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ABSTRACT

The damped nonlinear vibrations of thin-walled cylindrical shells with initial geometric imperfections are studied using Donnel -type nonlinear shallow-shell equations in which the appropriate damping and inertial terms are introduced. Analytical procedures similar to those of Ref. [1] are used here.

The results indicate that both the initial geometric imperfections and the magnitude of damping have significant influences on the nonlinear vibration behaviour of the shells. The results obtained so far provide qualitively satisfactory explanations of the available experimental results.

LIST OF SIMBOLS

A	Nondimensional amplitude function of the assumed driven mode
$A_0(t)$	Slowly varying amplitude function of driven mode
$\bar{A}$	Average value (over one period) of $A_0(t)$
B	Nondimensional amplitude function of the assumed companion mode
$B_0(t)$	Slowly varying amplitude function of companion mode
$\bar{B}$	Average value (over one period) of $B_0(t)$
C	Nondimensional amplitude function of the assumed vibration mode
c	Damping coefficient
E	Young's modulus
F	Total stress function $(=\hat{F} + \hat{\hat{F}})$
$\hat{F}$	Stress function of the fundamental state
$\hat{\hat{F}}$	Stress function of the dynamic state
F_C	Generalized excitation function; see equation (7-2)
F_D	Generalized excitation function; see equation (7-1)
$\bar{F}_d$	Average value of F_D (over one period)
$\bar{F}_D$	Nondimensional average value of $\bar{F}_d$
h	Wall thickness of shell
L	Length of the shell
$\ell_i, \ell_k, \ell_m, \ell_\ell, \ell_n, \ldots$	Normalized axial and circumferential wave parameters, defined in Appendix B of Ref [1]
L_D, L_H, L_Q	Linear differential operators, defined in Appendix A of Ref [1]
L_{NL}	Nonlinear differential operator, $L_{NL}(S,T) = S,_{xx}T,_{yy} - 2S,_{xy}T,_{xy} + S,_{yy}T,_{xx}$
$Q(x,y)$	Spatial distribution of the radial load; see equation (8)
$q(x,y,t)$	Radial loading applied to the surface of the cylinder
R	Radius of the shell
t	Time
W	Total radial displacement, $W = \hat{W} + \hat{\hat{W}}$
$\bar{w}$	Initial radial imperfection

$\hat{W}$	Radial displacement of the fundamental state
$\hat{\hat{W}}$	Radial displacement of the dynamic state
x,y	Axial and circumferential coordinate on the median surface, respectively
$\alpha_1, \ldots, \alpha_3$	coefficients in equation (14); defined by Eq. (14)
$\bar{\alpha}_1, \ldots, \bar{\alpha}_{13}$	Coefficients in equation (7-1); defined in Appendix A
$\hat{\alpha}_1, \ldots, \hat{\alpha}_7$	Coefficients in equation (15-2); defined in Appendix B
$\beta_1, \ldots, \beta_{10}$	Coefficients in equation (12) - (16); defined in Appendix C of Ref [1]
$\bar{\beta}_1, \ldots, \bar{\beta}_{10}$	Coefficients in equation (7-2); defined in Appendix A
$\hat{\beta}_1, \ldots, \hat{\beta}_7$	Coefficients in equation (15-3); defined in Appendix B
γ	Nondimensional damping coefficient
$\delta_{n\ell}$	Kroneckers delta function $\begin{cases} 0 & n \neq \ell \\ 1 & n = \ell \end{cases}$
ε	Small parameter defined in Ref [3], $(\frac{n^2 h}{R})^2$
ν	Poisson's ratio
ξ	Aspect ratio defined in Ref [3], $\frac{m\pi/L}{n/R}$
δ_1	Nondimensional amplitude of the axisymmetric imperfection
δ_2	Nondimensional amplitude of the asymmetric imperfection
$\hat{\delta}_0, \hat{\delta}_1, \hat{\delta}_2$	Nondimensional amplitudes of the radial displacement for the fundamental solution
$\bar{\rho}$	Specific mass density, defined in Appendix B of Ref. [1]
ϕ	Phase angle of the driven mode
$\bar{\phi}$	Average value (over one period) of ϕ
ψ	Phase angle of the companion mode
$\bar{\psi}$	Average value (over one period) of ψ
χ_1	Frequency parameter of the driven mode, $(\chi_1 = \Omega_s \tau + \bar{\phi})$

χ_2	Frequency parameter of the companion mode, ($\chi_2 = \Omega_s\tau + \bar{\psi}$)
ω	Vibration frequency
Ω	Nondimensional frequency, $\omega R \sqrt{\frac{2\bar{\rho}}{E}}$
Δ	Difference of the phase angles, $\phi - \psi$
$\bar{\Delta}$	Average value (over one period) of Δ
$()_{,x}$	Differentiation with respect to the variables following the comma

INTRODUCTION

In engineering the study of damped vibrations is of great importance since any realistic structure have some inherent material damping. The results available so far show that the damping has a pronounced influence on the nonlinear vibration of shells [2,3,4].

One of the earliest contributions to the theory of damped nonlinear ring/shell vibration was made by Evensen in 1964, who introduced for the first time the effect of damping in the nonlinear vibration analysis of rings [2]. The phenomenon called 'gap' by Evensen was found in his analysis.

A more meticulous analysis was performed by Ginsberg in 1973 [3]. This work is an extension of his earlier analysis of the undamped nonlinear vibration of infinite long shells to the case of finite long shell with damping. In his analysis a modal expansion approach was used. Also, many of the discrepancies in Evensen's analysis were corrected. Ginsberg's results showed that (a) the damping has a large influence on the coupled mode response, and (b) the presence of even small damping could completely alter the frequency-amplitude relationship. The 'gap' shown in Evensen's results for rings were not found in Ginsberg's analysis.

Chen analyzed the damped nonlinear vibrations of cylindrical shells by applying a systematic perturbation procedure to the Donnell shallow-shell equations in 1972 [4]. His results for the companion mode are quite similar to those of Ginsberg but for the driven mode response the agreement is not satisfactory. This was attributed to large values of damping used in Chen's analysis [4], but Chen did not agree with it.

As mentioned in [1], one of the conclusions that can be drawn from the results of previous studies is that although some basic characteristics of the damped vibration behaviour of shells have been derived analytically and also verified experimentally, there are certain situations where considerable disagreement still exists between results obtained by different analytical procedures and also between theoretical predictions and experimental evidence.

The objective of the present analysis is to investigate the effect of the initial geometric imperfections on the damped nonlinear vibration of shells. This objective is the natural extension of [1]. The emphasis of the current work is placed on the influence of initial geometric imperfections on the coupled mode response for which no solution is as yet available. In addition, the authors also intend to study discrepancies between the results of earlier studies and to get a reasonable explanation, if it is possible.

The same analytical procedure and assumptions as those used in Ref [1] are used in the present analysis. For the sake of brevity they are used without any explanation except for new ones. Similarly, the details of the solution of the fundamental state will not be repeated here, as well as stability analysis of solutions. Only the analysis for the damped dynamic state is presented in the present paper. Interested reader in the analyses of the fundamental state and stability can refer to Ref. [6] and [7]., respectively.

ANALYSIS

The mathematical model of the vibrating imperfect stiffened cylindrical shell with damping was arrived at by introducing the appropriate terms for the imperfections, the inertia and the damping into the nonlinear Donnell-type orthotropic imperfect shell equations.

The model consists of the following two coupled partial differential equations in the unknown functions W and F,

$$L_H(F) - L_Q(W) = \frac{1}{R} W,_{xx} - \frac{1}{2} L_{NL}(W, W + 2\bar{w}) \qquad (1\text{-}1)$$

$$L_Q(F) + L_D(W) = \frac{1}{R} F,_{xx} + L_{NL}(F, W + \bar{w}) - \bar{\rho} h W,_{tt} - chW,_t + q = 0 \qquad (1\text{-}2)$$

where all of the symbols have the same meaning as in equations (8-1) and (8-2) in Ref. [1], except c, which is the material damping coefficient.

Using the assumption that the displacement W and the stress function F of the shell during vibration under an axial compressive load and due to the lateral excitation can be expressed as a linear superposition of two independent states of displacement and stress, one can obtain the following two sets of equations. One of them is the set of governing equations of the fundamental static state (geometrically nonlinear)

$$L_H(\hat{F}) - L_Q(\hat{W}) = -\frac{1}{R} \hat{W},_{xx} - \frac{1}{2} L_{NL}(\hat{W}, \hat{W} + 2\bar{W}) \qquad (2\text{-}1)$$

$$L_Q(\hat{F}) + L_D(\hat{W}) = \frac{1}{R} \hat{F},_{xx} + L_{NL}(\hat{F}, \hat{W} + \bar{w}) \qquad (2\text{-}2)$$

while another is the set of governing equations of the nonlinear dynamic state due to small but not, infinitesimal vibrations about the fundamental state:

$$L_H(\hat{\hat{F}}) - L_Q(\hat{\hat{W}}) = -\frac{1}{R} \hat{\hat{W}},_{xx} - \frac{1}{2} L_{NL}(\hat{\hat{W}}, \hat{W} + 2\bar{W}) - \frac{1}{2} L_{NL}(\hat{\hat{W}}, \hat{W}) - \frac{1}{2} L_{NL}(\hat{\hat{W}}, \hat{\hat{W}}) \qquad (3\text{-}1)$$

$$L_Q(\hat{\hat{F}}) + L_D(\hat{\hat{W}}) = \frac{1}{R} \hat{\hat{F}},_{xx} + L_{NL}(\hat{F}, \hat{\hat{W}}) + L_{NL}(\hat{\hat{F}}, \hat{W} + \bar{w}) + L_{NL}(\hat{\hat{F}}, \hat{\hat{W}}) - ch\, \hat{\hat{W}},_t - \bar{\rho} h\, \hat{\hat{W}},_{tt} + q \qquad (3\text{-}2)$$

As mentioned above, only the dynamic state is analyzed in the present work.

