yielding the same result. He found that an instability will arise only when \( r_1 < 0.5 \), and that stability requires that \( N \) does not exceed a certain critical value \( N_{\text{crit}} \) defined by,

\[
N_{\text{crit}} = \frac{1}{0.5 - r_1}
\]

These solutions have been extended to specify the volumetric flow parameter corresponding to quasi-shear pattern of the flow. One can readily express the volumetric flow per unit width in the cross-sectional area of the film as,

\[
\dot{V} = \int_0^h u \, dy
\]

Substituting Newton's definition of viscosity and the viscosity ratio...
\( \mu_1/\mu \) from equation (3.7) one obtains,
\[
q = \frac{\int (\frac{u^2}{\mu}) \, dU}{(\frac{1}{\gamma - 1}) \, \mu} = \frac{1}{(1 + \gamma) \, \mu}
\]  
(3.12)

When equation (3.5) is substituted for \( \gamma \), and the integral is evaluated in the form,
\[
\frac{1}{\mu} \ln \left[ \frac{1}{1 + (r,-0.5) N} \right] \int \frac{u_1/u_2 + \frac{1}{\gamma} \ln \left( \frac{1}{1 + (r,-0.5) N} \right)}{\left( \frac{1}{1 + (r,-0.5) N} \right)}
\]
one obtains the following expression for the volumetric rate of fluid flow,
\[
q = \frac{1}{2} \left( u_1 + u_2 \right) h + (r,-0.5) u_2 h \left[ 1 + (r,-0.5) \right]
\]  
(3.13)

where
\[
A = 0.5 (r,-2/N) \left[ \ln \left( 1 + (r,-0.5) N \right) \right] \right]
\[
\frac{1}{(\arctanh \, Z_1 + \arctanh \, Z_2)}
\]

It may be noticed that in the expression for the rate of flow the first term on the right hand side represents isoviscous, isothermal Couette flow and the second a thermal correction term. The latter term vanishes for \( N = 0 \). Thus the non-dimensional rate of flow can conveniently be re-expressed in either of the following forms:
\[
q = \frac{0.5 (u_1 + u_2) h}{u_2 h} + (r,-0.5) - A
\]  
(3.14a)
or
\[
q = \frac{u_2}{u_1 h} + 0.5 \, u_2 h - 2 (r,-A)
\]  
(3.14b)

the latter form being preferred for expressing the results in a graphical form on a logarithmic scale. \( Q \) may be considered a correction factor to the isothermal flow in that it accounts for thermal effects.

For conveniently using these results in the analysis of the elastohydrodynamic films yet to be treated in Chapter 11 we can introduce the flow parameter \( V \), which can be obtained by multiplying equation (3.14b) throughout by \( \sqrt{N_1}/2 \). Then,
\[
V = \sqrt{N_1} \left[ \frac{Q}{2} \left( \frac{u_1}{k} \right)^{\frac{1}{2}} \frac{q - u_2 h}{u_1 h} - (r,-A - N_1) \right]
\]  
(3.14c)

The above expression is most general for non-isothermal shear flows in that it involves the velocity of surface 1 as well. If the slower surface is stationary, \( h \) vanishes and the resulting expression represents the effect of viscous dissipation due to pure sliding. Then the above expression can be given in the convenient form:
\[
V = (8k/\mu) \left( u_{av} \right)^{\frac{1}{2}}
\]  
where \( u_{av} = q/h \).

3.3 Results and Discussions

The solutions provided in the previous section in terms of various thermodaydrodynamic parameters are generally applicable to viscous flows under pure shear as well as quasi-shear conditions. They are also applicable to the flow problems under various thermal boundary conditions including the entire range of the heat partition ratio \( r,- \), specified through \( 0 \leq r,- \leq 1 \). For the range \( 0 \leq r,- \leq 1 \) the heat flux involved is wholly generated inside the film, and for \( r,- < 0 \) and \( r,- > 1 \) it means to be supplied from sources external to the film cross-section concerned. In accordance with the findings of Blok [3.5], the present range of heat partition ratio can be qualitatively divided into two regimes, namely that regime of \( r,- < 0.5 \) and that of \( r,- \approx 0.5 \). An investigation of the above solutions reveals that the published theoretical data of Davis and Laurence [3.3], Nihoul [3.10] etc. are to be classified in a very limited portion of the second regime. Conclusions of the above investigators can also be drawn from the present solution. The first of the above regimes, i.e. \( r,- < 0.5 \), calls for special attention. In this particular regime, as pointed out by Blok, there appears to exist a certain thermally important phenomenon that may cause a limitation to the flow and may lead to thermally precipitous events.

In practice it might turn out to be too difficult, if not impossible, to control both the temperature (or viscosity) and temperature gradient at a given surface. Thus a control of both \( N_1 \) and \( r,- \) might pose certain difficulties. But such a control can also be performed by replacing \( r,- \) by an equivalent wall temperature difference that can readily be related to \( r,- \) through the equations (3.8) and (3.9) in the form,
\[
\beta (T_2 - T_1) = \ln \left( 1 + (r,-0.5) N_1 \right) \right]
\[
\frac{1}{(\arctanh \, Z_1 + \arctanh \, Z_2)}
\]

or say (see definition (3.1)),
\[
\frac{1}{u_1/h} = \frac{1}{u_2/h} \cdot \frac{1}{(r,-0.5) N_1}
\]  
(3.15b)

3.3.1 Regime of thermal instability according to Blok

In the regime of heat partition ratio \( r,- < 0.5 \) the solutions obtained in the previous section do not exist over the whole operating regime specified through the range \( 0 \leq N_1 \leq \infty \), but only in a finite portion of that range. The extent of this finite range depends on \( r,- \) and seems to be realisable in practice. The range is given by the expression (3.11), namely,
\[
Q_{SN}_{1, \text{crit}} = 1/(0.5 - r,-)
\]

For all values of \( r,- < 0.5 \), it may be observed that, at such critical values of \( N_1 \), expressions (3.5), (3.10) and (3.14b) indicate corresponding infinitely large values for the solution. It may also be observed that such a behaviour of the solutions given by the above expressions can be contributed to the term,
\[
\arctanh \, Z_1 + \arctanh \, Z_2
\]

and to the characteristic fact that \( \arctanh \, (\pm 1) = \infty \). In fact, this term has led to the concept of a critical \( N_1 \)-value, \( N_1, \text{crit} \).
For the range of the heat partition ratio $r < 0.5$, the conclusions drawn by Blok (3.5) may be recalled. In this range of operation the shear stress asymptotically reaches a maximum value already at finite sliding speeds; see expression (3.5) and Figure 3.3. The second term on the right-hand side increases to infinitely large magnitude at certain definite value of $N_s$, namely $N_s,\text{crit}$; beyond which the thermally unstable operating regime mentioned in relation (3.11) is assumed to be prevalent.

An analysis by Joseph and Sparrow [3.11] of a class of problems to which the present one belongs provides an upper bound to the range of existence of a solution. They refrain from commenting on the physical significance of their approximated solutions. Thus a detailed discussion is now to follow.

Observing that physically there is sense only in considering positive values of $N_s$, $\beta$ being positive for lubricating oils, there exist no analytical solutions for the range of $N_s$ exceeding $N_s,\text{crit}$: Blok proposes that the regime beyond $N_s,\text{crit}$ may yet be reached in physical reality by employing sufficiently high sliding speeds. For instance for a mineral lubricating oil with a value of 56 cP/°C for $\frac{\mu_1}{\Delta T}$ in $N_s$ a speed not higher than 5 m/s would do.

For mineral oils in an elastohydrodynamically operating conjunc-tional zone the value $\frac{\mu_1}{\Delta T}$, which is simply equivalent to the opposite of the slope $-\frac{\mu_1}{\Delta T}$ of the viscosity-temperature curve, may under the influence of the local film pressure even reach much higher values. In fact, the entire viscosity-temperature-pressure relationship plays an important role in determining the critical sliding velocity corresponding to the lowest critical value $N_s,\text{crit}$ prevailing somewhere in such a film. In deciding whether or not critical conditions will occur in an elastohydrodynamic film the distribution of the surface viscosity, as well as that of the heat partition ratio need to be known. It would appear that the determination of the distribution of $r_s$ is, in principle, no more difficult than that of the distribution of the surface viscosity (film pressure and surface temperature). In fact the former distribution is closely bound up with that of the surface temperature (think of the flash temperature boundary condition). $r_s$ in any film cross-section may, however, be assumed equivalent either to the mean value for the whole conjunction zone, or, even to a better accuracy, to that commensurate with both surface temperatures in the film cross-section under consideration. With modern techniques of measurement an estimate of the latter may prove feasible. A theoretical estimate, at least for a simplified elastohydrodynamic pressure distribution based on dry contact pressure due to Hertz, can be carried out with reasonable ease as shown in Chapter 11 of the present thesis.

The question that may immediately arise concerns the physical existence of the critical condition. Blok considers only one possible evidence of the thermal instability, the one in gear scuffing tests conducted by Ibrahim and Cameron (3.12). A sudden collapse of the film at comparatively high speeds observed by these investigators might perhaps have been due to implosion initiated by evaporation, or say cavitation, somewhere in the conjunction zone of the film.

Due to lack of more direct evidence it would appear appropriate to investigate the existence of such an instability solely on theoretical grounds. For liquids with an exponential viscosity-temperature relationship we have already noticed that the existence of a thermally critical value, $N_s,\text{crit}$, implies an infinitely large non-dimensional shear stress and total heat flux given by expressions (3.5) and (3.10), respectively (see also Figure 3.3). In addition, it also implies some lim-
are hard to conceive according to Dyson. The thermal instability proposed by Blok will then never occur in any physically realisable system with finite non-zero values of the viscosity at both the surfaces. He proposes thermal boundary conditions in terms of measurable external parameters such as $T_1$ and $T_2$; in other words, to replace $r_1$ by the equivalent of temperature difference given by expression (3.15a). This does not change in any way the general conclusion that it is hard to physically realise the region of thermal instability. It may, however, be conceived that the steady state solution in the sense of $N_2$ does not exist for any selected level of $r_1$. The quantity $r_1$ is an internal parameter and adjusts itself to the operating conditions such as the surface temperature difference. It may prove difficult to perform any direct control on this quantity, except when making the surface concerned of some such heat insulating material as glass for which $r_1$ must assume a positive value much smaller than 0.5, and thus in the critical $r_1$-range.

Alternatively, if one employs the surface temperature difference as a variable, it may easily be realised that the critical states, at which expression (3.15b) vanishes, are prevalent at infinitesimally high temperatures $T_1$. Such high temperatures yield negligibly small $N_2$ values for all sliding speeds $u_2$. Only very large $u_2$ values would give rise to the required critical value of $N_2$. Such values of $u_2$ are hard to expect in practice, and before these values are approached the liquid would evaporate and/or decompose. Further, assumptions of Newtonian behaviour and an exponential viscosity-temperature relationship may also break down.

It has been made clear from the above discussion that the region beyond $N_2$ crit is thus a 'forbidden' range, in that neither $N_2$ nor $r_1$ can be imposed on the viscous flow. The next point of importance concerns the question of a close approach to the critical state, of course without subjecting the flow to a state where either the viscosity ratio $\mu_1/\mu_2$ becomes zero or the temperature difference becomes infinitely large. In order to analyse the possibility for the flow to approach the critical state, the other parameters decisive for the appearance of the thermal instability have to be investigated. It has been observed that at such a critical state the non-dimensional quantities for the shear stress, the total heat flux $\phi$ and the correction factor $Q$ to isothermal rate of flow can be expressed as functions of $r_1$ and $N_2$. These three parameters as well as $\mu_1/\mu_2$ can also directly be obtained by simply introducing the viscosity ratio $\mu_1/\mu_2$ in the previous solutions obtained with $r_1$ and $N_2$ as parameters. If one carries out such a transformation, or cyclically interchanges the indices 1 and 2, all the curves for the complete range of $0 < N_2 < \infty$ corresponding to various values of $r_1 < 0.5$ reduce to those for the complete range of $0 < N_2 < \infty$ and these correspond to various values of $r_2 < 0.5$. One may then readily observe that the critical state discussed above is a consequence of the fact that at which $N_2$ reaches very large values for all $r_2 > 0.5$. Any flow in the thermally unstable regime now corresponds to that beyond $N_2 = \infty$. It may then be concluded that the earlier discussion on the physically unrealistic nature of the solution beyond $N_2$ crit for $r_2 < 0.5$ has further been substantiated.

It may also be concluded that the behaviour of the solution for all $r_2 < 0.5$ can equally well be evaluated using the solution for the corresponding problem with $r_2 > 0.5$, the latter covering the complete range of physically realisable values of $N_2$, namely $0 < N_2 < \infty$. Thus the analysis of the solution both for a general value of the heat partition ratio $r_1 \neq 0.5$ representing non-symmetric thermal boundary conditions and for the equipartition $r_1 = r_2 = 0.5$, viz., symmetric thermal boundary conditions, can be carried out as shown in the following section.

After having proved the physical insignificance of the thermal instability suggested by Blok [3.3], I will now turn over to a physical nature of the solution obtained for all heat partition ratios $r_1 \neq 0.5$. I will further specify the regime of a thermal instability of a different kind which has so far been established only for flows with the equipartitioning of the heat, i.e., $r_1 = r_2 = 0.5$. Such an analysis will serve as a basis to the later analysis of the quasi-shear flows where the dissipation effects due to pressure gradients also will be accounted for. In addition the characteristic behaviour of the solution for the non-isothermal shear flows in the aforementioned operating regimes will be shown to imply to pressure flows as well, at least in terms of certain influential quantities comparable to those used in the present analysis.
3.3.2 Traction limited by maximum available shear stress

Thermal processes characterised by a maximum to the solution in terms of some operational parameter have been of continued interest in literature. For non-linear source distributions along with temperature dependent diffusion coefficients, Joseph and Sparrow [3.11] obtained bounds to the maximum intensity of the diffusive source similar to \( \Gamma_1 \) in the present problem. Their bounds are only upper ones, close to the actual maximum intensity, at least for a quadratic type of source distribution. This limitation to the intensity can be related to the physical nature of the materials. Some materials commonly employed in engineering practice may display a thermally affected breakdown when their thermal state is changed due to the internal friction in viscous fluids, dielectric heating in insulating materials, fission in nuclear rods and electric current in a conducting wire; this has already been pointed out in Chapter 1. In all these processes the rate of generation of heat increases approximately in an exponential or some other non-linear form. A failure to keep the process under control might result in catastrophic breakdowns due to excessive increase of temperatures and an ultimate deterioration of the materials concerned.

Steady and safe operation with the above-mentioned processes is possible only if the influential quantities are maintained below some definite permissible level. A careful study is thus required to specify this maximum level in order to maintain the process in a steady state. Specifying the maximum allowable temperature to safeguard the material from deterioration may then also be required.

The nature of the present solution for heat partition ratios \( r < 0.5 \) has already been discussed in the previous section. In the remainder of the operating regime i.e. \( r > 0.5 \), it can be readily shown that the non-dimensional maximum (also called ‘limiting’) shear stress always remains finite and depends on \( \Gamma_1 \). These maximum (limiting) values occur at some finite sliding speed, corresponding with practically realisable values of \( N \), (see Figure 3.3). The magnitude of the limiting shear stress for any selected value of \( N \) can be evaluated as shown below. The corresponding values of \( N \) can be evaluated, though not always in an explicit form. Results for somewhat simplified thermal boundary conditions, namely for uniform heat transfer coefficients at the surface, have been worked out by Pascovici [3.13].

By differentiating equation (3.5) with respect to \( N_1 \), equating it to zero, and then rearranging the terms one may obtain,

\[
\left( \text{arctan} x_1 + \text{arctan} x_2 \right) = \frac{\sqrt{2}}{V H (x_1 + x_2)} \left( \frac{x_1}{x_1 + x_2} + \frac{x_2}{x_1 + x_2} \right)
\]

(3.17)

Since for arbitrary values of \( x_1 \) an analytical solution in closed form is hard to obtain, this equation has been numerically evaluated separately for various values of \( x > 0.5 \) so as to find the value of \( N_1 \) at which the dimensionless shear stress parameter \( \Gamma_1 \),

\[
\Gamma_1 = I \cdot \log (A / \mu u)
\]

reaches a limiting value. We shall demonstrate in what follows that analytical solutions can be obtained for two particular cases of heat partition ratio: one with identical heat partition ratio \( r_1 = r_2 = 0.5 \), and another with \( r_1 = 0 \) and \( r_2 = 1.0 \). For the former set \( r_1 = r_2 = 0.5 \), i.e. equi-

...
Metal forming processes. On the other hand, in hydraulic drives where advantageous as to lower frictional losses in bearings, gear drives and in metal forming processes. The highest possible traction is desired the limitation might prove a handicap. However, a compromise might be struck through suitably controlling the heat partition ratios. This can be achieved by a proper selection of the material or even by optimized cooling or heating of the surfaces involved.

A similar discussion can be applied to the results obtained for $r_2$ in terms of $N_2$, while employing $r_1<0.5$. Again the assumption of the heat transfer due wholly to cross-conduction implies $r_1 r_2=1.0$, which involves that $r_1$ and $r_2$ are cyclically interchangeable in the various foregoing expressions. That is, if the variable $N_1$ is simultaneously replaced by $N_2$, then the set of curves for $r_1>0.5$ is identical with that for $r_1<0.5$. Then the shear stress represented in Figure 3.4 has to be correspondingly redefined using the viscosity at surface 2, viz.:

$$\tau = \frac{1}{r} \left( \frac{\mu}{h} \right)^{\frac{1}{2}}$$

This cyclic interchangeability of the solutions holds valid also for the distribution of velocity, temperature or viscosity, heat flux and volumetric rate of flow which are yet to be analysed in the next section.

It may be noticed from Figure 3.3 that the occurrence of a limiting shear stress at some characteristic values of $N_1$ in the regime $r_1>0.5$ is not quite obvious from the curves for $r_1<0.5$. Such an occurrence becomes visible only when one re-expresses the results using the aforementioned cyclic interchangeability. If the non-dimensional parameter $\Gamma_1$ is expressed using the viscosity $\mu_1$ at the surface temperature, then the variation of $\Gamma_1$ is indeed single-valued for all that $r_1<0.5$. In the section below, however, the complete regime of $r_1<0.5$ and $r_2>0.5$ will be investigated in order to demonstrate the characteristic behaviour of the solution. It is needless to say that $r_1<0.5$ can easily be re-expressed, if so desired, using the results of $r_1>0.5$.

### 3.3.3 Double-valuedness of distributions of velocity and temperature

The existence of double-valued solutions for distributions of velocity and temperature has already been discussed for a given shear stress under specific thermal boundary conditions. This aspect can readily be illustrated through the present calculations.

Heat partition ratios $r_1$, less than 0.5 give rise to monotonically increasing non-dimensional shear stress $\Gamma_1$; for any given $\Gamma_1$ there exists only one distribution of velocity of flow and equally one of temperature distribution inside the film. But when the heat partition ratio equals or exceeds 0.5, then according to equations (3.18a) and (3.18b), there are double values of $\eta_1$ and $\eta_2$ for any $\Gamma_1$, below its maximum. Thus for a given value of the shear stress $\Gamma_1$ and $r_1>0.5$ there can exist two values of $N_1$. Correspondingly, for any given oil there can exist two values for sliding speed $u_h$ corresponding with those of $N_1$. (see also equation (3.6b)). Figure 3.3 shows the double-valuedness of $N_1$ in the range of $r_1>0.5$, as well as the physically unreasonable limiting values of $N_1$, that cannot be imposed on the film, and certainly not be surpassed, in the range of $r_1<0.5$. Correspondence of both the shear stress parameter $\tau_1$ and $N_1$, with a classical thermohydrodynamic dimensionless parameter, Brinkman number, has been shown in Appendix 3A1.

Velocity distributions at selected values of $N_1$ less than $N_1 \text{crit} = (0.5-r_1)$ have been given in Figures 3.5a and 3.5b. It may be observed that for very low values of $N_1$, the velocity distribution is almost equal to that for isothermal flow, the viscous heating effects still being small.

For $r_1<0.5$ and for all practical values of $N_1$, the velocity of flow is everywhere in the film, except of course at the two bounding surfaces, larger than that for the isothermal case. However, for low positive values of $r_1$ the velocity near surface 1 is smaller than that near surface 2 as well as the isothermal case. Maximum deviation of the maximum from the isothermal value occurs close to the mid-plane of the film, but offset toward the faster surface. For low values of $N_1$ the distribution of the viscous dissipational heat deviates most towards the centre, although not by a large amount (see $\eta_1^M$ and $\eta_1^A$ against $r_1$). It may also be noted that for values of $r_1$ approaching the equipartitional value of 0.5, the fluid will be transported at velocities approaching those of the nearby surfaces (see curve 5 in Figure 3.5b), and the maximum viscous dissipation will take place away from those surfaces.

For boundaries having the same temperature, i.e. equipartitioning of heat, when $r_1=r_2=0.5$, see Figure 3.6, the conclusions drawn for the cases $r_1$ close to 0.5 remain valid. The velocity at midplane is identical for all values of $N_1$ with that for the isothermal case. Maximum deviation of the velocity occurs midway from the centre of the film to either surface. In fact, most of the viscous dissipation then occurs close to the centre, and for very large values of $N_1$ it concentrates in a small fraction of the film cross-section only.

For the thermal operating regime $r_1>0.5$, the velocity distribution may be observed for all that $r_2<0.5$ when one replaces $r_1$ cyclically by $r_2$ having the same value. This is clear from comparing Figure 3.7a with Figure 3.5b for $r_1=0.491 - 0.51$, so that $r_2=0.51$. For $r_1>0.5$ the entire velocity distribution lies within the equipartitional one, whilst the maximum deviation shifts towards the slower surface, increasing heat partition ratio, (see Figure 3.7b). For large values of $r_1$, most of the viscous dissipation takes place in a thin layer of the fluid close to surface 2. This is easily noticed for large values of $N_1$; see Figure 3.7c. The results discussed for $r_1<0.5$ are in agreement with those of Mahdavian and Wilson [3.14]. One may notice from figures for $r_1=0.5$ and $r_1>0.5$ that for any given value of the shear stress lower than the maximum attainable, there are two velocity distributions, one or more or less approaching the isoviscous curve and the other deviating most, especially in the low levels of the shear stress chosen. These results for the velocity distribution thus display the double-valuedness of the solution in terms of the shear stress given by the parameter $\Gamma_1$. For the limiting shear stress value, however, there is a unique curve in accordance with the uniqueness of the solution.

The question of validity of these two different velocity curves for a given shear stress has remained a controversy till now. Some investigators have not even noticed this aspect, see Mahdavian and Wilson [3.14] etc. We shall delve into this question and the definition of the workable range of $r_1$, and $N_1$. (3.6b)) Figure 3.3 shows the double-valuedness of $N_1$ in the range of $r_1>0.5$, as well as the physically unreasonable limiting values of $N_1$, that cannot be imposed on the film, and certainly not be surpassed, in the range of $r_1<0.5$. Correspondence of both the shear stress parameter $\tau_1$ and $N_1$, with a classical thermohydrodynamic dimensionless parameter, Brinkman number, has been shown in Appendix 3A1.
as the other half near surface 2 is entrained by the corresponding faster surface. For the equipartitional case, Figure 3.9, there is an identical effect on two halves of the film, and for the cases of \( r_{j} > 0.5 \) already at low values of \( N_j \) the velocity is reduced relative to the isothermal value, Figures 3.10a and 3.10b. It is readily seen from the foregoing results that for a specified shear stress parameter \( \tau_j \), there exist two values of \( N_j \) and correspondingly two distributions of velocity. A similar analysis of the results which have not been provided here due to lack of space, indicates that for the same set of \( \Gamma_j \) and \( N_j \), there exist two corresponding distributions of temperature and viscosity.

### 3.3.4 Limitation to volumetric rate of flow

Expressions obtained earlier for the correction factor \( Q \) to isothermal volumetric rate of flow, see expression (3.14b), have been used to show the flow variation as influenced by the heat partition ratio as well as by the sliding parameter. Figure 3.11 has been drawn for heat partition ratios ranging from negative to large positive values. The same results have been re-expressed in terms of the flow parameter \( V \) in Figure 3.12. Here the parameter \( V \), as against to the correction factor \( Q \), represents the volumetric rate of flow under non-isothermal conditions. It may be observed that for \( r_j > 0.5 \) increasing \( N_j \) results in increased flow. At values of \( N_j \) approaching the corresponding critical values, the flow parameter rapidly increases and ultimately becomes asymptotically infinite. This behaviour of the flow rate is similar to the unlimited increase of shear stresses for \( r_j < 0.5 \) depicted in Figure 3.3. Equipartition of heat between the surfaces represented by \( r_j = r_{j} = 0.5 \) results in a constant correction factor of 1.0. The results representing the behav-

---

**Figure 3.5** Sliding parameter \( N_j \) covering a wide range and the corresponding (a) velocity distribution for \( r_j = 0.49 \), (b) and (c) heat flux distribution for the same values of \( N_j \) as in (a).
Figure 3.7: The two distributions of velocity and heat flux existing in the range $r \geq 0.5$ for the two $N_1$ values then corresponding with one and the same $r_1$ value. The velocity distributions (a) and (b), for $r_1 = 0.051$ and twice, respectively. The heat flux distributions $\phi^*$, (c), for the same $r_1$ value as in (b).

One of $V_1$ (see expression (3.14c)) with $N_1$ also depict that of the average velocity with the same $N_1$. The results for the regime $r \geq 0.5$, however, do not show any double-valuedness for increasing $N_1$, in contrast to those for the near steady in Figure 3.3. The correction factor $Q$ asymptotically decreases to zero at infinitely large values of $N_1$.

A few conclusions may now be drawn from the behavior of these solutions. For equilibration of the heat, $r=r_1=0.5$, both the volumetric rate of flow and the average velocity remain equal to the corresponding value for isothermal flow. Then, can also be obtained from Figure 3.9, where the peak in the heat flow from the fundamental solution is shown.

Contrary to the behavior of these results, the temperature asymptotically decreases to $r_1=0.5$, indicating a continuous decrease of the shear flows and the average velocity with an increase of $N_1$. This phenomenon may be attributed to the fact that the fluid layer closer to the surface 3.
The heat partition ratio $r_1 = 0.50$ follows the velocity of this surface, whilst the remaining fluid tends to remain at the velocity of surface 1. At very large values of $N_j$ only a very thin fluid layer follows the velocity of surface 2, as shown in Figures 3.7a and 3.7b.

In the case of the conjunctural zone of elastohydrodynamic films, the local variation of $N_j$ takes place due to a change of the fluid viscosity under the influence of the local variation of both pressure and temperature, the latter variation being describable by the flash temperature theory. This local variation of $N_j$ also prevails in lubricant films in sliding bearings, in the workzone of metal deformation processes etc. Since this variation of $N_j$ is generally not appreciable, neither will the correction factor show any considerable local variation.

In order to visualize the required heat partition ratio at a changed sliding parameter $N_j$ to maintain a constant rate of flow, the results displayed in Figure 3.11 have been re-expressed in a somewhat different form. Such a representation employs the correction factor already provided through equation (3.14b) as the parameter. The characteristic behaviour of the solution might then be noticed through Figure 3.13 to be similar to that shown for the correction factor Q to the rate of flow in Figure 3.11. The domain of the instability due to Blok can readily be noticed in the range of the heat partition ratio $r_j<0.5$. This domain has been bound by the locus of the critical state which is characterized by $N_j_{critical}$ corresponding to the range $r_j<0.5$; see Figure 3.13. At the equipartitioning of heat, however, $Q$ is given by a constant value of unity.
3.4 CRITERIA FOR THE STABILITY OF FLOW

It has already been proved that the thermal instability predicted by Blok [3.5] for flows with asymmetric thermal boundaries where \( r_1 < 0.5 \) may not arise in practice. It has, however, been pointed out previously that a thermal instability of a different kind may yet be expected to arise. Such an instability has been proved to exist for thermally symmetric shear flows, with equipartitioning of the heat, by several investigators, namely by Joseph [3.16], and Vanderborck and Platten [3.7].

The word 'inviscid' is referred to fluid flows in the absence of viscosity effects. Any stability criteria established for the flow of an
If an instability is established in the 'inviscid\textsuperscript{a} limit', i.e. inviscid instability at infinitely high Reynolds numbers, with non-vanishing wave number and phase velocity, the disturbance with the same wave number is unstable in the real fluid when the Reynolds number is even low. Thus Joseph's\textsuperscript{[3.16]} results obtained in an 'inviscid limit' were incomplete since these did not refer to real fluids. Vanderborck and Platten\textsuperscript{[3.7]} have extended Joseph's analysis on the instability of the velocity distribution to finite values of the Reynolds number. Sukanek, Goldstein and Laurence\textsuperscript{[3.8]}, and Leben and Nguyen\textsuperscript{[3.17]} have, in addition to extending Joseph's\textsuperscript{[3.16]} and\textsuperscript{[3.18]} analysis to finite Re, explicitly studied the influence of a two-dimensional disturbance on the coupled momentum equation with the energy equation, the latter involving the convection term. It may here be included that lubricant films involving real liquids operate at finite, low Reynolds numbers. Any analysis of instability is thus relevant to the present work when it does not refer exclusively to very high Reynolds numbers.

### 3.4.1 Basic features of instability

Before we go into the analysis of the results by the aforementioned investigators we may as well acquaint ourselves with some basic features and their physical interpretation in terms of stability criteria. Such criteria are established through a consideration of an assumed disturbance on the velocity profile, the temperature profile, or both of the velocity and temperature profiles of the main flow. As to the main flow, both profiles might already be prevailing in a steady state, or in a state of development with time that, if undisturbed, will ultimately lead to a steady state.

While the classical stability analysis deals extensively with the velocity profile of an isoviscous, isothermal flow, the later analyses deal with the temperature profile that is independent of the velocity profile; in other words, the later analyses consider the energy equation uncoupled from the momentum equation. The most general analysis should, however, include both the velocity and temperature profiles which result from the coupled energy and momentum equations. An extensive stability analysis covering all the above-mentioned separate analyses has been carried out by Sukanek et al.\textsuperscript{[3.8]}. They divide their results on instability into four different modes as explained in Appendix 3.A2. 'Inviscid mode' results from the presence of an inflection point in the velocity profile; while the viscosity effects are taken into account, the instability associated with the aforementioned velocity profile independent of the temperature profile is called 'viscous mode'.

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\textsuperscript{a}inviscid fluid can in principle be used as a guide-line for studying the stability problem of a real viscous fluid. Thus a study of the stability problem for a real fluid may be obtained through considering a modification of the results of 'inviscid instability' by the effect of viscosity as proposed by Heisenberg; see Lin\textsuperscript{[3.15]}. Heisenberg has demonstrated that the effect of viscosity is generally destabilising at very large Reynolds numbers. For instance a disturbance with wave length 2\pi/\alpha, where \alpha is the wave number of a neutral disturbance, neutrally stable in the inviscid case, is unstable when viscous forces are considered.
mainly confined to the 'inviscid' limiting case, and very little being known about the finite Re case. Some results may be found, from the work of Sukanek et al. on the instability at finite Re where they included convection terms in order to carry out the analysis with respect to a two-dimensional disturbance. Nevertheless, their investigation is not directed towards the convective instability problem because their main flow excludes velocity due to convection terms; even the viscous dissipation effects alone would lead to an unstable flow already at finite Re. For the present study of steady flows only the characteristic results of their stability criteria will be made use of, in order to demarcate the regime of stable from that of unstable operation under the influence of unequal partitioning of the heat. Then the stability criteria can broadly be based on two distinctive features, one being connected with the flow and the other with the thermal mechanism. The maximum (limiting) shear stress criterion discussed in the previous section is important at least from two points of view. Firstly, in the inviscid limit (Re = ∞) the velocity profile is stable only up to this point of limiting shear stress and unstable beyond, Joseph [3.16]. Secondly, the temperature distribution becomes unstable at the same point of limiting shear stress for the flows of viscous fluids operating at finite Reynolds numbers [3.18].

Maximum temperature in the film increases along the first branch due to increased shear stress and continues to do so along the second branch even though shear stress there is on the decrease. For all non-linear fluidity-temperature relationships, including that of the exponential type, this behaviour holds true. The shear stress parameter \( \tau_1 \), including its maximum, i.e. its limiting value, is a function of film thickness \( h \). The corresponding Reynolds number, based on the constant viscosity of the liquid in a flow with the same uniform distribution of shear, depends also on the film thickness. The limiting shear stress criterion choosen already at a low Reynolds number (Re) seems to be quite suitable chosen. It can then be concluded that the thermal instability established at the point of limiting shear stress can occur at finite Reynolds numbers which might well be representative of lubricant flows.

Sukanek et al. [3.8], and Lebon and Nguyen [3.17] carry out a thorough analysis of the stability criteria of the flow. They employ the equipartitioning of the heat and equal boundary temperatures. Sukanek et al. recognised the instability predicted by Joseph [3.18] as due primarily to thermal effects. They established that the 'inviscid' type of instability occurring at the point of limiting shear stress [3.16] also implies that for real liquids it might take place at finite values of Reynolds numbers. For, under such operating conditions the involved sliding velocities, and thus the Brinkman numbers defined in Appendix 3.4, are finite.

Similar conclusions have been drawn by Vanderborck and Platten as well, at least with respect to the instability of the thermal mode. They have presented the results for Brinkman numbers higher than those corresponding with the limiting shear stress parameter \( \tau_1 \). Quite likely these authors have encountered numerical instability in the computer

\[ \text{Re} = \frac{\nu \cdot h}{\mu} \]

It is, however, clear from the results that for low Br, the critical Re is very high. Further the critical Re decreases considerably with increased Prandtl number Pr. The values of Pr for mineral and other lubricating oils being rather high, it may be concluded that the linear stability of the flows of liquids of our present interest is not so predominant, at least in the range of high Br. In addition, the numerical accuracy of the results of Lebon and Nguyen is not high, whilst Sukanek et al. have encountered instability of the numerical procedure at high values of Re already for Brinkman numbers smaller than the aforementioned ones. Such a numerical problem has also been encountered by Sukanek et al. for Brinkman numbers less than 15.

The results so far presented by the former investigators show, however, that the value of the Reynolds number concerned decreases with increasing Brinkman numbers.

By using the variational principle, Lebon and Nguyen [3.17] have analysed the linear stability of the viscous flow. They have been able to extend the region of stability analysis for this flow to low Brinkman numbers. They have compared their results with those of Sukanek et al. Their results have here been supplemented by those of Vanderborck and Platten, see Table 3.1. Since Lebon and Nguyen do not account for the convection effects, the Prandtl number does not appear in their results. The numerical comparison of the results in Table 3.1 on a common basis is difficult. It may, however, be noted that the influence of Pr is not so predominant, at least in the range of high Br. In addition, the numerical accuracy of the results of Lebon and Nguyen is not high, whilst Sukanek et al. have encountered instability of the numerical procedure at low Br. Thus the results on Table 3.1 are interpreted as providing only an order of magnitude at which an instability due to the thermal effects might arise. The aforementioned investigators employ a Reynolds number where the characteristic velocity is taken equal to the sliding velocity, namely,

\[ \text{Re} = \frac{\nu \cdot h}{\mu} \]

3.4.3 Instability of thermally asymmetric flows

There exist for every heat partition ratio \( \tau > 0.5 \) a set of thermal sliding parameter \( N \), or the Brinkman number Br and the shear stress parameter \( \tau_1 \), where the latter reaches its maximum value. Operating conditions for the present thermally asymmetric flows corresponding to the above set of \( N \) or Br and \( \tau_1 \) can also be analysed for the appearance of instability, as explained in the previous section for thermally symmetric flows. The present analysis is confined to quoting such sets of values from others, rather than rigorously proving the stability criteria for the present, thermally asymmetric flows. It has been assumed that the conclusions drawn by various investigators for the thermally symmetric flows with respect to the existence of the thermal instability hold also valid under somewhat similar condi-
Table 3.1 Critical values of Reynolds numbers Re at various Brinkman numbers Br (Br is equivalent to thermal sliding parameter N_x) for equipartitioning of the heat and for equal boundary temperatures.

<table>
<thead>
<tr>
<th>Br</th>
<th>Re</th>
<th>Q_{crit}</th>
<th>Pr</th>
<th>Re</th>
<th>Q_{crit}</th>
<th>Re</th>
<th>Q_{crit}</th>
<th>Modes of instability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>13665</td>
<td>1.60</td>
<td>1</td>
<td>9370</td>
<td>0.95</td>
<td>1</td>
<td>Thermal</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>7290</td>
<td>1.00</td>
<td>1</td>
<td>7570</td>
<td>1.00</td>
<td>5</td>
<td>Thermal</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4670</td>
<td>1.10</td>
<td>5</td>
<td>4340</td>
<td>1.15</td>
<td>5</td>
<td>Thermal</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>3835</td>
<td>1.20</td>
<td>10</td>
<td>3700</td>
<td>1.20</td>
<td>10</td>
<td>Thermal</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>4490</td>
<td>1.20</td>
<td>15</td>
<td>3540</td>
<td>1.25</td>
<td>15</td>
<td>Thermal</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>3625</td>
<td>1.25</td>
<td>15</td>
<td>3490</td>
<td>0.945</td>
<td>15</td>
<td>Thermal</td>
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<tr>
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<td>3455</td>
<td>0.955</td>
<td>15</td>
<td>1156</td>
<td>1.600</td>
<td>15</td>
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</tr>
<tr>
<td>19</td>
<td>1</td>
<td>3490</td>
<td>0.945</td>
<td>19</td>
<td>1028</td>
<td>1.535</td>
<td>19</td>
<td>Thermal</td>
</tr>
<tr>
<td>19</td>
<td>5</td>
<td>3455</td>
<td>0.955</td>
<td>19</td>
<td>1028</td>
<td>1.535</td>
<td>19</td>
<td>Thermal</td>
</tr>
<tr>
<td>19</td>
<td>50</td>
<td>1028</td>
<td>1.535</td>
<td>19</td>
<td>1028</td>
<td>1.535</td>
<td>19</td>
<td>Thermal</td>
</tr>
<tr>
<td>20</td>
<td>16000</td>
<td>0.185</td>
<td>50</td>
<td>20</td>
<td>16000</td>
<td>0.185</td>
<td>20</td>
<td>Thermal</td>
</tr>
<tr>
<td>25</td>
<td>1400</td>
<td>0.38</td>
<td>50</td>
<td>25</td>
<td>1400</td>
<td>0.38</td>
<td>50</td>
<td>Thermal</td>
</tr>
<tr>
<td>30</td>
<td>700</td>
<td>0.385</td>
<td>1</td>
<td>30</td>
<td>700</td>
<td>0.385</td>
<td>1</td>
<td>Thermal</td>
</tr>
<tr>
<td>30</td>
<td>176</td>
<td>0.406</td>
<td>5</td>
<td>30</td>
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<td>0.406</td>
<td>5</td>
<td>Inviscid</td>
</tr>
<tr>
<td>30</td>
<td>191</td>
<td>0.283</td>
<td>50</td>
<td>30</td>
<td>191</td>
<td>0.283</td>
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</tr>
<tr>
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<td>1.225</td>
<td>50</td>
<td>30</td>
<td>980</td>
<td>1.225</td>
<td>50</td>
<td>Thermal</td>
</tr>
<tr>
<td>40</td>
<td>280</td>
<td>0.55</td>
<td>1</td>
<td>40</td>
<td>280</td>
<td>0.55</td>
<td>1</td>
<td>Inviscid</td>
</tr>
<tr>
<td>40</td>
<td>44</td>
<td>0.8</td>
<td>0.8</td>
<td>40</td>
<td>44</td>
<td>0.8</td>
<td>0.8</td>
<td>Inviscid</td>
</tr>
<tr>
<td>40</td>
<td>116</td>
<td>0.473</td>
<td>1</td>
<td>40</td>
<td>116</td>
<td>0.473</td>
<td>1</td>
<td>Inviscid</td>
</tr>
<tr>
<td>40</td>
<td>68.2</td>
<td>0.645</td>
<td>5</td>
<td>40</td>
<td>68.2</td>
<td>0.645</td>
<td>5</td>
<td>Inviscid</td>
</tr>
<tr>
<td>40</td>
<td>64.2</td>
<td>0.623</td>
<td>50</td>
<td>40</td>
<td>64.2</td>
<td>0.623</td>
<td>50</td>
<td>Inviscid</td>
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<tr>
<td>80</td>
<td>1</td>
<td>31.8</td>
<td>0.850</td>
<td>80</td>
<td>1</td>
<td>31.8</td>
<td>0.850</td>
<td>Inviscid</td>
</tr>
<tr>
<td>80</td>
<td>5</td>
<td>27.7</td>
<td>0.975</td>
<td>80</td>
<td>5</td>
<td>27.7</td>
<td>0.975</td>
<td>Inviscid</td>
</tr>
<tr>
<td>80</td>
<td>50</td>
<td>22.9</td>
<td>0.960</td>
<td>80</td>
<td>50</td>
<td>22.9</td>
<td>0.960</td>
<td>Inviscid</td>
</tr>
<tr>
<td>600</td>
<td>1</td>
<td>1.67</td>
<td>1.581</td>
<td>600</td>
<td>1</td>
<td>1.67</td>
<td>1.581</td>
<td>Inviscid</td>
</tr>
<tr>
<td>600</td>
<td>5</td>
<td>3.92</td>
<td>1.527</td>
<td>600</td>
<td>5</td>
<td>3.92</td>
<td>1.527</td>
<td>Inviscid</td>
</tr>
<tr>
<td>600</td>
<td>50</td>
<td>4.01</td>
<td>1.496</td>
<td>600</td>
<td>50</td>
<td>4.01</td>
<td>1.496</td>
<td>Inviscid</td>
</tr>
</tbody>
</table>
Thus neither an interpolation nor an extrapolation may be carried out on the basis of the above-mentioned expression for $Re_{\text{crit}}$.

For heat partition ratios other than 0.5, the asymmetry is likely to impose a different characteristic behaviour of the flow. This might involve a change, with respect to the symmetric case, of the distributions of velocity and temperature upon which a disturbance of certain wavelength is superimposed for analysing stability. This might result in a stability criterion that is far more involved than that of the symmetric flows, those with equipartitioning of heat. There are, however, indications that the values of $Re_{\text{crit}}$ will decrease with increased asymmetry, Potter and Graeber [3.22].

Only approximate estimation of $Re_{\text{crit}}$ can be obtained through an analysis that is based on the proportionality of $Re_{\text{crit}}$ with respect to $\Gamma_1,\lim$. Such an estimation, based on the aforementioned expression, viz.,

$$Re = \Gamma_1,\lim \cdot \rho \cdot \left( \frac{k}{H} \right)^j,$$

is qualitative in nature in that it accounts only for the new value of $\Gamma_1$ for increased thermal asymmetry. An estimation of $Re_{\text{crit}}$ through $Re_{\text{crit}}$ for the equipartitioning in the neighbourhood of the limiting value of $\Gamma_1$ can, however, be performed for any arbitrary heat partition ratio $r_j$ by making use of that for the equipartitioning, i.e. $r_j = 0.5$. Then,

$$Re_{\text{crit}} = Re_{\text{crit},\lim} \cdot r_j,\lim$$

This expression consists only of the material properties of the liquid concerned and of the film thickness. The proportional reduction of $Re_{\text{crit}}$, so estimated, have been indicated for a few selected heat partition ratios in Table 3.3. These estimates have been achieved by making use of the available data for the limiting value of $\Gamma_1$ and both for equipartitioning of the heat and for $r_j > 0.5$. It may be observed from expression (3.22) that an increase in thermal asymmetry tends to reduce the value of $Re_{\text{crit}}$ in direct proportion to $r_j$ as compared with the value $Re_{\text{crit}}$ for the thermally symmetric case. The listed values of $N_1,\lim$ are those in the neighbourhod of the limiting value for $\Gamma_1$. We may conclude, in the absence of any proof more rigorous than that presented here, that the critical values of the Reynolds number would be lower for the thermally asymmetric pattern of plan-parallel flows. A really accurate assessment of the values of $Re_{\text{crit}}$ for these thermally asymmetric flows cannot be achieved only through a detailed stability analysis which, however, is beyond the scope of the present work.

### Table 3.2 Thermal and kinematic state of the film at which $\Gamma_1$ is at a maximum (limiting value) and the instability is expected to arise at finite Reynolds numbers.

<table>
<thead>
<tr>
<th>$r_j$</th>
<th>$\mu_r/\mu_0$</th>
<th>$\beta(\Gamma_2 - \Gamma_1)(\mu_r/\mu_0)$</th>
<th>$N_1,\lim$</th>
<th>$\Gamma_{1,\lim}$</th>
<th>$\Theta^*$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>1.0000</td>
<td>0.0</td>
<td>3.2767</td>
<td>18.2138</td>
<td>1.8745</td>
<td>8.0</td>
</tr>
<tr>
<td>0.51</td>
<td>1.8121</td>
<td>0.4661</td>
<td>3.7964</td>
<td>15.8955</td>
<td>1.7990</td>
<td>7.1726</td>
</tr>
<tr>
<td>0.55</td>
<td>1.5938</td>
<td>0.4661</td>
<td>3.7964</td>
<td>15.8955</td>
<td>1.7990</td>
<td>7.1726</td>
</tr>
<tr>
<td>0.60</td>
<td>1.9652</td>
<td>0.7655</td>
<td>2.7372</td>
<td>9.6512</td>
<td>1.4370</td>
<td>4.6462</td>
</tr>
<tr>
<td>0.70</td>
<td>2.4726</td>
<td>0.9053</td>
<td>2.5040</td>
<td>7.3362</td>
<td>1.2354</td>
<td>3.2452</td>
</tr>
<tr>
<td>0.80</td>
<td>2.8199</td>
<td>1.3011</td>
<td>2.9412</td>
<td>6.0664</td>
<td>1.1050</td>
<td>2.7216</td>
</tr>
<tr>
<td>0.90</td>
<td>3.0772</td>
<td>1.1240</td>
<td>3.1011</td>
<td>5.1929</td>
<td>1.0104</td>
<td>2.3025</td>
</tr>
<tr>
<td>1.00</td>
<td>3.2766</td>
<td>1.1868</td>
<td>3.2767</td>
<td>4.5534</td>
<td>0.9373</td>
<td>2.0000</td>
</tr>
<tr>
<td>2.00</td>
<td>4.1259</td>
<td>1.4173</td>
<td>5.1679</td>
<td>2.0839</td>
<td>0.6088</td>
<td>0.8788</td>
</tr>
<tr>
<td>5.00</td>
<td>4.6082</td>
<td>1.5278</td>
<td>11.0229</td>
<td>0.8018</td>
<td>0.3692</td>
<td>0.3306</td>
</tr>
</tbody>
</table>

The behaviour of the solution for circular shear flows in an annulus is similar to that of plan-parallel shear flows; such an analysis has not here been provided due to lack of space. There exists a maximum (limiting value) to the non-dimensional shear stress for equipartitioning of the heat and at a definite value of $N_1$. This non-dimensional parameter

$$\frac{\tau - \mu \left( \frac{d^2 u}{dy^2} \right) - \frac{u_0}{H} \int_0^H \frac{d^2 u}{dy^2} dy}{\mu_r/\mu_0}$$

rearranging terms and introducing $\Gamma_1$ and $N_1,\lim$, it may be shown that,

$$\Gamma_1,\lim = \frac{\sqrt{\mu_r/\mu_0}}{\int_0^H (\mu_r/\mu_0) dy}$$

or, rather:

$$\sqrt{\mu_r/\mu_0} = \mu_r/\mu_0.$$

This expression implies that the parameters specifying the shear stress and the sliding velocity are simply related through the average viscosity $\mu_r/\mu_0$ given by the term on the right hand side.

Then, one may replace, if necessary, the sliding parameter $\mu_r/\mu_0$ through $[\Gamma_1(\mu_r/\mu_0)]^2$ in the aforementioned expression for $Re$. Thus it is seen that the Reynolds number $Re$ relates to $\sqrt{\mu_r/\mu_0}$ or to $[\Gamma_1(\mu_r/\mu_0)]$. This also means that $Re_{\text{crit}}$ should monotonically increase with $N_1$ in the case of the equipartitioning of the heat for which $Q$ is unity throughout. However, this is not evident from the tabulated results of Sukanek et al. The reason seems to be that the stability analysis involving the wave number yields results obviously far from the simple form expected from the above relationship between $Re$ and $N_1$. Thus neither an interpolation nor an extrapolation may be carried out on the basis of the above-mentioned expression for $Re_{\text{crit}}$. 

$$Re_{\text{crit}}$$

For heat partition ratios other than 0.5, the asymmetry is likely to impose a different characteristic behaviour of the flow. This might involve a change, with respect to the symmetric case, of the distributions of velocity and temperature upon which a disturbance of certain wavelength is superimposed for analysing stability. This might result in a stability criterion that is far more involved than that of the symmetric flows, those with equipartitioning of heat. There are, however, indications that the values of $Re_{\text{crit}}$ will decrease with increased asymmetry, Potter and Graeber [3.22]. Only approximate estimation of $Re_{\text{crit}}$ can be obtained through an analysis that is based on the proportionality of $Re_{\text{crit}}$ with respect to $\Gamma_1,\lim$. Such an estimation, based on the aforementioned expression, viz.,

$$Re = \Gamma_1,\lim \cdot \rho \cdot \left( \frac{k}{H} \right)^j,$$

is qualitative in nature in that it accounts only for the new value of $\Gamma_1$ for increased thermal asymmetry. An estimation of $Re_{\text{crit}}$ through $Re_{\text{crit}}$ for the equipartitioning in the neighbourhood of the limiting value of $\Gamma_1$ can, however, be performed for any arbitrary heat partition ratio $r_j$ by making use of that for the equipartitioning, i.e. $r_j = 0.5$. Then,

$$Re_{\text{crit}} = Re_{\text{crit},\lim} \cdot r_j,\lim$$

Thus,

$$\Delta Re = Re_{\text{crit}} - Re_{\text{crit},\lim}$$

This expression consists only of the material properties of the liquid concerned and of the film thickness. The proportional reduction of $Re_{\text{crit}}$, so estimated, have been indicated for a few selected heat partition ratios in Table 3.3. These estimates have been achieved by making use of the available data for the limiting value of $\Gamma_1$ and both for equipartitioning of the heat and for $r_j > 0.5$. It may be observed from expression (3.22) that an increase in thermal asymmetry tends to reduce the value of $Re_{\text{crit}}$ in direct proportion to $r_j$ as compared with the value $Re_{\text{crit}}$ for the thermally symmetric case. The listed values of $N_1,\lim$ are those in the neighbourhood of the limiting value for $\Gamma_1$. We may conclude, in the absence of any proof more rigorous than that presented here, that the critical values of the Reynolds number would be lower for the thermally asymmetric pattern of plan-parallel flows. A really accurate assessment of the values of $Re_{\text{crit}}$ for these thermally asymmetric flows cannot be achieved only through a detailed stability analysis which, however, is beyond the scope of the present work.
the surfaces by,

\[ \text{R} \]

be given using the viscosity \( \mu \) of the fluid at the temperature of one of the surfaces by,

\[ \text{Ta} = 2 \left( \frac{R_1}{R} \right)^2 \frac{R_1 (\omega_1 - \omega_2)}{\mu_1 \rho \omega_1 R_1} \frac{R_1}{h} \]

where \( R_1 \) is the radius of the inner cylinder, and \( \omega_2 \) and \( \omega_1 \) are the an-

Table 3.3 Estimated transitional Reynolds numbers \( \text{Re}_{tr} \) for mineral oil VITREA 100 at various heat partition ratios, \( r_2 > 0.5 \), where film thickness \( h \) is to be substituted in meters. Estimated proportional reduction \( \text{Re} \) of the transitional Reynolds numbers for equipartitioning of the heat, \( r_1 = r_2 = 0.5 \).

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>( N_1 )</th>
<th>( \Gamma_{1,\text{lim}} )</th>
<th>( B_r )</th>
<th>( V )</th>
<th>( \text{Re}_{tr} )</th>
<th>( \Delta \text{Re} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>18.2138</td>
<td>1.8745</td>
<td>15.3465</td>
<td>2.1339</td>
<td>1.0162 ( x 10^6 ) h</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.51</td>
<td>15.8955</td>
<td>1.7990</td>
<td>14.1827</td>
<td>1.9513</td>
<td>0.9293 ( x 10^6 ) h</td>
<td>0.9597</td>
</tr>
<tr>
<td>0.55</td>
<td>11.8770</td>
<td>1.5968</td>
<td>11.5498</td>
<td>1.6035</td>
<td>0.7637 ( x 10^6 ) h</td>
<td>0.8518</td>
</tr>
<tr>
<td>0.60</td>
<td>9.6512</td>
<td>1.4370</td>
<td>9.5847</td>
<td>1.3939</td>
<td>0.6683 ( x 10^6 ) h</td>
<td>0.7666</td>
</tr>
<tr>
<td>0.70</td>
<td>7.3632</td>
<td>1.2534</td>
<td>7.1415</td>
<td>1.1667</td>
<td>0.5556 ( x 10^6 ) h</td>
<td>0.6590</td>
</tr>
<tr>
<td>0.80</td>
<td>6.0663</td>
<td>1.0150</td>
<td>5.6231</td>
<td>1.0322</td>
<td>0.4916 ( x 10^6 ) h</td>
<td>0.5895</td>
</tr>
<tr>
<td>0.90</td>
<td>5.1929</td>
<td>1.0104</td>
<td>4.5857</td>
<td>0.9385</td>
<td>0.4469 ( x 10^6 ) h</td>
<td>0.5390</td>
</tr>
<tr>
<td>1.00</td>
<td>4.5534</td>
<td>0.9373</td>
<td>3.8366</td>
<td>0.8676</td>
<td>0.4132 ( x 10^6 ) h</td>
<td>0.5000</td>
</tr>
<tr>
<td>2.00</td>
<td>2.0839</td>
<td>0.6088</td>
<td>1.2688</td>
<td>0.5991</td>
<td>0.2662 ( x 10^6 ) h</td>
<td>0.3248</td>
</tr>
<tr>
<td>5.00</td>
<td>0.8018</td>
<td>0.3692</td>
<td>0.3341</td>
<td>0.3385</td>
<td>0.1612 ( x 10^6 ) h</td>
<td>0.1969</td>
</tr>
</tbody>
</table>

Non-isothermal flows can better be studied in terms of a corrected Re which accounts for an average viscosity in the cross-section concerned instead of the viscosity at temperature of one of the surfaces. The corrected Reynolds number Re can be given, using the average viscosity in the cross-section, by,

\[ \text{Re} = \frac{\text{Ta}}{1 - \frac{r_1 \mu_1}{\mu}} \]

For fluids assumed to be free from changes in viscosity and density with temperature the critical Ta is known to be 3389.95, and the corresponding Reynolds number Re a critical value of 41.17, both values corresponding with a critical wave number of 3.117. These values are valid for small clearance to diameter ratios, and equipartitioning of the heat with equal boundary temperatures. For flows enclosed between surfaces of unequal temperatures, there are indications of smaller values of the critical Ta, and thus of the critical Re, Li [3.23].

Available results on the stability analysis of Li [3.23], however, are limited to thermal sliding parameters \( N_1 \), less than 4 for equipartitioning of the heat. For flows with adiabatic inner boundary, that is with \( r_1 = 0 \), \( N_1 \) values investigated lie close to 4.55 which corresponds to the limiting shear stress level. The corresponding critical Re-values are about 13 and 10 respectively. One may expect a further decrease of the critical Re with increased \( N_1 \) values in both the cases of \( r_1 = 0 \) and \( r_1 = 0 \), no investigation in this regard can be found in literature. The influence of thermal boundary conditions with heat partition ratios \( r_1 \) other than 0.5 and 0 on the critical Re remains speculative.

### 3.5 CONCLUSIONS

The foregoing analysis of non-isothermal shear flows influenced by viscous-shear heating of liquids following the exponential viscosity-temperature relationship has yielded some important results about instability of fluid flow under certain operating conditions. The regime of heat partition ratios \( r_1 < 0.5 \), where \( r_1 \) refers to the surface with respect to which the thermal sliding parameter \( N_1 \) is defined, need not lead to any thermal instability of the type suggested by Blok [3.5]. There may still prevail a different kind of instability which requires a due consideration.

Maximum shear stress, and thus maximum enforceable torque for the plan-parallel shear flows, operating with \( r_1 \geq 0.5 \), is limited to a finite value and this occurs at reasonably low thermal sliding speeds represented by the parameter \( N_1 \). The maximum (limiting) shear stress available in the operating regime \( r_1 \geq 0.5 \) is highest for equipartitioning of the heat, i.e. \( r_1 = r_2 = 0.5 \). This stress decreases with increased \( r_1 \), or say, increased thermal asymmetry. Reduced values of the non-di-
mensional shear stress $\Gamma_1$ at high values of $N_1$ may be interpreted for the quasi-shear type of thermo-elastohydrodynamic films in sliding contacts to affect lubrication beneficially in so far as frictional losses are concerned. This may also mean that the conventional design calculation methods, being based on isothermal flows, overestimate the losses in bearings, gear drives and in metal deformation processes such as strip and wire drawing. Flows with heat partition ratios $r_j \geq 0.5$ are characterized by double-valuedness of velocity and temperature distributions for all shear stresses lower than the maximum attainable at the $r_j$ value concerned. These distributions, however, are single valued in terms of thermal sliding parameter $N_j$ or Brinkman number $Br$ as shown in Figure 3.3. There will be a decrease in the volumetric flow rate with increased asymmetry at high levels of viscous dissipation, i.e., at large values of $N_1$. For flow conditions the transition of the liquid tends to be concentrated in a thin layer adjoining the surface absorbing the smaller amount of heat.

The existence of a maximum $\Gamma_1$ for a thermal asymmetry, as specified by the deviation of $r_j$ from 0.5, is quite significant for the safety and stability of the flow. The flow is stable and high Reynolds numbers can safely be achieved for all values of $N_1$ that are less than the corresponding transitional value. At low values of $N_1$, stability is highly sensitive towards the Prandtl number. In fact, when the latter is increased the critical Reynolds number is much reduced. This is especially important in lubrication practice, the usual lubricants possessing comparatively high Prandtl numbers. At higher values of the thermal sliding parameter $N_j$, however, the influence of $Pr$ becomes less predominant. The critical shear stress level $\Gamma_1$, or the Reynolds number may attain finite values depending on the film thickness.

The flow might become unstable at all values of $N_1$ exceeding the one corresponding to the transition point at $\Gamma_1, \lim$. The Reynolds number $Re$ at which the instability appears even decreases further in this range of increasing values of $N_j$. Stability criteria for the problems with thermal boundary conditions $r_j \geq 0.5$ can only qualitatively be established using the extensive data available for equipartitioning at the heat, or say for $r_j = r_j = 0.5$. This observation is based on the fact that the former problems can be reduced to the latter one by means of linear transformation of coordinates. It is also assumed that the nature of the flow at the above-mentioned transition point for $r_j = r_j = 0.5$ remains the same at the corresponding points for $r_j \geq 0.5$. The velocity and temperature distributions have neutral solutions of zero wave number in the present case, indicating the sensitive nature of the flow towards disturbances at these transition points; the distributions undergo instability beyond. Not only at, but also beyond the state given by $\Gamma_1, \lim$, the critical $Re$ rises to be finite.

As well as beyond, the transition point the critical Reynolds numbers for the flows with asymmetric thermal boundary conditions might be lower than those for thermally symmetric flows. The corresponding values of $N_j$ are also lower for increased thermal asymmetry. The reduction of the critical $Re$ at any given value of $Pr$ is directly proportional to that of the shear stress parameter $\Gamma_1$. For given material properties and a selected value of $N_j$, the danger of the appearance of an instability of either a velocity or a temperature profile is still dependent on the film thickness. This appears as a separate influential quantity in the critical Re. The instability associated with the temperature profile, i.e., of a thermal nature, is important in the regime beyond the transitional $N_j$ where the shear stress reaches a maximum. In this regime the instability might arise at very low Reynolds numbers, especially for fluids with high Prandtl numbers. For flows with asymmetric thermal boundary conditions differing from this simple type no quantitative results on stability criteria are available. For these flows only for the temperature distribution can the existence of an instability be postulated. So far it is only this limited analysis that predicts the present instability likely to appear at values of $N_1$, exceeding those at which the shear stress parameter $\Gamma_1$ reaches its limiting value.

3.6 REFERENCES

4.1 INTRODUCTION

Recent investigations on the behaviour of non-isothermal flows have shown some interesting results with respect to the thermal processes involved. There are indications that these processes can be beneficial as to reducing friction and thus the energy loss of lubricant films. On the other hand, when thermally important design details are not accounted for, thermal breakdown may occur as a consequence of certain inherent thermal instabilities; see Blok [4.2], Dakshina Murthy [4.3] and Sukanek et al. [4.4].

An analysis of the thermal processes concerned is strongly based on certain thermally affected properties of the lubricant, namely viscosity, and its internal energy increase by viscous dissipation when subjected to large shear stresses. It suffices to discuss the above dissipation effects of the lubricant in terms of shear stresses which are generated by sliding and/or by the pressure gradients acting on the film.

Also in the case of metal working processes operating under full film lubrication do thermal processes play an important role. Both dissipation effects, i.e. one due to viscous shear of the lubricant and the other due to plastic deformation of the working material, are indispensable to the analysis of these processes. An interesting survey has been made by Wilson [4.5].

Earlier works in the field of metal working processes accounted only for the heat due to plastic deformation while evaluating the film thickness distribution in the work zone; see eg. [4.6]. They excluded the effects of viscous dissipation in the lubricant. Recent work of Mahdavian and Wilson [4.7] is more representative of actual practice in that it does include the viscous dissipation in formulating the flow equations and thus the variation of film thickness in the work zone. This work is an extension to that of Dow, Kannel and Bupara [4.8] who have neglected the variation of viscosity across the film.

The above investigators, references [4.7] and [4.8], have accounted for the variation of surface temperature of the workpiece from the inlet up to the exit of the die. Their analyses are, nevertheless, far from complete since they do not reveal the thermal limitations involved. It will be shown in the analysis to follow, on the basis of the previously obtained results [4.3], that certain operating conditions might be severe in practice. Such severe operating conditions can also be specified for more general thermal boundary conditions involving unequal partitioning of the heat between the workpiece and the die.

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*The contents of this chapter were originally presented in the Sixth Leeds-Lyon Symposium on Tribology, "Thermal Effects in Tribology", held in Lyon, 18-21 September 1979, [4.1].
It may already here be stated that in the conjonctural zone of lubricating films in metal forming the pattern of fluid flow conforms better with shear flows than with pressure flows. The sensitivity of fluid flow rather towards sliding velocities than pressure gradients, at least in the work zone, opens up the possibility of applying the results of Chapter 3 to predict any thermal instabilities that may prevail in such lubricant films.

In the present work, the quasi-shear nature of the lubricating film in the plastically deformed work zone has been analysed for hydrostatic metal extrusion and the wire drawing process. Herebelow the operating regime of such processes as dealt with by Mahdavian and Wilson (4.7), and Snidle et al. (4.9), will be discussed for the possibility of thermal instability in the film under the more general type of the thermal boundaries constituted by the die and the workpiece surfaces.

4.2 ASSUMPTIONS ON MATERIAL PROPERTIES AND TYPE OF HEAT TRANSFER

The assumptions listed in Chapter 3 will be applied also to the present analysis in so far as the lubricant film is concerned. They are listed below for clarity.

1. The lubricant is assumed Newtonian,
   \[ \tau = \mu \frac{du}{dy} \]
2. The viscosity-temperature relationship is of the exponential type,
   \[ \mu = \mu_1 \exp \left[ -\theta (T - T_1) \right] \]
   where \( \mu_1 \) is a reference viscosity chosen at some standard temperature \( T_1 \) of surface 1 and under the local pressure prevailing in the film cross-section to be considered.
3. The dominant mechanism of heat development in the film is by viscous shear, and this is locally given per unit volume and unit time by,
   \[ \dot{q}_v = \tau \frac{du}{dy} = \mu \left( \frac{du}{dy} \right)^2 \]
4. The major process of heat transfer is by cross-conduction across the film, and convection heat transfer may be neglected. This is given by,
   \[ -k \frac{\partial T}{\partial y} \]

4.3 GENERATION OF HEAT IN METAL DEFORMATION PROCESSES

After having developed expressions for hydrodynamic and thermal parameters for the quasi-shear pattern of flow, Chapter 3, we must now analyse metal deformation processes. We also have to specify the conditions under which the problems studied in (4.7) and (4.9) offer results valid for the development of thermal instability in plastohydrodynamic lubricant films. For completeness we shall have to consider the general problem of heat generation by plastic deformation of the workpiece as well as that by viscous dissipation in the lubricant, represented by \( \dot{q}_{tot,P} \) and \( \dot{q}_{tot,v} \), respectively. Then the total amount of heat generated in any cross-section at a distance \( x \) from the apex of the working region, see Figure 4.1, can be given by,

\[ \dot{q}_{tot} = \dot{q}_{tot,P} + \dot{q}_{tot,v} \]

Figure 4.1 Non-isothermal flow of lubricant in hydrostatic extrusion.

Now one may visualize that some part of the heat flux is transmitted to the workpiece and the rest to the die. Under the assumption of a thin film, such as typical of plastohydrodynamic situations, the heat transfer takes place nearly wholly by cross-conduction, so that the transfer of heat by fluid convection will be ignored. Heat partition between the workpiece and the die in any film cross-section will be in proportions \( r_p \) and \( r_d \) that conform with an earlier definition for the heat partitioning,

\[ k \frac{\partial T}{\partial y} \bigg|_{b} - k \frac{\partial T}{\partial y} \bigg|_{d} = (r_p + r_d) h_{tot,v} \]

In formulating expression (4.1b), the heat developed by viscous dissipation within the fluid film has been considered to play a dominant role on the heat transfer; in contrast, it has been assumed that the heat generation by plastic deformation of the workpiece will in the first place create the relevant boundary temperatures at the lubricant-metal interface and, secondly, evoke definite temperature gradients at the boundaries. The latter temperature gradients define the extent of heat transfer from the film to the workpiece, and consequentially from the film to the die. One may further consider that the fluid at the boundaries of the workpiece and the die assumes the local surface temperature. One may also consider that temperatures in any film cross-section have reached an equilibrium level and there is no 'solid convection'. Here 'solid
convection' is characterized by the kind of heat transfer that is typical of moving solid bodies carrying away the heat from a stationary heat source. A number of investigations on thermal behaviour of thick hydrodynamic films and of thin elastohydrodynamic films are to be found in papers like those of Kannel and Zugaro [4.10], Cheng [4.11], etc. These analyses take into account an approximated surface temperature distribution identical to that of a simplified flash temperature theory in the form shown by Blok [4.12] to be sufficiently close to that following from the exact theory. It is assumed in the present analysis that a superposition of these surface temperatures on those that result from the plastic deformation of the workpiece is permissible, if required in future. The analysis to be carried out later in this chapter for thermal instability in plastohydrodynamic films is confined to the latter, i.e. the temperature increase due to plastic deformation and excludes the former, i.e. the surface temperatures resulting from the 'solid convection'. For the present, the analysis carried out in Chapter 3 can be applied to the situation of heat transfer under more general thermal boundary conditions involving unequal partitioning of the heat, i.e. \( t_b \neq 0.5 \) and \( t_d \neq 0.5 \).

For the sake of simplicity of the mathematical analysis the heat developed in the plastically deformed metal is assumed to be uniformly distributed across the workpiece cross-sections. Thus the temperature in any cross-section of the metal extrude will be uniform. The surface temperature of the workpiece equals the uniform bulk temperature in that cross-section, but will vary from inlet to outlet of the work zone. In the portion of the cross-section extending into the film, the heat generation need not be considered uniform. Heat development by viscous dissipation in such an extended cross-section in many cases has been found to be of the same order of magnitude as that developed by plastic deformation of the material involved.

In the regime of operation in which we are here interested, namely that representing a considerable viscous dissipation in the film, we will now analyse the non-isothermal flow of lubricant; we will also include high sliding speeds as well as low material reduction. The present, fairly general treatment will be based on the assumption of uniformity of the heat partition ratio over the complete work zone.

In order to incorporate the aforementioned local temperature variation along the boundary due to the plastic deformation process, we may employ the workpiece surface as the reference surface with respect to which the thermohydrodynamic and kinematic quantities will be established. Such a choice of the moving boundary as a frame of reference instead of the stationary surface of the die is converse to what Blok [4.2] employed in his original analysis. Nevertheless, the analysis becomes somewhat simpler in the present frame of reference when it comes to defining the heat partition ratio at arbitrary positions \( x \),

\[
t_b \theta_{tot} = \frac{k[1 + (r_b - 0.5)N]}{u_b}
\]

and the corresponding thermal sliding parameter:

\[
N_b = \frac{u_b k^2}{\theta_b}
\]

It has already been shown in Chapter 3 that a thermally critical state is conceivable for non-isothermal shear flow whenever the heat partition ratio \( r_b \) is less than 0.5. If one evaluates the operating conditions leading to these critical states, it may be conceived that such conditions should also define a limitation to lubrication in metal deformation processes.

4.4 LUBRICATION PROCESS IN THE WORK ZONE

Considering a uniform heat flux generation across any cross-section of the workpiece during its plastic deformation, the corresponding tempera-
ture along its boundary can be given by,

$$T_b = T_{bi} - \frac{\sigma_0}{pc} \ln \left( \frac{x_i}{x} \right)^2$$  \hspace{1cm} (4.4)

where $\sigma_0$ is the yield limit of the workpiece under uniaxial load, $pc$ is the volumetric specific heat of the workpiece material and the subscript $i$ refers to the operating condition at the inlet to the work zone. Rearranging the above expression and employing the temperature coefficient of viscosity $b$ one may obtain the temperature in a non-dimensional form,

$$b \ln \left( \frac{x_i}{x} \right)^2 = -2K \ln X$$  \hspace{1cm} (4.5)

It is well known that the surface velocity increases with the reduced workpiece thickness; thus considering its incompressibility, the velocity of the workpiece can be given by,

$$u_b = u_{bi} \left( \frac{x_i}{x} \right)^2 = u_{bi}/X$$  \hspace{1cm} (4.6)

We consider that under non-isothermal conditions the flow at the inlet is already fully developed. For the operating regime to be dealt with here there exists a stable flow, at least at the inlet, as long as the heat partition ratio $\tau_b$ and the thermal sliding parameter $N_b$ do not fall in the regime of thermal instability. While the former is assumed equal to the uniform value throughout the workzone, the latter can be given by,

$$N_b = \frac{\beta u_b x_i^2}{k}$$  \hspace{1cm} (4.7)

where $u_b$ is the sliding velocity of the workpiece with the die being stationary.

Solutions are first obtained considering no effects of friction at the boundaries; in other words, the coefficient of friction is assumed to be zero. The theory may then be extended to approximate the stress field in the workpiece for any assumed coefficient of friction. This process may have to be repeated if one desires to obtain an improved accuracy. By leaving out such an iteration technique, the procedure detailed below can be considered to be a first approximation. The pressure at any position is given by (4.7),

$$p = c_e + c_b \left[ 1 + \ln \left( \frac{x_i}{x} \right) \right] = c_e + c_b \left[ 1 - 2 \ln X \right]$$  \hspace{1cm} (4.8)

where $c_e$ is the extrusion pressure. If the viscosity-pressure-temperature relationship is expressed in the form:

$$\mu = \mu_b \exp \left[ \frac{\alpha}{T - T_{bi}} \right]$$  \hspace{1cm} (4.9)

where $\mu_b$ is the viscosity at any reference temperature $T_b$ and at the atmospheric pressure. By substituting this expression for the viscosity referred to the surface temperature of the workpiece the thermal sliding parameter in equation (4.7) can be obtained in the form:

$$N_b = \mu_b \exp \left[ \frac{\alpha}{T - T_{bi}} \right] \frac{\sigma_0}{pc} \left( \frac{x_i}{x} \right)^2$$

Using expression (4.5) for the temperature increase, and expression (4.8) for the pressure,

$$N_b = N_{bi} \exp \left[ G \left( \frac{c_e + \mu_b}{pc} + 1 \right) \right] \frac{x_i}{x}^2$$  \hspace{1cm} (4.10)

where $G = c_e/c_o$ and $G = c_e/c_o$. Further $N_b$ represents the thermal sliding parameter at the inlet and it does not include pressure effects on the viscosity. For convenience, the inlet thermal sliding parameter corrected for the extrusion pressure, i.e. $N_{bi} \exp G$ has been employed as a variable in the analysis to follow. However, the previously defined parameter $N_{bi}$ can be expressed in the form:

$$N_{bi} = \mu_b \exp \left[ G \left( \frac{c_e + \mu_b}{pc} + 1 \right) \right]$$

4.4.1 Film thickness distribution

If the correction factor $Q_i$ at the inlet corresponds to a specified thermal sliding parameter $N_{bi}$, the continuity of flow:

$$q = q_{bi} x_i$$

demands from equation (4.3):

$$Q u_l x_i = Q_{bi} u_{bi} x_i$$

Making use of expression (4.6) for the velocity an expression for the non-dimensional film thickness can be obtained in the form,

$$H = \frac{x_i}{x} \left[ \frac{\pi h}{Q_{bi} x_i} \right] X$$  \hspace{1cm} (4.11)

This expression is similar to that of Mahdavian and Wilson (4.7) but it takes into account through $Q_i$ the non-isothermal condition at the inlet. The calculation of non-dimensional film thickness $H$ at any position along the work zone reduces to the calculation of $Q^*$ for the corresponding sliding parameter. Assuming that the pressure does not vary along the work zone, the pressure gradient effects on the non-isothermal flow of the lubricant can be neglected. Then the flow can be analysed on the basis of the quasi-shear consideration of Chapter 3.

4.4.2 Velocity distribution

Velocity distribution in any film cross-section is represented using equation (3.6b) by,

$$u = u_b - \left( \frac{r_0 + 2/3h}{\sqrt{r_0^2 + 2/3h^2}} \right) \left( \frac{1 - r_0}{1 - r_0 + 2/3} \right) \left( \frac{1 - r_0 + 2/3}{1 - r_0 + 2/3} \right) + \frac{1 - r_0}{1 - r_0 + 2/3} \left( \frac{1 - r_0 + 2/3}{1 - r_0 + 2/3} \right)$$  \hspace{1cm} (4.12)

where $u = (u_b - u)/u_b$, and $y = y/h$. 

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4.4.3 Viscosity distribution

Viscosity distribution in any film cross-section can readily be expressed using the velocity distribution given in equation (4.12),

$$\frac{u_b}{u_d} = 1 + U (r_b - 0.5 U) N_b$$  \hspace{1cm} (4.13a)

where the viscosity ratio for the workpiece and the die surface is,

$$\frac{U_b}{U_d} = 1 + (r_b - 0.5 U) N_b$$  \hspace{1cm} (4.13b)

Viscosity distribution along the workpiece surface can be given by,

$$\frac{u_b}{u_d} = \exp \left[ -\frac{1}{2} (r_b - T_b) \right] = x$$  \hspace{1cm} (4.13c)

4.4.4 Shear stress distribution

The relative variation of shear stress along the film with respect to that at the inlet can be represented using the local value of the shear stress in a non-dimensional form, viz.,

$$\tau_1 = \left( \frac{\tau_b}{\tau_{1i}} \right) \left( \frac{u_b}{u_{1i}} \right)^{1/3} \left( \frac{u_b}{u_{1i}} \right)^{1/3} \tau^{1/3}$$

or, introducing the independent variables for $\left( \frac{\tau_b}{\tau_{1i}} \right)$,

$$\tau_1 = \left( \frac{u}{u_{1i}} \right)^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \tau^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \tau^{1/3}$$

Expression 4.14 represents the relative variation of coefficient of friction along the film, since the pressure is nearly constant in the work zone, viz.,

$$\frac{\tau_1}{\tau_{1i}} \approx \frac{\tau}{\tau_{1i}}$$

4.4.5 Heat flux distribution

If the heat flux distribution $\phi_v$ in any film cross-section is given by the expression:

$$\frac{G}{K} \phi_v = \frac{r_b}{r_{1i}} (1 + U (r_b - 0.5 U) N_b)$$

its relative variation along the film becomes,

$$\frac{\phi_v}{\phi_{vi}} = \frac{r_b}{r_{1i}} \left( \frac{G}{K} \right) \left( 1 + U (r_b - 0.5 U) N_b \right)$$

$$\frac{\phi_v}{\phi_{vi}} = \left( \frac{u}{u_{1i}} \right)^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \tau^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \tau^{1/3}$$

Total heat flux $\phi_{tot,v}$ in any film cross-section is provided by the expression:

$$\phi_{tot,v} = \frac{r_b}{r_{1i}} \left( \frac{G}{K} \right) \left( 1 + U (r_b - 0.5 U) N_b \right)$$

$$\phi_{tot,v} = \left( \frac{u}{u_{1i}} \right)^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \tau^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \left( \frac{u}{u_{1i}} \right)^{1/3} \tau^{1/3}$$

4.5 LIMITATION TO THE LUBRICATION PROCESS

We have already seen that the range of heat partition ratio $r_b<0.5$ leads to thermal conditions under which the correction factor $Q$ to the isothermal rate of flow may reach infinitely large values. Referring to equation (4.11) it may be observed that such values of $Q$ show a dangerously low value of film thickness which might ultimately become zero when the thermal sliding parameter reaches a critical value given by,

$$N_c = 1/(0.5 - r_b)$$

Thus the thermal instability as proposed by Blok [4.2] is realizable when the condition specified below is satisfied,

$$1/(0.5 - r_b) > N_c \exp \left[ G (c_E + 1) \right] x^{2(k-G-2)}$$

or, by rewriting:

$$N_c \exp \left[ G c_E \right] \text{crit} = \left[ [0.5 - r_b]^2 \right] x^{2(k-G-2)}$$

Each value of $r_b<0.5$ gives rise to a definite critical $N_c$ at some position $X$ along the film for any given values of $G, K$ and $N_c \exp G c_E$, the last depicting the effective sliding parameter corrected to the extrusion pressure. This value of $N_c$ is the upper limit for a thermally steady state to prevail in the lubricant film. For convenience of one single representation for all $r_b<0.5$ the above condition is given in the form,

$$F_1 = (0.5 - r_b) \cdot N_c \exp G c_E \left[ G \cdot x^{2(k-G-2)} \right] - 1$$

\hspace{1cm} (4.17)
\[ F \left( 0.5 - r_b \right) N_l \exp \left( G \left( c_r + 1 \right) \right) \mid_{\text{crit}} = X^{\left( -2(0.5 - r_b) \right)} \]  

(4.18)

\[ F = \left( 0.5 - r_b \right) N_l \exp \left[ a \left( \rho_0 \left( c_r + 1 \right) \right) \right] \mid_{\text{crit}} \]

Figure 4.3 Upper limit to thermally steady state for all \( r_b < 0.5 \) and selected values of \( G \) and \( K \) including those from references [4.7] and [4.9].

