Metric Properties of Denjoy’s Canonical
Continued Fraction Expansion

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Abstract. Metric properties of Denjoy’s canonical continued fraction expansion are studied, and the natural extension of the underlying ergodic system is given. This natural extension is used to give simple proofs of results on mediant convergents obtained by W. Bosma in 1990.

1. Introduction

It is quite well-known—see e.g. [9]—that every real number \( x \) can be written as a regular continued fraction (RCF) expansion

\[
x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = [a_0; a_1, a_2, \ldots],
\]

where \( a_0 = \lfloor x \rfloor \), and the \( a_i, \ i \in \mathbb{N}_+ := \{1, 2, \ldots\} \), are positive integers. In case \( x \) is irrational, the expansion (1) is unique and infinite; in case \( x \) is rational, (1) is finite and two possible expansions exist.

Apart from the RCF expansion (1), very many other continued fraction expansions of \( x \) exist. One such expansion, Denjoy’s canonical continued fraction expansion ([5]), has hardly attracted any attention. Let \( x \) be a real number with RCF expansion (1), and let \( d_0 \in \mathbb{Z} \) be such, that \( d_0 \leq x \). Then it was shown in [10] that \( x \) has an expansion of the form

\[
x = [d_0; (1,0)^{d_0-d_1}, (1,0)^{a_1-1}, 1, (1,0)^{a_2-1}, 1, \ldots],
\]

where \( (1,0)^k \) is an abbreviation for the string \( 1,0,1,0,\ldots,1,0 \) consisting of \( k \) pairs \( (0,1) \), which is empty if \( k = 0 \). Such a continued fraction expansion is called a canonical continued fraction (CCF) expansion of \( x \).
In [10], a map \( T_d : [0, \infty) \to [0, \infty) \) is studied, which `generates' a unique CCF expansion for every \( x > 0 \). This `Denjoy-map' is given by \( T_d(0) = 0 \), and

\[
T_d(x) = \frac{1}{x} - d(x), \quad x > 0,
\]

where

\[
d(x) = \begin{cases} 
1, & \text{if } x \in (0, 1], \\
0, & \text{if } x \in (1, \infty).
\end{cases}
\]

The digits (or partial quotients) \( d_n = d_n(x) \) of \( x > 0 \) are now given for \( n \in \mathbb{N}_+ \) by

\[
d_n(x) = d(T_d^{n-1}(x)) \quad \text{whenever } T_d^{n-1}(x) > 0.
\]

Using \( T_d \), we find (see equation (14) further on) that \( x > 0 \) given by (1) has as CCF expansion

\[
x = [0; d_1(x), d_2(x), \ldots].
\]

In [10] it has been shown that \(([0, \infty), B, \mu, T_d)\) is an ergodic system, where \( \mu \) is a \( \sigma \)-finite, infinite \( T_d \)-invariant measure with density

\[
f(x) = \frac{1}{x} 1_{(0,1]}(x) + \frac{1}{1+x} 1_{(1,\infty)}(x), \quad x \in \mathbb{R}_+:=(0, \infty).
\]

Furthermore, it has been shown in [10] that the CCF-convergents \( p_n/q_n \) of \( x \), which are obtained by taking finite truncations in (2), consist of the RCF-convergents \( P_n/Q_n \) of \( x \) and the RCF-mediants of \( x \). The latter are defined as

\[
\frac{aP_n - 1 + P_{n-1}}{aQ_n - 1 + Q_{n-1}} \quad \text{for all integers } a \text{ satisfying } 1 \leq a \leq a_n - 1.
\]

One `shortcoming' of the CCF algorithm is that every RCF-convergent \( P_n/Q_n \) of \( x \) appears \( a_n+1 \) times as a CCF-convergent of it. There are several algorithms yielding the RCF-convergents and mediants, see for instance [6] or [11], where such algorithms together with the underlying ergodic systems are described. In [11], the RCF-convergents and mediants of any \( x \in [0, 1) \) are `generated' in the same order, without the duplication of the RCF-convergents as in the case of the CCF expansion of \( x \). The underlying transformation \( S : [0, 1] \to [0, 1] \) in [11] is defined as

\[
S(x) = \begin{cases} 
x, & x \in \left[0, \frac{1}{2}\right), \\
\frac{1-x}{x}, & x \in \left[\frac{1}{2}, 1\right],
\end{cases}
\]

and there is a \( \sigma \)-finite, infinite \( S \)-invariant measure \( \nu \) with density \( 1/x, x \in \mathbb{R}_+ \). Moreover, Ito showed in [11] that the dynamical system \(([0, 1], B, S, \nu)\) is ergodic.
It is easy to check that

\[ S(x) = \begin{cases} 
T_d^2(x), & x \in \left[0, \frac{1}{2}\right), \\
T_d(x), & x \in \left[\frac{1}{2}, 1\right],
\end{cases} \]

i.e., \( S \) can be seen as a \textit{jump transformation} of \( T_d \). Hence, the ergodic properties of \( S \) can easily be carried over to \( T_d \). Note that \( T_d^2 \) is used to avoid duplication of RCF-convergents. Since, as noticed in [10],

\[ T(x) = T_d^{2(k-1)+1}(x), \quad x \in \left[\frac{1}{k+1}, \frac{1}{k}\right), \quad k \in \mathbb{N}_+, \]

where \( T \) is the usual Gauss map underlying the RCF expansion, i.e., \( T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor \), \( x \in (0, 1) \), ergodic properties of \( T_d \) can also be obtained from the ergodic properties of the RCF expansion.