The assumptions of the present analysis are the same as those used in Ref. [1] for the imperfection mode, the fundamental (static) mode and the dynamic response mode. They are

$$\bar{w} = \delta_1 h \cos\ell_i x + \delta_2 h \sin\ell_k x \cos\ell_n y \qquad (4)$$

$$\hat{W} = \hat{\delta}_0 h + \hat{\delta}_1 h \cos\ell_i x + \hat{\delta}_2 \sin\ell_k x \cos\ell_n y \qquad (5)$$

and

$$\hat{\hat{W}} = Ah \sin \ell_k x \cos\ell_\ell y + Bh \sin\ell_k x \sin\ell_\ell y + \ell_\ell^2 \frac{Rh}{4}$$

$$[A^2 + B^2 + 2A\delta_{n,\ell}(\hat{\delta}_2 + \delta_2)] \sin^2 \ell_m x \tag{6}$$

where the $\bar{w}$, $\hat{W}$ and $\hat{\hat{W}}$ are the amplitudes of the initial imperfection, the fundamental (static) behaviour and the dynamic response, respectively.

Through appropriate operation and the use of Galerkin's procedure one obtains two coupled nonlinear differential equations for A(t) and B(t)

$$\bar{\alpha}_1 \frac{d^2A}{dt^2} + \bar{\alpha}_2 \frac{dA}{dt} + \bar{\alpha}_3 A + \bar{\alpha}_4 \frac{d^2C}{dt^2}[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \bar{\alpha}_5 \frac{dC}{dt}[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] +$$
$$+ \bar{\alpha}_6 A^2 + \bar{\alpha}_7(A^2+B^2) + \bar{\alpha}_8(A^2-B^2) + \bar{\alpha}_9 A^3 + \bar{\alpha}_{10}(A^2+B^2)A +$$
$$+ \bar{\alpha}_{11}(A^2+B^2)A^2 + \bar{\alpha}_{12}(A^2+B^2)^2 + \bar{\alpha}_{13}(A^2+B^2)^2 A = F_D \tag{7-1}$$

$$\bar{\beta}_1 \frac{d^2B}{dt^2} + \bar{\beta}_2 \frac{dB}{dt} + \bar{\beta}_3 B + \bar{\beta}_4 \frac{d^2C}{dt^2} + \bar{\beta}_5 B \frac{dC}{dt} + \bar{\beta}_6 AB + \bar{\beta}_7 AB + \bar{\beta}_8(A^2+B^2)B +$$
$$+ \bar{\beta}_9(A^2+B^2)AB + \bar{\beta}_{10}(A^2+B^2)^2 B = F_C \tag{7-2}$$

where the $\bar{\alpha}_1$, $\bar{\alpha}_2$, ..., $\bar{\alpha}_{13}$ and $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_{10}$ are coefficients which are defined in Appendix A and F_D and F_C are generalized dynamic forces.

The details of derivations of equations (7) are given in Appendix C.
As in Ref. [1], we chose q to be fixed in space and harmonic in time:

$$q(x,y,t) = Q(x,y) \cos\omega t \tag{8}$$

and then choose Q(x,y) to be symmetric with respect to y and have a zero average value. This results in

$$F_C = 0$$

and

$$F_D = 2 \int_0^L \int_0^{2\pi R} \frac{Q(x,y)\{\cos\ell_\ell y \sin\ell_k x + \frac{\ell_\ell^2 Rh}{2}[A+\delta_{n,\ell}(\delta_2+\hat{\delta}_2)]\sin^2\ell_m x\}\cos\omega t}{\pi RL} dxdy \tag{9}$$

It is obvious that the coupled nonlinear differential equations (7) cannot be solved exactly. An approximate solution can be obtained by the procedure known as the method of averaging. The unknown functions A(t) and B(t) are taken to be of the form

$$A(t) = A_0(t) \cos(\omega t + \phi) \tag{10-1}$$

$$B(t) = B_0(t) \sin(\omega t + \psi) \tag{10-2}$$

where ϕ and ψ are the phase angles between the driven mode and the companion mode respectively and the excitation. These angles are functions of time.

Substituting equations (10) into equations (7) and then applying the method of averaging yield the approximate solutions for A(t) and B(t)

$$A(t) = \bar{A}(t)\cos(\omega t + \bar{\phi}) \text{ (driven mode)} \tag{11-1}$$
$$B(t) = \bar{B}(t)\sin(\omega t + \bar{\psi}) \text{ (companion mode)} \tag{11-2}$$

where $\bar{A}$, $\bar{B}$, $\bar{\phi}$ and $\bar{\psi}$ are the average value of the $A(t)$, $B(t)$, $\phi(t)$ and $\psi(t)$ over one period, respectively. They can be obtained by solving the following simultaneous nonlinear algebraic equations for a given average excitation $\bar{F}_D$, damping γ and forcing frequency Ω:

$$-\Omega^2\bar{A}\,\{1 + \beta_1[\bar{A}^2 - \bar{B}^2\cos 2\bar{\Delta} + 2\,\delta_{n,\ell}\,(\delta_2 + \hat{\delta}_2)^2]\} + \beta_2\bar{A} - \gamma\Omega\bar{A}\bar{B}^2\beta_1\sin 2\bar{\Delta} + \beta_3\bar{A}^3 + 2\beta_4\bar{A}\bar{B}^2\,(1 - \tfrac{1}{2}\cos 2\bar{\Delta}) + \beta_5\bar{A}\,[5\bar{A}^4 + 4\bar{A}^2\bar{B}^2\,(\tfrac{3}{2} - \cos 2\bar{\Delta}) + 2\bar{B}^4\,(\tfrac{3}{2} - \cos 2\bar{\Delta})] = \bar{F}_D\cos\bar{\phi} \tag{12-1}$$

$$\{\bar{A}\bar{B}^2[\beta_1\Omega^2 - \beta_4 - 2\,\beta_5\bar{A}(\bar{A}^2 + \bar{B}^2)]\}\sin 2\bar{\Delta} - \Omega\gamma\,\{2\bar{A} + \beta_1[\bar{A}^3 - \bar{A}\bar{B}^2\cos 2\bar{\Delta} + 4\bar{A}\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\} = \bar{F}_D\sin\bar{\phi} \tag{12-2}$$

$$-\Omega^2\bar{B}\{1 + \beta_6(\bar{B}^2 - \bar{A}^2\cos 2\bar{\Delta})\} + \beta_7\bar{B} + \Omega\gamma\bar{A}^2\bar{B}\beta_6\sin 2\bar{\Delta} + \beta_9\bar{B}^3 + 2\beta_8\bar{A}^2\bar{B}(1 - \tfrac{1}{2}\cos 2\bar{\Delta}) + \beta_{10}\bar{B}\{5\bar{B}^4 + [4\bar{A}^2\bar{B}^2 + 2\bar{A}^4]\,(\tfrac{3}{2} - \cos 2\bar{\Delta})\} = 0 \tag{12-3}$$

$$\{\bar{A}^2\bar{B}\,[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}\sin 2\bar{\Delta} + \Omega\gamma\{2\bar{B} + \beta_6[\bar{B}^3 - \bar{A}^2\bar{B}\cos 2\Delta]\} = 0 \tag{12-4}$$

where $\Omega = \omega R\sqrt{\dfrac{2\bar{\rho}}{E}}$, is the generalized nondimensional frequency;

$\gamma = cR\sqrt{\dfrac{1}{2\bar{\rho}E}}$, is the generalized nondimensional damping;

$\bar{\Delta} = \bar{\phi} - \bar{\psi}$, is the "average difference of phase angles";

$$\bar{F}_D = \frac{4R\int_0^L \int_0^{2\pi R} Q(x,y)\,\{\sin\ell_k x\,\cos\ell_\ell y + \dfrac{\ell_\ell^2 Rh}{2}\,[A + \delta_{n,\ell}(\hat{\delta}_2 + \delta_2)]\,\sin^2\ell_m x\}\,dxdy}{\pi LEh}$$

, is the generalized average excitation;

and β_1, β_2, ..., β_{10} are coefficients which are given in the Appendix C of Ref. [1].

The details of derivations of equations 12 are given in Appendix C. The analysis is carried out for two separate cases.

<u>CASE 1 Single mode response</u> ($\bar{A} \neq 0$, $\bar{B} = 0$)

As can be seen $\bar{B} = 0$ is a possible solution of equations (12). In this case equations (12) become

$$-\Omega^2\bar{A}\,\{1 + \beta_1\bar{A}^2 + 2\beta_1\delta_{n,\ell}\,(\delta_2 + \hat{\delta}_2)^2\} + \beta_2\bar{A} + \beta_3\bar{A}^3 + 5\beta_5\bar{A}^5 = \bar{F}_D\cos\bar{\phi} \quad (13\text{-}1)$$

$$-\Omega\gamma\,\{2\bar{A} + \beta_1\bar{A}^3 + 4\beta_1\bar{A}\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2\} = \bar{F}_D\sin\bar{\phi} \quad (13\text{-}2)$$

A single equation governing the amplitude-frequency relationship of the single mode response is obtained by first squaring both equations, then adding them and finally using the identity

$$\sin^2\bar{\phi} + \cos^2\bar{\phi} = 1$$

This yields

$$\alpha_1\Omega^4 + \alpha_2\Omega^2 + \alpha_3 = 0 \quad (14)$$

where

$$\alpha_1 = \bar{A}^2\{1 + \beta_1\,[\bar{A}^2 + 2\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\}^2$$

$$\alpha_2 = -\,2\bar{A}^2\,\{1 + \beta_1\,[\bar{A}^2 + 2\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\}\,\{\beta_2 + \beta_3\bar{A}^2 + 5\beta_5\bar{A}^4\} + \gamma^2\bar{A}^2\,\{2 + \beta_1\bar{A}^2 + 4\beta_1\delta_{n,\ell}(\hat{\delta}_2 + \delta_2)^2\}^2$$

$$\alpha_3 = \bar{A}^2\,\{\beta_2 + \beta_3\bar{A}^2 + 5\beta_5\bar{A}^4\}^2 - \bar{F}_D^2$$

It is obvious that one can obtain not only the damped response for various damping and excitation levels but also the free vibration (or undamped response) if one lets the damping and excitation terms vanish in equation (14).