### 4.6 RESULTS AND DISCUSSIONS

#### 4.6.1 Thermal instability according to Blok in the range \( r_b < 0.5 \)

Figure 4.3 displays the limitation to the lubrication process discussed in Section 4.5 for various values of \( G \) and \( K \) along the work zone. One can obtain from expression (4.17) the critical thermal sliding parameter at the inlet which when exceeded would lead to a thermal instability of the type discussed in Chapter 3, Section 3.3.3. The position of initiation of the thermal instability along the work zone and the corresponding thermal sliding parameter are listed in Table 4.1 for a particular case of \( r_b = 0.4 \). At the positions tabulated there the effective sliding parameter reaches the critical value of \( 1/(0.5 - r_b) = 10.0 \).

Table 4.1 Position of thermal instability as proposed by Blok [4.2] and the corresponding thermal sliding parameter at inlet for \( r_b = 0.4 \), \( G = 4 \) and \( K = 3 \):

<table>
<thead>
<tr>
<th>( r_b )</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{\text{exp}} \left( G \rho_0 \right) )</td>
<td>0.183156</td>
<td>0.048013</td>
<td>0.008545</td>
<td>0.000750</td>
</tr>
</tbody>
</table>

Table 4.2 Position \( X \) of thermal instability using \( G = 4 \), \( K = 3 \) for a selected set of \( r_b \) and \( N_{\text{exp}} \left( G \rho_0 \right) \). For cases marked with an asterisk \( N_{\text{crit}} \) occurs well before the inlet; thus there is no stable film.

<table>
<thead>
<tr>
<th>Heat partition ratio ( r_b )</th>
<th>0.04</th>
<th>0.40</th>
<th>0.80</th>
<th>1.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{\text{exp}} \left( G \rho_0 \right) )</td>
<td>0.1327</td>
<td>0.1948</td>
<td>0.2545</td>
<td>0.3059</td>
</tr>
<tr>
<td>0.01</td>
<td>0.1596</td>
<td>0.6159</td>
<td>0.8054</td>
<td>0.9673</td>
</tr>
<tr>
<td>0.10</td>
<td>0.6159</td>
<td>0.9040</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>1.00</td>
<td>0.9040</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Results for a few more values of the heat partition ratio \( r_b \) have been provided in Table 4.2. In that table the position \( X \) has been specified at which the aforementioned critical state might appear for a selected thermal sliding parameter \( N_{\text{exp}} \left( G \rho_0 \right) \) at the inlet to the work zone. It may be observed that the position \( X \) of the critical state moves upstream for any specified \( r_b \) with an increased value of \( N_{\text{exp}} \left( G \rho_0 \right) \). A similar trend may also be observed with a decreased heat partition ratio in the range \( r_b < 0.5 \) for any selected value of \( N_{\text{exp}} \left( G \rho_0 \right) \). The critical state is expected to appear already upstream of the inlet to the work zone for lower values of \( r_b \) provided the thermal sliding parameter \( N_{\text{exp}} \left( G \rho_0 \right) \) is sufficiently large at the inlet. Such critical states have been marked with an asterisk in Table 4.2. A physical interpretation of the above-mentioned critical state appearing along the work zone seems to be made only on the basis of an experimental investigation; see also Chapter 3, Section 3.3.1.

Distributions of film thickness, shear stress, etc. can now be evaluated along the film for any set of inlet conditions, \( G \) and \( K \). These distributions may, however, have to be drawn separately for various values of \( r_b \), since these depend on the distribution of \( Q \). All these curves for certain operating conditions will tend to asymptotic values corresponding to the previously mentioned critical state as shown in Figures 4.4 and 4.5. For a given set of \( K \) and \( G \) the region of \( r_b < 0.5 \) gives also an upper limit to the thermal sliding parameter at the inlet, which exceeded will not develop any hydrodynamic film in the plastic work zone. Any value less than the upper limit will yield the thermal instability somewhere inside the film. This may be observed from Figure 4.3. At such positions along the film where the instability will appear the viscosity ratio, \( \mu_0/\mu_\infty \), and the parameter \( Q \) reaches infinitely large values. This fact reflects itself in reducing the film thickness to very low values and increasing the shear stress or coefficient of friction to infinitely large values. Then nothing may remain of the lubricant film and an implosion may result.
4.6.2 Thermal and flow instability in the range $\eta < 0.5$

Thermally stable conditions that are free of the instability as proposed by Blok are feasible in the range of $\eta < 0.5$. Thermal boundary conditions are, i.e. representing the equipartitioning of heat, $\eta = 0.5$, readily seen to yield a constant unit correction factor $Q$ to isothermal rate of flow throughout the film. Then the expressions for various thermohydrodynamic quantities reduce to,

$$Q^* = 1$$
$$H = X$$

(4.19a)

$$\frac{\tanh z}{1 - \tanh z} = \frac{X}{1 + 0.5 \eta \nu}$$

(4.19b)

$$Q_{\text{tot},v} = X^c (1 + 0.5 \eta \nu \lambda) \tanh z$$
$$Q_{\text{tot},vi} = (1 + 0.5 \eta \nu \lambda) \tanh z$$

(4.19c)

The corresponding numerical results have been shown in Figure 4.6. Relationship (4.19a) indicates that the film thickness is linearly dependent on the position along the film and is influenced neither by the heat generation due to plastic deformation in the workpiece nor that due to viscous dissipation in the film. Thus the flow is solely dependent on the mechanism of continuity, and the isothermal property of the film for the whole zone is retained. One may also notice that none of the material properties $G$, $\nu$, and $K$ have any influence on the film thickness. Shear stress and heat flux distributions along the film, however, are influenced by the corresponding thermal boundary conditions as well as by the material properties. Film thickness, shear stress and heat flux will all follow a thermally stable state, and the thermal instability discussed so far does not arise in the whole of the work zone. It may be noticed that the somewhat strange trend of the curves for $\eta = 0.00001$ and $0.01$ in Figure 4.6a is attributed to the nature of the solution for shear flows, already discussed in Chapter 3. The remaining curves show a monotonic trend corresponding to the region of low $\eta$ values. At very low values of $X$, which are not commonly employed in metal deformation processes, the shear stress reaches very low and the heat flux relatively large values.

In the operational regime covered by $\eta < 0.5$ there may, however, exist a different type of instability which has been discussed in Chapter 3, Section 3.4. This instability may be characterised by one of four modes: 'inviscid', 'thermal', 'viscous' or 'coupling'. These modes of instability have also been shown there to appear whenever the thermal sliding parameter exceeds that corresponding to the maximum (limiting) shear stress, also called the 'transition point'. The corresponding Reynolds numbers might be as low as those that can physically be real-
HEAT PARTITION RATIO

1. $r_b = 1.00$
2. $r_b = 0.60$
3. $r_b = 0.40$
4. $r_b = 0.20$
5. $r_b = 0.10$

Figure 4.6 Distribution of shear stress and heat flux along the work zone for $G=4$, $K=3$, $r_b=0.5$ and a few selected values of inlet sliding parameter $N_{\exp} G_0 E$.

Figure 4.5 Thermal and kinematic condition of the film for $G=4$, $K=3$, $N_{\exp} G_0 E = 0.1$ and a wide range of $r_b$: a) film thickness, b) shear stress and c) heat flux.

The results indicate that the transition point moves upstream for increased values of either $r_b$ or $N_{\exp} G_0 E$. At very low values of $N_{\exp} G_0 E$, there is no danger of the existence of the instability since the transition point appears at very low values of $X$ which are not commonly employed in metal deformation processes. However, at high values of $N_{\exp} G_0 E$, the corresponding $X$ values are found to be as large as those commonly found in practice.
transition point occurs already at upstream of the inlet; these have been indicated in Table 4.3 by an asterix. Thus there may exist two branches for the values of X lower than those specified below, and for the cases marked with an asterix in the entire work zone. Some selected parameters are: G=4, K=3 and various values of $r_f$.

<table>
<thead>
<tr>
<th>$N_{\exp} G_G^*$</th>
<th>$X$</th>
<th>$H$</th>
<th>$r/f_1$</th>
<th>$h_{loc}$</th>
<th>$h_{loc}/v_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_f=0.50; R_{f*}=18.2138$</td>
<td>0.00001</td>
<td>0.1762</td>
<td>0.1762</td>
<td>2.4922</td>
<td>83.1386$\times 10^3$</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.3574</td>
<td>0.3574</td>
<td>0.9238</td>
<td>27.6750</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.8118</td>
<td>0.8118</td>
<td>0.7641</td>
<td>2.5490</td>
</tr>
<tr>
<td>$r_f=0.31; R_{f*}=15.0952$</td>
<td>0.00001</td>
<td>0.1803</td>
<td>0.1842</td>
<td>2.4505</td>
<td>71.3350$\times 10^3$</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.3501</td>
<td>0.3819</td>
<td>0.8123</td>
<td>23.6530</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.8039</td>
<td>0.8039</td>
<td>0.7722</td>
<td>2.5141</td>
</tr>
<tr>
<td>$r_f=0.00; R_{f*}=9.6512$</td>
<td>0.00001</td>
<td>0.1905</td>
<td>0.2184</td>
<td>2.1893</td>
<td>37.4490$\times 10^3$</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.4156</td>
<td>0.4846</td>
<td>0.7254</td>
<td>12.8080</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.9004</td>
<td>0.944</td>
<td>0.8230</td>
<td>1.6549</td>
</tr>
<tr>
<td>$r_f=1.00; G_G^* = 4.5534$</td>
<td>0.00001</td>
<td>0.2221</td>
<td>0.2734</td>
<td>1.6088</td>
<td>13.4180$\times 10^3$</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.7012</td>
<td>0.8297</td>
<td>0.6231</td>
<td>3.1963</td>
</tr>
</tbody>
</table>

4.7 CONCLUSIONS

Solutions developed for thermal instability in non-isothermal shear flows can readily be extended to the analysis of the lubrication as influencing the thermal effects of metal deforming. No measured results have, however, been found in literature specifying the operating conditions leading to the above-mentioned instability.

Results of dimensionless form for film thickness, shear stress and heat flux distributions have been presented for a few selected values of the heat partition ratio $r_f$ and for the inlet parameter $N_{\exp} G_G^*$. 0.00001 and 0.1 in Figures 4.4 and 4.5, respectively. It may be observed that the qualitative nature of the results is identical to that expected from references [4.2] and [4.3]. From the inlet to the exit of the work zone, the reduction in the material cross-section tends to increase the surface velocity of the workpiece. On the other hand, the resulting variation of the sliding parameter is solely influenced by the surface velocity and the viscosity of the lubricant.
In view of the above-mentioned thermal instabilities due to the presence either of the critical state for \( r < 0.5 \), or of the 'transition point' to the thermal sliding parameter for \( r > 0.5 \), the lubricant film in the work zone has preferably to be subjected for operation at low thermal sliding parameters. Thus, one may have to carefully select the inlet sliding parameter and the heat partition ratio in order to maintain a safe full film in metal deformation processes. The latter may often be controlled by preferential cooling of the involved workpiece and die.

4.8 REFERENCES


Chapter V

NON-ISOTHERMAL SHEAR FLOWS WITH SLOTTE’S μ-T RELATIONSHIP

5.1 INTRODUCTION

There exist a number of viscosity-temperature relationships for liquids commonly employed in the lubrication practice. Although a few of them are accurate enough to predict the viscosity change due to a change in temperature, their regime of accuracy is limited, and the accuracy decreases rapidly outside this regime. It is thus of importance to employ a proper relationship in the range of temperature in which one expects mass and energy transport processes to take place. This aspect has been discussed in somewhat greater detail elsewhere, see Chapter 2. It may here be pointed out that a kinematic process connecting the viscous and thermal changes is assumed due mainly to temperature changes, and the liquid is assumed to remain Newtonian.

The regime of accuracy of some specific models representing the viscosity-temperature relationship, along with their characteristic representation, has been explained in Chapter 2. It has also been explained there that some models are easy to work with while studying the involved kinematic process. Slotte’s relationship is one among them, and it is sufficiently accurate in a wide temperature range. In such a range it is proved to be more accurate than the exponential relationship used in Chapter 3. It also offers a mathematical simplicity because of its two parametric nature, and the flow and temperature distributions as influenced by viscous dissipation effects can almost equally be easily evaluated as in the case of the exponential relationship. Slotte’s relationship can be represented by,

\[ \mu = \frac{\nu_1}{(1 + \frac{T - T_0}{A})^m} \]

where \( m \) and \( A \) are characteristic parameters. By a proper choice of these parameters almost all the liquids, at least those complying with the Newtonian behaviour, and used in the engineering practice, can be described for their change of viscosity with temperature. With \( m=1 \) the above relationship becomes a linear one for fluidity – the inverse of viscosity – with the temperature. As it has already been pointed out in Chapter 2, \( m=\infty \) represents the exponential relationship through the classical definition of the exponential function \( e^x \), i.e.

\[ \lim_{m \to \infty} \left( 1 + \frac{z}{m} \right)^m = e^z \]

Replacing \( z \) by \( (T-T_0) \) in the above expression one may notice that the previously used \( \theta \) in the exponential relationship can be shown to be a particular case of \( \theta \) which is termed as the scaling temperature. This scaling temperature is related to the viscosity and viscosity gradient at the previously defined reference temperature \( T_r \) through,

\[ \theta = \left[ \frac{1}{\mu} \frac{d\mu}{dT} \right] \frac{m}{T_r} \]

Employing \( \theta \) in the solutions yet to be developed, one may provide a common ground for a comparison of the results using Slotte’s with those already obtained in the previous chapter using the exponential relationship.

These two cases involving \( m=1 \) and \( m=\infty \) have acquired a considerable importance, in the analysis of both shear and pressure viscous flows, by various investigators. This interest may easily be attributed to the comparative simplicity the mathematical analysis involving the momentum and the energy equation would acquire through the introduction of these two relationships. For other positive values of \( m \), however, the analysis becomes too involved. Analytical solutions for these cases are not readily available except for a few cases such as \( m=1/3, 2 \text{ and } 3 \). A general solution for a complete range of \( m \), viz. \( 0<m<\infty \) is not available in a closed form. In the present chapter numerical solutions have been provided, and also they have been compared with those available, although only in terms of special functions, for a few specific cases of \( m \).

5.2 SOLUTIONS TO VARIOUS THERMOHYDRODYNAMIC QUANTITIES

5.2.1 Influential quantities in non-dimensional form

In what follows only steady states will be analysed. The analysis has been carried out here on parallel lines with that of Chapter 3 based on the exponential viscosity-temperature relationship. All the assumptions made in Section 3.2.1 hold also valid, except for the viscosity-temperature relationship which is here assumed to follow Slotte’s type.

In order to introduce the dimensionless variables namely a thermal sliding parameter and a temperature parameter we make use of the equivalent viscosity-temperature coefficient \( \theta \) as in the exponential relationship at any reference temperature \( T_r \) at surface 1, namely,

\[ \theta = \left[ \frac{1}{\mu} \frac{d\mu}{dT} \right] \frac{m}{T_r} \theta \]

where \( m \) and \( A \) are coefficients in Slotte’s relationship expressed here again for convenience,

\[ \mu = \frac{\nu_1}{(1 + \frac{T - T_0}{A})^m} \]

where \( \nu_1 \) is the viscosity at the temperature \( T=T_0 \) of surface 1. Further on we introduce the dimensionless temperature \( f=-\theta (T-T_0) \) to obtain,

\[ \mu \theta = \left[ \frac{1}{\mu} \frac{d\mu}{dT} \right] \frac{m}{T_r} \theta \]

It may here be noted that the above expression is in a convenient form, and any comparison of the analysis and the results to follow with those of the exponential viscosity-temperature relationship becomes easy to perform. For ease of reference an equivalent form of the exponential re-

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The relationship is expressed, with \( \theta \equiv (T - T_0) \), as,

\[ \frac{\theta}{\theta_1} = e^{\frac{\theta}{\theta_1}} \]

We also define the thermal sliding parameter involving the viscosity at surface 1 as,

\[ \eta_1 = \frac{\theta}{\theta_1} \]

which is identical to that used in the analysis based on the exponential viscosity-temperature relationship, namely

\[ \eta_1 = \frac{\theta}{\theta_1} \]

As already pointed out by Blok [5.1] this parameter can be related to a new thermal sliding parameter \( \eta_{31} \) by making use of Slotte's exponent \( m \),

\[ \eta_{31} = \left(1 + \frac{1}{m}\right) \eta_1 \]

which again reduces to \( \eta_{31} - \eta_1 \) whenever \( m \to \infty \). Assuming that there exists a constant shear stress across the film the shear stress parameter may be defined as,

\[ \Gamma = \frac{t}{H} \left(1 - \frac{u_0^2}{u_1^2}\right) \]

which is identical with that defined for the exponential relationship.

Now having defined dimensionless parameters representative of the viscosity, temperature, and the operating conditions through \( \eta_1 \) we proceed to give solutions to the thermohydrodynamic problem corresponding to plan-parallel shear flows. It is interesting to study the maximum shear stress that can be imposed on such a plan-parallel film when the liquid selected might be represented using Slotte's relationship, namely

\[ \frac{\theta}{\theta_1} = e^{\frac{\theta}{\theta_1}} \]

or in the non-dimensional form,

\[ \frac{d^2 \theta}{dy^2} + \theta^2 \eta(0) = 0 \]

where

\[ \eta = \left(1 + \theta/\theta_1 \right)^m \]

The boundary conditions to be used here comply with those on Figure 5.1.

**Temperature and viscosity distributions**

It may be conceived that for the heat partition ratio \( r_1 \), referred to surface 1 confined to the range \( 0 < r_1 < 1 \), the maximum temperature occurs within the hydrodynamic film*. At the position of such a maximum temperature \( \theta_H \) there prevails a velocity \( u_H \) given by,

\[ u_H = r_1 u + \left(r_1 - r_0\right) \]

It may further be shown on similar lines as before, see Blok [5.1], that the viscosity distribution across the film becomes,

\[ \frac{\eta}{\eta_1} = \left(1 + u(1 - 0.5u)/\eta_1 \right)^m \]

It may also be conceived that the maximum temperature for the heat partition ratios outside the range \( 0 < r_1 < 1 \) occurs at one of the surfaces. These heat partition ratios, however, do not impair the present analysis of the thermohydrodynamics of the film.

---

*It may also be conceived that the maximum temperature for the heat partition ratios outside the range \( 0 < r_1 < 1 \) occurs at one of the surfaces. These heat partition ratios, however, do not impair the present analysis of the thermohydrodynamics of the film.
It may now be shown that for the maximum temperature occurring at surface 2, the above expressions reduce to simple forms,

\[(u_i / u_0)^{1/n} = 1 + (r_1 - 0.5)N_{s1},\]  

\[r_i = [1 + (r_1 - 0.5)N_{s1}]^{1/n} - 1\]  \((5.3c)\)

The range of heat partition ratio \(r_1 \leq 1\) or \(r_1 \leq 0\) will also yield the results pointed out above. The range \(0 < r_1 < 1\) will yield a maximum temperature somewhere within the film depending upon the value of \(T_1\). It may again easily be shown that for the above complete range of \(r_2\), i.e. \(-\infty < r_2 < +\infty\), expressions (5.3c) and (5.3d) for viscosity and temperature reduce to,

\[
\begin{align*}
(u_i / u_0)^{1/n} & = [1 + 0.5r_1^2N_{s1}]^{1/n} - 1 \\
\end{align*}
\]  \((5.3e)\)

Shear stress

It can now be shown that the shear stress in a dimensionless form be represented for the presently formulated problem as in Blok's [5.1] work. In order to do so, we start with the Newton's definition of viscosity, namely,

\[\tau = \mu \frac{du}{dy}\]

Separating the variables and integrating in the corresponding range,

\[
\int_0^y dy = \int_{u_1}^{u} (u/r) \, du
\]

For a pure shear flow we already know that the shear stress in the cross-section is independent of \(y\); then substituting for \(\mu\) from the above analysis, and noting that, \(du = u \, dU\)

the above expression for the whole of the cross-section reduces to,

\[
\frac{1}{2} \mu \frac{du}{dy} = \int_0^y \left[ 1 + r_1 N_{s1} \frac{u - 0.5N_{s1}u}{u} \right]^{1/n} - 1 \, du
\]

With the introduction of a new variable \(t\) in the place of \(U\), the above expression can be cast into a form that can easily be compared with that for the shear stress evaluated using the exponential viscosity-temperature relationship. Further on it becomes easy to compare the magnitude of the maximum attainable shear stress when various liquids with differing Slotte's exponent in the range \(0 < \omega < \infty\) are considered. Let

\[t = \left(1 + 0.5r_1^2N_{s1}\right)^{1/2}N_{s1}/2\]  \((5.4a)\)

or

\[u = r_i \left(1 + 0.5r_1^2N_{s1}\right)^{1/2}N_{s1}/2\]  \((5.4b)\)

thus

\[\frac{du}{dy} = \left(1 + 0.5r_1^2N_{s1}\right)^{1/2}N_{s1}/2\]  

Then

\[
\int_0^y \left[ 1 + r_1 N_{s1} \frac{u - 0.5N_{s1}u}{u} \right]^{1/n} - 1 \, dt = \int_0^y \left(1 + 0.5r_1^2N_{s1}\right)^{1/2}N_{s1}/2 \, dt
\]

The limits of integration reduce to,

\[u = 0, \quad t_1 = \frac{1}{\left(1 + 0.5r_1^2N_{s1}\right)^{1/2}}\]

\[u = 1, \quad t_2 = \frac{1 - 1/r_1}{\left(1 + 0.5r_1^2N_{s1}\right)^{1/2}}\]

One can also express \(t_2\) in terms of \(t_1\), then,

\[t_2 = (1 - 1/r_1) \cdot t_1\]

The non-dimensional form for the shear stress \(\tau_1\) as referred to surface 1 then reduces to,

\[
\tau_1 = \frac{t_1 \cdot \left(\frac{dN}{dt}\right)}{1 + r_1 N_{s1} \frac{u - 0.5N_{s1}u}{u}} = \frac{\chi h_{s1} \left(\frac{u_{s1}}{1 + r_1}\right)^1}{(1 + r_1 N_{s1})^{1/n}}
\]

\[\left(\frac{2m}{m^2}\right)^{1/2} \left(1 + \frac{r_1 N_{s1}}{1 + r_1}\right)^{1/n-1} \frac{dt}{t_2 (1 - t^2)^{1/n}}
\]

Velocity distribution

It may be shown on similar lines as those followed up for the evaluation of shear stress, that when the integral is extended over a portion of the cross-section, see equation (5.5), one also obtains,

\[
\int_0^y \left(\frac{dN}{dt}\right) = \left[1 + r_1 N_{s1} \frac{u - 0.5N_{s1}u}{u}\right]^{1/n} - 1 \, dt
\]

An implicit relationship for the velocity distribution using the above expression along with equation (5.5) reduces to a simple form as
follows,

\[
(1 - \tau) = \int_0^1 \frac{dt}{\int_0^1 \frac{dt}{(1-t)^{m+1}}} \tag{5.6}
\]

Although the velocity expressed in an implicit form above consists of involved integrals they may be simplified at least for certain values of \( m \). The velocity may be evaluated in an explicit form for such cases. This process, however, involves evaluating the above integrals in a closed form, if only in terms either of trigonometric functions in the case of \( m=1 \) or of Jacobian functions for other values of \( m \). These solutions will be shown in the sections to follow.

Heat flux distribution

The local heat flux developed per unit time and per unit volume due to viscous dissipation can be evaluated using the definition,

\[
\dot{q}_v = \frac{\tau v}{U}
\]

and can be expressed using equation (5.3a) for \( v \) in the convenient form,

\[
\dot{q}_v = (k_B /k) h^4 \left( 1 + \frac{u}{u_0} \right) \left[ \frac{1}{1 + (r_t - 0.5)u_0 - 0.5u} \right] \left( \frac{1}{n} \right) \times \int_0^1 \frac{dt}{(1-t)^{m+1}} \tag{5.7}
\]

Further on the total heat flux in any film cross-section can directly be evaluated using the expression already developed for the non-dimensional shear stress. This is done simply by multiplying \( T_1 \) by \( N_{s1} \), since, \( \dot{q}_{\text{tot}} = T_1 u_0 \), or,

\[
\dot{q}_{\text{tot}} = T_1 u_0
\]

Volumetric rate of flow

The volumetric rate of flow can now be evaluated using the expressions so far obtained for the velocity and viscosity distributions in any film cross-section under the specified heat partition ratio and thermal sliding parameter. Starting from the expression for the rate of flow,

\[
q = \int_0^1 u dy
\]

and replacing the coordinate \( y \) by the local velocity \( u \) with the help of Newtonian definition for viscosity, and employing the viscosity distribution from equation (5.3a), it may be shown that,

\[
q = \frac{\nu u_0}{T} \int_0^1 \frac{U (u_0/u) du}{1 + \left( 1 + r_t N_{s1} - 0.5u_0, u_0 \right)^{(r_t-0.5)u_0} - 0.5u} \int_0^1 \frac{dt}{(1-t)^{m+1}} \tag{5.2}
\]

With the introduction of the integration variable \( t \), we obtain

\[
q = \frac{\nu u_0}{T} \int_0^1 \frac{U (u_0/u) dt}{1 + \left( 1 + r_t N_{s1} - 0.5u_0, u_0 \right)^{(r_t-0.5)u_0} - 0.5u} \int_0^1 \frac{dt}{(1-t)^{m+1}} \tag{5.9}
\]

Substituting for the shear stress from expression (5.5) and simplifying,

\[
q = \nu u_0 \left( r_t (u_0/u_0) - 0 \right) \tag{5.10a}
\]

where

\[
h = (r_t^2 + 2/N_{s1}) \int_0^1 \frac{dt}{(1-t)^{m+1}}
\]

The integral in the numerator can be evaluated in a closed form and \( B \) can further be simplified to:

\[
h = (r_t^2 + 2/N_{s1}) \int_0^1 \frac{dt}{(1-t)^{m+1}} - \int_0^1 \frac{dt}{(1-t)^{m}}
\]

The expression for the volumetric rate of flow can also be written down in a form similar to that shown in the case of the exponential viscosity-temperature relationship (see equation 3.14b),

\[
q = (u_0/u_0) h + (r_t - 0.5)u_0 \int_0^1 \frac{dt}{(1-t)^{m+1}}
\]

or, in one of the non-dimensional forms representing a correction to the isothermal flow,

\[
q = \frac{0.5(u_0/u_0) h}{(r_t - 0.5)} - B
\]

or

\[
q = \frac{0.5 u_0 h}{(r_t - 0.5)} - 2(r_t - B)
\]

Expression (5.10a) can, however, be rearranged and expressed in a slightly different form in order to use in the non-isothermal EHD analysis yet to be carried out in Chapter 11. This has been carried out on lines similar to those followed in Chapter 3. Multiplying equation (5.10a) throughout by \( \sqrt{N_{s1}/2} \), one obtains,

\[
q = \frac{\nu u_0}{T} \int_0^1 \frac{U (u_0/u) dt}{1 + \left( 1 + r_t N_{s1} - 0.5u_0, u_0 \right)^{(r_t-0.5)u_0} - 0.5u} \int_0^1 \frac{dt}{(1-t)^{m+1}}
\]

Introducing the previously defined thermal sliding parameter \( N_{s1} \), and terming the above quantity as flow parameter \( V \), viz.,

\[
V = \left( \frac{\nu u_0}{T} \right) \left( 1 + r_t \right) \left( (u_0/u) h - 0 \right) \int \frac{dt}{(1-t)^{m+1}}
\]

Thus, the expression for the volumetric rate of flow can also be written down in a form similar to that shown in the case of the exponential viscosity-temperature relationship (see equation 3.14b),

\[
q = (u_0/u_0) h + (r_t - 0.5)u_0 \int_0^1 \frac{dt}{(1-t)^{m+1}}
\]

or, in one of the non-dimensional forms representing a correction to the isothermal flow,

\[
q = \frac{0.5(u_0/u_0) h}{(r_t - 0.5)} - B
\]

or

\[
q = \frac{0.5 u_0 h}{(r_t - 0.5)} - 2(r_t - B)
\]
In the above mentioned expression the term \( u_i = 0 \) if surface 1 is stationary. It may be noticed that these expressions for the non-dimensional volumetric flow using Slotte's viscosity-temperature relationship is almost identical to the one derived using the exponential relationship. The only parameter that differs from the previous one is \( B \). A comparison of the flow parameter obtained in the above mentioned two methods, reduces to a simple one involving \( A \) and \( B \). The evaluation of \( B \) indeed involves solving the definite integral that has previously appeared in expression (5.5) for the non-dimensional shear stress. This process, however, does not involve solving the indefinite integral that has appeared in the implicit expression for the velocity distribution. It will soon be shown that the evaluation of the definite integral in a closed form is limited only to a certain range of operational parameters \( r_j \) and \( N_j \). For the rest of the range the integral has to be evaluated by numerical methods.

### 5.2.3 Thermal boundary conditions and thermal steady states

It may be conceived that a control on the heat partition ratio can be achieved only by indirect means such as a preferential cooling of the bounding surfaces across the film. In engineering practice a large variation in \( r_j \) might be achieved by a proper choice of materials with differing thermal conductivities both for the liquid and the solid boundaries the temperature gradients at the interface in these two bodies would be identical. However for a continuity of heat flow, and for the absence of temperature jumps one can employ the thermal boundary conditions at the liquid-solid interface at surface 1 as follows,

\[
\gamma = 0, \quad T_1 = T_s, \quad \frac{dT_1}{dy} = \frac{dT_s}{dy} = r_i \text{hot}
\]

where subscripts 1 and s refer to the liquid and solid respectively. For isothermal boundary condition \( T_i \) is maintained at a constant value and all the heat developed in the film cross-section is rapidly carried away by cross-conduction. A schematic representation of the specification of thermal boundary condition has been specified in Figure 5.1.

The surface temperature does not undergo any increase, and \( r_i \) above might assume unity or 1/2 depending upon the process of heat conduction. If all the heat developed in the film cross-section is taken away by only one surface, say at \( y = 0 \), then \( r_i = 1 \). If only half the heat is taken away by this surface, then \( r_i = 1/2 \). Conversely the heat partition ratio at surface 2 will be \( r_i = 0 \) and 1/2 respectively. In both of the above cases the surface temperature at \( y = 0 \) is readily conceivable to remain at isothermal level. An adiabatic state on the surface at \( y = 0 \) is also conceivable when \( r_i = 0 \) (or \( r_i = -1 \)).

Intermediate cases of \( r_i \) differing from 0 or 1/2 are imaginable. In addition to another specific case of adiabatic boundary at \( y = 0 \) resulting in \( r_i = 0 \), various other cases of \( r_i \) lying in the range 0 < \( r_i \) < 1 are conceivable. Even the range of \( r_i \) such as \( r_i = 0 \) and \( r_i = 1 \) can also be conceived if the heat developed in the solids along the same film cross-section overrides that in the film, and the flow of heat takes place from one of the surfaces through the film cross-section to the other surfaces. When we confine the thermal analysis in the present investigation only to the liquid, the steady state for the film is observed to exist if the rate of heat generation within the film cross-section does not exceed the rate of heat removal at the solid boundaries. The condition for the steady state to prevail depends on the value of thermal boundary condition to be prescribed through the parameter \( r_i \) and also on the value of the sliding parameter. The latter might also be replaced, if necessary, by a selected value of the shear stress parameter \( \gamma \).

In the classical thermal analysis one commonly encounters ideal isothermal and adiabatic conditions prescribing the boundary heat transfer. Mixed problems in which the heat transfer might not take place in either of the above idealised forms has been analysed by Gruntfest [5.2]. He carried out numerical computations for the thermal boundary conditions of the slab of the working fluid regulated through the enclosing slabs of insulation. The outer boundaries of the insulation are held at the initial temperature. By employing identical volumetric heat capacity of the fluid and solids, and allowing the thermal conductivity to vary relative to that of the fluid, he analysed the maximum temperature as a function of the thermal boundary conditions. It has been shown by Gruntfest that the influence of changes in the thermal boundary conditions on the response of a fluid overrides the influence of changes in the shear stress.

It may be observed from his results that the stresses for which the steady states exist may be varied by properly manipulating the thermal boundary conditions. To the knowledge of the present investigator the steady cases for various thermal boundary conditions are not available, except in a fragmentary form by Joseph and Sparrow [5.3]. The previous analysis of Chapter 3 using the exponential relationship can now be extended using Slotte's viscosity-temperature relationship.

### 5.2.4 Occurrence of critical states and thermal instability

Having observed that the heat partition ratio \( r_i \) may assume any real values in the range \(-\infty < r_i < \infty\), we may note that \( N_j \) is always positive and Slotte's exponent \( m \) should also be positive to cover the liquids that might be employed in engineering practice including that of lubrication. Keeping in mind the various non-dimensional hydrodynamic parameters developed in the previous section, we might observe that the evaluation reduces to that of the integral, namely,

\[
\int_t^{t_i} \left( \frac{m}{t} \right)^{\frac{1}{m-1}} dt
\]

where \( t \) and \( t_i \) have been specified in terms of the operational parameters \( r_j \) and \( N_j \) in Section 5.2.2. The two extreme values of \( t_j \) i.e. \( t_1 \) and \( t_2 \) have been specified in terms of the non-dimensional velocity \( U \). It may also be observed that for positive definite values of \( m \), the exponent of the denominator of the integrand attains positive fractional value in the range of \( n \) the integrand is real only for positive values of the term \( 1\left(1-t^j\right) \) and complex for negative values. It is thus apparent that the integrand should assume real values in the whole range of integration specified through \( t_j \) and \( t_2 \). For \( t_j \) representing the condition \( U = 0 \), it is readily seen that the integrand satisfies the condition of real valuedness. For \( t_2 \) representing \( U = 1 \), however, the integrand assumes real values only in a definite range of \( r_i \) and \( N_j \) as specified below. For the integrand to assume non-complex values, the following integral to represent physically realisable situation of a thermo-hydro-
dynamic flow, it is seen that,
\[(1-t^2) = \frac{(r_1-0.5)^2}{(1+0.5r_1^2)N_1} > 0\]
For the particular case of \(U=1\), the above inequality gives rise to,
\[1 + (r_1-0.5)N_1 > 0\]
For the operational range of thermal sliding parameter, namely \(0<N_1<\infty\), representing the whole regime of viscous dissipation in shear flows, it is seen that the above condition might be violated for the range of \(r_1<1/2\) whenever \(N_1\) takes values exceeding those given by,
\[N_1, \text{crit} = 0.5-r_1\]
The critical value of \(N_1\), namely \(N_1, \text{crit}\) is established in the analysis employing the exponential viscosity-temperature relationship. On a similar discussion as that presented in the latter case, it may be concluded that thermally steady states are existent only in the range,
\[N_1 < \frac{1}{0.5-r_1}\]
where \(N_1\) is independent of \(m\). For a comparison of results for various fluids covering the Slotté's exponent \(0<m<\infty\) including that corresponding to exponential relationship, it will be convenient to re-express this expression in a form pointed out before, namely \(N_1/N_1,x(m/(m+1))\), thus reducing the inequality to,
\[N_1 < \frac{N_1,x}{m/(m+1)}\]
In the section to follow we will, however, deal with the question of solutions to various thermohydrodynamic quantities and we will also maintain that the solutions will be valid in the range of thermally steady states defined above.

5.3 SOLUTIONS IN CERTAIN DEFINITE OPERATING REGIMES

For various values of \(r_1\) covering the physically realisable range, and for the corresponding physically realisable values of \(N_1\), we may notice that the limits of integration in the previously obtained integral lie in the range \(-1 \leq t_2 \leq 1\). A detailed specification of the range of \(t_1\) and \(t_2\) corresponding to various values of \(r_1\) can be listed as follows.

5.3.1 The region of heat partition ratio, \(r_1 \leq 1/2\)

In this region of \(r_1\) it is expected that the shear stress parameter \(\Gamma_1\) given by equation (5.5) will remain finite in the complete range of the thermal sliding parameter \(N_1\). Infinitely large values for \(N_1\) are conceivable. The range of the integration limits \(t_1\) and \(t_2\) can thus be established for various combinations of \(r_1\) and \(N_1\). Recalling the definition of parameters \(t_1\) and \(t_2\),
\[t_1 = \frac{N_1}{1+N_1} \quad t_2 = \frac{N_1}{1+N_1}\]
we can further establish the range of \(t_1\) and \(t_2\) relevant to the evaluation of the solution as follows.

1. For \(N_1, \text{crit} = 0\), we have isoviscous flows, and obviously viscous dissipation effects do not play any role in influencing the hydrodynamics; \(t_1=0, t_2=0\).
2. For \(N_1, \text{crit} \rightarrow \infty\), the limit \(t_1\) will remain constant at unity and the other limit lies in the range \(-1 \leq t_2 \leq 1\). A few cases of special interest are those corresponding to isothermal boundary condition at surface 1, namely \(t_1=1/2, t_2=1, t_1=0, t_2=1, t_1=m, t_2=1\).
3. For any positive and real value of \(N_1\), \(0<N_1<\infty\), we have the representative region of viscous dissipation in engineering applications. For such values of \(N_1\) we have \(0<t_2<1\) for all real values of \(r_1\), namely \(0.5<r_1<\infty\). Correspondingly \(t_2\) assumes \(-1 \leq t_1 \leq 1\). The particular case of \(r_1=0\) gives rise to an extremum of \(t_1=1\) and \(t_2=1\).

5.3.2 The region of heat partition ratio, \(r_1>1/2\)

In this region the thermal sliding parameter that can be prescribed is limited only to a critical value mentioned in Section 5.2.4. Since there is no solution to steady state problem in the region of \(N_1\) exceeding the critical value given by,
\[N_{1, \text{crit}} = \frac{1}{0.5-r_1}\]
we will now establish the range of \(t_1\) and \(t_2\) for which the solution has to be sought for.

1. For \(N_1, \text{crit} = 0\) \(t_2\) will attain \(-1\) for all values of \(r_1\) in the above range. \(t_1\) will, however, attain a value given by \(t_1=r_1/(1-r_1)\). This value would, however, yet lie in the range \(-1 \leq t_1 \leq 1\).
2. For \(N_1, \text{crit} = \infty\) in the physically realisable range, namely \(N_1 < N_{1, \text{crit}}\), \(t_1\) and \(t_2\) will both attain values corresponding to the specified \(r_1\) and can be given by the general form already specified before. They will yet be falling in the range \(-1 \leq t_2 \leq 1\).
A few particular cases of special interest are adiabatic surface $l$, i.e., $r_1=0$, and $r_2=-\infty$. In the former case, $t_2=0$ for all allowable positive values of the thermal sliding parameter, and $t_2$ would assume $-1$ for $N_{sl}$ approaching the maximum allowable value; in general, $t_2$ would assume $-1$ already at $N_{sl}$ close to zero.

5.3.3 Solutions in the range of operation covered by $-1 \leq t_2 < t_1 \leq 1$

For certain operating conditions in which the regime is confined to $-1 \leq t_2 < t_1 \leq 1$, the evaluation of the solution reduces to the evaluation of the integral,

$$\int_0^{\infty} \frac{e^{-x/N_{sl}}}{(1-t_1^2)^{n}} \, dt \left[ \frac{1}{n} \right]$$

(5.13)

Only for the case $w=1$ the integral can be evaluated in a closed form. It may again be noticed that although the integrand becomes infinite at this value of the upper limit, the integral will, however, be finite as long as the exponent is less than unity. The latter condition on the exponent implies that $m$ should be prescribed with only finite values which are positive for the fluids we are presently concerned with. Thus the case of $m=0$ representing the exponential viscosity-temperature relationship is devoid of the solution for the integral provided below. An analytical solution for the above definite integral can be obtained in terms of the Gamma functions as shown below.

$$\int_0^{\infty} \frac{e^{-x/N_{sl}}}{(1-t_1^2)^{n}} \, dt \left[ \frac{1}{n} \right] = 2 \frac{2m}{2n+1} \left[ \frac{1}{2} \right] \left[ \frac{1}{2} \right] \left[ \frac{1}{2} \right]$$

(5.14)

Numerical results covering the range of $0 \leq m \leq 1$ have been tabulated for convenience in Table 5.1. The results in that table also correspond to the cases wherever $t_2=-1$. This is attributed to the fact that the above integral is symmetric towards the origin. The results correspond only to a few special cases of the heat partition ratio $r_1$ and the thermal sliding parameter $N_{sl}$. These are listed in Table 5.2. One may observe that the application of the results is limited only to a narrow operating range. It may be recalled that the exponential viscosity-temperature relationship used in Chapter 3 gave rise to infinitely high values for the non-dimensional qualities such as shear stress, total heat flux and volumetric flow rate for $r_1=0$ and $N_{sl}=N_{sl,crit}$. In contrast to such results, it is seen that for finite $m$-values representing most of the liquids, including lubricating mineral oils, all the above-mentioned non-dimensional quantities attain positive and finite values. A detailed discussion on this aspect will be offered in the sections to follow.

Table 5.1 Definite integral involved in the solutions for certain sets of $r_1$ and $N_{sl}$ shown in Table 5.2. Also results for shear stress $\Gamma_1$, total heat flux $\phi_1$, correction factor to the isothermal rate of flow $Q$ and flow rate $V$ at the critical state for $r_1=0$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$m/(m+1)$</th>
<th>$\int_0^{\infty} \frac{e^{-x/N_{sl}}}{(1-t_1^2)^{n}} , dt \left[ \frac{1}{n} \right] \frac{2m}{2n+1} \frac{1}{2} \frac{1}{2} \frac{1}{2}$</th>
<th>$\Gamma_1$</th>
<th>$\phi_1$</th>
<th>$Q$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1/4</td>
<td>1.1971</td>
<td>0.8465</td>
<td>1.1138</td>
<td>0.3928</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
<td>1.5708</td>
<td>1.5708</td>
<td>2.2214</td>
<td>1.2732</td>
<td>0.6366</td>
</tr>
<tr>
<td>2</td>
<td>2/3</td>
<td>2.1027</td>
<td>2.4280</td>
<td>3.9649</td>
<td>1.4227</td>
<td>0.8237</td>
</tr>
<tr>
<td>5/2</td>
<td>5/7</td>
<td>2.3636</td>
<td>2.8250</td>
<td>4.7752</td>
<td>1.4808</td>
<td>0.8849</td>
</tr>
<tr>
<td>3</td>
<td>3/4</td>
<td>2.6221</td>
<td>3.2114</td>
<td>5.8233</td>
<td>1.5255</td>
<td>0.9342</td>
</tr>
<tr>
<td>4</td>
<td>4/5</td>
<td>3.1346</td>
<td>3.9650</td>
<td>7.0928</td>
<td>1.5951</td>
<td>1.0088</td>
</tr>
<tr>
<td>5</td>
<td>5/6</td>
<td>3.6464</td>
<td>4.7075</td>
<td>8.5946</td>
<td>1.6455</td>
<td>1.0621</td>
</tr>
<tr>
<td>19</td>
<td>19/10</td>
<td>5.6619</td>
<td>7.5962</td>
<td>14.4128</td>
<td>1.7662</td>
<td>1.1848</td>
</tr>
<tr>
<td>199</td>
<td>199/100</td>
<td>51.1730</td>
<td>72.0068</td>
<td>143.2916</td>
<td>1.9542</td>
<td>1.3749</td>
</tr>
</tbody>
</table>

Table 5.2 The range of operation to which the solutions of Table 5.1 may be prescribed to.

<table>
<thead>
<tr>
<th>$r_1$</th>
<th>$t_2$</th>
<th>$t_1$</th>
<th>$N_{sl}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>$N_{sl,crit}$</td>
</tr>
<tr>
<td>1/2</td>
<td>-1</td>
<td>1</td>
<td>$N_{sl,crit} \times \infty$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$N_{sl,crit} \times \infty$</td>
</tr>
</tbody>
</table>

5.4 Solutions for Various Liquids with Slotte's Relationship

We will provide solutions in terms of the previously mentioned non-dimensional quantities wherever possible in an analytical form. Such cases will, however, be limited only to a few finite values of Slotte's exponent $m$. Some of these are available from the previous investigators whereas a few more have been developed in the present analysis. The solutions available from publications include those in a closed form for $m=1$ and 2, the former in terms of trigonometric functions and the latter in the form of elliptic integrals, see for instance Joseph [5-4]. Solutions obtained for the energy balance equation representing the reaction in nuclear fuel elements in the form of plates seem, at a first glance, to be applicable to the present problem involving viscous dissipation in the energy balance equation. Analytical solutions thus developed by Axford [5-5] are based on theory of groups and the method of solution.
seems to be elegant. During the process of obtaining such a closed form solution the independent and dependent variables are simultaneously transformed using the fact that the non-linear energy balance equation admits certain local transformation groups, which are called Lie groups. Once the precise nature of the group invariance properties of the non-linear energy balance equation has been established, it may easily be determined whether the reduced canonical form of the equation is amenable to direct integration by quadratures.

Such a determination of analytical solutions, including those beyond which no steady state solutions are existent, although only for the reaction of the nuclear fuel elements in the form of plates, has been worked out by Axford. This has been carried out for various combinations of coefficients that indicate the source strength and change of thermal conductivity with temperature. The solutions thus obtained and tabulated correspond to those we are presently interested in the analysis of non-isothermal viscous flows. They show certain deficiencies, namely that for positive values such as 2, 3 etc., of Slotte's parameter \( m \), the maximum permissible shear stress parameter is not available for a constant thermal conductivity of the involved material. The existence of such solutions, namely for \( m=2 \) has long been established by Kaganov [5.6] and Joseph [5.4]. In our present analysis we have succeeded in establishing the solutions with general thermal boundary conditions involving heat partition ratio for temperature and velocity distributions as well as for the other thermohydrodynamic quantities given in Section 5.2.2. Our results comply with those of Joseph as detailed below. Since certain questions of discrepancy are not yet resolved with respect to Axford's results, we confine ourselves to the analytical solutions suggested by others, evaluated independently in the present analysis as well as those evaluated using numerical integration techniques. For compactness the solution only to the definite integral

\[
\int_{t_1}^{t_2} \frac{dm}{(1-t_1^2)^{m+1}} = 2 \arcsin t + C, \quad t > 1
\]  

will be presented. The thermohydrodynamic quantities can further be readily obtained. In addition to such analytical solutions, numerical solutions for a few selected values of Slotte's exponent \( m \) will also be provided. These numerical solutions have also been compared with the former analytical solutions. The accuracy of the numerical integration has found to be satisfactorily good.

Apart from the rather trivial case of isoviscous flows where the viscosity of the fluid is considered to be independent of temperature, or what amounts to Slotte's exponent \( m=0 \) the following case of \( m=1 \) has previously been analysed.

5.4.1 For Slotte's exponent \( m=1 \), linear fluidity function

With Slotte's exponent \( m=1 \) the fluidity function becomes linear where the viscosity is inverse of the viscosity. Solutions for equilibrium temperature and velocity distributions have been shown to exist in the form of single periodic functions by Regirer [5.8], Schlichting [5.9], Carslaw and Jaeger [5.10], Winter [5.11] and Joseph [5.4]. This analysis for \( m=1 \) has a somewhat limited application since the liquids employed in engineering practice generally do not follow this viscosity-temperature relationship. There exist, however, a few liquids such as silicone oil that follow fairly closely this relationship (see Chapter 2 for details of Slotte's exponent including that representing approximately the linear fluidity function). The governing energy equation takes a simple form that can also directly be evaluated using the available analytical solutions, and this form becomes for the present \( m=1 \), using equation (5.2b):

\[
\frac{d^2T}{dt^2} + \frac{\rho c}{\rho_0} (1+\beta) = 0
\]  

(5.16)

Thermal boundary conditions can be specified on identical lines as before either in a 'one-point' or in a 'two-point' form. We will presently maintain the 'one-point' form provided in Figure 5.1, which can easily be cast into a 'two-point' form by straightforward transformation discussed in Chapter 3. Solutions for the present linear differential equation are available only for the idealised thermal boundary conditions of isothermal and adiabatic type; we will, however, show on the basis of the convenient form of the integral we have already proposed through expression (5.11) that the solution can equally easily be directly established even for non-idealised thermal boundary conditions.

5.4.1.1 Solutions to various thermohydrodynamic quantities

Since the indefinite integral (5.13) is the well known trigonometric function,

\[
\frac{d}{dt} = \arcsin t + C, \quad t > 1
\]

the solution to the definite integral for any definite values of \( t_1 \) and \( t_2 \) lying in the range \(-1 \leq t_1 < t_2 \leq 1\) can be given by,

\[
t = \sin \left[ (1-Y)\arcsin t_1 + Y \arcsin t_2 \right]
\]  

(5.17)

Expanding \( t \) from expression (5.4), the velocity distribution can be obtained in an explicit form, namely,

\[
U = \frac{r_0}{r} \left[ 1 - \left( 1 - \frac{2}{r_{H_0}} \right)^{\sin \left[ (1-Y)\arcsin t_1 + Y \arcsin t_2 \right]} \right]
\]  

(5.18)

The solution to the temperature and viscosity distribution can readily be written down in a non-dimensional form once the velocity distribution is evaluated using expression (5.18). Such solutions will also involve the parameters representing the operational regime, namely \( r_0 \) and \( H_0 \). Thus for the rather simple fluidity function representing the viscosity-temperature relationship through a hyperbolic one, these solutions can
be expressed, namely in the form,
\[ \theta = \left[ 1 + \frac{\nu(r,1-0.5)\eta_0}{\eta_0} \right]^{1/2} - 1 \]  
(5.19)
\[ \nu_i/\nu = \left[ 1 + \frac{\nu(r,1-0.5)\eta_0}{\eta_0} \right]^{1/2} \]  
(5.20)
Further the solution in terms of non-dimensional quantities for the other operational characteristics of the film that have been mentioned before can also be listed as follows. The non-dimensional form for the shear stress takes a simple form,
\[ \Gamma_1 = \arcsin t_1 - \arcsin t_2 \]  
(5.21)
The non-dimensional heat flux distribution and the total heat flux become,
\[ \phi^* = (\arcsin t_1 - \arcsin t_2)^2 \left( 1 + \frac{r^2\eta^2_0}{2} \right) \cos \left( (1-Y)\arcsin t_1 + Y \arcsin t_2 \right) \]  
(5.22)
\[ \phi^* = \sqrt{\eta^2_0/2} (\arcsin t_1 - \arcsin t_2) \]  
(5.23)
The correction factor to the isothermal volumetric rate of flow in non-dimensional form reduces to,
\[ Q = 2 r_1 (1-\eta_0) \]  
(5.24)
where
\[ B_9 = \left( 1 + \frac{2}{r^2\eta^2_0} \right)^{1/2} \frac{\int t_2 \left( 1 - t^2 \right)^{1/2}}{(\arcsin t_1 - \arcsin t_2)} \]
which can be shown to reduce to a simple form through the introduction of the analytical solution to the integral in the known form, i.e.,
\[ B_9 = \left( 1 + \frac{2}{r^2\eta^2_0} \right)^{1/2} \frac{(1-t^2)^{1/2} - (1-t^2)^{1/2}}{\arcsin t_1 - \arcsin t_2} \]  
(5.25)
An alternative form to the volumetric rate of flow, i.e., the flow parameter \( V \) can readily be obtained from expression (5.10b), namely,
\[ V = \left[ \frac{\rho_0}{\rho_1} B_9 \right]^{1/2} \]  
(5.26)
If one employs the governing equation (5.16) to obtain the solutions from Carslaw and Jaeger [5.10], then the distribution of the temperature and viscosity across the film can be expressed in an equally convenient form in terms of the shear stress parameter \( \Gamma_1 \),
\[ \theta = \cos \frac{\Gamma_1 Y}{\cos \Gamma_1} \]  
(5.27)
5.4.1.2 Occurrence of critical states
The critical state above which thermally steady states may not be existent can be shown, as discussed before, to prevail in the regime \( r \leq 1/2 \). This has already been established in terms of \( N_8 \), independent of \( m \), see equation (5.12a). The corresponding thermal sliding parameter for the particular case of \( m=1 \) gives rise to \( N_8 = N_8/2 \) from the definition. The distribution of non-dimensional velocity, temperature, viscosity and heat flux at the above-mentioned critical state can be expressed as follows,
\[ \nu = r_1 - (1-r_1) \sin \left( 1-Y \right) \arcsin t_1 - Y n/2 \]  
(5.28)
\[ \theta = \left[ 1 + \frac{1}{(1-2r_1)} \left( t_1^2 - (1-t_1)^2 \sin \left( (1-Y)\arcsin t_1 - \frac{Y}{2} \right) \right) \right]^{1/2} - 1 \]
\[ \nu_1/\nu = \left[ 1 + \frac{1}{(1-2r_1)} \left( t_1^2 - (1-t_1)^2 \sin \left( (1-Y)\arcsin t_1 - \frac{Y}{2} \right) \right) \right]^{1/2} \]
\[ \phi^* = \left( \frac{\cos \frac{\Gamma_1 Y}{\cos \Gamma_1}}{\sqrt{1-2r_1}} \right)^{1/2} \arcsin t_1 - \frac{Y}{2} \]
Then the maximum shear stress and the corresponding total heat flux, the correction factor \( Q \) and flow parameter, all in non-dimensional form, can be expressed in terms of the transformed variables \( t_1 \) and \( t_2 \). At the critical state where the thermal sliding parameter \( N_8 \) approaches the value given by expression (5.12a) the variable \( t_2 \) becomes -1 and \( t_1 = 2r_1/(1-2r_1) \) from its definition. The above mentioned solutions in non-dimensional form monotonically approach the following values:
\[ \Gamma_1 = \arcsin t_1 \]
\[ \phi^* = \frac{1}{\sqrt{1-2r_1}} \left( \frac{\arcsin t_1}{1-r_1} + \frac{\eta_0}{2} \right) \]
\[ Q = 2 (r_1 - 3r_1^2) \]
where
\[ B_9 = \left( 1 - r_1 \right)^{1/2} \frac{(1-t_1^2)^{1/2} - (1-t_1^2)^{1/2}}{\arcsin t_1 - \arcsin t_2} \]
and
\[ V = \sqrt{\eta_0} \frac{Q}{2\sqrt{2}} \]
The above mentioned non-dimensional parameters $I^*$, $Q$, and $V$ against $N_s$, covering the complete operational regime, have been presented in Figure 5.2a-d for a few selected values of $r$. The solutions for the critical state have also been tabulated in Table 5.3. The magnitude of $T_s$ corresponding to the critical state increases with increased heat partition ratio $r_1$; the former asymptotically approaches $T$ at $r_1$ close to $1/2$ and at very high values of the thermal sliding parameter $N_s$. It may also be observed that the heat flux parameter $\phi$ continues to increase, and for the heat partition ratio $r_1$ approaching $1/2$ it becomes infinitely large at high values of $N_s$. It may be seen that for the adiabatic condition at either of the surfaces, i.e. $r_1=0$ or $1.0$, the value of the maximum steady state shear stress $T_1=\pi/2$; this is equivalent to one-half the value of $T_1$ for the case of the isothermal boundaries, i.e. $r_1=1/2$. A discussion towards the characteristic nature of the solutions has, however, been provided in Section 5.5 yet to follow.
i.e. \( r_1 = 1/2 \), the solution can also be written down from the known analysis by various investigators mentioned before. Their solutions have, however, been expressed in terms of the non-dimensional shear stress parameter \( \tau_1 \). In the present analysis we have employed the non-dimensional shear stress parameter in the place of their non-convenient thermal sliding parameter. It has already been observed that the evaluated solutions are, however, single-valued in terms of the thermal sliding parameter and thus are preferred in order to compare the behaviour of the solutions for the fluids whose Slotte's exponent \( m \) might differ from unity. Confining presently to the general analysis with the thermal sliding parameter as a variable, the solutions for equi-partitioning of the heat can be seen to reduce to a simple form. The transformed variables become identically the same, namely,

\[
\tau_1 = \tau_2 = \left(1 + \frac{8}{R_0^2}\right)^{-\frac{1}{2}}
\]

Velocity, temperature and viscosity, and heat flux distributions in non-dimensional form across the film reduce to the following.

\[
\nu = 0.5 \left[ 1 - \left(1 + \frac{8}{R_0^2}\right)^{-\frac{3}{2}} \right] \sin \left[ \left(1 - r_2^2\right) \arcsin \tau_1 \right]
\]

\[
\theta = \left[ 1 + 0.50(1 - 0) R_0 \right] - 1
\]

\[
\nu_1/\nu = \left[ 1 + 0.50(1 - 0) R_0 \right]^{-\frac{1}{\mu}}
\]

\[
\phi^* = 2 \arcsin \tau_1 \left(1 + \nu_1/\nu\right) \cos \left[ \arcsin \tau_1 \right]
\]

Shear stress, heat flux and volumetric rate of flow in non-dimensional form at any specified thermal sliding parameter are given for the equi-partitioning of heat below.

\[
\Gamma_1 = 2 \arcsin \tau_1
\]

It may be observed that the correction factor to the isothermal volumetric rate of flow is uniformly unity for all values of the thermal sliding parameter. Thus the flow is not influenced by any variations in the thermal sliding parameter for isothermal boundary conditions. The flow parameter \( V \), however, is solely dependent on the thermal sliding parameter.

5.4.2 For Slotte's exponent \( m=2 \), quadratic fluidity function

Slotte's exponent \( m=2 \) represents a quadratic fluidity function. Solutions for such a viscosity-temperature relationship considerably differ from the solutions discussed in the previous section, namely with Slotte's exponent \( m=1 \). This aspect might be observed from the results to be presented in what follows.

This empirical relation for the fluidity, also called Poiseuille relationship, is rather widely used due to its mathematical simplicity. In addition to some mineral oils pointed out in Chapter 2 water complies quite accurately with this relationship. As referred to a temperature of \( 10^\circ \text{C} \), Lamb [5.12] gives the values of constants with which the evaluated fluidity is found to be accurate to three decimal places for the complete temperature range extending from freezing to boiling point at atmospheric pressure. The relationship for the fluidity is given by,

\[
\mu_1/\mu = 1 + \alpha_1 (T-T_1) + \beta_1 (T-T_1)^2
\]

where for water \( \alpha_1=0.03368/\text{C}^2 \), \( \beta_1=0.000221/\text{C}^2 \) and \( \nu_1=0.001307 \text{Ns/m}^2 \) at the reference temperature of \( 10^\circ \text{C} \). The constants do not fit accurately with the quadratic form that is expressed using Slotte's relationship in equation (5.1a), namely,

\[
\mu_1/\mu = \left[ 1 + \frac{T-T_1}{a} \right]^{-\frac{1}{m}}
\]

which has been used in Chapter 2. The scaling temperature, namely,

\[
\frac{1}{2} \frac{d^2}{dt^2} (A \nu_1)
\]

is simply equivalent to \( \alpha_1 \), i.e. \( 0.03368/\text{C}^2 \) for water. On the other hand the scaling temperature for any fluid with \( m=2 \) in Slotte's relationship as shown in Chapter 2 becomes \( 2/(A+2) \). One may readily recall that the coefficient of the quadratic term with the introduction of new variables for water as well as for the fluids with \( m=2 \), can be shown to take the values as shown in Table 5.4.
Table 5.4 Characteristic values connected with the quadratic fluidity function or Slott's relationship with m=2.

<table>
<thead>
<tr>
<th>Fluidity</th>
<th>Water</th>
<th>Fluid with m=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>0</td>
<td>2/[(\pi),\iota] + 0.015</td>
</tr>
<tr>
<td>( (T-T_0) )</td>
<td>( 1/[(\pi),\iota] + 0.156 )</td>
<td></td>
</tr>
</tbody>
</table>

5.4.2.1 Solutions to various thermo-hydrodynamic quantities

Having noted that quite some liquids follow Slott's relationship with \( m=2 \), among which water can be conceived to follow approximately so, we may now proceed to evaluate the solutions that have been sought for. Joseph [5.4] has shown that the temperature distribution can be evaluated in terms of the shear stress. He has also worked out the maximum permissible shear stress, beyond which no solutions are possible. The liquid studied is water and idealised thermal boundary conditions of isothermal and equal temperature type have been analysed.

We proceed to establish independently the solutions for the thermohydrodynamic parameters stipulated in Section 5.2.2 in terms of the thermal sliding parameter \( N_s \) and the complete range of heat partition ratio \( \delta \). As pointed out before the question of obtaining the solutions simply reduces to that of solving the integral given by equation (5.11) which for \( m=2 \) reduces to,

\[
\int_{t_0}^{t} \frac{dt}{(t-t_0)^3} = \frac{3}{2} \left[ \cos \Psi \right] + \frac{1}{2} \cos \Psi \\
(5.11)
\]

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\[
\int_{t_0}^{t} \frac{dt}{(t-t_0)^3} = \frac{3}{2} \left[ \cos \Psi \right] + \frac{1}{2} \cos \Psi \\
(5.11)
\]

where \( \Psi \) is the amplitude and \( x \) is the modulus. \( x = 0.965926 \). Then the solution can be given in the form of tabulated functions; see Byrd and Friedman [5.13]. Then the elliptic integrals for which the tabulated solutions are given, reduce to those specified on the right hand side of the following.

\[
P(Y,x) = Y F(Y,x) + (1-Y) F'(Y')x \\
(5.40)
\]

If one is interested in simply obtaining the integral in equation (5.39) it is preferable to work out the function \( F(Y,x) \) in the above mentioned form through evaluating the component functions \( F(Y,x) \) and \( F'(Y,x) \). On the other hand, if one aims at obtaining the velocity distribution in an explicit form, it is convenient to re-express the above form of the solution \( F(Y,x) \) in terms of Jacobian elliptic functions which are single-valued in the argument \( \Psi \). Such a process is found to be rather straightforward, and the numerical work involving the interpolation, whenever required, seems to be simpler than that corresponding to the similar process of the interpolation of the involved elliptic integrals. The corresponding Jacobian function, with the replacement of \( F(Y,x) \) by \( \Psi \), becomes

\[
\cos \Psi = 
(5.41)
\]

One can readily obtain the distribution of velocity from the above solution in the following form by making use of the original variables of \( t \) as specified in equation (5.4b).

\[
U = - \sqrt{2/N_s (1+0.5r^2(N_s^2)^{1/2})} \\
(5.42)
\]

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\[
\cos \Psi = 
(5.41)
\]

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\[
U = - \sqrt{2/N_s (1+0.5r^2(N_s^2)^{1/2})} \\
(5.42)
\]
be shown, within the accuracy of interpolation of the tables, identical

For Slotte's exponent $m=3$ and $1/3$ the integral to be solved is of the form, 

$$ F(w_i, x) = \frac{1}{1-w_i^2} \int \frac{1}{\sqrt{1-w_i^2 \ell^2}} \, dt $$

One may also notice that the symmetric nature of the solution implies $F(-w_i, x) = F(w_i, x)$. A discussion concerning the regime $r_i \geq 0.5$ will be offered at a later stage. An attempt to analytically evaluate the maximum permissible shear stress above which no solutions exist has revealed that this is hard to obtain in a closed form.

### 5.4.2 Occurrence of critical states

Critical states have been treated on general terms for the liquids with finite $m$ values in Section 5.2.4. These states are existent for all values of $r_i<0.5$, although only in their limited significance. The critical states are characterised by a zero value of the ratio for the viscosities at the walls. The shear stress level is finite at the same critical state, and remains finite as long as Slotte's exponent $m$ is finite.

For a few selected values of the heat partition ratio $-1 < r_i < 1/2$ the solution in the form of equation (5.39) has been worked out, and the critical state has been listed in Table 5.5. Detailed results covering a wide operating regime and the locus of the critical state have been provided for $T$ and $V$ in Figures 5.3a and 5.3b. For these cases, as pointed out in Section 5.3.2, $t_2=1$ and the corresponding solution for $F(Y, x) = 1.847092$. The solution to the corresponding integral of equation (5.39) with $t_2=1$ can be shown, within the accuracy of interpolation of the tables, identical to that provided in Table 5.1 using the Gamma functions. The present requirement of obtaining the solutions reduces to that of evaluating the elliptic integral $F(Y, x)$. One may also notice that the symmetric nature of the solution implies $F(-Y, x) = F(Y, x)$. A discussion concerning the regime $r_i \geq 0.5$ will be offered at a later stage. An attempt to analytically evaluate the maximum permissible shear stress above which no solutions exist has revealed that this is hard to obtain in a closed form.

### 5.4.3 For Slotte's exponent $m=3$ and $1/3$

A number of mineral, paraffinic and hydrocarbon liquids comply closely with $m=3$ in Slotte's relationship as seen in Chapter 2. Fluids with $m=1/3$ are rather uncommon, and in fact the entire list of liquids

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[5.4]

![Figure 5.3a](image)

Figure 5.3a Shear stress variation with thermal sliding parameter for liquids with Slotte's exponent $m=2$.

![Figure 5.3b](image)

Figure 5.3b Flow parameter with thermal sliding parameter for liquids with Slotte's exponent $m=2$.

commonly employed in lubrication practice does not contain any liquid with $m=1$. The solution has, nevertheless, been evaluated for $m=1/3$ in this section because of its similarity with that for $m=3$. Both these solutions have a mathematical simplicity similar to that of $m=2$ which has been analysed in Section 5.4.2.

For Slotte's exponent $m=3$ the integral to be solved is of the form,

$$ \int_{t_2}^{t_2} \frac{dt}{(t_2^2-t^2)^{1/2}} $$

This integral is also of the elliptic integral type with the modulus $x = 1/\sqrt{2}$. The form of the solution is similar to that given in equation
\( \frac{d}{dt} - \sqrt{2} F(Y, x) \)  

where \( F(Y, x) \) denotes the elliptic integral of the first kind. The corresponding Jacobian function becomes,

\[ \cos \theta \]

where \( \cos \theta = cn(w, x) = (1-L_z)^{1/2} \)

An explicit relationship for \( t \) may be obtained, as before in the case of \( m=2 \), by reading the corresponding inversion of the elliptic integral, viz. for \( cn \),

\[ t = \left| \frac{1-\left( \cos \theta \right) }{1} \right| \]

or, for the velocity distribution,

\[ U = r_1 - \left[ \frac{2}{R_1} \left( 1 + 0.5 r_1^2 \right) \right] \left| \frac{1-\left( \cos \theta \right) }{1} \right| \]

The critical states are also existent for all \( r_1 < 0.5 \), and their characteristic nature is identical to that discussed in Section 5.4.2.3. Similar to the case of \( m=2 \), \( \gamma \) is found to reduce to the complete elliptic integral of the first kind \( K(x) = F(1/2, x) \).

This can readily be read from the tables listed in Table 5.6. It may be noted that \( r_1 \) takes a different value as influenced by Slotte's exponent \( m=3 \).

The results obtained for the critical state have been shown as limiting values of \( T_j \) with \( N_s \) for some representative values of \( r_1 < 0.5 \) in Figure 5.4a. The locus of the critical state for a wide range of the heat partition ratio \( r_1 < 0.5 \) has also been shown. It may be observed that the critical shear stress parameter \( \gamma \) below which there exist solutions for the velocity and temperature distribution continues to increase with increasing \( r_1 \) values. Already at \( r_1 = 0.4 \) this maximum \( \gamma \) begins to diminish, and continues to do so until it gradually approaches that of \( r_1 = 0.5 \). Such a behaviour of the maximum \( \gamma \) approaching that of \( r_1 = 0.5 \) implies that at the heat partition ratio close to \( r_1 = 0.5 \) the behaviour of \( \gamma \), as expected, does not considerably deviate from that of the isothermal, equal boundary temperature case. An advantage inherent in the present method of evaluating the solutions can be noticed by referring to the locus specified in Figure 5.4a; the latter has been obtained by a simple evaluation using the tabulated elliptic integrals. As will be observed in Section 5.5 a similar locus cannot easily be obtained using the numerical integration techniques, since these techniques encounter certain difficulties especially at very low values of \( r_1 \).

Results for \( \gamma^*, Q \) and \( V \) have been provided in Figures 5.4b, 5.4c and 5.4d. But a discussion on these results has been presented in Section 5.5. For Slotte's exponent \( m=1/3 \) the integral to be solved is of the form:

\( \int_{0}^{1} \frac{d}{dt} - \sqrt{2} F(Y, x) \)
form,  
\[ \frac{L}{t} \frac{dc}{dt} = \left(1 - t^2\right)^{\frac{1}{2}} \]  
(5.46a)

The solution in its well known form consists of both the elliptic integral of the first kind \( E(\tau, x) \) and that of the second kind \( E(\tau, x) \) with their modulus \( x^{-1/2} \), and this solution can be specified through,  
\[ \int_0^1 \frac{dt}{\sqrt{1 - t^2}} = \pi \left( \frac{1}{\sqrt{2}} \right) \]  
(5.46b)

The integral on the right hand side can be worked out to be equivalent to,  
\[ \frac{L}{x^2} \left[ E(\tau, x) - (1 - x^2)^2 \right] \]  
(5.47)

Then one obtains similar to expression (5.40),  
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\[ \left( \frac{dU}{dt} \right) = \left(1 - t^2\right)^{\frac{1}{2}} \]  
(5.48)

One may replace the left hand side by \( \omega \) for simplicity, and then may obtain the Jacobian function as before, namely,  
\[ \cos \omega = \rho \]  
(5.49)

The previously established value of \( \tau = -1 \) and for all \( \tau < 0.5 \), one may find \( E(\tau, x) \) on lines similar to that followed in the case of solutions for \( m = 2 \) and \( m = 3 \). \( E(\tau, x) \) is found to reduce to the complete elliptic integral of the second kind, i.e. \( E(\tau, x) \). This can be read from the tables to be 1.390644. For \( \tau > 0 \), as shown before \( \tau = -1 \), and \( \tau = 0 \), then the integral given in equation (5.46a) reduces to 1.298140 which is nearly identical to the value that is specified in Table 5.1 using the Gamma functions. The inaccuracy might be attributed to the inaccuracy with which the tabulated values of the Gamma functions as well as those of the elliptic integrals can be interpolated.

The explicit relationship for \( t \) and that for the velocity distribution remain identical to that specified by equations (5.41) and (5.42). In the process of evaluating the value of \( t \) and then \( U \), one may encounter a tedious interpolation procedure which now concerns the evaluation of the first and second order elliptic integrals at the same amplitude.

The critical states corresponding to a few selected values of the

\begin{table}
<table>
<thead>
<tr>
<th>( t )</th>
<th>( E(\tau, x) )</th>
<th>( E(\tau, x) )</th>
<th>( E(\tau, x) )</th>
<th>( E(\tau, x) )</th>
<th>( E(\tau, x) )</th>
<th>( E(\tau, x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
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<td>0.6</td>
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<tr>
<td>0.7</td>
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<tr>
<td>0.8</td>
<td>0.8</td>
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<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
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<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
\end{table}

heat partition ratio \( r_i < 0.5 \) have been listed in Table 5.7. Results including the locus of the critical state have been shown for \( \tau_i \) in Figure 5.5. An interesting feature of the present case of \( m = 1/3 \) may be observed from the listing of \( \tau_i \). In contrast to the previous two cases, namely \( m = 2 \) and \( m = 3 \), the present distribution of \( \tau_i \) with \( r_i \) indicates a monotonic increase with an increase of \( r_i \) in the range \( -\infty < r_i < 0.5 \). Such a behaviour, although at first sight does not look far different from that of liquids with \( m = 1 \), may be distinguished as will explained later in Section 5.5.1 in that even for higher than \( r_i > 0.5 \) the magnitude of \( \tau_i \) continues to increase with the sliding parameter \( N_{sl} \). There exists no limitation to \( \tau_i \) that can be imposed in the range \( r_i > 0.5 \).

5.4.4 For Slotte's exponent \( m \) in the range \( 3 < m < \infty \)
significant one, at least as it concerns liquids used in the lubrication practice. The remaining range \(3 < m < 10\) is yet as important as that analysed so far in the preceding sections, viz. \(1 < m < 3\). Many lubricants listed in Chapter 2 fall in this range of \(m\).

Solutions in an analytical form are hard to find for the liquids falling in this range of \(m\), i.e., \(3 < m < 10\). In order to study the behaviour of the solutions, numerical integration techniques have been attempted.

A numerical quadrature formula in double precision has finally been successfully employed. This involves a high degree polynomial which yields results better than 1% accurate with respect to the solutions obtained in terms of the special functions already provided in Section 5.4.1 through 5.4.3 for a few values of Slotte's exponent \(m\). The same numerical technique has yielded the results of similar accuracy, although only for the heat partition ratio \(r < 0.5\), even in the range of \(3 < m < 10\), see Section 5.3.3. After having carried out these comparative studies on its accuracy the above mentioned numerical integration technique has been employed to evaluate solutions in the complete range of \(m\), i.e., \(0 < m < \infty\).

The numerical integration technique consisted of selecting a few values of the heat partition ratio in the range \(-2 < r < 5\), and specifying the critical values of \(N_0\) for \(r < 0.5\). Then the integration was carried out from very low values of \(N_0\), viz. 0.000001, up to the close vicinity of the critical \(N_0\) values in the case of \(r = 0.5\). For \(r > 0.5\) this range of \(N_0\) was extended up to 10, i.e. well beyond the values of \(N_0\) that might exist in practice, see Chapter 11. The locus of solutions to the critical state given by critical \(N_0\) values for \(r < 0.5\) has been evaluated up to very low values of \(r\). The solutions thus obtained have been cast into a form representing the previously mentioned non-dimensional quantities for the shear stress \(\tau\), the heat flux \(q\), the correction to the isothermal rate of flow \(Q\) and the flow parameter \(\gamma\). The above range of \(r\) and \(N_0\) covered by the solution is extensive and bounds most of the operating conditions encountered in the lubrication practice. The evaluation of velocity, temperature and heat flux distribution across the film at various selected values of \(r\) and \(N_0\) has separately been carried out, and this has been briefly in Chapter 7 while discussing the experimental results obtained for some mineral oils from tests on a specially built Couette type viscometer.

### 5.5 Results and Discussions

Solutions obtained both in terms of special functions as in the case of Slotte's exponent \(m=1/3\), 1, 2, 3 and by numerical integration techniques significantly deviate from those obtained using the exponential relationship in Chapter 3. It may also be recalled from Section 5.1 that the exponential relationship is a special case of Slotte's exponent \(m=1\) reaching an infinitely high value.

In literature one may find an extensive study of the solution using the exponential relationship, and a less extensive one using the linear fluidity function, the latter being a special case of Slotte's with the exponent \(m=1\). In literature solutions for the other values of \(m\) are hard to find. These are only fragmentary, and concern \(m=2\) analysed by Joseph [5.4]. In all these investigations thermal boundary conditions are assumed to be of the idealised type such as isothermal and adiabatic. As discussed in Chapter 3, Section 3.3, general thermal boundary conditions covering a wide range of operation are better representative of the practical situation than the idealised ones. Such boundary conditions have been included, as before in Chapter 3, in the present analysis as well. A representative form of these boundary conditions, covering the idealised ones, consists of the temperature gradient at the boundary, and its convenient variant may be considered as the heat partition ratio \(r\).

As in the case of the analysis using the exponential relationship, here also one may replace the boundary conditions consisting of the temperature gradient by those consisting of the temperature at the boundary. In contrary to equation (3.16a) of Chapter 3 Slotte's relationship yields an involved relationship, obtained through the substitution of the viscosity-temperature relationship given by equations (5.1b) and (5.3c). This is implicit in boundary temperatures at surfaces 1 and 2, i.e.,

\[
\frac{1}{1+0.5} \left[ 1 + \left( r_1 - 0.5 \right) N_0 \right]^{-\frac{5}{3}}
\]

One may employ an alternative form where the generality of the representation of the thermal boundary conditions is least reduced. Such a form consists of viscosities at the boundaries and is far simpler in its form than equation (5.49a). This form is a direct reduction from equation (5.3c), and may be given by,

\[
\frac{\mu_1}{\mu_2} = \left[ 1 + \left( r_1 - 0.5 \right) N_0 \right]^{-\frac{5}{3}}
\]

Either of the above two alternative forms may be employed in place of the heat partition ratio \(r\) while analysing the solutions for various thermal sliding parameters or for various shear stress levels. In literature one encounters some non-isothermal analyses using mainly the hyperbolic fluidity function, i.e., Slotte's relationship with \(m=1\), for various levels of surface temperature differences, see Wilson and Mahdavian [5.15]. Such an analysis for other viscosity-temperature relationships, such as Slotte's with finite values of \(m\) in the range \(1 < m < \infty\), is, to the knowledge of this author, not available. It may then be con-
cluded that the results obtained using either analytical solutions or numerical techniques are of importance, whether they are obtained in terms of the heat partition ratio \( r \) or in terms of an alternative parameter involving the surface temperatures or the viscosities at the same surfaces.

The Reynolds numbers of these non-isothermal flows can be specified on lines similar to those mentioned in Chapter 3. One may introduce the average velocity of such flows by making use of either the correction factor to isothermal flows \( Q \) or the flow parameter \( V \) expressed in terms of the operational parameters \( r \), and \( m \), as well. In this regime of the solutions presented for the exponential viscosity-temperature relationship, \( r \) increases with increasing values of \( r \) until a maximum is reached. Beyond this maximum the limiting shear stress continues to decrease and finally it approaches the curve representing the isothermal boundaries with equipartitioning of heat, i.e. \( r = 0.5 \). This behaviour of the limiting shear stress, namely the decreasing nature, is also shown in Figure 5.2a, where the critical state, increases with increasing values of \( r \) until a maximum, holds valid for all liquids of interest in the lubrication practice listed in Chapter 2. For the rather uncommon liquids with \( 0 < r < 1 \) there is a continuous increase of \( r \) with increasing \( r \). The curve approaches that for the equipartitioning of heat, i.e. \( r = 0.5 \), which for \( r = 0.5 \) attains an asymptotic maximum value only at infinitely large values of \( r \), see Figure 5.2a.

