Discovering an infinite prime flow by finding a Proximal Orbit Dense flow

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“Discovering an infinite prime flow by finding a Proximal Orbit Dense flow”

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1 Introduction

In this document, the discovery of an infinite flow that is prime is examined, following [2]. Most major results presented in this document are taken from this paper. Results from other papers will have their citation with them. Constructing finite prime flows is very easy, but it is not evident that there exist infinite flows that are prime. Indeed, in order to construct such flows, a lot of known theory is used, along with results that are shown in [2]. The purpose of this document is to give insight in how this construction works.

An explicit example of an infinite prime flow will be constructed. This is done by taking the irrational rotation on the circle, construct from this a Sturmian flow on the bisequence space, and then introduce a delay on this Sturmian flow. In this introduction, these three flows will be defined precisely, and definitions of minimality and primality will be given. For notational convenience, \((X,T), (Y,S), (Z,S), (N,R)\) etc. will denote general flows. Upcoming examples of particular interest will be the flows \((K,T_\alpha), (X,\sigma), (M,\sigma)\) and \((Y,\Psi)\).

1.1 Definitions

By a flow \((X,T)\) we mean a continuous invertible mapping \(T : X \rightarrow X\) on a compact Hausdorff space \(X\). Compactness and invertibility is not always assumed in general, but as we investigate in the constructions and results in [2] and these properties are assumed there, we also do it here.

Definition 1.1. A flow \((X,T)\) is minimal iff any closed invariant set is either the whole space \(X\) or the empty set. That is, if \(A \subseteq X\) closed, with \(T(A) \subseteq A\), then \(A = X\) or \(A = \emptyset\).

An equivalent formulation is: the orbit \(O_T(x) = \{T^n x| n \in \mathbb{Z}\}\) of any point \(x \in X\) is dense in \(X\). Note that by orbit, we mean forward and backward orbit. Since for any \(x \in X\), the orbit closure \(\overline{O_T(x)} \subseteq X\) is a nonempty closed invariant set, hence \(\overline{O_T(x)} = X\) if \((X,T)\) is minimal. On the other hand, if \((X,T)\) is not minimal, then let \(A\) be an invariant nonempty closed proper subset of \(X\) and pick \(x \in A\). Now, \(\overline{O_T(x)} \subseteq A\) and hence the orbit of \(x\) is not dense in \(X\). It follows that the orbit of \(x\) is dense in \(X\) for any \(x \in X\) if and only if \((X,T)\) is minimal. This formulation will be used several times in this document.

One might note that in the definition of minimality \(T(A) \subseteq A\) may be replaced with \(T(A) = A\). For minimal flows, this is evidently true. For nonminimal flows, let \(T(A) \subseteq A\) be a closed invariant set. The family of sets \(\{T^n A| n \in \mathbb{N}\}\) enjoys the finite intersection property and so

\[
B = \bigcap_{n \in \mathbb{N}} T^n A \neq \emptyset.
\]

by compactness. We have \(T(B) = B\), so one might as well use this \(B\) to show that a flow is not minimal.

A stronger notion than minimality is total minimality.
Definition 1.2. A flow \((X,T)\) is called totally minimal if the flows \((X,T_p)\) are minimal for all \(p \in \mathbb{Z}^+\).

There is an equivalent formulation to this definition in terms of factors of a flow. For this, first recall the notion of a homomorphism.

Definition 1.3. A homomorphism is a continuous mapping \(\pi : (X,T) \to (Y,S)\) for which \(\pi \circ T = S \circ \pi\).

Now, we define what a factor of a flow entails.

Definition 1.4. A flow \((Y,S)\) is a factor of the flow \((X,T)\), iff there exists a surjective homomorphism \(\pi : (X,T) \to (Y,S)\). Furthermore, in such cases \((X,T)\) is called an extension of \((Y,S)\).

We formulate the equivalent notion of total minimality. A rotation on a finite number of points is any flow \((X,T)\) of the form

\[ X = \{x_0, x_1, \ldots, x_{p-1}\}, \quad T(x_i) = T(x_{i+1 \mod p}). \]

Proposition 1.5. A minimal flow \((X,T)\) is totally minimal iff it does not have a rotation on a finite number of points as a factor.

Proof. For a minimal flow \((X,T)\) that is not totally minimal, there is a least \(1 < p \in \mathbb{Z}^+\) for which \((X,T_p)\) is not minimal. Let \(A\) be a nonempty closed proper subset such that \(T_p(A) = A\).

Since \((X,T)\) is minimal, we have

\[ X = \bigcup_{n \in \mathbb{Z}} T^n A = A \cup TA \cup \cdots \cup T^{p-1} A. \quad (1) \]

We choose \(A\) such that the sets \(T^j A, j = 0, \ldots, p-1\) are mutually disjoint. For, suppose \(A \cap T^j A \neq \emptyset\) for some \(i > 0\), then define \(A' = A \setminus T^j A\).

We show that \(A'\) is a nonempty closed invariant set as well. To show closedness, note that the boundary of \(A, \partial A = \overline{A} \cap \overline{A^c}\), is a closed invariant set under \(T^p\):

\[ T^p(\partial A) = T^p(\overline{A} \cap \overline{A^c}) = T^pA \cap T^p\overline{A^c} = A \cap \overline{T^pA^c} = A \cap \overline{A^c} = \partial A \]

Now, \(\bigcup_{i=0}^{p-1} T^i(\partial A)\) is a closed \(T\)-invariant subset of \(X\). Hence by minimality, it is either the empty set or the whole space \(X\). Suppose the latter is the case, then by Baire’s Category Theorem, at least one of the \(T^i(\partial A)\) has nonempty interior. But as

\[ T^i(\partial A) = T^iA \cap T^i\overline{A^c} = T^iA \cap (T^iA)^c = \partial(T^iA), \]

these are all boundaries. Note that \(T^iA\) is closed for \(i = 0, \ldots, p-1\), and it follows that all these boundaries do not have interior points. Hence \(\bigcup_{i=0}^{p-1} T^i(\partial A) = \emptyset\) and we conclude that \(T^iA\) is closed and open (clopen) for \(i = 0, \ldots, p-1\).

Now

\[ A' = A \setminus T^iA = A \cap (T^iA)^c \]

is closed as \(A\) is closed and \(T^iA\) is open.
$A' \neq \emptyset$, since in that case, $A \subseteq T^s A$, and then

$$T^{p-i}T^i A = T^p A = A \subseteq T^s A,$$

which means that $T^i A$ is a proper closed invariant subset for $T^{p-i}$. But $p$ was the least for which $(X,T^p)$ is not minimal, hence a contradiction, $A'$ is nonempty. Furthermore,

$$T^p A' = T^p (A \setminus T^i A) = T^p A \setminus T^{p+i} A = A \setminus T^i A = A',$$

and

$$A' \cap T^i A' = A \setminus T^i A \cap T^i A \setminus T^{2i} A = \emptyset.$$

Now, one repeats this procedure until all $T^j A$ are mutually disjoint for $j = 0\ldots p-1$.

Now, define $\pi : (X,T) \to (\{0,\ldots, p-1\}, i \mapsto i+1)$ as

$$\pi(x) = i \quad \text{for } x \in T^i A.$$

Note that $\pi$ is well defined due to (1) and the mutual disjointness. Clearly, $\pi$ is a surjective homomorphism showing a rotation on $p$ points as a factor. Conversely, if $\pi : (X,T) \to (Y,S)$ is a homomorphism with $(Y,S)$ a rotation on $p > 1$ points, then $A := \pi^{-1}(y)$ for some $y \in Y$ is a closed proper subset of $X$ that is invariant under $T^p$.

Now we turn to primality of minimal flows. Proceeding, interpret a ‘trivial’ flow to be the 1-point flow $Y = \{y\}$, $S(y) = y$.

**Definition 1.6.** A minimal flow $(X,T)$ is prime if for any given surjective homomorphism $\pi : (X,T) \to (Y,S)$, where $(Y,S)$ is a nontrivial flow, $\pi$ is an isomorphism.

Thus a flow is prime, if it has only itself (up to isomorphism) and the trivial flow as a factor.

One now might wonder whether we can drop the minimality assumption for primality. It turns out that there are easy examples of flows that satisfy the primality condition when the minimality condition is dropped. An example is given in the next section. The construction of an example of an infinite prime flow (which is minimal) is also presented in the next section. The verification that this example is indeed prime is intricate, this is the main investigation of this document. We are going to discover a prime flow by proving that this flow is Proximal Orbit Dense (POD).

**Definition 1.7.** Let $(X,T)$ be a flow. Two points $x, y \in X$ are called positively (negatively) asymptotic iff there is for any $\epsilon > 0$ there is a positive (negative) $N \in \mathbb{Z}$ such that whenever $n > N$ ($n < N$),

$$d(T^n x, T^n y) < \epsilon.$$

If $x, y$ are positively and negatively asymptotic, they are called doubly asymptotic. $x, y$ are called asymptotic if either one of the cases hold.

Slightly weaker than asymptoticity is proximality.
Definition 1.8. Let \((X, T)\) be a flow. Two points \(x, y \in X\) are called positively (negatively) proximal iff there is an increasing (decreasing) sequence \((n_j)_{j \in \mathbb{Z}}\) such that
\[
\lim_{j} T^{n_j} x = z \quad \lim_{j} T^{n_j} y = z
\]
for some \(z \in Y\). If \(x, y\) are positively and negatively proximal, they are called doubly proximal. \(x, y\) are called proximal if either one of the cases holds.

Definition 1.9. A flow \((X, T)\) is a proximal orbit dense (POD) flow if it is totally minimal, and whenever \(x, y \in X\) with \(x \neq y\), then for some \(n \neq 0\), \(T^n y\) is proximal to \(x\).

It will be shown in Section 3 that the POD condition implies that a flow is prime.

1.2 Examples

In this section, the particular flows of interest \((K, T_\alpha), (X, \sigma)\) and \((Y, \Psi)\) are defined. These flows will be used throughout this document. The first two examples are merely a handout to getting familiar with flows.

1.2.1 Easy examples

Finite flows

The easiest examples of flows one might come up with are finite flows. The 1-point flow is trivially (totally) minimal, but not prime. We have also already seen the rotation on a finite number \(p\) of points as a finite flow. These rotations are minimal. They clearly cannot be totally minimal (except the trivial flow), since \(T^p = \text{Id}\), the identity. A rotation \((X = \{x_1, \ldots, x_p\}, T)\) is prime if and only if the number of elements \(p\) is prime. For, suppose \(p = qr\), \(q, r > 1\), then let
\[
Y = \{0, 1, \ldots, q - 1\}, \quad S(y) = y + 1 \mod q
\]
and
\[
\pi : X \to Y, \quad \pi(x_i) = i \mod q.
\]

\((Y, S)\) is a nontrivial factor of \((X, T)\), hence \((X, T)\) is not prime. Conversely, if a homomorphism \(\pi\) exists which is not an isomorphism, then \(p = |\pi^{-1}(y)| \cdot |Y|\) (in fact, pick any \(y \in Y\)), hence \(p\) is not prime.

Properties of rotations are easily verified and not that hard to understand. But as we have seen, in the concept of total minimality, these flows do play part.

An example of a nonminimal flow that would be prime

Identify the point 1 = 0, and let
\[
N = \{\ldots, 15/16, 7/8, 3/4, 1/2, 1/4, 1/8, 1/16, \ldots, 0\}
\]

\[
R(a) = \begin{cases} 
(\frac{q-1}{q^2} & a = \frac{p-1}{p}, \quad p = 4, 8, \ldots \\
1 & a = \frac{1}{p^2}, \quad p = 2, 4, \ldots \\
0 & a = 0
\end{cases}
\]
We have a nonempty proper closed invariant subset \{0\} which is not the whole space, hence the flow is not minimal. Moreover, the orbit of every point except 0 is dense in \(N\). Now suppose there is a nontrivial surjective homomorphism
\[
\pi : (N, R) \to (Y, S),
\]
that is, \((Y, S)\) is a nontrivial flow. Let \(y_0 := \pi(0)\), this has to be a fixed point in \((Y, S)\). Note that \(\pi^{-1}(y_0) = \{0\}\), since if any other point would be in this inverse image, this would make the flow \((Y, S)\) trivial. Now let \(y \in Y, y \neq y_0\). Suppose
\[
\pi^{-1}(y) \supset \{x_1, x_2\}
\]
for some \(x_1 \neq x_2 \in N - \{0\}\), that is, \(\pi\) is not an isomorphism, there is a point which is not one-to-one. Now note that there is some \(0 \neq n \in \mathbb{Z}\) such that \(R^n x_1 = x_2\). It follows that \(S^n y = y\), that is, there is a periodic point in \((Y, S)\). As there is no cycle in \((N, R)\), this homomorphism cannot exist. Hence \(\pi\) has to be an isomorphism and the flow \((N, R)\) can be considered as a ‘non-minimal prime’ flow.

1.2.2 The flows of interest

Now, we introduce the three flows that are used to prove that an infinite prime flow exists. The latter two flows are both constructed based on the preceding one.

The irrational rotation on the circle: \((K, T_\alpha)\)

We define the flow \((K, T_\alpha)\) to be
\[
K = [0, 1) \quad (= \mathbb{R}/\mathbb{Z})
\]
\[
T_\alpha(x) = x + \alpha \mod 1
\]

Although this is addition modulo 1 on the unit interval, the mapping \(x \mapsto e^{2\pi i x}\) makes clear that this flow can be thought of as a rotation of the circle by the angle \(\alpha\).