CASE 2 Coupled-mode response ($\bar{A} \neq 0$, $\bar{B} \neq 0$)

Another possible case in the solution of equations (12) is $\bar{A} \neq 0$ and $\bar{B} \neq 0$, namely the damped coupled-mode response.

For the damped couple-mode response a direct simultaneous solution of equations (12) is too cumbersome. A further simplification can be carried out as follows. Initially one solves equations (12-3) and (12-4) for $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ in terms of $\bar{A}$ and $\bar{B}$, respectively. Then one uses the identity $\sin^2 2\bar{\Delta} + \cos^2 2\bar{\Delta} = 1$, which results in a single equation with the unknowns $\bar{A}$ and $\bar{B}$.

Next one back-subtitutes for $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ in equations (12-1) and (12-2) and then uses the identity $\sin^2\bar{\phi} + \cos^2\bar{\phi} = 1$, which yields a second equation with the unknowns $\bar{A}$ and $\bar{B}$.

The amplitude-frequency relationship of damped, coupled response then can obtained by solving these two nonlinear algebraic equations simultaneously for given values of damping, imperfection and excitation. The equations can be expressed in the following form:

$$< -\Omega^2\bar{A}\ \{1 + \beta_1[\bar{A}^2 - \bar{B}^2 \cos 2\bar{\Delta} + 2\delta_{n,\ell}\ (\hat{\delta}_2 + \delta_2)^2]\} + \beta_2\bar{A} - \gamma\Omega\bar{A}\bar{B}^2\beta_1 \sin 2\bar{\Delta} +$$
$$+ \beta_3\bar{A}^3 + 2\beta_4\bar{A}\bar{B}^2\ (1 - \frac{1}{2}\cos 2\bar{\Delta}) + \beta_5\bar{A}\ [5\bar{A}^4 + 4\bar{A}^2\bar{B}^2\ (\frac{3}{2} - \cos 2\bar{\Delta}) +$$
$$+ 2\bar{B}^4(\frac{3}{2} - \cos 2\bar{\Delta})]>^2 + <\{\bar{A}\bar{B}^2[\beta_1\Omega^2 - \beta_4 - 2\ \beta_5\ (\bar{A}^2 + \bar{B}^2)]\} \sin 2\bar{\Delta} +$$
$$- \Omega\gamma\ \{2\bar{A} + \beta_1\ [\bar{A}^3 - \bar{A}\bar{B}^2 \cos 2\bar{\Delta} + 4\bar{A}\delta_{n,\ell}\ (\hat{\delta}_2 + \delta_2)^2]\}>^2 - \bar{F}_D^2 = 0 \qquad (15\text{-}1)$$

and

$$\hat{\alpha}_1\bar{B}^{12} + \hat{\alpha}_2\bar{B}^{10} + \hat{\alpha}_3\bar{B}^8 + \hat{\alpha}_4\bar{B}^6 + \hat{\alpha}_5\bar{B}^4 + \hat{\alpha}_6\bar{B}^2 + \hat{\alpha}_7 = 0 \qquad (15\text{-}2)$$

or

$$\hat{\beta}_1\bar{A}^{12} + \hat{\beta}_2\bar{A}^{10} + \hat{\beta}_3\bar{A}^8 + \hat{\beta}_4\bar{A}^6 + \hat{\beta}_5\bar{A}^4 + \hat{\beta}_6\bar{A}^2 + \hat{\beta}_7 = 0 \qquad (15\text{-}3)$$

where

$$\sin 2\bar{\Delta} = - \gamma\Omega\{2[\beta_6\Omega^2 - \beta_8 - \beta_{10}(4\bar{B}^2 + 2\bar{A}^2)] + \beta_6[\beta_7 - \Omega^2 + \bar{B}^2(\beta_9 - \beta_8) +$$
$$+ 2\beta_8\bar{A}^2 + \beta_{10}(\bar{B}^4 + 4\bar{A}^2\bar{B}^2 + 3\bar{A}^4)]\}/\ S_d \qquad (16\text{-}1)$$

$$\cos 2\bar{\Delta} = \{\Omega^2\gamma^2\beta_6(2 + \beta_6\bar{B}^2) + [\Omega^2(1+\beta_6\bar{B}^2) - \beta_7 - \beta_9\bar{B}^2 - 2\beta_8\bar{A}^2 +$$
$$- \beta_{10}(5\bar{B}^4 + 6\bar{A}^2\bar{B}^2 + 3\bar{A}^4)][\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}/\ S_d \qquad (16\text{-}2)$$

$$S_d = (\bar{A}\beta_6\gamma\Omega)^2 + \bar{A}^2[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)][\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + 2\bar{B}^2)] \qquad (16\text{-}3)$$

Here: $\hat{\alpha}_1$, $\hat{\alpha}_2$, ..., $\hat{\alpha}_7$ and $\hat{\beta}_1$, $\hat{\beta}_2$, ..., $\hat{\beta}_7$ are functions of $\bar{A}$ and $\bar{B}$ respectively and are given in Appendix B.

One can get solutions for the driven mode $\bar{A}$ by numerically solving equations (15-1) and (15-2) simultaneously for given amplitudes of the companion mode $\bar{B}$ or get solutions for the companion mode $\bar{B}$ from equations (15-1) and (15-3) for the given amplitudes of the driven mode $\bar{A}$.

DISCUSSION OF THE RESULTS

Equations (15-1) and (15-2) and equations (15-1) and (15-3) are two sets of nonlinear algebraic equations for the two unknowns $\bar{A}$ and $\bar{B}$. A direct solution for $\bar{A}$ and $\bar{B}$ as functions of $\bar{F}_D$, Ω and γ is quite difficult. Therefore the normal procedure is to calculate $\bar{B}$ from equation (15-2) for given values of $\bar{A}$, Ω and γ (or to calculate $\bar{A}$ from equation (15-3) for given $\bar{B}$, Ω, and γ), then calculate the generalized excitation $\bar{F}_D$ from equation

(15-1) upon substituting $\bar{B}$ (or $\bar{A}$) for given values of $\bar{A}$ (or $\bar{B}$), Ω and Υ. By cross-plotting the results, it is possible to obtain curves for $\bar{A}$ vs. Ω and $\bar{B}$ vs. Ω for constant $\bar{F}_D$. For guaranteeing the necessary accuracy of solutions, the Newton-Raphson procedure is used in the present analysis, which takes the results obtained from cross-plotting as the starting values.

The isotropic shell used by Evensen, called ES herein is used. Its characteristic parameters are

$$\varepsilon = \left(\frac{n^2 h}{R}\right)^2 = 0.01$$

$$\xi = \frac{\pi R/n}{L/k} = 0.1$$

$$\nu = 0.3$$

The wave numbers of vibration are chosen such that

a. They satisfy the accuracy requirements of Donnell's equations, namely the circumferential wave number should be greater than 4;
b. They would constitute lower order modes which can be excited easily into the nonlinear region to make experimental verification possible

In the present analysis the mode k=1, ℓ=5 was selected. Various values of i and n were selected depending on the different coupling conditions.

A series of computations was performed for damped single and coupled mode response. In order to facilitate understanding, the discussions of these numerical results are divided into five categories.

a. Influence of Damping and Excitation

Figures* 1 to 4 show the amplitude-frequency relationships of a perfect shell for different values of damping and excitation. One can draw the conclusion that the damping has a dramatic influence on the behaviour of the coupled mode response after comparing the present results with those of the reference [1]. The presence of even very small damping changes the shape of the undamped coupled mode response completely. As can be seen from Figures 1 and 2, increasing the damping can quickly round-off the coupled-mode response peak. Similar results have been found by Ginsberg [13].

An interesting fact can be deduced from the results shown in figures 1, 3 and 4, namely, that increasing the amplitude of excitation or decreasing the value of damping can disrupt the stability of the coupled-mode response peak. This fact has been found by Chen [4] in his careful experiments fifteen years ago, but has not been predicted by theoretical analysis before.