In this paper we first discuss some metric properties of the CCF expansion. Then we determine the natural extension of \((\mathbb{N}, \mathbb{B}, \mu, T_d)\), and use it to study the metric properties of the so-called \textit{approximation coefficients} \( \theta_n = \theta_n(x) \) of \( x \), defined by

\[ \theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right|, \quad n \in \mathbb{N} := \mathbb{N}_+ \cup \{0\}, \]

where \( p_n/q_n, n \in \mathbb{N} \), are the CCF-convergents of \( x \). In particular, the metric properties of the RCF-mediants are also studied. We show that the results from Bosma [2] can be obtained in a direct and elegant way.

### 2. Metric properties

#### 2.1. The Perron-Frobenius operator

Let \( L^1(\mu) \) and \( L^\infty(\mu) \) denote the usual \( L^1 \)- and \( L^\infty \)- Banach spaces on \((\mathbb{R}_+, \mathbb{B}_+, \mu)\) with \( \mathbb{B}_+ = \text{Borel } \sigma \)-algebra in \( \mathbb{R}_+ \).

According to the general theory (cf., e.g., [1], p. 33), the Perron-Frobenius operator \( U \) of \( T_d \) under \( \mu \) takes \( L^1(\mu) \) into itself and satisfies the equation

\[ \int_{\mathbb{R}_+} (Uh)g \, d\mu = \int_{\mathbb{R}_+} (g \circ T_d)h \, d\mu \]

for any \( h \in L^1(\mu) \) and \( g \in L^\infty(\mu) \). Clearly, (3) implies that

\[ \int_{\mathbb{R}_+} (U^n h)g \, d\mu = \int_{\mathbb{R}_+} (g \circ T_d^n)h \, d\mu \]
for any \( n \in \mathbb{N} \). Let \( \nu_h(A) = \int_A h \, d\mu \) for \( h \in L^1(\mu) \) and \( A \in \mathcal{B}_+ \). Then \( Uh \) can be expressed as a Radon-Nikodým derivative, namely, \( d(\nu_h \circ T_d^{-1})/d\mu \). It is easy to check that we thus have

\[
Uh(s) = \left[ \frac{1_{[0,1]}(s)}{s+1} h \left( \frac{1}{s} \right) + \left( 1_{(1,\infty)}(s) + \frac{s1_{[0,1]}(s)}{s+1} \right) h \left( \frac{1}{s+1} \right) \right]
\]

for \( \mu \)-almost all \( s \in \mathbb{R}_+ \) and any \( h \in L^1(\mu) \). Clearly, \( U1 = 1 \), even if the constant functions do not belong to \( L^1(\mu) \). Actually, \( Uh \) thus defined makes sense for any function \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) but, of course, without satisfying (3) for such an \( h \).

The Gauss problem for the CCF transformation \( T_d \) can be approached in terms of the Perron-Frobenius operator \( U \) (cf. [9], Ch. 2, for the case of the RCF expansion.) This problem amounts to the asymptotic behavior of \( m(T_d^{-n}(A)) \) as \( n \to \infty \) for probability measures \( m \ll \lambda \) (Lebesgue measure on \( \mathcal{B}_+ \)) and \( A \in \mathcal{B}_+ \). It is immediate from (4) by taking \( g = 1_A \) that

\[
\int_A U^n h \, d\mu = \int_{T_d^{-n}(A)} h \, d\mu.
\]

Hence

\[
m(T_d^{-n}(A)) = \int_A U^n \left( \frac{p}{f} \right) \, d\mu
\]

for any \( n \in \mathbb{N} \) and \( A \in \mathcal{B}_+ \), where \( p = dm/d\lambda \). Therefore, as in the case of the RCF expansion, the asymptotic behavior of \( m(T_d^{-n}(A)) \) depends on the asymptotic behavior of \( U^n \) as \( n \to \infty \). However, in our case here things are much simpler due to the properties of the \( T_d \)-invariant measure \( \mu \). Indeed, by Proposition 1.1.3 in [1], the transformation \( T_d \) is conservative since incompressible, that is, \( A \in \mathcal{B}_+ \) and \( T_d^{-1}(A) \subset A \) imply \( A = T_d^{-1}(A) \mod \lambda \). [It is easy to check that \( T_d^{-1}(A) \subset A \) only holds for either \( A = \emptyset \) (the empty set) or \( A = \mathbb{R}_+ \).] Next, since \( T_d \) is conservative and ergodic, by [1], Exercise 2.2.1, p. 61, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} h \circ T_d^k = 0 \quad \mu\text{-a.e.},
\]

hence \( \lambda \)-a.e., for any \( h \in L^1(\mu) \). In particular, for \( h = 1_A \) with \( A \in \mathcal{B}_+ \) and \( \mu(A) < \infty \), we deduce that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A \circ T_d^k = 0 \quad \mu\text{-a.e.}
\]