It is interesting to note that at the critical state reached at finite values of \( r \), the heat flux distribution remains finite similar to the velocity distribution. Total heat flux \( G \), correction to isothermal rate of flow \( Q \), and flow parameter \( V \) given by equations (5.8), (5.10a) and (5.10b) also remain finite as shown in Tables 5.3, and 5.4 through 5.7. For the particular case \( m = 3 \) the variation of these three parameters has been presented in Figures 5.4a, 5.4c and 5.4d. Similar results can easily be obtained using the analytical solutions for the cases with \( m = 1/3 \), 1 and 2. As already discussed in the previous section, the numerical integration technique presently considered gives rise to certain difficulties at the critical states, especially for very low values of \( r \). Such difficulties have partly been overcome for the cases with \( m = 1/3 \) and 2, whereas the same have been completely overcome by employing a high accuracy computation of the special functions involved in the solutions for \( m = 1 \) and 3.

At the above mentioned critical state the minimum viscosity appearing at the position of the maximum temperature, see equation (4.3e), falls within the film for all positive non-zero values of \( r \). It has already been pointed out in Chapter 3 that such a minimum viscosity appears at the boundary of surface 1 for all values of \( r \). This regime of \( r \), as well as that of \( 0 < r < 1 \), for a steady-state heat transfer in the film is representative of the heat supply from sources external to the concerned film cross-section. In the regime \( 0 < r < 0.5 \) the ratio of the viscosity at surface 1 to the minimum is finite at the critical state, and is found by substituting the value of the limiting \( N_{B \cdot 1} \).

The results obtained for the shear stress \( \tau_1 \) using the numerical technique covering a wide range of \( r \), have been presented for Slotte's exponent \( m = 1/3 \), 1, 2, 5, 9, and 99 in Figures 5.5, 5.3a and 5.6 through 5.8. The general feature of the solution for positive finite values of \( m \) in the range \( 1 < m < \infty \), which has previously been discussed in Section 5.4, may be observed in these figures as well. For the case \( m = 99 \) the shear stress \( \tau_1 \) reaches high values at the critical values of \( r \). This

\[ (u_1 / u_2)_{\text{max}} = \frac{\pi^2 r_1}{1 - 2r_1} \]  

These limiting viscosity values at the critical states for the previously selected set of \( r \) values have been presented in Tables 5.5 through 5.7 for the \( m \)-values considered in these tables.

The results obtained for the shear stress \( \tau_1 \), using the numerical technique covering a wide range of \( r \), have been presented for Slotte's exponent \( m = 1/3 \), 1, 2, 5, 9, and 99 in Figures 5.5, 5.3a and 5.6 through 5.8. The general feature of the solution for positive finite values of \( m \) in the range \( 1 < m < \infty \), which has previously been discussed in Section 5.4, may be observed in these figures as well. For the case \( m = 99 \) the shear stress \( \tau_1 \) reaches high values at the critical values of \( r \). This

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behaviour of the solution, especially for very high values of the expon­
ent m, is similar to that of the exponential viscosity-temperature re­
lationship. On the other hand, the solution for the case m=1/3 has a
similarity with that of the isothermal flow, namely that the shear
stress continues to increase with increased N1 for the equipartitioning
of heat. The locus of the critical state for r1<0.5 also indicates a
continuous increase with N1. It may be mentioned here that the pre­
sently employed numerical technique has proved to be valid even when
non-integer values of m are employed, such as that which has already
been studied through Figure 5.5, i.e. m=1/3. Other thermohydrodynamic
parameters, such as the distribution of velocity and temperature, can
also be evaluated using the solution obtained through the presently de­
veloped numerical technique. These, however, have been presented in
Chapter 7 where a comparison has been carried out between the experimen­
tally measured quantities for various liquids that are characterized by
Slotte's relationship.

Further on, the Reynolds number Re of non-isothermal flows of liquids
following Slotte's viscosity-temperature relationship can be worked out
using either of equation (5.50a) or (5.50b) for the complete range of
the heat partition ratio -0.5<r1<0.5. Re remains finite even at the cri­
tical state where the shear stress reaches its maximum value for any
prescribed r1, at least for finite values of m. For the fluids with
finite m the flow parameter V is finite, and thus Re as given by equa­
tion (5.50b). Only at very high values of m, V and thus Re become
finite, similar to that evidenced for the exponential relationship in
Chapter 3. Reynolds numbers for a few liquids have been worked out and
included in Table 5.8: water with m=2 and a mineral oil, VITREA 100 with
m=3. It may thus be observed that the critical state is prevalent al­
ready at finite Reynolds numbers. In fact these numbers cannot be ex­
ceeded as evident from the non-existence of steady-state solutions in

<table>
<thead>
<tr>
<th>r1</th>
<th>-2.0</th>
<th>-1.0</th>
<th>-0.5</th>
<th>0.0</th>
<th>0.4</th>
<th>0.49</th>
<th>0.495</th>
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<td>900.0</td>
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<td>0.025</td>
<td>0.050</td>
<td>0.075</td>
<td>0.100</td>
<td>0.125</td>
<td>0.150</td>
</tr>
<tr>
<td>3</td>
<td>0.0125</td>
<td>0.026</td>
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<td>0.054</td>
<td>0.067</td>
<td>0.080</td>
<td>0.093</td>
</tr>
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</table>
such regions. Such numbers are the higher the smaller are $r_j$ and $m$. It is thus conceivable that lubricant films operating at low Reynolds numbers might approach the above mentioned critical state.

The type of thermal instability as predicted by Blok, for the fluids with the exponential viscosity-temperature relationship, may not be ruled out on the basis such as physically unrealistic infinitely large values of shear stress. On the other hand another relevant, but the most important, consideration namely that the viscosity ratio reaching high temperatures would be existent if the above mentioned thermal boundary condition $r_j<0.5$ is prevalent although only at some local regions of the boundary concerned. However the resultant critical state can never be exceeded, because there exist no steady state solutions in such operating regimes.

Most of the reported measurements in literature generally lack a satisfactory information on thermal boundary conditions imposed. The reported scuffing failures are an indirect evidence that certain extreme thermal state of the film might have been responsible for the otherwise safe hydrodynamic film formation in the corresponding lubricated regions.

### 5.5.2 Results in the regime $r_j>0.5$

Analytical results obtained using the linear fluidity function, as well as using the relationship with Slotte's exponent $m=1/3$ indicate a continuous increase of the non-dimensional shear stress $\Gamma_j$ with $N_j$, for all $r_j<0.5$ and the values of $m$ increased, say through the sliding velocity. If such a situation is not directly observed, this may be attributed either to the fact that in practice such values of $r_j<0.5$ cannot strictly be maintained, or to the occurrence of the Newtonian behaviour. It is also important to keep in mind that in Chapter 7 where a comparison of these results with those experimentally obtained has been carried out.

The aforementioned numerical technique can also be used to extend the results for the evaluation of velocity and temperature distributions, including the range in which these are double-valued in the physically realizable range of $N_j$. The heat flux $\phi$, the correction to the isothermal rate of flow $Q$ and the flow parameter $V$, can also be evaluated for the same set of $r_j$, covering a wide range, i.e. $0.5<r_j<1.0$. The characteristic behaviour of these results remains somewhat similar to that for the liquids following the exponential relationship as discussed in Chapter 3. These results for various Slottle's exponent $m$ have been presented in Figures 5.2 and 5.4. It may there be observed that the characteristics of these quantities of $N_j$, are identical to those representing the regime $r_j<0.5$. This may be observed through the NC1 parameter $N_j$ which represents the regime $r_j<0.5$ are identically equal to those representing the regime $r_j>0.5$. This may be observed through the NC1 parameter $N_j$ which represents the regime $r_j<0.5$ are identically equal to those representing the regime $r_j>0.5$.

Further, it has been observed that the solutions obtained for the regime $r_j<0.5$ are identical to those representing the regime $r_j>0.5$. This may be observed through the NC1 parameter $N_j$ which replaces $N_j$ with the corresponding parameters referred to surface 2, namely with $N_{j2}$ and $\Gamma_j$ which parameters consist of the viscosity of the liquid referred to the temperature at surface 2.

### 5.5.3 Criterion on the stability of flow

It may be recalled that for the heat partition ratio lying in the range $r_j<0.5$ the thermal instability as predicted by Blok might be feasible in a somewhat limited sense as discussed in Section 5.5.1.

We may now proceed to investigate the existence of the thermal instability of a different kind that may prevail for all $r_j>0.5$. We have observed during the analysis of plane shear flows using the exponential viscosity-temperature relationship in Chapter 3 that this instability may be prevalent on the second branch of the shear stress.

It has also been pointed out there that the first appearance of...
this instability might correspond to the thermal sliding parameter at the maximum shear stress level, i.e. the transition point.

It is evident from the characteristic nature of the solution for the liquid in Figure 5.2 and for the complete range of $r > 0.5$, that the maximum shear stress $T_1$ is finite and this is reached at an infinitely high thermal sliding parameter $N_s$. In essence all the results for the present linear fluidity relationship correspond to the first branch of the solution, and there exists no second branch. Thus the thermal instability of a second kind discussed in Chapter 3 may not arise at all as discussed in the paragraphs to follow.

The above conclusion on the thermal instability has been substantiated by Joseph [5.14] basing himself on Lin's [5.4] analysis and used a velocity profile of the form shown in Section 5.4.1.2 for isothermal boundaries. He has demonstrated that certain antisymmetric velocity profiles would not tolerate either neutral or self-excited oscillations. Such velocity profiles can be conceived in the case of unequal heat partition ratios. There also exists a point of inflection in the velocity profile. Yet the thermal instability that can be conceived might solely be of the 'inviscid' type represented by Reynolds number $Re$ becoming infinitely high. Consequently the instability of the hydrodynamic type plays an important role, as pointed out by Pu and Joseph [5.17]. It may be concluded that the instability due to thermal effects has no relevance to the flows operating under finite $Re$ of the liquids with Slotte's exponent $m$ in the range $0 < m < 1$.

The rate of the flow of liquids with $m=1$ with respect to the thermal instability of the second kind may, however, be conceived to be different. To the knowledge of the author, there have been no studies carried out towards the instability of such liquids. It may be observed that such liquids at least for the liquids to which the solution have been extended in an analytical form, can perhaps be performed. The general treatment of the problem would then lie on lines identical with those employed for the liquids following the exponential relationship, see Suknav et al. [5.18], and Vanderborch and Platten [5.19]. Due to the lack of results one may proceed to draw conclusions of only a qualitative nature towards the thermal instability under discussion.

In the inviscid limit ($Re \to \infty$) the velocity profile has been proved to be stable only up to the point of maximum shear stress and unstable beyond. Joseph [5.20] has pointed out the hydrodynamic instability might appear at the operating conditions resulting in such a maximum shear stress. For lubricant films of viscous fluids, the previously mentioned instability of the 'inviscid mode' is still feasible already at low $Re$; see Chapter 3.

The stability connected with the previously mentioned 'thermal mode' in Chapter 3 is of great importance since thermal effects might become predominant already at low Re. Instability of the temperature distribution may be assigned to flows operating on the second branch of the solution, since the maximum shear stress condition, and thus the transition point, can occur already at low Re if the film thickness and the material properties are suitably chosen. The results obtained by Suknav et al. [5.18] and Vanderborch and Platten [5.19] indicate a minimum characteristic nature of the solution, viz. an instability due primarily to thermal effects.

The second branch of the solution can easily be realised in practice already at low values of the thermal sliding parameter $N_s$. This second branch shifts towards lower $N_s$ for increased heat partition ratio above $r > 0.5$ as shown in Figures 5.3a, 5.4a and 5.6 through 5.8. The corresponding maximum shear stress decreases resulting in a decreased $Re$ for a liquid with a specified Slotte's exponent $m=1$, see Table 5.9. The $Re$ values evaluated using equation (5.50a) have been included. In the absence of any computed results, such as those of Table 3.1 for liquids with the exponential relationship, it may here be concluded that the critical $Re$ at which the present thermal instability of the second kind sets in decreases with increased $N_s$. Thus the films operating along the second branch of the solution are likely to undergo a thermal instability at $Re$ which might be feasible in lubrication practice. With the appearance of such an instability whatever may happen to the physical existence of the film has not yet been demonstrated before to the knowledge of the author. In the present investigation certain attempts have been made to experimentally determine the behaviour of the flow and the thermal state of such films as will be shown in Chapter 7.

5.6 CONCLUSIONS

Non-isothermal flow analysis of liquids following Slotte's viscosity-temperature relationship has provided a link between isothermal-isoviscous flows on the one hand and the flows of liquids with exponential relationships on the other. In accordance with the generality with which Slotte's relationship can be expressed to include the viscosity-temperature relationship for various liquids, as shown in Chapter 2, solutions for various thermohydrodynamic quantities can be readily obtained using this relationship.

Results obtained in the complete range of $0 < m < \infty$ can be classified into two distinct groups, in addition to those for the special case of $m=0$ for which Slotte's relationship degenerates into the exponential one. The first group in which $0 < m < 1$ is characterised by a monotonic in-

<table>
<thead>
<tr>
<th>$m$</th>
<th>$N_s$</th>
<th>$Re$</th>
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</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.984</td>
<td>0.722</td>
</tr>
<tr>
<td>1.5</td>
<td>0.781</td>
<td>0.519</td>
</tr>
<tr>
<td>2.0</td>
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<td>0.062</td>
</tr>
<tr>
<td>2.5</td>
<td>0.288</td>
<td>0.153</td>
</tr>
<tr>
<td>3.0</td>
<td>0.205</td>
<td>0.082</td>
</tr>
</tbody>
</table>

Table 5.9 Flow parameter $V$ and Reynolds number $Re$ at the position of the maximum shear stress (transition point) for various liquids; heat partition ratio $r > 0.5$.
crease of the shear stress given by the non-dimensional quantity \( \tau^* \) with an increased thermal sliding parameter \( \eta_3 \) for all values of the heat partition ratio \( \eta_1 \) or surface temperature difference. In the limiting case of \( \eta_1 \), the existence of an asymptotic value \( \eta_3 \) at indefinitely high values of the thermal sliding parameter \( \eta_1 \) for \( \eta_1 \leq 0.5 \), whereas for \( \eta_1 > 0.5 \) this increase is limited by a maximum value. The solutions for velocity and temperature remain single-valued throughout the operating regime of \( \eta_1 \) and \( \eta_3 \).

The second group, on the other hand, in which \( \eta_1 \leq \infty \) is characterised by the behaviour of the solution that is already known for liquids following the exponential viscosity-temperature relationship discussed in Chapter 3. This group is also characterised by the existence of a limiting shear stress for \( \eta_1 < 0.5 \) at the critical thermal sliding parameter at which the liquids with the exponential relationship have indicated an indefinite increase of the shear stress. The limiting shear stress value is obtained with an increased non-linearity of the viscosity-temperature relationship due to higher values of \( \eta_1 \) for any selected value of \( \eta_1 \). The limiting value reaches an indefinitely high magnitude as \( \eta_1 \) is further increased to very high values. In the regime \( \eta_1 > 0.5 \) the maximum attainable shear stress \( \tau^* \) decreases with an increased non-linearity of the viscosity-temperature relationship. This implies a decreased friction loss in the film constituted of a liquid with a strongly non-linear viscosity dependence on temperature. Increased thermal non-symmetry in the regime \( \eta_1 > 0.5 \) also results in a decreased maximum shear stress for a liquid with any specified \( \eta_1 \) value.

Thermal instability criterion established by Blok especially for liquids with the exponential viscosity-temperature relationship seems to be applicable to the liquids with finite values of the Blott's exponent \( \eta_1 \) if it was originally meant for. This stability criterion applicable to the thermal boundary condition \( \eta_1 < 0.5 \) may not be dismissed on the basis of physically unrealisable shear stress, heat flux and rate of temperature. However, if these kinematic and thermal parameters evaluated in Chapter 3 are suitably chosen, an increase either in \( \eta_1 \) or in \( \eta_3 \) results in a decreased \( \text{Re} \), in accordance with the decrease of the maximum shear stress. A quantitative information in this regard covering the entire second branch of the shear stress curve is believed to be useful in analysing flows such as in lubricant films.

## REFERENCES


Chapter VI

A HIGH-SPEED COUETTE VISCOMETER WITH ROTATING HEAT PIPE*

6.1 INTRODUCTION

The heat pipe and its variant the thermosyphon, have been extensively investigated in the last two decades for their industrial utility. In its first application the stationary heat pipe has been successfully employed in the absence of gravitational fields, namely in space crafts. Since that time the heat pipe has found its application in many engineering fields. It has been put into operation with a working fluid recirculated either by capillary action or by gravitational forces. A brief survey of the operating regime of various working fluids, and of the broad working principles for various engineering applications has been made by Finley [6.1]. An application to cooling of electronic components, of nozzle guide-vanes of gas turbines and of various components in mechanical engineering production processes under thermally important considerations has also been listed.

Rotating heat pipes have not yet found as extensive an application in mechanical engineering constructions as the stationary ones. This can mainly be attributed to information about their design details being too limited to account for the various conditions under which they are to operate. Owing to their working principle rotating heat pipes do not require wicks for recirculation of the condensate. On the other hand their operation is strongly influenced by the rotational speeds they are subjected to.

The rotating heat pipe is a closed hollow shaft (see Figure 6.1) made up generally of a heat conducting material such as steel or copper and is filled with a certain amount of working fluid. Evaporator and condenser regions are generally interconnected by an adiabatic section. While the evaporator interior might be selected either circular as in various experiments conducted so far, or conical as by Daniels and Al-Jumaily [6.2], the condenser interior has to be conical in order to provide the needed axial component of the centrifugal forces to transfer the condensate back to the evaporator section. In general, a very small taper is sufficient to initiate the recirculation already at low rotational speeds.

Under the application of a heat load, which in general will be non-uniformly distributed and need be intense enough to prevent overheating damage to the component to be cooled, the working fluid will vaporize and will fill the entire interior space of the heat pipe, thus extending into the condenser region. With the cooling of the condenser wall the vapour will condense and will in the presence of the centrifugal forces be thrown onto that wall. The axial component of the centrifugal force

*More design details may be found in a paper, which will shortly be published, giving guide lines to designers of rotating type heat pipes.
will drive the condensed fluid back to the evaporator along the tapered wall, thereby completing the cycle of operation.

In order to specify an efficient and optimum design of a rotating heat pipe various investigators have carried out theoretical and experimental analyses. While their analyses provide complete design details for a single given set of heat load and rotational speed, they are far from complete when the heat loads and rotational speeds to be met by a heat pipe vary in a wide range. Concerning these widely varying operating regimes, the previous investigators have so far been successful only to lay down certain incomplete design criteria. Their investigations include primary goals such as avoiding burning out of the working fluid when it is not present in a quantity sufficient for the applied heat loads. On the other hand these investigations have not offered useful criteria for the maximum amount of the working fluid that might be permitted for an efficient operation of the heat pipe, or what amounts to the same thing, on the smallest heat load that can be introduced when some of the best working fluids such as water are employed. Lack of information in such a heat load regime makes a designer's task uncertain about the success of operation while the heat pipe is employed for cooling of rotating shafts especially at low heat loads.

The present investigation is aimed at designing a rotating heat pipe for small heat loads and high rotational speeds which have not been covered by the previous investigators. The criteria which will be laid down towards the selection of a working fluid under such operating conditions are believed to be general enough for designing similar heat pipes for other applications. The heat pipe for which the design details will be given in the present chapter, will culminate in a well cooled and rotating shaft maintained at a uniform surface temperature so as to investigate non-isothermal shear flows of lubricant films in a Couette viscometer.

One of the first theoretical analyses carried out on the operation of a rotating heat pipe with laminar film condensation was by Balback [6.3]. He obtained solutions for the condensate film thickness and heat transfer coefficient. Later extensions were made by Newton [6.4] to include the effects of vapour pressure drop and interfacial vapour drag for very small cone angles. An approximate analysis by Marto [6.5] has covered the range of cone angles normally employed in the rotating heat pipes. This analysis gives satisfactory results for the design of a rotating heat pipe with water, ethyl alcohol and freon 113 as working fluids. A wide range of operational speeds, cone angles, condenser internal diameters and exterior heat transfer coefficients have been included as operating variables. His experimental results were more optimistic than those of the previous investigators. Marto's methods have been proved to be somewhat less than satisfactory, at least in the performance of a heat pipe designed by Bos [6.6] using water as the working fluid. A recent analysis by Daniels and Al-Jumaily [6.2] extends the original Musself type analysis of film condensation and includes the effect of vapour drag which generally becomes significant at high heat fluxes, and thus at high vapour velocities. They established an agreement between theoretically predicted and experimentally measured results for:

1. proper charge of working fluid, and
2. vapour in a near-dry saturated condition.

One of the important results relevant to heat pipe design is that for the effect of rotational speed on the rate of heat transfer. This has been given as a function of the saturation temperature of the vapour for various speeds with Arcton 113 as the working fluid by Daniels and Al-Jumaily [6.2] and with water by Marto [6.5].

It may be observed that for a given saturation temperature the heat transfer rate increases with increased speed. This may be attributed to the increased centrifugal forces which would improve the condensate flow rate. In addition there will be a resulting reduction in condensate film thickness leading to an increased heat transfer coefficient through decreased thermal resistance in the condensate film. For very intense exterior heat withdrawal the heat transfer coefficient would vary proportionately with \( \nu \), whereas \( \nu \) is the angular velocity, i.e. the angular velocity at very high rotational speeds would increase the vapour velocity to a large value inducing excessive vapour drag at the film-vapour interface thus ultimately causing the heat pipe to fail functioning satisfactorily.

The performance of a rotating heat pipe is strongly influenced by the physical properties of the working fluid. Water with a high latent heat of vaporization and high thermal conductivity can transfer high heat fluxes at saturation temperatures common with that of alcohol and Arcton. For all these liquids the heat transfer performance is improved by increased operating temperature brought about either by saturation temperature or by the difference of between this temperature and condenser wall temperature. There is, however, no considerable change in the heat pipe performance with a change of the charge of the working fluid.
6.3 DESIGN OF THE HEAT PIPE

6.3.1 Estimation of the film thickness distribution

There exist a few design calculation methods which give a reasonably good approximation of the amount of working fluid needed for given heat loads and speeds as well as the dimensions of the heat pipe. These dimensions are generally preselected depending on the permissible space where the heat pipe is intended for use.

Design data suggested by Daniels and Al-Jumaily [6.2] include, among other assumptions, a uniform condenser wall surface temperature which generally is not met in practice, and an iteratively adapted saturation temperature difference \( \delta_5 \). The total volume (charge) of working fluid required for a desired operation is evaluated in terms of two separate components: the liquid of the condensate film along the inner walls and the saturated vapour filling the rest of the heat pipe volume.

An iterative computer programme has been developed to evaluate the film thickness of the condensate and the vapour content for the dimension of the heat pipe specified in Table 6.1. Various sets of heat load and speed evaluated for a typical mineral oil accounting for viscous dissipation in the Couette type viscometer concerned, and as specified in Table 6.2 and Figure 6.2 have been selected as the required operating conditions of the heat pipe. The results obtained from the computer programme indicate a different film thickness distribution along the condenser inner surface for every set of operating conditions and for every working fluid. An average value of the film thickness in the condenser has been calculated as an indication of the volume of the liquid component present on the wall of the condenser, viz.:

\[
\frac{1}{L} \int_0^L \delta \, dx
\]

where \( \delta \) is the film thickness at the position \( x \) along the condenser of length \( L \) as shown in Figure 6.1. The volume of the liquid in the rest of the heat pipe can be calculated assuming a constant film thickness equal to that at the end of the condenser section. The vapour contribution has been evaluated using assumed values of the saturation temperature difference \( \delta_5 \) of the vapour. When the calculated heat load deviated from that selected, the saturation temperature difference was iterated until there was an agreement between the selected and the evaluated heat load. It may then be realized that the charge, especially the contribution due to vapour content, is dependent on the saturation temperature difference adapted from iteration.

Various results have been obtained for the heat load and speed conditions listed in Table 6.2 and depicted in Figure 6.2. The results obtained for the theoretical heat transfer rates and the corresponding saturation temperature differences with Arcon 113 as the working fluid, are included.

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<th>Heat load, N/m/s</th>
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<th>Centrifugal acceleration, g</th>
<th>Shear stress at liquid-vapour interface, ( \tau )</th>
<th>Friction drag number, ( D_r )</th>
<th>Two-phase Reynolds number, ( Re_{tp} )</th>
<th>Friction drag number, ( D_r )</th>
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<td>159.26</td>
<td>3.20</td>
<td>0.0262</td>
<td>10.86</td>
<td>0.0015</td>
</tr>
<tr>
<td>150.0</td>
<td>3.133</td>
<td>119</td>
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<td>3.20</td>
<td>0.0262</td>
<td>10.86</td>
<td>0.0015</td>
</tr>
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<td>3.20</td>
<td>0.0262</td>
<td>10.86</td>
<td>0.0015</td>
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<td>41</td>
<td>39.73</td>
<td>3.20</td>
<td>0.0262</td>
<td>10.86</td>
<td>0.0015</td>
</tr>
</tbody>
</table>

Table 6.1 Specification of heat pipe dimensions and the operating conditions.

Table 6.2 Specified operational heat loads and rotational speeds for a typical mineral oil, and the corresponding theoretical estimates of the saturated temperature difference and the average film thickness with Arcon 113 as the working fluid. Also included:

- Average Reynolds number
- Two-phase Reynolds number
- Average vapour velocity
- Shear stress at liquid-vapour interface
- Friction drag number

<table>
<thead>
<tr>
<th>Heat load, N/m/s</th>
<th>Heat flux, W/m²</th>
<th>Rpm</th>
<th>Centrifugal acceleration, g</th>
<th>Shear stress at liquid-vapour interface, ( \tau )</th>
<th>Friction drag number, ( D_r )</th>
<th>Two-phase Reynolds number, ( Re_{tp} )</th>
<th>Friction drag number, ( D_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>51.33</td>
<td>1.366</td>
<td>477</td>
<td>8.60</td>
<td>1.46</td>
<td>0.0084</td>
<td>14.75</td>
<td>0.0756</td>
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<tr>
<td>130.0</td>
<td>3.133</td>
<td>119</td>
<td>54.15</td>
<td>2.97</td>
<td>0.0393</td>
<td>13.35</td>
<td>0.0756</td>
</tr>
</tbody>
</table>
present operating regime. This is attributable to the gradual increase of $Nu$ being accompanied by an increase of condensate mass flow rate and a corresponding, albeit slight, increase in saturation temperature difference (cf. the definition of the Nusselt number). Further, it may easily be seen that the increase in the two-phase Reynolds number with increased heat load and rotational speeds may be attributed to the corresponding increase of the vapour velocity.

### 6.3.2 Selection of the working fluid

Comparatively speaking, the heat loads involved in the present work are low and rotational speeds are high. Among the nine working fluids investigated for their suitability and tabulated in Table 6.3 propane and Arcton 113 proved to be most suitable for the reasons to follow.

'Property group' is a criterion of merit of a working fluid for the heat pipe. This criterion is often used for the selection of a working fluid; the higher the value of the property group the better is the suitability of a particular fluid. Although water came out to be the best fluid under the consideration of the property group, for the reasons yet to be discussed below Arcton 113 has been selected.

It has been found from the computed results in Table 6.3 that propane offers the thickest films along the entire inner walls of both the condenser and the evaporator; Arcton 113 coming a good second. In the heat pipe actually constructed the interior surface finish of the copper pipe was far from fine. We aim at employing fluids that offer thicker films, and thereby at reducing the adverse effects such a comparatively poor surface finish on the performance of the heat pipe. Although propane seems to be suitable with an average film thickness as large as about 0.1 mm at the low heat loads and high surface speeds concerned, this

---

### Table 6.3 Physical properties and a suitability study on certain liquids as working fluids for the rotating heat pipe

<table>
<thead>
<tr>
<th>Working Liquid</th>
<th>Density $\text{g}_c$ $\text{cm}^{-3}$</th>
<th>Viscosity $\nu$ $\text{cSt}$</th>
<th>Condensate Mass Flow Rate $\dot{m}_c$ kg/s</th>
<th>Condenser Temperature $T_c$ °C</th>
<th>Evaporator Temperature $T_e$ °C</th>
<th>Total Heats $Q$ kJ/s</th>
<th>Total Surface Area $A$ m$^2$</th>
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</thead>
<tbody>
<tr>
<td>Propane</td>
<td>0.69973 at 35°C</td>
<td>0.006909 at 35°C</td>
<td>0.009500 at 35°C</td>
<td>0.003158 at 35°C</td>
<td>0.013034 at 35°C</td>
<td>0.00855 at 35°C</td>
<td>0.00324 at 35°C</td>
</tr>
<tr>
<td>Arcton 113</td>
<td>0.7020 at 35°C</td>
<td>0.007017 at 35°C</td>
<td>0.009500 at 35°C</td>
<td>0.003158 at 35°C</td>
<td>0.013034 at 35°C</td>
<td>0.00855 at 35°C</td>
<td>0.00324 at 35°C</td>
</tr>
</tbody>
</table>

---

### Figure 6.4

Nusselt number $Nu$ against two-phase Reynolds number $Re_{\text{v}_{\text{L}}}$ for the operating conditions tabulated in Table 6.2.
fluid has not been selected because of its low boiling point of -43 °C and the correspondingly increased difficulty to fill it into the heat pipe under vacuum. On the other hand the liquid ultimately chosen, Arcton 113, could be expected to operate about equally well at the same heat load and speed conditions, its average film thickness along the condenser inner wall being expected to be still comparatively large, that is, about 0.08 mm.

6.3.3 Estimation of the total charge (fill) of the working fluid

Total charges (fills), i.e. the sum of the aforementioned individual charges of the liquid and the vapour have been evaluated for every selected set of heat loads and speeds. The tabulated results in Table 6.2 show that the total fills so obtained for Arcton 113 differ for every set of operating conditions. Since such variable charges are not practically realisable for a wide range of operating conditions, the maximum charge which corresponds with the most severe condition has been selected for every working fluid investigated.

For a selected constant charge, the previously evaluated variation of saturation temperature difference with various heat loads, as well as the variation of average Nusselt numbers with two-phase Reynolds numbers shown in Table 6.2, no longer hold good. A theoretical estimate of the aforementioned quantities for that constant charge can only be made on the basis of other operational parameters that are still open and have yet to be adapted from experiments. For the present such an analysis will not be carried out because of its semi-empirical nature.

6.4 CONSTRUCTION AND OPERATION OF THE HEAT PIPE

6.4.1 Construction of the heat pipe

The overall dimensions including the taper angle have already been summarized in Table 6.1. Figure 6.5 shows the constructional details, including the two end closures of the heat pipe, in one of which a quick return valve has been installed for the vacuum connections as well as for filling the working fluid. The two end connections also carry precision shafts designed to mount the heat pipe in aerostatic bearings.

One end of the shaft is connected to a motor drive through a tachometer arrangement while the other end carries a silver slip ring assembly capable of connecting the thermocouples. The heat pipe shell, a copper cylinder, has been embedded with six copper-constantan thermocouples and has also been fitted with a pressure gauge to check, at regular intervals, the vacuum in the heat pipe when this is stationary. The central rod inside the shaft carries a thermocouple connected to record the saturation temperature while the heat pipe is in operation. The heat pipe was then dynamically balanced to ensure that at the intended high rotational speeds it would not introduce intense vibrations that might be detrimental to the operation of the bearings.
6.4.2 Filling of the heat pipe

Figures 6.5 and 6.6 show the assembled heat pipe fitted with thermocouples and a pressure gauge. Care has been taken to clean the inner walls of the heat pipe and end connections thoroughly. The first cleaning stage was performed with a solution of equal parts of ethyl alcohol and a fifty percent aqueous solution of sodium hydroxide warmed to 80 °C, followed by rinsing with distilled water. In the second stage, 10 percent sulphuric acid solution was used. The cleaning was completed with thoroughly rinsing the same parts with distilled water several times.

After having connected the thermocouples and the pressure gauge to the measuring instruments, the heat pipe was immersed in a cooling bath connected to a cooler capable of reaching about -40 °C. The quick return valve of the heat pipe was connected to a vacuum pump through a reservoir containing a charge of Arcton 113 intended for filling into the heat pipe, see Figure 6.7. The reservoir is, however, separately connected with valves \( V_2 \) and \( V_3 \) which facilitate the connection between the reservoir and the rest of the vacuum connection. The filling of the heat pipe was then carried out as follows.

First of all the cooler was set in operation and the heat pipe was cooled down to -30 °C in the cooling bath. With both valves \( V_3 \) and \( V_4 \) closed and \( V_2 \) open the vacuum was drawn to 2 mbar with the help of the vacuum pump. With both \( V_2 \) and \( V_3 \) closed and \( V_4 \) open the pump was operated to draw vacuum in the reservoir and to extract the dissolved air from the working fluid. Then the pump was switched off. With both valves \( V_2 \) and \( V_4 \) closed and \( V_3 \) open the desired amount of working fluid was allowed to escape in vapour form from the reservoir and finally to condense into liquid form inside the heat pipe. This transfer of the liquid from the reservoir to the heat pipe was stimulated by slightly warming up the reservoir. When the desired amount of the working fluid had thus been transferred from the reservoir, the valves \( V_3 \) and \( V_4 \) were closed, and finally the heat pipe was disconnected.

During the above filling process the temperature was monitored with the thermocouples embedded in the heat pipe wall, and the pressure at various stages of filling was recorded with the pressure gauge.

6.4.3 Calibration of the heat pipe

The heat pipe was mounted in aerostatic bearings; the latter were then aligned for concentricity with respect to the two shaft ends of the heat pipe. A cylindrical electric heating element was then suspended around the evaporator over a width of 100 mm. A clearance of about 1 mm was maintained between the inner wall of the heating element and the outer surface of the evaporator. A condenser jacket consisting of ports for cool-liquid connections and insulated on its outer wall was then mounted around the condenser, as schematically shown in Figure 6.8. Lip seals
assisted by pressurised air facilitated a suspension for the jacket around the condenser and nearly free from contact.

The heat pipe was then run at the desired rotational speeds and heat loads, the latter being monitored through the electric heating element. Cool-liquid circulation was maintained around the condenser until a steady-state heat transfer prevailed in the whole of the heat pipe. Further temperature recordings of the cool liquid at the inlet to, as well as the outlet from, the condenser section were made with the help of the thermocouples.

The heat pipe was calibrated for operation at various sets of operating conditions, covering a wide range of speeds and heat loads such as would be encountered in the tests yet to be run with oils for their viscous dissipation. To make the results as generally applicable as possible, a few rotational speeds have first been selected in a range from 460 up to 3000 rpm. In that rotational speed range the total heat load has been varied between 50 and 300 watt, which corresponds with about 10 to 65 kW/m² (cf. Figure 6.9).

Saturation temperature differences were then evaluated at every set of operating conditions from the measured sets of temperatures. The corresponding average Nusselt numbers and approximated two-phase Reynolds numbers were evaluated from the measured quantities and the known physical properties of the working fluid.

The results have been displayed in Figures 6.9 and 6.10. Previously evaluated theoretical values may be considered only approximate since the charge of the working fluid which corresponds with some prescribed set of operating conditions there is different in contrast to a constant charge in the present heat pipe for all the operating conditions employed so far. It may be observed from Figure 6.9 that the performance curves indicate saturation temperature difference to increase for increased rotational speeds and to be practically insensitive toward heat flux (cf. the approximated performance curves which are nearly straight horizontal lines). At low heat intensities and low speeds the saturation temperature difference is somewhat lower, and at high intensities and high speeds it is somewhat higher than the above-mentioned near constant values. The results presented in Figure 6.9, although corresponding with relatively small heat intensities and high rotational speeds,
display a trend similar to the results of previous investigators, which relate mainly to higher heat intensities and lower rotational speeds. Figure 6.10 is a non-dimensional representation of the performance characteristics of the heat pipe using the experimental results shown in Figure 6.9. Lower Nusselt numbers $Nu$ representing a smaller heat transfer coefficient prevail at low two-phase Reynolds numbers $Re_v$, involving a comparatively small vapour drag at the condensate film-vapour interface. Both $Nu$ and $Re_v$ vary almost proportionately except in the range of high $Re_v$. High $Re_v$ thus correspond with relatively low heat intensities, where $Nu$ becoming less dependent on $Re_v$. One may then conclude that at high $Re_v$, the heat transfer coefficient becomes less dependent on $Re_v$. The heat pipe will then transfer the increased heat loads by increased vapour velocity. Saturation temperature differences will then be high as may be seen from Figure 6.9, which implies that the cooling of the condenser should be more intense. There are indications from published data that at very high $Re_v$, $Nu$ tends to decrease indicating a lower heat transfer; at extremely high $Re_v$ there would be no heat transfer at all since $Nu$ becomes negligibly small.

From Figures 6.9 and 6.10 it has proved possible to predict the cooling characteristics of the heat pipe to be employed for the Couette viscometer concerned.

6.5 CONSTRUCTION OF THE COUETTE VISCOMETER

Viscous dissipation of liquids subjected to high shear stresses gives rise to generation of heat. Pure shear flows of mineral oils discussed in Chapters 3 and 5 will in the next chapter be taken up for an experimental investigation. It is intended to subject those shear flows to comparatively large shear rates through employing high sliding speeds, and to relatively well controlled thermal boundary conditions. Thus a Couette viscometer has been built to test the mineral oils enclosed in an annular space between its stator (an outer member) and its rotor (an inner member). Its design details, including those of the components mentioned herebelow, have been provided by Bos [6.6].

When the rotor of the Couette viscometer is built as a part of the evaporator section of the previously described heat pipe, its surface temperature can be well controlled, also as to its uniformity. It has already been pointed out that in a wide range of heat loads and rotational speeds the evaporator surface temperature can successfully be maintained at a very nearly constant level. The range of operational conditions met by the heat pipe described in the previous sections have been found to satisfy those corresponding to rotational speeds and the viscous dissipation (heat load) which are required to be met by the Couette viscometer.

To make the stator as adiabatic as possible it was made of a precision glass tube well insulated on its outer wall and suspended around the evaporator section, thus replacing the cylindrical electrical heating element previously used for calibrating the heat pipe. The glass tube has been so positioned as to maintain a uniform gap of 1 mm around the evaporator outer surface. This glass tube is housed in a precision hollow cylinder suspended in aerostatic bearing assembled inside the outer stainless steel housing (see Figure 6.5). The clearance space between the glass tube (stator) and the heat pipe (rotor) in the test section is kept pressurized by connecting it to an oil supply under pressure. The ends of this test section are sealed with aerostatically operated lip seals which are mounted on the stator: during operation of the Couette viscometer they provide for contact-free sealing of the oil.

Figure 6.11 Components of the oil sealing unit.

Figure 6.12 Major components of the Couette viscometer displayed for clarity.

The aforementioned criteria for the selection of the various components as well as their operational characteristics will be discussed in detail in Chapter 7. The suitability and the principles of operation of the present Couette viscometer will also be discussed there.
A rotating heat pipe can be constructed so as to be operable throughout the entire range of heat load and rotational speed of the Couette viscometer concerned. With the help of this rotating heat pipe uniform and constant temperatures can be maintained on the outer surface of the rotor under various operating conditions.

Various volatile liquids can be used in the heat pipe for the above-mentioned operating conditions. Arcton 113 is one of the best working fluids for the operating conditions at hand. Successful performance can be achieved with a thin film of Arcton 113 covering the entire inner wall of evaporator, condenser and adiabatic sections. A thin film of about 0.1 mm of Arcton 113 would be satisfactory for heat loads not exceeding about 350 watt and at rotational speeds as high as 3000 rpm. Saturation temperature differences measured from the experiments are approximately equal to those expected from the theoretical estimation of Daniels and Al-Jumaily [6.2]. These temperature differences do not exceed about 6°C indicating that the aforementioned heat loads of about 350 watt can easily be transferred from the test section without introducing any appreciable change in the temperature of the evaporator surface.

The evaporator section of a heat pipe can readily be employed as an isothermal rotor in a Couette viscometer. An adiabatic stator can be built around the rotor so as to enclose the oil to be tested for the effects of viscous dissipation in the clearance space. Such a Couette viscometer can also be employed for the measurement of tangential velocity distribution of the test oil using laser-Doppler anemometry. Finally, it can be used for the measurement of the viscous drag in the oil at various rotational speeds of the rotor.

6.7 REFERENCES


7.1 INTRODUCTION

It has been theoretically established in Chapters 3 and 5 that at the high sliding speeds (shear rates) to be aimed at the behaviour of shear flows will be strongly influenced by thermal effects. Consideration of the heat generation by viscous dissipation and the concurrent heat transfer by cross-conduction leads one to conclude, on theoretical grounds, that there exists a maximum to the shear stress which cannot be exceeded with increased sliding speeds.

For any given shear stress less than the above maximum there exist double-valued shear rates or thermal sliding parameters for all liquids which follow a non-hyperbolic viscosity-temperature relationship, such as the Slotte's with the exponent m greater than unity. This becomes also evident when the shear stress is expressed in a non-dimensional form involving the viscosity at the temperature of the surface with a heat partition ratio greater than 0.5. The results when re-expressed with reference to the other surface show that the heat partition ratio attains values less than 0.5: they also indicate that the shear stress in a non-dimensional form, while expressed using the viscosity corresponding to the temperature of the latter surface, undergoes an asymptotic increase already at finite values of the sliding parameter (see Chapter 3 for details).

Very little experimental work has so far been carried out to demonstrate the above behaviour of the flow. Kingsbury [7.1] was the first to carry out some experiments using a "flash method". The method consists of speeding up the rotor very fast to the desired final speed, and to measure as quickly as possible the frictional torque exerted on the stator. His intention was to determine the shear stress in the near-absence of heating during the very short time involved in the flash method. This method is even now practiced to establish the shear stress dependence on the shear rate at known, uniform temperatures and viscosities. Kingsbury, however, proclaimed that the flash method was meant to carry out the measurements before the thermal process had developed to any considerable degree, the heat transfer from the stator and rotor to the surroundings then being negligible. Blok [7.2] showed that in the (transient) starting-up stage the two partitioning ratios differ from those in the steady state. In the former stage the then prevailing partitioning ratios can be controlled through a suitable choice of the materials of the rotor and stator as to their thermal properties, whilst the latter steady state can be controlled through a preferential cooling of the surfaces of these materials. In the starting-up stage the shear stress (as measured) is the same as that for a uniform temperature (and viscosity) across the film, so that only in that stage, but not in the
steady state which is reached later, no corrections are needed for the actual non-uniform temperatures and viscosities: these corrections are negligible since the non-uniform temperatures and viscosities prevail only in a thin layer of the liquid close to the surfaces involved. Thus Blok's differentiation between the starting-up stage and the steady state placed Kingsbury's results in the proper perspective.

Some exploratory results were obtained by Hägg [7.3] through his measurements and the theory they were meant to verify. Thus the latter results which Golubev compared his experimental results. Hence the thermal boundary conditions were largely suppressed to a tolerable minimum. Thus the thermal boundary conditions were largely suppressed to a tolerable minimum. Thus the thermal boundary conditions were largely suppressed to a tolerable minimum.

The liquids employed by him namely castor oil and cylinder oil would not have followed the hyperbolic viscosity-temperature-temperature relationship. However, his results covered only the initial part of the shear stress versus sliding speed curve (see Figure 3.1 of Chapter 3). So no conclusive remarks could be drawn by Hägg on the characteristic behaviour of the shear stress in the region of really high sliding speeds.

Some exploratory results were obtained by Golubev [7.4] to conduct experiments in a wide range of sliding speeds. His results indicated a maximum to the shear stress, but did not indicate any decrease with further increase of sliding speeds. These results were qualitatively in line with the theoretical results of Pavlin [7.5]. Golubev was not able to trace the decreasing behaviour of the shear stress with increasing velocity because either he did not extend his experiments to high enough sliding speeds, or he was not in a position to exercise a control on thermal boundary conditions. The liquids employed by him namely castor oil and cylinder oil would not have followed the hyperbolic viscosity-temperature-temperature relationship. This can readily be achieved by employing a temperature sensitive viscous liquid and/or high sliding speeds. To comply with the former, Sukanek and Laurence [7.6] have employed the Arochlor 1260 series of chlorinated biphenyls manufactured by Shell Company. For the present investigation a mineral oil series Vitrea manufactured by Shell has been employed. This oil is available in a wide range of viscosity grades and does not contain any VI improvers (polymeric compounds). Vitrea oils have regularly been used as calibration fluids because of their Newtonian behaviour over a wide range of shear rates. It may be recalled that mineral oils have been extensively tested over a long period in various types of viscometers. At least base oils are known to remain Newtonian in a wide range of shear rates up to about 103 sec⁻¹, and shear stresses not exceeding about 102 N/m². This has been corroborated by various investigators, recently by Wright and Mather [7.8], who used specially constructed viscometers with carefully selected experimental methods in which the thermal effects were successfully suppressed to a tolerable minimum.

Vitrea oils, in addition satisfy the requirements of high viscosity, and relatively high values of βₚ. Three of these oils have been selected for the study of both their shear stress variation with various sliding speeds, and their velocity distribution, using a special laser-Doppler method. Their physical properties, such as densities and viscosities, have been listed in Table 7.1. Their viscosity was found to be quite sensitive to temperature. Oils 2A and 3A exhibit a rather high sensitivity, as is evidenced by their high value of Slotte's exponent m, i.e. 4.65 and 4.69, respectively.

Thermal conductivity of the same oils as influenced by temperature has been provided in Figure 7.2. Density in Figure 7.3. Other physical properties of these oils may be found in "Shell Oils for Industry". For these oils the Slotte's exponent m and coefficient a have been evaluated for the rather wide temperature range of 20-75 °C. Further, the equivalent βₚ-values have been evaluated for each oil using the method already explained in Chapter 2. These βₚ-values have been included in Table 7.1.

7.2 SELECTION OF TEST OILS

Behaviour of the flow under thermal effects can best be studied through a measurement of shear stress. Both decreasing shear stress with increased sliding speeds or shear rates for a heat partition ratio equal to or exceeding 0.5, and asymptotically increasing shear stress for the heat partition ratio less than 0.5 can be achieved only at high sliding speeds or shear rates. High shear stresses can either be expressed in terms of Brinkman number or a similar thermal sliding parameter N, which will be defined below. It is the intention to achieve high values of the thermal sliding parameter, i.e.

\[
\frac{R}{C} \left[ \frac{\dot{\gamma}}{\dot{\gamma}^*} \right] \approx k
\]

where the temperature coefficient, B, of viscosity relates to exponential viscosity-temperature relationships, and should be replaced equivalently by βₚ in the case of Slotte's relationship. This can readily be achieved by employing a temperature sensitive viscous liquid and/or high sliding speeds. To comply with the former, Sukanek and Laurence [7.6] have employed the Arochlor 1260 series of chlorinated biphenyls manufactured by Monsanto Company.

For the present investigation a mineral oil series Vitrea manufactured by Shell has been employed. This oil is available in a wide range of viscosity grades and does not contain any VI improvers (polymeric compounds). Vitrea oils have regularly been used as calibration fluids because of their Newtonian behaviour over a wide range of shear rates. It may be recalled that mineral oils have been extensively tested over a long period in various types of viscometers. At least base oils are known to remain Newtonian in a wide range of shear rates up to about 10³ sec⁻¹, and shear stresses not exceeding about 10² N/m². This has been corroborated by various investigators, recently by Wright and Mather [7.8], who used specially constructed viscometers with carefully selected experimental methods in which the thermal effects were successfully suppressed to a tolerable minimum.

Vitrea oils, in addition satisfy the requirements of high viscosity, and relatively high values of βₚ. Three of these oils have been selected for the study of both their shear stress variation with various sliding speeds, and their velocity distribution, using a special laser-Doppler method. Their physical properties, such as densities and viscosities, have been listed in Table 7.1. Their viscosity was found to be quite sensitive to temperature. Oils 2A and 3A exhibit a rather high sensitivity, as is evidenced by their high value of Slotte's exponent m, i.e. 4.65 and 4.69, respectively.

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Table 7.1 Physical properties, such as densities and viscosities at various temperatures, for the oils tested on the Couette viscometer. Also included are Slotte's parameters $n$ and $A$, and the viscosity-temperature coefficient $B$, evaluated at the temperature of 20 °C. Oils 1A, 2A and 3A correspond to Shell Vitrea 100, 320 and 460 respectively.

<table>
<thead>
<tr>
<th>Oil</th>
<th>Temperature °C</th>
<th>Density, kg/m$^3$ (at 23.4 °C)</th>
<th>Viscosity cf</th>
<th>Slotte's coefficients using data at 20 °C and 75 °C</th>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n$ $A$ $B$ $1/°C$ $1/°C$ $1/°C$ $1/°C$ $1/°C$</td>
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<td>1A</td>
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7.3 OPERATIONAL (AND DESIGN) FEATURES OF THE COUETTE VISCOMETER

7.3.1 Suitability of the Couette viscometer

Various types of viscometers have been in use already over several decades but almost none is universally applicable to measuring the viscosity of liquids at high rates of shear whilst allowing for corrections for the then occurring thermal effects. In fact, one of the major difficulties to ascertain whether a liquid is Newtonian or non-Newtonian, especially under high rates of shear, is the presence of the thermal effects due to viscous dissipation. As has been pointed out in Chapter 3, various investigators have falsely concluded to non-Newtonian behaviour by not properly correcting for the present effects, if at all.

Capillary viscometers are quite unsuitable for accurate measurement of viscosity at high rates of shear, i.e. in the presence of high thermal effects. Moreover, shear rates as well as shear stresses in these viscometers are non-uniform in any cross-section. Less popular but better suiting the present purpose are rotational viscometers. For a detailed survey of design and applicability of various types of viscometers the reader is referred to the book by Van Walzer et al. [7.9].

The rotating type, cone and plate viscometers, are not capable of yielding high rates of shear, and moreover suffer from poor thermal control over the surfaces bounding the film. In these two respects definitely Couette type rotating viscometer offers brighter prospects. Couette viscometers have been used for a long time to account for thermal effects in the flow of oils. These viscometers have been designed for studies concerning either non-Newtonian effects in regimes where thermal effects may be ignored or to correct for thermal effects in operating regimes where the liquids are known to remain Newtonian.

One of the most suitable designs is due originally to Barber, Muenger and Villforth [7.10] and is characterized by interchangeable inner cylinders. This offers a possibility to select a gap between the inner and outer cylinders in a wide range from 0.5 μm to 125 μm, and is capable of yielding shear rates exceeding 10$^5$ sec$^-1$ without a need for thermal corrections [see also Manrique and Porter [7.11]].

Most of the previous investigators have used the above-mentioned refined Couette type of viscometers to study the non-Newtonian behaviour of polymer solutions, blends of mineral oils with polymers, greases, paints, inks etc. In order to minimise the thermal effects they have...
employed very small gaps leading to low temperature gradients at low ro­
tating speeds, yet satisfying the requirement of high rates of shear. In their fairly successful attempts at satisfying all these needs, Bar­
er et al. have adapted the diameters and wall thickness of the inner
and outer cylinders such as to obtain equipartition of heat transfer
from the film to the boundaries. Easily operable gaps between the rotor
and the stator, a good control over boundary temperatures or temperature
gradients, along with allowing for ample time before a measurement is
made, are some of the achievements realized by the design of the present
Couette viscometer already explained in Chapter 6.

7.3.2 Time available for measurements

As mentioned before, one of the important factors which strongly influ­
ences any measurements at high shear rates, is the duration of the tran­
sient stage involved in achieving the required shear rates before the
fluid in the steady state would develop any considerable amount of heat
due to viscous dissipation. In the apparatus built by Williams, Carey
and Hamstock [7.12] this delay has been reduced to a minimum, that is,
to less than 0.01 sec for obtaining a shear rate of \(1.2 \times 10^5\) sec\(^{-1}\). The
 corresponding rate of change of shear rate was as high as about \(5 \times 10^6\)
sec\(^{-2}\) for liquids such as mineral oils and glycerin. Such high accelera­
tions followed by fast measurements have yielded results almost free of
thermal effects.

In order to stimulate the temperature gradients to attain higher va­
values thereby promoting the cross-conduction to at least one of the boun­
daries, here the inner one, the present Couette viscometer has been de­
signed for a comparatively thick film. Such a thick film also facili­
tates measurement of the velocity distribution in the film cross-section
through the laser-Doppler method. Although the amount of heat generated
in such a film is large, a well-controlled heat transfer has been ac­

tained to the rotating inner cylinder which has been built especially
for this purpose using the principles of a rotating heat pipe (see Chap­
ter 6). By selecting an insulated glass outer cylinder the opposite
surface has been made as nearly adiabatic as possible so that the heat
pipe in general withdraws by far the greater part of the heat.

Short acceleration times are not here required since the present in­
vestigation is directed towards the study of thermal effects as most in­
fluenced by the viscous dissipation in the steady state. For the suit­
ably selected mineral oils explained in the previous section, the non-
Newtonian effects are still negligible up to rates of shear of at least
\(10^5\) sec\(^{-1}\).

7.3.3 Gap size

One of the major problems involved with a Couette viscometer of very
small gap size concerns the differential thermal expansion of the rotor
and stator. A careful selection of the materials for these components
is required to maintain as uniform a gap as possible and to avoid any
seizure under the worst conditions. In the present viscometer this
Pure plane shear flows are hard to realize. As the ratio of the clearance to the diameter of the film in an annular gap is chosen smaller, the circular flow increasingly simulates the simpler plane shear flow. It has already been pointed out in Chapter 3 that the characteristic behaviour of the solution of the circular shear flow is similar to that of the plane shear flow, and as the ratio \( R^* \) of the outer to inner diameter of the film approaches unity, this behaviour is identical for the two types of flows. But there is some concern about the circular shear flow when it comes to stability of flow. That is, it is well known that circular shear flows may suffer from the formation of Taylor vortices. The corresponding Reynolds numbers at which Taylor vortices appear are rather low; it is suitable to discuss the state of flow in terms of the Reynolds numbers while comparing plan-parallel shear flows with the circular ones. Only at very high Reynolds numbers does the flow become non-laminar and ultimately turbulent.

7.3.5 Plane or circular shear flow

Centering of the rotor is another bottleneck in rotating viscometers with very small gaps. Williams et al. [7.12] point out that certain inaccuracies in their measurements can be attributed to eccentricity of the rotor even though only by a very small amount. Clear indications of the influence of eccentricity on the measured shear stresses have been discussed by Sukanek and Laurence [7.6]. Their experimental values of shear stress are lower than the theoretically expected ones. However, in their experimental apparatus they employed an adiabatic inner wall and a well cooled outer wall. Whether these particular thermal boundary conditions are met at all in practice for a comparison of their experimentally obtained results with the theoretical ones, is a question that needs careful scrutiny. Their concern about the lower shear stresses relates to the eccentric position of the rotor. As it will be pointed out later while discussing the experimental results, one may indeed expect shear stress for an eccentric rotor to be lower than that for a centered one. In the present viscometer the eccentricity could be maintained at the low value of 10 \( \mu \)m. This was the accuracy with which the hollow inner cylinder of copper could be produced in the workshop. Further, high-accuracy self-centering aerostatic bearings employed on either end of the inner cylinder reduced any further inaccuracy of centering to a minimum. Further it was expected that accurate centering would probably avoid the fluid rupture at high shear rates as observed by Williams et al [7.12].

7.3.4 Centering of the rotor and the stator

Centering of the rotor is another bottleneck in rotating viscometers with very small gaps. Williams et al. [7.12] point out that certain inaccuracies in their measurements can be attributed to eccentricity of the rotor even though only by a very small amount. Clear indications of the influence of eccentricity on the measured shear stresses have been discussed by Sukanek and Laurence [7.6]. Their experimental values of shear stress are lower than the theoretically expected ones. However, in their experimental apparatus they employed an adiabatic inner wall and a well cooled outer wall. Whether these particular thermal boundary conditions are met at all in practice for a comparison of their experimentally obtained results with the theoretical ones, is a question that needs careful scrutiny. Their concern about the lower shear stresses relates to the eccentric position of the rotor. As it will be pointed out later while discussing the experimental results, one may indeed expect shear stress for an eccentric rotor to be lower than that for a centered one. In the present viscometer the eccentricity could be maintained at the low value of 10 \( \mu \)m. This was the accuracy with which the hollow inner cylinder of copper could be produced in the workshop. Further, high-accuracy self-centering aerostatic bearings employed on either end of the inner cylinder reduced any further inaccuracy of centering to a minimum. Further it was expected that accurate centering would probably avoid the fluid rupture at high shear rates as observed by Williams et al [7.12].

7.4 Monitoring of the experiments

Constructional details including those of the constant temperature rotor based on the principles of a heat pipe have been provided in Chapter 6, and in somewhat greater detail by Bos [7.13]. A schematic diagram of the test section of the Couette viscometer has been provided in Figure 7.4. The rotor ran with a negligible eccentricity with respect to the precision stator made of glass. The stator and one of the ends of the rotor were aerostatically suspended in a single stainless steel enclosure thereby achieving a reasonable degree of concentricity in the annular space between the stator and the rotor. The other end of the rotor was aligned in an aerostatic bearing with the help of a specially built apparatus.

7.4.1 Monitoring of the rotor temperature

The temperature in the test section of the rotor has been monitored by observing the temperatures read from the thermocouples imbedded in the thin wall of the rotor. Since the rotor has been manufactured of 99.9% pure copper, and its wall is a mere 3 mm thick, the radial temperature gradients were assumed to be negligible. Thus the temperatures indicated by the thermocouples having a bead size of 0.8 mm and being embedded close to the outer surface, represented the surface temperatures of the rotor fairly accurately. Notwithstanding the axial temperature gradients along the rotor, due to the heat pipe not performing perfectly, the temperature in the test...
section could be maintained at a fairly uniform level of 20 °C by a proper control of the cooling in the condenser section of the heat pipe. During the measurements the uniformity of the distribution of the surface temperature in the test section of the rotor could be successfully maintained within ±2 °C, the lowest temperature occurring at the end of the test section towards the condenser and the highest at the other end. Only at very high operational speeds of the rotor, i.e. exceeding 2800 rpm, did the non-uniformity of temperature in the test section slightly exceed the above value of 2 °C. The difficulty to maintain the tolerable value of ±2 °C could be attributed to the inadequate cooling capacity of the Ultra Kryostat cooler used. Further, during these high rotational speeds it was discovered that the operating conditions deviated from those for which the heat pipe was originally designed and calibrated. It is thus admitted that, notwithstanding the careful precautions taken in the design stage, the heat pipe in its entirety, did not fully come up to expectations. It is the shortcomings of this system that prevented the author from verifying the present thermohydraulodynamic theory far beyond the maximum shear stresses thereby predicted. Unfortunately, time available proved too short for improving the heat pipe so as to enable the author to continue the experiments in the manner originally aimed at.

7.4.2 Steady-state conditions in the test section

The investigation was originally intended to study the steady-state behaviour of thermally affected plan-parallel Couette flow and thus to keep the convection effects due to the flow of liquid in the axial direction along the annular gap to a minimum. This requirement, however, had to be relaxed somewhat during the measurements at the highest rotational speeds of the rotor reached. But at lower speeds the amount of heat transfer due to convection effects was indeed negligible. Control on the convection effects has been achieved by allowing only a small amount of rate of flow from the inlet at the middle of the test section and, towards the outlet side, by connecting the ends of the test section to the sump. The slip-ring unit with a brush-lifting attachment was from IDM Electronics, Inc. This ultra-thin nickel-teflon thermometer in flat-grid configuration is known to provide a fast response on electrically insulated surfaces like glass. The filament had a thickness of only 50 μm, and thus was considered to represent the surface temperature of the wall sufficiently accurately. With the help of the Keithley digital display multimeter in conjunction with the present filament, the surface temperature could be assessed to an accuracy better than 0.05 °C.

7.4.3 Monitoring of the flow of oil in the test section

Maintaining a very nearly uniform steady state in the test section requires negligible flow through the test section. For the reasons discussed in the previous section, especially valid at lower rotational speeds and to a limited extent at higher speeds, the rate of flow through the test section had to be measured. This was done either with the help of an F&P rotoflowmeter connected at the inlet side of the test section and, through the test section, towards the exit close to the other end connecting the ends of the test section to the sump. The temperature of the supply oil to both the middle of the test section and the ends of the test section where the air-operated lip seals have been built-in, was measured with pressure gauges. The temperature of the oil at these two supply connections to the test section was measured with the help of copper-constantan thermocouples. The temperature of the supply oil had been maintained at ±2 °C by thermostatically cooling the oil sump to a lower temperature, thereby preventing possible over heating of the oil during experimentation.

7.5 Calibration of the instrumentation

The thermocouples employed in the viscometer were made of copper-constantan and were found to yield reproducible results even after their presence over a long period in the hostile environment of the test oil at somewhat elevated temperatures, or in that of the cooling liquid, namely methanol, at temperatures as low as -30 °C. The thermocouples were used to measure the temperature with the help of the digital display multimeter Keithley 177 to an accuracy of 0.25 °C. This accuracy, however, was limited to 0.5 °C by the Phillips multipoint recorder to which the thermocouples embedded in the wall of the rotor were connected. The slip-ring unit with a brush-lifting attachment was from IDM Electronics Ltd. This unit, employed to connect the thermocouples in the rotor to the recorder, had negligible influence on the signal quality by virtue of its resistance being lower than 40 milliohms between the slip-ring terminal and the brush block terminal. Temperature measurements of the inner, glass wall of the stator were carried out with the help of a filament resistance thermometer of BNL Electronics, Inc. This ultra-thin nickel-teflon thermometer in flat-grid configuration is known to provide a fast response on electrically insulated surfaces like glass. The filament had a thickness of only 50 μm, and thus was considered to represent the surface temperature of the wall sufficiently accurately. With the help of the Keithley digital display multimeter in conjunction with the present filament, the surface temperature could be assessed to an accuracy better than 0.05 °C.
7.5.2 Torque measurement

Torque measurement was carried out by connecting the strain gauge on the torque arm to a Wheatstone bridge on a Pekel electronic strain indicator. The calibration, however, was carried out by mounting a vertical loading arm at the mid-section of the stator unit. By applying dead weights on a string connected to this loading arm, set to pass over a pulley, and noting the corresponding strain on the indicator the calibration curve was established. At low loads the accuracy to which the shear stress could be evaluated in the film cross-section was better than 5 N/m², but at higher loads the accuracy was lower. Even the latter accuracy was sufficed, being about 0.8 percent of their full scale values.

7.5.3 Heat transfer from the rotor

Heat transfer from the test section was accomplished by means of the rotor being built on the principle of a rotating heat pipe. Details concerning the design and construction may be found in Chapter 6. Cooling of the condenser section was achieved by the circulation of a coolant, namely methanol. The heat transfer from the surface of the heat pipe outside the test section and the condenser section was negligible, this surface being coated with an insulation layer of "plasticoat". Thus the heat transfer measured by the coolant circulating around the condenser section may be considered to be fairly representative of the heat developed by viscous dissipation in the oil film. A flow meter was used to estimate the rate at which the coolant was supplied to the cooling section. This meter, one of Fischer & Porter, and having a flat rotor, was found to give a fairly accurate estimate of the rate of flow. The temperature of the coolant both at the inlet and the outlet was recorded by making use of copper-constantan thermocouples connected to a Fluke digital multimeter.

7.5.4 Other measuring instruments

The accuracy of the measurements performed was assured by carrying out checks in various regimes of operation of the viscometer. For instance, the measurement of temperatures through thermocouples was monitored by independently recording the temperature of the oil sump and the Ultra-Kryostat cooler. The motor speed was read through an AMF/Venner digital counter connected to a proximity transducer. This transducer was triggered by means of a toothed wheel mounted on the main shaft connecting the rotor to the electromotor. The motor speed was found to fluctuate by only 2 rpm provided the operation were in the steady state, and the annular gap between the stator and the rotor were free of any air bubbles. The usual, rather narrow range of operation, the oil flow through the test section was measured using a Fischer & Porter precision bore flow meter, this being calibrated with steel balls either 2 mm or 3 mm diameter. Beyond this range, however, a direct measurement of the oil passage was measured at the return connections to the oil sump.

7.6 Operation of the viscometer

It has already been pointed out that the annular gap employed in the viscometer is 1 mm, and that this rather large gap limits the shear rate that can be obtained without subjecting the rotor to unduly high rotational speeds. For the present investigation a shear rate of about $2.5 \times 10^4$ sec⁻¹ was originally chosen, the constructional details of the viscometer being adapted correspondingly.

However, during the operation of the viscometer it was found that the attainable shear rates amounted to about $1.2 \times 10^4$ sec⁻¹. This shear rate, although somewhat lower than aimed at originally, nevertheless sufficed for reaching the regime of operation in which the thermal effects on the flow pattern could be analysed successfully. As will be discussed for the results to follow, the regime of operation covered could indeed be extended well beyond that at which the non-dimensional shear stress with respect to the rotor surface would theoretically pass a maximum. It is further noteworthy that throughout the complete regime of operation that could here be covered through the mineral oil stocks, such as Shell Vitrea in the absence of polymeric additives, remained Newtonian up to about $10^5$ sec⁻¹ (or shear stresses up to about $10^6$ N/m²), and only at higher shear rates their viscosities tend to become lower due to the non-Newtonian behaviour. Moreover, for the present purpose of analysing the measurements for viscous dissipation effects the operating regime was found to be satisfactorily wide. It may finally be pointed out that the rather large film thickness was chosen with a view of adequately carrying out the laser-Doppler measurement for the velocity distribution across the film. As will be seen in the next section, such measurements were facilitated by film thickness being several times larger than the radial thickness of the measuring volume that could be established by a pair of intersecting laser beams.

7.6.1 Operating procedure

Operation was carried out only after a few preliminary requirements had been met. These included the supply of air, at the designed pressure of 6 atm., to the main bearings situated at both ends of the rotor, and at much lower pressure to the aerostatic lip seals at either end of the test section as well as the condenser section of the rotor. As a measure of safety the main bearings were constantly kept under observation on an oscilloscope for a possible spurious contact between the rotor and the bearing shell. The oil to the middle of the test section was supplied mostly in the experiments at the low pressure of about 0.1 atm., and to the sealing region at the end of the test section at a pressure not exceeding the same 0.1 level. The coolant was supplied from the Ultra-Kryostat cooler, at a temperature of about 15 °C and at a low rate of flow. After having made sure that the test section was completely filled with the test oil, and all air bubbles present in this section were vented out through a valve-controlled escape passage, the initial temperatures of all the thermocouples were recorded. The strain indicator for the measurement of torque was connected to an HP x-y recorder driven by a time base unit.
Then the electromotor was switched on and the rotor was first rotated at low speeds corresponding to a surface speed less than 1 m/s. The viscous dissipation then being small enough not to noticeably influence the isothermal flow. The generation of heat, although still being small, yet could be observed by the change of temperature indicated by the thermocouples embedded in the wall of the rotor as well as by the resistance thermometer fixed on the inside wall of the glass stator. The steady-state condition, especially that of the temperature on the stator surface, was found to have established itself after a period as long as about 20 minutes. This may be attributed to the thermal inertia of the heat pipe which is known to prevail at low heat loads.

The speed was increased at small steps of about 1 m/s and at each speed a continuous surveillance was maintained to make sure that the test section kept completely filled with the oil, and that no air bubbles were present. To achieve this the supply oil pressure in some of the experiments had temporarily to be increased to higher levels. At surface speeds exceeding about 3 m/s the supply oil pressure at the middle and at the end of the test section had even to be increased beyond 0.25 atm. In order to maintain the test section free of air bubbles. The supply oil pressure at the aerostatic lip seals, however, had to be maintained at a somewhat lower value so as to safeguard their operation, avoiding contact between them and the rotor. Operation of the seals could be observed to be safe by the presence of air in the exit oil from the test section.

A difference in the supply pressure of the oil at the middle and the end of the test section adjoining the seals would lead to convection effects, thus violating the preset theory which accounts only for the cross-conduction of the heat. But an estimate by measurement of the amount of oil exit, revealed the convective heat transfer to be negligibly small, at least in the initial, transient, operating regime: at higher speeds this convective heat transfer proved considerable. The convection effect could, however, be avoided by subjecting the test section to the same uniform pressure throughout, i.e. by supplying the test oil both in the middle and at the ends of the test section at one and the same pressure.

7.6.2 Cooling of the rotor

The rotor constructed on the principle of a rotating heat pipe was found highly suitable for maintaining the boundary of the rotor at the desired uniformly distributed temperature of 20 °C. Although the temperature of the rotor surface along the test section was not entirely uniform, its mean value could well be maintained at the above-mentioned level. With a change in the rotor speed, however, the amount of cooling of the condenser section of the rotor had to be adjusted to the concurrent change in the heat load in the test section. This adjustment was performed either by increasing the amount of circulation of the coolant or by setting the cooler to lower temperatures. At the highest heat loads encountered during operation the circulation of the coolant, methanol, had to be increased to as high as 0.05 m³/sec and the temperature had to be lowered to about 30 °C. At these heat loads the rotor surface temperature in the test section could barely be maintained at the above desired value of 20 °C. This has meanwhile been attributed to the performance of the heat pipe not coming up to such heat loads, these proving considerably higher than those for which the heat pipe was originally designed.

7.6.3 Steady-state operation

After the rotor was set to run at a chosen speed the steady-state condition was found to establish itself at shorter times of about 10 seconds of low speeds and with less viscous oils such as Vitrea 100 (oil 1A). However, at higher speeds and with more viscous oils, especially for Vitrea 460 (oil 3A), the time required by the flow to establish a thermally and kinematically steady state was more than one minute. The tests were conducted by stopping the experiment in question after this had reached a steady state at the newly set speed, and after the corresponding settings had been made on the x-y recorder. Then sufficient time of several minutes was allowed for the oil to again reach its new uniform temperature in the test section. Following this the rotor was suddenly brought to the previously set speed by employing a short acceleration time on the frequency regulator feeding the electromotor. Simultaneously the torque recorder was switched on, and the time required for the flow to reach a steady state was recorded. Repeating this procedure after allowing different rest times, the minimum time required to obtain the maximum initial torque could be evaluated. This maximum torque indicated the end of the kinematic process of the flow consisting of a negligible amount of temperature increase in the film. The thermal state would take a long time to become steady, whilst compared to this time the acceleration time would be quite short. This process of kinematic and thermal development can be noted from Figure 7.5 where for convenience a steady-state recording of the measured torque has been in-
7.7 RESULTS AND DISCUSSION

Specifications of various tested oils, i.e. Vitrea 100, 320 and 460, the latter two oils being slightly diluted with Vitrea 100, have already been presented in Section 7.2. These tested oils have been coded by 1A, 2A and 3A for convenience.

The basic data have been re-expressed in terms of the operational parameters discussed in chapters 3 and 5. For convenience the thermal sliding parameter \( N \) both as referred to the rotor surface and to the stator surface has been presented. These two values of the parameter, however, incorporate the same sliding speed, i.e. the difference in velocity of the rotor from the zero one of the stator. It may be noticed that the parameter \( N_2 \) as referred to the viscosity at the rotor surface temperature and \( N_1 \) as referred to that at the stator surface temperature also incorporate properties of the oil other than viscosity, viz.:

\[
N_{1,2} = \frac{v_{1,2}}{k} ; \quad N_1 = (1+\beta) N_{1,2}
\]

Here the viscosity \( v_{1,2} \) refers to the local surface temperature of either the stator or the rotor. The thermal conductivity \( k \), and the viscosity-temperature coefficient \( \beta \) have, however, been employed as constant and only insignificantly influenced by the temperature.

The non-dimensional shear stress \( \Gamma_{1,2} \), notwithstanding it may be considered uniform across the film, has been separately expressed with respect to either the rotor or the stator. In doing this, shear stress deduced from the measured torque at various steady-state operating conditions has been employed. Further, the above-mentioned values of \( v_{1,2} \), \( k \) and \( \beta \) along with the nominal film thickness of 1 mm have been used to evaluate the values of \( \Gamma_{1,2} \) under these conditions. The results have been expressed in a graphical form in Figures 7.6, 7.7 and 7.8 for the non-dimensional shear stress, viz.

\[
\Gamma_{1,2} = \frac{\tau}{v_{1,2}}
\]

where subscripts 1 and 2 again refer to the stator and the rotor surface respectively.

7.7.1 Shear stress in the regime of low sliding speed

From Figures 7.6, 7.7 and 7.8 it may be noticed that the measured shear stress values indicated through \( \Gamma_{1,2} \) agree reasonably well with the theoretically expected ones using two particular types of thermal boundary conditions: \( r_2=1 \) and its converse \( r_1=0 \). The agreement is good in the operating regime covered by low values of the thermal sliding parameter. However this agreement breaks down at high sliding; the cause for such a breakdown will be discussed in the section to follow.

Further, one may expect a reduction of shear stress, at least of its average around the circumference, in the case of an eccentric position of the rotor with respect to the outer stator. Such a reduction is well known in the analysis of journal bearings, these operating invariably with a certain amount of eccentricity under various loading conditions. Detailed analyses of the resulting torque for various degrees of eccentricity can be found in textbooks on hydrodynamic bearings. One of the approximate analyses provided by Pinkus and Sternlicht [7.15] indicates that for any given operating conditions the effect of the eccentricity ratio \( \varepsilon \) is to reduce the torque by the ratio:

\[
\frac{2(1-e^2)}{(2+e^2)^2}
\]

But, for an eccentricity of only 10 \( \mu m \), as it prevails as a maximum in the present viscometer, the reduction of the torque which might be expected is negligible. This reduction can be found to be a mere 0.01 per cent which increases to about 0.06 per cent for the worst case of eccentricity, i.e. when accounting also for the radial clearance of 15 \( \mu m \) of the air bearings supporting the rotor. It may thus be concluded that a correction on the measured shear stress due to the presence of eccentricity can be ignored from the analysis.

The agreement between the theoretically expected and the experimentally measured results for all the oils tested assures a satisfactory accuracy of the measurement: somewhat lower experimental shear stresses at low \( N \) values for oil 1A look as though attributable to the eccentricity of the rotor with respect to the stator. But this cannot be
true: the eccentricity has been found to be far too small to influence the measured results to any considerable extent. On the contrary, the results obtained for oils 1A and 2A do not display a lower experimental shear stress, at least not up to the operating conditions corresponding to the maximum shear stress \( T_2 \) with respect to surface 2, which might be attributed to this eccentricity. Thus the observed aforementioned deviation can be prescribed to the lack of accuracy with which the experimental measurements could be carried out. Further, the barely predictable operation of the air operated lip seals in terms of a parasitic torque may well have played a part too.

At somewhat higher values of the sliding parameter \( N_s \) due to increased sliding speeds the experimental points in Figures 7.6, 7.7 and 7.8 deviate from the theoretical curve No. 11 for the heat partitioning ratio \( r_2 = 1.0 \). Since no observable detachment of the test oil from the rotor surface could be recorded, it is believed that the increased heat loads due to increased visous dissipation at the corresponding increased rotational speeds have led to: (i) an escape of heat at the stator surface due to its insufficient insulation (it is hard to obtain a perfectly adiabatic surface); (ii) overloading, especially at very high

\[
N_{1,2} = \left( 1 + \frac{1}{m} \right) \frac{\beta_1 u_1^2}{c}, \quad N_{2,1} = \beta_2 u_2^2 c; \quad T_{1,2} = \frac{c}{u_1} \frac{\beta_2}{\beta_1}.
\]

Figure 7.7 Measured non-dimensional shear stress \( T_2 \) with the sliding parameter \( N_s \) (also \( N_2 \)). Oil tested: oil 2a (Vitrea 320). Solid lines indicate theoretically evaluated results for \( m=4.65 \).

\[
N_{1,2} = \left( 1 + \frac{1}{m} \right) \frac{\beta_1 u_1^2}{c}, \quad N_{2,1} = \beta_2 u_2^2 c; \quad T_{1,2} = \frac{c}{u_1} \frac{\beta_2}{\beta_1}.
\]

Figure 7.8 Measured non-dimensional shear stress \( T_{2,1} \) with the sliding parameter \( N_{2,1} \) (also \( N_{1,2} \)). Oil tested: oil 3a (Vitria 450). Solid lines indicate theoretically evaluated results for \( m=4.69 \).

heat loads, of the heat pipe beyond the operational range to which it was originally designed.

7.7.2 Shear stress in the regime of high sliding speed

Up to sliding speeds conforming with the \( N_s \)-parameter where at the heat partitioning ratio \( r_2 = 1.0 \) the actual non-dimensional shear stress \( T_2 \) theoretically reaches its maximum as depicted by curve No. 11, the experimental values follow fairly closely. Accordingly, the same is true of the actual non-dimensional shear stress \( T_2 \) with respect to the theoretical curve No. 1 for \( r_2 = 0 \).

However, beyond this regime of the maximum shear stress, the experimental results show a considerable deviation from the theoretically expected trend. The magnitude of \( T_2 \) with respect to the rotor surface continues to increase with increased sliding speed parameter \( N_s \). When values of the heat partitioning ratios \( r_1 \) and \( r_2 \) are assigned to the ex-
Experimental points by interpolation between the curves in Figures 7.6, 7.7 and 7.8, it is seen that the present trend corresponds with a continual decrease of the actual partitioning ratio \( r_2 \) of the rotor from its expected value of zero towards that, 0.5 of the equipartitioning. Conversely, the present trend corresponds with a continual increase of the actual partitioning ratio \( r_1 \) of the stator from its expected value of unity towards that, 0.5 of the equipartitioning. In this operating regime fully flooded flows in the test section had to be ensured by supplying the oil at a somewhat increased pressure; the axial flow thus induced a certain amount of heat transfer by convection.

This convective heat transfer has, however, been maintained at a low level by supplying the oil at the end of the test section, i.e. to the space between the test section and the air-operated lip seals, at a pressure nearly equal to that of the oil supplied to the middle of the test section. In certain experiments where the pressure of supply oil at the middle and at the ends of the test section were considerably different, the measured shear stress values were corrected as follows: The heat transfer due to convection of the oil leaving the end of the test section was measured. It was then assumed that this heat transfer, if absent, would increase the heat content of the oil enclosed in the test section, thereby reducing the shear stress by the corresponding amount. The measured shear stress values, thus, were reduced by the above-mentioned amount.

It was observed that, if only qualitatively, the trend of the shear stresses was not affected by the convective heat transfer of the axial flow. This trend must therefore be attributed to other shortcomings of the Couette viscometer when it is operated at high sliding speeds and high heat loads, namely at sliding speeds exceeding those where the shear stress parameter displays its maximum for \( r_2 = 1.0 \).