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*Notice that the frequencies in the present figures were normalized by dividing by the frequency of free vibration (linear theory) of the perfect unloaded shell.

b. Influence of Asymmetric Imperfections

The response-frequency relationships of a shell with asymmetric initial imperfections are shown in Figures 5 to 9, where the damping and excitation are held constant. It can be observed that asymmetric imperfections have a significant influence on the coupled-mode response if the coupling condition $n=\ell$ is satisfied. Increasing the amplitude of the asymmetric imperfection can quickly decrease the region where the coupled-mode response occurs. The asymmetric imperfection also changes the stability characteristics of the coupled-mode response. But as it can be seen from Fig. 9 the influence of the asymmetric imperfections on the coupled-mode response is minimal if the coupling condition $n=\ell$ is not satisfied. It should be noted that the influence of the asymmetric imperfections on the shape of the single mode response-frequency curve is also minimal.

c. Influence of Axisymmetric Imperfections

Figures 10 to 13 display the amplitude-frequency relationships of a shell with axisymmetric imperfections. As can be seen (from figures 10 to 12), if the coupling condition $i=2k$ is satisfied then also the axisymmetric imperfection has a strong influence on the nonliner vibration behaviour, otherwise the influence is quite slight, as shown in Fig. 13. Notice that in the case of $i=2k$, the left bifurcation point is now on the lower branch of the associated single mode response curve rather than on the upper branch as in the case of an asymmetric imperfection. In addition the axisymmetric imperfection changes the stability characteristics of the coupled-mode response curves in the case of $i=2k$.

d. Influence of Combined Imperfections

Figures 14-17 show the amplitude-frequency relationship of shell ES with the combined imperfections ($\delta_1 \neq 0$, $\delta_2 \neq 0$). It is seen from Figures 14 and 15 that the axisymmetric imperfection has a stronger influence on the nonlinear vibration than the asymmetric imperfection when they both have the same order of magnitude of the amplitude. In these cases, the characteristics of response is basically those of a shell with axisymmetric imperfection alone (see figure 11). Another fact that should be noted in Figures 14 and 15 is that the influence of the combined imperfections on the nonlinear vibration is not simply the superposition of the influence of the asymmetric and axisymmetric imperfection on the vibration, because the nonlinearity involved. For instance, the region of the coupled mode response in the case of combined imperfections is larger than the region in the case of asymmetric imperfection alone (see figures 6 and 15) but less than that in the case of axisymmetric imperfection alone (see figures 11 and 15).

The effects of the coupling condition $i=2k$ and $n=\ell$ are shown in figures 16 and 17. It is obvious that the influence of the combined imperfections on the nonlinear vibrations is reduced to that of the single mode imperfection when only one of the coupling conditions is satisfied.

e. Influence of Axial Compressive Load

Investigation of the effect of an axial compressive load on the nonlinear vibrations of a shell is of importance in engineering since many shells used in practice carry an axial compressive load. Figures 18 to 20 indicate the dynamic shell behaviour for the cases of $\lambda = 0.1$, $\lambda = 0.3$ and $\lambda = 0.5$ for perfect shells respectively. By studying these figures one can draw the conclusion that the axial compressive load has the following influence on the nonlinear vibration:

- It 'amplifies' the nonlinearity of the vibration, as mentioned in [1],
- It increases the amplitude of response, which is equivalent to decreasing the damping,
- It increases the region of the coupled mode response.

As can be seen, the presence of an axial compressive load does not change the vibration behaviour.

CONCLUSIONS

The nonlinear flexural vibration of perfect and imperfect thin-walled cylindrical shells with damping are analyzed by using Donnell's nonlinear shell equations. Numerical solutions are obtained by applying Galerkin's method together with the method of averaging. The study yield the following conclusions:

a. A good agreement between the present and Ginsberg's [3] analysis is obtained. The "gap" found in Evensen's analysis [2], which is the major difference between Evensen and Ginsberg is not found in the present analysis. In the present author's opinion the "gap" resulted because Evensen neglected the negative values of cos $2\bar{\Delta}$, in his ring analysis. Therefore, one can now say that no qualitative difference exists between the results of the different solution procedures: (a) Galerkin's method (Evensen [2,5]) and the present analysis, (b) the small parameter perturbation (Chen [4]), and (c) the special perturbation procedure (Ginsberg [3]).

b. The general characteristics of the damped response of perfect shells found by Ginsberg are confirmed by the present analysis, namely
   - the damping has a pronounced influence on the coupled-mode response. Increasing damping can completely eliminate coupled-mode response peaks,
   - damped response of a perfect shell can be divided into five regions, as shown in Fig. 1. In region (3) both the single mode and the coupled-mode responses are unstable. The coupled mode response peak in the region (4) is stable,
   - the single mode response between two bifurcation points is unstable.

   One of the extra results obtained by the present analysis is that the stability of the coupled-mode response in region (4) is not always stable. It depends on the magnitude of damping or excitation, as shown in Figs. 3 and 4.

c. Initial geometric imperfections have a significant influence on the damped vibrations of either the single or coupled-mode responses under certain coupling conditions. The general influence of imperfections is quite similar to that of damping. That is, increasing amplitude of imperfections can quickly eliminate the coupled-mode response. In addition, the presence of initial geometric imperfections changes the stability characteristics of the solutions. It is noted that the influence of combined imperfection modes cannot be obtained simply by superposition of the individually determined axisymmetric and asymmetric imperfection modes.

d. Axial compressive loads have an influence on the nonlinear vibration of perfect shells. Such loads amplify the nonlinearity of vibration, increase the amplitudes of response as well as the region of coupled-mode response. It does not change the vibration behaviour when λ is less than the 'critical point' of the load.

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REFERENCES

1. Liu, D.K. and J. Arbocz — "Influence of Initial Geometric Imperfections on undamped Nonlinear Vibrations of Thin-Walled Cylinders" Rept. LR-457, Dept. of Aerospace Engineering, TH-Delft, The Netherlands, 1986

2. Evensen, D.A. — "Nonlinear Flexural Vibrations of Thin Circular Rings" NASA TR R-277, Jan. 1966

3. Ginsberg, J.H. — "Large Amplitude Forced Vibrations of Simply Supported Thin Cylindrical Shells" J. Appl. Mech. June 1973, pp. 471-477

4. Chen, J.C. — "Nonlinear Vibration of Cylindrical Shells" Ph. D. Thesis, 1972, Dept. of Aeronautics, California Institute of Technology, Pasadena, Calif.

5. Evensen, D.A. — "Nonlinear Vibrations of Circular Cylindrical Shells" In: Thin-Shell Structures (Editors: Y.C. Fung and E.E. Sechler), Prentice-Hall, Englewood Cliffs, New Jersey, 1974, pp. 133-155

6. Hol, J.M.A.M. — "The effect of General Imperfections on the Natural Frequencies of Axially Compressed Stiffened Cylinders" Afstudeerverslag. Dept. of Aerospace Engineering, TH-Delft, The Netherlands, 1983

7. Liu, D.K. and Arbocz, J. — "The Stability Analysis of Nonlinear Vibrations of Imperfect Thin-Walled Cylinders" Rept. LR-461, Dept. of Aerospace Engineering, TH-Delft, The Netherlands, 1986

APPENDIX A

$\bar{\alpha}_1 = \bar{c}_1$

$\bar{\alpha}_2 = ch^2$

$\bar{\alpha}_3 = \bar{c}_2$

$\bar{\alpha}_4 = \bar{c}_3$

$\bar{\alpha}_5 = \frac{3}{4}ch^3\ell_\ell{}^2R$

$\bar{\alpha}_6 = \bar{c}_4$

$\bar{\alpha}_7 = \bar{c}_5$

$\bar{\alpha}_8 = \bar{c}_6$

$\bar{\alpha}_9 = \bar{c}_7$

$\bar{\alpha}_{10} = \bar{c}_8$

$\bar{\alpha}_{11} = \bar{c}_9$

$\bar{\alpha}_{12} = \bar{c}_{10}$

$\bar{\alpha}_{13} = \bar{c}_{11}$

$\bar{\beta}_1 = \bar{d}_1$

$\bar{\beta}_2 = ch^2$

$\bar{\beta}_3 = \bar{d}_2$

$\bar{\beta}_4 = \bar{d}_3$

$\bar{\beta}_5 = \frac{3}{4}ch^3\ell_\ell{}^2R$

$\bar{\beta}_6 = \bar{d}_4$

$\bar{\beta}_7 = \bar{d}_5$

$\bar{\beta}_8 = \bar{d}_6$

$\bar{\beta}_9 = \bar{d}_7$

$\bar{\beta}_{10} = \bar{d}_8$

where $\bar{c}_1$, $\bar{c}_2$, ..., $\bar{c}_{11}$ and $\bar{d}_1$, $\bar{d}_2$, ..., $\bar{d}_8$ are coefficients defined in C of Ref. [1].

APPENDIX B

$$\hat{\alpha}_1 = 100\,\beta_{10}^4$$

$$\hat{\alpha}_2 = 440\beta_{10}^4\bar{A}^2 - 40\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 100\beta_{10}^3(\beta_6\Omega^2-\beta_8)$$

$$\begin{aligned}\hat{\alpha}_3 = {} & 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 25\,\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 + 40\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \\ & - 40\beta_{10}^3(\Omega^2-\beta_7) + 21\,\gamma^2\Omega^2\beta_6^2\beta_{10}^2 + \bar{A}^2[80\beta_8\beta_{10}^3-128\beta_{10}^3(\beta_6\Omega^2-\beta_9) + \\ & - 340\beta_{10}^3(\beta_6\Omega^2-\beta_8)] + 780\beta_{10}^4\bar{A}^4\end{aligned}$$

$$\begin{aligned}\hat{\alpha}_4 = {} & 40\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 4\beta_{10}(\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8) + \\ & - 10\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) + 2\gamma^2\Omega^2\beta_6\beta_{10}[12\beta_{10}-5\beta_6(\beta_6\Omega^2-\beta_8) - 2\beta_6(\beta_6\Omega^2-\beta_9) + \\ & + \beta_6(\beta_9-\beta_8)] + \bar{A}^2[8\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 60\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 128\beta_{10}^3(\Omega^2-\beta_7) + \\ & + 88\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) - 80\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) + \\ & + 52\gamma^2\Omega^2\beta_6^2\beta_{10}^2] + \bar{A}^4[256\beta_8\beta_{10}^3 - 160\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 408\beta_{10}^3(\beta_6\Omega^2-\beta_8)] + \\ & + 720\beta_{10}^4\bar{A}^6\end{aligned}$$