If \(\alpha\) is rational, say \(\alpha = p/q\), then
\[
T_q^a(x) = x + q\alpha \mod 1 = x + p \mod 1 = x
\]
for all \(x \in K\). In this case, \((K, T_\alpha)\) is not minimal, since
\[
\{x, T_\alpha x, \ldots, T_q^{q-1} x\}
\]
is a closed invariant subset. It follows that for rational \(\alpha\), this flow is not totally minimal nor prime.

Let’s choose \(\alpha \in K\) irrational. For a sake of notational convenience, we notate an \(i\)'th iterate just as \(i\alpha\), and drop the \(\mod 1\) part. For instance, the forward orbit of 0 just becomes \(O(0) = \{n\alpha\}_{n \in \mathbb{N}}\), and so on.

For \(\alpha\) irrational, the orbit of any point becomes dense in \(K\). This follows from Dirichlet’s approximation Theorem. This theorem states that for any \(n \in \mathbb{N}\), there is a \(k \in \mathbb{Z}\) such that
\[
-1/n < k\alpha < 1/n.
\]
Hence let for any \( \epsilon > 0 \) this \( n \in \mathbb{N} \) be large enough such that \( 1/n < \epsilon \), and let \( k \in \mathbb{Z} \) be as in the theorem. Then \( \{jk\alpha | j \in \mathbb{Z} \} \) is \( \epsilon \)-dense in \( K \). This makes the orbit of the point \( 0 \in K \) dense. As any other orbit is just a rotation of the orbit of \( 0 \), these orbits are dense as well.

Hence, the flow \((K, T_\alpha)\) is minimal. Since \( 2\alpha, 3\alpha, \ldots \) are all irrational numbers, the flows \((K, T_{n\alpha})\) for \( n \in \mathbb{Z}^+ \) are all minimal. It follows that \((K, T_\alpha)\) is totally minimal.

\((K, T_\alpha)\) is not prime. Consider the flow \((K, T_{2\alpha})\), we show that this is a proper factor of \((K, T_\alpha)\). Define \( \pi : (K, T_\alpha) \to (K, T_{2\alpha}) \), \( \pi(x) = 2x \mod 1 \)

and verify

\[ \pi(T_\alpha x) = 2(x + \alpha) = 2x + 2\alpha = T_{2\alpha}(\pi(x)), \]

hence \( \pi \) is a homomorphism. However, it is not an isomorphism, as (e.g.) \( \pi^{-1}(0) = \{0, 1/2\} \).

**Sturmian flow on the symbolic sequences: \((X, \sigma)\)**

Let \((K, T_\alpha)\) be as above with an irrational angle \( \alpha \). Now choose some \( \beta \in K \), \( \beta \neq 0 \).

Let \( f : K \to \{0, 1\} \),

\[ f(\gamma) = \chi_{[0, \beta]}(\gamma), \quad \gamma \in K \]

that is, the characteristic function of the interval \([0, \beta]\), on the circle. We construct the bisequence \( x_0 \in \{0, 1\}^\mathbb{Z} \) as follows:

\[ x_0(n) = f(n\alpha) \quad \text{for } n \in \mathbb{Z}. \]

So, we iterate the point \( 0 \) in positive and negative direction, and check whether each iterate lies in the interval \([0, \beta]\). These indicator function results are all put in one element, which is the bisequence \( x_0 \). This is called the itinerary of the point \( 0 \). See Figure 1.

![Figure 1: Rotation on the circle. Some iterates land in the interval \([0, \beta]\). In this case, for \( n = \ldots, 0, 8, \ldots \) we have \( n\alpha \in [0, \beta] \).](image)

Note that \( n\alpha = T_\alpha^n(0) \), and in fact we have the following,

\[
 x_0 = (\ldots, f(T_{-2\alpha}^{-1}(0)), f(T_{-1\alpha}^{-1}(0)), f(0), f(T_\alpha(0)), f(T_{2\alpha}(0)), \ldots) \\
 = (\ldots, f(-2\alpha), f(-\alpha), 1, f(\alpha), f(2\alpha), \ldots).
\]
With the element $x_0$, we construct a flow $(X, \sigma)$. We define $\sigma$ to be the left shift on the space of bisequences, that is, for $x \in \{0, 1\}^\mathbb{Z}$,

$$\sigma x(n) = x(n + 1) \text{ for } n \in \mathbb{Z}. $$

In choosing the space $X$ on which the flow exists, we do not take the whole space $\{0, 1\}^\mathbb{Z}$, but we would like to have a space in which the $\sigma$-orbit of $x_0$ is dense, and also complete in the sense that it is closed under taking limits. This flow might then be minimal. Hence we start by defining a (natural) metric on $\{0, 1\}^\mathbb{Z}$ in the following way; we will have points close together whenever a large block of indices around 0 coincide. For this, define

$$d(x, y) = 2^{-k} \text{ where } k = \min\{|j| : x(j) \neq y(j)\}$$

We define our set $X$ to be the $\sigma$-orbit closure of $x_0$ with respect to this metric, that is, the space

$$X = \overline{\sigma(x_0)}. $$

Note that $\sigma$ is an invertible continuous mapping, and $X$ is compact Hausdorff.

The flow $(X, \sigma)$ is called the Sturmian flow of type $(\alpha, \beta)$. By construction, $X$ is topologically transitive, that is, there is a point that is dense, and clearly $x_0$ is dense. It indeed turns out that $X$ is minimal. $X$ is also totally minimal, but not prime. These properties will be derived in Section 2.

As a last remark, a basic result from ergodic theory is that the density of 1’s in $x_0$ is $\beta$. More precisely,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} x_0(i) = \lambda([0, \beta]) = \beta,$$

where $\lambda$ is the Lebesgue measure on $K$. One might wonder how ‘wild’ the sum in the above equation behaves, in the sense of how far it may diverge from $\beta$ on average. As a side result, the answer to this is found in Theorem 2.14.

**Delayed flow on the symbolic sequences: $(Y, \Psi)$**

After this rather elaborate construction of the Sturmian flow $(X, \sigma)$, we introduce the third example based on this previous flow. This flow, denoted $(Y, \Psi)$, is of particular interest, since it turns out to be prime. It has an even stronger property, namely, it is proximal orbit dense. In section 3, the POD property is shown, along with total minimality, which follows from topological weak mixing. Since these proofs are intricate enough to have a section of their own, we merely provide the construction of the flow $(Y, \Psi)$ here, and postpone further investigation until Section 3.

Let $B = \{x \in X | x(0) = 1\}$, and let $A$ be a homeomorphic copy of $B$ such that $X \cap A = \emptyset$, and $\phi : B \to A$ a homeomorphism. Let $Y = X \cup A$ and define $\Psi : Y \to Y$ by

$$\Psi(y) = \begin{cases} 
\phi(y) & \text{if } y \in B \\
\sigma(\phi^{-1}(y)) & \text{if } y \in A \\
\sigma(y) & \text{if } y \in X - B
\end{cases}$$

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One may interpret this construction as if a delay is introduced. When \( y \in B \), it is 'kept there' for one iterate, more explicitly, \( \Psi(y) \in A \), but \( A \) is just a homeomorphic copy of \( B \). See Figure 2. After visiting \( A \), \( \Psi(y) \) is sent to \( \sigma(y) \), i.e., \( \Psi(\Psi(y)) = \sigma(y) \) for \( y \in B \). We have \( \Psi(y) = \sigma(y) \) for \( y \in X - B \), so the delay is only introduced on points in \( B \), that is, bisequences with a 1 at the 0'th coordinate. Recall that this 1 just indicates that the 0'th index of the itinerary of a point on the circle \( K \) apparently lies in the interval \([0, \beta]\). Loosely speaking, one may view this as that the delay is introduced on the interval \([0, \beta]\) on the circle.

![Figure 2: A graphical presentation of the flow \((Y, \Psi)\). For elements in \( B \) we have \( \Psi^2 = \sigma \). For \( X - B \), \( \Psi = \sigma \). The drawing is taken from [2], but is modified.](image)

### 1.3 Outline of the discovery of an infinite prime flow

The following theorem is the main result that is proven in this document. We will give an overview of the proof, highlighting several results that are proven in the rest of this document. This overview gives relevance to the results that are established in the upcoming sections.

**Theorem 1.10.** The flow \((Y, \Psi)\) is a prime flow.

The proof is established by showing that \((Y, \Psi)\) is a POD flow. By Theorem 3.14, primality follows. To show that the POD condition holds, three results are proven in Section 3. We will state the results here without proof.

**Theorem 3.6.** \((Y, \Psi)\) is topologically weakly mixing and minimal.

The function \( \tau: Y \to K \) is defined in Section 3.

**Proposition 3.9.** Let \( x, y \in Y \), \( x \neq y \), and \( \tau(x) - \tau(y) = k\alpha \) for some \( k \geq 0 \). Then for some \( n \neq 0 \), \( x \) is asymptotic to \( \Psi^n y \).

**Proposition 3.13.** For \( \tau(x_1) - \tau(y_1) \notin \mathbb{Z}\alpha \), there is some \( n \neq 0 \) such that \( \Psi^n y_1 \) is proximal to \( x_1 \).

Topological weak mixing implies total minimality, this is shown in Theorem 3.7. Hence, from Theorem 3.6 it follows that \((Y, \Psi)\) is totally minimal. Furthermore, Propositions 3.9 and 3.13 show the proximality property for all \( x, y \in Y \). It follows that \((Y, \Psi)\) is a POD flow, hence it is also prime.

In proving Proposition 3.9, Theorem 2.7 is used. In proving Proposition 3.13, Corollary 2.13 is used. This corollary follows from Theorem 2.8.
2 Results on flows in symbolic dynamics

In this section, the flow \((X, \sigma)\), as constructed in the introduction, is investigated. Recall that the flow is constructed from a rotation on the circle with irrational angle \(\alpha\) and the indicator function on the interval \([0, \beta]\). First we discuss what the elements in the space \(X\) look like. These properties are summarized in theorem 2.7, which is originally proven in [7]. In the second part, a new flow \((M, \sigma)\) is introduced. This is done in order to establish a corollary about the irrational rotation on the circle. The corollary will be used later in the proof that the flow \((Y, \Psi)\) is proximal orbit dense. We finish this section with a separate result, namely a generalization of a result by Kesten [3].

2.1 Elements of the flow \((X, \sigma)\)

We now identify two types of elements in the bisequence space \(X\). As it turns out, \(X\) consists precisely out of these elements, uncountably many of one and only countably many of the other.

In the introduction, the orbit of the point \(0 \in K\) led to an element \(x_0 \in X\). The subscript in the latter is chosen conveniently, we define in the same manner

\[ x_s := (\chi_{[0,\beta]}(s + n\alpha))_{n \in \mathbb{Z}}, \]

that is, the itinerary of \(s \in K\) in the same manner as \(x_0\) followed from the orbit of 0, namely applying the indicator function of the interval \([0, \beta]\) on every iterate. As an example, \(\sigma x_0 = x_\alpha\).

Assume that \(\beta \notin \mathbb{Z}\alpha\). Then, the orbits of the boundary points do not coincide. Let the set \(E\) be the union of the orbits of the boundary points of the interval \([0, \beta]\),

\[ E := \mathbb{Z}\alpha \cup (\beta + \mathbb{Z}\alpha). \]

For \(s \in E\), let

\[ \bar{x}_s(n) = \begin{cases} x_s(n) & \text{for } n \in \mathbb{Z} - \{n_0\} \\ 0 & \text{for } n = n_0 \end{cases} \]

where \(n_0 \in \mathbb{Z}\) is such that \(s + n_0\alpha = 0\) or \(\beta\). It is the iterate at which the point \(s\) hits indeed the boundary of the interval. Note that this \(n_0\) is unique. The iterates cannot visit the same point twice since \(\alpha\) is irrational and by the assumption \(\beta \notin \mathbb{Z}\alpha\), only one of the two boundary points is visited in the orbit of any \(s \in E\). Note that \(x_s\) and \(\bar{x}_s\) differ only at one coordinate. In fact, \(1 = x_s(n_0) \neq \bar{x}_s(n_0) = 0\), and the two elements coincide on all other indices. We are now ready to describe all the elements in \(X\).

**Proposition 2.1.**

\[ X = \{x_s : s \in K\} \cup \{\bar{x}_s : s \in E\}. \]

The proof of this proposition follows from the following lemma’s.

**Lemma 2.2.** Coordinatewise convergence implies convergence in the metric on \(X\). That is, if \(\sigma^{n_i}x_0(i)\) converges to, say, \(a_i \in \{0, 1\}\) for all \(i \in \mathbb{Z}\), then \(\sigma^{n_i}x_0 \to x\), with \(x(i) = a_i\) for \(i \in \mathbb{Z}\).
Proof. Fix $\epsilon > 0$. Recall that for convergence in the metric on $X$, two elements are arbitrarily close when a sufficiently large block of indices around the index 0 correspond. Hence, let $N_\epsilon \in \mathbb{N}$ be sufficiently large. For any index $i \in \mathbb{Z}$, the convergence of $\sigma^n x_0(i)$ just states that there is a number $J_i \in \mathbb{N}$ such that

$$\sigma^n x_0(i) = a_i \text{ for all } j > J_i.$$ 

That is, the coordinates are eventually constant. Now let

$$J = \max_{-N \leq i \leq N} J_i.$$ 

Then, for all $j > J$,

$$\sigma^n x_0(i) = a_i \text{ for } -N_\epsilon \leq i \leq N_\epsilon.$$ 

Hence $d(\sigma^n x_0, \sigma^nx_0) < \epsilon$ whenever $j,k > J$. This implies that the sequence converges. Evidently, the limit is $x$ as in the lemma. \hfill \square

Lemma 2.3. $\{x_s : s \in K - E\} \subseteq X$.