The only explanation remaining for the discrepancy between the theoretical and the measured shear stresses would appear to lie in failure of achieving the set goal of an idealisation partitioning of heat of \( r_2 = 1.0 \), i.e. of the rotor absorbing all the heat developed by viscous dissipation. The actual values of the heat partition ratio under the various operating conditions can be evaluated by interpolation of these values through the position of the corresponding points for the measured shear stresses with respect to the adjacent theoretical curves in the figures concerned. Now, these interpolated values can be verified using the measured viscometer wall temperatures. Such an evaluation is already mentioned in Section 5.2.2. of Chapter 5, and may be repeated in its original implicit form,

\[
(\frac{\tau}{\mu})^{1+\frac{1}{m}} - 1 = (r_2 - 0.5)(1 + \frac{1}{m}) N_1 \nu_1
\]

This expression can readily be converted to the explicit form that is here required, viz. the heat partition ratio with respect to the rotor surface:

\[
r_2 = \left[ (\frac{\tau}{\mu})^{1+\frac{1}{m}} - 1 \right] / [(1 + \frac{1}{m}) N_1 \nu_1] + 0.5
\]

Conversely, the heat partition ratio with respect to the stator surface can be given by:

\[
r_1 = \left[ (\frac{\tau}{\mu})^{1+\frac{1}{m}} - 1 \right] / [(1 + \frac{1}{m}) N_1 \nu_1] + 0.5
\]

The actual values of the heat partition ratio can thus be evaluated for each of these oils for every one of its pairs of temperature data. Such an evaluation using the above expression indicates a trend of this heat partition ratio \( r_2 \) over a wide operating range covered by the sliding parameter \( 0.1 < N_2 < 80 \). The heat partition ratio \( r_1 \) thus estimated for the various oils tested has been depicted in Figure 7.9: also the corresponding \( r_2 \) values have separately been estimated. In Figure 7.9 also the sum \( r_1 + r_2 \) of the estimated heat partition ratios has been depicted for every pair of temperature data from the viscometer. It may be noticed from the trend of the sum-values \( r_1 + r_2 \) that at low sliding speeds given by low \( N_2 \) the sum-values are higher by about 5-10 per cent than the theoretically expected value of unity; at higher values of \( N_2 \), the sum-values exceed by only 5-10 per cent. The above-mentioned deviation of the sum-values may be attributed to the experimental inaccuracy involved, especially in the measurement of the wall temperature of the stator under the influence of the inevitably small amount of convective heat transfer due to the passage of the test oil.

For oil 1A at experimental shear stresses (see Figure 7.6) indicate that \( T_c \) values at low sliding speeds, i.e. at equally small values of \( N_2 \), approach their maximum values. Only at very high values of \( N_2 \) do...
they display a decreasing trend. This trend of $T_2$ can be observed by comparing with the theoretically obtained locus of the maximum $T_2$ in the range of $1.0>r>0.5$. While the experimentally obtained $T_2$-values fall to the left of the locus at low $N_r$-values, they fall to the right at very high $N_r$-values.

In contrast to oil 1A the experimental shear stresses for oils 2A and 3A (see Figures 7.7 and 7.8) indicate that the actual $T_2$-values at high values of $N_r$ fall to the right of the theoretical locus of the $T_2$-maxima. Thus this trend depicts, like those obtained for oil 1A, a decreased $T_2$ for increased $N_r$-values whilst due consideration is given to the realistic $r_1$-value prevalent at each experimentally obtained result. It may then be concluded that the decreasing trend of $T_2$ beyond its maximum with increased thermal sliding parameter $N_r$ has been experimentally realised in accordance with that theoretically expected in Chapters 3 and 5.

Higher $N_r$-values than the present ones could not be achieved on the Couette viscometer as provided with the present heat pipe; rotor speeds above about 2500 rpm resulted only in a slight increase of $N_r$ for the reasons to follow. The rotor surface could not be cooled sufficiently to maintain it at around 20 °C. This was due to the limited cooling capacity of the Ultra-Kryostat cooler, as well as to inefficiency of the heat pipe in this regime of operation. In fact the cooling capacity then required, proved to exceed 350 watt for which the heat pipe was originally designed. At the highest speeds obtained the mean surface temperature of the rotor in the test section could not be maintained at values lower than about 23 °C. The result was a considerable decrease in the oil viscosity thereby more or less offsetting the effect of an increase in speed so far as the value of the thermal sliding parameter is concerned. The higher rotor speed thus resulted only in a small net increase in $N_r$. Correspondingly there was a decrease in the heat partition ratio $r_2$ and a concurrent increase in $T_2$. A gradual decrease of $r_2$ with $N_r$ can be observed through the increasing trend of $r_2$, displayed in Figure 7.9 (note that $r_2$ and $r_1$ can be interchanged wherever necessary, when using their interdependence $r_2=1-r_1$ to the experimental accuracy of the sum-values ($r_1+r_2$) depicted in the same figure 7.9).

### 7.7.3 Establishment of kinematic and thermal processes

The establishment from the initially transient thermal states towards thermally steady-state operation of the flow in the viscometer can be visualized through observing the temperature data obtained from the measurements. This procedure has already been discussed in Section 7.6.3. There it has been pointed out that the steady-state condition could be reached only over periods exceeding one minute. Such periods proved shorter when the heat generation due to viscous dissipation was small. Such a difference in these periods are evident when comparing Figure 7.10a for $N_r=660$ rpm, with $N_r$ at which the heat partition ratio $r_2$, can be observed through the increasing trend of $r_2$ displayed in Figure 7.9 (note that $r_2$ and $r_1$ can be interchanged wherever necessary, when using their interdependence $r_2=1-r_1$ to the experimental accuracy of the sum-values ($r_1+r_2$) depicted in the same figure 7.9).

Figure 7.10 Thermal development in the film towards a steady state. Oil 3A and preset rotor speeds: (a) 660 rpm (b) 937 rpm (c) 1436 rpm. Note that the ordinate corresponds to the shear stress given by 9.926 N/m per $\mu$-strain.

depicted in Figure 7.10b. When increasing the sliding speed the peak gets higher, as is evident by comparing Figure 7.10c for 1436 rpm and
Figure 7.3 for 2217 rpm. As already discussed in Section 7.6.3 these peaks indicate the establishment of the kinematic process to take place under the initially nearly isothermal conditions, at least whenever the aforementioned periods are short. In all the experiments performed so far the peaks are reached in a relatively short period of time, the acceleration time employed on the frequency regulator controlling the laser power being short, however. In the more generally applicable technique the Doppler frequency shift is obtained through the scattered light of incident laser beam. This system derives its name from the procedure of handling the signal frequency, namely in that the scattered light is mixed with that of the reference beam. A photodetector is here required to respond only to the difference in the frequency. A more popular one of the kinds of operations is the dual beam system in which two laser beams of equal intensity are made to cross in the fluid where the velocity is to be measured. These beams interfere with each other and form a fringe pattern in a definite measuring volume. A single particle passing through this measuring volume will produce Doppler cycles that are directly related to the number of fringes. A photodetector is here required to respond to the Doppler frequency.

All standard laser measurement techniques have an 180° deg. direction ambiguity: they cannot distinguish whether a flow is in a particular direction or exactly in the opposite direction. Frequency shifting has been employed to overcome this ambiguity. One obtains from this frequency shift technique a 'zero offset', in which a stationary particle scatters light at the same frequency as that of the original coherent light. This zero offset in reference beam mode gives rise to a zero difference frequency, and in dual beam mode means no Doppler cycles at all. The signal frequency is that the scattered light pattern scatters the laser light at a lower frequency, and in the opposite direction at a higher frequency.

Although laser-Doppler techniques have already found a wide application in the measurement of laminar and turbulent flows of less viscous fluids, its usage in the measurement of velocity fields of mineral oils in tribologic applications has so far been rather limited. This may be explained by the severe difficulties involved when such measurements have been performed in the rather thin lubricant films that are typical of the application in hand. Laser-Doppler technique suitable for measuring volumes of at least several micrometers in cross-section, do not yield sufficient resolution for the velocity distribution in such lubricant films. Seeming of the oil and suitable optical arrangement as well as of the signal processing fall far from the standards so far reached in the measurements with other fluids such as water and air.

Herebelow it will be shown how to overcome the above-mentioned difficulties for film thicknesses down to about 1 mm and for velocities of flow as high as about 8.5 m/s.

7.8 MEASUREMENT OF DISTRIBUTIONS OF FLUID VELOCITY USING THE LASER-DOPPLER METHOD

7.8.1 Introduction

Velocity measurements in fluid flow can nowadays be carried out more efficiently than ever before. The success is to be attributed to the laser-Doppler technique which has in recent years replaced pitot tubes and hot-wire anemometers. This technique has been received warmly by scientists and engineers alike because of its intrinsic advantage that the velocity can be measured 'at a point' without disturbing the flow. The technique is not necessarily confined to a one-dimensional measurement; it can be used to measure even a two-dimensional velocity field as well.

The technique is based on the light-scattering property of moving small contaminant particles present in the fluid flow. These particles produce a Doppler frequency shift of the scattered light from a high coherent light of a laser. Since the difference between the velocity of the coherent laser light and that of the moving particles is very large, Doppler shift is generally small. Moreover, the actual frequency of the coherent light as well as that of the scattered light is far too high for a direct measurement on a photodetector and the commonly used electronization.

7.7.4 Significance of the data towards the plane Couette flow

A matter of concern in interpreting the results discussed so far is inherent in the nature of present kind of Couette flow, i.e., with what accuracy the cylindrical configuration conforms with the theoretical analysis of the plane Couette flow. This matter has already been treated in detail in Chapter 3 although only for the exponential viscosity-temperature relationship. But an analysis based on Slotte's more generally applicable relationship is rather involved. Therefore in what follows the results of the former relationship will be used to discuss the experimental results in at least a qualitative manner. Further, we shall have to account for the fact that the actual flow pattern in the plane Couette flow is of the cylindrical kind where the flow pattern in the theory has been assumed to be plane. However, at least in the regime of the maximum shear stress and beyond, the difference between these two kinds of flows is negligible, being within the accuracy of the measurements. Thus a theoretical interpretation of the measured data on the basis of the assumption of a plane Couette flow is considered justifiable.

7.8 MEASUREMENT OF DISTRIBUTIONS OF FLUID VELOCITY USING THE LASER-DOPPLER METHOD

7.8.2 Present laser-Doppler system

A frequency shift called Doppler signal is generated by a particle dispersed in a fluid flow. This signal is obtained by means of a laser beam as a light source which is treated and aligned by a suitable optical arrangement. The scattered light containing the signal is collected at a photomultiplier and processed in signal processors. Principles of scattering of light by the particles and formation of fringes are well known in optical theory. Optimisation of the final optics based on
these principles has been carried out separately (see De Barbanson [7.16]). Only the results have been presented due to lack of space (see Appendix 7.A1).

The system here selected is a dual-beam mode in back-scatter configuration (see Figure 7.11). This system seemed to be the most appropriate for the measurement of the rather high velocities of flow and the use of mineral oils. The dual beam system is easy to use, and it is known to give the best signal-to-noise ratio. In this system two laser beams of equal intensity are made to cross at the position where the velocity of flow is to be measured. If the two beams are in phase they give rise to a maximum intensity, and when they are opposing a minimum one. Two-dimensional wave fronts of the light of the two beams thus produce alternative light and dark fringes. The fringe distance \( d_f \) depends on the wavelength of the laser light \( \lambda \) and the angle between the laser beams \( \phi \), and is given by,

\[
d_f = \frac{\pi}{\lambda} \sin \phi
\]

Figure 7.11 General arrangement of back-scatter in dual-beam configuration, and definition of the optical parameters.

There exists an ellipsoidal measuring volume (see Figure 7.12) at the intersection of the two laser beams, its dimension depending on \( \lambda \), the focal length \( f_3 \) of the lens and the diameter of the beam leaving the laser \( D_1 \). \( f_3/D_1 \) corresponds with all points of intensity greater than \( 1/e^2 = 0.135 \) of the centre-line intensity. The cross-sectional diameter of the beam at the focal point is given by (Durrani and Greated [7.17]),

\[
d_f = \left( \frac{6}{\pi^2} \right)^{1/2} f_3/D_1
\]

The dimensions of the ellipsoidal measuring volume can then be estimated since the refractive index of the oil and that of the glass under consideration do not differ considerably. These dimensions shown in Figure 7.12 are then:

\[
l_m = d_f \sin \phi, \quad d_m = \frac{d_f}{\cos \phi}
\]

It may be noticed from these expressions for \( l_m \) and \( d_m \) that minimum measuring volume size is realizable whenever \( d_f \) is chosen as small as possible. Now, small \( d_f \)-values may be achieved by employing a short focal length \( f_3 \), or for any given laser, a large original beam diameter (see the second last formula). In the present investigation very small measuring volumes have in fact been achieved through such an adequate choice of both \( f_3 \) and \( D_1/e_2 \), the dimensions of the measuring volume have been supplied in Appendix 7.A1.

7.8.2.1 Pre-shift frequency and optical arrangement

Frequency shift of the two first-order beams for the dual-beam system is realized by means of a high-efficiency rotating grating, specified in Appendix 7.A1. This grating offers a high intensity of about 34\% for each of the two first order beams against that of only 3\% from a low efficiency grating with respect to the original beam intensity.

The rotating grating consisted of a small transparent disc mounted on the shaft of a brushless DC electric motor. The disc is etched with several tracks of radial lines of a uniform width and depth, and uniform angular spacing. The grating functions both as a beam splitter and as an optical frequency shifter giving rise to a pre-shift frequency between the two beams emerging from the grating concerned.

The frequency shift in the first-order diffracted beam is given by \( f_p = NF_p \), where \( N \) is the total number of lines of the grating and \( f_p \) is the rotational frequency of the disc. The difference in frequencies of a pair of dual beams thus is \( 2f_p \). The diffraction angle can be obtained from the wave length \( \lambda \) and the line-spacing width \( d_r \) of the grating, i.e.

\[
\sin \phi = \frac{\lambda}{d_r}
\]

The optical arrangement has been sketched in Figure 7.11 and the major features have been specified in Appendix 7.A1. In order to obtain a very small measuring volume an object lens of focal length of 55 mm has been selected.
7.8.2.2 Selection of laser

Laser are sources of well-nigh coherent light and offer a monochromatic radiation of wave length depending on the material employed for radiation. For relatively high seeding, by particles held in suspension in the test liquid, and velocities, He-Ne lasers of 5 mW power are adequate, at least in the dual-beam forward scatter method. Doppler signals, however, become poor when these low powered He-Ne lasers are used in back-scatter configuration. For the latter requirement, and when long focal length lenses are used, high powered lasers up to 15 mW or even higher are found suitable. He-Ne lasers, although available with higher power, are costly. Nevertheless, the latter kind has been selected, being most suited to acquire the scattered light on a photomultiplier (for its specification see Appendix 7.A1).

7.8.2.3 Seeding of the fluid

Impurities in the form of solid particles held in suspension in liquids bring about an essentially continuous generation of a Doppler signal, and suffice for low-velocity measurements. For high-velocity measurements, however, the requirements on such particles become more stringent. The quality of the output data depends then strongly on the concentration and type of particles introduced in the flow.

It has been found from literature that the performance of different types and sizes of particles when suspended in water and in air is well documented. This documentation is, however, rather incomplete for seedings particles to be used in the less transparent mineral oils. A systematic investigation, into the effect of both the types and sizes of such particles has therefore to be instituted independently for the present work.

Titanium di-oxide (TiO₂) particles of about 0.2 μm diameter brought into suspension with a thorax mixer finally proved to be satisfactory for the high fluid velocities employed in the present investigation. The velocities to be measured cover a wide range, a careful selection of the minimum cycles per Doppler burst has to be made. A well-proven rule is to select the number of cycles per burst to be sampled less than one-half of the number of cycles across the center of the measuring volume.

7.8.4 Data acquisition and processing of signals

The tracker processor TPD 1077, the one used at low fluid velocities, is the most versatile type, providing a combination of tracking, filter and frequency-to-voltage converter. This instrument measures the time interval of 8 or, alternatively, 16 cycles of the Doppler signal. The laser-Doppler measuring arrangement has been schematically presented in Figure 7.13 and whilst a photograph has been shown in Figure 7.14.

A microcomputer together with a multi-programmer constituted the central instrumentation in carrying out the experiments, retrieving and processing the signal, and also providing an output signal to the printer. In the experiments the multiprogrammer also displaces the object lens L₁ (see also Figure 7.12) thereby displacing the measuring volume in the flowing fluid. This was performed as steps of 25 μm across the 1 mm-thick oil film. At every position an average of 10 successive measured signal values were processed and registered on the printer. The measurements were started from a position on the rotating copper cylinder wall and continued at other positions until the glass wall was encountered, the measured frequency there equating the preshift value.

---


**TSI stands for Thermo-Systems Inc., Minnesota, U.S.A.
The maximum frequency, and thus the signal output, occurred on the moving copper wall, of which the surface speed was set at a certain selected value. Measured frequencies at all the successive positions were corrected by deducting the preshift frequency: this correction was performed by registering the signal output at the stationary glass wall which corresponds to the preshift frequency, and by applying this signal output as a correction to the measured signal value at every other position. The ratios with respect to the maximum outputs were thus assessed. The ensuing velocity distribution is expressed by,

\[ u = \frac{u}{u_0} \]

where \( u \) is the local velocity of the particle, and thus that of the flow, and \( u_0 \) that of the copper wall.

### 7.8.5 Results and discussion

Laser-Doppler velocity measurements were carried out on the Couette viscometer with all the three Shell Vitrea base oils 1B, 2B and 3B. These base oils were seeded with titanium di-oxide (TiO₂), as explained in Section 7.8.2.3. Their physical properties in the presence of TiO₂, such as densities and viscosities, have been specified in Table 7.2.

<table>
<thead>
<tr>
<th>Oil</th>
<th>Temperature °C</th>
<th>Density, kg/m³</th>
<th>Viscosity, cP</th>
<th>Slotte's coefficients ( m ) and ( A ) at 20°C, 32°C and 75°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1B</td>
<td>66.3</td>
<td>877</td>
<td>259.91</td>
<td>3.96, 35.24, 0.0717</td>
</tr>
<tr>
<td></td>
<td>74.0</td>
<td></td>
<td>213.9</td>
<td></td>
</tr>
<tr>
<td>2B</td>
<td>22.1</td>
<td>895</td>
<td>958.5</td>
<td>4.55, 33.15, 0.0826</td>
</tr>
<tr>
<td></td>
<td>44.4</td>
<td></td>
<td>212.9</td>
<td></td>
</tr>
<tr>
<td>3B</td>
<td>22.45</td>
<td>891</td>
<td>1026.86</td>
<td>4.61, 34.92, 0.0839</td>
</tr>
<tr>
<td></td>
<td>66.35</td>
<td></td>
<td>197.70</td>
<td></td>
</tr>
<tr>
<td></td>
<td>74.0</td>
<td></td>
<td>53.32</td>
<td></td>
</tr>
</tbody>
</table>

The measurements have been performed in accordance with Section 7.8.4. A few of the typical results have been displayed in Figures 7.16 through 7.20. It may be noticed that the quality of the results in the greater part of the film cross-section is reasonably good, except near the walls. The poor quality near the walls can be attributed to the ma...
Figure 7.15 Variation of dynamic viscosity with temperature of the test oils 1B, 2B and 3B.

suring volume obtained at the intersection of the two laser beams being comparatively large. The smallest measuring volume that could be obtained with the present optical system gave rise to the rather large radial measuring depth of 93 \( \mu \text{m} \). Since the laser-Doppler technique does not discriminate between the various velocities of the particles within the measuring volume, establishing the definite position of the measuring volume on the wall with the help of the object lens \( L_3 \) (see Figure 7.11) proved to be somewhat inaccurate. Establishing the position of the moving copper wall proved even more difficult than that of the stationary glass wall.

Figure 7.16 depicts a nearly isothermal, that is nearly triangular/pattern of flow. In fact in this test the viscous dissipation was inappreciable, due to low sliding speed of 2.5 m/s involved. The velocity distribution is nearly a straight line and agrees with the theoretically expected trend. At increased speeds, and thus at increased thermal sliding parameters \( N_2 \), the velocity distribution becomes somewhat concave towards the horizontal axis, which represents the film cross-section (see Figures 7.17a and 7.17b). At higher sliding speeds, and correspondingly increased \( N_2 \)-values, the velocity distribution becomes increasingly concave. Nevertheless the concavity is not so predominant for oil 1B as compared to that for oils 2B and 3B (see Figures 7.18 and
The latter oils with their higher values of \( m = 4.55 \) and 4.61, respectively, as against \( m = 3.96 \) for oil 1B, do produce more viscous dissipation. Accordingly, their velocity distributions deviate considerably from the straight line trend of the isothermal flows.

The velocity distributions depicted in Figures 7.17 through 7.19 for \( N_2 \)-values in the range 4.2<\( N_2 \)<6.8 indicate a qualitative trend similar to the theoretical one in Figure 3.7b for oils with an exponential viscosity-temperature relationship, the heat partition ratio \( r_2 \) of the rotor surface being put equal to unity. It may be shown that for the oils following Slotte's relationship with finite \( m \)-values the trend of theoretical velocity distributions would qualitatively remain the same. But it may be expected that the higher the \( m \)-value the greater the concavity of profiles of velocity distributions, thus coming closer to that for the exponential viscosity-temperature relationship, \( m = \infty \).

In contrast, this concavity, however, increases with a decrease in the heat partition ratio \( r_2 \), the one with respect to the rotor surface. In the high-speed regime the heat partition ratio \( r_2 \) has, in fact, already been shown in Section 7.7 to decrease with increased rotor speeds and thus to becoming increasingly lower than unity. Therefore the actual velocity profiles need not give rise to a substantial decrease in their concavity, when rotor speeds attain the present highest level.

Further the concavity at the position of the maximum velocity, that is at the rotor surface, may also be suppressed because of the averaging technique involved in the signal processor. In fact the velocity would be recorded as an average over a radial depth of 93 \( \mu \)m, so that the measured velocity distribution would be slightly different from the actual one.

Only very few laser-Doppler measurements could be carried out at sliding speeds (rotor surface speeds) exceeding 6.8 m/s. Figure 7.20 depicts the velocity distribution at 2016 rpm, the rotor surface speed being 8.44 m/s and the corresponding \( N_2 = 12.70 \) for oil 1B. An increased concavity of the velocity profile may now be observed. The signal quality already has become, however, somewhat poor at this high rotor surface speed. For oils 2B and 3B, the signal could not even be retrieved at this speed and at still higher rotor speeds not even for oil 1B. Thus an investigation of the velocity distribution at \( N_2 \) values higher than the aforementioned value, 12.70, could not be carried out.

7.9 CONCLUSIONS

Searches for non-Newtonian behaviour of liquids have in the past been performed through various methods. One of these is the "flash method" in which the measurements of frictional torque are made as quickly as at all possible after speeding up the rotor to the desired circumferential speed. On the contrary, thermal effects have been measured by allowing sufficient time for a steady state to establish itself. Couette viscom
i.e. at comparatively high sliding speeds and thus at thermal steady-state has been reached.

Shell Vitrea oils which are known to remain Newtonian up to shear stresses as high as about $10^5$ N/m$^2$, prove suitable for a study of thermal effects. Further, they are available in a wide viscosity range, and also cover a reasonably wide range of exponents $m$ in Slotte's viscosity-temperature relationship. In the present work these oils have indeed been found to be suitable for studying thermal effects due to viscous dissipation, including the variation of frictional torque with sliding velocity, and thus with shear rates; further, assessing the deviation of the present, actually non-isothermal velocity profiles from the theoretical, triangular isothermal ones.

The measured frictional torques, and the shear stresses derived therefrom, agree reasonably well with the theoretically predicted values, the latter accounting for viscosity variation with temperature according to Slotte's relationship, and also for realistic thermal boundary conditions at the stator and rotor surfaces. Further, up to the operating regime where the shear stresses theoretically attain a maximum, agreed well with the theoretically predicted values when related to the viscosity at the rotor surface temperature. However, shear stresses beyond that regime, i.e. at comparatively high sliding speeds and thus at the correspondingly high values of the thermal sliding parameter, deviated from the theoretically predicted ones. In the corresponding high-speed regime the heat partition ratios could not be held at their intended values, $r_1=1.0$ and $r_2=0.5$. This lack of control in the present regime could be attributed to heat loads that were unexpectedly higher than these for which the cooling arrangement had been designed for.

The laser-Doppler technique proved successful for registering the velocity distribution of the flow of mineral oils in the present Couette viscometer. With the test oil under study seeded with a very small amount of TiO$_2$ (about 3-6x10$^7$ per cent by volume), the velocity distribution across the film could accurately be registered up to a circumferential speed of the rotor of 8.44 m/s. The signal processors available satisfactorily registered these velocity distributions. The experimentally obtained velocity distributions of Shell Vitrea oils displayed the trend theoretically predicted. With increased viscous dissipation due to increased sliding speeds, the velocity distribution showed considerable deviations from the isothermal, triangular one. But these deviations tallied, at least qualitatively, with theoretical predictions.

Summarizing, the aforementioned results have corroborated the expectation that the non-isothermal flow behaviour of viscous liquids, such as mineral oils, when subjected to high sliding speeds or shear rates may indeed differ considerably from the isothermal flow behaviour, both as to thermal and kinematic effects and including those of the thermal boundary conditions imposed. Further, the present laser-Doppler technique proved successfully applicable to registering velocity distributions of non-isothermal flows as these arise at comparatively high sliding speeds. Future refinements of the present equipment, when aiming at still higher sliding speeds, should be concentrated upon a suitably redesigned rotor in so far as it serves as a rotating heat pipe.

7.10 REFERENCES

8.1 INTRODUCTION

There are indications from literature that there exists certain similarity between the thermohydrodynamic process of steady pressure flows in a plan-parallel film and those in the axisymmetric configuration (Poiseuille) such as in pipes and capillaries as to their thermohydrodynamic behaviour. Analytical solutions can be found for the latter pressure flows, at least with certain viscosity-temperature relationships for the liquids involved. These solutions can be shown to be useful in analyzing the plan-parallel flows for their analogy with the pipe flows. The characteristics of some of these solutions can be extended to the plan-parallel flows for which the analytical solutions cannot easily be derived. Such an extension serves as a basis for simplifying the extensive numerical work that would otherwise be needed.

We have noticed from the general introduction, Chapter I, that the axisymmetric flows of Newtonian fluids have attracted the interest of theoreticians. They have extensively treated these flows in pipes, and have obtained solutions for various thermohydrodynamic quantities. These investigators have started with the equations for momentum and energy, the latter involving heat generation due to viscous dissipation of the liquid involved and heat transfer due solely to cross-conduction. Thus, no convection term has been included in the energy equation employed by the investigators listed below. Although solutions to the pipe flows of fluids following a linear fluidity function for the relationship between the viscosity and temperature were obtained by Philippoff [8.1] in 1942 and critically analysed by Hausenblas [8.2] in 1950, the solutions for a non-linear fluidity function such as the exponential viscosity-temperature relationship were obtained by Kearsley [8.3] only at a much later date. His results indicate the existence of a limiting pressure gradient that cannot be surpassed with the pipe flows, and also that the limitation is mainly due to the thermal process involved. Similar solutions have been obtained also by other investigators, see Boxtandzhiyan et al. [8.4], Nihoul [8.5], Sukanek and Laurence [8.6] and others. Kearsley’s results have been verified by Joseph [8.7] for flow stability under the thermal effects and later more extensively by Platten and Vanderborck [8.8] in 1976.

The axisymmetric flows of a Newtonian liquid with an exponential viscosity-temperature relationship have been shown by Sukanek and Laurence [8.6] to exist for all values of assumed average velocity or rate flow. The results of Nihoul [8.5] indicate an existence of such flows only in a limited range of the pressure gradients involved. We will show in what follows that the solutions for these pipe flows can be expressed in terms of a dimensionless pressure gradient parameter specifying the thermal state of the fluid at the axis of the pipe. In the present analysis we use the exponential viscosity-temperature relationship as well as Slote’s relationship with the exponent m=1. The results to be devel-
oped in this chapter are meant to culminate in a comparison with the plan-parallel pressure flows, such as those in the elastohydrodynamic films, to be treated in the next chapter.

8.2 SOLUTIONS USING THE EXPONENTIAL RELATIONSHIP

8.2.1 Solutions in terms of viscosity at the wall

A steady flow of an incompressible Newtonian fluid with a constant thermal conductivity \( k \) can be expressed in terms of conservation of both linear momentum and energy, the latter involving the heat generation by viscous dissipation and the heat transfer by cross-conduction, viz.

\[
- \frac{1}{r} \frac{d}{dr} \left( ru \frac{du}{dr} \right) = g
\]

(8.1)

with the boundary conditions,

\[
r = 0, \quad \frac{du}{dr} = 0 \quad (8.2a)
\]

and

\[
r = R, \quad u = 0 \quad (8.2b)
\]

whilst

\[
\frac{k}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = -\nu \left( \frac{du}{dr} \right)^2
\]

(8.3)

Figure 8.1  *Coordinate system and boundary conditions for the axisymmetric Poiseuille flow in pipes.*

with the boundary conditions shown in Figure 8.1, viz.,

\[
r = 0, \quad \frac{dT}{dr} = 0 \quad (8.4a)
\]

\[
r = R, \quad T = T_1 \quad (8.4b)
\]

where in addition to the previously defined nomenclature, \( r \) is the radial coordinate from the axis of the pipe of radius \( R \), and \( g \) is the pressure gradient \((-dp/dx)\) in the flow direction \( x \). If the viscosity-temperature relationship of the exponential type is employed where \( \nu_1 \) is the viscosity at the wall temperature \( T_1 \), then

\[
\nu = \nu_1 \exp \left[ -\beta (T - T_1) \right]
\]

where the exponent \( \beta \) is, in general, a constant value for any given liquid.

Introducing the non-dimensional quantities,

\[
Y = \frac{r}{R}
\]

\[
\theta = \beta (T - T_1)
\]

\[
K = \frac{\beta g^2 R^2}{4k} \nu_1
\]

\[
u = \frac{u}{u^*}
\]

where \( u^* \) is a characteristic velocity of isoviscous flow under the pressure gradient \( g \) concerned, and defined by,

\[
u^* = \frac{g R^2}{2P}
\]

Equation (8.1) when integrating once, and accounting for condition (8.2a),

\[
\frac{d}{dY} \left( Y \frac{du}{dY} \right) + K Y^2 e^\theta = 0
\]

(8.5)

The boundary conditions corresponding to this equation reduce to,

\[
Y = 0, \quad \frac{du}{dY} = 0 \quad (8.7a)
\]

and

\[
Y = 1, \quad \theta = 0 \quad (8.7b)
\]

A further simplification of equation (8.7) gives rise to,

\[
\frac{d^2 e}{dY^2} + \frac{25}{4} Y^2 e^\theta = 0
\]

(8.8)

An analytical solution to the differential equation (8.8) for the temperature distribution in the cross-section of the pipe has been worked out by several investigators through various methods. A convenient method of solution originally worked out by Chambré [8.9] can be shown to be applicable to this equation. The solution can thus be obtained for the dimensionless temperature \( \theta \) as shown in Appendix 8.A1, namely,

\[
\theta = 2 \ln \left[ \frac{Y + K}{\gamma Y + K} \right]
\]

(8.9)

where \( \gamma \) is double-valued in terms of \( K \) and is given by,

\[
\gamma = 16 - K \pm 4 \sqrt{16 - 2K}
\]

(8.10)

The distribution of viscosity and velocity, and the rate of flow can be worked out using the above solution. These can also independently be obtained in terms of a pressure gradient parameter involving the viscosity of the liquid at the axis of the pipe. Such an analysis will be
carried out in the following section; it is meant to show the validity of the solution on the one hand, and the single-valuedness of the same solution on the other. The analysis is also meant to display its versatility towards obtaining the solution for plan-parallel pressure flows yet to be treated in the next chapter.

8.2.2 Solutions in terms of viscosity at the pipe axis

A steady flow of an incompressible Newtonian liquid with a constant thermal conductivity $k$ expressed in terms of conservation of both linear momentum and energy by equations (8.1) and (8.3) will here be analysed again. The pertinent set of boundary conditions for the analysis to follow consist of those from equations (8.2a) and (8.2b) as well as of "one point" type given by,

$$r = 0, \ T = T_H, \ \frac{d\theta}{dr} = 0$$  \hspace{1cm} (8.11)

Let us introduce the non-dimensional groups,

$$\theta = \frac{T - T_H}{\delta}, \ \frac{r}{R}, \ u^* = \frac{u}{u_H},$$

where $u^*$ is a characteristic velocity defined with respect to the viscosity of the fluid at the pipe axis,

$$u_H = \sqrt{\frac{k}{\rho}},$$

the subscript $H$ referring to the position of maximum temperature occurring at the pipe axis. For axisymmetric distributions of temperature and velocity such as those in pipe flows, at this position the velocity reaches a maximum. There the local shear stress and the local heat flux are zero.

Further on we shall be using a parameter $B$ that represents the thermal state of the fluid at the pipe axis and includes also the pressure gradient and the material properties. This parameter is similar to $K$ used in Section 8.2.1, and is given by,

$$B = \frac{8 k^{1/3}}{R \mu_H^2},$$

where $\mu_H$ has been defined in accordance with the previous assumption that it is independent of the temperature. The above mentioned governing equations then reduce to the following forms. The momentum equation becomes,

$$\frac{d\theta}{dr} = -\frac{\theta}{Y}$$  \hspace{1cm} (8.12)

whilst substitution in the energy balance equation yields,

$$\frac{d}{dr} \left( \frac{\theta^2}{Y} \right) + 12 B Y^2 \theta = 0$$  \hspace{1cm} (8.13a)

The latter equation can further be reduced to the form,

$$\frac{d^2\theta}{dr^2} + \frac{1}{Y} \frac{d\theta}{dr} + 6 B Y^2 \theta = 0$$  \hspace{1cm} (8.13b)

with the boundary conditions from equation (8.11) reducing to,

$$Y = 0, \ \theta = 0, \ \frac{d\theta}{dr} = 0$$  \hspace{1cm} (8.14)

Equation (8.13a) is identical with equation (8.6) treated before except for $\delta$ and $K$ which are now replaced by $\theta$ and $B$. The solution for the non-dimensional temperature $\theta$ can be obtained from equation (8.13b) on lines similar to those explained in Appendix 8.A1. Since $\theta = 0$ at the pipe axis,

$$\theta = -2 \ln (1 + BY^2)$$  \hspace{1cm} (8.15)

The physical meaning of the parameter $B$ will be explained in Section 8.2.4 yet to follow. The temperature and viscosity at the pipe wall can further be expressed by,

$$\theta = -2 \ln (1 + B)$$  \hspace{1cm} (8.16)

and

$$\frac{u^*}{u_H} = (1 + B)^2$$  \hspace{1cm} (8.17)

8.2.3 Other thermo-hydrodynamic parameters

Having established a basic solution, i.e. for the dimensionless temperature $\theta$, various thermo-hydrodynamic quantities such as the velocity distribution, the rate of flow, etc., can now be worked out in terms of the pressure gradient parameter involving the viscosity of the liquid at the channel axis. First of all, the viscosity distribution can readily be reduced using the assumed exponential relationship between the viscosity and temperature. Then,

$$\mu^* = (1 + B Y^2)^2$$  \hspace{1cm} (8.18)

Further, one may obtain the following thermo-hydrodynamic quantities which serve to be useful while discussing the physical significance of the solution covering a wide operating regime. Velocity distribution:

$$u = \frac{R^2}{2 u^*} \left[ \frac{1}{1 + B} - \frac{Y^2}{1 + B Y^2} \right]$$

or, in a non-dimensional form:

$$\frac{u}{u^*} = (1 + B Y)^2$$  \hspace{1cm} (8.19)

where $u^*$ is the isoviscous value, viz.,

$$\frac{u^*}{u^*_H} = \frac{8 k^{1/3}}{\mu_H^2} (1 - Y^2)$$

Volumetric flow rate:

$$q = \frac{8 \mu_H^2}{4 \pi} \int_0^R 2 \pi r \ \mu \ dr$$

or, in a non-dimensional form,

$$q/q^*_H = (1 + B)$$  \hspace{1cm} (8.20a)
where \( q_{\text{is}} \) is the isoviscous value, viz.,

\[
q_{\text{is}} = \frac{g R^2}{8 \mu_1}
\]

Average velocity:

\[
u_{av} = \frac{q}{\pi R^2} = \frac{g R^2}{8 \mu_1} (1 + B)
\]

where \( q R^2/8 \mu_1 \) is the average velocity \( u_{av, \text{is}} \) for the isoviscous case, so that

\[
\frac{u_{av}}{u_{av, \text{is}}} = (1 + B)
\]

(8.20b)

Velocity at the pipe axis:

\[
u_H = \frac{g R^2}{8 \mu_1} \left[ \frac{1}{1 + B} \frac{1}{\sqrt{1 + B}} \arctan \sqrt{B} \right] u_{av}
\]

(8.21)

8.2.4 Relationship among various input parameters

Referring to Section 8.2.1 for the solutions to the steady flow in terms of \( K \) involving the viscosity \( \mu_1 \) at the wall temperature of the pipe, namely

\[
K = \frac{g R^2}{8 \mu_1}
\]

we observe that the temperature at the pipe axis may be expressed using expression (8.9) in the form,

\[
\gamma_H = 2 \ln \left( \frac{\gamma}{K + 1} \right)
\]

where \( \gamma = \sqrt{1 + 2K} \) and \( x = 1 + B \)

The viscosity at the pipe axis can be expressed using the parameter \( K \) by,

\[
u_H/\mu_1 = (1 + \gamma)/2
\]

Comparing this with expression (8.17) it may be noticed that the parameters \( \gamma/K \) and \( B \) are identically equal to each other.

Comparing the two parameters, namely \( K \) with \( B \) in which the axial temperature has been taken for reference, it may be observed that these two are related through the viscosity ratio mentioned before, namely in the form,

\[
B = (\mu_2/\mu_1) K/32
\]

(8.22)

One may further on show that the average velocity given by equation (8.20b) can be cast in such a form that it displays a characteristic behavior identical with that of \( B \). With the help of the average velocity

under, isoviscous condition, viz.,

\[
u_{av, \text{is}} = \frac{g R^2}{8 \mu_1}
\]

the average velocity under non-isothermal conditions can be written by rearranging equation (8.20b), in the form,

\[
V = \left( \frac{1 + B}{B} \right) u_{av} = \frac{\gamma H}{4} \left( 1 + B \right) = V^2 B
\]

(8.23)

It may be noticed that the above parameter \( V \) has been defined independent of the pressure gradient, but involving the average velocity \( u_{av} \) influenced by the thermal effects, and thus the thermal state of the film cross-section. On the contrary, the parameters \( K \) and \( B \) involve only the viscosity of the fluid at the wall temperature and at the pipe axis where the temperature is maximum, respectively.

Some of the investigators have employed in their analyses a parameter involving the Brinkman number such as that shown in Appendix 8.A2, i.e. \( \gamma H B \). It has been shown in that appendix that such a parameter can be shown equivalent to \( 2B \). It may now be observed, by comparing this parameter involving the Brinkman number with \( V \) given by equation (8.23), that the former parameter is simply the square of \( V \).

One may then proceed to analyse the solutions, as shown in the following section, in terms of any of the above parameters, namely \( K \) or \( B \); a corresponding operation of the flow can be achieved by controlling either the pressure gradient which appears both in \( K \) and \( B \), or the rate of flow to which \( V \) is linearly related.

8.2.5 Uniqueness of solutions

Results obtained for the flow with respect to the isoviscous value, i.e. \( q/q_{\text{is}} \), have been expressed in terms of \( K \) in Figure 8.2a. The ratio \( q/q_{\text{is}} \) can also be termed as a "correction factor" to the isothermal rate of flow. This factor provides a means of correcting the rate of flow, and thus the viscosity measurements such as in capillary viscometers. The flow increases with an increased input parameter \( K \) along the first branch of the curve. There exists a limitation to any further increase of \( K \) beyond \( K = 8 \). The flow, however, continues to increase even with decreased values of \( K \) along the second branch of the curve. Thus for a given pressure gradient parameter \( K \) there exist two values of the flow, and correspondingly two velocity and temperature distributions in the cross-section of the pipe.

One may visualize the relationship between \( u_{av} \) (or the rate of flow, \( q = \pi R^2 u_{av} \)) and the pressure gradient \( g \) imposed by reploting Figure 8.2a. The ordinate has been selected to be \( V \) from equation (8.23) this has been evaluated using the pressure gradient parameter \( K \) and its variant \( \gamma \) represented through \( B \). The abscissa of Figure 8.2a can be re-expressed in terms of \( 2K \) which is directly proportional to the pressure gradient \( g \) and does not contain any numerical factor.

The results thus obtained for the flow parameter \( V \) have been displayed in Figure 8.2b. It is rather easy to conceive the variation of the average velocity with the pressure gradient. However, the parameter \( V \) can be controlled by varying the rate of flow with which a fluid can be forced through a pipe at a constant rate, such as in the operation of the Instron capillary rheometer, see Van Wazer [8.12]. In principle any desired value of \( V \) can be enforced in such experiments. A corresponding
\[ Q = \frac{q}{q_{is}} \]

Figure 8.2a The rate of flow with respect to the isoviscous value, \( q/q_{is} \) against the pressure gradient parameter \( K \).

Figure 8.2b Variation of the flow parameter \( V \) with the pressure gradient.

A similar behaviour of the solution in terms of \( B \) instead of \( V \) can be observed if one performs a control, by some means, on the maximum temperature occurring at the pipe axis. The analogy involved in the two processes, one by enforcing a control on \( V \) and the other by controlling the maximum temperature to obtain the same behaviour of \( K \), is consistent with the relationship mentioned in expression (8.23). This maximum temperature can indefinitely be increased, and so \( B \), whereas correspondingly the resulting term such as \( K \) will also increase until the definite maximum value of \( B \) is reached, and further on it will only decrease, see Figure 8.3. One may in principle achieve high values of \( B \) by supplying the fluid to the pipe at a high temperature and subjecting the pipe wall to cooling. Such a method has been suggested by Kaganov [8.10] but no corresponding measurements have yet been published.

The single-valuedness of solution, such as \( u_{i}/u_{1} \), is obvious from Figure 8.4. If one desires to express the solutions in terms of the pressure gradient and the viscosity at the wall temperature, the above parameter \( B \) can be replaced by the corresponding set of parameters, viz.,

\[ E = 32 \frac{B}{(1 + B)^{2}} \]

and

\[ \gamma = 32 \left( \frac{B}{1 + B} \right) \]

The variation of the viscosity ratio \( u_{i}/u_{1} \) in expression (8.17) can readily be retrieved from Figure 8.2a, by simply squaring the ordinate \( q/q_{is} \) given by expression (8.20a), as a function of \( K \). For the ease of visualising the distribution of temperature and viscosity with the pressure gradient, the results for \( u_{i}/u_{1} \) have been displayed in Figure 8.5. It is interesting to note that the double-valuedness of the solution is characterised by the fact that in terms of the rate of flow, or say \( V \), the present solution is single-valued whilst in terms of the pressure gradient, or say \( K \), it is double-valued.
obvious when one employs an input parameter such as K. In experiments with pressure-gradient assisted flow of a liquid supplied at the temperature of the isothermal wall, one is expected to achieve steady flow measurements along the first branch of the curve, see Figures 8.2a and 8.2b. In a sense the second branch is physically non-realisable according to Nihoul [8.5]. His comments have to be viewed with caution. According to the works of the other investigators to be discussed later in Section 8.4.2.2 the existence of the second branch cannot be denied.

8.3 SOLUTIONS USING $m=1$ IN SLOTTE'S RELATIONSHIP

In the analysis to follow we employ the somewhat general viscosity-temperature relationship due to Slotte which has been extensively treated in Chapter 2. This relationship is given by,

$$\eta = \eta_0 \left[ 1 - \frac{\tau}{T - T_0} \right]^{m-1}$$

where $\eta_0$ is the viscosity at the reference temperature $T_0$, $m$ is an exponent, and $\eta_0$ is a coefficient dependent on the chosen grade temperature and is comparable with $\eta$ used in Section 8.2.2. This relationship is directly comparable with the exponential relationship through the introduction of a dimensionless temperature term $T'$, i.e.

$$\frac{\eta}{\eta_0} = 1 + \frac{\tau}{T - T_0}$$

For $m=1$ the relationship between fluidity, the reciprocal of viscosity, and temperature reduces to a linear one.

The pressure gradient parameter $B$ can be maintained in its original form except for the viscosity-temperature coefficient $B$ which will now be replaced by $\eta_0$. Further on the governing equations for momentum and energy somewhat simplify in that the exponential term for the temperature, namely $\exp(0)$, reduces to a linear term in $T'$. Analytical solutions for the temperature and velocity have been obtained in terms of Bessel functions by Regirer [8.11], Hausenblas [8.2] and others.

The solutions can be evaluated on lines similar to those for the flows of liquids following the exponential relationship. Such solutions, for the various thermohydrodynamic quantities mentioned in the previous section, have been worked out in terms of the parameter $B$. For lack of space they have not been presented here. After all the solutions have been presented in a graphical form in Figures 8.3 and 8.4 The results from Figure 8.4 have also been re-expressed in Figure 8.5 in terms of $K$. It may be observed that for $m=1$ there is a limitation also to $B$ to which the flow is subject. This is an asymptotic value of $B=0.7625$ and agrees well with that pointed out by Joseph [8.7]. This value occurs at $K=0$ and represents the ratio of the viscosity at the pipe axis to that at the wall approaching zero. Such regions of operation are devoid of physical importance because the flow would probably then have reached a turbulent condition. The solutions are single-valued in terms of $B$ as seen from Figure 8.4, but they are double-valued in terms of $K$ as seen from Figure 8.5.

Solutions with $m$ in the range $1 < m < \infty$, representing liquids commonly used in engineering practice, can be worked out in terms of $B$ only numerically. Such results might be considered useful in interpreting the measurements for the viscosity of liquids such as in capillary viscometers.

8.4 DISCUSSION

Of course, the analysis of the pipe flows shown in the previous sections is not of direct relevance to our present investigation of the plan-parallel pressure flows. Nevertheless it is useful for exploring the more or less related numerical solutions for those plan-parallel flows for which an exact solution is not available.
Employing the above solutions in the momentum equation and in the expression for viscous dissipation for the pipe flows, one can further evaluate shear stress and heat flux distribution as well as wall shear stress and total heat flux in any film cross-section.

The following analysis of the results has been carried out for the solutions obtained with the exponential viscosity-temperature relationship. Although double-valued in \( K \), the temperature distribution in a pipe cross-section is single-valued in \( B \) as shown in Figure 8.6a and 8.6b. Certain selected values of \( K \) in those figures have been shown to yield double-values of \( B \) and correspondingly two different distributions of temperature. The distribution of velocity has been shown in Figure 8.7 in a convenient non-dimensional form similar to that of the flow parameter \( V \). The same set of \( K \) for the distribution of \( V \) has been employed as that for the temperature distribution. Referring to these figures for the temperature and velocity distribution it may be observed that the results are single valued in \( B \).

The distribution of \( q/q_i \) or \( u/R \) is given by expressions (9.20a) and (9.20b) as a single valued quantity linearly related to \( B \). It may now be concluded, from all these results, that the solutions thus obtained in terms of \( B \) are unique.

The solutions obtained above are an alternative to Kearsley's [8.3] and Nihoul's [8.5] solutions for the exponential viscosity-temperature relationship, and these solutions incorporate the thermal state of the fluid at the axis of the pipe through \( Y \). More important than this is that they provide velocity and temperature distributions as unique solutions of the pressure gradient parameter \( B \). This uniqueness is in accordance with that pointed out by Sukanek and Laurence [8.6] for pipe flows. Referring to the solutions listed in Section 8.2.4 one may notice that a local parameter such as the velocity at the pipe axis and an integrated value along the film cross-section such as the rate of flow are also unique in \( B \).

The above solutions with the exponential relationship can be studied at the operating condition where \( K \) reaches its maximum value of \( B \), and correspondingly \( B \) unity. The values for the dimensionless groups for maximum velocity and minimum viscosity may be evaluated at the pipe axis, viz.

\[ U_H = 3.6357 \]

\[ U_i/R_H = 4 \]

The corresponding flow is twice that for the isothermal value whereas the flow parameter \( V \) reaches a value of 1.4142.
8.4.1 Achieving flows in a wide operating range

In principle, any parameter that represents some local or total condition of the thermal or hydrodynamic state of the fluid in the film cross-section lends itself as an input parameter for obtaining solutions that are unique as to the remaining parameters, which also describe the thermo-hydrodynamic condition of the flow. It should in principle be possible to obtain flows through the complete range of B, i.e. by controlling one or even more of the influential quantities involved. As pointed out before, this may be achieved through a variety of methods. One of the methods consists of employing a positive displacement pump of the piston-cylinder type to perform a control on the throughput. Another method is that of supplying the fluid at a high temperature to the pipe and subjecting the pipe wall to cooling. A number of designs of various viscometers in which a control on these operating conditions can be enforced have been listed by Van Maer et al. [8.12]. Such a control on any of the influential quantities has been rejected by Nihoul [8.5] as physically impossible. However, in the light of the above discussion it would appear that his conclusion on the physical non-existence of the second branch of the solution is untenable. The physical significance of a second branch of the solution is already apparent from the study of elasto-hydrodynamic lubricant flows, see Chapter 11. A limitation of the hydrodynamic type to the flows along the second branch of the solution has not yet been fully established. There are only some indications that this second branch lies already in the region where hydrodynamic instabilities might develop leading to a flow failure even at low Reynolds numbers, see [8.13].

8.4.2 Instability of the flow

For viscous pipe flows of liquids with the exponential viscosity-temperature relationship there is no controversy on the stability of the solution along the first branch, see Figure 8.2b. A study on stability of the solution along the second branch was initiated by Platten and Vanderborck [8.8]. With the availability of their preliminary results it looks that the conclusion drawn previously by Joseph [8.7] are in agreement with certain aspects discussed in the following sections. The solutions for liquids with $\eta_1$ in Slotte's viscosity-temperature relationship are somewhat similar to those of liquids following the exponential relationship. An analysis of the stability has not been carried out on these solutions. A point of inflection in the velocity profile obtained from either of the relationships may however be found to appear at the wall already at low values of B as shown in the following Section.

8.4.2.1 Instability on the first branch of the solution

Joseph [8.7] analyses the thermo-hydrodynamic flows in pipes for the Rayleigh kind of hydrodynamic instability, i.e. by using the occurrence of a point inflection in the velocity profile as a criterion. This criterion is known to indicate instability, although so far only for isoviscous flows in pipes, as pointed out by Tollmien in 1935. Joseph argues that Kearsley's proposition of an instability due to the presence of a point of inflection in the velocity profile must be regarded with caution. He suggests that in pipes such flows are not susceptible to instability, dissipation-induced effects making it impossible to generate profiles showing a point of inflection in a manner analogous to Rayleigh's inflection point criterion. For the significance of Rayleigh's criterion the reader is referred to Moore [8.14]. For non-isoviscous flows, however, a point of inflection at the wall appears already on the first branch of the solution, see Figure 8.5. Referring to Nihoul's work [8.5], or also to the present solution given by equation (8.19) and graphically represented on Figure 8.7, it may be seen that such a point of inflection appears at a comparatively low value of $B=0.4$, thus $\gamma=0.5$, and the corresponding $B=0.1428$. Corresponding hydrodynamic parameters for the flow amount to,

$$q/q_0=1.1428$$
$$\nu=0.5345$$

and those for the viscosity and velocity at the pipe axis to,

$$\mu/\eta_0=1.3061$$
$$\nu_0=1.1186$$

An extensive numerical study has been carried out by Platten and Vanderborck using a variational technique that is tantamount to a self-consistent local potential method. By carrying out computations for flows showing a point of inflection in the velocity profile, they conclude that these flows are stable toward infinitesimal disturbances of both the axisymmetric and the non-axisymmetric type. They establish the existence of even better stability than that calculated numerically for isoviscous pipe flows. It may thus be concluded that Kearsley's suggestion that a hydrodynamic instability would invariably arise whenever there is a point of inflection in the velocity profile, is not acceptable, at least not in general.

For fluids with $\eta_1$ in Slotte's relationship for viscosity variation with temperature, the occurrence of the point of inflection in the velocity profile can easily be established. This is done through determining whether the second derivative of the velocity at some point vanishes. An evaluation using the known analytical solution for the temperature or viscosity distribution in the form of zero order Bessel functions can be carried out. The point of inflection appearing first at the wall yields an identity of the Bessel function of zero order and that of the first order. The corresponding value of $B$ can easily be worked out from the tables for Bessel functions, and is found to be 0.2573. The corresponding thermo-hydrodynamic parameters have been listed below for convenience.

$$q/q_0=1.3955$$
$$\nu=0.7173$$
$$K=4.5111$$

The viscosity and velocity at the pipe axis are given by,

$$\mu/\eta_0=1.8251$$
$$\nu_0=1.6329$$
To the author's knowledge no specific details about the stability aspects are yet available for fluids with Blotte's exponent \( n \). It may thus be presumed that the flow behaviour might not considerably differ from what has been explained for the fluids with the exponential viscosity-temperature relationship. For \( n \) there exists a maximum value of \( R^4 = 6630 \) which cannot be exceeded, and it corresponds to \( \theta = 0.32 \). With increased values of \( B \) along the second branch of the solution \( K \) continues to decrease. However, \( K \) vanishes at a finite value of \( \theta = 0.7625 \). At the corresponding operating conditions the velocity at the pipe axis and the rate of flow reach infinitely large values. Whether the operating conditions can be maintained until such rates of flow are reached is a question yet to be resolved. It may, however, be realised that well before such high rates of flow are achieved, the flow might undergo a transition from laminar to turbulent state due to the presence of high Reynolds numbers. Then the solutions obtained with the consideration of the laminar flow no longer hold valid.

8.4.2.2 Instability on the second branch of the solution

A preliminary study by Joseph [8.7] carried out on the energy equation alone and using the method of the maximum principle indicates an instability of a hydrodynamic type on the second branch. It may be observed that his results are only limited in scope since he has not accounted for the coupling of the energy equation with the momentum equation. Indeed a stability study for the flow is considered to be complete only when the coupling of the velocity and temperature is accounted for.

Such a study involving the thermal effects, although only for pipe flows, has also been carried out by Platten and Vanderborck [8.8] by means of their afore-mentioned numerical methods. However, only a limited amount of data is available from their computations. Their results would indicate that this second branch is always unstable. Due to disturbances at varying degrees, as represented by various combinations of the axial and azimuthal wave numbers, instability would prevail at all Reynolds numbers. Their instability occurs at Reynolds numbers the lower the smaller the axial wave numbers and the larger the azimuthal wave numbers. Thus a critical state would exist at a maximum value of \( \theta \) corresponding to the transition from the first to the second branch of the solution. This existence, along with the presence of an instability already at low Reynolds numbers, indicates that any dimensions of the pipe might yield the instability provided the pressure gradient and the material properties have properly been selected.

8.4.2.3 Experimental results on stability of flow

Since Osborne Reynolds established that the flow would be laminar, and stable if a characteristic number associated with the flow was confined to sub-critical values, this subject has been of interest to many investigators. The significance of this characteristic "Reynolds" number, and its critical value of about 2000 has been confirmed by Prandtl and many others. In their extensive measurements Fox, Lessen and Bhat have investigated the possibility of an instability to develop under non-axisymmetric and azimuthally periodic disturbances. The measurements have been carried out in the fully developed flow region. They have noticed the instability at Reynolds numbers higher than 2130 when the flow was subjected to a frequency due to the first mode of a azimuthally periodic disturbance. The critical Reynolds number at which the instability occurred was observed to decrease with an increased frequency.

The measured critical values are far lower than the values later evaluated theoretically by Platten and Vanderborck [8.8]. The doubt raised by Fox et al. [8.15] on the reliability of these results seems to be genuine. Although the above Reynolds number seems, at first sight, of much higher magnitude than that of our interest, for oils the above value might reduce to a relatively low level because of their considerably higher values of viscosity as shown in what follows. Referring to equation (8.20b) we may notice that for the same pressure gradient, pipe diameter, and assuming that the \( B \) value remains the same for water with its nearly quadratic fluidity function, any characteristic velocity is inversely proportional to the viscosity. Thus the Reynolds number is inversely proportional to the square of the viscosity. Then one may easily conceive through the details provided in what follows that the above critical Reynolds number for any viscous oil as the working liquid would reduce to a very low value.

Unfortunately the details given by Fox et al. [8.15] about their experimental conditions and the data obtained are too limited. So no conclusions can be drawn on the magnitude of the other thermal or hydrodynamic parameters corresponding to the critical Reynolds number where turbulent behaviour of their flows has appeared.

For pipe flows the distribution of the shear stress may be obtained explicitly as a first integral of the appropriate equation of momentum as already given in equation (8.1). The distribution of shear stress proves to be independent of viscosity and is given by,

\[
\tau_{\varepsilon}(r) = \frac{g R^2}{8 \mu_1} (1 + B)
\]

The average velocity has already been expressed by equation (8.20b) viz.,

\[
\bar{u}_v = \frac{g R^2}{8 \mu_1} (1 + B)
\]

Expressing the Reynolds number refined for non-isothermal, non-isoviscous flow by introducing the viscosity referred to the wall temperature, viz.,

\[
Re = \frac{\rho \, u \, R}{\mu_1} = \frac{\rho \, g R^2 (1 + B)}{8 \mu_1}
\]

and introducing this number in the definition of \( K \) and finally rearranging,

\[
K (1 + \gamma \omega)^2 = \frac{16 \, B \, Re^2 \, u^2}{k \, c^2 \, R^2}
\]

or using the definition of \( B \),

\[
B = \frac{\delta \, Re^2 \, u^2}{k \, c^2 \, R^2}
\]

For a given value of \( B \), material properties and pipe radius the critical \( Re \) varies inversely with \( \sqrt{\nu} \). In Table 8.1 the material properties of such an oil have been compared with those of water, and the resulting critical \( Re \) has been worked out. For the mineral oil with a low viscosity of 100 cp the decrease of the critical \( Re \) would be by a factor 670.
would undergo an instability already at a Reynolds number of approximately unity. A non-linear viscosity-temperature relationship of the exponential type indicates an interesting behaviour of the flow. In terms of a variable parameter K involving the viscosity of a liquid at the temperature of the wall in addition to the pressure gradient and other material properties, there prevails a limitation to the solution for the distribution of temperature and velocity. There exists a maximum to this variable parameter beyond which no solutions are possible, this value being K=8. For all values of K lower than this limiting value there exist double-valued solutions.

Solutions obtained in terms of similar material properties and some thermal or hydrodynamic reference state of the fluid inside the pipe do not indicate any limitation. These solutions are unique throughout the complete regime of operation including that at the operating conditions corresponding to the above-mentioned limiting value of K; they can be evaluated in terms of a parameter B which consists of the viscosity of the liquid at the temperature referred to the pipe axis, instead of that at the pipe wall. The above-mentioned limiting operating condition in terms of K, viz. K=8, can be represented by B=1. Alternatively, this limiting condition can also be specified through a parameter that accounts for the average velocity in the pipe cross-section, i.e. \( V=1.4142 \). However, the flow is unstable for the solution lying beyond the above-mentioned values of B and V. This instability prevails even at low Reynolds numbers provided the pressure gradient and material properties are suitably combined along with the pipe radius. Very small pipes such as capillaries will require quite large temperature or viscosity gradients to produce the instability for any selected maximum temperature difference.

It would appear that with non-isothermal, non-isoviscous flows the appearance of a point of inflection in the velocity profile for a flow at any point along the first branch of the solution does not necessarily involve any hydrodynamic instability. Such pipe flows would appear even more stable than the isoviscous pipe flows for the same Reynolds number and disturbances of the same wave numbers. The analytical solutions obtained for the non-isoviscous pipe flows, along with the study of the thermal instability, are useful in establishing similar characteristics of the viscous plan-parallel pressure flows. For the latter no analytical solutions are available except for that with \( m=1 \) in Slotte's viscosity-temperature relationship. To the author's knowledge stability studies accounting for the viscous effects have not been reported. The studies so far reported take into account only an assumed velocity profile which does not seem to incorporate the characteristic double-valuedness of the solutions. Improved solutions, although only numerically for most of the cases, can be obtained over the complete regime of operation for the plan-parallel pressure flows. The above-mentioned characteristics of the solutions can also be shown to exist for these solutions as well.

### Table 8.1 Material properties at atmospheric pressure and 10 °C.

<table>
<thead>
<tr>
<th></th>
<th>water</th>
<th>mineral oil</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k ), W/s°C</td>
<td>0.58</td>
<td>0.14</td>
</tr>
<tr>
<td>( \mu ), cP</td>
<td>1.307</td>
<td>100.0</td>
</tr>
<tr>
<td>( \beta_p ), 1/°C</td>
<td>0.03368</td>
<td>0.065</td>
</tr>
<tr>
<td>( \rho ), kg/m³</td>
<td>1000</td>
<td>900</td>
</tr>
<tr>
<td>Re(critical)</td>
<td>2130</td>
<td>1.42</td>
</tr>
</tbody>
</table>

It may thus be observed that the flow of the mineral oils in pipes would undergo an instability already at a Reynolds number of approximately unity.

### 8.5 CONCLUSIONS

Analytical solutions obtained for viscous axisymmetric flows in pipes with a non-linear viscosity-temperature relationship of the exponential type indicate an interesting behaviour of the flow. In terms of a variable parameter \( K \) involving the viscosity of a liquid at the temperature of the wall in addition to the pressure gradient and other material properties, there prevails a limitation to the solution for the distribution of temperature and velocity. There exists a maximum to this variable parameter beyond which no solutions are possible, this value being \( K=8 \). For all values of \( K \) lower than this limiting value there exist double-valued solutions.

Solutions obtained in terms of similar material properties and some thermal or hydrodynamic reference state of the fluid inside the pipe do not indicate any limitation. These solutions are unique throughout the complete regime of operation including that at the operating conditions corresponding to the above-mentioned limiting value of \( K \); they can be evaluated in terms of a parameter \( B \) which consists of the viscosity of the liquid at the temperature referred to the pipe axis, instead of that at the pipe wall. The above-mentioned limiting operating condition in terms of \( K \), viz. \( K=8 \), can be represented by \( B=1 \). Alternatively, this limiting condition can also be specified through a parameter that accounts for the average velocity in the pipe cross-section, i.e. \( V=1.4142 \). However, the flow is unstable for the solution lying beyond the above-mentioned values of \( B \) and \( V \). This instability prevails even at low Reynolds numbers provided the pressure gradient and material properties are suitably combined along with the pipe radius. Very small pipes such as capillaries will require quite large temperature or viscosity gradients to produce the instability for any selected maximum temperature difference.

It would appear that with non-isothermal, non-isoviscous flows the appearance of a point of inflection in the velocity profile for a flow at any point along the first branch of the solution does not necessarily involve any hydrodynamic instability. Such pipe flows would appear even more stable than the isoviscous pipe flows for the same Reynolds number and disturbances of the same wave numbers.

The analytical solutions obtained for the non-isoviscous pipe flows, along with the study of the thermal instability, are useful in establishing similar characteristics of the viscous plan-parallel pressure flows. For the latter no analytical solutions are available except for that with \( m=1 \) in Slotte's viscosity-temperature relationship. To the author's knowledge stability studies accounting for the viscous effects have not been reported. The studies so far reported take into account only an assumed velocity profile which does not seem to incorporate the characteristic double-valuedness of the solutions. Improved solutions, although only numerically for most of the cases, can be obtained over the complete regime of operation for the plan-parallel pressure flows. The above-mentioned characteristics of the solutions can also be shown to exist for these solutions as well.

### 8.6 REFERENCES


Chapter IX
NON-ISOTHERMAL PLAN-PARALLEL PRESSURE FLOWS

9.1 INTRODUCTION

Only limited published information is available on critical thermo-hydrodynamic behaviour of plan-parallel flows of the most general kind. The available information on the pure shear flow, however, does cover a wide range of operating conditions as shown in detail in Chapters 3 and 5.

Plan-parallel pressure flows are common in lubricant films operating with pure rolling under elasto- or plasto-hydrodynamic conditions. If the viscous dissipation involved is considerable the flows may well undergo, at least in certain operating regime, a strong deviation from the isothermal behaviour. These flows might even encounter an instability of a thermal or hydrodynamic type, such as suggested by various investigators mentioned in Chapter 8. Solutions for plan-parallel thermally affected pressure flows can in principle be obtained in a manner analogous to that used for the axisymmetric flows in pipes. To achieve this one starts with the equations for momentum and energy. The energy equation accounts for heat generation due to viscous dissipation and, for the ease of analysis, for heat transfer only through cross-conduction. Heat transfer due to convection has been neglected like in the analyses of thermally affected pure shear flows in Chapter 3 and 5, and axisymmetric pure pressure flows in Chapter 8.

Assuming a constant value for the thermal conductivity of the liquid i.e. independent of temperature and pressure, the existence and behaviour of solutions for various viscosity-temperature relationships have been studied. For instance, Hausenblas [9.1] obtained analytical solutions for the flow of a liquid with a linear fluidity function, i.e. with exponent \( m = 1 \) in Slotte's relationship. He obtained these solutions in terms of fractional Bessel functions. However, he seems not to have noticed the limiting pressure gradient of a finite value that might occur even at low rates of flow, and beyond which no solution exists.

Of much greater interest are the solutions to liquids with a non-linear viscosity-temperature relationship such as Slotte's with the exponent \( m \) exceeding unity and including the exponential one where \( m = 0 \). Early attempts to obtain the solutions for liquids with a viscosity-temperature relationship involving a quadratic fluidity function, i.e. \( m = 2 \), also by Hausenblas [9.1], did not reveal the characteristic double-valuedness of the solution due perhaps to the method of solution followed. Kaganov [9.2] was the first to point out, if only qualitatively, double-valuedness as characteristic of viscosity-temperature relationships involving non-linear functions.

For boundary conditions with equal temperatures of the two opposite solid surfaces Kaganov [9.2] attempted to delimit the existence regime of thermal plan-parallel pressure flows. He has bound the solutions both from above and below on the basis of the maximum principle. Similar solutions were obtained also by Joseph [9.3]. These investigators have employed a quadratic fluidity function as well as the exponential
viscosity-temperature relationship. They have expressed the solutions in terms of a parameter containing the pressure gradient to which the flow is subject and the viscosity of the liquid corresponding to the wall temperature. This parameter would take higher values with increased pressure gradient but would be limited by a maximum value which cannot be surpassed. There exist no solutions beyond this limiting value for this parameter. In the physically significant range of the parameter there exist, however, a second solution which involves an increase in rate of flow and maximum temperature at the axis of symmetry when the value of the present parameter is decreased.

This double-valuedness of the solution in terms of the present parameter is similar to that obtained for the pipe flows analysed in Chapter 8. This behaviour is somewhat similar to that of the thermally affected pure shear flows analysed in Chapters 3 and 5. Numerical solutions obtained for the exponential relationship by Aggarwal and Wilson [9.4] cover a wide operating regime. Their solutions, however, do not explicitly indicate any double-valuedness since these have been obtained in terms of a parameter containing both the pressure gradient and the rate of flow. A limitation to the solution in terms of a parameter containing the pressure gradient but free of the rate of flow has thus been overlooked by the latter investigators.

Literature on stability of plan-parallel pressure flows under the influence of thermal effects is scarce. Joseph [9.3] analyses the hydrodynamic stability of a solution found when the associated velocity and temperature distributions may be uncoupled. However, his analysis is far from complete. It might even give an erroneous picture of reality as was pointed out by Sukanek et al. [9.5] even though they concentrated mainly upon thermally affected pure shear flows.

In the present analysis solutions will be developed for liquids with non-linear viscosity-temperature relationships of Slotte's type with various finite values of the exponent \( m \), and including the limiting case of an exponential relationship, \( \eta \propto \exp(m \theta) \). For completeness, a solution for liquids with the linear fluidity function will also be established. These solutions will be evaluated, as has been done for the pipe flows in Chapter 8, in terms of a parameter involving the pressure gradient and the viscosity at the axis of symmetry of the flow. It will be shown that the solutions for the plan-parallel pressure flows are qualitatively similar to those for the axisymmetric pressure flows.

9.2 GOVERNING EQUATIONS

The present study deals with the flow analysis that corresponds, at least under rolling condition, to elastohydrodynamic lubricant films, as well as the elastohydrodynamic lubricant films that may exist in the strip rolling process of metal working. We maintain the assumptions listed in Chapter 3, namely a steady-state flow of an incompressible liquid with a constant thermal conductivity, \( k \). The heat generation is considered to take place due to viscous dissipation and the internal heat transfer due only to cross-conduction.

For the plan-parallel pressure flow to take place under the influence of a pressure gradient \( \frac{\partial p}{\partial y} \) in the direction of flow, \( x \), the momentum balance may be specified by the distribution of shear stress \( T \) in the form,

\[
\frac{\partial T}{\partial y} = -g \frac{\partial u}{\partial y}
\]

where \( g \) is independent of \( y \). The shear stress distribution in any film cross-section can be obtained by integrating equation (9.1a) along the coordinate \( y \) specified in Figure 9.1, i.e.,

\[
T = -g u + C
\]

The constant of integration \( C \) can be shown to become zero whenever the velocity distribution is symmetric, i.e. whenever the velocity gradient \( du/dy = 0 \) at the channel axis, \( y = 0 \).

For a thermally steady-state condition to prevail in any film cross-section under the above-mentioned assumptions concerning the thermal process, the governing energy equation can be represented, in the foregoing for the pure shear and the axisymmetric pressure flows, by,

\[
k \frac{\partial^2 T}{\partial y^2} + \phi = 0
\]

where \( \phi \) represents the heat generation in the film due to viscous dissipation per unit volume and per unit time due to the presence of a local rate of shear \( du/dy \). For Newtonian liquids \( \phi \) is given by,

\[
\phi = \mu \left( \frac{du}{dy} \right)^2 - \frac{r^2}{\mu}
\]

Boundary conditions to energy equation (9.2) will be chosen similar to those employed in the analysis of the axisymmetric pressure flows in pipes already treated in Chapter 8. So, they will represent profiles symmetric both for the velocity and temperature. This symmetry has been chosen in order to demonstrate the analogy between the axisymmetric and the present plan-parallel flows, both operating under some pressure gradient. The present thermal boundary conditions may be specified by,

\[
y = 0, T = T_i, \frac{\partial T}{\partial y} = 0
\]

or, alternatively, by,

\[
y = \frac{1}{2} h, T = T_i
\]

By introducing the expression for the shear stress from equation (9.1b) in the energy equation, one may proceed to seek a solution complying with the above boundary conditions. Such a solution can be worked out, as detailed in the following sections, for a liquid following any gen-

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Figure 9.1 Co-ordinate system and thermal boundary conditions employed for the plan-parallel pressure flow.
eral viscosity-temperature relationship such as Slotte's, including its particular variant, the exponential one.

9.3 SOLUTIONS WITH THE EXPONENTIAL RELATIONSHIP

9.3.1 Governing equations

The liquid is assumed to follow the exponential viscosity-temperature relationship that has been dealt with in Chapter 2, viz.,

\[ u = u^\circ \exp \left[ - \beta (T - T^\circ) \right] \]  

(9.4a)

where \( u^\circ \) is the viscosity at the reference temperature \( T^\circ \) and \( \beta \) is the viscosity-temperature coefficient. Introducing a non-dimensional quantity for the temperature, viz.,

\[ T = T - T^\circ, \quad \beta = \beta \left( T - T^\circ \right) \]

(9.6)

where \( \beta \) being independent of temperature (cf. Chapter 2), need no longer be related to any standard zero-level temperature such as \( T^\circ \) at the axis of the channel. The above relationship for the viscosity may be expressed in a simple form, i.e.,

\[ u = u^\circ \frac{\beta}{T} \]  

(9.4b)

The governing equations for momentum and energy as well as the boundary conditions can be non-dimensionalized through,

\[ Y = \frac{y}{h}, \quad \frac{\partial}{\partial Y} \]  

(9.7a)

Then the shear stress at any position \( Y \) in a non-dimensional form,

\[ \tau^\circ = \frac{\beta}{k} \left( \frac{\partial u}{\partial Y} \right) \]  

(9.5)

whilst the energy equation reduces to,

\[ \frac{\partial \theta}{\partial Y} + \frac{\partial}{\partial Y} \left( \frac{\partial \theta}{\partial Y} \right) = 0 \]  

(9.6)

with the boundary conditions,

\[ Y = 0, \quad \theta = 0, \quad \frac{\partial \theta}{\partial Y} = 0 \]  

(9.7a)

Since the present \( 0-Y \) relationship is symmetrical with respect to the origin, \( Y=0 \), these boundary conditions can, if so desired, be replaced by,

\[ Y = \frac{1}{2}, \quad \theta = \theta^\circ \]  

(9.7b)

It may be noticed that the above form of the energy equation, i.e., equation (9.6), is similar to that for the axisymmetric flows in pipes with \( \frac{1}{2} \) replacing \( R \). However, contrary to the availability of analytical solutions to the previous equation, there are no such solutions to the present one. Thus numerical solutions will be developed in what follows.

9.3.2 Solutions to the equations

Attempts have been made by various investigators to obtain the solution to equation (9.6) in a closed form. Such attempts include among other methods, the maximum principle operation for the particular case of two equal boundary temperature conditions. Kaganov [9.2] applied such an operation on an assumed corresponding integral equation for the temperature distribution in terms of the Green's function. He was successful in bounding the solution both on the upper and lower sides, and thereby to discuss the qualitative nature of the solution. Joseph [9.3] has recovered the results of Kaganov and has improved the estimate of the limiting value of a parameter involving the pressure gradient, i.e., similar to the following:

\[ K = \frac{\beta^2 \rho h}{k} \]  

(9.7)

beyond which no solution exists. Here \( u^\circ \) is the viscosity of the liquid at the wall temperature \( T^\circ \). He has employed also the maximum principle operation and has bound the solution between assumed parabolas of the first and the fourth degree.

Other investigators have followed numerical techniques and often have encountered difficulties in carrying out the computation throughout the widest range that might be significant in practical applications. Stoffel [9.6] has employed a numerical integration technique to obtain the solution for a flow problem involving a general type of thermal boundary conditions including two particular cases: one involving uniform and equal temperatures at both the surfaces, and the other involving an adiabatic surface. He has, however, encountered difficulty with the computation already at low levels of wall shear stress and the consequent viscous heating. He suggests that the failure of the numerical technique due to a stability limit involved coincides with the physical process due to the thermal effect that might arise in practice. A similar failure to obtain the solutions in a wide operating regime on the basis of a perturbation technique has been pointed out by El-Atiny [9.7]. The failure of these investigators may now be attributed with certainty to the improper selection of a parameter in which the pressure gradient and the rate of flow appear jointly. True, they expressed their analytical formulation in terms of the aforementioned pressure gradient parameter, viz. \( K \). They have, however, provided the results in terms of a different parameter involving both the pressure gradient and the rate of flow. This parameter has presumably been introduced by these investigators to avoid the numerical difficulty they would have experienced with the selection of \( R \) as the input parameter. They have, however, failed to comment upon this, and anyway have missed the limiting value of \( K \), which is to be pointed out here below. Indeed, thermally affected pressure flow in plan-parallel channels can analytically be shown to be establishable only for \( K \)-values, or say for pressure gradients, not exceeding a critical limit. A similar phenomenon occurs with axisymmetric thermally affected pressure flows as well (cf. Chapter 8).

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The second order differential equation (9.6), along with the boundary conditions in equation (9.7a), constitute a "one-point" boundary value problem. Such a problem can be solved by numerical techniques of the simple "marching" type. For the present analysis we have employed a special case of Milne's method (see Kuzn [3.8]), which is known to be suitable for the analysis of second order differential equations which do not contain the first derivative. The method involves a quadrature integration technique involving predictor-corrector formulae.