$$\begin{aligned}\hat{\alpha}_5 = {} & [64\beta_{10}^2+\beta_6^2(\beta_9-\beta_8)^2 + 6\beta_6^2\beta_{10}(\beta_7-\Omega^2) - 16\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) - 16\beta_6\beta_{10}(\beta_9-\beta_8) + \\ & - 8\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) + 2\beta_6^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)]\gamma^2\Omega^2 + (\Omega\gamma\beta_6)^4 + 4\beta_{10}^2(\Omega^2-\beta_7)^2 + \\ & + (\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8)^2 - 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \\ & - 10\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2 + \\ & + \bar{A}^2\{4\beta_6\beta_{10}\gamma^2\Omega^2[3\beta_6\beta_8 + 2\beta_6-(\beta_9-\beta_8) + 4\beta_{10}-\beta_6(\beta_6\Omega^2-\beta_9) - 3\beta_6(\beta_6\Omega^2-\beta_8)] + \\ & + 88\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 16\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 16\,\beta_8\beta_{10}^2(\Omega^2-\beta_7) + \\ & - 4\beta_{10}(\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8) + 16\,\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \\ & - 12\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) + 20\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2\} + \\ & + \bar{A}^4\{42\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 16\beta_8^2\beta_{10}^2 + 14\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 + \\ & - 160\beta_{10}^3(\Omega^2-\beta_7) + 72\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) +\end{aligned}$$

$$- 176\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8)\} +$$

$$+ \bar{A}^6\{320\beta_8\beta_{10}^3 - 96\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 200\beta_{10}^3(\beta_6\Omega^2-\beta_8)\} + 380\beta_{10}^4 \bar{A}^8$$

$$\hat{\alpha}_6 = \{2\gamma^2\Omega^2[\beta_6^2(\beta_9-\beta_8)(\beta_7-\Omega^2) - 16\beta_{10}(\beta_6\Omega^2-\beta_8) + 2\beta_6(\beta_9-\beta_8)(\beta_6\Omega^2-\beta_8) +$$

$$- 4\beta_{10}\beta_6(\beta_7-\Omega^2) + 2\beta_6(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \beta_6^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)] + 4\beta_6^3\Omega^4\gamma^4 +$$

$$- 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) + 2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{A}^2\{4\gamma^2\Omega^2[16\beta_{10}^2 + 3\beta_{10}\beta_6^2(\beta_7-\Omega^2) + \beta_6^2\beta_8(\beta_9-\beta_8) - 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) +$$

$$- 2\beta_6\beta_{10}(\beta_9-\beta_8) - 4\beta_6\beta_8\beta_{10} - 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) - \beta_6^2\beta_8(\beta_6\Omega^2-\beta_8)] +$$

$$+ 8\beta_{10}^2(\Omega^2-\beta_7)^2 - 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 16\beta_8\beta_{10}(\beta_7-\Omega^2)(\beta_6\Omega^2-\beta_8)+$$

$$- 12\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2 - 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{A}^4\{2\beta_6\beta_{10}\gamma^2\Omega^2[12\beta_6\beta_8 + 3\beta_6(\beta_9-\beta_8) - 4\beta_{10} + 3\beta_6(\beta_6\Omega^2-\beta_8)] +$$

$$+ 72\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\Omega^2-\beta_7) +$$

$$+ 16\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 6\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) - 16\beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8)+$$

$$+ 24\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 12\beta_{10}(\beta_6\Omega^2-\beta_8)^3\} +$$

$$+ \bar{A}^6\{12\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 32\beta_8^2\beta_{10}^2 - 36\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 96\beta_{10}^3(\Omega^2-\beta_7) +$$

$$+ 24\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 144\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9)\} +$$

$$\bar{A}^8\{192\beta_8\beta_{10}^3 - 24\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 36\beta_{10}^3(\beta_6\Omega^2-\beta_8)\} + 120\beta_{10}^4\bar{A}^{10}$$

$$\hat{\alpha}_7 = \{\gamma^2\Omega^2[4(\beta_6\Omega^2-\beta_8)^2 + \beta_6^2(\beta_7-\Omega^2)^2] + 4\beta_6^2\gamma^4\Omega^4 + (\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8)^2\} +$$

$$\bar{A}^2\{4\gamma^2\Omega^2[\beta_6^2\beta_8(\beta_7-\Omega^2) - 4\beta_{10}(\beta_6\Omega^2-\beta_8)] - 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) +$$

$$- 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\Omega^2-\beta_7)\} +$$

$$+ \bar{A}^4\{2\gamma^2\Omega^2[8\beta_{10}^2 + 2\beta_6^2\beta_8^2 - \beta_6^2(\beta_6\Omega^2-\beta_8)^2 + 3\beta_6^2\beta_{10}(\beta_7-\Omega^2)] +$$

$$+ 4\beta_{10}^2(\Omega^2-\beta_7)^2 + 4\beta_8^2(\beta_6\Omega^2-\beta_8)^2 + 16\beta_8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) +$$

$$- 6\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2 - (\beta_6\gamma\Omega)^4 - (\beta_6\Omega^2-\beta_8)^4\} +$$

$$+ \bar{A}^6\{4\beta_6\beta_{10}\gamma^2\Omega^2[3\beta_6\beta_8 + 2\beta_6(\beta_6\Omega^2-\beta_8)] + 24\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) +$$

$$- 16\beta_8\beta_{10}^2(\Omega^2-\beta_7) - 16\beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8) + 12\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 8\beta_{10}(\beta_6\Omega^2-\beta_8)^3\}+$$

$$\bar{A}^{8}\{\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 16\beta_8^2\beta_{10}^2 - 15\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 48\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) +$$
$$- 24\beta_{10}^3(\Omega^2-\beta_7)\} +$$
$$+ \bar{A}^{10}\{48\beta_{10}^3\beta_8 - 4\beta_{10}^3(\beta_6\Omega^2-\beta_8)\} + 20\beta_{10}^4\ \bar{A}^{12}$$

$$\hat{\beta}_1 = 20\beta_{10}^4$$

$$\hat{\beta}_2 = \{48\beta_8\beta_{10}^3 - 4\beta_{10}^3(\beta_6\Omega^2-\beta_8) + 120\beta_{10}^4\bar{B}^2$$

$$\hat{\beta}_3 = \{\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 16\beta_8^2\beta_{10}^2 - 15\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 48\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) +$$
$$-24\beta_{10}^3(\Omega^2-\beta_7)\} + \bar{B}^2\{192\beta_8\beta_{10}^3 - 36\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 24\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} +$$
$$+ 380\beta_{10}^4\bar{B}^4$$

$$\hat{\beta}_4 = \{4\beta_6^2\beta_{10}\gamma^2\Omega^2[3\beta_8+2(\beta_6\Omega^2-\beta_8)] + 24\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) - 16\beta_8\beta_{10}^2(\Omega^2-\beta_7) +$$
$$- 16\beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8) + 12\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 8\beta_{10}(\beta_6\Omega^2-\beta_8)^3\} +$$
$$+ \bar{B}^2\{12(\beta_6\beta_{10}\gamma\Omega)^2 + 32\beta_8^2\beta_{10}^2 - 36\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 96\beta_{10}^3(\Omega^2-\beta_7) +$$
$$+ 24\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 144\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9)\} +$$
$$+ \bar{B}^4\{320\beta_8\beta_{10}^3 - 200\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 96\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} + 720\beta_{10}^4\bar{B}^6$$

$$\hat{\beta}_5 = \{\gamma^2\Omega^2[16\beta_{10}^2 + 4\beta_6^2\beta_8^2 + 6\beta_6^2\beta_{10}(\beta_7-\Omega^2) - 2\beta_6^2(\beta_6\Omega^2-\beta_8)^2] - (\beta_6\gamma\Omega)^4 +$$
$$+ 4\beta_{10}^2(\Omega^2-\beta_7)^2 + 4\beta_8^2(\beta_6\Omega^2-\beta_8)^2 + 16\beta_{10}\beta_8(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) +$$
$$- 6\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) - (\beta_6\Omega^2-\beta_8)^4\} +$$
$$+ \bar{B}^2\{\gamma^2\Omega^2[24\beta_6^2\beta_8\beta_{10} + 6\beta_6^2\beta_{10}(\beta_9-\beta_8) - 8\beta_6\beta_{10}^2 + 6\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8)] +$$
$$+ 72\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\Omega^2-\beta_7) +$$
$$+ 16\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 6\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) - 16\ \beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8) +$$
$$+ 24\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 12\beta_{10}(\beta_6\Omega^2-\beta_8)^3\} +$$
$$+ \bar{B}^4\{42\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 16\beta_8^2\beta_{10}^2 + 14\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 +$$
$$- 160\ \beta_{10}^3(\Omega^2-\beta_7) + 72\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) +$$