By dominated convergence, this clearly implies

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} m(T_d^{-k}(A)) = \int_{\mathbb{R}_+} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A \circ T_d^k \, p \, d\lambda = 0
\]

for any \( A \in \mathcal{B}_+ \) such that \( \mu(A) < \infty \).
This shows that if the limit
\[ \lim_{n \to \infty} m \left( T_d^{-n}(A) \right) \]
exists, then it should be 0; and if so then, from (5),
\[ 0 = \lim_{n \to \infty} \int_A U^n \left( \frac{p}{f} \right) \, d\mu \geq \int_A \liminf_{n \to \infty} U^n \left( \frac{p}{f} \right) \, d\mu, \]
hence
\[ \liminf_{n \to \infty} U^n \left( \frac{p}{f} \right) = 0 \quad \mu\text{-a.e.} \]
on any \( A \in \mathcal{B}_+ \) such that \( \mu(A) < \infty \). This actually means that
\[ \liminf_{n \to \infty} U^n \left( \frac{p}{f} \right) = 0 \quad \lambda\text{-a.e. in } \mathbb{R}_+. \]
In any case one can assert that we always have
\[ \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k \left( \frac{p}{f} \right) = 0 \quad \lambda\text{-a.e. in } \mathbb{R}_+. \]
Recall that \( p \) above is any non-negative element of \( L^1(\lambda) \).

Coming back to equation (7), let \( A \in \mathcal{B}_+ \) be such that \( \mu(A) < \infty \). Let \( A^c = \mathbb{R}_+ \setminus A \).
Since \( 1_A + 1_{A^c} = 1 \), we have
\[ \frac{1}{n} \sum_{k=0}^{n-1} 1_A \circ T^k_d + \frac{1}{n} \sum_{k=0}^{n-1} 1_{A^c} \circ T^k_d = 1. \]
It follows from (7) that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{A^c} \circ T^k_d = 1 \quad \mu\text{-a.e.} \]
This means that \( \mu \)-almost all orbits \( (x, T_d(x), T^2_d(x), \ldots) \), \( x \in \mathbb{R}_+ \), hit \( A^c \) with relative frequency asymptotical to 1. Also, similarly to (8) we have
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} m \left( T^{-k}_d(A^c) \right) = 1. \]
Hence if the limit
\[ \lim_{n \to \infty} m \left( T_d^{-n}(A) \right) \]
extists, then it should be 1.
We conjecture that both limits (9) and (13) do not exist, so that (8) and (12) appear to be the solution of Gauss’ problem for \( T_d \).

**Remark.** The last conclusion holds, too, mutatis mutandis, for many other transformations with \( \sigma^- \)-finite, infinite invariant measure as, e.g., Ito’s transformation \( S \).

### 2.2. A Markov chain and an infinite order chain associated with the CCF expansion

We shall consider in this subsection the CCF expansions of irrationals in the unit interval. We shall consider in this subsection the CCF expansions of irrationals in the unit interval. For any \( n \in \mathbb{N}_+ \), we have

\[
\frac{1}{d_1 + T_d(x)} = \frac{1}{d_2 + \frac{1}{d_3 + \frac{1}{\ddots + 1 + T_d(x)}}}
\]

(14)

for any \( n \in \mathbb{N}_+ \) and any \( x \in \Omega \), where \( \Omega \) is the set of irrationals in \([0, 1] \). Clearly, \( d_1 = 1 \).

(When confusion cannot arise, we suppress the argument \( x \) in the notation.) Next, for the CCF-convergents \( p_n/q_n, n \in \mathbb{N}_+ \), the equations

\[
p_n = d_n p_{n-1} + p_{n-2}, \quad q_n = d_n q_{n-1} + q_{n-2},
\]

(15)
do hold for any \( n \in \mathbb{N}_+ \) with \( p_{-1} = 1, q_{-1} = 0 \), and \( q_0 = 1 \). Hence all \( q_n, n \in \mathbb{N} \), are positive for any \( x \in \Omega \). It follows immediately from (14) and (15) that

\[
x = \frac{(d_n + T_d^n(x)) p_{n-1} + p_{n-2}}{(d_n + T_d^n(x)) q_{n-1} + q_{n-2}} = \frac{p_n + p_{n-1} T_d^n(x)}{q_n + q_{n-1} T_d^n(x)}
\]