Denoting the differential equation in a general form, viz.:

\[ \theta'' = f(Y, P, \theta) \]

For any specified value of \( A \) one can specify the predictor formula by,

\[ \theta_{n+1} = \theta_n + \Delta \theta \frac{\theta''}{A^2} \left( \theta''_n + 2 \theta''_{n+1} + \theta''_{n-1} \right) \]

and the corrector formula by,

\[ \theta_n = 2 \theta_{n+1} - \theta_{n+2} + \frac{A^2 \theta''}{12} (\theta''_{n+1} + 10 \theta''_{n+2} + \theta''_{n+3}) - \frac{1}{240} (A^2)^2 \theta''(Y) \]

(9.8)

where \( \Delta \theta \) is the step length and primes indicate the order of differentiation. The last terms in both the predictor and corrector formulae are the error terms which have separately to be evaluated in order to investigate the accuracy of the numerical technique. The error terms consist of the sixth power of the step length. By halving once the step length the error term can in principle be reduced by a factor of 64. A computation with a step length of 0.01 has yielded sufficiently low error terms so that, for any specified value of \( A \) one can specify the predictor formula by,

\[ \theta_{n+1} = \theta_n + \Delta \theta \frac{\theta''}{A^2} \left( \theta''_n + 2 \theta''_{n+1} + \theta''_{n-1} \right) \]

(9.9)

The velocity distribution across the film cross-section can be worked out using the Newtonian definition for the viscosity, viz.,

\[ \tau = \mu \frac{du}{dy} \]

(9.12)

The resultent wall shear stress difference in a non-dimensional form, i.e. \( \tau^* \), can be expressed as shown in equation (9.5), by,

\[ \tau^* = \frac{h}{h} \frac{d \theta}{dy} \]

(9.13a)

Substituting \( \psi \) from definition (9.4b) for the exponential viscosity-temperature relationship, and rearranging the terms, the velocity distribution can be expressed in dimensionless form by,

\[ \psi = \frac{2}{(\theta'' \theta''')} \left( \psi - \frac{\psi}{\theta''} \right) \int_0^\theta \psi \theta'' \theta''' \]  

(9.13b)

The velocity distribution can also be expressed, if required, using the "isothermal" velocity at the channel axis, the one occuring when the flow would take place at a uniform viscosity. Then, with

\[ u_{is, max} = \frac{g}{h} \theta'' \]

(9.14)

another convenient non-dimensional form follows:

\[ u_{is, max} = \frac{8}{(u' \theta''') \theta''} \theta'' \theta''' \]

(9.15)

Volumetric rate of flow \( q \) can be obtained by making use of the above expression for the distribution of the velocity,

\[ q = \int_{-h/2}^{h/2} u dy \]
where \( q \) can also be expressed with respect to the rate of flow, \( q_{is} \), as it occurs in plan-parallel isoviscous pressure (Poisueille) flows, 
\[
q_{is} = \frac{q}{h^3/12} \mu_l
\]
so that, the "correction factor" to the isothermal rate of flow can be expressed by,
\[
q = q_{is} \left( 12 \left( \frac{\mu_l}{\mu} \right) \left( \int_0^h Y dy - 2 \int_0^h Y^2 dy \right) \right)
\]
(9.14b)
The distribution of the heat flux \( \dot{q} \) and an integrated total value \( \dot{q}_{tot} \) across the film cross-section can be evaluated using the definition for the viscous dissipation as follows.

\[
\dot{q} = \frac{1}{2} \rho U L v^2 (9.14a)
\]

or, in a convenient non-dimensional form,

\[
\dot{q}^* = \left( \frac{6}{k} \right) h^2 \frac{\dot{q}}{\mu} = \frac{9}{2} \frac{Y^2}{Y^0} (9.15)
\]

The latter expression can also be expressed in a non-dimensional form:

\[
\dot{q}^* = \left( \frac{6}{k} \right) h \dot{q}_{tot} = \frac{9}{2} \int_0^h \left( \frac{Y^2}{Y^0} \right) dy (9.16a)
\]

The integral in expression (9.16a) can, however, be expressed in terms of integrals that have previously been used to obtain other hydrodynamic quantities. The present expression then takes the form,

\[
\dot{q}^* = \frac{9}{2} \int_0^h \left( \frac{Y^2}{Y^0} \right) dy (9.16b)
\]

For ease of comparison with isothermal flows, the velocity distribution and the rate of flow can also be expressed in terms of \( \mu_l \) instead of \( \mu_H \). Formulae (9.13a) and (9.14a) can then be re-expressed as follows:

\[
\mu_l = \left( \frac{8}{k} \right) h \frac{1}{\mu} \int_a^b \mu H = \left( \frac{8}{k} \right) h \frac{1}{\mu} V = \left( \frac{8}{k} \right) h \frac{1}{\mu} q = \left( \frac{8}{k} \right) h \frac{1}{\mu} \left( \int_0^h Y dy - 2 \int_0^h Y^2 dy \right) (9.17)
\]

The above expressions have been developed for the plan-parallel pressure flow with stationary walls. For the problems involving the rolling condition, viz. identical surface velocities of the walls, these expressions have to be generalized. The additional terms due to such a rolling condition with the surface velocity \( \mu_l \), reduce for the velocity to,

\[
\mu_l = \left( \frac{8}{k} \right) h \frac{1}{\mu} \left( \int_0^h Y dy - 2 \int_0^h Y^2 dy \right) (9.18)
\]

\[
\dot{q}_r = \left( \frac{8}{k} \right) \frac{1}{12} \mu_l (9.19a)
\]

\[
\dot{q}_r = \left( \frac{8}{k} \right) \frac{1}{12} \mu_l (9.19b)
\]

Temperature distributions for various selected values of \( P \) have been evaluated according to the numerical technique explained in the previous section. For ease of comparison with the results obtained for the axis-symmetric flows in Chapter 8, these results have been re-expressed in terms of the viscosity ratio \( \mu_H/\mu \) given by equation (9.4b), i.e.

\[
\mu_H/\mu = \frac{1}{P} (9.19b)
\]

A graphical representation of this ratio has been presented in Figure 9.2. It is seen that this ratio monotonically decreases with increasing values of \( P \). It may be noticed that the trend of these results is similar to that of the axisymmetric thermally affected pressure flows, those in pipes as discussed in Chapter 8 (cf. Figure 8.4). For comparison the latter results have also been presented in Figure 9.2. The abscissa for the axisymmetric flows, 2048B, has been chosen to be similar to \( P \). The former differs in that diameter 2R takes the place of film thickness \( h \). Figure 9.2 clearly indicates the uniqueness of the solution for the plan-parallel flows for a wide range of \( \mu_H \), or say of operating tempera-
Further these results are an extension to those of Stoffel [9.6] who investigated only a limited operating regime. The present results are even more accurate than those bounded by the method of the maximum principle by Kaganov [9.2] and Joseph [9.3], and agree well with the previous ones of Aggarwal and Wilson [9.4] which were obtained by another evaluational method.

Now, when expressed in terms of the previously mentioned parameter $K$, the distribution of temperature can be shown to be double-valued, as follows. This parameter is similar to $K$ employed in the analysis of the axisymmetric flows, in that it consists of the known viscosity $\mu_1$ at the temperature of the channel wall as the influential viscosity, namely,

$$K = \frac{\beta e_k h}{h_k} \left( \frac{\mu_1}{\mu_1} \right) P$$

Conversion of the parameter $K$ is now seen, from the previously defined parameter $P$ to be achievable by making use of the viscosity ratio $\mu_2/\mu_1$. The results presented in Figure 9.3 indicate that for any given value of $K$ which does not exceed a maximum value there exist two values of $P$.

Of $P$. Such a relationship of $K$ with $P$ is similar to that of $K$ with $B$ presented in Chapter 8 for the axisymmetric flows; see Figure 8.3. The double-valuedness of the temperature $\theta_1$, which is related to the viscosity through the relationship (9.4b), is apparent from the results presented in Figure 9.4 for the variation of the viscosity ratio $\mu_1/\mu_2$ with $K$. This figure is qualitatively similar to Figure 8.5 in Chapter 8 corresponding with axisymmetric flows.

In addition to the temperature and viscosity, the velocity and shear stress are also double-valued in $K$. This can be observed by transforming expressions (9.12) and (9.13a) to a form involving this parameter. Then the shear stress at the wall given by equation (9.12) can be expressed involving the known viscosity $\mu_1$,

$$T_1 = h \left( \frac{\beta}{\mu_1} \right) \frac{1}{\kappa} - \sqrt{K}$$

and the velocity distribution may be converted from equation (9.13a) to,

$$\bar{v} = \left( \frac{\beta}{\mu_1} \right) \left( \frac{\mu_2}{\mu_1} \right) - \left( \frac{\mu_2}{\mu_1} \right) \sqrt{K} \int_0^1 e^0 y' d\eta'$$

It may be noticed that the double-valuedness of $T_1$ is apparent through the appearance of $K$ there: as mentioned before, for any given value of $K$ there exist two values of $P$. But it is not so apparent with $U$ since this expression involves in addition to $K$ also the viscosity ratio $\mu_2/\mu_1$; this ratio appears both as an independent parameter and as an implicit one in the integrand through $e^0 y' \mu_2/\mu_1$. It has already been noticed through Figure 9.2 that $\mu_2/\mu_1$, or alternatively $\mu_1/\mu_2$, is single-valued in $P$. It may be conceived, provided the integral is also single-valued in $P$, that the appearance of $K$ in the expression for $U$ implies the double-valuedness of the latter.

The volumetric rate of flow and the total heat flux, however, are also single valued in $P$. The behaviour of the former of the two quantities with $K$ is identical, as will be shown in the next section, with that for the axisymmetric pressure flows treated in Chapter 8. The rate of flow can be given in a convenient form, i.e.,

$$\left( \frac{\beta}{\mu_1} \right) \left( \frac{1}{\kappa} - h \right) \frac{1}{\kappa} \left( \frac{\mu_2}{\mu_1} \right) - 2 \left( \frac{\mu_2}{\mu_1} \right) \sqrt{K} \int_0^1 e^0 y' r' d\eta'$$

where the second term on the right-hand side represents the maximum velocity occurring at the channel axis. This expression for the rate of flow also represents the average velocity of the non-isothermal flow concerned, i.e. the flow parameter $V$. Its evaluational results will, however, be presented in the next section. It may then be concluded
that the solutions presently obtained for the plan-parallel pressure flows are characteristically identical with those in the case of axisymmetric pressure flows and which also show the double-valuedness concerned.

The foregoing conclusions are more accurate than those established by Kaganov [9.2] and Joseph [9.3]. These investigators, having only bounded the solutions and missing any further quantitative information, were only able to qualitatively indicate their trend. Further, the present solutions are an extension to Stoffel's [9.6], at least for the flows of the particular case of uniform and equal temperatures of the boundaries.

9.3.4.2 Results in a convenient form

The distributions of temperature and velocity can now be computed and expressed in a convenient form. This has been achieved by expressing the whole regime of operation by a single parameter \( w \) consisting of the previously defined parameter \( P \) and the non-dimensional coordinate \( r \).

\[ w = P \cdot \frac{r}{Y} \]

The value of this new parameter at the two boundaries \( Y_1 \) and \( Y_2 \) bounding the film is designated by \( w_1 \) and \( w_2 \). For a flow in a channel between \( Y_1 = -(1/2) \) and \( Y_2 = (1/2) \) and characterized by a definite value of \( P \), the distributions of temperature and velocity can be represented by corresponding ones of \( w \). One can also introduce \( W = (w_2 - w_1) \) which is identical to \( P \). Then the results yet to be evaluated can be expressed either in terms of \( W \) or \( P \), namely \( P = (w_2 - w_1) \). Thermally and kinematically symmetric flows may be conceived to be enclosed by the boundaries at uniform temperature and velocity. Then \( w_2 = w_1 = (P/2) \), and the viscosity ratio \( \frac{H}{\mu} = 1 \) at the axis of the channel.

The above convenient single representation can be obtained through a "segmenting" technique shown in Figure 9.5. For any specified value of \( P = P_1 \) there exists a corresponding dissected \( W = P_1 \) which establishes the boundaries at \( w_1 \) and \( w_2 \) corresponding to \( Y_1 = -(1/2) \) and \( Y_2 = 1/2 \) as shown in Figure 9.5a. The flow represented by such a value of \( P \) gives rise to the viscosity distribution \( \frac{H}{\mu} \) which varies between 1.0 at the axis and a definite value of \( \left( \frac{H}{\mu} \right) \) at the boundaries of the channel, see Figure 9.5b. The temperature, velocity, and various other thermal and hydrodynamic quantities can correspondingly be evaluated on similar lines. For any other specified value of \( P = P_2 \) there exists a corresponding flow gives rise to a different \( \left( \frac{H}{\mu} \right) \) at the boundaries whereas that at the axis remains unity, see Figure 9.5b.

Flows are characterized by a selected value of \( P \) that does not exceed the limit specified by \( \mu \). The corresponding limiting value of \( W \) for the distribution of viscosity \( \frac{H}{\mu} \), can thus be analysed through one single representation shown in Figure 9.5b. Such a representation replaces the bulk of the results that otherwise would have been separately displayed by an entire family of curves with \( P \) as their parameter.

The distribution of viscosity \( \frac{H}{\mu} \) which can easily be re-expressed in terms of the temperature, and of velocity have been presented in Figure 9.2, and Figures 9.3a and 9.3b, respectively. For small values of \( P \) the viscosity distribution shows a trend very much like a quartic curve and the velocity distribution like a parabola. For somewhat higher values of \( P \) both the distributions become increasingly parabolic.
The velocity distribution can be expressed in any of the various dimensionless forms such as \( U_H \) or \( \phi \). For the present the former form given by equation (9.13a) has been used. In order to make the results as general as possible, the aforementioned concept of expressing the results in terms of \( \phi \) has been employed. For the convenience of reading the velocity profiles the origin has been selected at the apex coinciding with the axis of the channel, at least for the present case of the symmetric boundary conditions. The velocity distribution for any specified value of \( P \) can thus be obtained in terms of the corresponding \( \phi \) from Figure 9.6a. It may be noticed that the flow is existent and the velocity profile is uniform at even large values of \( \phi \). At very large values of \( \phi \) the velocity profile increasingly approaches the triangular shape and of a delta function.

In order to maintain some conformity with results to be discussed in the next chapter for various thermal boundary conditions, a convenient common basis has been chosen. That is the velocity distribution \( U_H \) has been reduced by making use of the maximum velocity that corresponds to the particular case \( U_H \) is nearly zero. The maximum velocity is only a scaling quantity of \( U_H \) at \( \phi \approx 1.0 \), the corresponding parameter \( P \), and thus \( \phi \), being infinite. For simplicity, however, a finite value of \( \phi = 17.28 \) implying \( \phi = 8.64 \) has been selected where \( \phi \) approaches zero. This value of \( \phi \), i.e. \( \phi_{\text{max}} \), is again only a scaling quantity, and has been used to normalize the results to the convenient form of Figure 9.6b.

For illustration consider an operating condition specified by \( \phi = 3.0 \). For the present isothermal boundary conditions \( \phi = 8.64 \) the corresponding velocity distribution \( U_H \) can be read from Figure 9.6a. This value is found to be \( U_H = 0.99 \), and thus the reduced value of the velocity \( U_H = 0.542 \) occurring at \( \phi = 8.64 \). These values can be found to agree with those represented in Figure 9.6b.

The variation of flow with respect to the parameter \( P \) has been plotted in Figure 9.7a using equation (9.14b). With increased \( P \) the flow deviates considerably from that under the isoviscous condition as evidenced by the values of \( Q \) higher than unity. Such a deviation can also be called a "correction factor" to the flow due to non-isothermal effects. At about \( P = 1000 \) the flow becomes about 6 times that in the isoviscous case. At very high values of \( P \) the flow becomes infinitely large which can be evidenced by the monotonic increase of \( Q \).

The variation of rate of flow can also be expressed using the average velocity of the non-isothermal flow under the influence of any prescribed pressure gradient. This can be obtained through re-expressing the parameter used in Figure 9.7a: the abscissa can be selected to be the only dimensionless group containing the pressure gradient, further involving only measurable material properties such as the viscosity at the wall temperature; the ordinate from equation (9.18), i.e. the average velocity and the same measurable material properties. The results have been displayed in Figure 9.7b. It may be noticed that the rate of flow is double-valued in terms of the pressure gradient as expected. There exists a transitional point beyond which the parameter \( K \) is not permissible. The first branch of the solution yields lower average velocities, and thus lower rate of flow, than that given by the second branch. Along
the second branch the average velocity and rate of flow become increasingly large even at reduced values of the parameter $K$.

For a comparison also the results of Aggarwal and Wilson [9.4] have been presented in Figure (9.7b). Their numerical results which have been read to the graphical accuracy show good agreement with those from the present investigation. The latter results, however, cover a wider range than theirs.

For positive values of the pressure gradient, i.e. $g = -d\rho/dx > 0$, the parameter $V_K$, and correspondingly the flow parameter $V$, become negative. Such pressure gradients may be conceived to produce back flow. In specific problems involving rolling, i.e. equal velocity of the walls of the channel, the effect of the back flow is to reduce the rate of flow. This might be conceived through the consideration of a positive displacement due to the rolling action of the liquid enclosed, which would be partly offset by the back flow due to pressure gradient effects. Further on, it may also be observed that the aforementioned trends of the results, including that of $Q$ and $V$ given by Figures 9.7a and 9.7b, can be conceived to be identical with respect to an imposed wall shear stress. This can be conceived by referring to the expression provided in Section 9.3.3, namely,

$$Q = -V_K$$

Thus the results so far expressed in terms of $K$ can be conceived to remain unchanged with respect to an imposed $Q$.

9.4 SOLUTIONS WITH SLOTTE'S RELATIONSHIP

9.4.1 Governing equation

One of the most general viscosity-temperature relationships, which represents a wide variety of liquids, has been pointed out in Chapter 2 to be Slotte's relationship. It has there also been pointed out that some of the synthetic oils such as silicone oil follow the linear fluidity function, which corresponds with Slotte's exponent $m = 1$, whereas the mineral oils can accurately be represented over a wide range of temperatures by higher values of $m$, namely,

$$\mu = \frac{\mu_0}{\kappa} \psi^m$$

where $\mu_0$ is the viscosity, $\psi$ being defined using the coefficient $\beta_a$ and the exponent $m$, by

$$\psi = 1 + \frac{\beta_a}{m} (T - T_0)$$

$\beta_a$ is considered to depend on the reference temperature $T_0$. The governing equation for the momentum can be expressed using the non-dimensional parameters defined already in the foregoing,

$$Y = \frac{y}{h}$$

$$P = \frac{\beta_a}{k} h^n$$

so that,

$$\Gamma^n = \frac{h}{k} \frac{\beta_a}{\mu} \frac{1}{\kappa} Y$$

In order to facilitate a comparison of the results yet to be obtained with those based on the exponential relationship, the parameter $P$ can be replaced, wherever necessary, by,

$$P_a = (1 + \frac{1}{m^n}) P$$

It may be noticed that the above alternative forms, i.e. $P$ and $P_a$, are similar to the operational parameters $N$ and $N_a$ that in Chapter 5 have already been specified for the shear flows. However, the results to follow have been expressed in terms of $P$ in order to facilitate a comparison of various results in the whole range of $m$, i.e. $1 \leq m < \infty$. The governing equation for the energy balance reduces to,

$$\frac{d^2 Y}{\alpha^2} + \frac{1}{m} \frac{d^2 Y}{\alpha} = 0$$

The boundary conditions reduce to,

$$Y = 0, \quad \psi = 1, \quad \frac{dy}{d\alpha} = 0$$

and can be replaced, wherever desirable, by,

$$Y = 1/2, \quad \psi = \psi_0$$

Solutions to the governing equation (9.22) display a considerably wide characteristic behaviour depending on the value of $m$. For the present, non-isothermal flows of liquids that interest us, namely those covered by $1 \leq m < \infty$, these solutions are bound on one side by that for
the exponential relationship, i.e., Slotte's one with \( m = \infty \) as analysed in Section 9.3. On the other side they are bound by the solutions for the linear fluidity function, which is equivalent to Slotte's with \( m = 1 \). The solutions for all positive finite values of \( m \) greater than unity fall in the regime covered by these two extremes. Having established the behaviour of the solution at one extreme, i.e., for \( m = \infty \), we may turn over to the other, i.e., for \( m = 1 \) for which the solutions are known to exist in an analytical form.

The equally important case of isoviscous flows, although relevant from the theoretical point of view, has not been included.

9.4.2 Solution with \( m = 1 \) in Slotte's relationship

For the linear fluidity function, which is equivalent to Slotte's relationship with \( m = 1 \), the governing equation for the conservation of energy reduces to a simple form. Hausenblas [9.11] has extended and critically analysed the solutions that had previously been obtained by Philippoff [9.9]. Regirer [9.10] has independently worked out the solution in a form similar to that worked out by the earlier investigators, namely in terms of fractional Bessel functions. These investigators, however, have not analysed the solutions over a wide operating regime. Such attempts have been initiated by Hostantshdni and Stolin [9.11].

The results can be evaluated over a wide range either by extending the analytical solutions of Hausenblas and others, or by independently working out the present governing equation. To ensure an easy comparison with the results yet to be obtained for the other values of the exponent \( m \), we will pursue the latter method. It may be noticed that this method involves the operational parameter \( P \) in terms of the viscosity at the axis of the channel whereas the analysis of the previous investigators involved the viscosity at the temperature of the wall.

The governing equations can be expressed, for the particular case of \( m = 1 \) in Slotte's relationship, through replacing \( P \) by \( L \) for clarity, and introducing,

\[
\Gamma = \frac{P_0}{k \mu} - L Y \psi (9.24)
\]

with which,

\[
d_x^2 \psi + L \frac{dL}{dY} \psi = 0 \quad (9.25)
\]

The dimensionless temperature distribution \( \psi \) can be expressed in terms of the Bessel functions of the fractional order \( 1/4 \), viz.:

\[
\psi \sim \left( C_1 J_1(h \sqrt{\nu^4}) + C_2 J_2(h \sqrt{\nu^4}) \right) (9.26)
\]

where \( J_1(c) \) and \( J_2(c) \) are definable, with \( \nu = 1/4 \) and \( \nu = -1/4 \) respectively, through the general series expansion,

\[
J_n(c) = \sum_{k=0}^\infty \frac{(-1)^k (c/2)^{2k}}{k! \Gamma(k+1)} \quad (9.27)
\]

The constant coefficients \( C_1 \) and \( C_2 \) can be evaluated using the "one point" boundary conditions at \( Y = 0 \), i.e., \( \psi = 1 \) and \( d\psi/dY = 0 \). Such an evaluation results in *,

\[
C_1 = 0 \quad (9.28)
\]

\[
C_2 = \Gamma(1)(\sqrt{\nu^4}) \quad (9.29)
\]

The solution for the distribution of temperature now reduces to a simple form, namely,

\[
\psi = \frac{1}{\Gamma(1)(\sqrt{\nu^4})} J_1(h \sqrt{\nu^4}) \quad (9.30)
\]

From the definition of the viscosity-temperature relationship the distribution of the viscosity becomes,

\[
\frac{\mu}{\mu_0} = \psi \quad (9.31)
\]

The viscosity \( \mu \) at the wall can readily be expressed in terms of the minimum one \( \mu_0 \) at \( Y = 0 \), through \( Y = 1/2 \), viz.,

\[
\frac{\mu}{\mu_0} = \psi \quad (9.32)
\]

Other thermo-hydrodynamic parameters can be evaluated as shown before in Section 9.3.3. They will, first of all, be expressed in a general form as follows. Wall shear stress:

\[
\tau_w = \frac{1}{h} \int_0^h \psi Y \, dY \quad (9.33)
\]

Velocity distribution:

\[
u = \frac{1}{h} \int_0^h \psi Y \, dY \quad (9.34)
\]

Volumetric rate of flow:

\[
Q = \frac{1}{h} \int_0^h \psi Y \, dY \quad (9.35)
\]

Heat flux distribution:

\[
q^* = \frac{1}{h} \int_0^h \psi Y \, dY \quad (9.36)
\]

Total heat flux:

\[
q^* = \frac{1}{h} \int_0^h \psi Y \, dY \quad (9.37)
\]

*The expression for \( C_2 \) contains a Gamma function \( \Gamma \) with an argument 3/4.
An evaluation of the results for the foregoing thermohydrodynamic quantities involves that of the indefinite integral,

\[ \int_0^\infty \frac{\psi \, d\psi}{\sqrt{\omega^2 - \psi^2}} \]

as well as the definite integrals,

\[ \int_0^\infty \frac{\psi \, d\psi}{\sqrt{\omega^2 - \psi^2}} \]

and

\[ \int_0^\infty \frac{\psi \, d\psi}{\sqrt{\omega^2 - \psi^2}} \]

A method of evaluation in terms of fractional Bessel functions has been provided in Appendix 9.A1.

9.4.2.1 Results for various thermohydrodynamic quantities

Temperature distributions for various selected values of \( L \) might be obtained by using tables for the fractional Bessel functions, see Jahnke, Emde and Losch [9.12]. Results have, however, been obtained to a better accuracy, and at arbitrary values of the modulus of the Bessel function, by making use of standard programs on a digital computer. These results, in terms of the viscosity ratio \( \frac{\eta_{\infty}}{\eta} \), have been presented in Figure 9.2. There it may be observed that the viscosity distribution looks like a quartic curve; it does not contain a point of inflection in the whole range of the pressure gradients, as represented by \( L \leq 257.61 \), or say where \( K = 90.37 \). The double-valuedness of the solution in terms of \( K \) is apparent from Figures 9.3 and 9.4. The extreme solution for \( m = \infty \), that for the exponential relationship, has also been presented.

It may be recalled that the solutions for all viscometrically non-anomalous liquids, those represented by the range \( L < \infty \), fall in the regime bound by the two above-mentioned solutions, i.e. for \( m = 1 \) and for the exponential relationship. The qualitative and quantitative results corresponding to these liquids will be discussed in Section 9.4.3.

From Figure 9.3 it may be observed that along the first branch of the solution \( K \) increases with increasing \( L \). The maximum value of \( K = 257.61 \) for \( m = 1 \) cannot, however, be surpassed. This value is lower than that, \( K = 90.37 \), for \( m = \infty \) which corresponds with the exponential viscosity-temperature relationship. With a further increase of \( L \) the corresponding value of \( K \) decreases, as is shown by the second branch of the solution. A similar double-valuedness of the viscosity ratio \( \frac{\eta_{\infty}}{\eta} \) is apparent from Figure 9.4. The velocity distribution evaluated from equation (9.29a) has been presented in Figure 9.6a where the previously defined parameter \( v \) has yet to be maintained: the corresponding definition of \( w \) now involves \( L \), i.e.

\[ w \sim L^4 \]

On grounds similar to those of the previous discussion in Section 9.3.4.2 the results presented in Figures 9.6a and 9.6b can be conceived to be most general for the case \( m = 1 \). At low values of the parameter \( L \) the corresponding velocity distributions closely resemble a nearly parabolic. At higher values of \( L \) the distribution shows a point of inflection. This point appears first at the wall, and moves away towards the interior of the channel as the value of \( L \) is further increased. The appearance of the point of inflection in the velocity profile has a certain significance for the stability of the flow, both thermally and hydrodynamically, as to be discussed in Section 9.5.

It may be observed in Figure 9.6a that the velocity distribution for the fluids with Slotte's exponent \( m = 1 \) tends to coincide with that for \( m = \infty \) except at high values of \( w \) close to the maximum permissible value.

The velocity distribution is existent only for finite values of \( w \). The results have been reduced to a convenient form along lines similar to those for the fluids with the exponential relationship explained in Section 9.3.4.2. For the present value of \( m = 1 \) the scaling factors employed are \( \eta_{\infty} = 1.2510 \) and the corresponding \( W_{\text{max}} = 0.0637 \) where the viscosity ratio \( \frac{\eta_{\infty}}{\eta} \) reduces to zero, or say where \( \eta_{\infty} = \infty \) or \( \eta = 0 \). Also here can the illustrative method used in Section 9.3.4.2. be used to compare the correspondence of the results in Figures 9.6a and 9.6b. For the specified \( m = 3.0 \) leading to \( w_{\infty} = w_0 = 1.5 \), the non-dimensional velocity from Figure 9.6a becomes \( U_{\infty} = 0.971 \). The reduced value of the velocity is \( U_{\infty} = 0.7754 \) and occurs at \( w_{\infty} = w_{\infty} = 0.3744 \). These results are readily found to agree with those in Figure 9.6b.

The ratio \( q/q_{\text{max}} \) between the thermally affected rate of flow, \( q \), and the isothermal one, \( q_{\text{is}} \), which may be considered a "correction" factor \( Q \), has been evaluated from equation (9.30b) through the previously mentioned numerical technique for the evaluation of the two integrals concerned. The results have been presented in Figure 9.7a. There it may be observed that as compared with the results for the exponential relationship, the flow becomes infinitely large at the finite asymptotic value of \( L = 257.61 \). The results for the rate of flow have been re-expressed in Figure 9.7b in terms of the flow rate with the velocity isothermal one, \( q/f/q_{\text{is}} \), which may be considered a "correction" factor \( Q_{\text{is}} \).

The question whether the second branch of the solution for \( V \) in Figure 9.7b, can be reached at all, and the corresponding question of stability will be discussed in Section 9.5. The trend for \( m = \infty \) is similar to the exponential, critical, case, \( m = 1 \), except that the \( V \) values on the second branch for \( m = 1 \) are distinctly lower than those for the corresponding branch for \( m = \infty \).

9.4.2.2 Solution beyond the critical value of \( L \)

The solution for the temperature distribution of a liquid with the linear fluidity function, or say with Slotte's exponent \( m = 1 \), displays a similar behaviour compared with that for a liquid with the exponential relationship, \( m = \infty \). This may be analysed through the solution for the viscosity ratio, \( \frac{\eta_{\infty}}{\eta} \). This ratio reaches zero at the critical value of \( L = 257.61 \) which in Figure 9.3 has already been shown to be the critical, maximum attainable \( L \) value for \( m = 1 \). This value of \( L \) yields a zero value for \( K \). From the characteristics of the fractional Bessel functions it may, however, be noticed that solutions still exist mathematically for values of \( L \) exceeding the above-mentioned critical value. There will correspondingly exist negative values of both \( \eta_{\infty}/\eta \) and \( K \). Due to the oscillating nature of fractional Bessel functions these two parameters however also oscillate, and reach again positive values at some higher values of \( L \). With any further increase of \( L \) the solution, and correspondingly the distribution of \( \eta_{\infty}/\eta \), continues to oscillate as shown in Figure 9.8. In that figure the results obtained from the tabulated fractional Bessel functions and those from the numerical integration have both been presented. Both results show an identical trend.
Figure 9.8 Viscosity variation $\nu/\nu_0$ for a wide range of $L$ both from the fractional Bessel functions and from the numerical integration. The abscissa:

$$w_j = -\frac{1}{L} \left( B - s^2 h_V u_H \right) J$$

at least for reasonably high values of $L$, but deviate at very high values of $L$.

The nature of the solution indicating negative viscosity has been considered by various investigators to be devoid of any physical significance. The solutions corresponding to the physically significant operating condition are bound by the aforementioned critical maximum value of $L=257.61$. Then the positive solutions appearing in the range $L<257.61$, see Figure 9.8, cannot be considered to be feasible. This is due to the fact that the range $L<257.61$ cannot be reached at all in the presence of a physically insignificant region $L>257.61$. Further on, no solutions beyond $L_k$ may be considered significant, although in this operating regime the numerical method has yielded continuously increasing positive values of viscosity with increased $L$ values.

A different meaning has also been attributed to the existence of such a range as $L>257.61$, where the viscosity reaches negative values. There may then exist a turbulent transport of momentum as explained by Starr [9.13]. The flow might continue to take place under the influence of the Reynolds stress $T_R$.

$$T_R = \frac{u \delta u}{\nu}$$

where $\nu$ tends systematically to negative values. The eddy momentum of $T_R$ is directed from regions of smaller values of mean flow towards those of larger values. This action is exactly opposite to that of the frictional drag effect, and is not amenable to treatment by analogy to molecular viscosity.

This regime of operation implies a turbulent flow, and thus the negative viscosity effects of Starr have to be considered applicable only in a limited sense. These effects consist of the destruction of the turbulent regime, and a return of the flow to a laminar one. Such a return might in fact prevail if a recovery of sufficient kinetic energy has taken place from eddies to the mean flow. The eddies might finally die out and ultimately yield a mean flow that is existent under the molecular viscosity.

It looks, from the above discussion, that the negative viscosity effects can be superseded by imposing higher values of $L$. Then the flow might return to a state where the solutions bounded in the region $L<257.61$ hold valid. But in the process of subjecting the flow to take place in this regime, the assumptions of laminar motion would be exceeded. The analysis with which the behaviour of the solution is established in the region $L<257.61$, may not be valid at all. It may thus be preferable to employ the present analysis only to the region covered by $L$ less than 257.61.

### 9.4.3 Solutions using some selected positive integer values of $m$

Most of the Newtonian liquids employed in lubrication practice can be characterized for the viscosity dependence on temperature by Slotte's relationship, see Chapter 2, and also definitions (9.20a) or (9.20b), viz.,

$$\nu_0 = \frac{\nu}{\nu_0} u_0$$

where the dimensionless temperature is given by,

$$\gamma = 1 + \frac{\nu_0}{\nu} (T - T_R)$$

The bounds of the range, $1 < m < \infty$ provide bounds also to the solutions for the usual, intermediate values of $m$. The solutions to be developed here below will yield results useful in the analysis of the thermal state and the hydrodynamic behaviour of the flow.

Solutions for the general problem specified through equation (9.22) have only qualitatively been analysed by a few investigators for a few values of $m$. Hausenblas [9.1] has made an attempt to graphically obtain the solution for liquids with a quadratic fluidity function, that is for $m=2$. He has also only indicated the solutions for temperature, velocity and rate of flow in terms of a series expansion. He has, however, not performed any evaluations. Joseph [9.3] has recovered Kaganov's [9.2] results for liquids also with a quadratic fluidity function. The latter investigators have bound the solutions from above and below by means of the maximum principle. Joseph has indicated the dependence of wall shear stress on the viscosity at the axis of the channel. So, all these results are only qualitative. They are indicative of a maximum wall shear stress occurring at a finite pressure gradient and also of the presence of double-valued solutions.
These solutions have been obtained from the aforementioned set of governing equations in a manner similar to that explained in Section 9.3.2 for the exponential relationship. The method of numerical analysis is of the "marching" type explained before, and involves a quadrature formula containing a predictor and a corrector stated as in expressions (9.8) and (9.9). The starting values have been obtained, as before, using Taylor's expansion. The coefficients of Taylor's expansion reduce to a simple form, and the starting values can be estimated using:

\[ v = 1 - \frac{1}{2} \beta \theta^2 \]

The evaluation of the starting values has, however, been explained in Appendix 10.A1 of the next chapter.

9.4.3.1 Results for various hydrodynamic quantities

A numerical method due to Milne has yielded the solutions for various values of Slotte's exponent \( m \). The relationship between the viscosity ratio \( \nu / \nu \) and \( w \), has been shown in Figure 9.9 for a few integer values of \( m \), i.e. \( m = 2 \) and 5. A common abscissa involving \( P \) has been chosen where \( m \) is implicitly involved through the coefficient \( \beta \) in the definition of \( P \) as shown in Section 9.4.1. Representation of the results has been made in a convenient form covering the complete range of \( P \) that is physically realizable for the particular value of the exponent \( m \) to be considered.

The double-valuedness of the solution is apparent if one analyses the behaviour of the parameter \( K \) involving both the viscosity \( \nu \) at the channel wall and the pressure gradient \( g \):

\[ K = \frac{\beta \nu^p}{\nu} \]

Results covering a wide range of \( P \) in which the viscosity ratio \( \nu / \nu \) remains positive have been presented in Figure 9.10. An alternate representation with \( P \), and \( L = P \) with \( m = 1 \), would depict that the curve for the exponential case, i.e. \( m = 1 \), encloses all the others from below instead of from above as shown in Figure 9.10. Such a representation can be obtained, if desired, from the same figure.

There it may be observed that for any specified non-zero value of \( K \), which does not exceed a definite maximum, there exist two finite values of \( P \) corresponding to the two branches of the solution. The second branch approaches asymptotically finite maximum values of \( P \) at the highest possible amount of viscous dissipation for all liquids with finite \( m \). At such a limiting operating condition the parameter \( K = 0 \); also the viscosity ratio by definition, i.e.

At such operating conditions increased values of \( m \) result in increased \( P \). For liquids with the exponential relationship, however, the maximum permissible value of \( P \) is only infinite at the limiting condition \( K \) and there \( \nu / \nu \) becoming zero; such an investigation has not been presented here due to lack of space.

It is apparent that the operation of thermally affected plan-parallel pressure flows can easily be reached along the first branch by exercising a control on one or more influential quantities in the parameter \( K \). On the contrary, the operation along the second branch can only be realised through performing a direct or indirect control along the following lines: by imposing a very small value of \( \nu \), and thus a very high \( P \), or by obtaining a very large amount of throughput; see Figure 9.7b. A method of performing either of the above controls has already been mentioned in Chapter 8 (cf. Section 8.4.1). It may then be concluded that, in principle, it is possible to obtain a steady-state flow along either the first or the second branch of the solution.

Non-dimensionalized distributions of the velocity have been plotted on a common abscissa in Figure 9.11 as an extension to Figure 9.6b. It may be noticed that at values of \( w^* \) approaching unity, i.e. high values of \( P \) where \( \nu / \nu \) becomes small, the flow takes place mainly in the middle of the channel. Such a behaviour implies that the flow is least affected by thermal effects near the channel walls. This is especially so for liquids with high values of \( m \) in Slotte's relationship.
DISCUSSIONS ON THE STABILITY OF FLOW

This section will deal with the stability both from the thermal and hydrodynamic point of view. It will cover the complete range of viscosity-temperature relationships, that is, the range 1 ≤ m ≤ ∞ in Slotte's relationship. Thermal instability of the kind suggested by Blok (9.14) and discussed in detail in Chapter 3 (see Section 3.3.1 on shear flows) is not directly relevant to the present problem. This is due to the fact that the present problem is thermally symmetric in that the two boundaries show equal temperatures and, accordingly, equal partitioning of the heat generated. Such a problem is thus devoid of the thermal instability proposed by Blok.

The present instabilities belong either to the thermal or the hydrodynamic category (or mode), as specified in Chapter 3, Appendix 3.A2. One is due to effects of temperature distribution on the viscosity, and can be named 'thermal mode'; the other concerns the point of inflection in the velocity profile and is named 'inviscid mode'. The latter type of instability is qualitatively identical to the hydrodynamic instability since this satisfies the requirement of the inflection-point criterion that is basic to the existence of the instability in isoviscous flows.

9.5.1 Hydrodynamic instability

The information available on the flow instabilities of thermally affected plan-parallel shear flows may be considered fairly complete, see Sukanek et al. [9.5] and also see Chapter 3, Section 3.3.4. On the contrary, the results available in literature on a similar study of the stability of thermally affected plan-parallel pressure flows are scarce.

For plan-parallel pressure isoviscous flows, when ignoring temperature perturbations, Lebon and Nguyen [9.15] have established the critical Reynolds number Re to be 5727.2, which corresponds to a wave number $\alpha_c$ = 1.02. Similar results have also been obtained by Platten et al. [9.16]. However, hydrodynamic instability established with the assumption of isothermal, that is, isoviscous flows is considered to be of limited interest to the present study of thermally affected lubricant films.

9.5.2 Thermal instability

Thermal instability which might lead to a flow failure is highly important for the present investigation. Two aspects concerning the solution that we have previously obtained seem to provide criteria for the thermal instability, these will be discussed in what follows.

9.5.2.1 Maximum value of the pressure gradient parameter K

It has already been pointed out that the solutions for the distribution of velocity and temperature exist only when the pressure gradient parameter $K$ does not exceed a definite maximum value. This limiting value has been shown to depend on Slotte's exponent $m$. For increasing value of $m$ this value increases, as indicated in Table 9.1. It may, however, be noted that the tabulated results, especially for the exponential relationship, qualitatively agree with that predicted by Joseph [9.3] through the maximum principle. For all values of $K$ less than the limiting value the solutions are double-valued.
would have reached an infinitely high level, the latter represented by wall shear stress and $K$ decrease. In the limiting case of the zero value of $K$, the transition from the first to the second branch of the solution represents a critical state characterized by an instability already at low Reynolds numbers. Similar behaviour of the plan-parallel pressure flows can only be inferred due to its similarity with that of the pipe flows. It may thus be concluded that there exists some danger of an instability due to thermal effects already at the beginning of the second branch of the solution.

Table 9.1 Maximum attainable values of the parameter $K$ and the corresponding Reynolds numbers for different values of Slotte's exponent $m$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$D$</th>
<th>$P$</th>
<th>$K$</th>
<th>$V$ ($\mu/\mu_1$)</th>
<th>$Re$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>115.74</td>
<td>54.50</td>
<td>0.9956</td>
<td>0.4709</td>
<td>0.14693x10^6h</td>
</tr>
<tr>
<td>2</td>
<td>175.6</td>
<td>88.11</td>
<td>1.2321</td>
<td>0.3878</td>
<td>0.14693x10^6h</td>
</tr>
<tr>
<td>3</td>
<td>256.0</td>
<td>79.36</td>
<td>1.4220</td>
<td>0.310</td>
<td>0.14693x10^6h</td>
</tr>
<tr>
<td>4</td>
<td>323.20</td>
<td>90.37</td>
<td>1.673</td>
<td>0.2796</td>
<td>79.674x10^5h</td>
</tr>
</tbody>
</table>

The criterion that applies to the instability to be discussed below is attributed to the previously mentioned 'thermal mode'. On grounds similar to those used for thermally-affected shear flows in Chapter 3 the characteristic nature of the solution can be discussed. Along the first branch of the solution depicted in Figures 9.4 and 9.10 the velocity and temperature, as well as the wall shear stress, increase with increased value of $K$. This can be conceived firstly from the trend of the first two parameters in the form of $\mu/\mu_1$ and $\mu/\mu_1^2$ in Figures 9.11 and 9.9, and secondly by noting that the same parameters increase in magnitude with increased $\psi_1$ and $\psi_2$ through the corresponding values of $P$. Further, $K$ being an increasing function of $P$ along the first branch of the curves in Figure 9.10 and also related to the wall shear stress through $P_{\text{crit}}$ (see Section 9.3.1.1), the aforementioned flow parameters can be conceived to increase with increased $K$. Along the second branch, however, the velocity and the temperature continue to increase while the wall shear stress and $K$ decrease. In the limiting case of the zero values of the wall shear stress and $K$, the velocity and temperature values would have reached an infinitely high level, the latter represented by $\mu_1/\mu_{\text{crit}} = m$, see Figure 9.4. A discussion on this limiting operating condition corresponding to $\mu_1/\mu_{\text{crit}} = m$ or $\mu_1/\mu_{\text{crit}} = 0$ may be found in Section 9.4.3.1.

Table 9.2 Position of the point of inflection in the velocity profile for the complete range of $m$ in $1 \leq m \leq \infty$, and the corresponding hydrodynamic parameters.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$P$</th>
<th>$K$</th>
<th>$V$ ($\mu_1/\mu_{\text{crit}}$)</th>
<th>Critical Re</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>41.62</td>
<td>33.01</td>
<td>0.7499</td>
<td>0.7930</td>
</tr>
<tr>
<td>2</td>
<td>39.69</td>
<td>32.22</td>
<td>0.7370</td>
<td>0.8118</td>
</tr>
<tr>
<td>3</td>
<td>33.18</td>
<td>28.01</td>
<td>0.6810</td>
<td>0.8442</td>
</tr>
<tr>
<td>4</td>
<td>53.93</td>
<td>41.37</td>
<td>0.8412</td>
<td>0.7669</td>
</tr>
</tbody>
</table>

9.5.2.2 Point of inflection in the velocity profile

Great importance has been devoted in literature to the existence of a point of inflection in the velocity profile of plan-parallel pressure flows. It has long been known that the appearance of such a point of inflection of the inviscid (viscosity independent of temperature) velocity profile indicates an instability. Similar considerations, viz., those of the existence of a point of inflection in the velocity profile that is modified by viscous dissipation effects, have been used by various investigators to infer an instability. Due to such considerations this analysis can be identified somewhat equivalent to that of the hydrodynamic instability discussed in Section 9.5.1. In order to distinguish between these two analyses, namely the hydrodynamic and thermal both based on the point of inflection, the present instability under consideration can be identified as due to the 'inviscid mode', nevertheless under the influence of the thermal effects. However, in pressure flow in pipes the occurrence of a point of inflection, as has already been pointed out in Chapter 8, is rather unimportant, such a flow nevertheless being stable.

For a symmetric velocity profile, such as that in plan-parallel pressure flows with equal boundary temperatures, a drastic destabilisation has been proved by Vanderborck and Platten [9.17]. Their analysis is based on an assumed quartic-symmetric velocity profile with two points of inflection. The computed results indicate a low critical Reynolds number of even less than 800 corresponding to a wave number $c_{\text{crit}}$=1.0. As the points of inflection are forced to move towards the centre of the channel by increasing $P$, the Reynolds number at which the instability appears decreases, especially for high wave numbers. As long as the basic flow is of a stable type, the local instability produced by a perturbation might soon die away (see Joseph [9.18]). While the flow instability under the above arbitrarily assumed velocity profile is not expected to be prevalent, the steady solutions of a viscous flow need a careful analysis. Typical velocity profiles are a natural consequence of the dissipation effects, and they are not confined to a local region. They may stretch uniformly even throughout the length of the flow region. Then the influence of a steady velocity profile, including that with a point of inflection, on the stability of flow needs a careful investigation. Such an investigation, if carried out, might yield results showing the instability of flow due to the presence of points of inflection in the velocity profile. An equivalent type of flow instability due to the presence of a point of inflection in the velocity profile,
Although only for shear flows, had Sukane et al. [9.5] to conclude that the critical Reynolds numbers giving rise to an instability can even be quite low. For the plan-parallel pressure flow under study, there exist two points of inflection in the velocity profile, and this is quite different from the type of flow analysed by Sukane et al. To the author’s knowledge there is no published data available.

Somewhat relevant data are due to Mott and Joseph [9.19] for essentially plan-parallel flows. By considering the pressure flow between two concentric cylinders, and the corresponding velocity profile as affected by viscous dissipation, they specify the critical Reynolds numbers. For ratios of the outer to the inner cylinder close to unity the annulus flow approaches the plan-parallel pressure flow. The resulting critical Reynolds numbers, which correspond to the point of inflection in the velocity profile, are identical with those for the inviscid case, namely approximately 5800. Lacking detailed information, the presence of points of inflection in the velocity profile may in general not be wholly ignored. On the same grounds that have been used by Sukane et al. [9.5] to infer the instability of the shear flows, it may be concluded that there might exist a danger of an instability even for plan-parallel pressure flows; such an instability might arise for the latter flows already at rather low Reynolds numbers.

From the computed results we may easily determine pressure gradient parameters such as P, L and K, at which a point of inflection in the velocity profile appears. Such points of inflection are identified by,

\[ \frac{d^2u}{dy^2} = 0 \]

Lowest values of P at which the point of inflection appears have been determined from the tabulated results for the gradient of the temperature as shown in Appendix 9.A2. Such values of P for liquids in the range of Slotte’s exponent \( m = \infty \) can then be shown to be represented by,

\[ P_{inf} = \left( \frac{v}{L} \right)^{n} \]  

(9.34)

For the exponential viscosity-temperature relationship, or say, \( m = \infty \), the latter point of inflection appears at the channel wall for the pressure gradient that gives rise to \( \text{Kinf} = 53.93 \) or \( \text{Kinf} = 41.37 \). Table 9.2 indicates the position of the points of inflection and the values of the corresponding pressure gradient parameters for the complete range of m, i.e. 1 \( \leq m \leq \infty \). The results have been obtained through the same numerical procedure as the one for obtaining the previously mentioned solutions. An analytical form for P in the equality (9.34) providing the point of inflection in the velocity profile can, however, be readily obtained for the linear fluidity function, namely that with \( m = 1 \) in Slotte’s relationship. In this case \( L \) replaces P whilst \( \psi \) has already been evaluated in terms of the fractional Bessel functions as shown in equation (9.27). For the specific case of the point of inflection appearing at the wall, \( y = 1 \) in that equation must be replaced by \( 1/2 \). Then the gradient \( d^2u/dy^2 \) can be evaluated using known properties of Bessel functions and the value of L is obtained from Appendix 9.A2.

\[ L_{inf} = \frac{J_{1/2}(V/L)}{E_{1/2}(V/L)} \]  

(9.35)

where

\[ J_{1/2}(V/L) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2} \sin(xV/L) \, dx \]

The results thus agree with those obtained for the present \( m = 1 \) when using the numerical methods discussed in Section 9.4.1. The values of \( L_{inf} \) and the corresponding \( \text{Kinf} \) have been found to be 41.62 and 33.01.

A comparison of other results of the present kind of numerical evaluation namely those for \( m = 2 \) and \( m = \infty \), have been made with those obtained by Joseph through the maximum principle. The data for water at 10°C have been employed in the evaluation of \( \text{Re} \); the specifications for oil representing a typical lubricant have been taken from an oil manufacturer’s handbook. These data may be found in Section 8.4.2. The corresponding Reynolds numbers based on isoviscous flow can be reduced to an equivalent form; for the exponential relationship,

\[ \text{Re} = \frac{\mu \cdot \nu \cdot h}{\mu_0 \cdot \nu_0 (k/b) \cdot h \cdot v} \]

With Slotte’s relationship in the above expression should be replaced by \( \psi \). The critical values of Re at which there appears a point of inflection in the velocity profile have also been included in Table 9.2. Similar values of Re for the maximum attainable value of the parameter K have been tabulated in Table 9.1. It may be noticed that the critical Re contains thickness \( h \) of the liquid film as a free parameter. This implies that, with film thickness as they occur in hydrodynamic lubrication, the critical Re can attain very low values indeed. Such critical values of Re also imply that there need exist a meaningful inviscid limit represented by \( \text{Re} = \infty \) or even a limiting value of \( 577.7 \) representing a hydrodynamic instability; the critical Re might even be finite. In the above expression for Re there appears \( \mu \cdot \nu \cdot h \) this implies that the finite value of Re, due either to the point of inflection in the velocity profile or to the maximum permissible value K, might become very low provided the oil selected be viscous enough.

9.6 CONCLUSIONS

Analytical solutions obtained for plan-parallel pressure flows and accounting for the viscosity variation with temperature indicate a thermo-hydrodynamic behaviour similar to that of the axisymmetric flows in pipes. The solutions expressed in terms of a pressure gradient parameter \( P \) involving the minimum viscosity \( \mu_{0} \), that at the axis of the channel, exhibit a behaviour which is qualitatively comparable with that of the flows in pipes. The solutions are single-valued and available for a wide operating regime including the limiting one of infinitely high temperature, or say zero viscosity, at the channel axis, i.e. \( \mu_{0}/\mu_{1} \cdot 0 \).

High values of P are attainable at the limiting condition of \( \mu_{0}/\mu_{1} = 0 \) for liquids with increased exponent \( m \) of Slotte’s viscosity-temperature re-
relationship. Such limiting values of $P$ are finite for finite values of $m$; $P$ approaches an infinitely high value only for $m = m^*$, that is, with an exponential viscosity-temperature relationship. However, solutions are non-existent beyond a maximum value of the pressure gradient parameter $K$, which involves the viscosity at the wall and (in contrast to $P$) not at the axis of the channel. The maximum values of $K$ increase with increased Slotte's exponent $m$. They are, nevertheless, finite for all values of $m$ including $m^*$.

The solutions in terms of $K$ show a characteristic behaviour that is identical with that of the axisymmetric pressure flows in terms of $K$ dealt with in Chapter 8. They are also identical with those of shear flows in terms of the shear stress parameter used in Chapters 3 and 5. For all $m$-values, except $m = 1$, all these solutions are characterised by double-valuedness. The plan-parallel pressure flows, like the axisymmetric ones, exhibit double-valuedness also for the particular case of $m = 1$. These solutions differ from those of shear flows for the same particular case of $m = 1$. That is, the latter flows do show single-valued velocity and temperature distributions for any finite, whatever high shear stress level.

It may now be concluded that in plan-parallel pressure flows the instability due to viscous dissipation effects may well develop. On the basis of the analysis of Joseph [9.3] this may be expected to occur at operating conditions that yield a transition point to the solution corresponding to the maximum attainable value of $K$. His results may have to be considered to be somewhat limited in their applicability. After all Joseph ignored the coupling of the momentum and energy equations. An improved analysis, if only on a numerical basis, but accounting for the coupling, is needed. Considerable importance has been attached in literature to the existence of a point of inflection in the velocity profile. One of the most basic data for such an analysis are those of a plan-parallel limiting case of the pressure flow in the annulus between two concentric cylinders. The stability of thermally and kinematically symmetric flows is limited by finite but not very low critical Reynolds numbers. A detailed analysis has been carried out only for shear flows, i.e. by Sukanek et al. [9.5] and Vanderborck and Platten [9.17]. For the plan-parallel pressure flows such an analysis, once carried out, might yield very useful quantitative data for the operating conditions specifying the occurrence of thermal instability.

### REFERENCES


Chapter X
NON-ISOTHERMAL PLAN-PARALLEL GENERAL FLOWS

10.1 INTRODUCTION
Analyses concerning viscous dissipation effects in plan-parallel flows operating under combined shear and pressure gradient are rare in literature. These flows, here called 'general flows', may be conceived to consist of two component flows: a shear flow that can be ascribed to the motion of the two boundaries at unequal velocities, and a pressure flow that results from the pressure gradient imposed. The behavior of the general flows may then be conceived to reflect that of the component flows: a transition of the solution at certain operating region of operation as pointed out in Chapters 3, 5 and 9.

Solutions explicitly displaying the above-mentioned double-valuedness of the problem under study have not been found in literature, not even for the simple thermal boundary conditions of the two particular types discussed before: one, where both walls absorb equal amounts of heat at uniform and equal temperatures; the other, where one wall absorbs all the heat at some uniform temperature whilst the other is adiabatic. There appears only one such solution, i.e. for the special case, where the fluidity function is linear so that in Slotte's relationship the exponent $m=1$. This is due to Bostandzhyan and Stolin [10.1] and has been worked out as an extension to Regier's [10.2]. Even their solution, however, does not explicitly indicate the aforementioned double-valuedness. This may perhaps be attributed to the inherent nature of the component solution, namely that representing the single-valuedness of the pure shear flow, which has already been discussed in detail in Chapter 5.

Although Bostandzhyan and Stolin subjected their solution to an extensive analysis it is still incomplete in that they did not uncover the double-valuedness of the pressure flow component in terms of the present pressure gradient parameter, which involves the viscosity at the wall temperature. Their results indicate a limiting value of the pressure gradient parameter consisting of the minimum viscosity in the channel that cannot be surpassed by the flow. Their analysis, certainly in various special cases, seems to be in agreement with that of Haussenblas [10.3]. In fact, it accounts also for the most general thermal boundary conditions.

In the analysis to follow, the most general type of laminar, incompressible and inertialless flows in oil films will be covered. It will include the effects of motion of the two surfaces that enclose the film as well as those of the pressure gradients, if any. It will include also the most general thermal boundary conditions in a form explicit in terms of steady wall temperatures. Implicitly prescribed wall temperature-gradients that are useful to evaluate the heat partition ratios will also be dealt with in the analysis to follow. Attention will be drawn to the limiting operating conditions that may yield an instability of thermal or of hydrodynamic type. The physical significance of such an instability will also be discussed in the light of proposals of various investigators, at least separately for the component pure shear and pressure-gradient assisted flows. The findings in the present chapter, like those in Chapters 3, 5 and 9, are applicable not only to all successive flow cross-sections of plan-parallel flows when the distributions of the temperatures of both surfaces are uniform, but more generally even to each individual cross-section of lubricant films (e.g. thermoplastic/plastohydrodynamic ones) that are not plan-parallel, and of which the distributions of the surface temperatures are not uniform. The analysis and results to be presented cover all Newtonian liquids that are characterized by Slotte's viscosity-temperature relationship (for a detailed discussion of this relationship, see Chapter 2).

10.2 GOVERNING EQUATIONS

The analysis to follow covers the regime of sliding, including that of pure rolling, of elastohydrodynamic as well as plastohydrodynamic lubricant films. The thermal boundary conditions to be specified are general in nature, and include equipartitioning of the heat. The analysis, however, is similar to that presented for the plan-parallel pressure flows in Chapter 9.

The momentum balance equation may be specified in a differential form for the plan-parallel pressure flow including sliding of the bounding surfaces, i.e.,

\[
\frac{dU}{dy} = \frac{dp}{dx} \quad (10.1a)
\]

Since \((-dp/dx)g\) is independent of \(y\), integration with respect to \(y\) yields,

\[
\tau = \int g(y_g - y) \quad (10.1b)
\]

where \(y_g\), serving as an integration constant, can be suitably adapted to the following condition. First observe that Newton's definition of viscosity, viz.,

\[
\tau = \rho \frac{du}{dy} \quad (10.2a)
\]

can be written in the form,

\[
\frac{du}{dy} = \frac{\tau}{\rho} \quad (10.2b)
\]

Now with the consideration,

\[
\frac{1}{2} \int_1^b \left( \frac{du}{dy} \right)^2 dy = \int_0^u \frac{du}{dy} \quad (10.2c)
\]

the condition concerned follows from the integration of the right-hand side of equation (10.2b), viz,

\[
\frac{1}{2} \int_1^b \frac{du}{dy}^2 = \frac{1}{2} \left( \rho \frac{dp}{dx} \right) \int_0^u \frac{du}{dy} \quad (10.2d)
\]
Here $\eta_0$ is the minimum viscosity at any maximum temperature yet to be specified for the film cross-section to be considered. Once the relative fluidity $(U_H/\eta)$ has been determined as a function of $y$ for the viscosity-temperature relationship employed, and for the thermal boundary condition, $\eta_0$ may be adapted by evaluating the integral in conditions considered. This may be observed in this connection that at the point $y = y_R$ the shear stress vanishes by definition (see relationship (10.2a) or (10.2b), involving that $d\xi /dy = 0$ at $y = y_R$. At this same point $\eta_0$ the distribution of the fluid velocity $u(y)$ will show an extreme, and furthermore, the dissipation heat $q$ generated per unit volume and per unit time at this point is zero, defined (see relationship (10.2a) or (10.2b)), involving that $du/dy = 0$ at $y = y_R$. In such cases, although $y_R$ is then devoid of any physical significance, it still helps for analytical and computational purposes.

A similar observation can also be made with respect to still another device which is due to Hagg [10.4], namely that of a characteristic point $y = y_H$ where the distribution of temperature, and thus also that of viscosity, is at an extreme. Of course, Bostandzhiyan and Stolin's characteristic point $y = y_R$ and that of Hagg $y$ will in general coincide, except in the case of both thermally and kinematically symmetric flows (cf. Figure 10.1a). The general case of thermal and kinematic asymmetry has been shown in Figure 10.1b. First, observe that with the introduction of Bostandzhiyan and Stolin's characteristic point $y_R$, the dissipation heat $q = T/\eta$ generated per unit volume and per unit time can now be given by,

$$\frac{d}{dy} \left[ \frac{T}{\eta} \right] = 0$$

The governing energy equation for a steady-state flow of an incompressible liquid with a constant thermal conductivity $k$, under the assumption that the heat transfer takes place only by cross-conduction of the heat generated by viscous dissipation, can be formulated by,

$$\frac{d}{dy} \left[ \frac{k}{\eta} \right] = 0$$

Thermal boundary conditions are here suitably made to relate mainly, or even exclusively, to Hagg's point where the distributions of temperature and viscosity reach an extreme. In any case, at such a point $y_R$, the temperature attains a maximum. In fact, the referential viscosity $\eta_0$ mentioned before may be assumed to be the minimum one corresponding to this maximum temperature. The corresponding boundary conditions may be specified by,

$$y = y_R, \quad T = T_R \quad (or \quad T = 0)$$

Thermal boundary conditions are here suitably made to relate mainly, or even exclusively, to Hagg's point where the distributions of temperature and viscosity reach an extreme. In any case, at such a point $y_R$, the temperature attains a maximum. In fact, the referential viscosity $\eta_0$ mentioned before may be assumed to be the minimum one corresponding to this maximum temperature. The corresponding boundary conditions may be specified by,

$$y = y_R, \quad T = T_R \quad (or \quad T = 0)$$

Thermal boundary conditions are here suitably made to relate mainly, or even exclusively, to Hagg's point where the distributions of temperature and viscosity reach an extreme. In any case, at such a point $y_R$, the temperature attains a maximum. In fact, the referential viscosity $\eta_0$ mentioned before may be assumed to be the minimum one corresponding to this maximum temperature. The corresponding boundary conditions may be specified by,

$$y = y_R, \quad T = T_R \quad (or \quad T = 0)$$
although, like $T_B$, it may vary with the pressure in the film cross-section concerned. The viscosity-temperature relationship can be expressed using the dimensionless temperature $\Theta$, viz.:

$$u = u_B e^{-\Theta} \quad (10.6)$$

where $\Theta = \frac{T - T_H}{T_H - T_H}$. Here the viscosity $\mu$ is defined with respect to the reference temperature $T_H$ occurring at the above defined position $y-y_H$. Introducing the non-dimensional quantities, viz.:

$$Y = \frac{y}{h} \quad (10.7a)$$

and

$$\zeta = \frac{1}{h^2} (y - y_H) \quad (10.7b)$$

where $P$ is defined, as before, in terms of the pressure-gradient $g$ to which the flow is subjected and of the viscosity $\mu$, i.e.

$$g = \frac{\delta}{\delta y} \left( \frac{y^2}{h^2} \right) \quad (10.7c)$$

One may employ a convenient non-dimensional variable quantity $w$ which enables to obtain the results in a general form with which a wide range of operating conditions can easily be covered through one single set of results. In addition, this form also excludes the necessity of representing the results separately for various sets of thermal boundary conditions. The variable $w$ is defined by,

$$w = \zeta - \zeta_{HB} = \frac{1}{h^2} (Y - Y_H) \quad (10.8a)$$

where in accordance with definition (10.7b),

$$\zeta_{HB} = \frac{1}{h^2} (Y_H - Y_H) \quad (10.8b)$$

It may be noticed that the variables $w$ and $\zeta$ are somewhat similar. The former indicates the location of any point with respect to the position of maximum temperature in a film cross-section, and the latter, i.e. $\zeta$, the location with respect to the position of maximum velocity in the same film cross-section under consideration. The parameters have been also graphically displayed in Figure 10.1. In the analysis to follow, most of the solutions will be obtained in terms of $w$, except for one particular case where the use of $\zeta$ proves to be more convenient.

The governing momentum and energy equations can now be expressed in terms of $w$ as the independent variable,

$$\Gamma_w = \frac{1}{h^2} (w - w_H) \quad (10.9)$$

and

$$\frac{d^2}{dw^2} \left[ (w + \zeta_{HB}) \right] e^\Theta = 0 \quad (10.10)$$

The corresponding "one-point" set of thermal boundary conditions (10.5) reduces to,

$$w = 0, \Theta = 0 \quad (or \Theta = 1), \frac{d}{dw} w = 0 \quad (10.11)$$

The present formulation of the problem through expressions (10.9), (10.10) and (10.11) is more general than the symmetric one analysed in Chapter 9 for plan-parallel pressure flows. In fact, it accounts for the most general cases, those where the dissipative heat is no longer partitioned equally between the two bounding surfaces at $y=\pm h/2$, and where the latter surfaces may have different velocities, $u_2$ and $u_1$, respectively. It may also be noted that the surfaces No.2 and No.1 are now specified at the positions,

$$w_2 = \frac{1}{h^2} (1 - Y_H) \quad (10.12a)$$

and

$$w_1 = \frac{1}{h^2} (-1 - Y_H) \quad (10.12b)$$

Of course, the thermally symmetrical problem is a particular case of the above general formulation and it can readily be solved by observing that $Y_H=2=CUB=0$. An analysis of this particular case has, in fact, already been carried out in Chapter 9. Further, let us devote some attention to the curvature $d^2w/dw^2$ at $w=0$ of the curves $G(w)$ at their apices, i.e. at their maxima $\Theta=0$ at $w=0$. It may readily be found from the general differential equation, i.e. equation (10.10) that this curvature is given by,

$$\frac{d^2w}{dw^2} = \frac{c_n}{h^2} \quad (10.13)$$

Accordingly, in all the foregoing expressions where $c_n$ appears, one may replace the latter through the above expression in terms of the curvature, viz.

$$c_n = \frac{1}{h^2} \left[ \frac{d^2}{dw^2} w, \Theta = 0 \right] \quad (10.14)$$

The general differential equation, if so desired, can also be expressed by,

$$\frac{d^2}{dw^2} \left[ (w + \zeta_{HB}) \right] e^\Theta = 0 \quad (10.15)$$

It may also be observed that the choice of the apex curvature involves the choice of the difference $(Y_H-Y_H)$ between the characteristic coordinate of Hagg, and that of Bostandzhiyan and Stolin. For any liquid following the exponential viscosity-temperature relationship, various degrees of thermal and kinematic asymmetry can be studied by properly selecting the curvature at the apex.

10.3.2 Solutions to the equations

Solutions in analytical form for the general differential equation (10.10), have not been found in literature. The difficulty encountered in analytically obtaining such a solution is similar to that in the case of plan-parallel pressure flows in Chapter 9.

One of the important numerical integration techniques based on the quadrature principle and well suited to second-order equations namely
Milne's "Predictor-Corrector" method has been employed in the present analysis. This method has certain advantages somewhat similar to those of finite difference methods. The results at each step are determined as accurate as desired before proceeding with the following step. This method involves, as explained in Appendix 10.A1, predicting results at a new point by means of a formula which does not require the knowledge about the point concerned, but of four preceding "pivotal points". First at the new point concerned the original differential equation is applied to obtain the second derivative, and then using this derivative and a corrector formula, new results are calculated. The latter step is repeated several times until the successive estimation of the function shows a deviation deemed small enough. Some of the predictor-corrector methods utilise the information from the correction terms associated with the predictor and corrector expressions to terminate the numerical iteration process.

10.3.3 Formulation of a computational programme

Four starting values are required for the general problem, which involves the differential equation (10.10). They include the boundary values at \( w_0 \) and may not fall beyond the range covered by \( w_0 \) \& \( w_4 \), where \( w_0 \) and \( w_4 \) are the dimensionless coordinates specified in Appendix 10.A2. The computational programme thus selected on the basis of Milne's method consists of both a predictor, viz.:

\[
0_{n+1} = (1 - a_0 - a_1) \frac{2a_0}{6} \left[ 50y_0 + (1 - a_1) 50y_1 + \frac{17}{240}(\omega \theta) \right] (10.15)
\]

and a corrector, viz.:

\[
0 = 20 - 0 = \frac{(1 - a_0 - a_1) \frac{2a_0}{6} \left[ 50y_0 + (1 - a_1) 50y_1 + \frac{17}{240}(\omega \theta) \right]}{n+1} (10.16)
\]

The last terms in both the predictor and corrector formulae are error terms, whilst both can be evaluated by making use of the sixth differential of \( \theta \). This differential of \( \theta \) can be found in Appendix 10.A3.

The computation consists of an estimation of the solution for the dimensionless temperature \( \theta \) with the help of the above predictor formula followed by a refinement with the help of the corrector formula. The previously mentioned four starting values serve as initial estimates in a relatively narrow neighbourhood of the apex of the curve indicating the distribution of \( \theta \) with \( w \). For an estimate of the solution one only need have at hand both the solution and its second derivative in the preceding steps, and which are already known from the governing differential equation. For evaluating the error terms in both the predictor and corrector formulae, however, the sixth derivative is needed. It may be noticed that the first derivative is not involved during the computation. This is partly attributable to the nature of the governing differential equation in that it does not contain the first derivative. It may further be attributed to the characteristics of Milne's method which is here employed in arriving at the predictor and corrector formulae.

An evaluation of the first derivative of \( \theta \), is required especially when one intends to evaluate the heat partition ratio throughout the range of \( w \) being studied. This first derivative, wherever the solution for \( \theta \) has been evaluated as shown above, can be obtained through either of the following methods. One consists in employing the predictor and corrector formulae along lines similar to those used to evaluate \( \theta \). The second consists in an integration of the present differential equation (10.10).

The first derivative, and thus the heat partition ratio can, therefore, be evaluated by means of the numerical integration of the original differential equation, that is equation (10.10). Any quadrature method can be employed to perform the required integration of the form,

\[
\theta' = \frac{\partial \theta}{\partial w} (10.10)
\]

This second method of evaluating the first derivative \( \theta' \) may be considered to be straightforward. The accuracy expected of the results is not inferior to that of the previously explained predictor-corrector method.

10.3.4 Other thermohydrodynamic quantities

It is apparent from the previous section that the solution to the energy equation for thermally steady-states involving heat generation by viscous dissipation and heat transfer by cross-conduction can be obtained quite accurately using Milne's predictor-corrector method. Of course, the particular case of thermal and kinematic symmetry is included in the general solution. For such a symmetry \( \mu_0 \mu_0 = 0 \) and \( \mu_0 \mu_0 = 0 \), the latter implying that the curvature at the apex of the distribution of \( \theta \) (see equation (10.13)) reduces to zero.

The solution \( \theta \) obtained in Section 10.2.1., and its first derivative \( \theta' \), serve as basic data through which various thermohydrodynamic quantities, particularly those of interest in lubrication problems, can be evaluated. It is deemed convenient to express these quantities in terms of \( P \), the non-dimensionalized pressure gradient which involves the theoretically minimum viscosity \( \mu_0 \) (see definition (10.7c)). By doing so it is possible to cover the entire operating regime since this is single-valued in \( P \). Further on, the distribution of each of the thermohydrodynamic parameters involving \( P \) can be characterized by one single curve, which represents, as explained in Chapter 9, (cf. Section 9.3.4.2., and Figure 9.5.), the bulk of the results corresponding to various combinations of thermal and kinematic asymmetry. Such a characterization has, however, been withheld until a later section (cf. Section 10.3.5.), devoting the present section mainly to the formulation of various thermohydrodynamic quantities. A reduction of the solution to a form involving the viscosity at the temperature of one wall or the other, is straightforward. This can be achieved, if so desired by replacing \( \mu_0 \) by \( \mu_1 \) or \( \mu_2 \), corresponding to surfaces 1 and 2, and expressing the solutions in terms of \( K \) where, as before in Chapter 9:

\[
\frac{1}{k_{12}} = \frac{\mu_0}{\mu_1} \frac{1}{p} = \frac{1}{p} \frac{1}{h^2} \frac{1}{h^2} \frac{1}{h^2} (10.17)
\]

The shear stress parameter \( \Gamma \) (see definition (10.9)) consists, of course, in accordance with the Consideration of the general pattern of flow, of a pure shear and a pure pressure component. This has already been pointed out through equation (10.1b), namely,

\[
\frac{1}{k_{12}} = \frac{8}{(y_2 - y_1)} (10.3)
\]
The volumetric rate of flow can be evaluated by integrating once the expression for the velocity distribution. For the general flow under study, which includes thermal and kinematic asymmetry, the boundaries can be characterized simply by the values of \( u_1 \) and \( u_2 \) so that the operational value of \( \tau \) is given by \( (u_2 - u_1)^2 \). Accordingly the shear stress difference between the boundaries \( \tau_H \) can be given by,

\[
\tau_H = -\sqrt{\tau - \tau_g}
\]  

(10.18a)

where using definitions (10.12a) and (10.12b),

\[
\tau = (u_2 - u_1)
\]  

(10.18b)

Shear stress values at the wall 1 or 2 can be given in terms of the viscosity at the corresponding wall, viz.:

\[
\tau_{1,2} = \tau_{1,2} \cdot \frac{u}{k} \left( \frac{u}{k} \right) = -\left[ \int_{u_{1,2}}^{u} \frac{1}{u} \left( \frac{u}{k} \right) \right] \frac{1}{w}
\]

(10.19)

For the simple flows that are both thermally and kinematically symmetric, however, the values of 1 and 2 are identical. The velocity distribution can be obtained, as before for the plan-parallel flows analysed in Chapter 9, by making use of the Newtonian definition for the viscosity, viz.:

\[
(u_2 - u) = \int_{u_2}^{u} \frac{1}{u} \left( \frac{u}{k} \right) \right] \frac{1}{w}
\]

Substituting \( \tau \) and \( u \) from the previously obtained solutions for \( \tau_H \) and \( \theta \) and rearranging yields,

\[
\frac{\bar{u}_H}{\bar{u}} = \left( \frac{\bar{u}_H}{\bar{u}} \right) \left( \frac{\bar{u}_H}{\bar{u}} \right) = \left[ \int_{u_{1,2}}^{u} \frac{1}{u} \left( \frac{u}{k} \right) \right] \frac{1}{w}
\]

(10.20)

For the general pattern of flow under study where the difference between the surface velocities may be non-zero, a sliding speed parameter can be formulated using the above expression for the distribution of velocity. This can be done by extending the integration over the complete film thickness. Then the sliding speed parameter \( \nu_H \) can be expressed in the form,

\[
\nu_H = \left( \frac{\bar{u}_H}{\bar{u}} \right) \left( \frac{\bar{u}_H}{\bar{u}} \right) \int_{u_{1,2}}^{u} \frac{1}{u} \left( \frac{u}{k} \right) \right] \frac{1}{w}
\]

(10.21)

The volumetric rate of flow can be evaluated by integrating once the expression for the velocity distribution. A form that here can be considered convenient is,

\[
V = \left( \frac{\bar{u}_H}{\bar{u}} \right) \int_{u_{1,2}}^{u} \frac{1}{u} \left( \frac{u}{k} \right) \right] \frac{1}{w}
\]

(10.22a)

10.3.4 Other thermohydrodynamic quantities

It is apparent from the previous section that the solution to the energy equation for thermally steady-states involving heat generation by viscous dissipation and heat transfer by cross-conduction can be obtained quite accurately using Milne's predictor-corrector method. Of course, the particular case of thermal and kinematic symmetry is included in the general solution. For such a symmetry \( \mathcal{V}_H = 0 \) and \( \mathcal{V}_H = 0 \), the latter implying that the curvature at the apex of the distribution of \( \tau \) (see equation (10.13)) reduces to zero.

The solution for \( \theta \) obtained in Section 10.3.2., and its first derivative \( \theta^{(1)} \), serve as basic data through which various thermohydrodynamic quantities, particularly those of interest in lubrication problems, can be evaluated. It is deemed convenient to express these quantities in terms of \( \mathcal{P} \), the non-dimensionalized pressure gradient which involves the theoretically minimum viscosity, \( \mu_B \) (see definition (10.7c)). By so doing it is possible to cover the entire operating regime since this is single-valued in \( \mathcal{P} \). Further on, the distribution of each of the thermohydrodynamic parameters involving \( \mathcal{P} \) can be characterized by one single curve, which represents, as explained in Chapter 9, (cf. Section 9.3.4.2. and Figure 9.5), the bulk of the results corresponding to various combinations of thermal and kinematic asymmetry. Such a characterization has, however, been withheld until a later section (cf. Section 10.3.5.), devoting the present section mainly to the formulation of various thermohydrodynamic quantities. A reduction of the solution to a form involving the viscosity at the temperature of one wall or the other, is straightforward. This can be achieved, if so desired by replacing \( \mu_B \) by \( \mu_1 \) or \( \mu_2 \), corresponding to surfaces 1 and 2, and expressing the solutions in terms of \( \mathcal{P} \) where, as before in Chapter 9:

\[
\mathcal{P}_{1,2} = \left[ \frac{\mathcal{P}}{\mathcal{P}_{1,2}} \right] \frac{\mu_1}{\mu_2}
\]

(10.17)

The shear stress parameter \( \tau_H \) (see definition (10.9)) consists, of course, in accordance with the consideration of the general pattern of flow, of a pure shear and a pure pressure component. This has already been pointed out through equation (10.1b), namely,

\[
\tau = \tau_0 (\nu_2 - \nu)
\]

(10.3)
and has been expressed in a non-dimensional form through equation (10.9), viz.:
\[ \Gamma_H = -\sqrt{F}(Y - Y_B) \]
To make \( \Gamma_H \) correspond with some selected thermal boundary condition, the viscosity at the wall may be introduced to obtain the wall shear stress.

For the general flow under study, which includes thermal and kinematic asymmetry, the boundaries can be characterized simply by the values of \( \omega_1 \) and \( \omega_2 \), so that the operational values of \( F \) is given by \( (\omega_2 - \omega_1)^2 \). Accordingly the shear stress difference between the boundaries \( \Gamma_H \) can be given by,
\[ \Gamma_H = -\sqrt{F} - \omega_1^2 \]

where using definitions (10.12a) and (10.12b),
\[ \omega = (\omega_2 - \omega_1) \]

Shear stress values at the wall 1 or 2 can be given in terms of the viscosity at the corresponding wall, viz.:
\[ \Gamma_{1,2} = T_{1,2} \left( \frac{\mu_{1,2}}{\mu_1} \right) = \frac{1}{\mu_1} \int_{\nu_1}^{\nu_2} \sigma \left( \frac{\mu_{1,2}}{\mu_1} \right) \]

For the simple flows that are both thermally and kinematically symmetric, however, the values of 1 and 2 are identical. The velocity distribution can be obtained, as before for the plan-parallel flows analysed in Chapter 9, by making use of the Newtonian definition for the viscosity, viz.:
\[ (v_i - u) = \int_0^1 du - \int_{\nu_2}^{\nu_1} \sigma \]

Substituting \( \tau \) and \( \mu \) from the previously obtained solutions for \( \Gamma_H \) and \( \Theta \) and rearranging yields,
\[ \mu_H = \left( \frac{\mu_{1,2}}{\mu_1} \right) (v_i - u) = \int_0^1 \sigma \int_0^\nu_1 \Theta (\omega + \Theta_{HB}) dv \]

For the general pattern of flow under study where the difference between the surface velocities may be non-zero, a sliding speed parameter can be formulated using the above expression for the distribution of velocity. This can be done by extending the integration over the complete film thickness. Then the sliding speed parameter \( \nu_H \) can be expressed in the form,
\[ \nu_H = \left( \frac{\mu_{1,2}}{\mu_1} \right) (v_i - u) = \int_0^\nu_1 \Theta (\omega + \Theta_{HB}) dv \]

The volumetric rate of flow can be evaluated by integrating once the expression for the velocity distribution. A form that here can be considered convenient is,
\[ \nu_H = \left( \frac{\mu_{1,2}}{\mu_1} \right) \int_1^{\nu_1} \int_0^\nu_1 \Theta (\omega + \Theta_{HB}) dv' \]

Figure 10.2 Distribution of relative fluidity \( \nu_H/\nu \) with the parameter \( \omega \) using the exponential viscosity temperature-relationship for assumed values of \( \Theta_{HB} \) in the range \( 0 \leq \Theta_{HB} \leq 1.0 \).
Figure 10.2 covers the operating range completely in which the relative fluidity $U_{rr}/U_r$ reduces from unity to zero, the latter representing infinitely high temperatures at some position inside the film. The corresponding abscissa $w$ would then reach infinitely high values for the present exponential viscosity-temperature relationship.

The results presented in Figure 10.2 are very general in that they represent various sets of thermal boundary conditions for various degrees of thermal and/or kinematic asymmetry as they are specified through various non-zero values of the parameter $C_{ij}$. Although these results have been plotted only for positive values of $C_{ij}$ they can also be used for negative values of $C_{ij}$. The latter are simply the mirror image about the axis $w=0$ of the results shown in Figure 10.2.

The results for other thermohydrodynamic quantities can be evaluated through the integrals involving the relative fluidity $U_{rr}/U_r$. These integrals (see expressions (10.20) through (10.25)) have been numerically evaluated. The accuracy of such an integration has been found to be very good when it concerns the single integral, namely (see expressions (10.20), (10.21), (10.23) and (10.24)),

$$\int_{w_1}^{w_2} e^0 (w + C_{H_B}) \, dw$$

(10.29)

The accuracy is somewhat lower for the double-integral, namely (see expressions (10.22a), (10.25) and (10.28)),

$$\int_{w_1}^{w_2} \int_{w_1}^{w_2} e^0 (w + C_{H_B}) \, dw' \, dw$$

(10.30)

Results for the velocity distribution for a few selected values of the curvature at the apex have been presented in Figure 10.3. They have been provided in the aforementioned range of $0 < C_{HB} < 1.0$, and in a form convenient for a later use in that the distribution of velocity, viz.

$$U_{H} = \left( \frac{\partial U_{H}}{\partial w} \right) (w - u)$$

has been so expressed that it is zero when $U_{H}/U_1 = 0$.