$$
\begin{aligned}
& - 176\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8)\} + \\
& + \bar{B}^6\{196\beta_8\beta_{10}^3 - 408\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 160\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} + 780\beta_{10}^4\bar{B}^8 \\
\hat{\beta}_6 & = \{4\gamma^2\Omega^2[\beta_6^2\beta_8(\beta_7-\Omega^2) - 4\beta_{10}(\beta_6\Omega^2-\beta_8)] - 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) + \\
& - 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\Omega^2-\beta_7)\} + \\
& + \bar{B}^2\{4\gamma^2\Omega^2[16\beta_{10}^2 + 3\beta_6^2\beta_{10}(\beta_7-\Omega^2) + \beta_6^2\beta_8(\beta_9-\beta_8) - 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) + \\
& - 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) - 2\beta_6\beta_{10}(\beta_9-\beta_8) - 4\beta_6\beta_8\beta_{10}-\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8)] + \\
& + 8\beta_{10}^2(\Omega^2-\beta_7)^2 - 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + 16\beta_8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)+ \\
& - 12\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) - 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} + \\
& + \bar{B}^4\{4\gamma^2\Omega^2[3\beta_6^2\beta_8\beta_{10}+ 2\beta_6^2\beta_{10}(\beta_9-\beta_8) + 4\beta_6\beta_{10}^2-\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_9) + \\
& - 3\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8)] + 88\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 16\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) + \\
& - 16\beta_8\beta_{10}^2(\Omega^2-\beta_7) - 4\beta_{10}(\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8) + 16\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \\
& - 12\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) + 20\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2\} + \\
& + \bar{B}^6\{52\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 8\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 60\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 128\beta_{10}^3(\Omega^2-\beta_7) + \\
& + 88\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) - 80\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8)\} + \\
& + \bar{B}^8\{80\beta_8\beta_{10}^3 - 340\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 128\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} + 440\beta_{10}^4\bar{B}^{10} \\
\hat{\beta}_7 & = \{\gamma^2\Omega^2[4(\beta_6\Omega^2-\beta_8)^2 + \beta_6^2(\beta_7-\Omega^2)^2] + 4\beta_6^2(\gamma\Omega)^4+(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8)^2\} + \\
& + \bar{B}^2\{2\gamma^2\Omega^2[\beta_6^2(\beta_7-\Omega^2)(\beta_9-\beta_8) + 2\beta_6(\beta_6\Omega^2-\beta_8)(\beta_9-\beta_8) - 4\beta_6\beta_{10}(\beta_7-\Omega^2) + \\
& - 16\beta_{10}(\beta_6\Omega^2-\beta_8) + 2\beta_6(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \beta_6^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)]+ 4\beta_6^3(\gamma\Omega)^4+ \\
& - 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) + 2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} + \\
& + \bar{B}^4\{\gamma^2\Omega^2[64\beta_{10}^2 + \beta_6^2(\beta_9-\beta_8)^2 + 6\beta_6^2\beta_{10}(\beta_7-\Omega^2) - 16\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) + \\
& - 16\beta_6\beta_{10}(\beta_9-\beta_8) - 8\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) + 2\beta_6^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)] + \beta_6^4(\gamma\Omega)^4 + \\
& + 4\beta_{10}^2(\Omega^2-\beta_7)^2 + (\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8)^2 - 10\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + \\
& - 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)\} + \\
& + \bar{B}^6\{\gamma^2\Omega^2[2\beta_6^2\beta_{10}(\beta_9-\beta_8) - 24\beta_6\beta_{10}^2 - 10\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8) - 4\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_9)] + \\
& + 40\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)+ 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 4\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)^2+
\end{aligned}
$$

$$- 10\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{B}^8\{21\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 25\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 40\beta_{10}^3(\Omega^2-\beta_7) +$$

$$+ 40\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)\} +$$

$$- \bar{B}^{10}\{40\beta_{10}^3(\beta_6\Omega^2-\beta_9) + 100\beta_{10}^3(\beta_6\Omega^2-\beta_8)\} + 100\beta_{10}^4\ \bar{B}^{12}$$

APPENDIX C DERIVATION OF EQUATIONS (12)

Equations (12) are derived from equations (7) using the method of averaging which is often employed in nonlinear vibration problems.

To apply the method of averaging to equations (7), let

$$A = A_0(t)\cos[\omega t + \phi_0(t)] = A_0 \cos\chi_1 \quad \text{(C1-1)}$$

$$B = B_0(t)\sin[\omega t + \psi_0(t)] = B_0 \sin\chi_2 \quad \text{(C1-2)}$$

where A_0, B_0, ϕ_0 and ψ_0 are assumed to be slowly varying functions of time t, and

$$\chi_1 = \omega t + \phi_0(t)$$

$$\chi_2 = \omega t + \psi_0(t)$$

Taking the derivative of A and B respectively gives

$$\frac{dA}{dt} = \frac{dA_0}{dt}\cos\chi_1 - A_0\omega\sin\chi_1 - A_0\frac{d\phi_0}{dt}\sin\chi_1 \quad \text{(C2-1)}$$

$$\frac{dB}{dt} = \frac{dB_0}{dt}\sin\chi_2 + B_0\omega\cos\chi_2 + B_0\frac{d\psi_0}{dt}\cos\chi_2 \quad \text{(C2-2)}$$

Using the assumptions that A_0, B_0, ϕ_0 and ψ_0 are slowly varying functions of time gives

$$\frac{dA_0}{dt}\cos\chi_1 - A_0\frac{d\phi_0}{dt}\sin\chi_1 = 0 \quad \text{(C3-1)}$$

$$\frac{dB_0}{dt}\sin\chi_2 + B_0\frac{d\psi_0}{dt}\cos\chi_2 = 0 \quad \text{(C3-2)}$$

and

$$\frac{dA}{dt} = -A_0\omega\sin\chi_1 \quad \text{(C4-1)}$$

$$\frac{dB}{dt} = B_0\omega\cos\chi_2 \quad \text{(C4-2)}$$

The second derivatives $\frac{d^2A}{dt^2}$ and $\frac{d^2B}{dt^2}$ are then, computed from equations (C4)

$$\frac{d^2A}{dt^2} = -\frac{dA_0}{dt}\omega\sin\chi_1 - A_0\omega^2\cos\chi_1 - A_0\frac{d\phi_0}{dt}\omega\cos\chi_1 \quad \text{(C5-1)}$$

$$\frac{d^2B}{dt^2} = +\frac{dB_0}{dt}\omega\cos\chi_2 - B_0\omega^2\sin\chi_2 - B_0\frac{d\psi_0}{dt}\omega\sin\chi_2 \quad \text{(C5-2)}$$

Substituting the equations (C5), (C4) and (C1) into equations (7) yields

$$\bar{\alpha}_1\{-\frac{dA_0}{dt}\,\omega\sin\chi_1 - A_0\omega^2\cos\chi_1 - A_0\frac{d\phi_0}{dt}\,\omega\cos\chi_1\} + \bar{\alpha}_2\{-A_0\omega\sin\chi_1\} + \bar{\alpha}_3\{A_0\cos\chi_1\} +$$

$$+ \bar{\alpha}_4\,\frac{\ell_\ell^2 Rh}{2}\{(A_0\omega\sin\chi_1)^2 - A_0\frac{dA_0}{dt}\omega\sin\chi_1\cos\chi_1 - (A_0\omega\cos\chi_1)^2 - A_0^2\frac{d\phi_0}{dt}\,\omega\cos^2\chi_1 +$$

$$+ (B_0\omega\cos\chi_2)^2 + B_0\frac{dB_0}{dt}\omega\sin\chi_2\cos\chi_2 - (B_0\omega\sin\chi_2)^2 - B_0^2\frac{d\phi_0}{dt}\omega\sin^2\chi_2 +$$

$$+ \delta_{n,\ell}(-\frac{dA_0}{dt}\,\omega\sin\chi_1 - A_0\omega^2\cos\chi_1 - A_0\frac{d\phi_0}{dt}\omega\cos\chi_1)(\delta_2+\hat{\delta}_2)\}\ \{A_0\cos\chi_1 +$$

$$+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\} +$$

$$+ \bar{\alpha}_5\,\frac{\ell_\ell^2 Rh}{2}\,\{-A_0^2\omega\sin\chi_1\cos\chi_1 + B_0^2\omega\cos\chi_2\sin\chi_2 - A_0\omega\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_1\}$$

$$\{A_0\cos\chi_1 + \delta_{n,\ell}\,(\delta_2+\hat{\delta}_2)\} +$$

$$+ \bar{\alpha}_6\,A_0^2\cos^2\chi_1 + \bar{\alpha}_7\,\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\} + \bar{\alpha}_8\,\{(A_0\cos\chi_1)^2 +$$

$$- (B_0\sin\chi_2)^2\} + \bar{\alpha}_9\,(A_0\cos\chi_1)^3 + \bar{\alpha}_{10}\,\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}\,A_0\cos\chi_1 +$$

$$+ \bar{\alpha}_{11}\,\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}\,(A_0\cos\chi_1)^2 + \bar{\alpha}_{12}\,\{(A_0\cos\chi_1)^2 +$$

$$+ (B_0\sin\chi_2)^2\}^2 + \bar{\alpha}_{13}\,\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}^2\,A_0\cos\chi_1 = F_D\,\cos(\chi_1-\phi_0) \qquad \text{(C6-1)}$$

and

$$\bar{\beta}_1\{+\frac{dB_0}{dt}\,\omega\cos\chi_2 - B_0\omega^2\sin\chi_2 - B_0\frac{d\psi_0}{dt}\,\omega\sin\chi_2\} + \bar{\beta}_2\{B_0\omega\cos\chi_2\} + \bar{\beta}_3\{B_0\sin\chi_2\} +$$

$$+ \bar{\beta}_4\,\frac{\ell_\ell^2 Rh}{2}\{(A_0\omega\sin\chi_1)^2 - A_0\frac{dA_0}{dt}\,\omega\sin\chi_1\cos\chi_1 - (A_0\omega\cos\chi_1)^2 +$$

$$- A_0^2\frac{d\phi_0}{dt}\,\omega\cos^2\chi_1 + B_0^2\,\omega^2\cos^2\chi_2 + B_0\frac{dB_0}{dt}\,\omega\sin\chi_2\cos\chi_2 - (B_0\omega\sin\chi_2)^2 +$$