(16)

for all \( n \in \mathbb{N}_+ \) and \( x \in \Omega \). Also, it is easy to check that

\[
p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}, \quad n \in \mathbb{N}.
\]

For any \( n \in \mathbb{N}_+ \) and any \( i_k \in \{0, 1\}, 1 \leq k \leq n \), consider the admissible \( n \)-tuples \( i^{(n)} := (i_1, \ldots, i_n) \) with \( i_1 = 1 \), and where no two consecutive 0’s appear in the finite sequence \( i_1, \ldots, i_n \). Let \( I(i^{(n)}) \) denote the set of \( x \in \Omega \) for which \( d_k(x) = i_k, 1 \leq k \leq n \). Such a set is called a cylinder set of rank \( n \).

It is perhaps interesting to note that the number of admissible \( i^{(n)} \) ending in 1 is \( F_{n-1}, n \in \mathbb{N}_+ \), while that of admissible \( i^{(n)} \) ending in 0 is \( F_{n-2}, n \geq 2 \). Here the \( F_n \) are the Fibonacci numbers defined recursively by \( F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \in \mathbb{N}_+ \).

**Proposition 1.** For any \( n \in \mathbb{N}_+ \) a cylinder set \( I(i_1, \ldots, i_n) \) is either the set of irrationals in the interval with end points \( p_n/q_n \) and \((p_n + p_{n-1})/(q_n + q_{n-1})\) when \( i_n = 0 \) or the set of irrationals in the interval with end points \( p_{n-1}/q_{n-1} \) and \( p_n/q_n \) when \( i_n = 1 \). [Here, the \( p_n \) and \( q_n \), \( n \in \mathbb{N} \), are computed using the rules (15) with \( p_{-1} = 1, p_0 = 0, q_{-1} = 0, q_0 = 1, d_k = i_k, 1 \leq k \leq n \).]
We nevertheless give the details as there are some traps.

(a) If \( i_n = 0 \) then \( i_{n+1} = 1 \), so for \( x \in I(i_1, \ldots, i_n) \) we have
\[
T^n_d(x) = \frac{1}{1 + \ldots} < 1.
\]

Hence \( T^n_d(x) \) varies through irrational values between 0 and 1, and by (16) we are done.

(b) If \( i_n = 1 \) then either \( i_{n+1} = 1 \) or \( i_{n+1} = 0 \). If \( i_{n+1} = 1 \) then, as before, we get the irrationals in the interval with end points \( p_n/q_n \) and \((p_n + p_{n-1})/(q_n + q_{n-1})\) while if \( i_{n+1} = 0 \) we have
\[
T^n_d(x) = \frac{1}{0 + \ldots} > 1.
\]

That is, \( T^n_d(x) \) varies through irrational values between 1 and \( \infty \), and by (16) we get the interval with end points \((p_n + p_{n-1})/(q_n + q_{n-1})\) and \((p_n/q_n)\). It follows from the second equation in (15) that
\[
s_n = \frac{1}{d_n + s_{n-1}}, \quad n \in \mathbb{N}_+.
\]

Note that in our sequence \((s_n)_{n \in \mathbb{N}}\) the values 0 and 1 appear no longer if \( n \geq 3 \). Indeed, \( s_2 \) takes on the values 1 and 1/2 both with probability 1/2 while the possible values of \( s_3 \) are 1/2, 2/3, and 2 with probabilities 1/2, 1/6, and 1/3, respectively. Then equation (18) shows that \( s_n \neq 0 \) and 1 for any \( n \geq 3 \). Hence the random events \( \{s_n > 1\} \) and \( \{d_n = 0\} \) are equivalent for \( n \geq 3 \).

Abusing notation, let \( \lambda \) denote the Lebesgue measure on \( I \), too.

**Proposition 2.** For any \( n \in \mathbb{N}_+ \) we have
\[
\lambda(d_{n+1} = 1 \mid d_1, \ldots, d_n) = 1_{[0]}(d_n) + 1_{[1]}(d_n) \frac{s_n}{s_n + 1}
\]
and
\[
\lambda(d_{n+1} = 0 \mid d_1, \ldots, d_n) = 1 - \lambda(d_{n+1} = 1 \mid d_1, \ldots, d_n) = 1_{[1]}(d_n) \frac{1}{s_n + 1}.
\]

**Proof.** The equations are simple consequences of Proposition 1 and equations (17). For example, if \( d_n = 1 \) and \( n \geq 2 \), then
\[
\lambda(d_{n+1} = 1 \mid d_1, \ldots, d_n) = \frac{\lambda(I(d_1, \ldots, d_{n-1}, 1))}{\lambda(I(d_1, \ldots, d_{n-1}, 1))}
\]
\[
= \left| \frac{p_n - p_{n+1}}{q_n} \right| \left| \frac{p_{n-1} - p_n}{q_{n-1}} \right| = \left| \frac{q_n - 1}{q_{n+1}} \right|.
\]
\[
\frac{qn - 1}{qn + qn - 1} = \frac{sn}{sn + 1}.
\]