10.3.5.2 Segmenting technique for using the results

The previously mentioned convenient form of representation of the results in terms of $w$, viz.

$$w = \left( \frac{\partial U_{H}}{\partial w} \right) \left( \frac{\partial Y_{H}}{\partial w} \right)$$

may be supplemented through the segmenting which here below will be illustrated by means of Figure 10.2. For a thermal asymmetry, for example,

$$C_{HB} = \left( \frac{\partial U_{H}}{\partial w} \right) \left( \frac{\partial Y_{H}}{\partial w} \right) = 1.0$$

and a set of operating conditions, say by,

$$P = \frac{\partial U_{H}}{\partial w} = 81$$

which follows from $W(\partial U_{H}/\partial w) = 81$, the boundaries $w_1$ and $w_2$ can be located individually where

$$w_1 = P \left( \frac{1}{2} - Y_{H} \right); w_2 = P \left( \frac{1}{2} + Y_{H} \right)$$

Then the corresponding boundary temperatures can be read from Figure 10.2 provided one of the boundary conditions through either $U_{H}/U_1$ or $U_{H}$ has also been specified. This interpretation of the results can best be conceived through a few examples as follows.

Let $C_{HB}$ and $W$ be specified by the above-mentioned values; $C_{HB} = 1.0$, and $W = 3.0$. For simplicity if it is considered that $U_{H}/U_1 = U_{H}/U_2$ implying equal boundary temperatures, then no specification of the numerical value of either $U_{H}/U_1$, or $U_{H}/U_2$ is required for the present purpose of illustration. Then the boundaries of the film can be located at points $B_1$ and $B_2$, and the distribution $U_{H}/U_1$ by the segment $B_1CB_2$ of the $C_{HB}$-curve concerned as shown in Figure 10.2. The axis of the channel, that is the mid-position with respect to segment $B_1B_2$, is accordingly found to fall at $w = -0.50$. The velocity distribution corresponding to the above-mentioned operating conditions can be obtained by projecting the values of $w_1$ and $w_2$ read from Figure 10.2 upon Figure 10.3 for the velocity distribution. The segment $V_{C}C_{H}$ of the $C_{HB}$-curve for the value $C_{HB} = 1.00$ concerned in Figure 10.3 represents the corresponding velocity distribu-
The above examples lead to the conclusion that the solution for any selected value of $C_{HR}$ implies a definite criterion of non-symmetry through the above offset with respect to the channel axis. Flows characterized by even a different $P$ value, but following the above $C_{HR}$-curve concerned, again the one corresponding $w=1$. Boundary temperatures are read at $B_5$ and $B_6$ corresponding to the values of $w_1$ and $w_2$ obtained from segmenting in Figure 10.3. The corresponding velocity distribution is represented by the segment $B_5\text{CB}_6$ of the curve for $C_{HR}=1.00$ in Figure 10.2. It may be noticed that the position $w=0$ of the maximum temperature and that $w=-1$ of the maximum velocity are offset by an equal amount from the channel located axis at $w=-0.5$ in the preceding two examples.

The above examples lead to the conclusion that the solution for any selected value of $C_{HR}$ implies a definite criterion of non-symmetry through the above offset with respect to the channel axis. Flows characterized by even a different $P$ value, but following the above $C_{HR}$-curve will have the same above-mentioned offset of the maximum temperature and the maximum velocity. Thus flows of a definite thermal and kinematic asymmetry can for all $P$-values be represented by appropriate segments of one single curve for the viscosity and thus the temperature distributions, and by equally appropriate segment of another single curve as far as the velocity distributions are concerned. It has already been shown in Chapter 9 for symmetric plan-parallel pressure flows, and now from the above discussion for plan-parallel pressure flows accompanied by relative sliding of the bounding surfaces, that the distributions of relative fluidity in Figure 10.2 and those of the velocity in Figure 10.3 are very general and cover the entire regime $0.0 < \frac{\mu_1}{\mu_2} < \infty$.

The third, and the most general example, is characterized by unequal surface temperatures and velocities. For the previously chosen values of $C_{HR}=1.0$ and $P=3.0$, let either the temperature through $\frac{\mu_1}{\mu_2}$, or the velocity $U_3$ be specified at one of the surfaces. These can be located independently in Figures 10.2 (see point $B_3$) and 10.3 (see point $V_3$). The extent $W_3$ now locates the second surface at $B_3$ from Figure 10.2 and at $V_3$ on Figure 10.3. It may be noticed that the offset of point $B_3$ and $B_6$ along the ordinate represents unequal relative fluidities $\frac{\mu_H}{\mu_1}$ and $\frac{\mu_H}{\mu_2}$, and thus unequal surface temperatures. The offset of points $V_3$ and $V_4$ along the ordinate for $U_3$ represents sliding.

In practice a plan-parallel flow problem can generally be specified by temperatures and velocities of the walls, in addition to by other influential quantities such as the pressure gradient, film thickness and lubricant properties. Neither $C_{HR}$ nor $P$ will be known prior to the distributions of temperatures and velocities in the cross-section of the flow are obtained. Here $C_{HR}$ is equal to the offset of the apices of the two distributions (see definition (10.8b)), and $P$ involves the minimum viscosity $\mu_H$ (see definition (10.7c)). An approach to solve the flow problems, some of certain particular nature, with the help of the results presented in Figure 10.2 and 10.3 has been explained in what follows.

To start with, the quantity $K$, viz.

$$K = \frac{\beta^2}{\beta^2} = \frac{h^2}{h}$$

is calculated using the already given influential quantities including the viscosity at the walls, i.e. $\mu_1=\mu_2$ since the problem concerned involves equal boundary temperatures.

The above examples lead to the conclusion that the solution for any selected value of $C_{HR}$ implies a definite criterion of non-symmetry through the above offset with respect to the channel axis. Flows characterized by even a different $P$ value, but following the above $C_{HR}$-curve will have the same above-mentioned offset of the maximum temperature and the maximum velocity. Thus flows of a definite thermal and kinematic asymmetry can for all $P$-values be represented by appropriate segments of one single curve for the viscosity and thus the temperature distributions, and by equally appropriate segment of another single curve as far as the velocity distributions are concerned. It has already been shown in Chapter 9 for symmetric plan-parallel pressure flows, and now from the above discussion for plan-parallel pressure flows accompanied by relative sliding of the bounding surfaces, that the distributions of relative fluidity in Figure 10.2 and those of the velocity in Figure 10.3 are very general and cover the entire regime $0.0 < \frac{\mu_1}{\mu_2} < \infty$.

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The above examples lead to the conclusion that the solution for any selected value of $C_{HR}$ implies a definite criterion of non-symmetry through the above offset with respect to the channel axis. Flows characterized by even a different $P$ value, but following the above $C_{HR}$-curve will have the same above-mentioned offset of the maximum temperature and the maximum velocity. Thus flows of a definite thermal and kinematic asymmetry can for all $P$-values be represented by appropriate segments of one single curve for the viscosity and thus the temperature distributions, and by equally appropriate segment of another single curve as far as the velocity distributions are concerned. It has already been shown in Chapter 9 for symmetric plan-parallel pressure flows, and now from the above discussion for plan-parallel pressure flows accompanied by relative sliding of the bounding surfaces, that the distributions of relative fluidity in Figure 10.2 and those of the velocity in Figure 10.3 are very general and cover the entire regime $0.0 < \frac{\mu_1}{\mu_2} < \infty$.

The third, and the most general example, is characterized by unequal surface temperatures and velocities. For the previously chosen values of $C_{HR}=1.0$ and $P=3.0$, let either the temperature through $\frac{\mu_1}{\mu_2}$, or the velocity $U_3$ be specified at one of the surfaces. These can be located independently in Figures 10.2 (see point $B_3$) and 10.3 (see point $V_3$). The extent $W_3$ now locates the second surface at $B_3$ from Figure 10.2 and at $V_3$ on Figure 10.3. It may be noticed that the offset of point $B_3$ and $B_6$ along the ordinate represents unequal relative fluidities $\frac{\mu_H}{\mu_1}$ and $\frac{\mu_H}{\mu_2}$, and thus unequal surface temperatures. The offset of points $V_3$ and $V_4$ along the ordinate for $U_3$ represents sliding.

In practice a plan-parallel flow problem can generally be specified by temperatures and velocities of the walls, in addition to by other influential quantities such as the pressure gradient, film thickness and lubricant properties. Neither $C_{HR}$ nor $P$ will be known prior to the distributions of temperatures and velocities in the cross-section of the flow are obtained. Here $C_{HR}$ is equal to the offset of the apices of the two distributions (see definition (10.8b)), and $P$ involves the minimum viscosity $\mu_H$ (see definition (10.7c)). An approach to solve the flow problems, some of certain particular nature, with the help of the results presented in Figure 10.2 and 10.3 has been explained in what follows.
Figure 10.5 Relationship between the relative fluidity $U_H/U_I$ and $K$ representing various sets of operating conditions.

10.5. Since the quantity $K$ is related by definition to $w$ which has been used as abscissa in Figure 10.2, viz.

$$K^1 = \frac{U_H}{U_I} \cdot \frac{V}{w}$$

it may be observed,

$$K^1 = \frac{U_H}{U_I} \cdot \frac{V}{w}$$

where $w$ represents the position of wall 2 at $P_2^1/2$. We are in a position to construct the variation of the quantity on the left hand side by taking the product of $w$ with $(U_H/U_I)^1$, the quantity $U_H/U_I$, there being the corresponding ordinate at every point along the viscometric curve in Figure 10.5. The trend of the present relationship, which is symmetric with respect to the $U_H/U_I$-axis, corresponds with that for the $U_H/U_I$ relationship depicted in Figure 9.4 of Chapter 9. In the latter figure, $K$ has been used instead of $K$ as the abscissa. This explains why the maximum value of $K=90.37$ corresponds with the present extremes (transitions) for $(K/2)^1=1.54$.

For the problem at hand, i.e. the non-isothermal flow problem, specified through a definite value of $K$ which has already been evaluated, it is straightforward to locate both the rubbing surfaces at $y/h=1/2$ and $y/h=-1/2$ along the abscissa $2K/2$, say at $A_2$ and $A_1$. The corresponding ordinates for the $f$-value are now found: they are double-valued as expected on the basis of previous analysis of the plan-parallel flows in Chapter 9. The first value of the ordinate, i.e. at $C_1$, corresponds to the first branch $A_2$, whereas the second branch $A_3$, and the second branch $A_1$, is characterized by the ordinate at $C_2$. The distribution of the relative fluidity corresponding to the ordinate at $C_1$ is given by $A_2$, $A_3$, $A_1$, and $A_2$. The corresponding to the ordinate at $C_2$ is hard to visualize, because the former distribution given by $B_2CB_2$ passes through the transition points $A_1$ and $A_2$. This distribution can, however, be better visualized with the help of Figure 10.4 as follows.

The aforementioned points $A_2$ and $A_1$ located on the abscissa in Figure 10.5 can now be relocated on the abscissa in Figure 10.4. For such a relocation it is indeed necessary to make use of both the ordinates at $C_1$ and $C_2$ in the former figure. The relative fluidity $U_H/U_I$ corresponding to the first ordinate at $C_1$ along with the already-known $K$-value gives rise to one set of points on the abscissa, i.e. $A_1$ and $A_2$, which specify the boundaries at $y/h=1/2$ and $y/h=-1/2$ in terms of the variable $w$ which has already been defined by,

$$w = \frac{U_H}{U_I} \cdot \frac{V}{h}$$

Corresponding to the second ordinate at $C_2$, one obtains along similar lines the second set of points $A_3$ and $A_4$ in Figure 10.4.

It may now be visualized that for any given set of operating conditions which yield a definite value of $K$, there exist two distributions of the relative fluidity as shown in Figure 10.4. One depicted by $A_2$, $A_3$, and $A_4$ is characterized by the ordinate at $C_1$, and the variable given by the abscissa $A_2A_1$ or $A_1A_2$, and the other depicted by $A_3$, $A_4$, is characterized by the ordinate at $C_2$ and the variable $w$ by $B_2CB_2$. The uniqueness of the distribution can, therefore, be established with the help of Figure 10.4 and its abscissa and ordinates. Along similar lines, one may also establish the uniqueness of the volumetric rate of flow in terms of the abscissa covered by $W'$ and $W''$, or their respective $P$-values, viz. $P'$ and $P''$. The figure depicting the volumetric rate of flow against $P$ has already been presented in Chapter 9 (cf. Figure 9.7a). The double-valuedness of the distribution of relative fluidity involving the viscosity at the boundaries 1 and 2, i.e. $U_H/U_I$ has been illustrated for a particular chosen value of $K=45.43$. For this value of $K$, two distributions corresponding to double-solutions, namely along branches I and II have been worked out first by reading the two relative fluidities $U_H/U_I$ from Figure 10.5. Following this the distributions have been worked out using Figure 10.4, and depicted by curves 1 and 2 in Figure 10.6a. The distribution given by curve 1 is nearly uniform over a considerable width of the flow, and that by curve 2 indicates large dissipation effects over the entire width.

The velocity distribution in terms of $U_H$ involving the viscosity $\mu$ has been presented in Figure 10.7. Also these distributions given by $A_2$, $A_3$, and $B_2CB_2$ are unique when expressed with $w$ as the variable. The above, however, non-unique if expressed with $K\gamma/h$ as the variable. Also the double-valuedness of the distribution of velocity involving the viscosity at the boundaries 1 and 2, i.e. $U_H\gamma/h$, has been illustrated for the previously chosen particular value of $K=45.53$. The distributions have been worked out using Figure 10.7, and depicted by curves 1 and 2 in Figure 10.6b. Curve 1 is one-half of the parabola about its own axis as expected of the isoviscous flow. Thermal effects are still low for this distribution, but they become strong for that given by curve 2. The latter consists over a point of inflection, the significance of which will be discussed in section 10.5.2.
Distributions of relative fluidity and velocity from the set of operating conditions giving rise to the maximum permissible value of \( K \), i.e. \( K_{max} = 182.67 \), are single-valued in \( K \) as shown by the curves in Figures 10.6a and 10.6b. These distributions correspond to the solution at the maximum conceivable value of \( K \), i.e. \( K_{max} = 182.67 \). Curve 3 corresponds to the solution along branch II (see Figure 10.5) both for the chosen value of \( K = 45.53 \). Curve 3 corresponds to the solution at the maximum conceivable value of \( K \), i.e. \( K_{max} = 90.37 \).

The second branch of the distribution of relative fluidity covered by \( A_1r^2 \Delta t \) in Figure 10.5 proves to impose certain limitations to the flow from thermal considerations. Distributions along the second branch indicate excessively high temperatures inside the flow which cause decomposition or degradation of oils concerned. A detailed discussion, however, will be presented in a later section of this chapter.

10.3.5.4 The most general plan-parallel flows

The most general case of the fluid in plan-parallel configuration can be characterized by unequal temperatures and velocities of the surfaces concerned. Somewhat simple cases include: case 1, equal temperatures accompanied by sliding of the surfaces; case 2, unequal temperatures accompanied by equal velocities of the surfaces. All of the above cases yield asymmetric distributions of \( U_H \) and \( Q_H \), and thus non-zero values of \( \zeta \), as already discussed before. They can, however, be analyzed along lines similar to those explained in the foregoing.

Firstly, the quantity \( K \) is evaluated using the given operating conditions at surfaces 1 and 2:
\[
K_1 = \frac{L_H - s}{u_H}; \quad K_2 = \frac{L_H - s}{u_H}
\]

These two values of \( K \) will be generally different except for the aforementioned case 1 for which equal surface temperatures yield \( K_1 = K_2 \).

Secondly, an arbitrary point, say \( A_1 \), is chosen along a profile, e.g. curve 2, characterized by a definite parameter \( \zeta_B = 0.5 \) as shown in Figure 10.8. Following this, the extent \( W \) is evaluated for case 1 by taking the horizontal intercept \( A_1A_2 \) of the profile concerned; for the other cases a different procedure may have to be adapted. For the latter cases, it is necessary to make use of the specified unequal surface temperatures and to obtain the relative fluidity \( \zeta_2/\zeta_1 \). In conjunction with this \( \zeta_2/\zeta_1 \) it is necessary to employ the ordinate of the previously selected arbitrary point \( A_1 \), i.e. \( U_H/U_1 \) to obtain the relative fluidity \( W_H/W_1 \):
\[
\frac{W_H}{W_1} = \frac{U_H}{U_1} \cdot \frac{\zeta_2}{\zeta_1}
\]

The point concerned corresponding to this ordinate on the previously selected profile can be located, say at \( C_2 \) (see Figure 10.8). The extent \( W \) for the case under study is given by the horizontal projection, i.e. on the abscissa, of the segments \( A_1C_1 \) and \( C_2G_2 \).
Having evaluated the extent \( W \) for the case chosen, one may proceed to obtain either \( U_H/U_1 \) or \( U_H/U_2 \) using this extent \( W \) and the previously evaluated corresponding quantity \( K_1 \) or \( K_2 \), since

\[
U_H = K_1 U_1, \quad U_H = K_2 U_2.
\]

It may be noticed that the particular case of equal surface temperatures gives rise to one single value for both \( U_H/U_1 \) and \( U_H/U_2 \). For this particular case the latter ratio \( U_H/U_1 = U_H/U_2 \) may be compared with that given by the ordinate of point \( C_1 \). For the other cases either of \( U_H/U_1 \) or of \( U_H/U_2 \) may be compared with the corresponding ordinate of \( C_1 \) or \( C_2 \).

If the above comparison of the relative fluidity does not yield an agreement, the above-mentioned process of evaluation has to be repeated by selecting the point \( A_1 \) at a different ordinate along the same previously chosen profile, i.e. curve 2. If no agreement is found throughout the range of the profile, a profile corresponding to a different \( C_H^B \)-value may have to be chosen to repeat the aforementioned analysis.

For the particular choice of the ordinate of the point \( A_1 \) and the profile for which the above equality towards the relative fluidity is satisfied, the viscosity distribution is given by \( A_1 - A_2 \), for the aforementioned case 1 and \( A_1 - C_2 \) for the other cases. The abscissa \( w_1 \) and \( w_2 \) represent the position of walls 1 and 2 of the plan-parallel flow concerned. Velocity distributions \( U_H \) can then be obtained by relocating the set of abscissa \( w_1 \) and \( w_2 \) in Figure 10.9. Both the cases considered yield asymmetric distributions as shown in the latter figure: \( D_1, D_2, \) for the particular case 1 and \( D_1, D_2, \) for the most general case.

Thus the above-mentioned analysis yields the required solutions for the distributions of viscosities and velocities. It provides also the parameter \( C_H^B \) which represents the degree of asymmetry of the latter distributions. During the analysis one may encounter that the profiles for various \( C_H^B \)-values are not close enough, if not completely absent, to obtain the required equality of the relative fluidity. Interpolations of profiles for closer intervals of \( C_H^B \)-values, or even some extrapolations, may have to be made to acquire a proximity of the required equality.

One may obtain equivalent results as those obtained using \( U_H/U \)-distributions, but now with the help of the profiles for the distribution of velocity, i.e. \( U_H \). Such an alternate method has not been provided here due to lack of space. However, for the aforementioned particular case 2, an application of the alternate method would simplify the analysis considerably. It is preferable to use Figure 10.9 together with Figure 10.8 for this particular case as follows. An arbitrary point, e.g. \( E_1 \), along the left branch of a curve, say curve 2, characterized by a definite parameter \( C_H^B = 0.5 \) is selected. The extent \( W \) is read by locating a point, e.g. \( E_2 \), along the right hand branch of the same curve 2 and at the same ordinate \( U_H \), implying identical velocities of the surfaces. Following this the quantity \( P + W \) is evaluated.

Further the abscissa \( w_1 \) and \( w_2 \) corresponding to \( W \) in Figure 10.9 are relocated in Figure 10.8 to obtain \( U_H/U_1 \) and \( U_H/U_2 \). Either of these two quantities can be compared with that obtained using the parameters \( K_1 \) and \( K_2 \), which have been previously evaluated using the supplied data, and \( P \) just obtained. When no agreement is found, the above-mentioned procedure may have to be repeated over the entire range of the profile.
concerned. A different profile, and even an interpolated new one characterized by a definite $\text{CH}_B$ value may have to be chosen to satisfy the required agreement. When this agreement is achieved, the resulting distributions may be located along the segments $E, F, E$, for velocity in Figure 10.9 and $B, C, B$, for viscosity in Figure 10.8.

Indeed the evaluation in all of the cases of the asymmetric flows considered has been carried out using the data depicted with $w$ as the variable in Figures 10.8 and 10.9. Along lines similar to those explained in the aforementioned particular case of equal temperatures and equal velocities of the boundaries, it may be noted that the above evaluation yields unique distributions of $u_B/u_1$ and $u_B$. At the specified set of operating conditions with high values of the extent $W = W_T - W_F$, it may be expected that the distributions would correspond to the second branch of the solution (cf. Figure 10.3) if the latter solution is depicted with $\text{CH}_B$ against $x_2$, these being referred to surface 1 and 2. Figures similar to Figure 10.5 have here not been provided due to lack of space; the basic data to construct such figures, nevertheless, is already presented in Figure 10.8. Limitations on the flow along the second branch due to excessive high temperature may be conceived for the present plan-parallel asymmetric flows as well.

10.3.5.5 Results in a normalized form

A somewhat different representation of the results involves recasting Figure 10.8 in terms of a reduced quantity $u_B$. This has been achieved along lines similar to those employed in Section 9.3.4.2 of the previous chapter. The aforementioned quantity $u_B$ has been obtained by normalizing the velocity $u_B$ given by,

$$u_B = \left( \frac{\beta w}{\beta} \right) \left( u_g - u \right)$$

so that,

$$u_B^* = \frac{u_B}{u_B^{1.13}}$$

using the scale factor $u_B^{1.13}$, i.e. the maximum conceivable value of $u_B$, as it corresponds to the apex of the velocity profile. The latter factor $u_B^{1.13}$ depends on the maximum value of $W$ chosen, i.e. $W^{1.13}$. The method of reading these results expressed in Figure 10.10 and 10.11 is similar to that described in the previous chapter (cf. Section 9.3.4.2). It may however be noticed that the results cover the widest conceivable range of operation represented through $0 < \text{CH}_B < 1$. The thermal and kinematic asymmetry covered by the results is somewhat limited, viz. the curvature at the apex of the temperature profile given by expression (10.13) being confined to 0, 0.25, 0.50, 0.75 and 1.0. Both positive and negative values of $\text{CH}_B$ derived from the above set of values for the curvature have been introduced in the analysis. Other values of the curvature, including higher values than unity, can be employed, if desired. The qualitative nature of the results, however, would not change.

The results shown in Figures 10.10 and 10.11 indicate that at smaller values of $w^* = W/w^{1.13}$, following from smaller values of the extent $W = W_T - W_F$ and the quantity $P$, the shapes of the velocity profiles for the asymmetric thermal and kinematic boundary conditions remain qualitatively similar to the shape of the velocity profile for the symmetric case $\text{CH}_B = 0$.

But farther away from $w^* = 0$ there appear considerable deviations at higher values of $W$ and thus of $P$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1010}
\caption{Distribution of velocity normalized using $u_B$ as referred to surface 1 for the exponential relationship and for assumed positive values of $\text{CH}_B$ in the range $0 < \text{CH}_B < 1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1011}
\caption{Distribution of velocity normalized using $u_B$ as referred to surface 1 for the exponential relationship and for assumed negative values of $\text{CH}_B$ in the range $0 < \text{CH}_B < 1$.}
\end{figure}
The present results have been studied also in terms of an easily identifiable parameter: \( P \) involving \( \frac{h}{h_0} \) has been replaced by \( K \) involving \( u \), referred to surface \( 1 \), as in equation (10.17), viz.

\[
K = \frac{\frac{h^3}{h_0}}{u} = \left(\frac{h}{h_0}\right) P
\]

As expected from the symmetric flow analysed in Chapter 9 (cf. Figure 9.7b), also the asymmetric flows display a similar trend: the flow quantity \( V \), viz.

\[
v = \left(\frac{h}{h_0}\right)^\frac{1}{g} \frac{h}{h_0}
\]

would undergo a transition at a limiting value of \( K \), i.e. \( K_{lim} \). Further, it would continue to increase even with decreased \( K \). The results, however, have not been provided here due to their bulkiness. As against a high value of \( \frac{\text{Klim}}{K_1} = 9.506 \) for the symmetric flows (cf. Figure 9.7b in Chapter 9) under decreasing pressure gradients, i.e. increasing positive values of \( g \) since by definition \( g = (-dp/dx) \), the limiting values of \( K \) for thermally and kinematically asymmetric flows with \( \frac{\text{vlim}}{K_1} \) prove to be lower. This means that the limiting pressure gradients for these asymmetric flows are confined to values smaller than the one 9.506, for the symmetric flow. The corresponding flow parameter \( V \) shows an increasing trend with increased asymmetry, its value for the symmetric flow being 1.673. On the contrary, for the asymmetric flows given by \( \frac{\text{vlim}}{K_{lim}} \), the trend of both \( \frac{\text{vlim}}{K_{lim}} \) and the corresponding \( V \) is just opposite to the above-mentioned one. Increased asymmetry gives rise to increased values of \( \frac{\text{vlim}}{K_{lim}} \) and decreased \( V \)-values.

### 10.3.6 Thermal Reynolds equation relating thermohydrodynamic quantities

The results so far presented are far too involved for ready use in the analysis of lubrication processes. Such use would be facilitated if they could be represented by sufficiently accurately fitted "empirical" relationships.

Wilson and his associates [10.7, 10.6, 10.8] have made such an attempt in a series of articles. They carried out a fitting analysis of the general flow concerned on slightly different lines, namely by employing a thermal sliding parameter equivalent to \( H \), defined through an expression (10.24) and a non-dimensional temperature difference \( \Theta = R(T_2 - T_1) \). Any desired thermal and kinematic asymmetry is brought out in the analysis by a proper choice of their parameters \( U \) and \( N \). This is analogous to introducing asymmetries as explained previously in this section, viz., by choosing various non-zero values for \( \Theta \).

Now, Wilson and his associates condensed their basic results into the form of what they called the "thermal Reynolds equation". This equation relates the local pressure gradient to the rate of flow. In contrast to the classical Reynolds equation it accounts for the variation of the viscosity of the lubricant across the thickness of the lubricant film. With the intense shear heating generated in the film by fast sliding and/or high pressure gradients the classical assumption about viscosity being constant is far from justified. The lack of justification is only partly substantiated by the fact that the viscosity of the liquids concerned substantially decreases due to rather high temperatures caused by the intense shear heating. Then one must employ the more realistic thermal Reynolds equation, i.e. the one accounting for the thermal effects of the viscous dissipation in the various film cross-sections to be considered and which relates the local pressure gradient to the rate of flow. The latter two parameters, i.e. \( N \) and \( D \), can conveniently be represented by the previously formulated non-dimensional parameters, for instance \( K \) and \( (V_1 - V) \). The latter quantities have previously been provided through definitions (10.17), (10.25) and (10.26b),

\[
K = \left(\frac{\frac{h^3}{h_0}}{u}\right) P ; \quad V = \left(\frac{h}{h_0}\right)^\frac{1}{g} \frac{h}{h_0}
\]

and

\[
v = \left(\frac{h}{h_0}\right)^\frac{1}{g} \frac{h}{h_0}
\]

where \( u \) is the entraining velocity;

\[
\frac{\text{vlim}}{K_{lim}} = \frac{1}{h} (u_1 + u_2)
\]

By employing a uniform rate of viscous shear heat dissipation across the film, Wilson and his associates deduced a thermal Reynolds equation which proves easy to use in the thermal inlet zone analysis of elastohydrodynamic lubricant films. In formulating the thermal Reynolds equation they assume the variation of viscosity with temperature according to either the exponential relationship, or an hyperbolic one. For the present their results for the particular case of the exponential viscosity-temperature relationship will be discussed in what follows.

The thermal Reynolds equation obtained by Wilson and Wong [10.7], at least for the rolling problem with equal velocities and equal temperatures of the two rubbing surfaces, i.e. \( N_1 = 0 \) and \( D = 0 \), is expressed in the form:

\[
E^2 - \frac{\text{vlim}}{K_{lim}} \frac{1}{h} \left(\text{V}_1 - \text{V}\right) \frac{\Theta}{h} = \text{r}^{\frac{1}{2}} \frac{\Theta}{h} \left(\text{V}_1 - \text{V}\right) = 1 = 0
\]

(10.31)

where \( E \) and \( R \) can be related to the quantities employed in the present analysis. The "thermal flow parameter" \( E \) is defined, using equations (10.25) and (10.26b), by,

\[
E = \left(\frac{\text{vlim}}{K_{lim}}\right)^\frac{1}{h} \left(\text{V}_1 - \text{V}\right) = \text{V}_1 - \text{V}
\]

(10.32a)

where \( \Theta = \text{V}_1 - \text{V} \) and thus \( \text{V}_1 - \text{V} \) and \( E \) can be positive, negative, or zero. The parameter \( R \), i.e. a "thermal correction function" to the Reynolds equation, is defined by:

\[
R = \left(\frac{\text{vlim}}{K_{lim}}\right)^\frac{1}{h} \frac{\Theta}{h} \left(\text{V}_1 - \text{V}\right) = 1
\]

(10.32b)

Now \( R \) may be conceived as a correction factor to the conventional Reynolds equation for isothermal flows, viz.

\[
\text{h}_1 = \frac{h}{h_0} \left(\frac{1 - \frac{h^3}{h_0}}{\Theta (\text{V}_1 - \text{V})} \right)
\]

(10.32c)

Further, it may be readily observed that with negligible thermal effects \( R \equiv 1 \) for which equation (10.32b) reduces to equation (10.32c). In fact,
equation (10.32b) includes all the characteristics of the conventional isothermal Reynolds equation being in addition accounting also for thermal effects in the flow. In essence, for any thermally affected flow \( R \) simply relates the pressure gradient to the rate of flow, i.e. \( K \) to \( E \), through the so-called "flow criterion", which is commonly employed in solving the classical Reynolds (isoviscous) equation. The latter "flow criterion" given by:

\[
\frac{dp}{dx} = 12 \frac{\mu_1 (h_0 - E)}{h^3} \]

(10.32a)

satisfies the continuity of flow in that the rate of flow is one and the same value for all the successive film cross-sections (being thus independent of the coordinate \( x \) along the film). Therefore, an evaluation of the thermal Reynolds equation in general reduces to finding a relationship between \( K \) and \( E \), and thus between \( R \) and \( E \) as discussed in what follows. For simplicity Wilson and Wong represented their thermal Reynolds equation (10.31) by fitting thereto, to a reasonable approximation the following semi-empirical relationship between \( R \) and \( E \),

\[
R = \left[ 1 + 0.608 \left( \frac{E}{\mu_1} \right) \right]^{1.66}
\]

(10.33a)

so that

\[
\frac{dp}{dx} = 12 \frac{\mu_1 (1+0.608(V-u)^{1.66})}{h^3}
\]

(10.33b)

Aggarwal and Wilson [10.6] demonstrated the above semi-empirical thermal Reynolds equation given by expressions (10.33a) and (10.33b) to be accurate enough even when the assumption of uniform energy dissipation across the film is relaxed and is replaced by the exact one. A comparison of both this fitted formula and the graphical representation of their more rigorous results of \( R \) against \( E \) with the present results indicates that their results fall short of the accuracy especially in the region of the limiting pressure gradient, that is in terms of \( K_{lim} \). Their results depict a slightly lower value of \( K_{lim} \) namely 87.61 as against 90.36 obtained from our own numerical analysis. The corresponding values of the thermal flow parameter amount to 1.65 (implying \( h=0.4595 \)) and 1.67 respectively. Although this discrepancy would appear far from disturbing, it is felt that an accurate assessment of the critical state given by the limiting pressure gradient may well be crucial, at least when it comes to specifying the safe film thickness corresponding to this critical state.

The region of validity of the semi-empirical thermal Reynolds equation (10.33) is somewhat narrow, viz.:

\[ \mu_1 > 10 \]

Semi-empirical expressions for thermal Reynolds equation applicable to a wide range of operation, especially at very high values of \( E \), have been developed from our own results. Firstly, \( E \) has been related to \( K \)

\[
K = \frac{\mu_1 (1+0.608(V-u)^{1.66})}{h^3} \]

(10.34)

where \( C_1 \) and \( C_2 \) are constants yet left open for best fit. At moderately high values of \( E \), i.e. \( 10 \leq E \leq 25 \), the constants reduce to \( C_1 = 56.7062 \), \( C_2 = 1.2297 \). At very high values of \( E \), i.e. \( E > 25 \), the formula exhibits large temperature variations within the film as indicated by \( \mu_1 (u^2) \), and thus, in the present ranges of real high \( E \)-values expression (10.32b) for the thermal Reynolds equation can correspondingly be simplified in terms solely of \( \frac{V}{k} \).

More generally valid, and equally fitted expressions developed for the thermal Reynolds equation by Wilson and his associates as compiled by Aggarwal and Wilson [10.6] are here worth mentioning. The two surfaces are still maintained at equal temperatures but slide on each other so that \( h=0 \), they fit the thermal Reynolds equation by,

\[
R = \left[ 1 + 0.5 \left( \frac{E}{\mu_1} \right)^{1.70} + 0.2 \left( \frac{\mu_1}{h} \right)^{0.7} \right]^{-1}
\]

(10.35)

When the two surfaces involved are to undergo pure rolling, so that \( N_r=0 \), whilst their surface temperatures are unequal so that \( D>0 \), the thermal Reynolds equation can be approximated by,

\[
R = \left[ 1 + 0.5 \left( \frac{E}{\mu_1} \right)^{1.70} \right]^{-1}
\]

(10.36)

where

\[
N_r = \frac{2D(262 - 262')N_r^2 (u^2)^{1.70}}{h^3 (e-1)}
\]

However, for the most general case, which involves sliding of the surfaces as well as unequal surface temperatures, Wilson and his associates have not so far been able to fit a comparatively simple formula to the then governing thermal Reynolds equation.

Undeniably, the introduction of empirical formulae fitted sufficiently accurately to the thermal Reynolds equation considerably simplifies the otherwise complex problem of the flow of lubricant, which involves both the energy and momentum equations. Indeed, such fitted formulae have already been shown by previous investigators to simplify the analysis of elasto/plasto-hydrodynamic lubricant films. In an attempt to demonstrate such a simplification of their thermal inlet zone analysis Wilson and his associates have provided the aforementioned fitted formulae for a few particular cases of operating conditions. However, especially for extremely loaded lubricant films, i.e. those involving very high values of the flow parameter \( R \) where \( \mu_1 (u^2) \) may well approach zero, these formulae have proved to be far from accurate. The author's solutions in the present chapter would appear to open up a possibility for developing more accurately fitted thermal Reynolds equations than those provided by the previous investigators. It may prove even worthwhile to split up the entire operating regime covered by \( 0.4 \mu_1 (u^2) < 1 \) into separate regions and to fit a thermal Reynolds equation separately to each of these regions.
It should carefully be noted that the thermal Reynolds equations mentioned in the present section were all established only for the exponential viscosity-temperature relationship. However, similar equations can be developed also for the more realistic relationship, i.e. Slotte's relationship with finite values of the exponent $m$.

10.4 SOLUTIONS WITH SLOTTE'S RELATIONSHIP

10.4.1 Governing equations

For liquids commonly used in lubrication practice Slotte's exponent $m$ is positive. The value $m=\infty$ represents the exponential relationship, and $m=0$ the isothermal behaviour. Thus the latter value is a lower bound to the present, non-isothermal flow analysis. However, solutions can be obtained for temperature, velocity, rate of flow, etc. in the whole range $0 < m < \infty$ along lines similar to those followed in the case of the exponential relationship. The non-isothermal flows to be treated in this section are of the most general kind where shear effects are combined with those of pressure gradients, and where thermal boundary conditions of both symmetric and asymmetric types are considered.

First of all we recall Slotte's relationship involving the exponent $m$, viz.,

$$u = \nu_H \varrho^m$$

where $\nu_H$ is the reference viscosity at the maximum temperature $T_H$ in the film cross-section, and $\varrho$ is the non-dimensional temperature given by,

$$\varrho = 1 - \frac{\beta_s}{H} (T - T_H)$$

Here $\beta_s$ is the viscosity-temperature coefficient defined at the reference temperature $T_H$. The coefficient $\beta_s$ has been employed in the present analysis for the reasons already explained in Chapter 2: firstly, to cast Slotte's viscosity-temperature relationship in a convenient reference-invariant form (cf. Section 2.4), and secondly, to facilitate a comparison of the results yet to be obtained in this section with those already obtained using the exponential relationship.

With the introduction of non-dimensional quantities, namely

$$Y = \frac{Y}{H}$$

and as before in definition (10.7b),

$$\zeta = p^1 (Y - Y_B)$$

where $Y_B$ is the dimensionless representation of the position of maximum velocity, as introduced by Bostandzhyan and Stolin [10.1]. Further, $p$ is defined as in Section 10.3 where $\beta_s$ is replaced by $\beta_s$, viz.

$$p = \frac{\beta_s}{H} \varrho$$

We define, as before, the curvature at the apex of the temperature curve, i.e.

$$\frac{d^2 \varrho}{dw^2} |_{w=0} = \zeta_{HB}$$

or

$$\zeta_{HB} = \frac{1}{2} \left[ \frac{d^2 \varrho}{dw^2} |_{w=0} \right]^2$$

Here $\zeta_{HB}$ also represents the offset of the previously mentioned position $X_B$ from $Y_B$, the former being the one of Bostandzhyan and Stolin and the latter that of Hagg where the temperature reaches its maximum. The variable $w$ will be defined below. Then

$$\zeta_{HB} = \frac{1}{2} (Y_B - Y_B)$$

Further using the curvature $\zeta_{HB}$ the new coordinate $w$ is introduced, viz.

$$w = \zeta - \zeta_{HB}$$

The governing momentum and energy equations take forms similar to equations (10.9) and (10.10), namely

$$\rho = 1 - \frac{\beta_s}{H} (\varrho - \varrho_H)$$

(10.39)

$$\frac{d^2 \varrho}{dw^2} + (\varrho + \zeta_{HB})^2 \varrho = 0$$

(10.40)

As before, a convenient "one-point" set of thermal boundary conditions is given by,

$$w = 0, \varrho = 1 \quad (or \ \zeta = 1), \frac{d \varrho}{dw} = 0$$

(10.41)

The general analytical formulation of the non-isothermal flow through equations (10.39), (10.40) and (10.41) also includes that of Bostandzhyan and Stolin [10.1] for the linear fluidity function, $m=1$. This formulation includes also the most general set of thermal and kinematic boundary conditions at the walls situated at $Y=\pm 1/2$. A general formulation of the results can be obtained when considering the position of the walls to be given in terms of the present coordinate $w$ by,

$$w = \frac{1}{2} (1 - Y_H)$$

(10.42a)

and

$$w = \frac{1}{2} (-1 - Y_H)$$

(10.42b)

10.4.2 Solutions with $m=1$ in Slotte's relationship

Only for the particular case of the linear fluidity function, or $m=1$ in Slotte's relationship, are closed solutions available, i.e. in terms of fractional Bessel functions (Bostandzhyan and Stolin [10.1]).
range of operating conditions, including thermal and kinematic asymmetry, has been covered. The analysis is, however, not complete, at least concerning the double-valuedness of the solution in a certain part of the operating regime and limiting operating conditions when one employs the more convenient viscosity at the wall temperature instead of that at the position of the maximum temperature as an input parameter.

For the thermally and kinematically symmetric case, i.e. plan-parallel pressure flow, as already analysed in Chapter 9, the solutions have been worked out in terms of the aforementioned fractional Bessel functions. Solutions for the asymmetric case can also be obtained in terms of the same kinds of Bessel functions. To obtain these solutions we may employ the energy equation (10.40) for convenience with the previously defined variable \( \eta \):

\[
\eta' + \zeta = 0
\]

with the "one-point" thermal boundary conditions,

\[
\zeta = \eta_0, \quad \eta' = 1, \quad \frac{d\eta}{d\xi} = 0
\]

The solutions can readily be obtained in terms of fractional Bessel functions of the orders 1/4 and -1/4, viz.,

\[
\psi = V_{\text{NB}} \left[ c_1 J_{1/2} \left( \frac{\eta}{\xi} \right) + c_2 J_{-1/2} \left( \frac{\eta}{\xi} \right) \right]
\]

where \( c_1 \) and \( c_2 \) can be determined from the thermal boundary conditions (10.44) as shown in Appendix 10.A4, viz.:

\[
c_1 = \frac{J_{1/2} \left( \frac{\eta_0}{\xi} \right) - J_{-1/2} \left( \frac{\eta_0}{\xi} \right)}{J_{1/2} \left( \frac{\xi}{\eta_0} \right) - J_{1/2} \left( \frac{\eta_0}{\xi} \right) - J_{-1/2} \left( \frac{\xi}{\eta_0} \right) - J_{-1/2} \left( \frac{\eta_0}{\xi} \right)}
\]

\[
c_2 = \frac{J_{1/2} \left( \frac{\xi}{\eta_0} \right) - J_{-1/2} \left( \frac{\xi}{\eta_0} \right) + J_{1/2} \left( \frac{\eta_0}{\xi} \right) - J_{1/2} \left( \frac{\eta_0}{\xi} \right)}{J_{1/2} \left( \frac{\xi}{\eta_0} \right) - J_{1/2} \left( \frac{\eta_0}{\xi} \right) - J_{-1/2} \left( \frac{\xi}{\eta_0} \right) - J_{-1/2} \left( \frac{\eta_0}{\xi} \right)}
\]

The derivative of \( \psi \), i.e. \( d\psi/d\xi \) which will be required to evaluate the heat partition ratio at a later stage, is given by,

\[
\frac{d\psi}{d\xi} = \frac{\xi}{V_{\text{NB}}} \left[ c_1 J_{1/2} \left( \frac{\eta}{\xi} \right) - c_2 J_{-1/2} \left( \frac{\eta}{\xi} \right) \right]
\]

10.4.2.1 Other thermohydrodynamic quantities

The exact solution (10.45) thus obtained for the distribution of temperature will be used to determine the accuracy and the region of validity of the results yet to be obtained through the numerical technique explained in the following section.

Further, various thermohydrodynamic quantities such as velocity, heat flux, volumetric rate of flow and total heat flux can be evaluated using integrals containing fractional Bessel functions. These quantities will now be listed, but their dependence on operating conditions will be investigated only in the next section where they will be derived from the foregoing governing equation.

The wall shear stress:

\[
\tau_w = \frac{1}{h} \left( \frac{\eta}{\xi} \right)
\]

the velocity distribution:

\[
v = \frac{1}{h} \left( \frac{\eta}{\xi} \right) \int_0^\xi \psi \, d\xi
\]

the sliding speed parameter \( \eta \) can be expressed in the form:

\[
\eta = \left( \frac{1}{h} \frac{\eta}{\xi} \right) \int_0^\xi \psi \, d\xi
\]

the volumetric non-isothermal rate of flow:

\[
v_{\text{H}} = \left( \frac{1}{h} \frac{\eta}{\xi} \right) \int_0^\xi \psi \, d\xi
\]

the distribution of heat fluxes across the film cross-section to be considered,

\[
\psi = \left( \frac{1}{h} \frac{\eta}{\xi} \right) \int_0^\xi \psi \, d\xi
\]

and the total heat flux per unit area and per unit time:

\[
v_{\text{H}} = \left( \frac{1}{h} \frac{\eta}{\xi} \right) \int_0^\xi \psi \, d\xi = \frac{1}{h} \frac{V_{\text{H}}}{\xi} \int_0^\xi \psi \, d\xi
\]

The solutions for the distribution of velocity and the rate of flow in equations (10.48) and (10.50) can also be re-expressed in terms of \( \eta \), the viscosity at the wall. This can be achieved by making use of the relative fluidity \( \eta_0/\eta \). Then the velocity distribution \( v \) in terms of \( U_0 \) (see definition (10.44)):

\[
v = \left( \frac{1}{h} \frac{\eta}{\xi} \right) \int_0^\xi \psi \, d\xi = \left( \frac{1}{h} \frac{\eta_0}{\xi} \right) \int_0^\xi \psi \, d\xi
\]

and the non-isothermal rate of flow in terms of \( V_{\text{H}} \) (see definition (10.46)):

\[
v = \left( \frac{1}{h} \frac{\eta}{\xi} \right) \int_0^\xi \psi \, d\xi = \left( \frac{1}{h} \frac{\eta_0}{\xi} \right) \int_0^\xi \psi \, d\xi
\]

The quantity \( (\zeta_2-\zeta_1) \) (see definition (10.36b)) appearing in the aforementioned various thermohydrodynamic quantities is identical with that defined in terms of \( w \), i.e. of \( (w_2-w_1) \) (see definitions (10.42a) and (10.42b), where the latter is equivalent to the extent \( w \), viz.

\[
w = (w_2-w_1) = (\zeta_2-\zeta_1)\]
This will prove convenient when comparing the results obtained for the present particular case of $m=1$ with those of other values of $m$.

10.4.2.2 Results with $m=1$ in Slotte's relationship

The solution for the temperature distribution given by equation (10.45) accounts for thermal and kinematic asymmetry through $\zeta_{HB}$. An explicit representation of such an asymmetry is obvious if one replaces $\zeta$ by its equivalent involving the curvature at the apex of the curve for the temperature distribution, namely from equation (10.37b), i.e. which relates $\zeta$ and $w$ from definition (10.38), i.e.

$$w = \zeta - \zeta_{HB}$$

where from the previous definitions,

$$w = y^1(y - y_H); \zeta = y^1(y - y_B); \zeta_{HB} = y^1(y_H - y_B)$$

and where

$$y = \frac{y^1}{k}h^1$$

The solutions of the foregoing section for $m=1$ in terms of $\zeta$ can readily be re-expressed in terms of $w$ explicitly, whilst $\zeta_{HB}$ then appears as an explicit quantity representative of the degree of asymmetry concerned. Such a re-arrangement of these solutions will yield results that can be compared with those for various values of $m$ yet to be obtained in the next section. It will also facilitate the comparison of the results already evaluated for $m=\infty$, i.e. for the exponential viscosity-temperature relationship.

Milne's method of solution, which has been employed to analyse the non-isothermal flow of liquids following the exponential relationship, can equally well be employed for Slotte's relationship under study, i.e. $m=1$. His method somewhat simplifies the presentation of the results for $m=1$ and also facilitates a comparison with the results for the liquids with other values of Slotte's exponent $m$. Therefore, in the analyses to follow most results have been obtained through Milne's method after having verified the accuracy of these results for $m=1$ with that of those using fractional Bessel functions. The former accuracy indeed proved to be excellent even up to very high values of the quantity $w$. Only at extremely high values of $w$ does the accuracy become poor (see Chapter 9, cf. Figure 9.8). At such high values of $w$ the solution for temperature and thus for $\mu_{fl}/\mu$ oscillates as shown in Figure 10.12. There is, however, no need to study the solutions in the range beyond $w$-values depicting the first zero of the solution (see the reasons already advanced in Section 9.4.2.2 of Chapter 9, even though they have there been applied only to the symmetric case, i.e. the one with $\zeta_{HB}=0$). For the non-symmetric cases, or say for non-zero values of $\zeta_{HB}$, the qualitative nature of the solution remains unchanged. Thus, in the range of $w$-values that is covered in the present analysis, the accuracy of Milne's numerical method proves to be as good as that achieved by using the fractional Bessel functions.

For a wide range of curvatures at the apex of the curves for the various temperature distributions, some characteristic parameters have been tabulated in Table 10.1. The table provides data corresponding only to the maximum conceivable thermal loading, as it is specified by vanishing relative fluidity $\mu_{fl}/\mu=0$. The table also includes the previously ana-
lysed thermally and kinematically symmetric case, i.e. $C_{HB}=0$, of plan-
parallel pressure flows (see Chapter 9). Further, the results are most
general in that for various combinations of thermal and kinematic asymmetry,
or say for various $C_{HB}$-values, has a set of results been pre-
sented. The behaviour of the solutions for the entire range of operat-
ing conditions covered by the thermal states $\frac{\gamma H}{\gamma S_0}$, although only
for a few selected values of $\frac{\gamma H}{\gamma S_0}$ have been represented in Figures 10.13

$$\frac{\gamma H}{\gamma S_0}$$

through 10.15. It may be observed both from these figures and from Ta-
ble 10.1 that a finite value of $P=(w_2-w_1)^{\gamma H}$ the critical state $\frac{\gamma H}{\gamma S_0}=0$
and that this can not be exceeded in practice. This critical maximum
value of $P$, viz. $P_{\text{max}}$,

$$P_{\text{max}} = \frac{\frac{\gamma H}{\gamma S_0}}{\gamma H_{\text{max}}} \left( \frac{\gamma H}{\gamma S_0} \right)_{\text{max}} = u_{\text{max}}$$

is greatest for the thermally and kinematically symmetric case, $C_{HB}=0$, and
decreases with increased asymmetry specified by $C_{HB}>0$. The corresponding
non-dimensional maximum shear stress $\Gamma_{\text{max}}$ at the walls is simply,

$$\Gamma_{\text{max}} = \sqrt{\frac{\gamma H}{\gamma S_0}}$$

and also continues to decrease with increased asymmetry, as does the
sliding parameter $N_H$.

Distributions of viscosity, as represented through the relative
fluidity $\frac{\gamma H}{\gamma S_0}$, are nearly flat at small pressure gradient parameter $P$
which are located in the neighbourhood of $w=0$ (cf. $P=(w_2-w_1)^{\gamma H}$). This is
evident from the results provided in Figure 10.14 for $C_{HB}=0$, the parti-
cular case indicating from its definition a straight line at the apex of the
curves for the distribution of relative fluidity. With increasing
curvatures $C_{HB}$ the distributions of relative fluidity become increas-

$$\frac{\gamma H}{\gamma S_0}$$

Figure 10.13 Distribution of relative fluidity $\frac{\gamma H}{\gamma S_0}$ with the par-
parameter $w$ using Slotte's relationship with $m=1$ for
assumed negative values of $C_{HB}$ in the range
$0 \leq C_{HB} \leq 1.0$.
ing skewed, as depicted in Figures 10.13 and 10.14. At still higher curvatures, \( \psi_{HB} \approx 1.0 \), the distributions become more and more pointed at their apices (see Figure 10.15). Consequently, at high enough values of \( \tau \), close enough to \( \tau_{max} \), these distributions become triangular shaped and thus correspond to a delta function.

Also provided are the distributions of velocity in their normalized form, i.e. \( U^*/U_{Blas} \), as is the maximum conceivable dimensionless velocity referred to surface 1 and as it corresponds to the apex of the velocity profile. This form of representation of velocity distributions is sufficiently general for practical usage and shows the advantages already pointed out in Section 9.3.4.2 of Chapter 9. It may, however, be noticed that \( U_{B} \), \( U_{Blas} \) and \( U_{B} \) replace the previously used set of \( U_{H} \), \( U_{Bl} \), and \( U_{H} \). For problems with thermal and kinematic asymmetry, the former set proves to be the more convenient one in the present analysis of results. The results of symmetric flows of plan-parallel type treated in Chapter 9 remain nevertheless valid for the present two sets of normalized velocities, these sets then being identical because of the thermal and kinematic symmetry concerned.

The distributions of \( U_{B} \) indicate, as shown in Figure 10.16, a considerable amount of non-symmetry at reduced \( w^* \) falling close to the extremes of the range \(-0.5 \leq w^* \leq 0.5\), especially for \( \psi_{HB} \approx 0 \) where \( w^* \approx w/\psi_{HB} \). In such a range of \( w^* \), there prevail high values of \( \tau \) close to \( \tau_{max} \) following from \( \tau_{max} = \psi_{HB} \approx 1.0 \). The solutions beyond the limiting value \( \tau_{max} \), although existent, are of no physical significance. The details concerning the physical insignificance have already been presented in Chapter 9 (cf. Section 9.4.2.2).

For the curvatures given by \( \psi_{HB} \approx 0 \), the results depict a similar behaviour. These results have not been provided here due to lack of space.

The solutions beyond the limiting value of \( \tau \), i.e. \( \tau_{max} \), although existent, are of no physical significance. The details concerning the physical insignificance have already been presented in Chapter 9 (cf. Section 9.4.2.2).

10.4.2.3 Thermal Reynolds equation with \( m=1 \) in Slotte's relationship

A thermal Reynolds equation similar to that discussed in Section 10.3.6 for the exponential viscosity-temperature relationship, \( m=\infty \), can be formulated also for the present Slotte's relationship with \( m=1 \). With the assumption of heat generation due to viscous dissipation being uniform in the given film cross-section, Wilson and Mahdavian [10.9] have provided a thermal Reynolds equation for the particular operating conditions involving sliding and equal surface temperatures. In terms of the previously defined quantities \( E \) and \( R \), the former being defined using the coefficient \( \beta \) instead of \( \beta \), viz.:

\[
R = \frac{gh^2}{12\mu (gh-q)} - \frac{V^2}{12E} \quad E = \left( \frac{\mu U_{Blas}}{k} \right)^{\frac{1}{m}} (gh-q) \quad V = \frac{U_{Blas}}{12} \quad g = g_{0} \quad h = h_{0}
\]

from Figure 10.17 for the same values of \( \psi_{HB} \) as those employed in Figure 10.16.
10.4.3 Solutions with arbitrary positive integer values of m

It has already been pointed out in Chapter 2 that the variation of viscosity with temperature can for most lubricating oils be expressed sufficiently accurately by Slotte's relationship. For instance, at atmospheric pressure oils of silicone base can be characterized by \( m = 1 \), and those of fluorcarbon base and mineral oils by \( m = 2 \) to 5. For \( m = 2 \) Kagamov [10.10] and Joseph [10.11] have carried out analyses, if only for plan-parallel pressure flows with stationary walls. It has already been pointed out in Chapter 9 that the solutions obtained by these investigators are only qualitative in nature, in that they offer only upper and lower bounds to the complete solutions. Joseph's analysis is important since it specifies certain condition to be imposed on a flow to be thermally stable.

The governing equations (10.39) and (10.40) have already been formulated in Section 10.4.1. They are general enough to cover the complete range of \( m \), i.e. \( 1 \leq m < \infty \). Solutions for a few values of \( m \) in this range, excluding the two extreme cases of \( m = 1 \) and \( m = \infty \) which have already been dealt with, will be obtained in this section through the numerical technique previously employed in Chapter 9 for plan-parallel pressure flows as well as in Section 10.3 for general flows with \( m = \infty \). The latter analysis was carried out to obtain the dimensionless temperature as a function of the relative fluidity \( \frac{U}{\mu} \) and in terms of the quantity \( w \) for any specified curvature \( \frac{U}{\mu} \) at the apex of the curve for temperature distribution. The same analysis can be extended to the problem under study where a wide range of \( m \)-values, i.e. \( 1 < m < \infty \), in Slotte's relationship has to be covered.

10.4.3.1 Formulation of a computational programme for \( 1 < m < \infty \)

The method of solution in Section 10.3.2, including the evaluation of the starting values and the consideration of accuracy there discussed, hold good also for the present problems. Milne's method has been found to yield accurate results while the starting values are evaluated through Taylor's expansion in the neighbourhood of \( w = 0 \) as discussed in Appendices 10.A2 and 10.A3.

10.4.3.2 Results with positive integer values of m

The results thus obtained for the values \( m = 2, 3, 5 \), and 9 of Slotte's exponent \( m \) depict a trend which is qualitatively similar to that for the aforementioned extremes of Slotte's exponent \( m \), i.e. \( m = 1 \) and \( m = \infty \). At not too high values of \( w \), say in the range \(-1 < w < 1 \), the distribution of the relative fluidity, i.e. \( \frac{U}{\mu} \), is fairly flat for small values of curvatures at the apex, say up to \( \frac{U}{\mu} = 0.25 \). This is true of all liquids investigated, in a manner similar to the aforementioned two extreme cases. For illustration, results for only two values of \( m \), namely \( m = 2 \) and 5, have been provided in Figures 10.19a, 10.20a and 10.21a. At \( w \)-values farther away from the above-mentioned range, the viscosity ratio \( \frac{U}{\mu} \) drops less rapidly with increased \( m \)-axis. The latter ratio reaches zero only farther away along the abscissa, i.e. \( w \)-scale; the ratio tends very gradually to zero as depicted by the long-tailed ends of the corresponding curves. It is worth noting that the last behaviour is rather common
to all cases with \( m > 1 \) as against the previously discussed case of \( m = 1 \) for which the ends of the curves intersect the abscissa at a steep angle. At the corresponding values of \( \frac{\text{max}}{\text{min}} \), the family set of curves for \( m = 1 \) indicates near-adiabatic surfaces, its slope there being very small, whilst the latter case of \( m = 1 \) indicates a flow of considerable amount of heat to the boundaries. The former family, as already observed in Chapter 9, indicates that most of the dissipation effects are concentrated at the middle of the flow section.

The distribution of the normalized velocities for the complete operating regime, \( \frac{\mu H}{\mu B} \), have been presented for oils with \( m = 2 \) and 5 in

\[
0.00  \quad 0.25  \quad 0.50  \quad 0.75  \quad 1.00
\]

\[
0.00  \quad 0.50  \quad 0.7071  \quad 0.8660  \quad 1.00
\]

**Figure 10.19a** Distribution of relative fluidity \( \frac{H H}{B} \) with the parameter \( w \) using Slotte's relationship with \( m = 2 \) for assumed positive values of \( \zeta_{HB} \) in the range \( 0 < \zeta_{HB} < 1.0 \).

**Figure 10.19b** Distribution of velocity normalized using \( \frac{H}{B} \) as referred to surface \( 1 \) for Slotte's relationship with \( m = 2 \) and for assumed positive values of \( \zeta_{HB} \) in the range \( 0 < \zeta_{HB} < 1.0 \).

**Figure 10.20a** Distribution of relative fluidity \( \frac{H H}{B} \) with the parameter \( w \) using Slotte's relationship with \( m = 5 \) for assumed positive values of \( \zeta_{HB} \) in the range \( 0 < \zeta_{HB} < 1.0 \).

**Figure 10.20b** Distribution of velocity normalized using \( \frac{H}{B} \) as referred to surface \( 1 \) for Slotte's relationship with \( m = 5 \) and for assumed positive values of \( \zeta_{HB} \) in the range \( 0 < \zeta_{HB} < 1.0 \).

**Figure 10.21b** Distribution of relative fluidity \( \frac{H H}{B} \) with the parameter \( w \) using Slotte's relationship with \( m = 5 \) for assumed positive values of \( \zeta_{HB} \) in the range \( 0 < \zeta_{HB} < 1.0 \).

It may be observed that the results provided in this section cover a wide range of operating conditions. First, the entire regime of thermally affected flows, although only with a prescribed amount of thermal and kinematic asymmetry. Second, the entire regime of pressure gradients that can be imposed, i.e. at most reaching the maximum conceivable values of \( W \) or \( P \) which cannot be exceeded in practice. The corresponding limitations on sliding velocities and wall shear stress are apparent from expressions (10.49) and (10.55) respectively.
Figure 10.20b Distribution of velocity normalized using $U_{FB_{1,as}}$ referred to surface 1 for Slotte's relationship with $m=5$ and for assumed positive values of $\xi_{HB}$ in the range $0 \leq \xi_{HB} \leq 1.0$.

Figure 10.21a Distribution of relative fluidity $\mu_{FB_{1,as}}$ with the parameter $w$ using Slotte's relationship with $m=5$ for assumed negative values of $\xi_{HB}$ in the range $0 \leq \xi_{HB} \leq 1.0$.

*Owing to their bulkiness results for $E$, that is in terms of $V$, are not here provided. Tabulated results have for the present been stored on a computer for future use.*

10.5 DISCUSSION ON THE STABILITY OF THE FLOW

10.5.1 Instability due to point of inflection in the velocity profile

Disturbances in the flow tend to grow as to their energy level, and are determined by a criterion involving an energy stability limit in terms of a critical value $V_c$ for the kinematic viscosity. For the range of the stability limit $V_c$, there exists one stable and unique basic flow.
For $v = v_0$, the basic flow will become unstable. The nature of solutions in the range of the criterion $v_0$ is generally studied with the help of the family of stability, i.e., stability against disturbances of infinitesimal amplitude.

Linear theory of stability can be applied to divide the flow regime at high Reynolds numbers into isoviscous and viscous regions. In the viscous region the velocity distribution or the possibility of the occurrence of small disturbances changes. In such a region the results obtained using the linear theory of stability are highly sensitive to even small changes in the definition of the problem.

Hydrodynamic stability has long been studied, if only for isoviscous flow and through Rayleigh's inflection point criterion. According to this criterion there prevail neutral eigensolutions for Rayleigh's problem and an instability of the basic flow will occur if there exists a point of inflection. If Rayleigh's criterion is satisfied, the basic flow is always judged to be stable even in the limit $Re = \infty$, where $\alpha$ is the wave number. Whenever this criterion is satisfied there exists a band of wave numbers $\alpha$ for which the basic flow might become unstable.

Like the solutions of the linear theory of stability as they account for disturbances, the corresponding critical Reynolds numbers are very sensitive to the kind of disturbances assumed on some basic velocity profile. For a cubic basic velocity profile $F_0$ and Joseph [10.12] analyse the stability of plan-parallel pressure flows as a function of the parameter $\gamma = \frac{\delta}{\lambda}$ and for a viscosity-temperature relationship similar to Slotte's relationship with $m=1$. They accounted for temperature difference of the two walls. For thermally and kinematically asymmetric flows there appears first one point of inflection at an increased pressure gradient. At certain operating conditions, for instance due to still higher pressure gradients imposed, there may appear even two points of inflection in the velocity profile. For such operating conditions the discussions presented in Chapter 9 (cf. Section 9.5.2.2) are applicable.

For the flows with one point of inflection, however, there appear strange flow behavior only at higher values of $\gamma$. For $0 \leq \gamma \leq 1.6$ there appears to be no noticeable effect of the inflection point on the stability. Then the flow would behave almost inviscous. An asymmetry has a tendency to stabilise the flow. For the flows with $1.6 \leq \gamma \leq 3.2$ the role of the point of inflection seems to be significant on the stability of the flow. With increased pressure gradients, the critical Reynolds number monotonically decreases and the flow tends to be unstable at large wave numbers, as proposed by and Joseph [10.12], and Van derborck and Platten [10.13].

The physical significance of thus predicted instability of the flow due to the presence of the point of inflection in the velocity profile has been theoretically demonstrated unimportant by Joseph [10.14]. The instability concerned has been attributed to the arbitrary nature of the assumed velocity profile. The latter type of velocity profile may be compared with an asymptotic distribution of the velocity in a shear flow that tends to stabilise the motion, and thus results in higher critical Reynolds numbers.

The appearance of the point of inflection in the velocity profile and the corresponding instability need not be confined to the first branch of the solution in a manner similar to that established by Joseph [10.11], if only for the symmetric plan-parallel pressure flow. For asymmetric flows which are somewhat similar to shear flows, the point of inflection may appear also along the second branch of the solution.

---

* Solutions for asymmetric plan-parallel general flows undergo a transition where the quantity $K$ reaches a maximum, i.e., $K_{\text{max}}$. For details refer Section 10.3.4.5, and Section 9.3.4.1 of Chapter 9.
10.5.2 Critical Reynolds numbers corresponding to the inviscid mode

There exist critical Reynolds numbers giving rise to instability of the 'inviscid mode'. These may be evaluated basing on the isothermal flow considerations, viz.:

\[ \text{Re}_v = \frac{\rho u_{\text{av}} h}{v} \]

The above expression is valid for the exponential relationship. For Slote's relationship the Reynolds numbers may be evaluated by simply replacing \( \beta \) by \( \beta_v \).

A somewhat better estimation of the critical state at which the instability of the inviscid mode may appear is done using the average velocity corresponding to the non-isothermal flow influenced by the viscous dissipation effects. Then the corresponding Reynolds numbers may be given by,

\[ \text{Re}_v = \frac{\rho u_{\text{av}} h}{v} \]

Since the average velocity is not known in an analytical form, a general expression can be obtained for the present using the rate of flow given by expression (10.22) for the exponential relationship. Then the average velocity is given by,

\[ u_{\text{av}} = \frac{\beta}{\beta_v} \left( \frac{h}{h_{\text{av}}} \right)^{1/2} \]

Along lines similar to those mentioned for the symmetric plan-parallel pressure flows, the Reynolds number can be expressed by,

\[ \text{Re}_v = \frac{\rho h^{\frac{1}{m}}}{V} \]

For Slote's relationship the above expression for \( \text{Re}_v \) remains valid with the replacement of \( \beta \) by \( \beta_v \) wherever necessary.

10.5.3 Instability of thermal nature

Only one other type of instability which may be considered relevant from literature, is that arising purely from thermal effects. Joseph's analysis (10.14) is somewhat limited in nature, because it accounts only to an uncoupled diffusion equation. Any analysis that does not consider the coupled diffusion equation with the momentum equation may, after all, not yield any complete information on the stability of flow.

---

*The instability of 'inviscid mode' is not necessarily confined to isoviscous flows. It is prevalent even in viscous flows, and it originates at the point of inflection in the velocity profile. More details may be found in Chapter 3, Appendix 3.A2.*

10.6 CONCLUSIONS

Plan-parallel flows of the most general type are combined shear and pressure gradient assisted flows. These flows of which the two bounding (rubbing) surfaces may show sliding, including pure rolling as a particular case. Further, these surfaces may, again in general, show unequal temperature gradients, or say unequal heat partition ratios. This most general type of flows, therefore, is characterized by kinematic and thermal asymmetry. This asymmetry, in turn, may be accounted for in the mathematical formulation of the problem through the curvature of the temperature profile at its apex. This curvature provides also a measure of the offset of the apex of the velocity profile from that of the temperature profile. In the particular case of kinematic and thermal symmetry, the curvature reduces to zero; correspondingly, then the apices of the two profiles, viz. those of temperature and velocity, coincide and fall both at the axis of the channel.

Evaluation of the temperature distribution in the cross-section to be considered for the flow concerned is basic to the evaluation of the distributions of other thermohydrodynamic quantities such as velocity and heat flux; it is also required in the analysis of volumetric rate of flow and total heat flux in that cross-section. The evaluation of the temperature distribution, as well as that of temperature gradients, can accurately be carried out using Milne's predictor-corrector method for second-order ordinary differential equations. This method has yielded accurate results in a wide operating regime for all kinds liquids, as they are characterized by various values of the exponent \( m \) in Slote's viscosity-temperature relationship in the range \( 1 \leq m \leq 8 \).

In the study of flows thermally affected by viscous dissipation the exponential viscosity-temperature relationship, being the particular case of \( m = \infty \), has so far been the most widely employed one. The results that have here been obtained through Milne's method prove to be more accurate than those published by Aggarwal and Wilson [10.6]. In addition, the present results cover a far wider range of operation than that reported by these investigators. In fact they include also the extreme range of operation where \( \frac{\mu_1}{\mu_2} \), i.e. the ratio of the viscosity at the apex of the temperature distribution in the flow to that at its walls, assumes extremely small values.

When using the non-dimensional quantity related to pressure gradient \( P \), which includes the viscosity \( \mu_p \) (see definition (10.7c)), the present results indicate, like those of the symmetric plan-parallel flows, single-valuefulness of the rate of flow, and thus of the distributions of the velocity and the temperature, however, in terms of a similar quantity \( K \) (see definition (10.17)), which includes the viscosity at the channel
all these distributions are double-valued. Accordingly there exists a limiting value of K which cannot be surpassed. It would appear that both the existence and the importance of this limiting K has so far been overlooked by previous investigators; certainly when it comes to studying the most general kind of flows as it is crucial for lubricant films, that is the flows that are kinematically and thermally asymmetric.

The solutions that have numerically been obtained for all kinds of liquids characterized by $m \leq m_0$ are far too involved to apply in the analysis of non-isothermal lubricant films. The application is facilitated by employing the 'thermal Reynolds equation' in an accurately fitted empirical form. This empirical form relates the volumetric rate of flow to the imposed operating conditions including pressure gradient, sliding velocity and temperature differences (or partitioning of heat) of the bounding surfaces. The present solutions for the case of pure rolling with uniform and equal boundary temperatures yielding very high values of the rate of flow can be approximated with a single empirical thermal Reynolds equation. Solutions for the most general operating conditions and for various lubricating oils can in principle be approximated also with similar empirical equations.

Numerical solutions concerning liquids with finite values of the exponent m in Slotte's relationship display a similar trend as that with $m_0 \leq m$, for the exponential relationship. The particular case of $m=1$ can be studied with the help of either the Milne's numerical method or the rigorous one which results in the fractional Bessel functions. The solution for $m=1$ shows a finite value of B, as against an infinite value in the solution for the exponential relationship ($m_0=1$), at which $\eta_{	ext{th}}/U_T$ becomes zero. It also indicates a finite flow of heat to the walls at all operating conditions that result in $\eta_{	ext{th}}/U_T > 0$. One may therefore conclude that the flow of heat to the walls continues to take place even under the latter extreme operating conditions. So, in contrast to the case with $m=m_0$, for $m=1$ thermal effects need not be confined only to a narrow portion of the film close to the centre of the film. Indeed, in the latter case the thermal effects do spread over a considerable portion of the cross-section of the film.

As far as the stability of the flow is concerned, the kinematically and thermally asymmetric flows have not been investigated before. Therefore the conclusions drawn for the symmetric flows may be considered applicable to the present flows as well, at least in a qualitative sense. In this context, the transition point to the solution corresponding to the maximum conceivable value of K has certain significance. An instability of 'thermal mode' may prevail in the flows operating along the second branch of the solution beyond the transition point. Its significance, however, is limited as discussed in Chapter 3. Hydrodynamic instability of 'inviscid mode' is attributed to the point of inflection in the velocity profile of symmetric flows. Neither such an instability, however, is yet proven, nor the critical Reynolds numbers have been provided so far for the asymmetric flows. In the operating conditions under which the point of inflection might appear in the velocity profile can, nevertheless, be obtained from the present set of results.

10.7 REFERENCES


Plan-parallel films do not in general contribute to the load capacity of sliding/rolling element bearings. The load capacity is obtained from converging non-planar films due to hydrodynamic effects of the variation of viscosity with pressure along these films. This load capacity which in actual practice is generated nearly wholly by the pumping ("boosting") action in the inlet zone was the basic concept behind the isothermal theory of elastohydrodynamic lubrication, and enabled Grubin [11.1] (or rather the real originator, Ertel) to work out the theory for isothermal elastohydrodynamic films in a fairly simple form. This action may, like in conventional types of pumps, be described by the "pumping characteristic," which amounts to a relationship between the rate of flow (discharge) and the pressures generated. For the elastohydrodynamic film as a whole it is the rate of flow, \( q \), that is of primary importance since this determines the "flow criterion" to which the central film thickness, \( h_c \), may be put identically equal throughout the conjunctional zone, i.e. the zone directly following the inlet zone. The simplicity of Ertel-Grubin's analysis has raised considerable interest and stimulated certain investigators, who have used this concept to refine certain features which had not yet been considered in Ertel-Grubin's work, such as the so-called Petrusevich pressure spike and the constriction in the film profile near the exit of the film (see [11.2], [11.3]). Indeed, only thereafter have Dowson and Higginson [11.4] been able to account for such detailed features, if only at the expense of rather complex numerical evaluations and even though these were still confined to isothermal flow of the lubricating oil. Their analysis was soon found inadequate for explaining the experimentally known maximum in the trend of traction with sliding speed between the lubricated surfaces. Therefore a refined analysis that did account for thermal effects was essential.

In an attempt to include the effects of variation of local temperatures in lubricant films Crook in his series of papers ([11.5], [11.6], [11.7]) established that traction is strongly influenced by viscous-shear heating of the lubricating oil as it passes through the inlet, the conjunctional and finally the exit region. He also pointed out that, unless the tangential velocities of the rubbing surfaces with respect to the film are rather excessive, the effects of shear heating are only marginal in the inlet region so that the "pumping (boosting) characteristic" is usually hardly influenced by these effects. This involves that then the central film thickness, the one in the middle of the conjunctional zone, is affected by the viscous-shear heating only to a minor extent. His investigation on the thermal effects discovered has proved valuable, even though if applied only to moderately loaded elastohydrodynamic films. Further, it prompted other investigators to extend his analysis to heavily loaded films, and also to films subjected to fast sliding. A few of Crook's conclusions play a central role in the analysis to follow.

Heat generation due to shear heating is predominant from effects of pressure gradient caused by rolling (entraining) velocities the inlet zone and pure sliding in the conjunctional zone. While the rolling exerts a somewhat minor influence, the sliding proves to increase the maximum temperature in certain portions of the film to as high as 200°C already at moderate loads and speeds. Viscous oils, which might be used with an intention of obtaining a thicker film, yield surprisingly low frictional traction provided high speeds of sliding are employed. Further, heat transfer then takes place mainly by cross-conduction, the liquid convection of the heat along the film being negligible.

Conclusions drawn by the previous investigators on somewhat qualitative grounds, and there relating to the then assumed moderate loads and surface speeds, have in the present-day application been subjected to improved scrutiny for higher loads and surface (rolling) speeds. Present-day engineering designs of rubbing elements aim at heavier loads giving rise to a central film pressure up to several GPa (several tens of thousands bar) and surface speeds exceeding 100 m/s. What may happen to the elastohydrodynamic films under such extreme operating conditions is next to unknown. The non-isothermal analysis carried out in the foregoing chapters proves to be valuable for analysing such high-speed elastohydrodynamic films. Accordingly, certain limitations bound up with thermal effects, such as the limitation to the shear stress and thus to traction in shear flows and perhaps even the more crucial one concerned with the limitation to the pressure gradients, are expected to show up in the operation of thermo-elastohydrodynamic lubricant films.}

A schematic representation of elastohydrodynamic lubricating film for band-shaped contacts is depicted in Figure 11.1. A comparatively long inlet zone constitutes a film portion where the film thickness decreases monotonically and where the pressure distribution shows a monotonic increase with positive pressure gradients; following this comes nearly parallel conjunctional zone characterized by a small pressure drop; the pressure distribution undergoes large variations and displays the so-called "Petrusevich" pressure spike towards the exit; finally the exit zone characterized by both a sudden drop of pressure and a sudden widening of the gap. The figure depicts the results of analysis based on the basis of isothermal elastohydrodynamic calculation, i.e. the entire film and boundaries are considered to be at a uniform and equal temperature, or at least the effects of temperature on the viscosity are negligible, while the effects of pressure on viscosity are duly accounted for.
films will display distributions of pressure and film thickness different from those shown in Figure 11.1. Such films may well be strongly influenced by thermal effects in respects explained in the preceding chapters. At present it is well recognized that it is the thermal effects in so far as they occur in the inlet zone that are dominant in determining the magnitude of the film thickness in the conjunctional zone. Before going into details concerning these two quantities, i.e. the film thickness and thereby the viscous friction, we will devote attention to the changes the pressure distribution undergoes as a consequence of the thermal effects.

11.2.1 Pressures generated in thermo-EHD films

The pressure distributions in the inlet and conjunctional zones have a different nature: under operating conditions representative of practice the pressure distribution in the inlet zone, as it is generated by the boosting action, determines the overall level of film thickness in the conjunctional zone; that in the conjunctional zone is decisive for the load capacity of the elastohydrodynamic film. Thus a study of thermal effects on the pressure distribution both in the inlet and the conjunctional zones is directly relevant to the study of the performance of elastohydrodynamic films. In this connection it may be mentioned that the pressure distribution in the outlet region has only a limited significance.

Thermal effects give rise to considerable deviations of the actual, non-isothermal pressure distribution from that arising with isothermal flow. Examples of these two kinds of pressure distributions, both having been evaluated theoretically on the simplifying assumption of a uniform viscosity across the film (11.8), have been shown in Figure 11.2. These two pressure distributions may be compared only qualitatively since the present assumption is not always sufficiently representative. Only a more realistic assumption, then in conjunction with an adequate viscosity-pressure-temperature relationship and a satisfactory thermodynamic treatment of the flow of the lubricant concerned, can give a quantitatively correct picture of the pressure distributions. An analysis based on such refined models proves to be rather difficult in that it involves a set of governing equations that are non-linear. In view of such evaluational difficulties some of the measurements carried out by Schouten (11.9) and reproduced in Figure 11.3, as well as those of Kannel and Dow (11.10), and more recent ones by Janiczak and Hofman (11.11), are extremely valuable. However, the results available from such measurements are rather limited whilst their originators themselves do not provide a direct comparison with theoretically obtained isothermal pressure distributions. Even so, it may be concluded that the pressure distribution as affected by thermal effects in the conjunctional zone may well display a considerable deviation from the hypothetical isothermal one. Such a deviation, even when not resulting in a considerable change in load capacity, may well be distinctly influential as to other thermally affected quantities such as the rate of flow and the frictional losses, or say the generation of heat due to dissipation. Through the former quantity the film thickness in the conjunctional zone, and through the latter the temperatures in the film, may show considerable changes.

The pressure generation in the inlet zone, like the rate of flow, is promoted primarily by an increase in entraining velocity. In these respects sliding (slip) has only a secondary influence which only at sufficiently high sliding velocities becomes really important.

Thermal effects in the inlet region have so far been analysed mainly for various cases of pure rolling. Unfortunately, in general these analyses are incomplete in that they account inadequately for both the viscosity variation with temperature and the actual thermal boundary conditions. The lack of published information on more representative analyses, which may also include the effects of sliding, may be explained by their complexity.

While in the inlet zone the effect of pressure is to monotonically increase the viscosity of the lubricating oil as it passes through the successive cross-sections, the effect of temperature rise is opposite. Therefore, analyses neglecting thermal effects in the inlet zone are
Figure 11.3 Distribution of pressure along an elastohydrodynamic lubricant film under pure rolling with an entraining velocity $U = 6.28 \text{ m/s}$; Schouten [11.9].

deemed to over-estimate the boosting action, or say the rate of flow generated and thereby the critical film thicknesses, those in the conjunctional zone. Analyses including the latter effects do indicate a decreased boosting action as shown by Greenwood and Kauzlarich [11.12], Wilson and his associates ([11.13], [11.14]), and Goksem and Hargreaves [11.15]. Unfortunately, the lack of information in these papers on the pressure distribution in the inlet zone renders the interpretation in terms of the results of the preceding chapters next to impossible. For instance, there exists a need for improving such analyses through employing realistic thermal boundary conditions along the film for the successive film cross-sections. A refined theoretical analysis, although only for cases of pure rolling, will be given in Section 11.4 and results in delimiting the regimes of operation where thermal effects play a vital role in affecting the boosting action. An independent theoretical treatment for viscous friction affected by thermal effects will be presented in Sub-section 11.4.1.