$$- B_0^2\frac{d\psi_0}{dt}\,\omega\sin^2\chi_2 + \delta_{n,\ell}(-\frac{dA_0}{dt}\,\omega\sin\chi_1 - (A_0\omega^2\cos\chi_1 - A_0\frac{d\phi_0}{dt}\omega\cos\chi_1)(\delta_2+\hat{\delta}_2)\}$$

$$B_0\sin\chi_2 +$$

$$+ \bar{\beta}_5\,\frac{\ell_\ell^2 Rh}{2}\{-A_0^2\omega\,\sin\chi_1\cos\chi_1 + B_0^2\omega\sin\chi_2\cos\chi_2 - A_0\omega\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_1\}\,B_0\sin\chi_2 +$$

$$+ \bar{\beta}_6\,A_0B_0\sin\chi_2\cos\chi_1 + \bar{\beta}_7\,A_0^2B_0\sin\chi_2\cos\chi_1^2 + \bar{\beta}_8\,\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}$$

$$B_0\sin\chi_2$$

$$+ \bar{\beta}_9\,\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}\,A_0B_0\sin\chi_2\cos\chi_1 + \bar{\beta}_{10}\,\{(A_0\cos\chi_1)^2 +$$

$$+ (B_0 \sin\chi_2)^2\} B_0 \sin\chi_2 = 0 \tag{C6-2}$$

Both sides of equation (C6-1) are multiplied by $\cos\chi_1$, and the results are added to equation (C3-1) after the latter has been multiplied by $\omega\sin\chi_1$. This procedure yields

$$\bar{\alpha}_1 \{- A_0\omega^2\cos^2\chi_1 - A_0 \frac{d\phi_0}{dt} \omega\} + \bar{\alpha}_2\{- A_0\omega\sin\chi_1\cos\chi_1\} + \bar{\alpha}_3\{A_0\cos^2\chi_1\} +$$

$$+ \bar{\alpha}_4 \frac{\ell_\ell^2 Rh}{2} \{A_0^3\omega^2\sin^2\chi_1\cos^2\chi_1 - A_0^2 \frac{dA_0}{dt} \omega\sin\chi_1\cos^3\chi_1 - A_0^3\omega^2\cos^4\chi_1 - A_0^3 \frac{d\phi_0}{dt}$$

$$\omega\cos^4\chi_1 +$$

$$+ B_0^2A_0\omega^2\cos^2\chi_1\cos^2\chi_2 + A_0B_0 \frac{dB_0}{dt} \omega\sin\chi_2\cos\chi_2\cos^2\chi_1 - A_0B_0^2\omega^2\sin^2\chi_2\cos^2\chi_1 +$$

$$+ B_0^2A_0 \frac{d\psi_0}{dt} \omega\sin^2\chi_2\cos^2\chi_1 + \delta_{n,\ell}(-A_0 \frac{dA_0}{dt} \omega\sin\chi_1\cos^2\chi_1 - A_0^2\omega^2\cos^3\chi_1 +$$

$$- A_0^2 \frac{d\phi_0}{dt} \omega\cos^3\chi_1)(\delta_2+\hat{\delta}_2) + \delta_{n,\ell}(A_0^2\omega^2\sin^2\chi_1\cos\chi_1 - A_0 \frac{dA_0}{dt} \omega\sin\chi_1\cos^2\chi_1 +$$

$$- A_0^2\omega^2\cos^3\chi_1 - A_0^2 \frac{d\phi_0}{dt} \omega\cos^3\chi_1 + B_0^2\omega^2\cos^2\chi_2\cos\chi_1 + B_0 \frac{dB_0}{dt}\omega\sin\chi_2\cos\chi_2\cos\chi_1 +$$

$$- B_0^2\omega^2\sin^2\chi_2\cos\chi_1 - B_0^2 \frac{d\psi_0}{dt}\omega\sin^2\chi_2\cos\chi_1 - \frac{dA_0}{dt} \omega\sin\chi_1\cos\chi_1 - A_0\omega^2\cos^2\chi_1 +$$

$$- A_0 \frac{d\phi_0}{dt} \omega\cos^2\chi_1) (\delta_2+\hat{\delta}_2)^2\} +$$

$$+ \bar{\alpha}_5 \frac{\ell_\ell^2 Rh}{4} \{- A_0^3\omega\sin\chi_1\cos^3\chi_1 + A_0B_0^2\omega\sin\chi_2\cos\chi_2\cos^2\chi_1 - \delta_{n,\ell} A_0^2\omega\sin\chi_1\cos^2\chi_2$$

$$(\delta_2+\hat{\delta}_2) +$$

$$+ \delta_{n,\ell}(-A_0^2\omega\sin\chi_1\cos^2\chi_1 + B_0^2\omega\sin\chi_2\cos\chi_2\cos\chi_1 - \dot{A}_0\omega(\delta_2+\hat{\delta}_2)\sin\chi_1\cos\chi_1)$$

$$(\delta_2+\hat{\delta}_2)\} +$$

$$+ \bar{\alpha}_6 A_0^2\cos^3\chi_1 + \bar{\alpha}_7 \{A_0^2\cos^3\chi_1 + B_0^2\sin^2\chi_2\cos\chi_1\} + \bar{\alpha}_8 \{A_0^2\cos^3\chi_1 +$$

$$- B_0^2\sin^2\chi_2\cos\chi_1\} + \bar{\alpha}_9 A_0^3\cos^4\chi_1 + \bar{\alpha}_{10}\{A_0^3\cos^4\chi_1 + A_0B_0^2\sin^2\chi_2\cos^2\chi_1\} +$$

$$+ \bar{\alpha}_{11}\{A_0^4\cos^5\chi_1 + A_0^2B_0^2\sin^2\chi_2\cos^3\chi_1\} + \bar{\alpha}_{12}\{A_0^2\cos^2\chi_1 + B_0^2\sin^2\chi_2\}^2A_0\cos\chi_1 +$$

$$+ \bar{\alpha}_{13}\{A_0^2\cos^2\chi_1 + B_0^2\sin^2\chi_2\}^2A_0^2\cos^2\chi_1 = F_D\cos(\chi_1-\phi_0)\cos\chi_1 \tag{C-7}$$

In this state of the analysis, this equation is "averaged" by integrating over one period on χ_1 or χ_2. In the integration, A_0, B_0, ϕ_0 and ψ_0 are approximated by their average values $\bar{A}$, $\bar{B}$, $\bar{\phi}$ and $\bar{\psi}$, for example,

$$\int_0^{2\pi} A_0(t)\cos^2\chi_1 d\chi_1 \approx \int_0^{2\pi} \bar{A}\cos^2\chi_1 d\chi_1 = \bar{A}\pi$$

$$\int_0^{2\pi} A_0^3(t)\frac{d\phi_0}{dt}\cos^3\chi_1 d\chi_1 \approx \int_0^{2\pi} \bar{A}^3\frac{d\bar{\phi}}{dt}\cos^3\chi_1 d\chi_1 = 0$$

$$\int_0^{2\pi} F_D\cos\phi_0 d\chi_1 = 2\pi\bar{F}_d\cos\bar{\phi}$$

$$\int_0^{2\pi} A_0B_0^2\cos 2(\phi_0-\psi_0)d\chi_1 = 2\pi\bar{A}\bar{B}^2\cos 2\bar{\Delta}$$

where $d\bar{\phi}/dt$ and $\bar{\Delta}$ are the average value of $d\phi_0/dt$ and $\phi_0-\psi_0$, respectively.

When equation (C7) is averaged in this fashion, it becomes

$$\bar{\alpha}_1\{-2\pi\bar{A}\frac{d\bar{\phi}}{dt}\omega - \pi\bar{A}\omega^2\} + \bar{\alpha}_2\{0\} + \bar{\alpha}_3\{\pi\bar{A}\} + \bar{\alpha}_4\frac{\ell_\ell^2 Rh}{2}\{-\frac{1}{2}\pi\bar{A}^3\omega^2 - \frac{3}{4}\pi\bar{A}^3\frac{d\bar{\phi}}{dt}\omega +$$
$$+\frac{1}{2}\pi\bar{A}\bar{B}^2\omega^2(1+\frac{1}{2}\cos 2\bar{\Delta}) - \frac{1}{4}\pi\bar{A}\bar{B}\frac{d\bar{B}}{dt}\omega\sin 2\bar{\Delta} - \frac{1}{2}\pi\bar{A}\bar{B}^2\omega^2(1-\frac{1}{2}\cos 2\bar{\Delta}) +$$
$$+\frac{1}{2}\pi\bar{A}\bar{B}^2\frac{d\bar{\psi}}{dt}\omega(1-\frac{1}{2}\cos 2\bar{\Delta}) + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\{-\pi\bar{A}\omega^2 - \pi\bar{A}\frac{d\bar{\phi}}{dt}\omega\} +$$
$$+\bar{\alpha}_5\frac{\ell_\ell^2 Rh}{2}\{-\frac{\pi}{4}\bar{A}\bar{B}^2\omega\sin 2\bar{\Delta}\} + \bar{\alpha}_6\{0\} + \bar{\alpha}_7\{0\} + \bar{\alpha}_8\{0\} + \bar{\alpha}_9\{\frac{3}{4}\pi\bar{A}^3\} +$$
$$+\bar{\alpha}_{10}\{\frac{3}{4}\pi\bar{A}^3 + \frac{\pi}{2}\bar{A}\bar{B}^2(1-\frac{1}{2}\cos 2\bar{\Delta})\} + \bar{\alpha}_{11}\{0\} + \bar{\alpha}_{12}\{0\} +$$
$$+\bar{\alpha}_{13}\{\frac{5}{8}\pi\bar{A}^5 + \frac{1}{2}\pi\bar{A}^3\bar{B}^2(\frac{3}{2} - \cos 2\bar{\Delta}) + \frac{1}{4}\pi\bar{A}\bar{B}^4(\frac{3}{2} - \cos 2\bar{\Delta})\} = \pi\bar{F}_d\cos\bar{\phi} \qquad \text{(C-8)}$$