Proof. Choose $s \in K - E$ arbitrarily. We need to show that there is a sequence $(n_k)_k$ such that $\sigma^{n_k} x_0 \to x_s$. By minimality of the irrational rotation on the circle, let $(n_k \alpha_k)_{k \in \mathbb{N}}$ be convergent to $s$. Then, for any $n \in \mathbb{Z}$,

$$\lim_k n_k \alpha + n \alpha = s + n \alpha$$

All points $s + n \alpha \not\in \{0, \beta\}$, thus $f = \chi_{[0, \beta]}$ is continuous in all points of the orbit of $s$. We have

$$\lim_k f(n_k \alpha + n \alpha) = f(s + n \alpha)$$

but this is precisely

$$\lim_k \sigma^{n_k} x_0(n) = x_s(n).$$

for $n \in \mathbb{Z}$. Hence we have coordinatewise convergence. By Lemma 2.2, $\sigma^{n_k} x_0 \to x_s$ and the lemma follows. \hfill \square

Lemma 2.4. $\{x_s : s \in E\} \subseteq X$.

For $s \in E$, $f$ will be not continuous in one point, so we need to be a bit more careful here. The key step here is to choose the sequence $(n_k)_k$ a bit wiser.

Proof. Let $n_0 \in \mathbb{Z}$ be the index where $s + n_0 \alpha = 0$ or $\beta$. For all $n \in \mathbb{Z} - \{n_0\}$, $s + n \alpha$ is a point of continuity of $f$. For these $n$, any sequence $(n_k \alpha)_n$ approaching $s$ gives the desired coordinatewise convergence $\sigma^{n_k} x_0(n) \to x_s(n)$.

The remaining index is $n_0$. We have $x_s(n_0) = 1$, because $s + n_0 \alpha = 0$ or $\beta \in [0, \beta]$. We choose the sequence $(n_k \alpha)_k$ to approach $s$ from above in the 0-case and from below in the $\beta$-case. See also Figure 3. We then have $f((n_0 + n_k) \alpha) = 1$ for $k$ big enough, as $(n_0 + n_k) \alpha \in [0, \beta]$ for $k$ big enough. For this sequence, we have that

$$\lim_k f(n_k \alpha + n_0 \alpha) = f(s + n_0 \alpha).$$

and so,

$$\lim_k \sigma^{n_k} x_0(n_0) = x_s(n_0).$$

as desired. As convergence is verified for all coordinates we apply Lemma 2.2, conclude that indeed $\sigma^{n_k} x_0 \to x_s$ and the lemma follows. \hfill \square
Figure 3: By choosing an appropriate sequence, one shows for instance that $\bar{x}_0$ and $x_\beta$ are in $X$. For any $s \in E$, this is the way to show that $x_s \in X$ and $\bar{x}_s \in X$.

The last type of elements in $X$ are the ‘barred’ elements, the elements where one coordinate is changed from a 1 to a 0. The proof goes very similar as Lemma 2.4, but now, the point $s$ on the circle is approached from the outside of the interval.

Lemma 2.5. $\{\bar{x}_s : s \in E\} \subseteq X$.

Proof. Again, let $n_0 \in \mathbb{Z}$ be the index where $s + n_0 \alpha = 0$ or $\beta$. Let $(n_k \alpha)_k$ be a sequence approaching $s \in E$ from above in the $\beta$-case, from below in the 0-case. Then

$$(n_k + n_0) \alpha \notin [0, \beta]$$

for all $k$ large enough. Hence,

$$\sigma^{n_k}x_0(n_0) = f((n_k + n_0)\alpha) = 0$$

for all $k$ large. The coordinatewise limit also holds for all $n \neq n_0$, again by continuity. For the index $n_0$, we have a limit value 0 instead of 1. Indeed, this corresponds to the definition of $\bar{x}_s$. Apply Lemma 2.2 and the result follows.  

We arrive at the final lemma to prove the proposition.

Lemma 2.6. $\{x_s : s \in K\} \cup \{\bar{x}_s : s \in E\} \supseteq X$.

Proof. Let $x \in X$, that is, there is some sequence such that

$$(\sigma^{n_k}x_0)_k \to x \text{ for } k \to \infty.$$  

Fix $\epsilon > 0$ and choose $M$ so large that $\{i\alpha : |i| < M\}$ is $\epsilon$-dense in $K$. Let $k_1, k_2$ be large enough such that

$$\sigma^{n_{k_1}}x_0(i) = \sigma^{n_{k_2}}x_0(i) \text{ for } |i| < M.$$  

We claim that $|n_{k_1} \alpha - n_{k_2} \alpha| \leq \epsilon$. For this, suppose $|n_{k_1} \alpha - n_{k_2} \alpha| > \epsilon$. Then, there is an $|k| < M$ such that $ka \in (n_{k_1} \alpha, n_{k_2} \alpha)$, because of $\epsilon$-density. It follows that

\[
\begin{align*}
\chi_{[k_1 \alpha, \alpha + \beta]}(n_{k_1} \alpha) & \neq \chi_{[k_1 \alpha, \alpha + \beta]}(n_{k_2} \alpha) \\
\iff \chi_{[0, \beta]}(n_{k_1} \alpha - k \alpha) & \neq \chi_{[0, \beta]}(n_{k_2} \alpha - k \alpha) \\
\iff \sigma^{n_{k_1}}x_0(-k) & \neq \sigma^{n_{k_2}}x_0(-k)
\end{align*}
\]
However, for \(| - k | < M\) we have that \(\sigma^{n_0} x_0(-k) = \sigma^{n_0} x_0(-k)\). Hence a contradiction.

We conclude that \((n_0, \alpha)_k\) is also Cauchy and hence convergent to some \(s \in K\). Now we have the candidates \(x_s\) and \(\bar{x}_s\), and we still need to show that indeed \(x\) is one of these elements. Fix \(N \in \mathbb{N}\). For \(s \in K \setminus E\), either \(s + n\alpha \in (0, \alpha)\) or \(s + n\alpha \in (\beta, 1)\), for \(|n| \leq N\). As these intervals are open, we have for each \(n\) an \(\epsilon_n\) such that the ball \(B(s + n\alpha, \epsilon_n) \subset (0, \alpha)\) or \((\beta, 1)\) as well. Let \(\epsilon_N := \min_n \epsilon_n\).

Now, whenever \([n_k \alpha - s] < \epsilon_N\), then \(\sigma^{n_k} x_0(n) = x_s(n)\) for \(|n| \leq N\). We conclude that \(\sigma^{n_k} x_0 \to x_s\) as \(k \to \infty\).

For \(s \in E\), let \(n_0\) be such that \(s + n_0\alpha \in \{0, \beta\}\). This case goes similar to the above, there is only one index where the situation changes. Use above reasoning but add that \(k\) must be sufficiently large such that the coordinate at index \(n_0\) remains constant. Then, convergence follows. We have two cases:

- If \(x(n_0) = 0\), then \(x = \bar{x}_s\).
- If \(x(n_0) = 1\), then \(x = x_s\).

\[
\square
\]

**Proof of Proposition 2.1.** By Lemma’s 2.2 through 2.6.

Proposition 2.1 gives insight in the upcoming theorem, which is found in [7]. Most of the proof is established by Proposition 2.1, for the remaining parts of the proof we follow [7]. Recall that it is assumed that \(\beta \notin \mathbb{Z}\alpha\).

**Theorem 2.7.** The mapping \(\rho(\sigma^n x_0) = n\alpha\) can be extended to a homomorphism \(\rho: (X, \sigma) \to (K, T_\alpha)\), such that

(a) \(\rho^{-1}\{\gamma\}\) is a singleton unless \(\gamma \in E = (\mathbb{Z}\alpha) \cup (\beta + \mathbb{Z}\alpha)\),

(b) for each \(\gamma \in E\), \(\rho^{-1}\{\gamma\}\) consists of exactly two points,

(c) for \(\rho^{-1}\{0\} = \{x_0, \bar{x}_0\}\), it holds that \(\bar{x}_0(0) \neq x_0(0)\) but \(\bar{x}_0(n) = x_0(n)\) for \(n \neq 0\),

(d) if \(\rho(x) = \rho(x')\), then \(x\) and \(x'\) are doubly proximal, and in addition,

(e) \((X, \sigma)\) is minimal.

**Proof.** Evidently, define \(\rho(x_\gamma) = \gamma, \rho(\bar{x}_\gamma) = \gamma\). By the proof of Lemma 2.6, \(\rho\) is continuous and structure preserving, hence a homomorphism.

We now verify the properties stated in the theorem.

(a) For \(\gamma \notin E\), the only preimage is \(x_\gamma\), so indeed \(\rho^{-1}\{\gamma\}\) is a single point.

(b) For \(\gamma \in E\), we have \(\rho^{-1}\{\gamma\} = \{x_\gamma, \bar{x}_\gamma\}\), which are indeed two points.

(c) \(0 \in E\), and indeed \(\rho^{-1}\{0\} = \{x_0, \bar{x}_0\}\).

(d) For \(x = x'\), there is nothing to prove. The case \(\rho(x') = \rho(x)\) and \(x' \neq x\) happens precisely when \(x' = \bar{x}\). Let \((n_k)_k\) be an increasing/decreasing sequence of integers such that \((\rho(x) + (n_k \alpha))_k \to \gamma\) for some \(\gamma \notin E\). \(X\) is compact, so there is a subsequence such that

\[
\begin{align*}
z &:= \lim_k \sigma^{n_k} x \\
z' &:= \lim_k \sigma^{n_k} x'
\end{align*}
\]

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\]
both exist. Now,
\[
\rho(z) = \lim_k \rho(\sigma^n x) = \lim_k (\rho(x) + (n_k \alpha)_k) = \gamma
\]
\[
\rho(z') = \lim_k \rho(\sigma^n x') = \lim_k (\rho(x') + (n_k \alpha)_k) = \gamma
\]
Because \( \gamma \not\in E \), we must have \( z = z' \), which means that \( x \) and \( x' \) are doubly proximal.

(e) Choose \( x \in X \). We show that \( x_0 \in O(x) \), then it follows that \( X \subseteq \overline{O(x)} \) and in fact equality holds. Let \( s = \rho(x) \). Choose a sequence \((n_k)_k\) such that \( s + n_k \alpha \in (0, \beta) \) for all \( k \), and
\[
\lim_k s + n_k \alpha = 0
\]
\[
\sigma^n x \to x' \text{ for some } x' \in X.
\]
This can be done since \( X \) is compact. Now we show that \( x' = x_0 \).
\[
\rho(x') = \lim_k \rho(\sigma^n x) = \lim_k s + n_k \alpha = 0,
\]
so \( x' \in \{x_0, \bar{x}_0\} \). To decide which of the two it is, we look at the 0th index:
\[
x'(0) = \lim_k \sigma^n x(0) = \lim_k \chi_{[0,\beta]}(s + n_k \alpha) = \lim_k 1 = 1.
\]
Hence \( x = x_0 \), and we conclude that \((X, \sigma)\) is minimal as any orbit is dense.

In part (d), one notes that for \( s \in E, x_s \) and \( \bar{x}_s \) differ by only one index \((n_0)\). For a sequence \((n_k)_k\) going to \( \pm \infty \), this index is pushed to the upper (lower) tail of the bisquence. As the metric on \( X \) is such that points become closer when more coordinates close to 0 correspond, we see that \( x_s \) and \( \bar{x}_s \) are doubly asymptotic.

Note that in the proof of minimality of \((X, \sigma)\), Zorn’s lemma is not used. This is a slight improvement on the proof in [7]. Theorem 2.7 will be used in further proofs. The theorem provides that the following diagram commutes.

\[
\begin{array}{ccc}
(X, \sigma) & \xrightarrow{\rho} & (X, \sigma) \\
\downarrow \rho & & \downarrow \rho \\
(K, \tau_\alpha) & \xrightarrow{\tau_\alpha} & (K, \tau_\alpha)
\end{array}
\]

In Theorem 2.7 it was shown that \((X, \sigma)\) is minimal. We can also show that \((X, \sigma)\) is totally minimal. This proof is completely similar to that of the minimality of \((X, \sigma)\). For, let \( x \in X \) and \( p > 1 \) be arbitrary. We know that \( x = x_s \) or \( x = \bar{x}_s \) for some \( s \in K \). Since \((K, \tau_\alpha)\) is totally minimal, there is a sequence \((n_j p \alpha)\) converges. \( \rho(\sigma^{n_j} x_0) \to s \), so \( (\sigma^{n_j} x_0)_j \to x_s \) or \( \bar{x}_s \). By letting \((n_j p \alpha)\) getting from below or above, both \( x_s, \bar{x}_s \in \overline{O_\sigma(x_0)} \). Hence the orbit of \( x_0 \) is dense.

The remaining part, that the orbit of any \( x \in X \) is dense, goes similar to the proof of Theorem 2.7(e), and is omitted. As a last remark, note that \((X, \sigma)\) is not prime, as \((K, \tau_\alpha)\) is a nontrivial proper factor.
2.2 The bounded-unboundedness phenomenon

Now, we take a step back from the examples introduced in the introduction and first establish a result on arbitrary minimal flows. We will then apply this result to the irrational rotation on the circle \((K, T)\), hereby following [2].