We also state

**Proposition 3.** For any \( u \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \) we have

\[
\lambda(T_n d_1, \ldots, d_n) = \frac{1 - u}{1 + s_n u} \left( \frac{1}{0,1}(d_n) + \frac{1_{\{1\}}(d_n)}{s_n + 1} \right) 1_{[0,1]}(u) + \frac{1_{\{1\}}(d_n)}{1 + s_n u} 1_{(1,\infty)}(u).
\]

**Proof.** The equation, that generalizes the result in Proposition 2, is again a consequence of Proposition 1 and equations (16) and (17), so that we skip the details. \( \Box \)

**Remark.** Similar considerations can be made as for irrationals in \((1,\infty)\). Note that the invertible transformation \( \tau : x \mapsto 1/x, x \in (0,1) \), takes \( \Omega \) into the set \( \Omega' \) of irrationals in \((1,\infty)\) while Lebesgue measure on \((0,1)\) is taken to the probability measure \( \lambda\tau^{-1} \) on \((1,\infty)\) with density \( 1/y^2, y \in (1,\infty) \). If \( x' \in \Omega' \) then we have \( d_1(x') = 0 \), so that \( d_2(x') = 1 \), and \( d_n(x') = d_{n-1}(1/x') \) for any \( n \geq 2 \).

Proposition 2 shows that the sequence \((d_n)_{n \in \mathbb{N}^+}\) is a \([0,1]\)-valued process which can be called an infinite order chain, cf. [8], Section 5.5. We will return to it a little bit later.

As already noticed, we have \( s_n \neq 1 \) for \( n \geq 3 \). Then, by (18) and Proposition 2, the sequence \((s_n)_{n \geq 3}\) is a \((\mathbb{Q}^+ \setminus \{0,1\})\)-valued Markov chain on \((\Omega', \mathcal{B}_{[0,1]}, \lambda)\) with the following transition mechanism. From state \( s \in \mathbb{Q}^+ \setminus \{0,1\} \) the only possible transitions are to states \( 1/s \) and \( 1/(1+1/s) \) with probabilities

\[
\frac{1_{(0,1]}(s)}{s+1} \quad \text{and} \quad 1_{(1,\infty]}(s) + \frac{s 1_{(0,1]}(s)}{s+1}.
\]

More generally, we shall also consider an \( \mathbb{R}_+ \)-valued Markov process with the same transition mechanism and states \( s \in \mathbb{R}_+ \). Obviously, the transition operator \( U \) of such a process can be expressed as

\[
U h(s) = \frac{1_{(0,1]}(s)}{s+1} h \left( \frac{1}{s} \right) + \left( 1_{(1,\infty]}(s) + \frac{s 1_{(0,1]}(s)}{s+1} \right) h \left( \frac{1}{s+1} \right)
\]

for any function \( h \in L^\infty(\lambda) \) (or, equivalently, \( h \in L^\infty(\mu) \) since \( \lambda \equiv \mu \)). It is not a coincidence that the transition operator and the Perron-Frobenius operator of \( T_d \) under \( \mu \) have the same analytical expression (and both of them were denoted by the same letter). For a full explanation the reader is referred to [7], pp. 1–5, and Example (c), p. 6.

Concerning \( U \) as a transition operator of a Markov process we note the following.

(a) \( U \) is actually a typical operator dealt with in dependence-with-complete-connection theory. Cf. [8], especially Chapters 1 and 5. No known results can be used in our case to derive convergence of \( U^n f \) as \( n \to \infty \) for certain functions \( f \in L^\infty(\lambda) \).
(b) The existence of a stationary measure different from \( \mu \) for our Markov process is an open question. Such a measure would be a \( T_d \)-invariant one, leaving just the possibility of a \( \lambda \)-singular measure. Cf. [1], p. 45, on unicity of a \( \lambda \)-absolutely continuous invariant measure. We are unable to confirm or reject this possibility.

Coming back to the random sequence \( (d_n)_{n \in \mathbb{N}_+} \), we remark that by Proposition 2 the probability \( \lambda(d_{n+1} = 1) \) that \( d_{n+1} \) takes on value 1 is equal to

\[
\frac{1}{2} U^{n-3} h\left(\frac{1}{2}\right) + \frac{1}{6} U^{n-3} h\left(\frac{2}{3}\right) + \frac{1}{3} U^{n-3} h(2), \quad n \geq 3,
\]

where

\[
h(s) = 1_{(1, \infty)}(s) + \frac{s 1_{[0,1]}(s)}{s + 1}, \quad s \in \mathbb{R}_+.
\]

Even if one cannot precise the asymptotic behavior of \( U^n h \) as \( n \to \infty \) (see (a) above), we can get instead some information about the asymptotic relative frequencies of 0 or 1 on the trajectories of \( (d_n)_{n \in \mathbb{N}_+} \), not only for irrationals in \((0, 1)\) but even for irrationals in \( \mathbb{R}_+ \).