11.2.2 Modes of heat transfer in EHD lubricant films

A fundamental study on the increase of temperature along elastohydrodynamic films, and also on the traction involved when some amount of sliding of the rubbing surfaces accompanied the rolling, was worked out by Crook [11.16]. By accounting for the thermal effects he succeeded in clarifying certain discrepancies between his own experimental results and the theoretical ones of Dowson and Higginson [11.17] and those of Archard, Gair and Hirst [11.18], all of whom had neglected the thermal effects of viscous-shear heating.

Extensive measurements were carried out by Crook to further clarify certain features of thermal effects on elastohydrodynamic films. Other investigators, putting Crook's hypothesis to test, have later performed even more refined theoretical analyses. Notwithstanding their attempts Crook's clarifications on thermal effects have proved valid at least in a qualitative sense. In the light of information published later, Crook's [11.16] approach can be reviewed as follows.

Viscous shear of the oil in the film gives rise to frictional heat and subsequent rise of temperature. In every individual cross-section, throughout the entire length of the film, heat transfer takes place mainly by cross-conduction. Only in zones far removed from the very highest pressure may convection play any significant role. Indeed, quite a number of later investigations have considerably substantiated the relative importance of the present modes of heat generation and heat transfer. These indicate that only minor discrepancies occur between the solutions obtained by including convection and those obtained by excluding it.

An analysis of relative orders of magnitude of heat transfer due to cross-conduction and liquid convection by Blok* has yielded certain important conclusions which prove their worth in studying quite a variety of elastohydrodynamically self-acting lubricant films. In carrying out this analysis he included thermal boundary conditions of the pertinent realistic type, i.e. those arising from flash temperature theory. In fact, these boundary conditions arise naturally because of the relative tangential movement of at least one of the two rubbing surfaces with respect to the film, and thus with respect to the heat source which is constituted by the distribution of the viscous-shear heat fluxes in the film. The pertinent heat withdrawal through the moving rubbing surfaces into the rubbing bodies thus arises as a process that comes natural to lubricant films, and accordingly may suitably be termed 'solid convection', in contrast to the aforementioned and more familiar 'liquid convection' inside the film.

Blok's conclusions on the conjunctional zone are qualitatively in agreement with Crook's. He emphasises in addition, at least in the more critical cases where the rubbing velocities are on the high side, that the effects of liquid convection may be neglected with respect to those of 'solid convection'. Unfortunately, in the many investigations that Wilson and his associates ([11.13] and [11.14]) have performed on the basis of their 'thermal inlet zone' analysis, they have ignored not only

*Private communication of Professor H. Blok in which he analysed the relative importance of 'solid convection' as against 'liquid convection' through a study of various elastohydrodynamic lubricant films in lubricated traction drives, rolling-element bearings, toothed gears, and cams and tappets.
the effects of liquid convection but also those of the decisive solid convection. Therefore, although their results may well be valuable up to moderate rubbing velocities, these are no longer trustworthy for the more critical high velocities.

11.2.3 Traction and viscous friction in thermo-EHD films

A considerable amount of attention has been devoted by various investigators to the prediction of traction, i.e., friction force at the surface of a driven member, in a pair of lubricated rubbing members. The traction is zero when there exists a pure rolling motion between the members. It is non-zero only when there is a slip between the driven and the driving member. In all cases of rolling and/or sliding of the lubricated contacts there prevail frictional losses due to viscous shear of the lubricating oil employed.

The interest in traction reflects itself in the large number of publications that have appeared on this subject over the last two decades. The reason behind this can be traced back to several objectives. One is concerned with the performance of traction drives such as mechanical continuously-variable transmissions. A second objective concerns the safe operation of elastohydrodynamic films. A third may be exemplified as follows: since the films often operate with their thicknesses close to the heights of the asperities on the surfaces to be lubricated, there might occur occasional metal to metal contacts. An increased traction of these films in operation may often be interpreted as direct evidence of the occurrence of such contacts, which would yield a far higher traction than that arising from the liquid friction alone.

The fourth and most important objective behind the wide interest in traction of elastohydrodynamic films concerns the study of the effects of the thermal processes involved on the highest temperatures to be expected in the rubbing system to be considered, that is, the highest oil temperature in the film and/or the highest surface temperature of the two rubbing bodies. For instance, oil temperatures may become impermissibly high in that they cause permanent changes in the composition of the oil through chemical degradation, such as by oxidation or even carbonization, such as by evaporation of its more volatile constituents and/or by those effects of shear stresses which is also symmetric. Thus the net shear stress and rolling friction, which, however, tends to be rather small as compared with that generated in the inlet zone. So, under pure rolling conditions the rolling friction, or say the frictional power loss, will be determined mainly in the inlet zone.

When relative sliding of the surfaces is introduced the variation of traction with sliding speed shows non-monotonic trend, such as the one depicted in Figure 11.5. The region corresponding with low sliding speeds shows a linear trend because the liquid may be assumed to show a nearly Newtonian behaviour whilst moreover the thermal effects may still be ignored. In the next region, which covers only a narrow range of sliding speeds, traction starts to depend noticeably on thermal effects, but not yet to any appreciable extent. What is more important, this is conjectural by many investigators, to be in general the region where also non-Newtonian effects start to play an appreciable role.
3. Heat transfer inside the film by cross-conduction alone is sufficiently representative of elastohydrodynamic films.

The aforementioned theoretical analysis, while justifying the theoretical consideration, yet calls for further investigations: firstly, one accounting for the realistic thermal boundary conditions as they appear in a natural way due to the passage of the moving boundaries through a region of intense heat generation due to viscous shear; secondly, one covering operating conditions more severe than those investigated in the above analyses, namely those concerning higher rolling speeds, relative sliding of the boundaries, and intense loading, as they appear in elastohydrodynamic films under severe duty. In fact, it is on these lines that the present analysis will be extended in what follows.

11.2.4 Temperature distributions along and across thermo-EHD films

Crook [11.16], although at the time only under somewhat idealized conditions, already pointed out that temperature rises resulting from the heating by viscous-shear are duly to be accounted for. Under operating conditions such that the increase of the surface temperatures along the inlet zone does not exceed a very few degrees centigrade the effects of the non-uniformity of the distributions of these temperatures may be ignored. In other words, for predicting the rate of flow generated in that zone and thereafter from the film thickness representative of the conjunctional zone, only can the aforementioned assumption about the uniformity of the present distributions be maintained to a reasonable approximation. However, the higher rolling speeds, and the thicker oils which are often employed for establishing elastohydrodynamic films, give rise to much more non-uniformity. Greenwood and Kauzlarich [11.12] have shown that with such increased oils* along the inlet zone can be as high as 19 °C already at moderate rolling speeds. One may then infer that at the rather excessive rolling speeds to which certain rolling-element bearings are nowadays subjected, the increase of temperature along the inlet zone must be high.

The temperature increases in the conjunctional zone of elastohydrodynamic films may well be still higher, as has been brought out theoretically by various investigators (see for instance, Houpert [11.23]). For lightly loaded, moderate speed elastohydrodynamic films average temperature increases in the conjunctional zones are expected to exceed 40 °C. Maximum temperatures to yield such rather high average values may be considered to be quite high in view of the peak in the temperature distribution which prevails especially in thermally and kinematically non-symmetric flows (cf. Chapter 10). Measurements by several investigators, if only of average temperatures at the exit of elastohydrodynamic films, indicate similar temperature increases even for low-viscosity liquids; see for example Conry [11.24]. Kannel and Dow [11.10] report similar...

*Viscosity variation with temperature of these oils has been characterized by Greenwood and Kauzlarich [11.12] through Slotte’s relationship. Their analysis is one of the few which have used this relationship and accounts for Slotte’s exponent m=3 to A=8 to 22 °C. Such values are indeed characteristic of mineral oils (cf. Chapter 2).
reproduced in Figure 11.6. Distributions showing peak temperatures that are so much higher that they would become crucial in view of any thermal degradation of the oil have not so far been reported. So far, distributions of temperature at various film cross-sections have been neither evaluated nor experimentally determined.

\[ a = 0.305 \times 10^{-3} \text{ m} \]
\[ b = 0.74 \times 10^9 \text{ N/m}^2 \]
\[ \Delta = 6.28 \text{ m/s} \]

Figure 11.6 Distributions of surface temperatures along elastohydrodynamic lubricant films for various sliding speeds \((u_0 - u_p)\text{m/s};\) Schouten [11.9].

11.2.5 Film profile of thermo-EHD films

As already pointed out before, the pressure distribution in the inlet zone plays a decisive role in the formation of a film in the conjunctural zone. Accordingly, thermal effects in the former zone play an indirect role as to the magnitude of the film thicknesses in the latter zone. For safe operation of elastohydrodynamic films it is crucial that the minimum film thickness attains at least a few times the c.l.a. (centre line average) roughness value of the surfaces involved. In fact, the danger of metal-to-metal contact is the highest at the 'nip' close to the exit end of the conjunctural zone. It has long been suspected, i.e. by Dyson et al. [11.26], that the effects of viscous-shear heating in the inlet zone of elastohydrodynamic films under pure rolling and rolling with sliding would be responsible for the experimentally observed reduction of the actual film thickness as compared to that predicted for isothermal flow in the film. Early analyses by Crook [11.16], Cheng and Sternlicht [11.18], and others were performed for operating conditions where shear heating proved negligible and the consequent temperature increase in the inlet zone to be low. This was questioned by Greenwood and Kauzlarich [11.12] who showed on theoretical grounds that for elastohydrodynamic films operating under pure rolling the shear heating in the inlet zone will not invariably be negligible. They demonstrated that rolling speeds as low as a few metres per second gave rise to a considerable amount of heating of the lubricant in the inlet zone as to a subsequent reduction of the film thickness in the conjunctural zone by about 25 percent. Their results showed film thicknesses different from the experimental ones; they were slightly higher than those of Cameron and Gregory [11.27] but slightly lower than those of Dyson et al. [11.26]. Their calculations were only indicative because they had excluded the effect of the combined effect of temperature and pressure on the viscosity, and had used the pressure distribution corresponding to isothermal conditions to calculate the film thickness under non-isothermal conditions.

Their approximated pressure distribution is the main source of uncertainty in Greenwood and Kauzlarich's analysis, and presumably explains the disagreement between their theoretical film thicknesses and the experimental ones. In an attempt to resolve this disagreement through considering the more representative pressure distribution for non-isothermal conditions in the inlet zone Wilson and his associates formulated and solved their "thermal Reynolds equation" which has already been discussed in detail in Chapter 10 (see Section 10.3.6). Whilst assuming that the viscous heating due to slip was not significant as compared to that arising from rolling action, and thus confining themselves to heating affected by the pressure gradients, Wilson and Wong [11.28] had arrived at the "thermal Reynolds equation". March and Wilson [11.13] used a simple empirical form for this equation while studying the variation of elastohydrodynamic film thickness again under pure rolling. Although similar lines Goksem and Hargreaves [11.15] also evaluated the variation of central film thickness as affected by shear heating in the inlet zone.
zone. They, in fact, carried out an evaluation of the film thickness at the exit end of the inlet zone and assumed it to be equal to a uniform film thickness over the entire conjunctional zone. Their results indicate that the central film thickness shows a maximum for a particular set of loads and rolling speeds. However, it deviated considerably from the isothermal value especially at high rolling speeds of about 6 m/s and higher. Indeed, the agreement between their theoretical film thicknesses and the aforementioned experimental ones proved to be very good. The reliability of these results, in conjunction with the simplicity with which the non-isothermal film thickness can be evaluated, has prompted Blok [11.29] to propose procedures to evaluate for which at a given speed (DN-value) for roller bearings the oil could be chosen such as to optimize its inlet viscosity for as large a minimum film thickness as possible.

In view of the experimentally recorded high rises in surface temperature along the conjunctional zone, it is far from realistic to assume uniform temperatures. Already in the inlet zone these temperature rises might reach at least a few degrees centigrade, even when the sliding speeds are comparatively low. At higher rolling and/or sliding speeds they are expected to be sufficiently high to influence the pressure distribution such as to reduce the central film thickness. So, for predicting rolling friction and temperature increase along thermo-elastohydrodynamic films more accurately a refined analysis would appear desirable. Such an analysis should account for the non-uniformity of the distributions of the surface temperatures as it is by the aforementioned 'solid convection' caused by the motion of the rubbing bodies relative to the film.

A thermal effect of crucial nature which has so far received only limited attention in tribologic literature is the limitation to the shear stress (cf. Chapters 3 and 5). Indeed some maximum shear stress cannot be exceeded and accordingly imposes a limitation on traction drives such as mechanical continuously-variable transmissions. A determination of this limitation as influenced by the non-uniformity of the distributions of the surface temperatures, as it results from the aforementioned 'solid convection' while still assuming Newtonian behaviour of the oil, should be one of the major aims of the evaluation of thermal effects. The Newtonian assumption would, of course, reduce the complexity of the analysis and yield solutions which constitute a basis of comparison for judging whether or not the non-Newtonian, or even or piezoviscous, behaviour (see Houpert [11.30]) actually occurs under certain, more or less critical, operating conditions. Another thermal effect of more crucial nature which has presumably not even been recognized in tribology is the limitation in the form of a maximum to the pressure gradient which cannot be surpassed (cf. Chapters 9 and 10). Thermoelastohydrodynamic films in present-day advanced applications undergo increasingly large variations of pressure along the film, and even a very severe one especially in the neighbourhood of the Petreushevich pressure spike. The repercussions of such large variations of pressure, for instance as to the stability of the flow in the film, and therefore of the film itself, well deserve more attention in the future.

11.3 GOVERNING EQUATIONS

The present model for the thermo-elastohydrodynamic analysis consists of two lubricated elastically deforming circular cylinders rolling against each other under a given load as shown in Figure 11.7. An analysis of generated pressures and pressure gradients, temperatures, film shapes, surface friction, etc. is governed by the consideration of hydrodynamic, energy and elastic equations. Of these three basic equations the former two can be combined to one single equation as already shown in Chapter 10, namely that given by the "thermal Reynolds equation", viz.

$$\frac{dp}{dx} = \frac{12 \mu \dot{u} (h+h_c)}{h^3}$$  \hspace{1cm} (11.1)

where $\frac{dp}{dx}$ is the pressure gradient in the downstream direction $x$, i.e. along the direction of the entraining velocity $\dot{u}=(u_i+u_j)/2$, $\mu$ is the characteristic viscosity which is yet to be specified, $h_c$ is the uniform film thickness in the conjunctional zone, $h$ is the local film thickness and $R$ is the local thermal correction function.

The elastic equation interrelating the distribution of hydrodynamic pressures to the elastic deformation of the loaded cylinders is too involved for using in the present analysis. This equation can, however, be approximated from the Hertzian theory through replacing the distribution of hydrodynamic pressures by the equivalent distribution of Hertzian pressures which gives rise to the same load capacity as that resulting from the former distribution. Then the Hertzian profile which is a component of the film profile $h$ outside the conjunctional zone may be employed for the analysis to follow in the form (see Timoshenko and Goodier [11.31]):

$$h = h_c = \frac{4W}{E'Y}x$$  \hspace{1cm} (11.2a)

where $W$ is the load per unit length, $E'$ is the reduced modulus of elast-
ticity, viz.

\[ E' = \left[ \frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2} \right] \]  

(11.2b)
in which \( E_1 \) and \( E_2 \) are the moduli of elasticity and \( v_1 \) and \( v_2 \) are the Poisson's ratios of rubbing bodies no. 1 and 2, and \( \bar{X} \) is the reduced coordinate relating the abscissa \( x \) to the Hertzian semi-width of contact \( a \), i.e. \( x = a/\bar{X} \) (see Figure 11.7), in the form,

\[ \bar{X} = |x| (x^2-1)^{-1} - ln[|x| + (x^2-1)^{1/2}], |x| > 1 \]  

(11.2c)

\[ \bar{X} = 0, |x| < 1 \]  

(11.2d)

In addition to the aforementioned two governing equations, namely the thermal Reynolds equation and the simplified elastic equation (11.2a), there is a need to account for the realistic thermal boundary conditions at the rubbing surfaces that are specified by the flash temperature theory, viz.

\[ T_{1,2}(x) = T_{b1,2} + \frac{\sqrt{b_1/\beta v_1}}{\sqrt{1/\epsilon^2}} \int_{0}^{\epsilon} \frac{q_{\text{tot}}(x)}{\sqrt{\epsilon^2-x^2}} \, d\epsilon \]  

(11.3)

where \( \xi \) is the coordinate related to \( X \) through,

\[ \xi = X - \lambda \bar{X} \]

\( b = \sqrt{4\rho c} \) is the thermal contact coefficient of the solid rubbing material concerned in which \( k \) is the thermal conductivity, \( \beta \) is the density and \( c \) is the specific heat. In expression (11.3) \( u \) is the surface velocity, \( q_{\text{tot}} \) is the total heat flux in the film cross-section \( \xi \), and subscripts 1 and 2 refer to rubbing surfaces no. 1 and 2. Detailed mathematical analyses of the above-mentioned equations for the thermo-elastohydrodynamic theory will be treated in the sections to follow.

11.4 SIGNIFICANCE OF THE "THERMAL REYNOLDS EQUATION" IN THERMO-ELASTICITY THEORY

Let us first consider the conventional Reynolds equation, as it applies to isothermal lubrication, whether this be elastohydrodynamic or not. Because of the isothermal nature of this kind of lubrication, the energy equation is irrelevant whilst the distributions of the surface temperatures along the film cannot but be assumed uniform and equal to each other.

The theoretically simplest cases of isothermal lubrication are those where the flow in the film is isoviscous, the effect of pressure on viscosity being ignored so that viscosity is assumed uniform throughout the film. The uniform viscosity is then of course taken identical with the entry viscosity, \( \mu_p \), as it occurs for the given oil at the entry temperature and at the entry pressure, or say at zero gauge pressure. So, to determine this entry viscosity for various oils and various entry temperatures one needs to know only the isobaric viscosity-temperature relationship at atmospheric pressure for the oils concerned.

The most general cases of isothermal lubrication are the non-isoviscous ones where viscosity \( \mu(p) \) is allowed to vary only with the film pressures, \( p \), the temperature still being assumed uniform throughout the film. So, for these cases one needs to know only the isothermal viscosity-pressure relationships, \( \mu(p) \), for the various oils at the inlet temperature chosen. Like in the isoviscous cases the energy equation does not come into play at all. Further, as shown by Weibull in 1914, since Reynolds's equation is still valid for these non-isoviscous isothermal cases, it can be reduced to that for the isoviscous cases by introducing the "reduced pressure", \( p^* \), defined by,

\[ \frac{dp^*}{dp} = \frac{1}{\mu(p)} \]  

(11.4a)
or, say,

\[ p^* = p \frac{\mu(p)}{\mu(p_{\text{tot}})} \]  

(11.4b)

so that the actual pressure gradient, \( dp/dx \), may be expressed by

\[ \frac{dp}{dx} = \frac{dp^*}{dx} \]  

(11.4b)

whilst in the usual boundary conditions for the actual pressure distribution \( p(x) \) to be assessed, and which in general may contain \( p=0 \) and \( dp/dx=0 \), one may introduce instead \( p^*=0 \) and \( dp^*/dx=0 \), respectively. It may finally be observed that for the isothermal non-isoviscous cases, Reynolds's equation represents a relationship between the distribution of the actual film pressures, \( p(x) \), or rather that of the pressure gradients \( dp/dx \), and the equally actual film profile \( h(x) \), as the latter may be mechanically and thermally distorted in operation. When the actual film profile \( h(x) \) is band-shaped and known, for instance from measurements during operation, Reynolds's equation must be conceived as an ordinary differential equation from which, together with the boundary conditions imposed upon the pressure distribution \( p(x) \) sought, this distribution may be determined by integration along the film.

Now, for the most general cases where, such as in thermo-elastohydrodynamic lubrication, the flow in the film is to be considered non-isothermal, Reynolds's classical equation may usefully be replaced by a more general one. Then the previously defined "reduced pressure" for the present non-isothermal case may be obtained from,

\[ \frac{dp}{dx} = \frac{1}{\mu(p_{\text{tot}})} \]  

(11.5a)
or, say,

\[ p^* = p^0 \frac{\mu(p_{\text{tot}})}{\mu(p)} \]  

(11.5b)

so that the actual pressure gradient \( dp/dx \) may be given by,

\[ \frac{dp}{dx} = \frac{dp^*}{dx} \]  

(11.5b)

where \( \mu_{\text{tot}} \) is the "ambient" viscosity at the entry to the elastohydrodynamic film concerned. In fact, as has been shown by W.B.D. Wilson and associates, the classical Reynolds equation may under certain assumptions be replaced by a generalized one, i.e. by what they call the "thermal" Reynolds equation. These assumptions are that both the heat conduction along the film and the conventional heat transport inside the
film may, like the compressive heating, be neglected altogether with respect to the heat conduction across the film. Then the energy equation may be simplified such that for the individual film cross-section concerned it may be combined with both the classical Reynolds equation and the isobaric viscosity-temperature relationship for the local pressure to one single equation, the "thermal" Reynolds equation. The latter equation, like the classical one, is valid for every individual cross-section, thus again representing a relationship between the distribution of pressures \( p(x) \) (or rather that of the pressure gradients) and the actual film profile \( h(x) \). But since, in contrast to the classical Reynolds equation, the "thermal" one contains the two local surface viscosities \( \mu_1(x) \) and \( \mu_2(x) \) as quantities unknown a priori, some adequate kind of thermal boundary condition will have to be introduced for either rubbing surface. One of the approximate methods is due to Wilson and associates, which assumes a simplified thermal boundary condition consisting of uniform surface temperatures all along the elastohydrodynamic films concerned.

A more accurate and realistic specification of thermal boundary condition along either of the rubbing surfaces is deemed important for a more generally valid analysis of thermo-elastohydrodynamic lubrication. Such a kind of boundary condition is presented by that following in a straightforward manner from the "flash temperature theory". So, one of the two "flash temperature" boundary conditions interconnects the distribution of the flash temperatures \( T_f(x) = T_1(x) - T_2(x) \) to that of the heat fluxes \( \phi_{tot}(x) \) along surface No. 1, whilst the other interconnects \( T_f(x) = T_2(x) - T_0 \) to \( \phi_{tot}(x) \) (note that \( T_1(x) \) and \( T_2(x) \) stand for the two distributions of the actual surface temperatures, whilst \( T_0 \) denotes the bulk temperatures of surfaces Nos. 1 and 2, respectively). So, whereas the energy equation is no longer involved in any explicit manner when it comes to integrating the thermal Reynolds equation the viscosity-temperature-pressure relationship still is.

For the thermo-elastohydrodynamic film analysis to follow the classical isothermal Reynolds equation, which is generally valid for every successive cross-section throughout the film, viz.

\[
\frac{dp}{dx} = 12 \frac{u_1}{h_f} \frac{(h - h_f)}{h_f} R \tag{11.9b}
\]

In expressions (11.9a) and (11.9b) \( \delta \) is the entraining velocity \((u_1 - u_2) / 2\), \( h_f \) is the uniform film thickness in the conjunctional zone and \( h \) is the film thickness at any local position concerned along the film.

If in the cross-section at the transition of the inlet and the conjunctional zones the boundary condition is specified identical with that of Ertel-Grubin, i.e. the reduced pressure there being put equal to its asymptotic value \( 1/a \), then

\[
p_a^* = 1; \quad B = 0 \tag{11.7}
\]

While integrating equation (11.9b) for the pressure distribution it should be considered that in every given film cross-section, or say for the value of the pressure gradient \( dp/dx \) concerned in the "thermal" Reynolds equation, either of the two surface viscosities \( \mu_1(x) \) or \( \mu_2(x) \) be taken at the same local pressure \( p(x) \), the latter being however unknown a priori. So, for any given individual film cross-section, the pressure \( p(x) \) being constant throughout that cross-section, we need know only the isobaric viscosity-temperature relationship at that same (unknown) pressure so as to be able to express the two local surface viscosities \( \mu_1(x) \) and \( \mu_2(x) \) in terms of the local surface temperatures, \( T_1(x) \) and \( T_2(x) \), resulting from the aforementioned flash temperature theory.

It may be recalled that the thermal Reynolds equation, i.e. equation (11.9b), contains the thermal correction function \( R \), which represents the thermal effects on the flow. For non-isothermal flows \( R \) lies in the range \( 0 \leq R \leq 1 \) and approaches unity for isothermal flows for which the thermal effects may be considered negligible. Now, \( R \) depends on the assumed isobaric viscosity-temperature relationship characterizing the ac-

\[
\frac{dp}{dx} = e^{ap} \frac{dp}{dx} \tag{11.6d}
\]

Integration of expression (11.6d) yields the reduced pressure \( p^* \) in the form,

\[
p_a^* = 1 - e^{ap} \left( T_1^* - T_2^* \right) \tag{11.7}
\]

For convenience the second term on the right-hand side may be replaced by \( B \), viz.

\[
B = e^{ap} \left( T_1^* - T_2^* \right) \tag{11.8}
\]

For the thermo-elastohydrodynamic film analysis to follow the classical isothermal Reynolds equation, which is generally valid for every successive cross-section throughout the film, viz.
ual variation of the viscosity with temperature of the lubricating oil chosen. This dependence of $R$ in the thermal Reynolds equation concerned has already been discussed in Section 10.3.6 of Chapter 10. Further, whilst the so-called “elasticity” equation which interrelates the elastic distortion of the two rubbing surfaces and the film pressure distribution sought, has not yet come into play in the derivation of the thermal Reynolds equation, it must in the integration of this equation along the film.

11.5 THERMAL INLET ZONE ANALYSES OF THERMO-EHD FILMS

In order to simplify such analyses, Ertel-Grubin’s concept will here be employed. Physically his concept boils down to the pumping or boosting action of the inlet zone, which relates the rate of flow or discharge to pressure differences generated, and to other influential quantities.

The conjunctional region can be approximated by the Hertzian parallel one that results from the deformation of unlubricated surfaces when they are subjected to the same load as on the lubricated surfaces. Now introducing as an approximation that the deformed unlubricated surfaces are separated at some constant distance, say $h$, and that the deformed surfaces outside the conjunctional region have a shape identical with the one following from Hertz’s contact theory, we may expect to obtain a reasonable approximation to the distribution of pressure in the inlet zone, that of velocity of flow and shear stresses, etc. In the conjunctional zone, however, the film thickness is not actually uniform throughout, especially not at heavy loads and high speeds. Numerical results obtained by various investigators have, in fact, revealed this non-uniformity of the film thickness in that, e.g., near the exit zone there is a noticeable constriction or “nip” where the film thickness reaches a minimum. At this constriction, and immediately behind it, the pressure drops steeply to the ambient level.

Ertel-Grubin’s concept is a convenient one and has proved itself sufficiently accurate especially in inlet zone analyses, as already shown by various investigators. A recent verification for isothermal elastohydrodynamic films has been provided by Terrill [11.32]. This concept was originally applied only to isothermal elastohydrodynamic lubrication and thus resulted in overestimates of central film thickness, albeit not to any appreciable extent. Wilson and his associates ([11.13] and [11.28]) extended the application of Ertel-Grubin’s concept by including it in their thermal inlet zone analysis, thereby accounting for the thermal effects on the central film thickness. In what follows we will develop an analysis which, although retaining certain features of their analysis, will be refined in several respects.

In the present analysis the Hertzian solution to the band-shaped contact along two rubbing cylindrical rollers with radii of curvature $R$ and $R'$, and subjected to a load per unit length, $W'$, will be considered on the basis of Figure 11.7. The maximum Hertzian pressure at the axis of symmetry is then given by,

$$ P_H = \frac{8W'}{\pi E R'} \left( \frac{R + R'}{R R'} \right) $$

(11.10a)

where $a$ is the semi-width of Hertzian contact, and

$$ a' = \frac{R + R'}{R R'} $$

(11.5)

is the so-called “radius of conformity”. The semi-width of the Hertzian contact is given by,

$$ a = \left( \frac{8W'}{\pi E} \right)^{1/2} $$

(11.10b)

Introducing a non-dimensional group $W^*$ for load, viz.

$$ W^* = \frac{W'}{R^2} $$

(11.11)

the above expressions for $P_H$ and $a$ can be re-expressed in convenient forms, viz.

$$ P_H = \frac{W^*}{2 \pi R} $$

(11.12a)

and

$$ a = \left( \frac{8W^*}{\pi} \right)^{1/2} R $$

(11.12b)

The aforementioned “elasticity equation”, which interrelates the elastic distortion of the two rubbing surfaces and the Hertzian pressure distribution, contains in its general form an integral (see Timoshenko and Goodier [11.31]) which poses certain computational difficulties. A differential version of the elasticity equation facilitating the numerical analysis has been provided by Weber [11.33]. For the present, however, an approximated version relating the Hertzian deformed contour of the rubbing surfaces along the inlet zone to the imposed load $W'$ per unit length of the band-shaped contact has been employed, i.e. the second term on the right hand-side in the following expression:

$$ h = h_c + \frac{W'}{2\pi E} \bar{x} $$

(11.13a)

where $h$ is the actual (deformed) film profile, $h_c$ is the aforementioned uniform central film thickness in the conjunctional zone and $ar{x}$ is the reduced coordinate already given through equation (11.2c), viz.

$$ x = (x' - \bar{x}) \sqrt{1 - (x' - \bar{x})} \right) \right) $$

Through introducing the non-dimensional quantity $W^*$, the film profile $h$ may be given by,

$$ h = h_c + \frac{4W^*}{\pi} \bar{x} $$

(11.13b)

Expression (11.13b) characterizes the actual film profile in non-dimensional form along the inlet zone of elastohydrodynamic films.

Now, the coordinate $x$ can be re-expressed in terms of $\xi$, facilitating the analysis to follow, i.e.

$$ x = x - x_in $$

(11.14a)

where $x_in$ corresponds to the entry to the inlet zone at which $\xi$ reduces to,

$$ \xi = 0 $$

Further, the exit of the inlet zone, which is identical with the entry to the conjunctional zone, is given by

$$ \xi = 1 - x_in $$

(11.14b)
The film profile \( h \) in the inlet zone, which is given by expression (11.13b), can now be expressed in the convenient form,

\[
A \, \xi, \quad h = h_{c} + \frac{2}{n} \frac{u^{*} - \mathbf{R}}{R'}
\]

where \( \xi \) replaces \( \mathbf{x} \) and is given by,

\[
\xi = \left[ 1 - \xi_{o} \right] \left[ \left[ 1 - \xi_{o} \right] \xi^{-1} - 1 \right] - \ln \left[ 1 - \xi_{o} \right], \quad \left| 1 - \xi_{o} \right| > 1
\]

and, where

\[
h_{c} = \frac{4 \, u^{*} \eta^{*}}{R'}
\]

so that,

\[
h = \left[ h_{c} + \frac{2}{n} \frac{u^{*} - \mathbf{R}}{R'} \right]_{\text{entry}} - \left[h_{c} + \xi \right]_{\text{exit}}
\]

For analysing thermo-elastohydrodynamic films, expression (11.13b) (or alternatively expression (11.17)) will be employed along with the thermal Reynolds equation (11.9b). It will be shown in the analysis to follow that these two expressions can be combined to one single expression relating the distribution of hydrodynamic pressure in the film to the film profile throughout the inlet zone concerned. For solving the resulting "combined" expression, however, the flash temperature boundary condition at least along one of the rubbing surfaces has yet to be specified for characterizing realistic thermal boundary conditions at the rubbing surfaces concerned. An expression for the flash temperature condition as well as a method of solution for the thus resulting set of two expressions, viz. the "combined" expression and the flash temperature boundary condition, will be dealt with in the sections to follow.

11.5.1 Friction of EHD films for pure rolling (thermally symmetric case)

Only the problem of pure rolling, i.e. in the complete absence of sliding, of a counterformal pair of rubbing surfaces will here be studied. Then, in any film cross-section the shear stress \( \tau_{s} \) at either surface due only to the local pressure gradient \( dp/dx = -g \) is given by,

\[
\tau_{s} = \frac{dp \, h}{dx} \left( \frac{h}{2} \right)
\]

where \( h \) is the local film thickness, and where this shear stress (including its sign) represents the one resulting at the surface due to viscous forces of the oil adjacent to it. Friction force \( F \) per unit length at either of the rubbing surfaces due to viscous friction can then be evaluated in a convenient form by considering the entire length of the elastohydrodynamic film concerned using,

\[
F = \int_{\text{entry}}^{\text{outlet}} \tau_{s} \, dx
\]
11.5.2 Distribution of surface temperatures

In order to evaluate the distribution of the surface temperatures along the present films one may evaluate, first of all, the distribution of the total heat fluxes \( \dot{\Phi}_{\text{tot}} \) (see Appendix 11.A2). The total heat flux for a plan-parallel flow has already been evaluated in terms of the operating conditions specified in Chapter 10. For the present films (pure rolling) the total heat flux has been given in a convenient form by expression (9.16c) (see Chapter 9), viz.

\[
\dot{\Phi} = \frac{g}{h} \dot{\Phi}_{\text{tot}} = \sqrt{T \left( \frac{k}{h} \right)^3 g}
\]

where in the present cases one should take

\[
\sqrt{T} = \left( \frac{g}{h} \right)^3 \frac{1}{|s|} h^3
\]

The value of \( \dot{\Phi}_{\text{tot}} \), i.e.,

\[
\dot{\Phi}_{\text{tot}} = \frac{|s|}{q}
\]

is simply the product of the rate of flow \( q \) and the modulus of the local pressure gradient. Since the rate of flow has one and the same value in every successive cross-section over the entire length of the film, the distribution of the total heat flux is simply proportional to the distribution of the modulus of the pressure gradient along the film.

The local pressure gradient can be expressed in a convenient non-dimensional form by employing the previous definition (11.8) for \( B \), viz.

\[
b = e^{\alpha} \dot{B} \left( T_s - T_b \right)
\]

where \( T_s \) and \( T_b \) refer to the surface and bulk temperatures of either rubbing body, and where the suffices 1 and 2 may be deleted since we confine ourselves to the thermally symmetric films where not only the two bulk temperatures are equal but, accordingly, also the two distributions of the surface temperatures. Substituting this expression in equation (11.6b) while representing either of the rubbing surfaces through subscript \( s \),

\[
\frac{\partial B}{\partial \xi} = - \frac{1}{\partial \xi} \frac{\partial \dot{B}}{\partial \xi} = - \frac{\dot{B}}{\partial \xi} \left( \dot{B} - T_s - T_b \right)
\]

In order to evaluate the total heat fluxes in the various cross-sections along the inlet zone of the film, it is required to employ the rate of flow \( q \). This can be done conveniently in terms of the "thermal back flow parameter", \( E \), from expression (10.32a), namely

\[
E = \frac{g}{h} \left( \frac{k}{h} \right)^3 \frac{1}{|s|} (\dot{B} \dot{B} - T_s - T_b)
\]

where

\[
v = \frac{\dot{B}}{h} \left( \frac{k}{h} \right)^3 \frac{1}{|s|} \dot{B} \dot{B} - \frac{g}{h} \left( \frac{k}{h} \right)^3 \frac{1}{|s|} \dot{B}
\]

in which \( \mu_s \) is the local viscosity of the lubricant at the surface temperature and at the pressure in the cross-section concerned, and \( \dot{u} \) is the entraining velocity which here is equivalent to the surface velocity \( u \) of either of the rubbing bodies. The rate of flow can now be expressed in the following form,

\[
q = \left( \frac{k}{h} \right)^3 \frac{1}{|s|} \left( \frac{k}{h} \right)^3 \frac{1}{|s|} \dot{u}
\]

For the volumetric rate of flow to remain constant in every successive cross-section, so as to satisfy the continuity equation, the right-hand side should also be constant. Notwithstanding the individual quantities, i.e., \( \mu_s \), \( h \), \( E \) and \( \dot{u} \), vary, it suffices to evaluate the right-hand side only for one convenient cross-section, say for that at the entry to the inlet zone, and to use the same value in every successive cross-section.

The distribution of surface temperature along each of the rubbing surfaces can then be obtained by employing the total heat flux given by expression (11.22) as shown in Appendices 11.A3 and 11.A4 (see also equation (11.3)), namely

\[
T_s (\xi) = T_b + \frac{g}{h} \left( \frac{k}{h} \right)^3 \frac{1}{|s|} \dot{B} \dot{B} T_s \int_0^{1/\sqrt{c_e}} T_s \left( \frac{c_e}{1/\sqrt{c_e}} \right) dc_e
\]

where \( T_b \) represents the bulk temperature and \( b \) stands for the "thermal contact coefficient" of the material used for the rubbing bodies, being defined by

\[
b = \sqrt{\frac{T_s}{c_e}}
\]

and where \( k \) is the thermal conductivity, \( \rho \) the density and \( c \) the specific heat of the material per unit mass. An inversion to equation (11.26a) which proves to be more convenient in certain analyses can be given by,

\[
T_s (\xi) = T_b + \frac{g}{h} \left( \frac{k}{h} \right)^3 \frac{1}{|s|} \dot{B} \dot{B} \left( \frac{c_e}{\sqrt{c_e}} \right) \int_0^{1/\sqrt{c_e}} T_s (\xi') \left( \frac{c_e}{1/\sqrt{c_e}} \right) dc_e
\]

where the positive sign in the left-hand term refers to surface 1 and the negative sign to surface 2. Introducing expression (11.25) for the rate of flow as well as the pressure gradient given by expression (11.23) into expression (11.26a) for the distribution of surface temperatures \( T_s (\xi) \) at the boundary of either of the rubbing surfaces, one obtains:

\[
T_s (\xi) = T_b + \frac{g}{h} \left( \frac{k}{h} \right)^3 \frac{1}{|s|} \dot{B} \dot{B} \left( \frac{c_e}{\sqrt{c_e}} \right) \int_0^{1/\sqrt{c_e}} T_s (\xi') \left( \frac{c_e}{1/\sqrt{c_e}} \right) dc_e
\]

which relationship covers the entire inlet zone, or say the range \( 0 \leq \xi' \leq 1 \).

The modulus in the integrand of equation (11.27) stands for the heat flux which attains positive values for both the positive and the negative pressure gradients involved in \( dB/d\xi' \). One may obtain the flash temperature boundary condition (11.27) in terms of basic influential quantities and \( B \) by substituting definition (11.16b) for \( H' \), viz.:
given by expression (11.10b),

$$T_f(t) - T_s(t) - T_b = \frac{q}{\sqrt{\text{a} \cdot \text{b}}} - \int_0^t \left[ -\frac{\partial p}{\partial t} + \frac{\partial T_f}{\partial t} \right] \frac{4\text{c}'}{\sqrt{\text{c}'^2}} \, dt' \, (11.28b)$$

Along the following lines can expression (11.27) be re-expressed in terms wholly of basic influential quantities, thus reflecting their relative importance in the flash temperature boundary condition. Firstly, writing the definition for B from expression (11.8), viz.

$$B = e^{-\text{p} \cdot \text{t} \cdot \text{r} \cdot \text{i}} + \text{e} - \text{p} \cdot \text{t} \cdot \text{r} \cdot \text{i}$$

and substituting

$$\left[ \frac{1}{\text{B} \cdot 4\text{c}'} \right] = \left[ \frac{\partial p}{\partial t} \ln \left[ -\text{e}^{-\text{p} \cdot \text{t} \cdot \text{r} \cdot \text{i}} \right] \cdot \text{B} \cdot 4\text{c}' \right]$$

into equation (11.27), the condition concerned can be cast into the form

$$T_f(t) \times \left[ \frac{1}{\sqrt{\text{a} \cdot \text{b}}} \right] - \int_0^t \left[ -\frac{\partial p}{\partial t} + \frac{\partial T_f}{\partial t} \right] \frac{4\text{c}'}{\sqrt{\text{c}'^2}} \, dt' \, (11.28c)$$

It may be noticed that this expression shows explicitly, the combined effect of the distribution, $p(t')$, of the film pressures and that, $T_f(t')$, of the flash temperatures. This interrelationship can be represented also in the form,

$$\sqrt{\text{a} \cdot \text{b}} \cdot \left[ -\frac{\partial p}{\partial t} + \frac{\partial T_f}{\partial t} \right] \frac{4\text{c}'}{\sqrt{\text{c}'^2}} \, dt' = \int_0^t \left[ -\frac{\partial p}{\partial t} + \frac{\partial T_f}{\partial t} \right] \frac{4\text{c}'}{\sqrt{\text{c}'^2}} \, dt' \, (11.28d)$$

which should be satisfied identically for any film cross-section over the entire length of the film.

But it may be noted that for the computational technique, which will be treated later in this chapter, equation (11.27) is the most suitable one. Remarkably enough, this equation does neither involve the distribution of the viscosity $\text{u}_0$ nor distributions of the previously employed non-dimensional quantities $B$ and $V$. Accordingly, it is no need to evaluate the distributions of these quantities. It is admitted though, that their evaluation proves worthwhile, especially in the discussions of certain thermal limitations of the flow in the sections to follow.

### 11.5.3 Central film thickness of thermo-EHD films

In order to obtain the aforementioned friction force on the rubbing surfaces and the distribution of surface temperatures of elastohydrodynamic films under pure rolling in thermally symmetric cases, the distributions of pressure, pressure gradient and those of the thermal back flow parameter $R$ and the thermal rolling parameter $V$ are required to be evaluated. These distributions can be obtained through employing the previously analysed "thermal Reynolds equation" which has been dealt with in Chapter 10. While employing this equation, i.e. equation (11.9b), in order to satisfy the requirement of continuity of flow along the film for the rolling problem concerned Wilson and Wong [11.28] derive the following relationship (expression (10.31) of Chapter 10), viz:

$$E^2 - \frac{E}{E' + 1} \frac{\partial T_f}{\partial t} \left( \frac{\partial T_f}{\partial t} + 1 \right) = 0 \, (11.29)$$

where the thermal back flow parameter $E$ is given by,

$$E = \left( \frac{\text{h} \cdot \text{t} \cdot \text{r} \cdot \text{i}}{\text{h} \cdot \text{b} \cdot \text{t} \cdot \text{r} \cdot \text{i}} \right) \left( \text{h} \cdot \text{b} \cdot \text{t} \cdot \text{r} \cdot \text{i} \right) \, (11.30a)$$

and $R$ is defined in terms of the influential quantities for a thermally affected flow through:

$$R = \frac{1}{\text{h} \cdot \text{b} \cdot \text{t} \cdot \text{r} \cdot \text{i}} \left( \text{h} \cdot \text{b} \cdot \text{t} \cdot \text{r} \cdot \text{i} \right) \, (11.30b)$$

Further, $R$ is conceived to be a "thermal correction function" to the conventional Reynolds equation as already shown through expression (11.9b), viz.

$$\frac{\partial p}{\partial x} = -\frac{\text{h} \cdot \text{b} \cdot \text{t} \cdot \text{r} \cdot \text{i}}{\text{h} \cdot \text{b} \cdot \text{t} \cdot \text{r} \cdot \text{i}} \, (11.31)$$

As already explained in Chapter 10, $R$ lies in the range $0<R<1$ for non-isothermal flows, whilst it is unity for isothermal flows. Since relationship (11.29) does not in any direct way yield $R$ explicitly in terms of $E$, it is rather inconvenient for evaluating the film profile of thermo-elastohydrodynamic films. Simplified, nevertheless reasonably accurate, fitted relationships where $R$ does appear explicitly in terms of $B$ have already been provided by Wilson and associates, although only for films operating at comparatively low rubbing velocities of the rubbing surfaces. These relationships as well as an improved one, which has been independently obtained especially for high rubbing velocities, have already been provided in Chapter 10 (cf. Section 10.3.6).

In expression (11.31), $\text{u}_0$ is the local viscosity of the lubricant at the pressure and at the surface temperature in the cross-section concerned. By introducing the non-dimensional quantities $B$ for the reduced pressure, $H^2$ for the central film thickness and $\xi$ for the abscissa, and using expression (11.31), we obtain one of the non-dimensional versions conceivable of the "thermal Reynolds equation", viz.

$$-\frac{\partial B}{\partial \xi} = \frac{D \cdot \xi}{(H^2 + \xi^2)} \, (11.32)$$

where $\xi$ is the reduced abscissa defined by expression (11.16a), and $D$ is the "non-dimensional loading parameter" defined by,

$$D = \left( \frac{\text{u}_0 \cdot \text{t} \cdot \text{r} \cdot \text{i}}{\text{u}_0 \cdot \text{t} \cdot \text{r} \cdot \text{i}} \right) \left( \frac{\text{u}_0 \cdot \text{t} \cdot \text{r} \cdot \text{i}}{\text{u}_0 \cdot \text{t} \cdot \text{r} \cdot \text{i}} \right)$$

where $G^* = B^* \cdot a$ is the non-dimensional material parameter, and $U^*$ is the non-dimensional speed parameter, viz.

$$U^* = \frac{\text{u}_0 \cdot \text{t} \cdot \text{r} \cdot \text{i}}{\text{u}_0 \cdot \text{t} \cdot \text{r} \cdot \text{i}}$$
Integration of equation (11.32) can be carried out along lines similar to those of Murch and Wilson [11.13], and Goksem and Hargreaves [11.15]. But the present problem has been refined in that their simplifying and unrealistic condition of uniformity of the surface temperatures has been replaced by the flash temperature boundary condition (11.27). So, it has now to be accounted for, through the latter condition, that the thermal correction function \( R \) in equation (11.32) will vary in accordance with the variation of the surface temperatures. In fact, for deriving solutions for the distributions of surface temperature and pressure and for the film profile through numerical methods both equations (11.32) and (11.27) have to be solved simultaneously.

Then the resulting distributions of pressure and pressure gradient are made to comply with the flash temperature boundary condition.

The numerical evaluation may be started with some suitably assumed non-dimensional central film thickness \( H^* \), and be followed up by a numerical integration along the successive cross-sections. At every cross-section the local viscosity should be evaluated from the surface temperature and the pressure thus assessed, viz.:

\[
\nu_s = \nu_s^e \tag{11.33}
\]

It may be noted that by definition (11.8) \( B \) incorporates a viscosity-pressure-temperature relationship in which the isobaric viscosity-temperature relationship has been assumed to be exponential, the one represented by \( m = \infty \) in Slotte's relationship (see Chapter 2). For the more accurate isobaric relationship with some suitable finite \( m \)-value, usually in the range \( 2 < m < 5 \), one may readily replace the term

\[
-8F_T^e(C)T_p
\]

by

\[
\left[ 1 + \frac{\theta}{T_0} \right]^{-m} \tag{11.34a}
\]

while carrying out the analysis along similar lines. For thermal inlet zone analyses of elastohydrodynamic films for sufficiently low rubbing velocities, however, the exponential relationship proves sufficiently accurate. Since the analysis to follow is confined to such velocities, the isobaric exponential viscosity-temperature relationship has been maintained. Further, the local values of the "thermal correction function" \( R \) have to be evaluated using the above expression for \( \nu_s^e \). One may then use either Wilson and Wong's [11.28] fitted equation,

\[
R = \left[ 1 + 0.608 [E]^{1.64} \right]^{-1} \tag{11.34a}
\]

or, the similar, equally fitted and more accurate one of Mahdavian [11.34],

\[
R = \left[ 1 + 0.5 [E]^{1.72} \right]^{-1} \tag{11.34b}
\]

However, at high rolling speeds which give rise to high values of \( E \), the resulting low values of \( R \) prove to be far from representative. Then it proves necessary to employ the more accurate equation (10.34), viz.

\[
R = \frac{0.45 [E]^{1.813}}{128} \tag{11.34c}
\]

In all these expressions \( E \) is the "thermal back flow parameter", as defined by expression (11.39a), viz.

\[
E = \left\{ \frac{\nu_s^e}{\nu_s} \right\} \left[ \frac{h - h_s}{h} \right] \tag{11.35}
\]

The expression for \( E \) can further be simplified as follows. The local surface viscosity \( \nu_s^e \) is employed from expression (11.33) to simplify the local thermal rolling parameter \( V_T \), viz.

\[
V_T = \frac{8_{TS}}{k} \left[ \frac{\nu_s^e}{\nu_s} \right] \frac{\nu}{V S} \tag{11.36}
\]

where

\[
\nu_s^e = \frac{8_{TS}}{k} \frac{\nu}{V S} \tag{11.37}
\]

is the known value assumed by the thermal rolling parameter \( V_T \) at the entry to the inlet. Then expression (11.35) for \( E \) reduces to,

\[
E = \frac{V_T}{[R/(h - h_s)]} \tag{11.38}
\]

This local value of \( E \) in the film cross-section concerned has to be employed to obtain the corresponding local value of the thermal correction function \( R \) from either (11.34a), (11.34b) or (11.34c). Thereupon the "thermal Reynolds equation" (11.32) may be solved in conjunction with the expression for surface temperature, i.e. equation (11.27).

During the above process of evaluation a simultaneous analytical solution for both equations (11.32) and (11.27) proves difficult to achieve. A reasonably good approximation consists of solving numerically equation (11.32) at any cross-section by making use of \( T_p^e(C) \) obtained in the previous cross-section, or, alternatively an approximated gradient of the surface temperature estimated using various surface temperatures at the preceding cross-sections (cf. also expression (11.26c)). Further, \( T_p^e(C) \) can be evaluated at the cross-section concerned using the values of \( B \) and \( dB/\theta \) just obtained.

Along these lines the entire inlet zone can be analysed for any assumed trial value of \( H^* \) for all the successive cross-sections, thus obtaining a first approximation to the distributions of pressure and surface temperatures. Following Ertel-Grubin's boundary condition, i.e. the reduced pressure reaching its asymptotic (finite) value (so that \( B \) vanishes) at the exit of the inlet zone, \( B \) should vanish at \( \theta = 0 \). If with a trial value of \( H^* \) this does not prove to be so, a different value of \( H^* \) has to be assumed, and following this the numerical evaluation over the entire zone may have to be iterated until the above condition \( B = 0 \) is met. The present, iterative numerical integration proves to be sufficiently accurate as long as Ertel-Grubin's boundary condition is satisfied in that in proceeding along the thermal inlet zone the term \( B \) (see equation (11.8)) may not be overcompensated due to increase of surface temperatures as compared to that of pressure. In the comparatively mild thermo-elastohydrodynamic range so far covered in literature, both detailed theoretical calculations and experiments indi-
cate that the temperature increase along the rubbing surfaces in the inlet zone is rather low. This low temperature increase especially at the surface of the transitional cross-section, \( E \), i.e. that between the inlet and the conjunctional zone, then proved too small to overcompensate for the increase of pressure which in the present operating range proves to amount to comparatively large values. Thus Ertel-Grubin's condition can indeed be satisfied so that at least in the present mild operating range the present numerical method for the thermal inlet zone analysis would appear to be justifiable.

The numerical integration for estimating the friction force through equation (11.20a) proves to be straightforward once the distribution of pressure, and thus of \( p \), is known from the above-described evaluation. The central film thickness \( h_c \) in its non-dimensional form (see expression (11.16b)), resulting from the aforementioned iterative procedure can be worked out to obtain its dimensional value \( h_c \). Further, the present analysis readily yields the local values of \( E \) and \( V_r \) in every successive cross-section. Finally, other thermo-hydrodynamic quantities characterizing the distributions of the shear stress \( T_s \), the total heat flux \( Q_{tot} \), etc., already defined in Chapters 9 and 10 can, if desired, be obtained and the occurrence or non-occurrence of the limitations stipulated in the same chapters can be verified.

### 11.6 CONSIDERATIONS ON LIMITING PRESSURE GRADIENTS IN THERMO-ELASTOHYDRAULIC FILMS

It has been pointed out in Chapters 9 and 10 that there exists a limitation to \( K \) which cannot be exceeded for all flows induced by pressure gradients. \( K \) is a non-dimensional quantity which involves the pressure gradient \( g \), the viscosity \( \mu \) at the surface temperature, the film thickness \( h \) and the lubricant properties \( \beta \) and \( \kappa \), viz.

\[
K = \frac{\beta \mu}{\nu} \frac{h}{h^*} \quad \text{(11.39a)}
\]

In order to maintain the generality of the present analysis \( K \) has been replaced by its square-root, i.e. \( \sqrt{K} \), which has been represented for convenience through \( G \), viz.

\[
G = \frac{\sqrt{K}}{\mu} \frac{\mu}{h} \frac{d^2}{dx^2} h^2, \quad -9.506 < G < 9.506 \quad \text{(11.39b)}
\]

Quantity \( G \) is proportional to the pressure gradient \( dp/dx \). There exist accordingly two limiting values of \( G \), one being positive and the other negative. For any value of \( G \) between zero and either of its limiting values there exist two solutions which are characterized by two values for the flow parameter as already shown in Figure 9.7b: the figure has been generalized for the present purpose, with appropriate notations, in Figure 11.8. In fact, Figure 9.7b was originally drawn for stationary surfaces. But with moving walls, such as in the present elastohydrodynamic films with pure rolling, as characterized by some finite value of the thermal rolling parameter \( V_r \), the flow parameter \( V_r \) has to be correspondingly adjusted. The ordinate has thus been re-expressed in the form (cf. definition (11.24))

\[
E = \left( \frac{\mu}{\kappa} \right) \frac{h}{h^*} \left( \frac{\mu}{\nu} \right) \frac{d^2}{dx^2} h^2 = V_r - V
\]

where \( E \) is the thermal back flow parameter. It may also be noticed that the curve relating \( E \) to \( G \) falls in the first quadrant for positive values of the pressure gradient \( dp/dx \), and in the third quadrant for negative values. Further, the behaviour of the flow in successive cross-sections can be analysed in terms either of the parameter \( G \) or the thermal back flow parameter \( E \) along the following lines.

#### 11.6.1 Occurrence of transitional states

In general thermo-elastohydrodynamic films are non-parallel (certainly in their inlet zone), whilst furthermore the distribution of pressures and that of the surface temperatures are not known a priori. It proves that for a certain range of operating conditions in roller bearings (not too low speeds and lubricated with oils of viscosity grades usually specified) transitional states \((E = 1.675, G = 9.507)\) will occur in at least one cross-section somewhere in the inlet zone. But this cross-section is unknown, and neither is it possible to find out the cross-section where a limiting pressure gradient arises in such films, unless both film profile and the two aforementioned distributions have been
evaluated. Only in simpler cases, where the same transitional state may also occur, can a limiting pressure gradient be evaluated a priori. Such simpler cases have been dealt with in Chapters 9 and 10 and may be exemplified by cross-sections of flow of known uniform thickness and where the surface temperatures are kept uniform.

In cross-sections of a thermo-elastohydrodynamic film that are far upstream of the conjunctional zone (say at \( x = -\infty \)), for which fully flooded conditions is for analytical purposes usually considered as the entry to the inlet), the local film thickness \( h \) is much larger than the central value \( h_0 \). Therefore, in such cross-sections very large \( E \)-value must prevail. When considering roller bearings running at any but rather low speeds, and lubricated with oils having viscosity grades as conventionally specified, these \( E \)-values will distinctly exceed the transitional value 1.675 (cf. Figure 11.8). Further, at least in thermal inlet zone analyses, the exit of the inlet zone, in that it coincides with the entry to the conjunctional zone, may be characterized by putting the local film thickness \( h \) equal to \( h_0 \). So, there \( E \) vanishes altogether so that this film cross-section is characterized by the origin \((0, \ E=0)\) in Figure 11.8. So, in thermal inlet zone analyses one must usually reckon that in some cross-section, in between the entry and the exit of the inlet zone, the transitional state will be reached, i.e. the state characterized by \( G=9.507 \) and \( E=1.675 \) (note that \( dp/dx \) may be assumed positive and, therefore, \( g \) negative, throughout the inlet zone). It is noteworthy that in the transitional state, as it applies to thermally affected flows the comparatively small value is found for the thermal correction function, \( R \), viz.

\[
R = \frac{1}{12 \mu g} \frac{h^6}{h_0^4} \frac{dp}{dx} = \frac{G}{12E} \quad (11.40)
\]

Indeed, this transitional \( R \)-value 0.4735, which is valid for an exponential viscosity-temperature relationship (or say for Slotte's relationship with \( m=-1 \)) is smaller than the value of unity, which is valid for isothermal lubrication (say for Slotte's relationship with \( m=0 \)). Further, once the distributions of pressures and surface temperatures, and thus of the surface viscosities \( \mu_s \), should for any given rate of flow, \( q=\pi h_0^2 \), have been determined throughout the inlet zone, the particular cross-section where the transitional state occurs may be assessed. At the exit of the inlet zone, i.e. the entry to the conjunctional zone, \( h=h_0 \), and correspondingly \( E=0 \) (see equation (11.30)). In reality, in this cross-section \( h=h_0 \) in that it is larger, albeit only to a relatively small extent. So, there \( E=0 \). Only in cross-sections where pressure \( p \) is at an extreme will \( E \neq 0 \). One such cross-section occurs at about the centre of the conjunctional zone where \( p \) reaches a maximum that is about equal to the Hertzian one. Another such cross-section is the one where the Petrovitch spike occurs, if any. Still another such cross-section is that at the very exit of the outlet zone, where \( p=0 \) and \( dp/dx=0 \).

Further, over the entire inlet zone the film thickness decreases monotonically, and so must the term \( (h-h_0)^2/\mu \) in expression (11.32). Surface viscosity of the lubricant along the surface of the rubbing elements, as it is included in \( B(\zeta) \), tends to increase due to the monotonically increasing pressure \( p(\zeta) \) but to decrease due to the monotonically increasing surface temperature \( T(\zeta) \). Unfortunately, there is a paucity of data so far published on the distributions of pressure and of surface temperature in thermal inlet zone analyses for the thermo-elastohydrodynamic cases in hand, i.e. those with pure rolling. However, for any specified value of the thermal rolling parameter \( V_0 \) at the inlet, \( E \)

\[
E = \frac{V_0}{V_0} \frac{(h-h_0)}{h_0} \quad (11.41)
\]

In any cross-section downstream of the entry, the viscosity can be given by equation (11.33), viz.

\[
\mu(\zeta) = \mu(0) \exp(-\zeta/\kappa) \quad (11.33)
\]

Again in any cross-section the values of \( E \) and \( V_0 \) are related to their inlet values and the local viscosity-value \( \mu \) for the present rolling problem with equal boundary temperatures. The relative fluidity in the film cross-section corresponding to the present rolling parameter \( V_0 \) can be given using expression (11.37) in the form:

\[
E = \frac{V_0}{V_0} \frac{(h-h_0)}{h_0} \quad (11.41)
\]

In any cross-section downstream of the entry, the viscosity can be given by equation (11.33), viz.

\[
\mu(\zeta) = \mu(0) \exp(-\zeta/\kappa) \quad (11.33)
\]

Again in any cross-section the values of \( E \) and \( V_0 \) are related to their inlet values and the local viscosity-value \( \mu \) for the present rolling problem with equal boundary temperatures. Thus any prescribed value \( E \) at the entry to the inlet zone, i.e. through the thermal rolling parameter \( V_0 \) (see expression (11.36)), will ultimately reduce to zero at the exit of that same zone, where \( G=0 \) (cf. Figure 11.8). If the prescribed value lies somewhere on \( OA_1 \) this may for the present discussion be termed the "first branch" of the curve, as shown in Figure 11.8, the decrease of \( E \) along the inlet zone is monotonic. However, if this value lies above \( A_1 \) (this segment of the curve being termed the "second branch"), the decreasing value of \( E \) will pass through the transitional point \( A_1 \) at which the \( G \)-value will reach a maximum. For any imposed operating condition at the entry to the inlet that may influence \( E \), therefore, there cannot prevail a value of \( G \) larger than its limited one throughout the inlet zone. Further, it has already been pointed out in Chapters 9 and 10 that the flows representable on the first branch (corresponding to \( OA_1 \)) are stable against disturbances of any magnitude. It has also been established, albeit only qualitatively, that the flow along the second branch (above \( A_1 \)) may undergo certain limitations due to the appreciable thermal effects then involved. In fact (see Section 9.5.2 of Chapter 9), a certain danger of instability might then arise even for flows with moderate Reynolds numbers. Accordingly, for flows in the inlet zones of the non-isothermal elastohydrodynamic films the danger of instability, if any, may be investigated through the curve in the first quadrant in Figure 11.8. Thus flows of oils with low kinematic viscosities operating with an \( E \)-value at the entry to the inlet zone higher than that given by the transitional state are subject to a possible danger of instability if they are accidentally subjected to disturbances of adverse wave number. Since only limited data are available about the present matter, we shall confine ourselves to cases where the \( E \)-values at the entry to the inlet that do not exceed the transitional one, i.e. that corresponding to the limiting \( G \)-value.

The present flow constraint, being based on the limiting value of \( G \) and the transitional point throughout the inlet zone prescribes the flow to prevail only along the first branch. Therefore, the operational parameters \( h_0 \) and \( V_0 \) are limited to the aforementioned values. The corresponding range of the thermal correction function may then be prescribed through:

\[
1 \geq R \geq 2.9.4735
\]

The relative fluidity in the film cross-section corresponding to the transitional state, i.e. the ratio \( \mu(\zeta)/\mu_0 \) of the viscosity at the maximum temperature \( \mu_0 \) within the film to that viscosity at the surface reaches the low value of 0.2796 as has been already pointed out in Section 9.5 of Chapter 9.

\[
\mu(\zeta) = \mu(0) \exp(-\zeta/\kappa) \quad (11.33)
\]
11.6.2 Limitation to rolling friction of thermo-EHD films

The aforementioned flow constraint based on the limiting value of $G$ can be used also to assessing limitations to friction forces on lubricated rubbing surfaces. A qualitative trend of the results thus obtained will then be compared with the trend of the results provided by Goksem and Hargreaves [11.15]. These investigators have analysed the variation of rolling friction coefficient $K_t$, which may be defined by the ratio of the friction force $F$ to the load $W'$ on the rubbing surfaces (see expression (11.20c)), through a quantity $K_{w_0}L$:

$$K_{w_0}L = f \times D$$

Their results relating $K_{w_0}L$, the loading parameter $D$ and the parameter $L/D$ have been reproduced in Figure 11.9. The last parameter $L/D$ has been conceived by them as a measure of the susceptibility of the film to thermal effects.

Goksem and Hargreaves [11.15] inaccurately call the rolling friction, i.e. the friction force due to viscous shear at the rubbing surfaces, "traction". For pure rolling of the lubricated rubbing members, in view of the symmetry of the flow in the lubricant film then occurring, there cannot be any traction in the conventional sense of a force that can be transmitted tangentially from one rubbing surface to the opposite one. Their term "traction" should thus be interpreted as the "rolling friction" on the rubbing surfaces.

The distribution of shear stresses $t_{R1}$ on the surface of either of the rubbing members resulting from the fluid friction can be related to that of $G$ through:

$$t_{R1} = \frac{G}{k u \nu}$$

The distribution of $t_{R1}$ along the rubbing surfaces in the entire inlet Zone of thermo-elastohydrodynamic films can be analysed in terms of the distribution of $G$. Since the friction force at each of the rubbing surfaces is an integral of the shear stresses $t_{R1}$ shown through expression (11.19a), the variation of the friction force for each of the rubbing surfaces is an integral of the shear stresses as shown through expression (11.19a). Further, the aforementioned flow constraint based on the limiting value of $G$ can now be used also to analyse the friction thus obtained. The friction force attains a maximum value corresponding to the above-limiting value of $G$. These results correspondingly a limitation to the rolling friction coefficient $K_{w_0}L$. This limitation to $K_{w_0}L$ may also be observed in Figure 11.9 reproduced from Goksem and Hargreaves [11.15]. Friction forces under thermally moderate conditions are represented by the segments on the right-hand side of the curve given by $L=2.89$, whilst these forces under thermally more severe conditions are represented by the segments on the right-hand side of the curve. A somewhat different representation of theirs of the same results in Figure 11.10 shows that the above two sets of friction forces are represented correspondingly by the curves above and below the line $L=2.89$. Also lines representing a constant value of $V_10 = 2.8056$ have been depicted in Figures 11.9 and 11.10 (note that in these figures their non-dimensional thermal loading parameter $L$ is identical with the present authors' $V_10$). The frictional force is at a maximum at the same value of $V_10 = 2.8056$ and $L$. Thus our results substantiate those of the aforementioned investigators while yielding the above explanation for the maximum traction at $V_10 = 2.8056$. There exists only a minor difference between this numerical value and that provided by Goksem and Hargreaves [11.15], namely $L=2.89$ as shown in Figure 11.9. This difference may be attributed to different approaches that have been followed by the above investigators and in the present analysis. The actual value of the coefficient $K_{w_0}L$ will, however, be different if a realistic distribution of surface temperature is employed instead of the uniform one used by the aforementioned investigators.

Further, for any geometry, load, and set of material and lubricant properties, a characteristic quantity can be formed, viz.

$$V_10 = \frac{2}{g} \frac{6 k \nu^3}{B'g^2}$$

where $V_10$ is defined by expression (11.36) and $D$ through (11.32). This quantity is a measure of the magnitude of the effect of viscous-shear heating, especially in the inlet region, in band-shaped thermo-elastohydrodynamic films. The quantity $D$, however, represents the variable loading through the rolling speed. A limitation to rolling frictional force through the limiting value of $G$ is recognizable in Figure 11.9 where the friction coefficient, thus the friction losses in the elastohydrodynamic films, is seen to reach a maximum for any selected value of $V_10 = 2.8056$ and while $D$ is varied.

Figure 11.9 depicts rolling friction of thermo-elastohydrodynamic films in pure rolling. If a small amount of sliding between the rubbing surfaces is introduced, the trend of the results would still remain qualitatively the same. Indeed it has already been pointed out in Chapter 10 that also the most general flows induced by moving boundaries having
unequal temperatures possess limiting values of $G$. Limitations to $G$-values, $V_2$, may surface friction, are conceivable in view of limitations inherent in thermo-elastohydrodynamic films.

A verification of these limitations can be instituted through analysing reported experimental data. Those of Adams and Hirst [11.22] have been plotted by Goksem and Hargreaves [11.15] in a form that can readily be used for the present purpose of verification. Their figures, reproduced here in Figures 11.11 and 11.12, depict the variation of rolling friction with rolling speed. It may be observed that the maximum to the friction curve, which corresponds to the aforementioned limitation to $G$ in Figure 11.11, falls approximately at point A. The theoretical curve to the left of A represents the first branch of the solution in Figure 11.8 and that to the right of A represents the second branch. A similar behaviour of the results may be observed in Figure 11.12 where point B represents the position of the maximum to friction. As expected, the experimental results of Adams and Hirst [11.22] fall along the left branch of the aforementioned curves. So there they comply with the expected steady operation of non-isothermal elastohydrodynamic films. The results just about reach point B in Figure 11.12 but none extend beyond...
it. These results do not theoretically exclude a limitation to rolling friction due to thermal effects, although verification is admittedly incomplete due to lack of experimental data at higher rolling speeds, i.e. to the right of point A in Figure 11.11.

The results available in literature for still higher rolling speeds, include a certain amount of slip between the rolling elements. Experimental results of Johnson and Cameron [11.21], Conry [11.24], Anderson [11.36] indicate a steep increase of traction on the driven member in a pair of lubricated rubbing disks, even with a slight increase of slip between the rubbing elements for any given rolling speed. When continually increasing the slip, while maintaining some constant rolling speed, the traction reaches a maximum or ceiling, and beyond this the traction decreases monotonically. The results of Johnson and Cameron [11.21] indicate that experiments run with a supply of lubricant at various temperatures also show a 'ceiling' to the traction coefficient, i.e. to the ratio of traction force to normal load. The results of Conry substantiate this observation. A more thorough analysis of the results reported so far for the cases of rolling with slip, however, is beyond the scope of the present investigation of the cases of pure rolling.

The aforementioned results for pure rolling, therefore, reveal a maximum to rolling friction that depends on both the kind of lubricant and the entry temperature, or say on the entry viscosity. This maximum can be conceived to represent the aforementioned limitation to G due to the thermal effects involved. When high shear stresses should be involved in the flow, the lubricant may well show a non-Newtonian behaviour that would also affect the aforementioned maximum rolling friction. Yet, the overall trend of the rolling friction still remains similar to the one predicted through employing a Newtonian model for the lubricant involved. Further, the aforementioned experimental results reveal elastohydrodynamic films to be operable even in states represented on the second branch of the solution depicted in Figure 11.8. In fact, a similar conclusion can be drawn also from the oil-specifying charts provided by the manufacturers of rolling-element bearings. Whatever may happen to elastohydrodynamic films subjected to severe viscous-shear heating and operating in the present states, can further be illustrated along the following lines which have already been put forth in Chapters 9 and 10 (cf. Section 10.5).

Albeit so far only on theoretical grounds, limitations to thermally affected rates of film flow are conceivable, i.e. in view of some maximum permissible temperature of the kind of lubricant concerned, or say in view of the corresponding minimum viscosity as it may occur in some cross-sections of a thermo-elastohydrodynamic film. If low viscosities in such cross-sections give rise to high Reynolds numbers exceeding certain critical values, such as specified in Section 10.5.2, the flows concerned are known to be subject to an instability. The critical Reynolds numbers have been shown to be rather low and are concurrent with a point of inflection on the velocity profile in the film cross-sections concerned. There are indications from experiments on scuffing of gears (Ibrahim and Cameron [11.37]) that such thermally affected instabilities and a consequent collapse of the film might indeed appear in lubricating practice. Alternatively, if the pressures prevailing in these cross-sections are sufficiently high the viscosity will not reach really low values, notwithstanding the high temperatures involved. Then the aforementioned cases of high Reynolds numbers exceeding certain critical values may be expected not to occur in thermo-elastohydrodynamic film flows. However, if the elastohydrodynamic pressures are not very high, i.e. at low loads and high rolling speeds, then the viscosity increase may be overcompensated in certain cross-sections by the viscosity decrease due to viscous-shear heating. Then there prevail low viscosities in those cross-sections endangering the film by the aforementioned limitation.

11.6.3 Limitations to central film thickness of thermo-EHD films

An analysis of possible limitations to central film thickness, as it occurs in the conjunctional zone and as it is mainly determined by the thermally affected boosting (pumping) action in the inlet zone, can be carried out along lines similar to those for rolling friction. Central film thickness values due to the thermally affected flow on either side of the transition point A in Figure 11.8, i.e. along the first and the second branches, can be distinguished using the aforementioned flow constraint set up by the limiting value of G. These results do not theoretically exclude a steep increase of traction on the driven member in a pair of lubricated rubbing disks, even with a slight increase of slip between the rubbing elements for any given rolling speed.

Figure 11.13 Relationship between non-dimensional film thickness $H_c$, loading parameter $D$, and rolling parameter $L$ (Goksem and Hargreaves [11.15]).
Hargreaves [11.15]. The ordinate in this figure is the non-dimensional film thickness $H_c$ defined by equation (11.16b). The same authors have cross-plotted this figure to Figure 11.14 by introducing $L/D'$ instead of $L$ as the parameter and retaining $D$ and $H_c$ as the coordinates. The maximum film thickness in Figure 11.14 lies at $L=12.65$ which is higher than the one, $L=9.67$, shown by the theoretical curve in Figure 11.14. Experimental results of Adams and Hirst [11.22] have also been included in Figures 11.11 and 11.12. It so happens that all of their measured film thicknesses fall to the left of the abscissae of the points A and B for the maximum rolling friction in these figures. The film thickness values correspond, therefore, to the first branch of the solution in Figure 11.8, like the rolling friction-values analysed in the preceding subsection. The results just about reach the transitional value of $L=9.67$, which is represented by the aforementioned abscissa. Thus these experimental results for film thickness are not at variance with the theoretical limitation to the rate of flow due to thermal effects. Unfortunately, due to lack of experimental results at higher rolling speeds, this limitation could not so far be verified definitely.

The above conclusions may be considered as qualitative only, because the theoretical curves of Goksem and Hargreaves [11.15] are at best only partly representative. That is, their curves have been obtained on the assumption of uniform temperatures of the surfaces of the rubbing bodies. Only the more complex analysis presented in Section 11.4.3, once it is evaluated numerically, may be expected to yield theoretical film thicknesses that are sufficiently representative in that they account for the realistic thermal boundary conditions obtainable from the flash temperature theory. The results thus to be obtained may be expected to depict a trend different from that given in Figures 11.11 and 11.12.

The experimental results of Adams and Hirst [11.22], as depicted by Goksem and Hargreaves [11.15] in Figures 11.11 and 11.12, can be used for such an analysis as follows. The upper curves in these figures, which relate to the same oil but to two different entry temperatures, depict theoretically expected film thicknesses for various rolling speeds, the abscissa being expressed in terms of the product of the rolling speed and the viscosity of the lubricant at the entry to the inlet zone. The maximum to the film thickness can be observed to occur at a higher value of the abscissa than that for the maximum traction. The former maximum value corresponds to the $L=9.67$, shown by the theoretical curve in Figure 11.14. The results thus to be obtained may be expected to depict a trend different from that given in Figures 11.11 and 11.12.

The aforementioned limitation to the thermally affected rates of flow may be conceivable also from a consideration of film thickness that is similar to that for friction in the previous Section. It may thus be concluded, although so far only on the theoretical grounds presented in Chapters 9 and 10, that even under pure rolling are thermo-elastohydrodynamic films likely subject to instabilities due to thermal effects. But a really conclusive justification can be achieved only after having measured film thicknesses of elastohydrodynamic films at rolling speeds surpassing those indicated by points A and B in Figure 11.11 and 11.12, respectively.

Thermo-elastohydrodynamic films operating with sliding, and therefore with two unequal distributions of surface temperatures, can be theoretically investigated along lines similar to those used for films under pure rolling. Such an investigation can be instituted by employing an appropriate thermal Reynolds equation which accounts, like in the work of Wilson and Sheu [11.14], for unequal rubbing velocities and unequal temperatures at the boundaries in the film cross-section considered. Such an investigation would yield theoretical results for traction and film thickness of thermo-elastohydrodynamic films of a type more general than commonly encountered in literature, that is, for the particular films subject to pure rolling such as in rolling-element bearings.
Analyses of elastohydrodynamic films for any but low rubbing velocities are inadequate for estimating the film thickness in the conjunctional zone when they are based on the simplifying assumption of isothermal flow in the film. The same is true of estimates of the rolling frictional and the corresponding friction losses at the rubbing surfaces. Especially at higher rolling speeds and increased loads, thermal effects due to viscous-shear heating influence the flow considerably and tend to give rise to film thicknesses and rolling friction smaller than those following from that isothermal theory. Thus analyses that do account for the thermal effects are imperative for estimations of the magnitude of the inlet zone and thus the central film thickness of band-shaped thermo-elastohydrodynamic films. This same equation is equally suitable for interrelating the rate of flow and the corresponding pressure distribution in the conjunctional zone.

Published theoretical estimates of the film thickness and rolling friction of thermo-elastohydrodynamic films require refinements in terms of more realistic thermal boundary conditions. That is, instead of the distributions of surface temperatures that have so far been conveniently assumed uniform all over the length of such films, one should rather rely on the non-uniform distributions that follow from the flash temperature theory. Especially at high rubbing velocities the version of this theory for high Peclet numbers of the rubbing surfaces lends itself best to establishing such boundary conditions. An analysis of the inlet zone of thermo-elastohydrodynamic films under pure rolling of rubbing bodies with identical thermal properties shows the distribution of surface temperatures to be a function of that of the local pressures.

Complete analyses of thermo-elastohydrodynamic films thus require the simultaneous solution of the "thermal Reynolds equation" and the "flash temperature boundary conditions" for the successive film cross-sections throughout the entire extent of such films. Although thermal inlet zone analyses are somewhat simpler in that they primarily serve the limited objective of a reasonable estimate of the rate of film flow, or say of the central film thickness, they still require a simultaneous solution, albeit only over the extent of the inlet zone.

Nevertheless, the overall trend of the results for both film thickness and frictional losses can be analysed on the basis of published results which have previously been obtained through assuming a uniform distribution of temperatures along the rubbing surfaces. With increased rolling speeds both rolling friction at rubbing surfaces and film thickness show an equally increasing trend, but only up to a transition point which is followed by a decreasing trend. The transition point of the friction curve can be determined simply by considering that the "thermal back flow parameter" $E$ is highest at the entry to the inlet zone and tends towards zero at the exit of that same zone. Thus this transition point results solely from the occurrence of the transition state corresponding with $E$-value $E_{cr}$, at the entry to the inlet zone and does not depend on the distributions of pressures, pressure gradients or rates of flow in the region downstream of the entry. The transition value of $E$ at the inlet corresponds to the rolling parameter $V_r$ reaching its "limiting value" $V_{rlm}$. Rolling parameters $V_r$ higher than the limiting value give rise to $E$-values that are greater than the transitional value. Correspondingly the rolling friction shows a decreasing trend.

Thermo-elastohydrodynamic films operated under the corresponding high rolling speeds require a careful scrutiny. In fact, limitation to the flow due to thermal effects is conceivable of these films for which the viscosity increase due to the film pressures would be overcompensated in certain cross-sections by the viscosity decrease due to viscous-shear heating. Flows in such films are characterized by the second branch of the solution which has extensively been discussed in Chapters 9 and 10 and depicted in Figure 11.8. The corresponding flows are also likely to undergo severe heating locally at least in some portions of the lubricant film.

11.8 REFERENCES


Chapter XII

MAJOR FINDINGS, CONCLUSIONS AND SUGGESTIONS

12.1 SURVEY OF THE MAJOR FINDINGS INCLUDING SOME MAIN CONCLUSIONS

12.1.1 Slotte's viscosity-temperature relationship has been shown to imply with Pawlowski's concept of "referential invariance", i.e. through introducing the coefficient $A$ and maintaining the exponent $m$ while dispensing with the coefficient $B$. However, $B$ implicitly involves $A$, which, just like Roelands' Slope Index $S$, may be considered an index of "viscometric homology". This kind of homology runs more or less parallel with "structural homology" in that for any structurally homologous series of liquids $A$ shows one and the same value for all its members. Viscosity characteristics of various such structurally homologous series of oils used in lubrication practice have been analysed for their characteristic behaviour as to, e.g., the shear stress and also two different distributions of temperature and of heat flux. Correspondingly there exist two different rates of heat load in the special high-speed Couette viscometer described in Chapter 6 could be brought under control by a rotating heat pipe which imposes the uniform surface temperatures required at the rotor, i.e. at the moving boundary of the lubricant film. Among various volatile liquids investigated for their suitability as heat-pipe fluids, Arcton 113 has proved to function satisfactorily up to moderate heat loads of 350 watt and rotational speeds as high as 3000 rpm. The evaporator section of the heat pipe has been successfully integrated into the isothermal test section of the rotor of the Couette viscometer. Rendering the very nearly adiabatic stator has been realized by making it of glass.