For notational convenience, the denotation of flows is chosen such that the application comes natural. However, one must note that the flow \((K, T)\) is not necessarily the rotation on the circle, \(x_0\) is not a symbolic bisequence, etc.

Let \((K, T)\) be an arbitrary minimal flow on a compact Hausdorff space \(K\). Let \(A, B \subseteq K\) be subsets and \(x_0 \in K\). Let

\[
g(T^n x_0) = \chi_A(T^n x_0) - \chi_B(T^n x_0)
\]

be defined on the orbit of \(x_0\). In contrast to the interval \([0, \beta]\) earlier on, we now have two sets and corresponding indicator functions. Let \(G\) denote all points in \(K\) to which \(g\) cannot be extended continuously. In fact, this is the boundary of \(A\) and \(B\).

We are interested in the boundedness of the quantity

\[
N(n) = \sum_{i=0}^{n-1} g(T^i x_0),
\]

that is, whether the iterates get arbitrarily often in one set but not in the other, on the long run. This relates to an improvement on the mean ergodic theorem, which was posed in the introduction. With the aid of symbolic dynamics, a theorem solving this problem can be established. The improvement on the mean ergodic theorem will then follow as a corollary. Henceforth, let

\[
m_0(n) = g(T^n x_0) \text{ for } n \in \mathbb{Z},
\]

hence,

\[
m_0 \in \{-1, 0, 1\}^\mathbb{Z},
\]

so we are in a slightly different setting than with the flow \((X, \sigma)\). We proceed similarly; let \(\sigma\) be the left shift on this space of bisequences,

\[
\sigma m(n) = m(n+1) \text{ for } n \in \mathbb{Z},
\]

and take \(M\) to be the \(\sigma\)-orbit closure of \(m_0\). The metric on \(M\) is defined the same way as it was done on \(X\). Let \(h\) be the function

\[
h : M \to \{-1, 0, 1\}, \quad h(m) = m(0).
\]

This function is continuous. At last, we introduce the map \(\pi(\sigma^n m_0) = T^n x_0\) on the orbit of \(m_0\). Summarized in a diagram,

\[
\begin{array}{ccc}
(M, \sigma) & \overset{\sigma}{\longrightarrow} & (M, \sigma) \\
\downarrow \pi & & \downarrow \pi \\
(K, T) & \overset{T}{\longrightarrow} & (K, T)
\end{array}
\]

\[
\begin{array}{ccc}
 & & \\
\overset{g}{\longrightarrow} & \{ -1, 0, 1 \} & \end{array}
\]

Note that there is commutation on the orbit of \(m_0\), but not necessarily on whole \(M\). Also note that the function \(g\) is not continuous (henceforth it is yet only defined on the orbit of \(x_0\)). However, \(h\) is continuous, and \(h = g \circ \pi\) on the orbit of \(m_0\). The following theorem now solves the issue.

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Theorem 2.8. Assume the following:

1. \((M, \sigma)\) is minimal.
2. The map \(\pi(\sigma^n m_0) = T^n x_0\) can be extended to a homomorphism \(\pi : (M, \sigma) \rightarrow (K, T)\).
3. There is \(x_1 \in K\) whose orbit does not intersect \(G\).
4. There is \(z \in G\) with \(O(z) \cap G = \{z\}\).

Then \(N(n)\) is unbounded for \(n > 0\).

The proof of this theorem is quite elaborate. It heavily depends on a remarkable lemma, found in [2]. We first present this lemma, followed by several other claims needed in the proof. After that, the proof of Theorem 2.8 follows.

Lemma 2.9. For \((M, \sigma)\) an arbitrary minimal flow and \(h \in C(M)\), a necessary and sufficient condition for there to exist an \(f \in C(M)\) with \(f \circ \sigma - f = h\) is that for some \(m_0 \in M\), the sums \(\sum_{i=0}^{n} h(\sigma^i m_0)\) are bounded for \(n > 0\).

Proof. \(f\) satisfying the identity \(f \circ \sigma - f = h\) is called a coboundary. Note that for such an \(f\) we have

\[
\sum_{i=0}^{n} h(\sigma^i m) = f(\sigma^{n} m) - f(m)
\]

for \(m \in M\), thus, when the sum is unbounded, \(f\) is unbounded, but since \(M\) is compact, no such \(f\) can exist. By this, necessity is proven.

For sufficiency, assume boundedness of the sum for some \(m_0 \in M\). Define the following homomorphisms \(R, T_s : M \times \mathbb{R} \rightarrow M \times \mathbb{R}\) by

\[
R(m, t) = (\sigma^m, t + h(m))
\]

\[
T_s(m, t) = (m, t + s).
\]

Note now that \(R^n(m_0, 0) = (\sigma^n m_0, \sum_{i=0}^{n} h(\sigma^i m_0))\), for \(n > 0\). \(M\) is compact, and the sum is bounded, so set of the limit points of

\[
\{R^n(m_0, 0) | n > 0\}
\]

is also compact. Thus it contains a minimal subset \(N\), by Zorn’s Lemma. That is, a nonempty, closed set \(N \subseteq (M \times \mathbb{R}, R)\) which is invariant under \(R\), such that there is no smaller set \(N' \subset N\) having these properties.

Note that by minimality of \(M\), \(\{m|(m, r) \in N\} = M\). Now since \(N\) is a minimal set, for any \(m \in M\), there is a unique \(r_m\) such that \((m, r_m) \in N\). To show this, suppose \((m, r_m + \delta) \in N\) too. Then, by minimality \(\{R^n(m, r_m)| n > 0\}\) and \(\{R^n(m, r_m + \delta)| n > 0\}\) are dense in \(N\). Alternatively stated,

\[
N = \overline{O}_R((m, r_m)) = \overline{O}_R((m, r_m + \delta))
\]

or

\[
T_\delta N = N.
\]

Applying \(T_\delta\) over and over, we find that \(T_\delta^n N = N = T_\delta^n N\). Pick an increasing sequence \((n_k)_{k \in \mathbb{N}} \subset \mathbb{N}\) and consider the sequence in \(N\) of the form \((a_k)_{k :=}
\((m_k, r_m + n_k \delta)_k\). As \(N\) is compact, we may choose this sequence to converge. However, as \((n_k)_k\) is increasing, all points in \((a_k)_k\) lie at least \(\delta\) away from each other (in the second coordinate). Hence, we must have \(\delta = 0\), and thus \(r_m\) is unique for every \(m \in M\).

Now we define \(f(m) = r_m\). We need to verify that \(f\) is continuous and \(f \circ \sigma - f = h\). For continuity, let \(\sigma^{n_k}m_0 \to m\), now \(R^{n_k}(m_0,0) = (\sigma^{n_k}m_0, \sum_{i=0}^{n_k-1} h(\sigma^i m_0))\). This converges to an unique limit in the minimal set \(N\). The second coordinate gives us \(f(\sigma^{n_k}m_0) \to f(m)\). Now note that for \(M \ni m (= \lim n_k \sigma^{n_k}m_0\) for some sequence \((n_k)_n)\),

\[
\begin{align*}
 f(\sigma m) - f(m) &= \lim_k f(\sigma^{n_k+1}m_0) - \lim_k f(\sigma^{n_k}m_0) \\
 &= \lim_k \left( \sum_{i=0}^{n_k} h(\sigma^i m_0) - \sum_{i=0}^{n_k-1} h(\sigma^i m_0) \right) \\
 &= \lim_k h(\sigma^{n_k}m_0) \\
 &= h(m)
\end{align*}
\]

as desired.

Before heading forward to the proof of Theorem 2.8 we first make some claims, under the assumptions made in the theorem.

Let \(x \not\in G\), that is, \(g\) can be continuously extended to \(x\). Then for any \(m \in \pi^{-1}(x)\) we have

\[
m(0) = \lim_n \sigma^{a_n}m_0(0) = \lim_n g(T^{a_n}x_0) = g(x).
\]

for a sequence with \(T^{a_n}(x_0) \to x\). As \(g(x)\) is continuously extended, \(g(x)\) can be only one value. Hence, the value \(m(0)\) must be constant on \(\pi^{-1}(x)\). The following claim states that the converse also holds.

Claim 2.10. \(G = \{x \in K : m(0)\ assumes more than one value on \(\pi^{-1}(x)\}\}\).

That is, \(x \in G\) when in the preimages of \(x\) under \(\pi\), let’s say \(\{m_1, m_2, \ldots\}\), there are some \(m_i, m_j\) with \(m_i(0) \neq m_j(0)\).

Proof. Suppose \(x \in G\). Hence, we have sequences \((a_n)_n\) and \((b_n)_n\) such that

\[
\lim_n T^{a_n}x_0 = \lim_n T^{b_n}x_0 = x
\]

but

\[
\lim_n g(T^{a_n}x_0) \neq \lim_n g(T^{b_n}x_0)
\]

(provided that these limits even exist. For this, take subsequences, \(\{-1, 0, 1\}\) is compact). Note that

\[
\begin{align*}
\lim_n g(T^{a_n}(x_0)) &= \lim_n m_0(a_n) = \lim_n \sigma^{a_n}m_0(0), \\
\lim_n g(T^{b_n}(x_0)) &= \lim_n m_0(b_n) = \lim_n \sigma^{b_n}m_0(0)
\end{align*}
\]

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We may choose subsequences such that \( m_1 := \lim_n \sigma^{a_n} m_0 \) and \( m_2 := \lim_n \sigma^{b_n} m_0 \) exist, not only on the 0th coordinate, but in \( M \). Then, we conclude that there are \( m_1, m_2 \in \pi^{-1}(x) \), where \( m_1(0) \neq m_2(0) \). This is indeed what was needed.

Let \( f \) as in Lemma 2.9 applied to \( h \).

**Claim 2.11.** \( f \) may be taken integer-valued.

**Proof.** Note that if \( h = f \circ \sigma - f \), for some \( f \), then this also holds for \( f + c \), where \( c \in \mathbb{R} \) is some constant. For one element in \( M \) we may choose the value of \( f \), so we let \( f(m_0) = 0 \).

For \( m \in \mathcal{O}(m_0), m = \sigma^a m_0 \) for some \( a \in \mathbb{Z} \), and

\[
 f(\sigma^a m_0) = f(\sigma^a m_0) - f(m_0) = \sum_{n=1}^{a} f(\sigma^n m_0) - f(\sigma^{n-1} m_0)
 = \sum_{n=1}^{a} h(\sigma^{n-1} m_0)
 = \sum_{n=1}^{a} \sigma^{n-1} m_0(0) \in \mathbb{Z}
\]

since the summands are in \( \{-1, 0, 1\} \).

For \( m \in M \setminus \mathcal{O}(m_0) \), \( m \) is a limit of orbit points. As \( f \) is continuous and integer-valued on the orbit of \( m_0 \), \( f(m) \in \mathbb{Z} \) as well.

Note that the set \( G \) can be defined as

\[
 G = \{ x \in K : h \text{ assumes more than one value on } \pi^{-1}(x) \}.
\]

In the same sense, let \( F = \{ x \in K : f \text{ assumes more than one value on } \pi^{-1}(x) \} \).

**Claim 2.12.** \( F \) is closed.

**Proof.** Let \( (u_n)_{n \in \mathbb{N}} \to u \) be a sequence in \( F \). We need to show that \( u \in F \).

For each \( u_n \), we know that \( \pi^{-1}(u_n) \) has at least two points

\[
 m_{1,n}, m_{2,n} \in M
\]

with

\[
 f(m_{1,n}) \neq f(m_{2,n}).
\]

\( M \) is compact, and so we pass on to a subsequence \((a_n)_{n}\) such that we get convergent sequences

\[
 m_{j,a_n} \to m_j
\]

for \( n \to \infty, j = 1, 2 \).

As \( f \) is continuous, we have that \( f(m_{j,a_n}) \to f(m_j) \), but as these values for \( j = 1, 2 \) differ for every \( a_n \) and \( f \) is integer-valued, we must have \( f(m_1) \neq f(m_2) \) and so \( m_1 \neq m_2 \).
Also \( \pi \) is continuous, so
\[
\pi(m_1) = \lim_n \pi(m_{1,a_n}) = \lim_n u_{a_n} = u, \\
\pi(m_2) = \lim_n \pi(m_{2,a_n}) = \lim_n u_{a_n} = u.
\]
Hence \( m_1, m_2 \in \pi^{-1}(u) \), with \( f(m_1) \neq f(m_2) \) and we conclude that \( u \in F \), as desired.

Now we turn to the proof of Theorem 2.8.

\textit{Proof of Theorem 2.8.} Suppose that
\[
N(n) = \sum_{i=0}^{n-1} g(T^i x_0) = \sum_{i=0}^{n-1} m_0(i) = \sum_{i=0}^{n-1} h(\sigma^i m_0)
\]
is bounded and we will arrive at a contradiction. Note that boundedness of \( N(n) \) is exactly what we need to apply Lemma 2.9. Hence we apply Lemma 2.9 to the flow \((M, \sigma)\) with \( h \in C(M) \). By the lemma, there is some \( f \in C(M) \) with
\[
h(m) = m(0) = f(\sigma m) - f(m)
\]
for \( m \in M \). Let \( F \) and \( G \) be as above. Now, for \( x_1 \) in Assumption 3, \( \{m(0) : m \in \pi^{-1}(T^n x_1)\} \) is one point for all \( n \). Note that
\[\{m(0) : m \in \pi^{-1}(T^n x_1)\} = \{m(n) : m \in \pi^{-1}(x_1)\}.\]

Since all these sets are in fact single points, \( m(i) \) is one value for all \( i \in \mathbb{Z} \), where \( m \in \pi^{-1}(x_1) \). Hence \( \pi \) is one-to-one in the point \( x_1 \), and therefore \( x_1 \notin F \).