Let \( 0 < \varepsilon < 1 \) and \( M > 1 \) be arbitrarily fixed. It follows from equation (11) – take \( A = (\varepsilon, M) \) – that the sum of the relative frequencies of 0 and 1 when the values of \( T_d^k \), \( 0 \leq k \leq n - 1 \), are either less than \( \varepsilon \) or exceed \( M \) converges to 1 an \( n \to \infty \) while the values between \( \varepsilon \) and \( M \) bring no asymptotic contribution. Cf. equation (7). It thus appears that the most common situation is that where the values \( T_d^k(x), x \in \mathbb{R}_+, k \in \mathbb{N} \), are clustering near 0 or \( \infty \). We conjecture that there are no limiting relative frequencies of 0 or 1. See, however, [10], p. 237.

3. A natural extension, and its consequences

3.1. A natural extension. In the last two decades the concept of natural extension from ergodic theory has been very fruitful in the study of both arithmetic and metric properties of continued fractions; see, e.g., [14], [13], [3], [12], and [9]. A natural extension is a bijective system which “contains” the original system as a factor, and which is in some sense the “smallest’ system with this property. One cannot speak of the natural extension; usually, various natural extensions of a certain system exist, and these natural extensions are metrically isomorphic. In the natural extension for \(([0, \infty), \mathcal{B}, \mu, T_d)\), we want the second coordinate to act as “the past.” In view of this, we define a map \( T_d : L \to L \) (where \( L \) still needs to be defined) by

\[
T_d(x, y) = \left( T_d(x), \frac{1}{d(x) + y} \right).
\]

Setting

\[
L = ([0, 1) \times [0, \infty)) \cup ([1, \infty) \times [0, 1])
\]
one easily checks that a.e. $T_d : L \rightarrow L$ is a bijective map, see also Figure 1.

Now, let $\bar{\mu}$ be the $\sigma$-finite, infinite measure on $L$ with density
$$\frac{1}{(1 + xy)^2}, \quad (x, y) \in L.$$  
An easy Jacobian transformation shows that $\bar{\mu}$ is $T_d$-invariant, i.e.,
$$\bar{\mu}(T_d^{-1}(A)) = \bar{\mu}(A), \quad A \in B_L.$$

In [10] it was already noticed that if $T$ is the Gauss-map, i.e., $T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor, \ x \in (0, 1)$, then
$$T(x) = T_d^{2(k-1)+1}(x), \quad x \in \left[ \frac{1}{k+1}, \frac{1}{k} \right].$$
Nakada showed in [13] that the natural extension map $T$ of the Gauss-map $T$ is given by
$$T(x, y) = \left( T(x), \frac{1}{a(x)+y} \right), \quad (x, y) \in [0, 1) \times [0, 1],$$
where $a(x) = \lfloor \frac{1}{x} \rfloor$ for $x \in (0, 1)$. He also showed that $(\{0, 1\} \times [0, 1], B, \bar{\gamma}, T)$ is a $K$-system, hence certainly ergodic. Here $\bar{\gamma}$ is a $T$-invariant probability measure on $[0, 1) \times [0, 1]$ with density
$$\frac{1}{\log 2} \frac{1}{(1 + xy)^2}, \quad (x, y) \in [0, 1) \times [0, 1].$$
Nakada's system is 'sitting inside' the dynamical system $(L, B_L, \bar{\mu}, T_d)$ as an induced system. To see this, for $(x, y) \in [0, 1) \times [0, 1]$ define $r(x, y) = 0$ if $x = 0$, and
$$r(x, y) = \min\{k : k \geq 1, T_d^k(x, y) \in [0, 1) \times [0, 1]\} \in \mathbb{N}_0.$$
If \( x \neq 0 \). Define the induced transformation \( K : [0, 1) \times [0, 1) \to [0, 1) \times [0, 1) \) by
\[
K(x, y) = T^r(x, y), \quad (x, y) \in [0, 1) \times [0, 1).
\]

We have the following result, which is stated here without proof.

**Proposition 4.** For \((x, y) \in (0, 1) \times [0, 1]\) let \( k \in \mathbb{N}_+ \) be such that \( \frac{1}{k+1} \leq x < \frac{1}{k} \). Then
\[
r(x, y) = k \quad \text{and} \quad K(x, y) = T(x, y).
\]

As an immediate consequence we find, see e.g. [15], that \((L, B_L, \bar{\mu}, T_d)\) is an ergodic system.

Note that projecting on the first coordinate yields the ergodic system \(((0, \infty), B_{(0, \infty)}, \mu, T_d)\) considered in [10].