12.1.2 Non-isothermal shear flows of various liquids have been analysed for their characteristic behaviour as to, e.g., the shear stress under various sets of thermal boundary conditions and sliding velocities. It has been proved that with increasing sliding speeds a limiting shear stress cannot be exceeded. This limiting shear stress depends on the viscosity characteristics of the liquid chosen and the thermal boundary conditions imposed. For all liquids, except the hypothetical ones following the linear fluidity-temperature characteristic, or say the hyperbolic viscosity-temperature characteristic, the actual shear stress decreases when sliding speed increases beyond the transitional one which gives rise to the limiting shear stress. So, any shear flow yielding a shear stress lower than the limiting value may show either of two patterns that are characterized by two different distributions of velocity, and also two different distributions of temperature and of heat flux. Flows operating in the particular regime where the actual shear stress decreases with increasing sliding speed are known to be sensitive to certain thermally affected instabilities of either hydrodynamic or thermal type. Specifications of this regime have been evaluated for various kinds of liquids and it has been shown that the stronger the dependence of viscosity on temperature the lower is the sliding speed at which the regime concerned sets in.

12.1.3 The above analysis has been extended to study both the flow behaviour and the variation of film thickness of thermo-plastohydrodynamic lubricant films with various sets of thermal boundary conditions. For a selected plastically deforming billet material and for a chosen lubricating oil following the exponential viscosity-temperature relationship regimes of safe operation have been evaluated, that is, those not subject to the aforementioned thermally affected instabilities. It has been shown that here too the transitional sliding speeds are dependent on the thermal boundary conditions imposed. Theoretically evaluated film profiles and distributions of shear stress and of heat flux along the track are indicative of the existence of the aforementioned transition points and also of the portion of the film where the aforementioned thermally affected instabilities might occur.

12.1.4 The range of heat load in the special high-speed Couette viscometer described in Chapter 6 could be brought under control by a rotating heat pipe which imposes the uniform surface temperatures required at the rotor, i.e. at the moving boundary of the lubricant film. Among various volatile liquids investigated for their suitability as heat-pipe fluids, Arcton 113 has proved to function satisfactorily up to moderate heat loads of 350 watt and rotational speeds as high as 3000 rpm. The evaporator section of the heat pipe has been successfully integrated into the isothermal test section of the rotor of the Couette viscometer. Rendering the very nearly adiabatic stator has been realized by making it of glass.

12.1.5 Measurements of torque and through a well-adapted laser-Doppler technique also distributions of the velocity of flow could be successfully carried out on the special Couette viscometer. Up to the operating regime where the shear stresses theoretically attain a maximum, measured frictional torques and the shear stresses derived therefrom agreed reasonably well with the values predicted for Newtonian oils. The velocities of flow registered cover the wide range of 0.8-5.5 m/s. Summarizing, the aforementioned results have corroborated the expectation that the non-isothermal flow behaviour of viscous liquids when subjected to high sliding speeds or shear rates will, in agreement with theoretical expectation, differ considerably from the isothermal behaviour in that shear stress is subject to a limitation in the form of a maximum.

12.1.6 Pressure-induced axisymmetric non-isothermal flows in pipes and capillaries have also been investigated, if only theoretically. These flows display certain characteristics that are in essence similar to those of plan-parallel non-isothermal shear flows. For instance, the pressure gradient that can be imposed on the present flows is subject to an upper bound. This behaviour is similar to the limitation of the shear stress in the case of non-isothermal shear flows. For every imposed pressure gradient smaller than a limiting value there exist two different distributions of velocity and similarly two different distributions of temperature. Correspondingly there exist two different rates of flows. Axisymmetric non-isothermal flows up to the transition point corresponding to the limiting pressure gradient are stable. Beyond the transition point they seem to be susceptible to instabilities due to disturbances at least of certain sets of axial and azimuthal wave numbers.

12.1.7 A theoretical analysis of plan-parallel pressure-induced symmetric non-isothermal flows showed results analogous to those for axisymmetric but otherwise similar flows. Even more general plan-parallel
flows may be conceived when including not only pressure gradients but also moving boundaries and/or unequal and non-uniform surface tempera-
tures. Various types of liquids, as characterized by different values of
Slotte's exponent m, have been selected for the analysis. Evaluations of the corresponding temperature distributions have
successfully been carried out using Milne's predictor-corrector method
for the second-order ordinary differential equation governing the energy
balance in the flow. The results obtained for the particular case of
thermal and kinematic symmetry, i.e. distributions of temperature and
velocity symmetrical with respect to the axis of the plan-parallel chan-
nel, prove to be more accurate than those reported by previous investi-
gators. In addition they cover a far wider range of operation than that
reported hitherto.

12.1.8 The aforementioned general plan-parallel pressure flows, those
of the asymmetric type, display a characteristic behaviour similar to that
of both the symmetric plan-parallel flows and the axisymmetric
flows, namely, that there exists a limiting pressure gradient which
cannot be surpassed. Unfortunately, the solutions that have numerically
been obtained proved too involved to employ readily in the analysis of
non-isothermal lubricant films. However, their application may be facili-
tated by recasting empirically fitted results in the form of a "ther-
mal Reynolds equation". This empirical equation yields as one of its
major results the thermal correction factor for the volumetric rate of
flow as it is affected by the pressure gradient imposed. Various ther-
mo-hydraulic quantities such as rate of flow, total heat flux, etc.,
have been represented conveniently in terms of the solutions already ob-
tained through Milne's predictor-corrector method.

12.1.9 Only to a limited scope have investigations been reported in
literature on thermally affected instabilities of a hydrodynamic or
thermal type. A theoretically conceivable thermal instability has been
reported for plan-parallel pressure flows when operating in the region
beyond the aforementioned transition. Thus the accuracy with which this
transition can be evaluated for various lubricating oils is of great valu-
able. A theoretical investigation carried out in this regard for
symmetric pressure-induced plan-parallel flows has revealed that the
transition is different for different viscosity characteristics of oils,
as these may be specified through Slotte's exponent m. Although no
published material has been found on the instability of the most general
types of flows, i.e. those that are pressure-induced and characterized
by sliding boundaries and unequal surface temperatures, it is suspected
that these may also undergo an instability of the type conceivable for
symmetric flows. In any case the transition, as well as the flow regime
for which a point of inflection might appear in the velocity profile,
can be obtained from the present set of results.

12.1.10 The foregoing results for shear and pressure-induced flows, as
well as those for the most general type of flows, can readily be used in the
more accurate analyses of film thickness, traction and frictional
losses of elasto-hydrodynamic lubricant films where viscous-shear heating
and the consequent thermal effects are accounted for. Such analyses are
facilitated through using the aforementioned "thermal Reynolds equa-
tion". These analyses have been made representative of actual lubrica-
tion practice by employing the realistic "flash temperature" boundary
conditions, if only somewhat simplified ones that are representative of
cases of high Peclet numbers of the rubbing surfaces. Further, the pre-
sent analyses can easily be carried out along conventional lines, thus
estimating the pumping (or "boosting") performance of the inlet zone and
the corresponding central film thickness, i.e. by following the method
originally devised by Ertel (still erroneously attributed to Grubin)
for isothermal elasto-hydrodynamic films. The trend of results may be ex-
pected to remain qualitatively the same as that ensuing from the ana-
lyses where the two distributions of the boundary temperatures are con-
sidered equal and both uniform along the entire length of the film.
Quantitatively, however, the results are expected to deviate noticeably.
With increasing rolling speeds both friction and film thickness values
undergo a trend increasing up to a transition point and decreasing bey-
on. The transition point on the friction curve results from the occur-
rence of the transition point in the variation of the "thermal back flow
parameter" E which is bound up with the rolling speeds imposed. At
higher values of E, and thus at higher rolling speeds, the possibilities
for thermally affected instabilities of hydrodynamic or thermal type
exist is apparent for thermo-elasto-hydrodynamic films. But the actual
occurrence of such instabilities calls for experimental verification.

12.2 SUGGESTIONS FOR FUTURE RESEARCH

12.2.1 The present investigation has revealed that an accurate represen-
tation of viscosity-temperature-pressure relationship is essential to
analyse the flow behaviour of liquids subjected to large variations of
temperature and pressure such as in lubricant films between rubbing sur-
faces. Slotte's isobaric viscosity-temperature relationship, being "re-
ferentially invariant", proves to be suitable for the theoretical analy-
sis of the present flows. While reducing the analysis to a simpler one
this relationship characterizes "viscometrically homologous" group of
liquids by one single set of Slotte's parameter A and the isobaric expo-
nent m. There is, however, a need to establish for liquids the exponent
m at pressures higher than the ambient pressure.

12.2.2 "Thermal Reynolds equation" which is structurally similar to the
classical Reynolds equation, has been recognized valuable for calcu-
lating the major characteristics of elasto/plastohydrodynamic lubricant
films. A formulation of this equation should be carried out for various
oils that can be characterized by the parameters of Slotte's viscosity-
temperature relationship. The thermal Reynolds equation thus to be for-
mulated should be made to account also for the most general operating
conditions, including sliding of the two rubbing surfaces at unequal
and non-uniform temperatures. In addition, realistic thermal boundary con-
ditions along rubbing surfaces have to be specified for effectively
employing the thermal Reynolds equation so as to arrive at the actual
film profiles, traction, and frictional losses of elasto/plastohydrodynamic films.

12.2.3 A simultaneous analysis of the appropriate thermal Reynolds
equation in conjunction with the realistic thermal boundary condi-
tions, as those resulting from the "flash temperature theory" at least
for the cases of high Péclet numbers of the rubbing surfaces, may be
performed in a straightforward manner. The above analysis, if carried
out, will yield corrections to the classical isothermal calculations for
the minimum film thickness; these corrections will prove crucial espe-
cially when the film thickness concerned is critically small, i.e., when
approaching the centre line average (c.l.a) value of the roughness of
the rubbing surfaces. The analysis would then also substantiate that
lubricant films undergoing severe viscous-shear heating display exces-
sively high oil temperatures, which while causing strong reduction in
viscosity of the lubricating oil concerned reduce their load capacity
considerably.

12.2.4 Further when the temperatures involved should be critically
high, the lubricating oil concerned may well undergo changes in its mo-
lecular structure and a consequent degradation. Even in lubricant films
subjected to less severe operating conditions there exists a danger of
the occurrence of thermal or hydrodynamic instability, thus possibly of
collapse of these films. Thus an accurate evaluation of the heat gener-
ation in elasto/plasto-hydrodynamic films, and an analysis of the flow
pattern of the lubricant during its passage through the successive film
cross-sections of the inlet, the conjunctural and subsequently the exit
regions is deemed highly desirable.

12.2.5 An analysis along the present lines will provide an accurate
design guide-line for rolling/sliding element bearings which accounts
for viscous-shear heating, and which will assure safe operation even un-
der extreme operating conditions. The analysis would also provide a
possibility to specify the surface roughness of the rubbing surfaces and
of the capacity of the lubricant for protecting the rubbing surfaces
from occasional disruption of a full fluid film.

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### Appendix 2.A1 Various Viscosity Grades with Given Slope Index S₀

AIRT (The American Society for Testing and Materials) Viscosity Index
VI = 0 for the L-series and VI = 100 for the H-series have been shown by
Roelands [2.A1] to display a reasonably good agreement with his own Dy-
namic Viscosity Index DVI in the range of DVI = 0 to 100. Further he has
shown that his Slope Index S₀, being empirically related to DVI through
the relationship (2.33), is substantially constant over almost the en-
tire range of viscosity grades covered by one VI-value, or what amounts
to, for the range of liquids covered by one and the same DVI-value. Altern-
avely, in the commonly employed range of temperatures, say 20 to
150 °C, the Slope Index S₀ is not influenced to any considerable degree
by temperature. In order to overcome the restriction, if any, concerning
the temperature influence on S₀, and to enable a fair evaluation of the
viscosity-temperature qualities of different liquids, Slope Indices of
these liquids have been selected in comparable temperature ranges.

Liquids with a constant Slope Index have been analysed on lines simi-
lar to those explained in the previous analysis with arbitrarily assumed
viscosity grades. But now the dynamic viscosity corresponding to the
given series of liquids has been taken into account instead of an arbi-
trarily chosen viscosity grade. L- and H-series mineral oils belonging
to the ASTM standard reference series have been first considered in the
range of viscosities to which they were originally proposed by ASTM for
the VI-system.

Following the hypothesis put forward by Roelands [2.A1] about the
correspondance of VI and DVI, and about the empirical relationship of
the latter DVI with the Slope Index S₀ at atmospheric pressure have been found for these L- and H-series mineral oils using equation (2.33) to lie at 1.50 and 1.23, respec-
tively. In these S₀-values the third and higher decimal places have
been rounded off to the second decimal place in accordance with Roe-
lands' proposition that there is very little to gain from a considera-
tion of higher decimal numbers due to the empirical nature of the Slope
Index S₀. The results obtained towards Roelands' parameter G₀, Slotte's
parameters m and A, and also viscosity-values obtained at 100 °C and at
atmospheric pressure using both Roelands' and Slotte's relationships have
been tabulated in Table 2.A1. The coefficient S₀ calculated from ex-
pession (2.27) at the grade temperature T₀ = 40 °C has also been tabu-
lated. A presentation of the variation of m in a wide range of viscosity
grades has been made in Figure 2.A1. In the range of validity of
VI-scale, i.e., in the viscosity grade ranging between 10 and 70 cP, the
variation of m follows a straight line. Straight lines for L- and H-
series oils are not parallel, and they cover an approximate range of m-
values between 2.50 and 3.65. The corresponding \( S_g \)-values fall in the range covered by 0.02 < \( S_g < 0.045 \). It may also be observed from Table 2.A1 that the corresponding \( A \)-values remain unchanged over the viscosity grades covered by these \( C \)- and \( R \)-series.

n-paraffin oils, especially the members from \( n-C_5 \) up to \( n-C_{14} \), covering a wide range of viscosity grades in the entire temperature range from 0 to 150 °C, have been found by Roelands to agree with the Slope Index \( S_g = 1.00 \). It has also been pointed out by Roelands that other truly homologous groups of pure hydrocarbons, namely the \( n \)-alkanes, the \( n \)-alkyloctylcyclometanes and the \( n \)-allylbenzenes display the same value of the Slope Index. The results obtained, on similar lines as before in the temperature range 40-150 °C, for the latter pure hydrocarbons have been displayed in Table 2.A1 and Figure 2.A1. \( m \)-values again fall along a straight line in the range of viscosity grade applicable to the former hydrocarbons, namely up to about 20 cP. Accurate values of the Slope Index \( S_g \) for 40 pure hydrocarbons taken at random by Roelands from API Research Project have been listed in Table 2.A2, where it may be observed that the liquids with relatively high viscosity grade (towards the end of the table) do possess somewhat higher Slope Index than unity. Correspondingly the \( m \)-values approach as high as 4.00, and they fall along a curve somewhat convex toward the viscosity axis in the complete range of hydrocarbons covered by Table 2.A2.

**APPENDIX 2.A2 Various Slope Indices \( S_g \) with given viscosity grade**

Various other liquids such as fluorocarbons and mineral oils have been analysed using the data provided by Roelands towards their Slope Index \( S_g \) in the temperature range covered by 40 °C and 150 °C. Various liquids belonging to the same group do not display any constancy of the Slope Index \( S_g \), and in fact they cover also a wide viscosity grade.

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**Table 2.A1 Slotte's parameters \( m \) and \( A \), and the coefficient \( S_{gg} \) at the grade temperature \( T_g = 40 \) °C for oils with constant Slope Indices.** Oils have been selected from ASTM-tables for kinematic viscosity of mineral oils; n-paraffines belonging approximately represented through \( S_g = 1.00 \) have been arbitrarily chosen with respect to their viscosity grade.

<table>
<thead>
<tr>
<th>Type of liquid</th>
<th>( T_g ) (°C)</th>
<th>( \mu ) cP</th>
<th>( \alpha )</th>
<th>( \gamma )</th>
<th>( \beta_{gg} )</th>
<th>( m )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASTM, ( L )-series, ( S_g = 1.23 )</td>
<td>10</td>
<td>4.3504</td>
<td>1.116</td>
<td>91.52</td>
<td>3.42</td>
<td>0.0563</td>
<td>3.0139 15.99</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>5.9168</td>
<td>1.942</td>
<td>163.81</td>
<td>6.6873</td>
<td>0.0627</td>
<td>3.3124 15.99</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>7.3732</td>
<td>2.862</td>
<td>224.72</td>
<td>9.2703</td>
<td>0.0682</td>
<td>3.8412 15.99</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>8.8298</td>
<td>3.777</td>
<td>284.58</td>
<td>11.6009</td>
<td>0.0737</td>
<td>4.3707 15.99</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>10.2865</td>
<td>4.691</td>
<td>344.43</td>
<td>13.9098</td>
<td>0.0794</td>
<td>4.9007 15.99</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>11.7432</td>
<td>5.604</td>
<td>404.28</td>
<td>16.2035</td>
<td>0.0853</td>
<td>5.4317 15.99</td>
</tr>
<tr>
<td>n-paraffine, ( S_g = 1.00 )</td>
<td>1</td>
<td>2.0859</td>
<td>1.116</td>
<td>91.52</td>
<td>3.42</td>
<td>0.0563</td>
<td>3.0139 15.99</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.3504</td>
<td>1.116</td>
<td>91.52</td>
<td>3.42</td>
<td>0.0563</td>
<td>3.0139 15.99</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>6.6165</td>
<td>1.942</td>
<td>163.81</td>
<td>6.6873</td>
<td>0.0627</td>
<td>3.3124 15.99</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>8.8298</td>
<td>2.862</td>
<td>224.72</td>
<td>9.2703</td>
<td>0.0682</td>
<td>3.8412 15.99</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>10.2865</td>
<td>3.777</td>
<td>284.58</td>
<td>11.6009</td>
<td>0.0737</td>
<td>4.3707 15.99</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>11.7432</td>
<td>4.691</td>
<td>344.43</td>
<td>13.9098</td>
<td>0.0794</td>
<td>4.9007 15.99</td>
</tr>
</tbody>
</table>

---

**Figure 2.A1 Slotte's exponent \( m \) in the temperature range 40 °C to 150 °C for various lubricating oils presented in Tables 2.A1-2.A8. \( m \) has been depicted for a wide range of dynamic viscosities.**

Two liquids belonging to actual lubrication practice call for special consideration on almost the same reason as that pointed out by Roelands [2.A1]. He has extensively investigated these two liquids belonging to two different homologous groups of synthetic oils, namely linear polymethylsiloxane and poly(ethylene-propylene) glycol derivative, for their rather ambiguities while they are represented by the ASTM Viscosity Index. These two liquids seem to possess one and the same VI of 135 and a kinematic viscosity grade of 183 cSt at 55 °C. The ambiguity has been removed by Roelands by relating the viscosity-temperature relationship through his Slope Index \( S_g \); he has also shown that the same ambiguity could also be removed using his DVI. While analysed on aforementioned lines using their given viscosity at 37.8 °C, these two liquids give rise to two different sets of the parameters \( m \) and \( A \), and the coefficient \( S_{gg} \). Each of these two liquids thus can equally uniquely be represented through a set of \( m \) and \( A \) or \( m \) and \( S_{gg} \) in a way similar to that using Roelands Slope Index \( S_g \) or his DVI. The results similar to those shown for the previously analysed liquids in Table 2.A2 have now been provided in Table 2.A3.
Table 2.A2 Slotte's parameters m and A, and fitted viscosities at 100 °C using Slotte's and Roelands' relationships for 40 pure hydrocarbons. The liquids have been chosen from Roelands' experimental data (Roelands [2.A1], Table VIII-10).

<table>
<thead>
<tr>
<th>Type of oil, PNI No.</th>
<th>n</th>
<th>Viscosity cp at 40 °C</th>
<th>m</th>
<th>A, °C</th>
<th>Viscosity cp at 100 °C using relationships: Slotte's, Roelands' E variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear polyethylene</td>
<td>1.03</td>
<td>1.45</td>
<td>0.62</td>
<td>1.251</td>
<td>20.0675</td>
</tr>
<tr>
<td>Poly (ethylene- propylene)</td>
<td>1.00</td>
<td>1.00</td>
<td>0.51</td>
<td>1.450</td>
<td>76.207</td>
</tr>
<tr>
<td>Fluorocarbon I</td>
<td>0.99</td>
<td>2.31</td>
<td>0.37</td>
<td>1.247</td>
<td>11.1004</td>
</tr>
<tr>
<td>Fluorocarbon II</td>
<td>0.97</td>
<td>2.27</td>
<td>0.38</td>
<td>1.430</td>
<td>74.607</td>
</tr>
<tr>
<td>Fluorocarbon III</td>
<td>0.90</td>
<td>1.40</td>
<td>0.51</td>
<td>1.495</td>
<td>76.207</td>
</tr>
<tr>
<td>Fluorocarbon IV</td>
<td>0.97</td>
<td>2.27</td>
<td>0.66</td>
<td>1.768</td>
<td>28.165</td>
</tr>
<tr>
<td>Fluorocarbon V</td>
<td>0.94</td>
<td>3.94</td>
<td>0.56</td>
<td>1.646</td>
<td>24.944</td>
</tr>
<tr>
<td>Fluorocarbon VI</td>
<td>0.94</td>
<td>3.94</td>
<td>0.69</td>
<td>1.654</td>
<td>24.944</td>
</tr>
<tr>
<td>Fluorocarbon VII</td>
<td>0.94</td>
<td>3.94</td>
<td>0.69</td>
<td>1.654</td>
<td>24.944</td>
</tr>
<tr>
<td>Fluorocarbon VIII</td>
<td>0.94</td>
<td>3.94</td>
<td>0.69</td>
<td>1.654</td>
<td>24.944</td>
</tr>
<tr>
<td>Fluorocarbon IX</td>
<td>0.94</td>
<td>3.94</td>
<td>0.69</td>
<td>1.654</td>
<td>24.944</td>
</tr>
<tr>
<td>Fluorocarbon X</td>
<td>0.94</td>
<td>3.94</td>
<td>0.69</td>
<td>1.654</td>
<td>24.944</td>
</tr>
</tbody>
</table>

Fluorocarbon liquids possess Slope Indices $B_N$ varying in a rather wide range of 1.44 and 1.84. The results obtained for these liquids have been tabulated in Table 2.A4. While plotted on a graph sheet the n-values for these liquids do not seem to follow any predictable trend.

Appendix 2.A3: Mineral oils

Characteristic mineral oils have been tabulated by Roelands at increasing viscosity and Slope Index at regular intervals. These tabulated oils have been found to be typical of those used in lubrication practice; an analysis of their parameters may thus be considered useful for the study of rheological problems. The aforementioned oils cover a wide range of viscosity grades at 40 °C, that is, between 1 cp and 5000 cp, and $S_N$-values between 0.94 and 1.64. The results provided in Table 2.A3 vary in the approximate range $1.00 < S_N < 1.60$, and their m-values show a variation of Slotte's exponent $m$ in the approximate range $1.00 < m < 5.0$. These $m$-values while plotted in Figure 2.A1 depict a near straight-line variation, especially in the range of high viscosity grades.

Appendix 2.A4: Synthetic oils

The parameter $m$ and $A$, the coefficient $B_N$ at the temperature $T_{oc} = 100$ °C for typical fluorocarbon oils previously analysed by Roelands (2.A1, Table VI-7). Also included are fitted viscosities at 100 °C using Roelands' and Slotte's relationships.

Table 2.A3 Slotte's parameters m and A, and the coefficient $B_N$ at the temperature $T_{oc} = 100$ °C for two particular homologous groups of synthetic lubricating oils. Also included are fitted viscosities at 100 °C using Roelands' and Slotte's relationships.

<table>
<thead>
<tr>
<th>Compound</th>
<th>$m$</th>
<th>$A$, °C</th>
<th>$B_N$, °C</th>
<th>$m$</th>
<th>$A$, °C</th>
<th>$B_N$, °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear polyethylene</td>
<td>1.15</td>
<td>15.90</td>
<td>0.6395</td>
<td>19.58</td>
<td>0.6395</td>
<td>19.58</td>
</tr>
<tr>
<td>Poly (ethylene- propylene)</td>
<td>1.16</td>
<td>16.20</td>
<td>0.6395</td>
<td>19.58</td>
<td>0.6395</td>
<td>19.58</td>
</tr>
<tr>
<td>Fluorocarbon I</td>
<td>1.09</td>
<td>19.58</td>
<td>0.6395</td>
<td>19.58</td>
<td>0.6395</td>
<td>19.58</td>
</tr>
<tr>
<td>Fluorocarbon II</td>
<td>1.09</td>
<td>19.58</td>
<td>0.6395</td>
<td>19.58</td>
<td>0.6395</td>
<td>19.58</td>
</tr>
</tbody>
</table>

2.5 Show a variation of Slotte's exponent $m$ in the approximate range $1.00 < m < 5.0$. These $m$-values while plotted in Figure 2.A1 depict a near straight-line variation, especially in the range of high viscosity grades.

*Paratone*-blended mineral oils (a mineral-oil solution of polyisobutylene), also undercarded by ASTM Viscosity Index, like the aforementioned two synthetic oils, have been analysed, tabulated in Table 2.A6 and finally depicted also in Figure 2.A1. These polymer-blended mineral oils depict a near straight line relationship of m-values with increased polymer concentration, the latter being represented through an increased viscosity grade.

A sample group of saturated mineral oils with a rather low viscosity grade at 40 °C and represented by a near constant value of the Slope Index $S_N$ close to unity is reasonably well correlated by a straight line relationship with the viscosity grade. Their $m$-values fall in the approximate range of $1.00 < m < 2.40$, but exceed 2.40 especially in the viscosity grade exceeding about 20 cp (see Table 2.A7 and Figure 2.A1).

An analysis of the representative mineral oils compiled by Roelands has also been carried out and presented in Table 2.A7. Their $S_N$-values fall approximately in the range of $1.00 < S_N < 1.60$, and their $m$-values...
have been found to cover the range given by 1.0 < m < 4.0 in the temperature range covered by 40 °C up to 150 °C.

The agreement between the viscosity values obtained using Roelands' relationship and those obtained making use of the parameters m and A thus evaluated in Slotte's relationship is, on the whole, found to be satisfactory good. In order to quantify this agreement, the viscosity values obtained at 100 °C have been tabulated for every oil in the corresponding tables discussed herebefore. It may be noticed referring to the last three columns of these tables that Slotte's relationship slightly over-estimates the viscosity than does Roelands' relationship in the complete range of the viscosity grade covered by every group of oils.

APPENDIX 2.A4 Water

In order to study the applicability of Slotte's relationship to a typical illustrative liquid, water has been investigated. The variation of viscosity of water with temperature has been studied by various investigators, and it has been given in the form of a fluidity function of
where \( \nu \) is the viscosity at the reference temperature \( T_r = 20 \, ^\circ C \) given by:

\[
\nu(T) = \frac{\nu(T_r)}{1 + a(T - T_r) + b(T - T_r)^2 + c(T - T_r)^3}
\]

A more accurate relationship for water, however, is that provided by the National Bureau of Standards (NBS) (cf. Handbook of Chemistry and Physics [2.A4]). This relationship in the temperature range 20 °C - 100 °C is:

\[
\nu(T) = 10^{-5} \left( 1 + 0.667(T - T_r) + 0.407(T - T_r)^2 \right)
\]

The viscosity-temperature coefficient \( \beta \) can be evaluated at the same reference temperature and at atmospheric pressure, obtained by differentiating equation (2.Aa) with respect to \( T \). Table 2.A8 gives values of Slotte's parameter \( a \) and \( B \), and fitted viscosities at 150 °C (cf. Appendix 2.A4).

Table 2.A8

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>Slotted's Parameter a</th>
<th>Slotted's Parameter B</th>
<th>Fitted Viscosity at 150 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0335</td>
<td>0.0021</td>
<td>1.07</td>
</tr>
<tr>
<td>30</td>
<td>0.0369</td>
<td>0.0023</td>
<td>1.13</td>
</tr>
<tr>
<td>40</td>
<td>0.0403</td>
<td>0.0025</td>
<td>1.18</td>
</tr>
<tr>
<td>50</td>
<td>0.0437</td>
<td>0.0027</td>
<td>1.23</td>
</tr>
<tr>
<td>60</td>
<td>0.0471</td>
<td>0.0029</td>
<td>1.28</td>
</tr>
<tr>
<td>70</td>
<td>0.0505</td>
<td>0.0031</td>
<td>1.33</td>
</tr>
<tr>
<td>80</td>
<td>0.0539</td>
<td>0.0033</td>
<td>1.38</td>
</tr>
<tr>
<td>90</td>
<td>0.0573</td>
<td>0.0034</td>
<td>1.43</td>
</tr>
<tr>
<td>100</td>
<td>0.0607</td>
<td>0.0036</td>
<td>1.48</td>
</tr>
</tbody>
</table>

(continued)
The viscosity-temperature coefficient $\beta$ can then be estimated using its definition given by expression (2.20). At any temperature, say $T=20°C$ (the viscosity being 1.002 cP), $\beta$ reduces to 0.0245, at $T=100°C$ (the viscosity being 0.2481 cP). To 0.0166. Then Slotte's parameters from their definition, given by expressions (2.25) and (2.26) reduce to:

m = 5.022

$A=41.37°C$

The resulting Slotte's relationships of three- and two-parameter type in terms of $m$ and $A$, at $20°C$ have been found to be fairly accurate. For instance, at any selected temperature, say $T=40°C$, the above Slotte's relationships give rise to a viscosity, $\mu=0.6526$ cP, which agrees quite well with that tabulated by HBS, i.e. 0.6529 cP. Viscosity-values evaluated using HBS relationship given by expression (2.22) at $T=40°C$ is 0.6517 cP and that obtained using Roelands relationship with $\beta=1.44$ and $A=1.4521$ is 0.6300 cP. It may hence be concluded that Slotte's relationship of three- as well as two-parameter type yield far more accurate viscosity values than HBS relationship, although these were fitted using the latter relationship.

### APPENDIX 2.45 References

- [2.2] Lamb, H., Hydrodynamics, Cambridge Univ. Press, 1940.
Appendices to Chapter III

APPENDIX 3.A1 BRINKMAN NUMBER

Brinkman number, \( Br \), is a useful hydrodynamic dimensionless parameter that can conveniently be used to re-express our results. Being the product of the Prandtl and Eckert numbers, it incorporates fluid properties through the former and dissipation through the latter. \( Br \) represents, in a physical sense, the ratio of the total amount of heat produced by internal friction to the amount of heat that is transported from the fluid through the walls. This parameter \( Br \), which has been employed by various investigators in the past, can readily be used to recast the results that have been obtained in the present chapter.

Representation of \( Br \) is not yet universally standardised as to the influential quantities included in this parameter. While some investigators employ only the boundary conditions for the state of the fluid and the motion of the walls, others employ influential quantities corresponding to the extreme kinematic and thermal state within the film cross-section. The former representation of \( Br \) has a form more similar to that of the thermal sliding parameter \( N_j \) employed in the present chapter. Some investigators have, however, defined \( Br \) excluding the viscosity-temperature coefficient \( \beta \) in the case of the exponential relationship. The latter form, using the notation employed in this chapter, is,

\[
Br = \frac{u_s}{k(T - T_i)}
\]

where \( u_s \) is some such characteristic velocity as the sliding velocity of the two surfaces. The Brinkman number \( Br \), in the form used in the present chapter does include the coefficient \( \beta \) and is given by,

\[
Br = \frac{u_s}{k(T - T_i)}
\]

where \( \beta \) takes the place of some such reference temperature as \( T_i \) in the former definition of \( Br \). This latter definition of \( Br \) is seen to be identical with that of \( N_j \). In comparing the results presented in this chapter against those of others this fact may be used to advantage.

There is yet another form of \( Br \) that is rarely used in literature. This form incorporates the deviation of the actual behaviour of the fluid with respect to the hypothetical case of constancy of the viscosity. It has proved convenient in the analysis of entry problems and convective heat transfer. It is closely related to the latter definition of \( Br \), viz.:

\[
Br = \frac{u_s}{k(T - T_i)}
\]

APPENDIX 3.A2

In certain kinds of problems the deviation of the fluid behaviour may be conveniently accounted for through some such characteristic temperature as the maximum \( T_H \), occurring within the film cross-section, viz.:

\[
Br^* = \frac{u_s}{k(T - T_i)}
\]

The last definition also indicates the proportional amount of viscous dissipation and the temperature development. Except for \( N_j \), greater than or equal to \( N_j \), in the range \( r_i < 0.5 \), the solution of Section 3.3.1, yields:

\[
Br^* = \left(1 - \frac{r_i}{r_i - 0.5}\right) \ln \left(\frac{r_i}{r_i - 0.5}\right)
\]

Thus, in the limiting case of vanishing sliding velocity,

\[
lm Br^* = \frac{1}{r_i}
\]

At the upper limit the whole range of \( r_i \) can be split up into two ranges, namely,

\[
lm Br^* = \frac{1}{r_i}
\]

\[
lm Br^* = \frac{1}{r_i}
\]

From the analysis for these two ranges of \( r_i \) in the corresponding operating range of viscous dissipation, it is again seen that the nature of the variation of the shear stress parameter \( \Gamma_i \) with \( Br^* \) is almost identical with that with \( N_j \), except at \( r_i = 0 \) where \( Br^* \) reaches infinitely large values for all \( N_j \). For \( r_i > 0.5 \) \( Br^* \) is double-valued with respect to any prescribed parameter \( \Gamma_i \). But the variation of \( Br^* \) with respect to the selected parameter \( \Gamma_i \) is different from that of \( N_j \). (Compare Figure 3.A1 with Figure 3.3). The variation of \( \Gamma_i \) with \( Br^* \) is exceedingly steep, reaching infinitely large values even with a slight increase of \( Br^* \) (see Figure 3.A1 and particularly curve No. 1 for \( r_i = 0.05 \)). Operating regime in terms of \( Br^* \) becomes somewhat wider as \( r_i \) is approaching 0.5, from below, as may be seen from curve No. 4 for \( r_i = 0.4 \) in the same figure. For \( r_i = 0.495 \), in the present somewhat wider range of \( Br^* \), the parameter \( \Gamma_i \), reaches a peak and then falls down, thereafter to increase steeply to infinitely large values at \( Br^* \) close to the latter's particular critical value. This behaviour is similar to that of the shear stress parameter \( \Gamma_i \) with \( N_j \), for the same values of \( r_i \). At very low viscous dissipation, or \( N_j < 1 \), an increase of \( r_i \) from 0.05 to 1.0 reduces the value of the Brinkman number, see Figure 3.A2.

Referring to Figures 3.A1 and 3.A2 it may be noticed that in the range 0.05 \( r_i \), \( r_i \), the Brinkman number \( Br^* \) is almost independent of \( N_j \), as well as of \( \Gamma_i \). It there depends nearly solely on \( r_i \). In other words, in this operating regime \( Br^* \) is not indicative of the amount of viscous dissipation in the way characterized by the parameter \( N_j \). Further, the near constancy of \( Br^* \) represents, on the basis of the definition of the \( Br^* \), a uniform viscous dissipation and the temperature development in the whole of the operating regime lying up to that given by the critical condition. This is apparent from curves 1, 2, 3 and 4 in Figure 3.A2.
Figure 3.1 Shear stress parameter $T_1^*$ for the complete set of $r_1$, covering a wide range of Brinkman numbers $Br^*$. However, with higher values of $r_1$, especially $r_1 \geq 0.5$, the thermal behaviour of the flow undergoes a drastic change. The near constancy of $Br^*$ is limited to operating conditions leading to low viscous dissipation, being represented by low values of $N_1$. With an increased viscous dissipation, i.e. at higher values of $N_1$, the temperature of the liquid does not increase in a proportional manner. The values of $Br^*$ increase monotonically with $N_1$. At high values of $N_1$ the temperature increase remains almost constant, as is indicated by the nearly linear relationship between $Br^*$ and $N_1$.

It may further be observed that the regime of thermal instability proposed by Blok is evident also from the presently evaluated results. In this 'forbidden' regime of operation the Brinkman number $Br^*$ is equally non-existent.

Appendix 3.A2 Specification of the regime of instability

A classification of several modes of thermal or flow instability has been set up by Sukanek et al. [3.A1] who based it on features characterizing the solutions obtained by them. A short survey of their results has been provided in what follows; for more details their publication may be referred to. Other investigators, also dealing with stability criteria at a finite $Re$, have refrained from classifying their various solutions; Lebon and Nguyen [3.A2], and Vanderborck and Platten [3.A3]. The latter investigators have specified the lowest critical $Re$ at which the instability, irrespective of its kind, might arise. Their results are expected to be quantitatively comparable with the lowest critical $Re$ among Sukanek et al.'s different modes at various values of $N_1$ or $Br$.

Modes of instability of flow

1. Inviscid mode

The present mode is characterized by laminar flow with a point of inflection in the velocity profile. The point of inflection may appear in the middle of the film for equipartitioning of the heat, and also in isoviscous flow. It shifts away for asymmetric thermal boundary conditions. Obviously, velocity profiles of the non-linear type, such as those evaluated for the non-isothermal flows in Section 3.2.4, do possess a point of inflection at a sufficiently high $N_1$ or $Br$. The instability, if it takes place with the present mode, originates at the point of inflection, where the second derivative of the velocity is zero. This instability first appears at $N_1$- or $Br$-values corresponding to the limiting shear stress parameter $T_{1_{\text{lim}}}$. Although no information on the magnitude of $Re$ at these $N_1$ or $Br$ values is readily available, it is expected to be finite at an incipient such instability. It is, however, certain from the results of the previous investigators that the critical $Re$ at which this instability is initiated, decreases at increasing value of $N_1$ or $Br$. This is attributed to increased skewness of the velocity profile. Most of the early investigators in their search for a flow instability have used the stability criterion of 'inviscid' fluids ignoring viscosity effects. Accordingly they theoretically predicted infinitely high $Re$ since no point of inflection in the velocity profile would then be found. With the introduction of an exponential viscosity-temperature relationship along with a linear temperature distribution, Deardorff [3.A4] found the critical $Re$ to be finite. Later investigators generalized the velocity profile to a cubic function and considered the fluid to undergo a change in fluidity according to a hyperbolic relationship, Fu and Joseph [3.A5]. At high values of a parameter somewhat identical with $N_1$ or $Br$ they noticed a considerable reduction of the critical $Re$. 
A reasonably satisfactory study should account for the basic velocity profile of the type indicated in Section 3.3.4. The study should also concern the instability of flow with the coupled momentum and energy equations. In addition it should involve a two dimensional disturbance. Such a study has been carried out by a few investigators only for equi-partitioning of the heat. Their results, as listed in Table 3.1, indicate a drastic decrease of the critical Re at high $N_l$ values.

It may now be concluded that the earlier prediction by Joseph [3.A6] of a flow instability of the 'inviscid' mode is justifiable. Such an instability for real viscous fluids has thus been proven to appear already at low Re. It is a consequence of the non-linearity of the basic velocity distribution. Turbulent velocity profiles observed already at low values of Re between 600 and 1450 in some classical Couette flow experiments conducted about two decades ago can now be explained at least partly.

2. Viscous and coupling modes

The viscous mode results from an interaction of terms of the strongly non-linear variable viscosity with those of the velocity in the momentum equation. This mode may be attributed to the mechanism, also called 'diffusive' mechanism of viscosity, which alters the position of a fluid particle in the flow field will also alter its viscosity.

The coupling mode arises from the coupling of the stability equations and the viscous dissipation terms in the energy equation. At low $N_l$ or $Br$ values this mode is strongly influenced by the convective terms. At higher values of $N_l$ and $Br$, however, the influence of these terms decreases. This mode is determined by the viscous heating effects. However, the corresponding critical values of Re for the complete range of $Br$ investigated by Sukanek et al. are far too high to be attributable to the coupling mode.

3. Thermal mode

The thermal mode differs from the 'inviscid' one in that it mainly relates to the temperature distribution. It was originally worked out through a somewhat simplified treatment by Joseph [3.A7]. The pertinent stability criterion has been derived by decoupling the velocity and temperature distributions. Joseph confined himself to the stability of the temperature distribution, and did not consider that of the flow. He introduced only a one-dimensional disturbance into the diffusive equation. Thus the appearance of the secondary flow, which may be attributed mainly to instability of flow, could not be investigated.

As explained before, the stability criterion is determined by analysing the time response of some postulated disturbance. Such a disturbance may contain some definite wave speed and wave number. Through a series of related problems Joseph, [3.A7, 3.A6, 3.A8], established that with heat sources that vary non-linearly with temperature, there exists a neutral solution of the perturbation equation for a zero wave number disturbance already at some such finite characteristic shear stress as the limiting value. Along the second branch (see Figure 3.1) however, the eigen-value pertinent to the wave speed and the shear stress parameter considered has a negative slope. Since in the classical analysis this is a sufficient criterion for the development of instability, it may be concluded that the flow really becomes unstable. Such an analy-

APPENDIX 3.A3 REFERENCES


Appendix to Chapter VII

APPENDIX 7.1 MAJOR FEATURES OF THE OPTICAL SYSTEM USED IN THE LASER-DOPPLER METHOD

Laser:
Coherent Argon-ion laser, 0.3 watt (effective).
Wave length \( \lambda = 0.5145 \text{ \mu m} \).
Diameter of the beam \( b_0 (=D_1/e^2) = 1.2 \text{ mm} \).
Diameter of the beam leaving lens \( L_2, d_2 = 2.0 \text{ mm} \).

Rotating grating:
Number of radial lines \( N = 2^{14} = 16384 \).
Rotational frequency \( f_r = 54 \text{ Hz} \).
Resulting frequency difference \( f_s = 1.77 \text{ MHz} \).
Refractive angle \( 4.85^\circ \).

Refractive index:
Tested oils \( n_1 = 1.480 \) (relating to the sodium D-line at 20 \( ^\circ \text{C} \)).
Stator glass \( n_2 = 1.520 \) (relating to the sodium D-line at 20 \( ^\circ \text{C} \)).

Lenses (see Figure 7.11):
\( f_1 = 120 \text{ mm} \)
\( f_2 = 200 \text{ mm} \)
\( f_3 = 55 \text{ mm} \)

Measuring volume (see Figure 7.12):
Fringe spacing \( d_2 = 0.871 \text{ \mu m} \).
Thickness of the measuring volume \( d_m = 19 \text{ \mu m} \).
Length of the measuring volume \( l_m = 93 \text{ \mu m} \).
Number of fringes, 21.

Beam angles:
Half angle between beams before entering glass \( \phi = 17.18^\circ \).
Half angle between beams before entering oil \( \phi_m = 11.20^\circ \).
Distance between parallel, collimated laser beams where they enter the final lens system, \( s = 39.34 \text{ mm} \).

Appendices to Chapter VIII

APPENDIX 8.1 SOLUTION TO THE GOVERNING ENERGY EQUATION

The second order ordinary differential equation given by equation (8.8) can be shown to belong to the class of equations of the form,
\[
\frac{d^2}{dt^2} \frac{1}{\theta} \frac{d}{d\theta} \theta^2 \frac{d}{d\theta} \theta = 0
\]
This class of equations has known solutions for the following integer values of \( n \): 0, 1 and 2. When \( n=0 \) the equation complies with rectangular coordinates, \( n=1 \) with cylindrical, and \( n=2 \) with spherical coordinates. The present equation under study can be shown to belong to the cylindrical coordinate system when putting,
\[
\tilde{Y}^2 = \zeta
\]
The temperature derivatives with respect to the new variable \( \zeta \) become,
\[
\frac{d^2}{d\zeta^2} \theta = 2\sqrt{\theta} \frac{d}{d\zeta} \theta
\]

Substitution of the foregoing expressions and rearrangement of terms yield the above equation with \( n=1 \). This equation belongs to a class which represents problems such as the two-dimensional vortex motion of incompressible fluids, the space charge of electricity of a glowing wire and the distribution of mass of gaseous interstellar material under the influence of its own gravitational field. The present equation is a one-dimensional representation of the above problems in the cylindrical coordinate system for which an exact solution has been worked out by Chambré [8.11]. For details Ames [8.12] may be referred to. A convenient form of the solution as given by Chambré's method, and in terms of the dimensionless temperature \( \theta \), reads:
\[
\theta = 2 \ln \left[ \frac{\gamma + K}{\sqrt{\gamma^2 + K}} \right]
\]
where \( \gamma \) is double-valued in terms of \( K \) and is given by,
\[
\gamma = \sqrt{16 - 2K} - K
\]

APPENDIX 8.2 SOLUTION IN TERMS OF BRINKMAN NUMBER

\[\text{[Appendix 8.A2}\]
In the analysis of isothermal flows under the influence of viscous dissipation the Brinkman number $Br_p$ is commonly employed. This number can be expressed using the previously defined characteristic velocity for isoviscous flows, viz. $u^* = gR/(2U)$, namely in the form,

$$Br_p = \left(\frac{\mu}{k}\right) \frac{u^*}{r} = \frac{2k}{k \tau}$$

so that the foregoing parameter can be related to the Brinkman number as follows:

$$B_T Br_p - K$$

With this identity, the solution obtained in Appendix 8.A1 proves to be identical with that obtained by Sukanek and Laurence [8.A3]. In fact, one of the well-known methods expressing various thermohydrodynamic quantities is given by Sukanek and Laurence in terms of the Brinkman number as defined in terms of the average velocity, namely,

$$Br_v = \left(\frac{u}{k \tau}\right) \frac{u^*}{u}$$

where $u_{av}$, the average velocity of flow, depends on the thermal effects, and replaces $u^*$ used in $Br_p$. This velocity $u_{av}$ is defined in terms of the rate of flow $q$ by,

$$u_{av} = \frac{q}{\tau R^2}$$

Using the solution for the volumetric flow rate given by Nihoul [8.A4], or that independently obtained in Section 8.2.4, a parameter involving the Brinkman number $Br_v$ can be expressed as follows.

$$u_{av} = \left(\frac{q}{\tau V}\right) (1 + \gamma/K)$$

and

$$B_T Br_v = \frac{K}{\tau} (1 + \gamma/K)$$

By making use of the relationship between $K$ and $\gamma$ expressed in Appendix 8.A1, and the identity between $\gamma/K$ and $B$, it may be shown that $B_T Br_v/2$ is identical to $B$.

**APPENDIX 8.A3 REFERENCES**


A point of inflection in the velocity profile first appears at the wall at some critically high value of the pressure gradient parameter $P$. Such values can be determined, if only numerically, as shown herebelow.

Starting from the definition of Newtonian behaviour of liquids,

$$\tau = \frac{\mu}{\dot{\gamma}}$$

one obtains,

$$\frac{du}{dy} = -\sqrt{r} \left( \frac{k}{\theta} \right)^{1/2} \sum \frac{h}{k}$$

The second derivative is simply,

$$\frac{d^2u}{dy^2} = -\sqrt{r} \left( \frac{k}{\theta} \right)^{1/2} \left[ \sum \frac{h}{k} + \gamma \frac{du}{dy} \right]$$

This expression may further be simplified to obtain the point of inflection at an arbitrary coordinate $y_{inf}$, which does not necessarily lie on the wall,

$$y_{inf} = -\frac{1}{dP/dy_{inf}}$$

In accordance with the general representation of the results mentioned before in Section 9.3.4.2,

$$P_{inf} = -\frac{1}{dP/dy_{inf}}$$

For Slott's viscosity-temperature relationship in the range $1<\alpha<\infty$, the above relationship can further be reduced to,

$$P_{inf} = -\frac{1}{\alpha} \left( \frac{d\psi}{d\alpha} \right)_{inf}$$

Since the results have already been numerically evaluated for $\psi$ and $d\psi/d\alpha$, $P_{inf}$ can readily be obtained.

For fluids with $\alpha=1$ in Slott's relationship the present critical value of parameter $L$ can be worked out. This can be achieved by making use of the previous solutions for the dimensionless temperature $\psi$ in terms of fractional Bessel functions. First of all, the gradient $d\psi/dL$ can be evaluated using certain characteristics of the differentiation of Bessel functions. This result is,

$$\frac{d\psi}{dL} = -\frac{1}{2} \sum (\psi_L)^2 - \frac{1}{2} \sum (\psi_L)^2$$

where $j^*$ can be expressed in one of the following forms,

$$j^* = \sqrt{2} \sum J_{-1}(\psi_L) + J_{-2}(\psi_L)$$

or,

$$j^* = \sqrt{2} \sum J_{-1}(\psi_L) - J_{-2}(\psi_L)$$

The condition for the first appearance of the point of inflection at the wall can now be simplified to the form,

$$y_{inf} = -\frac{1}{L} \sum (\psi_L) + j^*$$

Appendices to Chapter X

APPENDIX 10.A1 NUMERICAL TECHNIQUES TO OBTAIN SOLUTIONS

Numerical techniques can, however, be applied to working out solutions to the general problem under study, i.e. given by differential equation (10.10). Methods for solving non-linear differential equations are generally selected on the basis of a priori knowledge about the structure of the solution. Such knowledge is more important in the case of asymptotic solutions at infinite values of the independent variable than in the case of solutions which do not show an asymptotic behaviour. In the former case the choice of the numerical method of solution is critical with respect to both stability and convergence. Difficulties encountered by some investigators who have attempted to analyse the problem of the present type may easily be identified. Stoffel [10.A1] while analysing the plan-parallel pressure flow has, in fact, encountered an instability of this numerical method. His failure to obtain the solution in a really wide operating regime may now be attributed both to improper selection of the input parameters and to his numerical method. Further details have already been discussed in Chapter 9. In addition to the problems connected with the stability and convergence of the numerical method the truncation error involved in numerical evaluation calls for a proper selection of the method. A large magnitude in the truncation error may generally be attributed to too few steps of iteration.

Method of Solution

Milne's method for the second-order ordinary differential equation in which the first derivative is absent seems to be most suitable; cf. Kunz [10.A2]. Other predictor-corrector methods for non-linear equations such as that due to Clenshaw and Olver [10.A3] (cf. Fox [10.A4]) are applicable to the more general type of second-order equations where the first derivative is not missing. The latter method does not seem to be particularly suitable for the present type of equation.

All in all, Milne's method is deemed the most suitable one. This is only a conjecture, no mathematic confirmation having been found. An independent confirmation, although not too difficult to carry out, is

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These predictor-corrector methods are suitable for the numerical analysis of non-isothermal flows in pipes, where the second-order differential equation does involve the first derivative. It is perhaps preferable to reduce the equation concerned to a set of first order equations and then to use any conventional suitable numerical technique.
time-consuming and has not therefore pursued for the present. Let it suffice to mention that such a confirmation can be carried out along lines already known to be suitable for first order differential equations. This has been done successfully by Hamming [10.A5], and Fox [10.A4]. They have devised their method as a special expedient for improving the numerical accuracy and have thereby offered an extension to the checking procedure originally introduced by Milne. A special feature of his method is the use of the difference between the truncation error of the predictor and the corrector formulae as a truncation error of the numerical integration procedure. This difference is used to eliminate part of the truncation error and to produce higher accuracy.

**Starting the integration**

The accuracy of a numerical solution depends strongly on that with which the starting values are estimated. For certain predictor-corrector methods this may also be somewhat critical as to the stability of the numerical technique. Taylor's method, Runge-Kutta's method and Picard's method of successive approximation are a few of many methods available. Each method has its own advantages and disadvantages. There are, however, no definite rules about their suitability for the various predictor-corrector methods. The former two methods are equally applicable and popular. The Runge-Kutta method requires reduction of the original differential equation to an equivalent set of first order equations and this method can readily be used when the initial conditions do not impose an indeterminate form for the derivative.* Taylor's method well suits the present second-order differential equation where the first derivative is absent. Elimination of coefficients and the region of validity of Taylor's method have been explained in Appendix 10.41.

**Accuracy, stability and convergence of the solution**

The truncation error is readily assessible through independent remainder terms in the predictor-corrector formulæ. Any improvement desired can be realized for the numerical technique either by iterating the solution or by reducing the step length.

Rounding errors and their propagation in the solution of second-order initial value problems are of certain importance. These are generally evaluated at every local point and treated as random variables. A common practice is to specify the mean and variance of these variables. Both truncation and rounding errors affect the calculation in successive steps. Any tendency of these errors to accumulate rapidly gives rise to instability of the solution. Stability of a solution is indicated by the approach of the asymptotic nature of the solution towards the true one of the differential equation when the number of steps is increased. If a solution approaches the true solution at each particular point as the interval tends to zero, then the demand for convergence is satisfied.

Any discussion of stability and convergence is rather difficult, particularly if the differential equation involved is non-linear, such as in most of the present analyses. Common types of instability encountered with a predictor-corrector method are attributed to spurious solutions whose behaviour is critical. These spurious solutions may not disappear as the step length is reduced to extremely small values, and may result in lack of stability and convergence. The stability analysis of the present method of Milne can be performed through the well-known general theory of Dahlquist (cf. Fox [10.A4]).

**APPENDIX 10.A2 STARTING VALUES FOR MILNE'S METHOD**

An evaluation of starting values to initiate a desired numerical technique is basic to the computational procedure. The accuracy of these starting values has a considerable influence on the accuracy of the results to be computed. Taylor's series expansion proves both easy to use and accurate in a reasonably wide range in order to obtain the required starting values. Prior to applying it the convergence of terms in the independent variable w may have to be verified.

Let the required solution of the given differential equation be,

\[ \theta = \varphi(w) \]

If \( w_0 \), \( w_1 \) is not a singular point of the function, the latter function can be expanded as follows:

\[ \theta = \varphi_0 + (w-w_0) \varphi_1 + \frac{(w-w_0)^2}{2!} \varphi_2 + \frac{(w-w_0)^3}{3!} \varphi_3 + \cdots \]

It is now required to determine the derivatives \( \varphi_0, \varphi_1, \varphi_2, \varphi_3 \), etc. which appear as coefficients in this equation. To achieve this the original second-order differential equation can be made use of, viz.:

\[ \varphi_2 = \varphi_0 + (w-w_0) \varphi_1 \]

where \( \varphi_0 \) is some function of both the dependent and independent variable, \( w \) and \( \theta \). Differentiating both sides once,

\[ \varphi_3 = \frac{\partial \varphi_2}{\partial w} = \varphi_1 + (w-w_0) \varphi_2 \]

and, further by repeated differentiation,

\[ \varphi_s = \varphi_0 + (w-w_0) \varphi_1 + \cdots \varphi_0 \]

where, the following recurrent formula may be used,

\[ \varphi_s = \frac{\partial \varphi_{s-1}}{\partial w} + \frac{1}{2!} \varphi_1 + \frac{1}{3!} \varphi_2 + \cdots + \frac{1}{s!} \varphi_s \]

Setting \( w = w_0 \), and \( \theta = \varphi_0, \varphi_1 = \varphi_0/\varphi_1 \), at \( w = w_0 \) the coefficients can be evaluated from:

\[ \varphi_s = \varphi_0 + (w-w_0) \varphi_1 + \cdots \varphi_0 \]

---

*In the case of a second-order differential equation where the first derivative is not missing, the "starter" formulæ used for Milne's first order equation can well be employed (cf. Runz [10.A2]). First, an equivalent set of first order equations is established, and then the "starter" formulæ are brought into play.
APPENDIX 10.A3 COEFFICIENTS IN THE TAYLOR SERIES

The convergence of the Taylor series is considered to be valid in the range here to be covered, \( w \leq w_n \). The second-order differential equation (10.10) for the cases with an exponential viscosity-temperature relationship can be expressed in the form:

\[
d^2v \over dw^2 + (v + \zeta_{HB})^2 e^v = 0
\]

with the "one-point" set of thermal boundary conditions,

\[
v = 0, \ \theta = 0, \ \frac{dv}{dw} = 0
\]

Successive differentiation of the above differential equation yields,

\[
\begin{align*}
\phi^{(3)} &= e^v \left[ -(v + \zeta_{HB})^2 v \right] \left[ 2 + (w\zeta_{HB})^2 v \right] \\
\phi^{(4)} &= e^v \left[ -1 + (w\zeta_{HB})^2 v^2 \right] \\
\phi^{(5)} &= e^v \left[ 7(w\zeta_{HB})^3 v^3 \right] - 6(w\zeta_{HB})^2 v^2 \phi^{(2)} + \left[ -6 + 4(w\zeta_{HB})^2 v^2 \right] \phi^{(1)} - 6(w\zeta_{HB}) (\phi^{(0)})^2 + \left[ - (w\zeta_{HB})^3 (\phi^{(1)})^2 \right] \\
\phi^{(6)} &= e^v \left[ (w\zeta_{HB})^2 v^2 \right] \left[ 27 - 4(w\zeta_{HB})^2 v^2 \right] + 42(w\zeta_{HB})^3 v^2 \phi^{(1)} + \left[ -12 + 11(w\zeta_{HB})^2 v^2 \right] \phi^{(1)} - 8(w\zeta_{HB}) (\phi^{(0)})^3 + \left[ - (w\zeta_{HB})^3 (\phi^{(1)})^2 \right]
\end{align*}
\]

and so on. Owing to the boundary conditions at \( w=0, \ \theta=0 \) these derivatives become:

\[
\begin{align*}
\phi^{(1)} &= 0 \\
\phi^{(2)} &= -\zeta_{HB}^2 \\
\phi^{(3)} &= -2\zeta_{HB} \\
\phi^{(4)} &= -2v\zeta_{HB} \\
\phi^{(5)} &= 7\zeta_{HB} \\
\phi^{(6)} &= \zeta_{HB}^2 \left[ 27 - 4\zeta_{HB}^2 \right]
\end{align*}
\]

Then the function estimated using the Taylor series reduces to:

\[
\phi = \frac{\zeta_{HB}}{2} \psi^2 - \frac{\zeta_{HB}^3}{3} \psi^3 - \frac{(2 - \zeta_{HB}^2)}{24} \psi^4 + \frac{7\zeta_{HB}}{120} \psi^5 - \frac{\zeta_{HB}^4}{720} \psi^6 + \ldots
\]

APPENDIX 10.A4 SOLUTIONS TO THE LINEAR FLUIDITY FUNCTION

Solutions for the particular case specified in Section 10.4.2 using a linear fluidity function can be worked out in terms of special functions. The aforementioned governing second-order differential equation for energy balance, i.e. equation (10.43):

\[
d^2\psi \over dt^2 + \alpha \frac{dv}{dt} = 0
\]

can be solved in terms of the fractional Bessel functions \( J_{\zeta} \) and \( J_{-\zeta} \), viz.,

\[
\psi = \sqrt{\zeta} \left[ c_1 J_{\zeta} \left( \frac{t^2}{2} \right) + c_2 J_{-\zeta} \left( \frac{t^2}{2} \right) \right]
\]

The coefficients \( c_1 \) and \( c_2 \) can be evaluated using the "one-point" thermal boundary conditions (10.44), viz.,

\[
\zeta = \zeta_{HB}, \ \psi = 1, \ \frac{dv}{dt} = 0
\]

For such an evaluation of \( d\psi/dt \) the properties of the Bessel functions may be used as follows:

\[
\frac{d}{dt} \left[ t^{-\zeta} J_{\zeta} \right] = -t^{-\zeta} J_{\zeta+1} \left( t \right)
\]

and

\[
\frac{d}{dt} \left[ t^{\zeta} J_{-\zeta} \right] = t^{\zeta} J_{\zeta-1} \left( t \right)
\]

Then introducing,

\[
t = \frac{\sqrt{\zeta}}{2}
\]

the solution for \( \psi \) can be given by:

\[
\psi = \left( 2t \right)^{\frac{1}{2}} \left[ c_1 J_{\zeta} \left( t^2 \right) + c_2 J_{-\zeta} \left( t^2 \right) \right]
\]

and its derivative with respect to \( t \):

\[
\frac{d\psi}{dt} = 2t \left[ c_1 t J_{\zeta} \left( t^2 \right) - c_2 t J_{-\zeta} \left( t^2 \right) \right]
\]

Applying the boundary conditions at \( z = z_{HB} \) and solving for the coefficients \( c_1 \) and \( c_2 \) in the last two expressions, we obtain in terms of the original variable \( z_{HB} \):

\[
c_1 = \frac{1}{\sqrt{z_{HB}}} \frac{1}{J_{\zeta} \left( z_{HB}^2 / 4 \right) J_{-\zeta} \left( z_{HB}^2 / 4 \right)} \left[ J_{-\zeta} \left( z_{HB}^2 / 4 \right) J_{\zeta} \left( z_{HB}^2 / 4 \right) + J_{-\zeta} \left( z_{HB}^2 / 4 \right) J_{\zeta} \left( z_{HB}^2 / 4 \right) \right]
\]

\[
c_2 = \frac{1}{\sqrt{z_{HB}}} \frac{1}{J_{\zeta} \left( z_{HB}^2 / 4 \right) J_{-\zeta} \left( z_{HB}^2 / 4 \right)} \left[ J_{-\zeta} \left( z_{HB}^2 / 4 \right) J_{\zeta} \left( z_{HB}^2 / 4 \right) + J_{-\zeta} \left( z_{HB}^2 / 4 \right) J_{\zeta} \left( z_{HB}^2 / 4 \right) \right]
\]
and
\[ J_\nu(\nu^{1/2}) \]

\[ C_2 = J_\nu(\nu^{1/2})/2 + J_\nu(\nu^{1/2})/2 \]

The solution \( \psi \) and its derivative \( d\psi/d\xi \) appear in all of the thermohydrodynamic quantities given by equations (10.48) through (10.54). In these quantities, as well as in the coefficients \( c_1 \) and \( c_2 \), the foregoing Bessel function is given by the series expansion,

\[ J_\nu(t) = \frac{1}{\Gamma(1+\nu)} \sum_{k=0}^{\infty} \left( -1 \right)^k (t/2)^k \frac{k!}{k! \Gamma(k+\nu+1)} \]

or, in the equivalent form, which is more convenient for the present computational purpose, including at \( t=0 \):

\[ J_\nu(t) = \left( t/2 \right)^\nu \frac{1}{\Gamma(1+\nu)} \sum_{k=0}^{\infty} \left( -1 \right)^k (t/2)^k \frac{k!}{k! \Gamma(k+\nu+1)} \]

The second derivative reduces to (cf. equation (10.43)):

\[ \frac{d^2 \psi}{d\xi^2} = C_1 \left[ c_1 J_\nu(\nu^{1/2}) - c_2 J_\nu(\nu^{1/2}) \right] \]

In the aforementioned thermohydrodynamic quantities (cf. Section 10.4.2.1) there also appear integrals involving \( \psi \). These integrals can be evaluated by making use of the Abramowitz and Stegun's [10.66] well-known integrals of the Bessel functions, although only in the form of summation,

\[ \int_0^\infty t^n J_\nu(t) \, dt = \frac{\pi}{\Gamma\left(n+\nu+1\right)} \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^k}{k! \Gamma(k+\nu+1)} J_n(\nu^{1/2}), \quad R(n+\nu+1) > 0. \]

The above integral can be reduced to a simple form by making use of a property of the Bessel functions for the two values \( \nu = \pm 1/4 \) of the fractional parameter \( \nu \) in the solution, for \( \nu = 1/4 \),

\[ \int_0^\infty t^{1/2} J_{1/4}(t) \, dt = \frac{\pi}{\Gamma(3/4)} \sum_{k=0}^{\infty} \frac{(2k+3/4)(2k+1/4)}{k! \Gamma(7/4+k)} J_{k+3/4}(t), \quad R(3/4+k) > 0, \]

and for \( \nu = -1/4 \),

\[ \int_0^\infty t^{1/2} J_{-1/4}(t) \, dt = \frac{\pi}{\Gamma(1/4)} \sum_{k=0}^{\infty} \frac{(2k+1/4)(2k+1/4)}{k! \Gamma(3/4+k)} J_{k-3/4}(t), \quad R(1/4+k) > 0. \]
FRICITION AND FILM THICKNESS OF THERMO-ELASTOHYDRODYNAMIC LUBRICANT FILMS

APPENDIX 11.A1 FRICTION IN THERMO-ELASTOHYDRODYNAMIC FILMS

The Hertzian displacement outside the conjunctional zone under the application of a load per unit width, \( W' \), can be shown \( [11.A1] \) to reduce in terms of the non-dimensional load \( W^* \) to (see Figure 11.7):

\[
4W'R'X
\]

\[ n - h = \frac{C}{\pi} \]

where the reduced coordinate \( X \) is given by,

\[
x = |x| (x^2 - 1)^{1/2} \]

\[
\frac{\pi}{|x| + (x^2 - 1)^{1/2}} \]

in which \( X = x/w \) is the abscissa along the film. Then the film profile \( h \), as far as it extends over the inlet zone, can be given by,

\[
4wR'X
\]

\[ h - h_c + \frac{C}{\pi} \]

where \( h_c \) is the central film thickness in the conjunctional zone. Friction force at either of the rubbing surfaces resulting from the shear stress due to viscous effects as given by expression (11.19a), viz.

\[
F = \frac{n}{\alpha} \int_{ou}^{in} \frac{dx}{dx} \]

where the limits of integration, \( ou \) and \( in \), represent exit and entry to the inlet respectively. In order to reduce this to a non-dimensional form \( p=P^*E' \) and \( h=HR' \) may be substituted to obtain:

\[
F = \frac{8wR'}{\alpha} \int_{ou}^{in} \frac{dx}{dx} \]

Following the method of Goksem and Hargreaves \( [11.A2] \), integration by parts yields:

\[
F = \frac{8wR'}{\alpha} \left( \frac{H}{ou} - \int_{ou}^{in} P^* \frac{dh}{dx} \right) \]

Since the pressure is zero at both the entry to the inlet and at the outlet of the film, the first term on the right-hand side reduces to zero. Then the expression for friction reduces to,

\[
F = \frac{n}{\alpha} \int_{ou}^{in} \frac{dx}{dx} \]

where the derivative \( dh/dx \) can be evaluated through the aforementioned values for \( H \) and \( X \), then

\[
\frac{dh}{dx} = \frac{8wR'}{\alpha} (X^2 - 1) \]

and \( P^* \) will be worked out as follows. Introducing the non-dimensional term \( B \) in the reduced pressure from expression (11.8), i.e.

\[
B = e^{-(pE^*)} \]

it can be shown that

\[
P^* = -\frac{1}{\alpha} \]

Substituting for \( P^* \) and \( dh/dx \) in the expression for friction along each of the rubbing surfaces, one obtains:

\[
F = \frac{8wR'}{\alpha} \int_{ou}^{in} \frac{dx}{dx} \]

For a symmetric Hertzian distribution of pressure in the conjunctional zone, however, the friction would be zero. Then, in the above expression the lower limit of integration \( ou \) can be replaced by \( X=1 \), i.e. the exit of the inlet zone which is the same as the entry to the conjunctional zone. Then

\[
F = \frac{8wR'}{\alpha} \int_{1}^{in} \frac{dx}{dx} \]

APPENDIX 11.A2 HEAT PARTITION RATIOS FOR THE TWO RUBBING SURFACES

Heat partition ratios \( r_1 \) and \( r_2 \) have previously been used (see Chapters 3 and 5) as the proportions of the total heat partitioned between two surfaces enclosing a fluid in flow which produces that heat. This ratio for surface 1 can be related to that for surface 2 since the total heat flux \( \phi_{tot} \) generated in any particular cross-section concerned is transferred entirely by cross-conduction to the two rubbing surfaces concerned, viz.

\[
r_1 \phi_{tot} + r_2 \phi_{tot} = \phi_{tot} \]

whilst,

\[
r_1 + r_2 = 1 \]

Distributions of the heat partition ratios along elastohydrodynamic films can be visualized as follows. Consider two rubbing members in pure rolling, materials of which have two different thermal diffusivities, viz.

\[
\chi_1 = \left( \frac{1}{\rho c_1} \right) \quad \chi_2 = \left( \frac{1}{\rho c_2} \right) \]

[Appendix 11.A1]

[Appendix 11.A2]
where \( k \) is the thermal conductivity, \( \rho \) is the density and \( c \) is specific heat at constant pressure of the material of either of the rubbing bodies concerned. For simplicity consider that the film thickness is uniform throughout the conjunct ional zone. Further, assume one and the same temperature for the two surfaces concerned at the inlet to the conjunct ional zone. It may then be conceived that in the successive cross-sections up-stream from the entry to the conjunct ional zone, each of the rubbing elements receives a certain amount of heat depending on its thermal diffusivity and the surface speed. Thus both heat partition ratios \( r_1 \) for surface 1 and \( r_2 \) for surface 2 can be conceived uniform throughout the conjunct ional zone of the two rubbing members when subjected to pure rolling. A similar trend of the heat partition ratio can be shown to exist for the inlet zone as well, and even when some sliding between the two rubbing surfaces is included.

**APPENDIX 11.A3 SURFACE TEMPERATURES IN TERMS OF HEAT PARTITION RATIOS**

A relationship between the aforementioned heat partition ratios and surface temperatures of each of the rubbing surfaces can be established using the flash temperature theory based on the "rod-model". For high speeds of rolling of the rubbing surfaces (Peclet numbers of the surfaces involved then being high), distributions of surface temperatures along the film in a thermally steady state can be given by,

\[
T_{1,2}^\infty (\xi) = T_{b,1,2}^\infty + \frac{\sqrt{\frac{2}{b_1,2}}}{\sqrt{\frac{2}{b_1,2}} b_1,2} \int_0^\xi \sqrt{\frac{2}{b_1,2}} \, d\xi'
\]

or alternatively, by its inversion:

\[
\frac{d^2 T_{1,2}^\infty}{d\xi'^2} = \frac{\sqrt{\frac{2}{b_1,2}}}{\sqrt{\frac{2}{b_1,2}} b_1,2} \int_0^\xi \sqrt{\frac{2}{b_1,2}} \, d\xi'
\]

where the positive sign on the left-hand term refers to surface 1 and the negative sign to surface 2. \( \xi \) is any local position along the flow, \( b \) is the thermal contact coefficient of the solid rubbing material concerned, i.e. \( b = \frac{V_h c_p k}{\rho c} \), and subscripts 1 and 2 refer to surfaces no. 1 and 2, respectively. Since the partitioning of the total heat flux \( \Phi_{\text{tot}} \) has already been shown to be uniform in every successive film cross-section throughout the conjunct ional zone, and in fact even in the inlet zone, the distribution of temperatures along each rubbing surface can then be given by,

\[
T_1^\infty (\xi) = T_{b,1}^\infty + \frac{r_1}{b_1,2} \int_0^\xi \sqrt{\frac{2}{b_1,2}} \, d\xi'
\]

\[
T_2^\infty (\xi) = T_{b,2}^\infty + \frac{r_2}{b_1,2} \int_0^\xi \sqrt{\frac{2}{b_1,2}} \, d\xi'
\]

It may readily be seen that the distributions of flash temperatures of the two rubbing surfaces given by the last terms on the right-hand side of the above two expressions are related to their heat partition ratios through the velocities and thermal contact coefficients involved.

Thus, in view of \( r_1 = 1 \),

\[
\frac{T_1^\infty (\xi) - T_2^\infty (\xi)}{T_1^\infty (\xi) - T_{b,1}^\infty} = \frac{r_2}{1 - r_1}
\]

The right-hand side of this expression yields a quantity independent of the abscissa \( \xi \). Therefore, it may be concluded that the proportional increase of temperatures at every cross-section along the surfaces is uniform. A few special cases will readily follow from the above relationship. For rubbing surfaces in pure rolling, i.e. \( u_1 = u_2 \), then:

\[
\frac{T_1^\infty (\xi) - T_2^\infty (\xi)}{T_1^\infty (\xi) - T_{b,1}^\infty} = \frac{r_2}{1 - r_1}
\]

For rubbing surfaces of similar materials moving with unequal velocities:

\[
\frac{T_1^\infty (\xi) - T_2^\infty (\xi)}{T_1^\infty (\xi) - T_{b,1}^\infty} = \frac{r_2}{1 - r_1}
\]

Finally, for rubbing surfaces of similar materials in pure rolling:

\[
\frac{T_1^\infty (\xi) - T_2^\infty (\xi)}{T_1^\infty (\xi) - T_{b,1}^\infty} = \frac{r_2}{1 - r_1}
\]

Since in the last particular case \( r_1 = r_2 = 0.5 \), it follows that \( T_1^\infty (\xi) = T_2^\infty (\xi) \) throughout; this particular case is also representative of the equipartitioning of heat, i.e. \( r_1 = r_2 = 0.5 \).

**APPENDIX 11.A4 HEAT PARTITION RATIOS OVER A WIDE OPERATING REGIME**

Surface temperatures along the film generally vary depending upon the properties of the rubbing materials, the velocity of the rubbing surfaces and the heat partition ratios involved as shown in Appendix 11.A3. Further, the heat partition ratios have been shown in Appendix 11.A2 to remain uniform throughout the length of the film, which in any film cross-section are related to the temperature gradient at the surface (see expressions (9.10a) and (9.10b) of Chapter 9), viz.

\[
\frac{\Delta T}{\Delta \xi} = \frac{r_1}{1 - r_1}
\]

The temperature gradient in the above-expression can be re-expressed also in terms of previously established parameters, namely the relative fluidity for each surface, \( \frac{\mu_1}{\mu} \) and the non-dimensional quantity \( w \), which for the exponential viscosity-temperature relationship have already been given in Chapters 9 and 10, viz.

\[
\frac{\Delta T}{\Delta \xi} = \frac{r_1}{1 - r_1}
\]

and

\[
W = \frac{r_1}{r_2} \left( \frac{\mu_1}{\mu} \right)
\]
A relationship between the above two quantities has already been depicted in Figure 10.4 (see Chapter 10). Then the gradient of the curve depicting \( \frac{\mu H}{\mu} \) with respect to \( w \) can be related to the aforementioned temperature gradient, viz.

\[
\frac{d(\mu H/\mu)}{dw} = \frac{d}{d\theta} \left( \frac{T - T_0}{\theta} \right) = \frac{\beta h}{\rho l} \frac{d}{dy} \left( \frac{\mu H}{\mu} \right)
\]

which can be further reduced for either of the surfaces to,

\[
\frac{d(\mu H/\mu)}{dw} \bigg|_1 = - \frac{d(\mu H/\mu)}{dw} \bigg|_2
\]

The particular case of equipartitioning of heat can be obtained as follows. Since the slope of the curve given by the left-hand side is uniform over a wide range of \( w \) and a corresponding wide range of \( (\mu H/\mu)_{1,2} \), e.g. \( 1 > (\mu H/\mu)_{1,2} > 0.1 \), and since the two branches of the curve on either side of the mid-section \( w=0 \) are symmetrical, it may be inferred that the corresponding term \( k(dT/dy) \) is also symmetrical for any given set of values for \( (\mu H/\mu)_{1,2} \), \( P \), \( \beta \), \( k \) and \( h \). This symmetry can then be exemplified by,

\[
\frac{d(\mu H/\mu)}{dw} \bigg|_1 \bigg|_1 = \left( \frac{\beta h}{\rho l} \right) \frac{1}{k} \left( \frac{k}{dy} \right)
\]

and correspondingly,

\[
k \frac{dT}{dy} \bigg|_1 = - k \frac{dT}{dy} \bigg|_2
\]

The latter symmetry readily yields on the basis of the definitions of \( r_1 \) and \( r_2 \):

\[ v_1 = v_2 \]

Since the heat transfer has been assumed to take place entirely by cross-conduction, the last equality yields,

\[ r_1 = r_2 = 0.5 \]

On the basis of the above analysis, it may then be concluded that the heat partition ratio for either surface remains uniformly 0.5 over a wide range of relative fluidity, i.e. \( 1 > (\mu H/\mu) > 0.1 \) and thus over a wide range of temperatures. This range of relative fluidity also corresponds to an equally wide range of operating conditions such as loads and rolling speeds.

**APPENDIX 11.A5 REFERENCES**


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**STELLINGEN behorende bij het proefschrift van H.B. Dakshina Murthy, M.Sc.**

1. The "thermal Reynolds equation" is a convenient analytical expedient for inlet analyses of thermo-elastohydrodynamic lubricant films. By ignoring the "flash temperature boundary conditions" its potentials have been only insufficiently explored.

2. Thermo-hydrodynamic instabilities of the flow in full fluid films in lubricated rubbing elements have in literature so far received much less attention than the thermo-mechanical instabilities that have long been recognized for solid surface layers of those elements themselves. The crucial nature of the former instabilities as to collapse of the lubricant films deserves further investigation.

3. Machine elements, if they were to be constructed on the basis of tribologically updated design methods rather than the conventional ones, may be expected to consume less energy in actual operation. Yet the scope of the former methods is deemed to remain insufficiently complete.

4. It would appear that hitherto too few investigators have been sufficiently engineering minded or experimentation minded to discern the existence of rates of flow that, paradoxically enough, increase upon decreasing the pressure drop along full fluid films.

5. The difficulty of assessing the partitioning of the viscous-shear heat among the two rubbing surfaces, between which this heat is generated in a full fluid film interposed, has so far been overestimated grossly.

6. As long as local temperatures in flows in lubricant films cannot be measured, experimental results of techniques such as laser-Doppler measurements of velocities of flow will continue to mislead researchers.

7. Archimedes is reputed to have maintained: "Give me a lever and a place to stand on, and I will move the world". A tribologist nowadays may boast: "Give me a few drops of oil and a hole to push them through, and I will move the world".

8. Cases of falsification and plagiarism in research are considered to be sins, yet remain unabated; the more cardinal cases among professors and their students are not even brought to light.

9. Seizing of power in universities - by the lineal representatives of the ancient monarchs- from the elected hierarchs of modern democracy, is another turning-point in the strife for slaying the "eternal-infant" education.

10. In Indian society the caste system is a permanent source of disturbance on which politically conveniently to project the socio-economical differences; it has there replaced other sources which belong to dynamically bent societies that are also constantly looking for scapegoats.

11. Meditation, Yoga, Devotion and Worship, which are the outcome of one's own thought and education, or say of what is called one's own culture, have been abused and made into a mere virtue of necessity.

12. 'Appropriate Technology' for developing countries is a misnomer to 'Material Saving Technology', and is a blessing in disguise to these countries.

13. To decide entering institutional marriage often is more short-sighted than other major decisions of life.