Now consider \( z \in K \) as in Assumption 4. \( z \in G \), so there is more than one point \( m_1(0) \neq m_2(0) \), for \( m_1, m_2 \in \pi^{-1}(z) \). By \((*)\) we know that
\[
f(\sigma m_1) = m_1(0) + f(m_1) \\
f(\sigma m_2) = m_2(0) + f(m_2)
\]
Assume \( f(\sigma m_1) = f(\sigma m_2) \), then it follows that \( f(m_2) \neq f(m_1) \) and hence \( z \in F \). On the other hand, assume \( f(m_2) = f(m_1) \), then \( f(\sigma m_1) \neq f(\sigma m_2) \) and it follows that \( Tz \in F \). So in any case, either \( z \) or \( Tz \) lies in \( F \).

Suppose \( z \in F \). Now Assumption 4 states that \( \{m(0) : m \in \pi^{-1}(T^n z)\} \) is a single point, for any \( n \in \mathbb{Z} - \{0\} \). Again, because
\[
\{m(0) : m \in \pi^{-1}(T^n z)\} = \{m(n) : m \in \pi^{-1}(z)\},
\]
and by \((*)\) we see that
\[
\vdots
\]
\[
m(-2) = f(\sigma^{-1}m) - f(\sigma^{-2}m) \tag{2a}
m(-1) = f(m) - f(\sigma^{-1}m) \tag{2b}
m(0) = f(\sigma m) - f(m) \tag{2c}
m(1) = f(\sigma^2 m) - f(\sigma m) \tag{2d}
m(2) = f(\sigma^3 m) - f(\sigma^2 m) \tag{2e}
\]
\[
\vdots
\]
for \( m \in \pi^{-1}(z) \). Because \( z \in F \), \( f(m_1) \neq f(m_2) \). But \( m_1(-1) = m_2(-1) \) is a single value by Assumption 4, so by equation 2b, \( f(\sigma^{-1}m_1) \neq f(\sigma^{-1}m_2) \). Repeating this argument, because \( m(n), n < 0 \) is a single value, we conclude \( f(\sigma^n m_1) \neq f(\sigma^n m_2) \) for \( n < 0 \). Hence, \( \{f(m) : m \in \pi^{-1}(T^n(z)) \} \) is more than one point for \( n < 0 \) and so \( T^n(z) \in F \). However, \( \{T^n(z) : n < 0 \} \) is dense in \( K \), and \( F \) is closed, so \( F = K \). But \( x_1 \notin F \), so we have a contradiction.

We remain to show the same for the case \( Tz \in F \). The argument goes similarly. \( \{f(m) : m \in \pi^{-1}(Tz) \} \) is more than one point, but this is the same as \( \{f(\sigma m) : m \in \pi^{-1}(z) \} \) is more than one point. Now, \( m(1) \) is a single point for \( m \in \pi^{-1}(z) \), so by equation 2d, \( f(\sigma^2 m) \) attains more than one value. Proceeding, we have that \( \{f(\sigma^n m) : m \in \pi^{-1}(z) \} \) is more than one point for \( n > 0 \). Hence, for \( n > 0 \), \( T^n z \in F \). Again this set is dense in \( K \), and by closedness of \( F \), \( F = K \), a contradiction.

Therefore, such an \( f \) cannot exist, and \( N(n) \) has to be unbounded. \( \square \)

Theorem 2.8 has a nice corollary for the irrational rotation on the circle \((K,T)\).

It is this corollary that is needed in the discovery of a prime flow.

**Corollary 2.13.** Let \( K = [0,1) \) denote the circle as a compact group with addition mod 1, and pick an irrational \( \alpha \in K \) and 0 \( \neq \beta \in K \). Fix \( \gamma, \gamma' \in K \) and set \( A = [\gamma, \gamma + \beta) \) and \( B = [\gamma', \gamma' + \beta) \). Then

\[
N(n) = \sum_{i=0}^{n} \chi_A(i\alpha) - \chi_B(i\alpha)
\]

is bounded for \( n > 0 \) precisely if \( \beta \in \mathbb{Z}\alpha \) or \( \gamma - \gamma' \in \mathbb{Z}\alpha \).

The proof of the corollary is given in Appendix A.

Note that when \( \gamma - \gamma' \in \mathbb{Z}\alpha \), there is a \( k \in \mathbb{Z} \) such that \( T^k A = B \). Thus by \( k \) iterations, the intervals coincide. On the other hand, when \( \beta \in \mathbb{Z}\alpha \), note that \( \chi_A - \chi_B = \chi_{[\gamma, \gamma')} - \chi_{[\gamma + \beta, \gamma' + \beta)} \), and there is a \( k \) such that \( T^k[\gamma, \gamma') = [\gamma + \beta, \gamma' + \beta] \), we again have that \( T^k A = B \). Precisely then, the quantity \( N(n) \) is bounded. Whenever there is no such \( k \) to ‘match up’ the intervals, it turns out that \( N(n) \) is unbounded. The author finds the proof of this ‘bounded-unboundedness phenomenon’ through symbolic dynamics rather inconvenient and has put great effort in establishing a more elementary, insightful proof of this phenomenon. Unfortunately, no results have been established yet, in this direction.

**2.3 A separate result**

We finish this section with a seperate result, which is a generalization of a result from Kesten [3]. This theorem is easily proven with Lemma 2.9. The very short and elegant proof is presented the same as it is in [2].

**Theorem 2.14.** Let \((K,T)\) be a minimal flow, and fix \( x_0 \in K \) and \( A \subseteq K \). Define \( m_0(n) = \chi_A(T^n x_0) \), and let \( M \) be the \( \sigma \)-orbit closure of \( m_0 \). If \((M,\sigma)\) is minimal, then

\[
\sum_{i=0}^{n-1} (\chi_A(T^i x_0) - \delta)
\]
is bounded for \( n > 0 \), only if \( e^{2\pi i \delta} \) is the eigenvalue of a continuous eigenfunction of the flow \((M, \sigma)\).

**Proof.** Assume that \( \sum_{i=0}^{n-1} (\chi_A(T^i x_0) - \delta) \) is indeed bounded. Then, we apply Lemma 2.9 with \( g(m) := m(0) - \delta \) for \( m \in M \), and we conclude that there exists an \( h \in C(M) \) with \( h(\sigma m) - h(m) = g(m) \) for \( m \in M \). On the orbit of \( m_0 \) this reduces to

\[
h(\sigma^{n+1} m_0) - h(\sigma^n m_0) = \chi_A(T^n x_0) - \delta
\]

or

\[
e^{2\pi i (h(\sigma^{n+1} m_0) - h(\sigma^n m_0))} = e^{2\pi i (\chi_A(T^n x_0) - \delta)} = e^{-2\pi i \delta}.
\]

Since \( \{\sigma^n m_0 \mid n \in \mathbb{Z}\} \) is dense in \( M \), and the exponent and \( h \) are continuous functions, the identity above holds for all \( m \in M \). We rewrite this as

\[
e^{2\pi i (-h(\sigma m))} = e^{2\pi i \delta} e^{2\pi i (-h(m))}.
\]

Hence, with \( H(m) = e^{-2\pi i h(m)} \), we have \( H(\sigma m) = e^{2\pi i \delta} H(m) \). That is, \( e^{2\pi i \delta} \) is an eigenvalue of the flow \((M, \sigma)\) with continuous eigenfunction \( H(m) \).

This theorem gives answer to the consideration in the introduction. If we apply Theorem 2.14 to the circle \((K, T_\alpha)\), then

\[
\sum_{i=0}^{n-1} (\chi_{[0, \beta]}(n\alpha) - \beta)
\]

is bounded only if \( e^{2\pi i \delta} \) is an eigenvalue of \((X, \sigma)\). The eigenvalues of \((X, \sigma)\) are \( e^{2\pi i k \alpha} \) for \( k \in \mathbb{Z} \). It follows that for \( \beta \not\in \mathbb{Z} \alpha \), the sum remains unbounded.
3 The flow \((Y,\Psi)\)

In this section it is shown that \((Y,\Psi)\) is a prime flow. The first subsection shows that this flow is a POD flow. In the second subsection, it is shown that the POD condition implies primality. Furthermore, some results on how POD flows relate to (other) prime flows are presented.

Recall that \(Y\) was constructed by creating a homeomorphic copy \(A\) of \(B = \{x \in X | x(0) = 1\} \subset X\), where \((X,\sigma)\) is the flow on the symbolic bisequences, generated by \(x_0 = (\chi_{[0,\beta]}(n\alpha))_{n \in \mathbb{Z}}, n\alpha \in K\), being a rotation on the circle. \(Y = X \cup A\) was then defined with \(\Psi : Y \to Y\) the mapping as in Section 3. Also recall that \(\rho : X \to K\), was a homomorphism of the symbolic space \(X\) onto the circle \(K\).

3.1 Proximal orbit density

Before heading to the POD property, we first obtain a preliminary result; namely \((Y,\Psi)\) is topologically weakly mixing minimal. We need some known theory to establish this.

**Definition 3.1.** Let \((X,T)\) be a flow. Then \((X,T)\) is called topologically weakly mixing if for any nonempty open subsets \(A,B,C,D \subset X\) there is an \(n \in \mathbb{Z}\) such that \(T^n A \cap C \neq \emptyset\) and \(T^n B \cap D \neq \emptyset\).

**Definition 3.2.** A flow \((X,T)\) is called isometric if there is a metric \(d\) on \(X\) such that for all \(n \in \mathbb{Z}\), \(d(x,y) = d(T^n x,T^n y)\) for \(x,y \in X\).

**Theorem 3.3.** Let \((X,T)\) be a minimal flow. Then either \(X\) has a non-trivial isometric factor, or \(X\) is topologically weakly mixing.

This theorem is quite deep; its proof is beyond the scope of this document. For a proof, see [5]. A more clear proof can be found at Terence Tao’s blog, 254A, Lecture 7 [8].

**Proposition 3.4.** Every minimal isometric system \((X,T)\) is a Kronecker system, i.e. isomorphic to an abelian group rotation \((K,x \mapsto x + \alpha)\).

Again, for a proof, see Terence Tao’s 254A, lecture 6 [8]. We can now prove the characterization of topological weak mixing that is used.

**Proposition 3.5.** A minimal flow \((X,T)\) is topologically weakly mixing iff \(\lambda = 1\) is the only eigenvalue of this flow, and consequently, the continuous eigenfunctions are the constants.

**Proof.** We show that for \((X,T)\) not topologically weakly mixing, there are eigenvalues \(\lambda = e^{i\alpha}, 0 \neq \alpha < 2\pi\).

By Theorem 3.3, let \((Y,S)\) be the isometric factor. In fact, by Proposition 3.4, let this \((Y,S) = (K,T_\alpha)\), where \(K = \{z \in \mathbb{C} : |z| = 1\}\), and \(T_\alpha(z) = e^{i\alpha}z\) for such an \(\alpha \in (0,2\pi)\) that this flow is indeed a factor of \((X,T)\). Also, let \(\rho : (X,T) \to (K,T_\alpha)\) be the factor map.

Now we note that the function

\[ f : K \to \mathbb{C} \quad f(z) = z \]
is a nonconstant eigenfunction for the eigenvalue $\lambda = e^{i\alpha}$. Indeed,

$$f(T_\alpha z) = f(e^{i\alpha}z) = e^{i\alpha}f(z)$$

for all $z \in K$. Hence $f$ is a nonconstant eigenfunction for $(K, T_\alpha)$. Now define

$$g : X \to \mathbb{C} \quad g = f \circ \rho.$$

Then,

$$g(Tx) = f(\rho(Tx)) = f(T_\alpha(\rho(x))) = e^{i\alpha}f(\rho(x)) = e^{i\alpha}g(x).$$

Hence, $g$ is an eigenfunction on $(X, T)$ for the eigenvalue $e^{i\alpha}$, which was to show. Conversely, Let $\lambda \neq 1$ and $f$ be such that $f(Tx) = \lambda f(x)$. It is easy to see that $|\lambda| = 1$. Note that $|f(T^n x)| = |\lambda^n f(x)| = |f(x)|$ for all $n \in \mathbb{Z}$, $f$ is continuous and $\{T^n x\}$ is dense, hence $|f(x)|$ is constant for all $x \in X$. We might set this modulus to 1 as well, and arrive at $f(X) \subseteq K$, where $K$ is again the unit circle in the complex plane. Now, the flow $(f(X), k \mapsto \lambda k)$ is a nontrivial factor of $(X, T)$, and it is the irrational rotation on the circle for $\lambda = e^{i\alpha}$ with $\alpha$ irrational, for rational $\alpha$, the flow is a rotation on finitely many points. In both cases, we find an isometric factor, hence by Theorem 3.3, $(X, T)$ is not topologically weakly mixing.

\[\square\]

**Theorem 3.6.** $(Y, \Psi)$ is topologically weakly mixing and minimal.

**Proof.** Minimality of $(Y, \Psi)$ follows from the minimality of $(X, \sigma)$. For, the $\sigma$-orbit of any point is dense in $X$. As $(Y, \Psi)$ is only taking a ‘detour’ through $A$, it follows that the $\Psi$-orbit of any $x \in X$ is dense in $Y$. Of course for $x \in A$, $\Psi^{-1}x \in X$, so these points also have a dense $\Psi$-orbit. We conclude that $(Y, \Psi)$ is minimal.