To obtain various results on the distribution of the RCF approximation coefficients \( \varphi_n \), \( n \in \mathbb{N} \), Bosma et al derived from the ergodic system \(((0, 1), B_{(0,1)}, \mu, T)\) the important result below; see [3], or Chapter 4 in [9], for a proof of this result.

**Theorem 1 ([3]).** For almost all \( x \) the two-dimensional sequence
\[
(T_n, V_n) = T^n(x, 0), \quad n \in \mathbb{N},
\]
is distributed over \([0, 1) \times [0, 1]\) according to the density function
\[
\frac{1}{\log 2} \frac{1}{(1 + xy)^2}, \quad (x, y) \in [0, 1) \times [0, 1).
\]

A similar result also holds in the present situation.

**Theorem 2.** For almost all \( x \) the two-dimensional sequence
\[
(t_k, v_k) = T^k_d(x, 0), \quad k \in \mathbb{N},
\]
is distributed over \( L \) according to the density function
\[
\frac{1}{(1 + xy)^2}, \quad (x, y) \in L.
\]

### 3.2. Approximation coefficients.

Following [12], or Chapter 4 in [9], one can show that for any irrational \( x > 0 \) and for \( k \in \mathbb{N} \) the approximation coefficients \( \theta_k \) satisfy
\[
\theta_{k-1} = \frac{v_k}{1 + iv_k}, \quad \text{and} \quad \theta_k = \frac{t_k}{1 + it_k}.
\]

where \((t_k, v_k) = T^k_d(x, 0), k \in \mathbb{N}\). Now, let \( x \) have RCF expansion (1), and suppose that \( a_{n+1} \geq 2 \) for some \( n \in \mathbb{N} \). Setting \( k(n) = a_0 + 2(a_1 - 1) + 1 + \cdots + 2(a_n - 1) + 1 \), it was
shown in [10] that
\[ \frac{p_k(n)}{q_k(n)} = \frac{p_k(n) + 2j}{q_k(n) + 2j} = \frac{P_n}{Q_n} \quad \text{for} \quad 1 \leq j \leq a_{n+1} - 1 \]
and
\[ \frac{p_k(n) + 2j - 1}{q_k(n) + 2j - 1} = \frac{j P_n + P_{n-1}}{j Q_n + Q_{n-1}} \quad \text{for} \quad 1 \leq j \leq a_{n+1} - 1. \]

In view of the “redundancy” (20), we will work with the ergodic system
\[ (\widetilde{\Omega} = [0, 1] \times [0, \infty), \mathcal{B}_{\widetilde{\Omega}}, \tilde{\mu}, S), \]
where \( S : \mathcal{B} \to \mathcal{B} \) is the induced transformation on \( \mathcal{B} \), i.e., for \((x, y) \in \mathcal{B}\) we define
\[ S(x, y) = \begin{cases} T_d(x, y), & \text{if } T_d(x, y) \in [0, 1] \times [0, 1], \\ T_d^2(x, y), & \text{if } T_d(x, y) \in [1, \infty) \times [0, 1]. \end{cases} \]

The next result follows at once from the dynamics of \( T_d \).

**Proposition 5.** Let \( x > 0 \) be a real number with RCF expansion (1) and CCF expansion (2), and suppose that \( a_{n+1} > 1 \). Furthermore, let
\[ k(n) = a_0 - d_0 + 2(a_1 - 1) + \cdots + 2(a_n - 1) + 1. \]

Then for \( 1 \leq j \leq a_{n+1} - 1 \) we have
\[ T_d^{k(n) + 2j - 1}(x, 0) = (t_{k(n) + 2j - 1}, v_{k(n) + 2j - 1}) \in (a_{n+1} - j - 1, a_{n+1} - j) \times [0, 1] \]
and
\[ T_d^{k(n) + 2j}(x, 0) = (t_{k(n) + 2j}, v_{k(n) + 2j}) \in [0, 1] \times [j, j + 1] = I_j. \]

It follows from Proposition 5 that if \( T_d^k(x, 0) = (t_k, v_k) \in I_B = [0, 1] \times [B, B + 1] \) for some \( B \geq 1 \), then \( p_k/q_k \) is equal to the RCF-convergent \( P_n/Q_n \) for some (unique) \( n \), while \( p_{k-1}/q_{k-1} \) is equal to the mediant
\[ \frac{BP_n + P_{n-1}}{BQ_n + Q_{n-1}} = \frac{L_m(B)}{M_m(B)} \]
for some \( m \in \mathbb{N} \), \( m = m(n) \). Hence, for each \( B \in \mathbb{N} \), the limiting distribution of the approximation coefficient \( \Theta_n^{(B)}(x) \) defined as
\[ \Theta_n^{(B)}(x) = \left( \frac{M_m(B)}{M_m(B)} \right)^n \left( x - \frac{L_m(B)}{M_m(B)} \right), \quad n \in \mathbb{N}, \]
can be derived for almost all \( x \). We thus have the following result.