Let $g$ be an eigenfunction

$$g(\Psi y) = \lambda g(y) \text{ for } y \in Y.$$

Let $h := g|_X$ be the restriction of $g$ to $X$, and define $u(x) = 1 + x(0)$ for $x \in X$. Now,

$$h(\sigma x) = \lambda u(x)h(x) \text{ for } x \in X.$$

Note that indeed $u(x) = 1, 2$, accordingly to whether $x \in B$ or not. By Theorem 2.7(d), $x_0, \bar{x}_0$ are doubly proximal, so let $(n_k)_k$ be an increasing sequence of positive integers, and $(m_k)_k$ be an decreasing sequence of negative integers such that

$$\lim_{k} \sigma^{n_k}x_0 = \lim_{k} \sigma^{n_k}\bar{x}_0, \quad \lim_{k} \sigma^{m_k}x_0 = \lim_{k} \sigma^{m_k}\bar{x}_0.$$ 

Note that $u(\sigma^n x_0) = u(\sigma^n \bar{x}_0)$ for $n \neq 0$. Now for $n > 0$,

$$\frac{h(\sigma^n x_0)}{h(x_0)} = \frac{h(\sigma x_0)}{h(\sigma^{n-1} x_0)} \cdots \frac{h(\sigma x_0)}{h(x_0)} = \lambda u(\sigma^{n-1} x_0) \cdots \lambda u(\sigma x_0) \lambda u(x_0)$$

$$\quad = \lambda^{n-1}\sigma^{n-1}x_0 \cdots \lambda u(\sigma x_0) \lambda u(x_0)$$

$$\quad = \frac{h(\sigma^n \bar{x}_0)}{h(\sigma^{n-1} \bar{x}_0)} \cdots \frac{h(\sigma \bar{x}_0)}{h(\bar{x}_0)} \lambda^{-u(\bar{x}_0)} \lambda^{-u(x_0)}$$

$$\quad = \frac{h(\sigma^n \bar{x}_0)}{h(\bar{x}_0)} \lambda^{u(x_0)-u(\bar{x}_0)}$$
and so, for $n_k \to \infty$ ($h$ is continuous),

$$\frac{1}{h(x_0)} = \frac{1}{h(\bar{x}_0)} \lambda^{u(x_0) - u(x_0)}$$

So we see that $h(x_0) = \lambda^{u(x_0) - u(x_0)} h(\bar{x}_0) = \lambda^{-1} h(\bar{x}_0)$.

Similarly it follows for $m < 0$ that

$$h(x_0) = \frac{h(\sigma^m x_0)}{h(\sigma^m \bar{x}_0)} = \frac{\lambda^{u(\sigma^m x_0)} \cdots \lambda^{u(\sigma^{1} x_0)}}{h(\sigma^m \bar{x}_0)} = \frac{\lambda^{u(\sigma^{m+1} x_0)} \cdots h(\bar{x}_0)}{h(\sigma^m \bar{x}_0)}$$

and so, for $m_k \to -\infty$,

$$h(x_0) = h(\bar{x}_0)$$

It follows that $\lambda = 1$, and since $(Y, \Psi)$ is minimal, $g$ is constant. We conclude that $(Y, \Psi)$ is topologically weakly mixing.

The above result is used to show that the flow $(Y, \Psi)$ is totally minimal.

**Theorem 3.7.** Let $(X, T)$ be a weakly mixing minimal flow on a compact metric space $X$. Then $(X, T)$ is totally minimal.

**Proof.** Suppose on the contrary that $(X, T)$ is not totally minimal. Then there is an integer $p > 1$ for which there is a nonempty closed invariant subset $A \neq X$, hence $T^p A \subseteq A$, where $A$ can be chosen such that $A \cap T^i A = \emptyset$ for $1 \leq i \leq p - 1$. We have

$$A \supset T^p A \supseteq T^{2p} \supseteq \ldots$$

Since $X$ is minimal,

$$\bigcup_{n \in \mathbb{Z}} T^n A = X = A \cup TA \cup \cdots \cup T^{p-1} A.$$  

and these are mutually disjoint sets. Now pick $C \subseteq A$, with $C$ open. Pick $D \subseteq T^k A$ for some $1 \leq k \leq p - 1$, with $D$ open. Since $(X, T)$ is weakly mixing, there is an $n \in \mathbb{Z}$ such that

$$T^n C \cap C \neq \emptyset \quad T^n C \cap D \neq \emptyset.$$  

This implies on one hand that $n = q_1 p$ and on the other hand that $n = q_2 p + k$ for integers $q_1, q_2$, a contradiction. Hence $(X, T)$ has to be totally minimal.

Now, we turn to proximal orbit density. Recall the definition of the POD condition:
**Definition 1.9.** A flow \((Y, \Psi)\) is a proximal orbit dense (POD) flow if it is totally minimal, and whenever \(x, y \in Y\) with \(x \neq y\), then for some \(n \neq 0\), \(\Psi^n y\) is proximal to \(x\).

It is now shown that \((Y, \Psi)\) is a POD flow. We establish the proof step by step with the upcoming lemma’s.

Let \(\theta : Y \to X\) be the projection, that is, 
\[
\theta(x) = \begin{cases} 
  x & \text{for } x \in X \\
  \Psi^{-1}x & \text{for } x \in A 
\end{cases}
\]
and let \(\tau := \rho \circ \theta\), which first projects \(Y\) onto \(X\) and then \(X\) onto the circle. \(\theta, \tau\) are not homomorphisms. In the following diagram, \(\rho\) is the only homomorphism.

\[
\begin{array}{c}
(Y, \Psi) \xrightarrow{\Psi \circ \Psi^2} (Y, \Psi) \\
\downarrow \theta \quad \downarrow \theta \\
(X, \sigma) \xrightarrow{\sigma} (X, \sigma) \\
\downarrow \rho \quad \downarrow \rho \\
(K, T_\alpha) \xrightarrow{T_\alpha} (K, T_\alpha)
\end{array}
\]

**Lemma 3.8.** Let \(x, y \in Y\) and suppose that \(\tau(x) - \tau(y) = k\alpha\) for some \(k \geq 0\). Then there is an integer \(m\) with \(k \leq m \leq 2k\) such that \(\tau(\Psi^m y) = \tau(x)\).

**Proof.** Since for any \(y \in Y\), \(\tau(\Psi^2 y) = \tau(y) + \lambda\alpha\) for \(\lambda\) either 1 or 2, we apply this some \(m_0\) times until
\[
\tau(\Psi^{2m_0} y) = \tau(y) + k_0\alpha,
\]
where \(k_0\) is either \(k\) or \(k + 1\). If \(k_0 = k\), pick \(m = 2m_0\). For \(k_0 = k + 1\), note that
\[
\tau(\Psi^{2m_0-1} y) = \tau(y) + k\alpha,
\]
hence, pick \(m = 2m_0 - 1\). The inequalities are self-evident. \(\square\)

**Proposition 3.9.** Let \(x, y \in Y\), \(x \neq y\), and \(\tau(x) - \tau(y) = k\alpha\) for some \(k \geq 0\). Then for some \(n \neq 0\), \(x\) is asymptotic to \(\Psi^n y\).

**Proof.** By Lemma 3.8, let \(m \geq 0\) be such that \(\tau(\Psi^m y) = \tau(x)\). Recall the set \(E = \mathbb{Z}\alpha \cup \beta + \mathbb{Z}\alpha\), the set of points where \(\rho : X \to K\) is not one-to-one. We make the distinction between \(\tau(x) \in E\) and \(\tau(x) \not\in E\).

Suppose \(\tau(x) \not\in E\). Then we know that
\[
\tau^{-1}(\tau(x)) = \begin{cases} 
  \{x, \Psi^{-1}x\} & \text{for } x \in A \\
  \{x, \Psi x\} & \text{for } x \in B \\
  \{x\} & \text{for } x \in X - B 
\end{cases}
\]

Hence we must have that \(\Psi^n y = x\), \(\Psi^n y = \Psi x\) or \(\Psi^n y = \Psi^{-1} x\). The proposition follows by choosing \(n \in \{m - 1, m, m + 1\}\) accordingly. Then, \(\Psi^n y = x\), and equal points are certainly asymptotic.
Now suppose $\tau(x) \in E$. Then, recall that $\rho^{-1}(x) = \{y, \bar{y}\}$, where $y(n) = \bar{y}(n)$ for all $n \in \mathbb{Z} - \{j\}$, and $j$ the integer such that $y(j) = 1$ and $\bar{y}(j) = 0$. Now,

$$\tau^{-1}(\tau(x)) = \begin{cases}
\{y, \Psi y, \bar{y}, \Psi \bar{y}\} & \text{for } y \in B, \bar{y} \in B \\
\{y, \Psi y, \bar{y}\} & \text{for } y \in B, \bar{y} \notin B \ (\text{this is when } j = 0) \\
\{y, \bar{y}\} & \text{for } y \notin B, \bar{y} \notin B
\end{cases}$$

Note that $y \notin B, \bar{y} \notin B$ is not included since that never happens. Now since $x, \Psi^m y \in \tau^{-1}(\tau(x))$, we need to verify that given any of the above situations, we find an $n$ such that $x$ and $\Psi^n y$ are asymptotic. As it turns out, we need to choose $n \in \{m - 1, m, m + 1\}$ correctly for the proposition to follow.

For $y \notin B, \bar{y} \notin B$, these two points are positively asymptotic if $j < 0$ and negatively asymptotic if $j > 0$. For, one observes that these points lie in the same set $X - B$. As all the iterates in the forward (backward) $\sigma$-orbit lie in the same ($B$ or $X - B$) set, points of the the forward (backward) $\Psi$-orbit lie in the same ($A, B$ or $X - B$) set as well. Now, $y, \bar{y}$ are asymptotic in $(X, \sigma)$, and hence the points are are also asymptotic in $(Y, \Psi)$. One chooses $n = m$, in this case. For $y \in B, \bar{y} \in B, \{y, \Psi y\}$ and $\{\bar{y}, \Psi \bar{y}\}$ are clear, since one picks $n = m + 1$ accordingly. Then we have equal points again and they obviously are asymptotic. $\{y, \bar{y}\}$ and $\{\Psi y, \Psi \bar{y}\}$ are positively (negatively) asymptotic when $j < 0 \ (j > 0)$, as by previous reasoning, the iterates of the forward (backward) orbit lie in the same set and are asymptotic in $(X, \sigma)$, hence also in $(Y, \Psi)$.

The remaining points to be verified asymptotic are $\{y, \Psi \bar{y}\}$ and $\{\bar{y}, \Psi y\}$. But now, by choosing $n = m + 1$, correctly, we are back to the previous case. Hence in this situation one chooses $n$ correctly and the proposition follows.

Note that for $y \in B, \bar{y} \notin B$ (i.e. $j = 0$), above reasoning applies and one finds $n \in \{m - 1, m, m + 1\}$ for the proposition once again.

By the above, we have established the POD condition for points $x, y \in Y$, with $x \neq y$ and $\tau(x) - \tau(y) \in \mathbb{Z} \alpha$, as asymptotic implies proximal. Now, we turn to the points $x_1, y_1 \in Y$ with $\tau(x_1) - \tau(y_1) \notin \mathbb{Z} \alpha$.

**Lemma 3.10.** Let $x \in Y$. For $n > 0$, define $c_n(x) = \sum_{k=1}^{n} \chi_X(\Psi^k x)$. Then we have

$$\sum_{j=0}^{c_n(x)} \theta x(j) = \chi_A(x) + \sum_{j=0}^{n} \chi_B(\Psi^j x).$$

Proving this lemma is a matter of careful counting.

**Proof.** Note that for $x \in X$, $\theta x = x$ and so

$$\sum_{j=0}^{c_n(x)} \theta x(j) = \sum_{j=0}^{c_n(x)} x(j) = \sum_{j=0}^{c_n(x)} \sigma^j x(0) = \sum_{j=0}^{c_n(x)} \chi_B(\sigma^j x) \ (n > 0).$$

Now notice that $c_n(x)$ counts how many times $x$ is in the set $X$ for $n$ iterates forward. In fact, $n - c_n(x)$ is the amount of times these iterates are in $A$. As
\[ \Psi^2 = \sigma \text{ for } x \in B \text{ (and hence } \Psi x \in A) \text{ and } \Psi = \sigma \text{ for } x \in X - B, \text{ we may as well take the sum as follows,} \]

\[ \sum_{j=0}^{c_n(x)} \chi_B(\sigma^j x) = \sum_{j=0}^{n} \chi_B(\Psi^j x) \quad (n > 0). \]

We have added \( n - c_n(x) \) terms. They correspond to the iterates that are in \( A \). Since \( \chi_A(x) = 0 \) for \( x \in X \), the lemma follows in this case. The remaining case is \( x \in A \). In that case, \( \Psi^{-1} x \in B \), and we apply above result.

\[ \sum_{j=0}^{c_n(\Psi^{-1} x)} \theta \Psi^{-1} x(j) = \chi_A(\Psi^{-1} x) + \sum_{j=0}^{n} \chi_B(\Psi^j \Psi^{-1} x). \]

Now,

\[ c_n(\Psi^{-1} x) = \sum_{k=0}^{n-1} \chi_X(\Psi^{k-1} x) = \sum_{k=0}^{n-1} \chi_X(\Psi^k x) = \sum_{k=0}^{n-1} \chi_X(\Psi^k x) = c_{n-1}(x) \quad (x \in A) \]

and \( \theta \Psi^{-1} x(j) = \theta x(j), \chi_A(\Psi^{-1} x) = 0 \) and

\[ \sum_{j=0}^{n} \chi_B(\Psi^j \Psi^{-1} x) = 1 + \sum_{j=0}^{n} \chi_B(\Psi^j x) = 1 + \sum_{j=0}^{n-1} \chi_B(\Psi^j x). \]

Combining these results in

\[ \sum_{j=0}^{c_{n-1}(x)} \theta x(j) = \chi_A(x) + \sum_{j=0}^{n-1} \chi_B(\Psi^j x). \]

As the intermediate result is for all \( n > 1 \), the lemma also follows for \( x \in A, n > 0 \).

**Lemma 3.11.** \( c_n(x_1) - c_n(y_1) \) is unbounded as a function of \( n \in \mathbb{Z}^+ \).

Suppose, for a contradiction, that

\[ \left| \sum_{j=0}^{n-1} \chi_X(\Psi^j x_1) - \chi_X(\Psi^j y_1) \right| \leq D \]

for all \( n \in \mathbb{Z}^+ \). We have in fact that

\[ \left| \sum_{j=0}^{n-1} \chi_B(\Psi^j x_1) - \chi_B(\Psi^j y_1) \right| = \left| \sum_{j=1}^{n} \chi_A(\Psi^j x_1) - \chi_A(\Psi^j y_1) \right| \]

\[ = \left| \sum_{j=1}^{n} \chi_X(\Psi^j x_1) - \chi_X(\Psi^j y_1) \right| \leq D \]

Fix \( P > 1 \). Now, there is some \( n > 1 \) such that \( \sum_{j=1}^{n} \chi_X(\Psi^j x_1) = P \). Furthermore, there is some \( n' > 1 \) such that \( \sum_{j=1}^{n'} \chi_X(\Psi^j y_1) = P \). These \( n \) and \( n' \) are not too far apart, we show that \( |n - n'| \leq 2D \).
For $x \in A$, note that $\Psi^{-1}x, \Psi x \in X$. This leads to

$$\sum_{j=1}^{n} \chi_A(\Psi^j z) \leq \lceil n/2 \rceil$$

for any $n > 1$, $z \in Y$. Now note that

$$|n - n'| = \sum_{j=1}^{n} 1 - \sum_{j=1}^{n'} 1$$

$$= \sum_{j=1}^{n} (\chi_A(\Psi^j x_1) + \chi_A(\Psi^j y_1)) - \sum_{j=1}^{n'} (\chi_A(\Psi^j y_1) + \chi_A(\Psi^j y_1))$$

$$= P + \sum_{j=1}^{n} \chi_A(\Psi^j x_1) - P - \sum_{j=1}^{n'} \chi_A(\Psi^j y_1)$$

$$= \min(n, n') \sum_{j=1}^{n} (\chi_A(\Psi^j x_1) - \chi_A(\Psi^j y_1)) + \sum_{j=\max(n, n') + 1}^{\max(n, n')} \chi_A(\Psi^j z)$$

$$\leq D + \frac{1}{2} (\max(n, n') - \min(n, n'))$$

$$= D + \frac{1}{2} |n - n'|.$$

where $z = x_1$ or $y_1$, accordingly. We do not need to consider the odd case, since $n$ and $n'$ can be chosen such that $\Psi^n x_1 \in X$ and $\Psi^{n'} y_1 \in X$. It follows that $|n - n'| \leq 2D$.

Now apply Lemma 3.10 to $x_1$ with $n$:

$$\sum_{j=0}^{c_n(x_1)} \theta x_1(j) = \chi_A(x_1) + \sum_{j=0}^{n} \chi_B(\Psi^j x_1).$$

Note that $c_n(x_1) = P$. Similarly, we apply the lemma for $y_1$ with $n'$:

$$\sum_{j=0}^{c_{n'}(y_1)} \theta y_1(j) = \chi_A(y_1) + \sum_{j=0}^{n'} \chi_B(\Psi^j y_1).$$

Note also that $c_{n'}(y_1) = P$. Combining the two we estimate

$$\left| \sum_{j=0}^{P} (\theta x_1(j) - \theta y_1(j)) \right| = \chi_A(x_1) + \sum_{j=0}^{n} \chi_B(\Psi^j x_1) - \chi_A(y_1) - \sum_{j=0}^{n'} \chi_B(\Psi^j y_1)$$

$$\leq 2 + \sum_{j=0}^{\min(n, n')} \left( \chi_B(\Psi^j x_1) - \chi_B(\Psi^j y_1) \right) + \sum_{j=\max(n, n') + 1}^{\max(n, n')} \chi_B(\Psi^j z)$$

$$\leq 2 + D + \sum_{j=\min(n, n') + 1}^{\max(n, n')} 1 = 2 + D + |n - n'|$$

$$\leq 2 + 3D$$ (3)
for \( z = x_1 \) or \( y_1 \), correctly. Since \( P \) was arbitrary, this is in contradiction with Corollary 2.13. To see a concrete counterexample, let \( \tau(x_1) = s \) and \( \tau(y_1) = t \). Then note that \( s - t \not\in \mathbb{Z}\alpha \). Let, for instance,

\[
\begin{align*}
x_1 &= (\chi_{[0,\beta]}(s + i\alpha))_{i \in \mathbb{Z}} \\
y_1 &= (\chi_{[0,\beta]}(t + i\alpha))_{i \in \mathbb{Z}}.
\end{align*}
\]

Then, of course

\[
\begin{align*}
x_1 &= (\chi_{[-s,-s+\beta]}(i\alpha))_{i \in \mathbb{Z}} \\
y_1 &= (\chi_{[-t,-t+\beta]}(i\alpha))_{i \in \mathbb{Z}}.
\end{align*}
\]

And by \( s - t \not\in \mathbb{Z}\alpha \) we have (for \( \beta \not\in \mathbb{Z}\alpha \)) that \( N(n) \) is unbounded. However, \( N(n) \) is bounded in inequality 3, hence a contradiction.

**Lemma 3.12.** For any \( \omega \in K \) there exist \( (x_2, y_2) \in \overline{O_{\Psi \times \Psi}(x_1, y_1)} \) with \( \tau(x_2) - \tau(y_2) = \omega \).

**Proof.** With induction, we show that \( \tau(\Psi^n x) = \tau(x) + c_n(x)\alpha, n \geq 1 \). For \( n = 1 \),

\[
\tau(\Psi x) = \begin{cases} 
\tau(x) + \alpha & \text{for } x \in A \\
\tau(x) & \text{for } x \in B \\
\tau(x) + \alpha & \text{for } x \in X - B
\end{cases}
\]

and \( \tau(\Psi x) = \tau(x) + c_1(x)\alpha \) follows. Now suppose \( \tau(\Psi^n x) = \tau(x) + c_n(x)\alpha \) for some \( n \), and consider

\[
\tau(\Psi^{n+1} x) = \begin{cases} 
\tau(\Psi^n x) & \text{for } \Psi^n x \in B \\
\tau(\Psi^n x) + \alpha & \text{for } \Psi^n x \not\in B
\end{cases}
\]

\[
\begin{align*}
&= \begin{cases} 
\tau(x) + \alpha \sum_{k=1}^{n} \chi_{\Psi^k x} & \text{for } \Psi^n x \in B \\
\tau(x) + \alpha \left( \sum_{k=1}^{n} \chi_{\Psi^k x} + 1 \right) & \text{for } \Psi^n x \not\in B
\end{cases} \\
&= \begin{cases} 
\tau(x) + \alpha \sum_{k=1}^{n+1} \chi_{\Psi^k x} & \text{for } \Psi^n x \in B \\
\tau(x) + \alpha \left( \sum_{k=1}^{n+1} \chi_{\Psi^k x} \right) & \text{for } \Psi^n x \not\in B
\end{cases} \\
&= \tau(x) + c_{n+1}(x)\alpha.
\end{align*}
\]

Now the rest of the proof is straightforward. Fix \( \omega \in K \). We have

\[
\tau(\Psi^n x_1) - \tau(\Psi^n y_1) = \tau(x_1) - \tau(y_1) + (c_n(x_1) - c_n(y_1))\alpha.
\]

\( c_n(x_1) - c_n(y_1) \) is unbounded and changes in increments of \( \pm 1 \), so its range contains either \( \mathbb{Z}^+ \) or \( \mathbb{Z}^- \). \( \mathbb{Z}^+\alpha \) and \( \mathbb{Z}^-\alpha \) are both dense in \( K \), so we find a sequence \( (n_j)_j \) such that \( \tau(x_1) - \tau(y_1) + (c_n(x_1) - c_n(y_1))\alpha \to \omega \). Let \( (\Psi \times \Psi)^{n_j}(x_1, y_1) \to (x_2, y_2) \) by passing on to a subsequence if neccessary. Since \( \tau \) is continuous, \( x_2, y_2 \) will have the desired property. \( \square \)

**Proposition 3.13.** For \( \tau(x_1) - \tau(y_1) \not\in \mathbb{Z}\alpha \), there is some \( n \neq 0 \) such that \( \Psi^n y_1 \) is proximal to \( x_1 \).

**Proof.** By Lemma 3.12, let \( (x_2, y_2) \in \overline{O(x_1, y_1)} \) such that \( \tau(x_2) - \tau(y_2) \in \mathbb{Z}^+\alpha \). Let \( (n_j)_j \) be the sequence such that

\[
\begin{align*}
\Psi^{n_j} x_1 &\to x_2 \\
\Psi^{n_j} y_1 &\to y_2
\end{align*}
\]
By Proposition 3.9, there is some \( n \neq 0 \) for which \( x_2 \) is asymptotic to \( \Psi^n y_2 \). This means that, for any \( \epsilon > 0 \) there is some \( N \) such that whenever \( m > N \),
\[
d(\Psi^{n+m} y_2, \Psi^m x_2) < \epsilon
\]
Fix some \( \epsilon > 0 \). Choose \( m \) large enough such that the above holds. Then for some \( j \) (depending on \( m \)) large enough,
\[
d(\Psi^{n+m+j} x_1, \Psi^m x_2) < \epsilon
\]
\[
d(\Psi^{n+m+j} y_1, \Psi^{n+m} y_2) < \epsilon
\]
because \( \Psi \) is continuous. Combining the three, we get
\[
d(\Psi^{n+m+n+j} y_1, \Psi^{n+m} x_1) \leq d(\Psi^{n+m+n+j} y_1, \Psi^{n+m} y_2) + d(\Psi^{n+m} y_2, \Psi^m x_2) + d(\Psi^{n+m+j} x_1, \Psi^m x_2) < 3 \epsilon
\]
In conclusion, for any \( \epsilon > 0 \) we find a \( k \) such that
\[
d(\Psi^{n+k} y_1, \Psi^k x_1) < \epsilon.
\]
This induces a subsequence where \( \Psi^n y_1 \) and \( x_1 \) lie close together, hence they are proximal.

The Propositions 3.9 and 3.13 show that for all points \( x, y \in Y, x \neq y \), there is some \( n \neq 0 \) such that \( x \) and \( \Psi^n y \) are proximal. The flow is also totally minimal by Theorem 3.6, hence we conclude that \( (Y, \Psi) \) is a POD flow.

### 3.2 POD flows are prime

Now, we can make the final steps in discovering an infinite prime flow. We prove that any POD flow is in fact prime. As we have constructed the flow \( (Y, \Psi) \) to be POD and this flow is infinite, the discovery of an infinite prime flow is complete.

**Theorem 3.14.** Let \( (X, T) \) be a POD flow. Then \( (X, T) \) is prime.

**Proof.** Let \( (X, T) \) be a POD flow and consider a non-trivial factor,
\[
\pi : (X, T) \rightarrow (Z, S) \quad \pi(Tx) = S(\pi(x)) \quad x \in X.
\]
Suppose that this (surjective) homomorphism is not an isomorphism (that is, suppose \( (X, T) \) is not prime). Then, there are some \( x, y \in X, x \neq y \), with \( \pi(x) = \pi(y) \). Moreover, since \( (X, T) \) is POD, there is an \( n \in \mathbb{Z} \) such that \( T^n x \) is proximal to \( y \). This means that for some \( z \in X \), there is a sequence \( (m_j)_{j \in \mathbb{N}} \) such that
\[
\lim_{j \to \infty} T^{m_j} y = z = \lim_{j \to \infty} T^{m_j} T^n x.
\]
But, because \( (X, T) \) is minimal, we may pick any \( z \in X \). By continuity we have
\[
\lim_{j \to \infty} \pi(T^{m_j} y) = \pi(z) = \lim_{j \to \infty} \pi(T^{m_j} T^n x).
\]
Where the latter part can be rewritten as
\[
\lim_{j \to \infty} \pi(T^{m_j} x) = \pi(T^{-n} z).
\]
But notice that
\[ \pi(T^m x) = S^m(\pi(x)) = S^m(\pi(y)) = \pi(T^m y) \]
for all \( m \in \mathbb{Z} \). Hence \( (\pi(T^m x))_{j \in \mathbb{Z}} \) and \( (\pi(T^m y))_{j \in \mathbb{Z}} \) are in fact the same sequence and thus have the same limit (by uniqueness of limits),
\[ \pi(T^{-n} z) = \pi(z) \]
Now, as \((X, T)\) is minimal and \((Z, S)\) is a factor, \((Z, S)\) must also be minimal and hence, this flow is a rotation on finitely many points (namely, some divisor of \(|n|\)). This contradicts the fact that \((X, T)\) is totally minimal. Hence, \(\pi\) is an isomorphism and it follows that \((X, T)\) is prime.