**Proposition 6 ([2]).** Let \( B > 1 \) be an integer.
(i) For any irrational number $x$ with RCF expansion (1) and for any $n \in \mathbb{N}_+$ such that $0 < B < a_n$ we have

$$\frac{B}{B+1} \leq \Theta_n^{(B)} \leq B+1.$$  

(ii) For almost all $x$ the sequence $(\Theta_n^{(B)}(x))_{n \geq 1}$ is distributed according to the density function

$$G^{(B)}(z) = \frac{1}{\log \frac{B+2}{B+1}},$$

where

$$G^{(B)}(z) = \begin{cases} G_0^{(B)}(z) = 0, & \text{if } z \leq \frac{B}{B+1} \\ G_1^{(B)}(z) = -1 + \frac{B+1}{B} - z - \log \left(\frac{B+1}{B}z\right), & \text{if } \frac{B}{B+1} \leq z < \frac{B+1}{B+2} \\ G_2^{(B)}(z) = \frac{z}{B(B+1)} + \log \left(\frac{B(B+2)}{(B+1)^2}\right), & \text{if } \frac{B+1}{B+2} \leq z < B \\ G_3^{(B)}(z) = 1 - \frac{z}{B+1} + \log \left(\frac{B+2}{(B+1)^2}z\right), & \text{if } B \leq z < B+1 \\ G_4^{(B)}(z) = \log \frac{B+2}{B+1}, & \text{if } B+1 \leq z. \end{cases}$$

PROOF. It follows from (19) that

$$\Theta_n^{(B)}(x) = \frac{V}{1+TV},$$

where $(T, V) = T_d^{k(n)+2B}(x, 0) \in I_B$. Since

$$\frac{\partial}{\partial t} \left( \frac{v}{1+tv} \right) = -\frac{v}{(1+tv)^2} < 0 \quad \text{and} \quad \frac{\partial}{\partial v} \left( \frac{v}{1+tv} \right) = \frac{1}{(1+tv)^2} > 0,$$

we find that $\Theta_n^{(B)}(x)$ attains its maximum and minimum on the boundary of $I_B$. Its maximum (which is $B+1$) is attained at $(0, B+1)$ while its minimum (which is $B/(B+1)$) is attained at $(1, B)$. This proves (i).

Now, $\Theta_n^{(B)}(x) \leq z$ if and only if $(t, v) \in I_B$ and

$$\frac{v}{1+tv} \leq z, \quad \text{i.e.,} \quad v \leq \frac{z}{1-zt}.$$  

It follows from Theorem 2 that

$$\lim_{n \to \infty} \frac{1}{n} \# \left\{ k; 1 \leq k \leq n, \Theta_n^{(B)}(x) \leq z \right\}$$
exists for almost all \( x \) and is equal to

\[ \frac{\overline{\mu}(D^{(B)}(z))}{\overline{\mu}(I_B)} , \]

where

\[ D^{(B)}(z) = \left\{(t, v) \in I_B; \ v \leq \frac{z}{1 - zt}\right\} . \]

Therefore, since \(\overline{\mu}(I_B) = \log \frac{B+2}{B+1}\), we are left with the computation of \(\overline{\mu}(D^{(B)}(z))\). We distinguish 5 cases; see Figure 2. As an example, we deal with the case where \( B \leq z \leq B + 1 \) (see Figure 3); the other cases can be treated in a similar way. Since

\[ \overline{\mu}(D^{(B)}(z)) = \overline{\mu}(I_B) - \int_0^{B+1} \left( \int_B^{B+1} \frac{dy}{(1 + xy)^2} \right) dx \]
and
\[
\int_0^{B+1} \left( \int_B^{B+1} \frac{dy}{(1+xy)^2} \right) dx = \int_0^{B+1} \left( \frac{B+1}{1+(B+1)x} - z \right) dx
\]
\[= \log \left( \frac{B+1}{z} \right) - 1 + \frac{z}{B+1},\]
we obtain
\[
\bar{\mu} (D(B)(z)) = 1 - \frac{z}{B+1} + \log \left( \frac{B+2}{(B+1)^2z} \right) = G^3(z).
\]

REMARK. In [2], Bosma obtained Proposition 6 directly from the ergodic system
\((0, 1) \times (0, 1), B, \gamma, T)\) underlying the RCF expansion, by cleverly combining two consecutive regular approximation coefficients. In this way, Bosma was able to express approximation coefficients of convergents and mediants in terms of the orbit \((T^n(x, 0))_{n \geq 0}\) in \([0, 1) \times [0, 1]\).

To be precise, Bosma (see (1.8) in [2], p. 423) found that
\[
\Theta^{(B)}_n = \frac{(1 - BT_{n-1}V_{n-1})}{1 + T_{n-1}V_{n-1}}.
\]

Independently, Brown and Yin also obtained in [4] this relation. The outlook in this latter paper is quite different from that in [2].
References


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