One might wonder how POD flows relate to (other) prime flows in general. The main paper of interest [2] gives answer to this consideration in the following manner. The proofs of these statements are not in the scope of this project and are thus omitted.

**Definition 3.15.** Two minimal flows \((X, T)\) and \((Y, S)\) are disjoint iff the product flow \((X \times Y, T \times S)\) is minimal.

Note that if one of the flows is a factor of the other, then clearly those flows are not disjoint. For POD flows, the converse also turns out to hold:

**Theorem 3.16.** Let \((X, T)\) be a POD flow. If a minimal flow \((Z, S)\) is not an extension of \((X, T)\), then \((Z, S)\) and \((X, T)\) are disjoint.

*Proof.* See [2], Theorem 4.3.

**Corollary 3.17.** Let \((X, T)\) be a POD flow. If \((Z, S)\) is a prime flow, then it is either isomorphic to \((X, T)\) or the two flows are disjoint.

*Proof.* See [2], Corollary 4.4.

This corollary points out that within the class of prime flows, POD flows are disjoint from any other type of prime flow. In this document, only one type of prime flow is considered, namely the POD flows. In terms of disjointness, POD flows stand by themselves with respect to other prime flows.
4 Conclusion

In the past three sections, we have seen the how an infinite prime flow can be found, following the method in [2]. The key element in this discovery is the construction of flows with the POD property. It has been shown that POD flows are prime, this is proven in Section 3.

In view of the POD property, the ‘hard’ part of the construction is actually showing that for any two points, some iterate of the one is proximal to the other. This part has been established in Section 3, in two propositions. In order to be able to prove these propositions, a lot of theory on symbolic dynamics was needed, this has been handled in Section 2. As a consequence, a nice corollary has been found, which is called the ‘boundedness-unboundedness phenomenon’ in this document. This corollary gives rise to a train of thoughts on how the rates of convergence of sums as in the Mean Ergodic Theorem behave when one considers the difference of two intervals rather than one. This looks like a very interesting phenomenon, more research in this direction is certainly possible. Moreover, with the aid of a lemma proven in Section 2, a generalization of a result by Kesten was proven with a short and elegant proof.

By showing the existence of a class of POD flows and as those flows are prime, there is a specific type of flows identified within the class of prime flows. At the end of Section 3, a result is stated on how these POD flows relate to other prime flows in general. A subsequent challenge might be to find a flow that is prime without the need of the elaborate construction of a POD flow. Whenever this flow is not isomorphic to a POD flow, these flows are disjoint. This disjointness could be a starting point in finding other non-POD prime flows.
A Proof of the ‘bounded-unboundedness phenomenon’

In this Appendix, we prove Corollary 2.13. This proof is elaborate, some parts are similar to the proof of Theorem 2.8.

Corollary 2.13. Let $K = [0,1)$ denote the circle as a compact group with addition mod 1, and pick an irrational $\alpha \in K$ and $0 \neq \beta \in K$. Fix $\gamma, \gamma' \in K$ and set $A = [\gamma, \gamma + \beta)$ and $B = [\gamma', \gamma' + \beta)$. Then

$$N(n) = \sum_{i=0}^{n} \chi_A(i\alpha) - \chi_B(i\alpha)$$

is bounded for $n > 0$ precisely if $\beta \in Z\alpha$ or $\gamma - \gamma' \in Z\alpha$.

The case that indeed either $\beta \in Z\alpha$ or $\gamma - \gamma' \in Z\alpha$ is shown directly. For the converse, we can apply Theorem 2.8, but not in all cases. Note that for $G$ in the theorem, we now have $G = \{\gamma, \gamma', \gamma + \beta, \gamma' + \beta\}$. To fulfil assumption 4, we must assume that either $\gamma + k_1\alpha \neq \gamma' + \beta$ for all $k_1 \in Z - \{0\}$, or that $\gamma' + k_2\alpha \neq \gamma + \beta$ for all $k_2 \in Z - \{0\}$. If there would be $k_1, k_2$ so that in both cases equality holds, then $\gamma' + \beta \in O(\gamma)$ and $\gamma + \beta \in O(\gamma')$, and we could not fulfil assumption 4, hence the theorem is not applicable. Note that in the case of $\gamma' = \gamma + \beta$, the theorem can still be applied since $G$ now consists of 3 points instead of 4, where the point $\gamma'$ fulfils assumption 4.

For the case $\beta \notin Z\alpha$, $\gamma - \gamma' \notin Z\alpha$, $\gamma + k_1\alpha = \gamma' + \beta$ and $\gamma' + k_2\alpha = \gamma + \beta$, for some $k_1, k_2 \neq 0$, we prove the corollary without the use of the theorem. This part of the proof looks similar to the proof of Theorem 2.8.

Proof. Case $\beta \in Z\alpha$ or $\gamma - \gamma' \in Z\alpha$

First, let’s assume $\gamma - \gamma' \in Z\alpha$, that is, there is a $k \in Z$ such that $T^k A = B$. Then,

$$|N(n)| = \left| \sum_{i=0}^{n} \chi_A(i\alpha) - \chi_{T^k A}(i\alpha) \right|$$

$$= \left| \sum_{i=0}^{n} \chi_A(i\alpha) - \chi_A(i\alpha - k\alpha) \right|$$

$$= \left| \sum_{i=n-k+1}^{n} \chi_A(i\alpha) - \sum_{i=-k}^{n} \chi_A(i\alpha) \right|$$

$$\leq |k|$$

When $\beta \in Z\alpha$, the argument goes the same. By noting that $\chi_A - \chi_B = \chi_{[\gamma, \gamma')} - \chi_{[\gamma + \beta, \gamma' + \beta]}$, and there is a $k$ such that $T^k[\gamma, \gamma') = [\gamma + \beta, \gamma' + \beta)$, then, similarly, $|N(n)| \leq k$.

Case $\beta \notin Z\alpha$ and $\gamma - \gamma' \notin Z\alpha$, and $\gamma + k_1\alpha \neq \gamma' + \beta$ for all $k_1 \neq 0$.

We verify the assumptions of the theorem. As usual, let $m_0$ be the symbolic bisequence corresponding to $0 \in K$, and $M$ be the $\sigma$-orbit closure of $m_0$. Let $h(x) = \chi_A(x) - \chi_B(x)$ for $x \in K$. We begin with verifying assumption 2.
To this end, let $\sigma^m m_0 \to m_1$ for $i \to \infty$. For $\pi$ to be extended continuously, we must have that the sequence $(n_i \alpha)_i$ converges in $K$. Let henceforth $\xi, \zeta + \delta$ be clusterpoints of the sequence and we show that $\delta = 0$.

By convergence in $M$, we certainly have coordinatewise convergence, $h(n_i \alpha + na) = \sigma^m m_0(n) \to m_1(n)$ for any $n \in \mathbb{Z}$. Now, $\xi + na, \zeta + \delta + na$ are both clusterpoints of the sequence $(n_i + n \alpha)_i$, and whenever $\xi + na, \xi + \delta + na \notin G$, we have that $h(\xi + na) = h(\xi + \delta + na)$ for all $n \in \mathbb{Z}$. These sets are dense in $K$, so we must have $\delta = 0$, since this function $h = \chi_A - \chi_B$ cannot be shifted over a $\delta > 0$, such that $h(x_i) = h(x_i + \delta)$ for a dense set $\{x_i | i \in \mathbb{N}\} \subset K$. Hence, $\pi$ can be extended to the whole domain $M$, and assumption 2 is verified.

For assumption 1, let again $\sigma^m m_0 \to m_1$, and we show that $m_0 \in \overline{O(m_1)}$. Let $(k_i \alpha)_i$ be such that $(\pi(m_1) + k_i \alpha)_i$ goes to 0 from above. Now note that

$$\lim_{i} \sigma^{k_i} m_1(n) = \lim_{i} h(\pi(m_1) + k_i \alpha + na) = h(0 + na) = m_0(n)$$

for $n \in \mathbb{Z}$, where the last equation follows since $h$ is right continuous and the sequence goes to 0 from above.

In Lemma 2.2 we already saw how to prove that coordinatewise convergence implies convergence in the metric on $M$. By doing so we establish $m_0 \in \overline{O(m_1)}$, and assumption 1 is verified.

Assumption 3 holds trivially. Since $G$ is finite, a countable amount of points in $K$ do not fulfill assumption 3 (that is, the orbits of the points in $G$). There remain an uncountable amount of points in $K$ to pick $x_1$ from.

For assumption 4, choose $z = \gamma$. Since $\gamma + k_1 \alpha \neq \gamma' + \beta$ for $k_1 \neq 0$, we have $\gamma' + \beta \notin O(\gamma)$. Since $\beta \notin Z\alpha$, we have that $\gamma + \beta \notin O(z)$. At last, since $\gamma - \gamma' \notin Z\alpha$, we have $\gamma' \notin O(z)$. We conclude that $O(z) \cap G = \{z\}$.

Hence, Theorem 2.8 is applicable, and it follows that $N(n)$ is unbounded.

Case $\beta \notin Z\alpha$ and $\gamma - \gamma' \notin Z\alpha$, and $\gamma' + k_2 \alpha \neq \gamma + \beta$ for all $k_2 \neq 0$. Note that assumptions 1 to 3 still hold, only for assumption 4 we now pick $z = \gamma'$. In the same manner we conclude that $O(z) \cap G = \{z\}$ and it follows that $N(n)$ is unbounded.

Case $\beta \notin Z\alpha$ and $\gamma - \gamma' \notin Z\alpha$, and $\gamma + k_1 \alpha = \gamma' + \beta$ for some $k_1 \neq 0$, and $\gamma' + k_2 \alpha = \gamma + \beta$ for some $k_2 \neq 0$.

This is the last case where, as argued above, the theorem is not applicable. We need a separate argument. We focus on the point $\gamma + \beta$, and we note that for $k = -k_2$, $\gamma - k\alpha$ is the same point. Now, $\{m(n)|m \in \pi^{-1}(\gamma + \beta)\}$ is one point for $n \notin \{0, k\}$. For $n \in \{0, k\}$ we have $\gamma + \beta + n\alpha \in G$.

Suppose we have $\lim_{i} \sigma^m m_0 = m$, with $m \in \pi^{-1}(\gamma + \beta)$ If $n_i \alpha$ approaches $\gamma + \beta$ from below, then $m_1(0) = 1, m(k) = 0$, since $h(n_i \alpha)$ is constant for $i$ large enough. On the other hand, if from above, then $m(0)(1) = m(k) = -1$ analogously. Hence,

$$m_2(0) - m_1(0) = m_2(k) - m_1(k) = 1,$$

or, more explicitly,

$$m_2(0) = 1, \quad m_2(k) = 0$$

and

$$m_1(0) = 0, \quad m_1(k) = -1$$
for \( \{m_1, m_2\} = \pi^{-1}(\gamma + \beta) \).

Now we proceed as in the proof of Theorem 2.8. We still have \( K \neq F \). If \( N(n) \) was te be bounded, by Lemma 2.9 there is an \( f \in C(M) \) satisfying (*). For this \( f \), we have

\[
(f(\sigma^{k+1}m_2) - f(\sigma^{k+1}m_1)) - (f(m_2) - f(m_1))
\]

\[
= \sum_{i=0}^{k} (f(\sigma^{i+1}m_2) - f(\sigma^{i}m_2)) - \sum_{i=0}^{k} (f(\sigma^{i+1}m_1) - f(\sigma^{i}m_1))
\]

\[
= \sum_{i=0}^{k} \sigma^i m_2(0) - \sum_{i=0}^{k} \sigma^i m_1(0) \quad \text{(by (*))}
\]

\[
= \sum_{i=0}^{k} m_2(i) - \sum_{i=0}^{k} m_1(i)
\]

\[
= 2 \quad \text{(by (**))}
\]

and again as in the proof of the theorem, we conclude that either \( \pi(m_1) \) or \( \pi(\sigma^{k+1}m_1) \) is in \( F \). If \( \pi(m_1) = \gamma + \beta \in F \), then for \( k > 0 \), we know that

\[
\{m(n)|m \in \pi^{-1}(\gamma + \beta)\}
\]

is one point for \( n < 0 \). Hence, the set

\[
\{\pi(m_1) + na|n < 0\}
\]

is in \( F \), but this set is dense in \( K \), and by closedness of \( F \) we have a contradiction. For \( k < 0 \), we know that the set (4) is one point for \( n > 0 \). Then the set (5) is in \( F \) and dense in \( K \), contradiction.

At last, when \( \pi(\sigma^{k+1}m_1) \in F \), then for \( k > 0 \), we know that (4) is one point for \( n > k + 1 \). Hence, the set

\[
\{\pi(m_1) + na|n > k + 1\}
\]

is in \( F \), but this set is dense in \( K \), and by closedness of \( F \) we have a contradiction. For \( k < 0 \), we know that (4) is one point for \( n > 0 \). Then the set (5) is in \( U \) and dense in \( K \), contradiction.

For this last case we hence conclude that \( N(n) \) is unbounded, which concludes the proof of the corollary.
References