It should be noted that the steady-state vibrations are studied in the present analysis, which means the average values $\bar{A}$ and $\bar{\phi}$ remain steady (i.e., constant) with time. In this case, the average derivatives $d\bar{A}/dt$, $d\bar{B}/dt$, $d\bar{\phi}/dt$ and $d\bar{\psi}/dt$ are identically zero, and equation (C8) can be reduced to

$$-\bar{\alpha}_1\bar{A}\omega^2 + \bar{\alpha}_3\bar{A} + \bar{\alpha}_4\frac{\ell_\ell^2 Rh}{2}\{-\frac{1}{2}\bar{A}^3\omega^2 + \frac{1}{2}\bar{A}\bar{B}^2\omega^2(1+\frac{1}{2}\cos 2\bar{\Delta}) - \frac{1}{2}\bar{A}\bar{B}^2\omega^2$$
$$(1-\frac{1}{2}\cos 2\bar{\Delta}) - \delta_{n,\ell}\bar{A}\omega^2(\delta_2+\hat{\delta}_2)^2\} + \bar{\alpha}_5\frac{\ell_\ell^2 Rh}{2}\{-\frac{1}{4}\bar{A}\bar{B}^2\omega\sin 2\bar{\Delta}\} + \bar{\alpha}_9\frac{3}{4}\bar{A}^3 +$$
$$+\bar{\alpha}_{10}\{\frac{3}{4}\bar{A}^3 + \frac{1}{2}\bar{A}\bar{B}^2(1-\frac{1}{2}\cos 2\bar{\Delta})\} + \bar{\alpha}_{13}\{\frac{5}{8}\bar{A}^5 + \frac{1}{2}\bar{A}^3\bar{B}^2(\frac{3}{2} - \cos 2\bar{\Delta}) + \frac{1}{4}\bar{A}\bar{B}^4$$
$$(\frac{3}{2} - \cos 2\bar{\Delta})\} = \bar{F}_d\cos\bar{\phi} \qquad \text{(C-9)}$$

In nondimensional form. equation (C9) is

$$
\begin{aligned}
&- \Omega^2\bar{A}\{1 + \beta_1(\bar{A}^2 - \bar{B}^2\cos 2\bar{\Delta} + 2\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2)\} + \beta_2\bar{A} - \gamma\Omega\bar{A}\bar{B}^2\beta_1\sin 2\bar{\Delta} + \\
&+ \beta_3\bar{A}^3+2\beta_4\bar{A}\bar{B}^2(1-\frac{1}{2}\cos 2\bar{\Delta}) + \beta_5\{5\bar{A}^5 + 4\bar{A}^3\bar{B}^2(\frac{3}{2} - \cos 2\bar{\Delta}) + 2\bar{A}\bar{B}^4(\frac{3}{2} - \cos 2\bar{\Delta})\} \\
&= \bar{F}_D\cos\bar{\phi}
\end{aligned}
\tag{C-10}
$$

where

$$\Omega^2 = \frac{2\bar{\rho}R^2\omega^2}{E}$$

$$\bar{F}_D = \frac{2R^2\bar{F}_d}{Eh^2}$$

and

$$\gamma = c/\sqrt{\frac{2E\bar{\rho}}{R^2}}$$

In a similar fashion, the equation (12-2) is obtained by

(1) Multiplying both sides of equation (C6-1) by sin χ_1

(2) Adding this result to equation (C3-1) after multiplying the latter by $-\omega\cos\chi_1$

(3) Averaging the final equation by the method of averaging.

These manipulations give equation (12-2).

Similarly, the equations (12-3) and (12-4) can be obtained by multiplying both sides of equation (C6-2) by $\sin\chi_2$ respectively $\cos\chi_2$, and then using the procedure mentioned above.

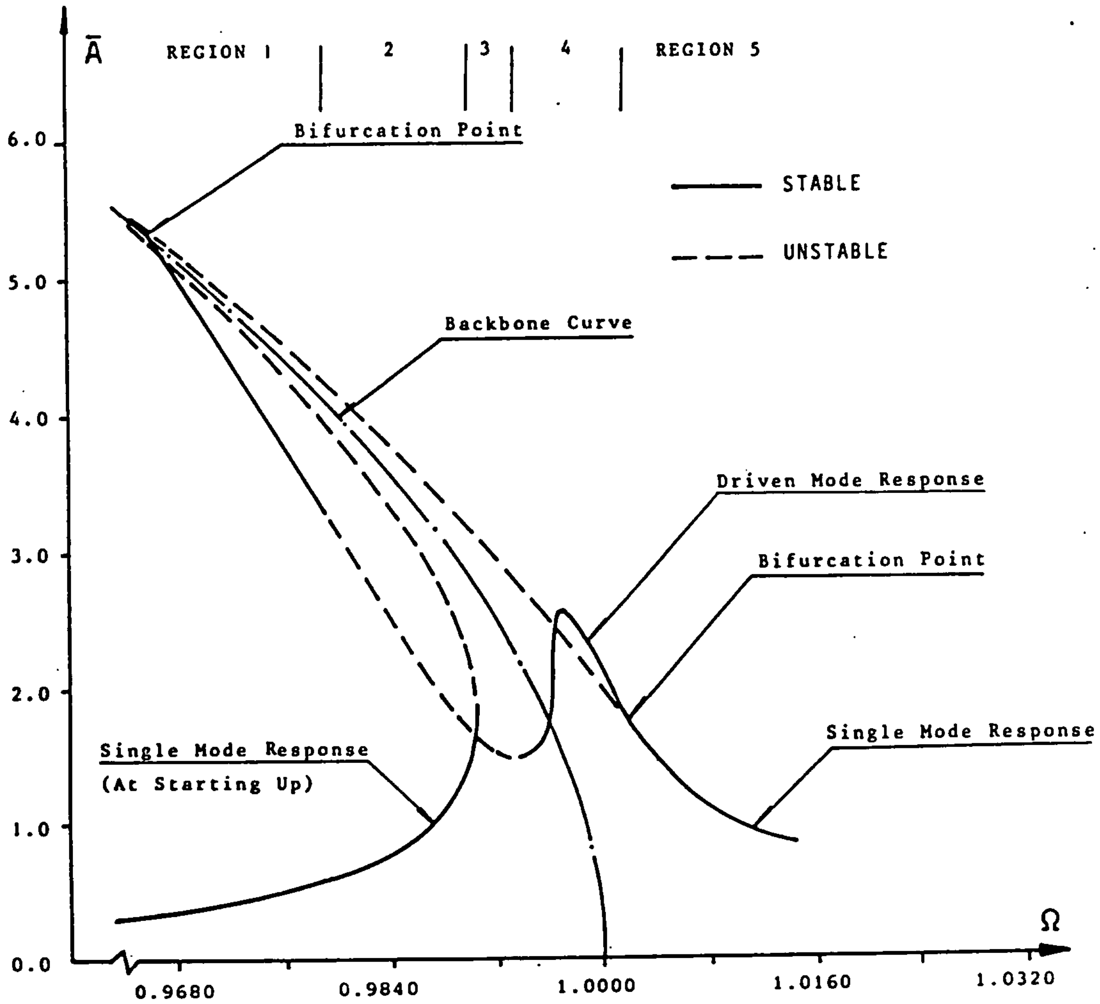

Fig. 1a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL.
DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

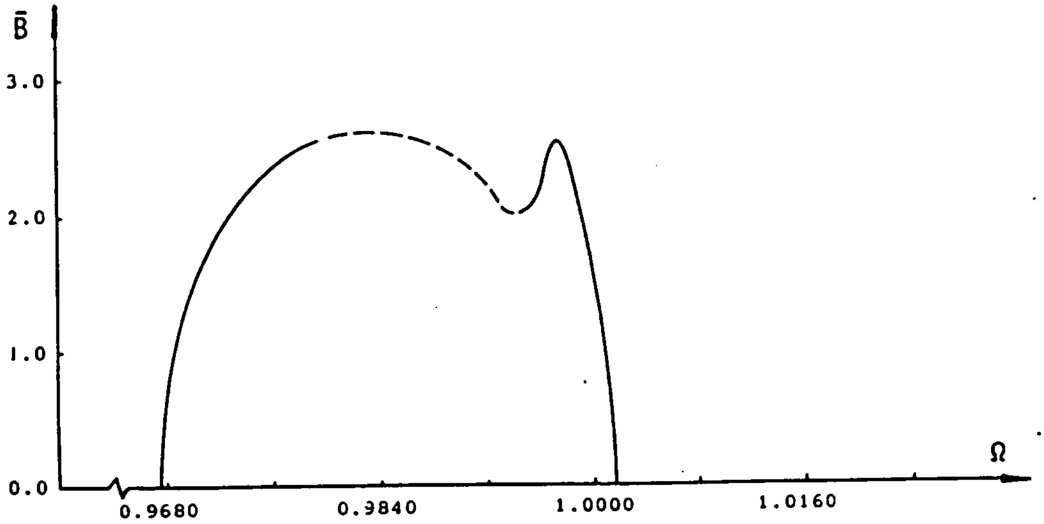

Fig. 1b COMPANION MODE RESPONSE.

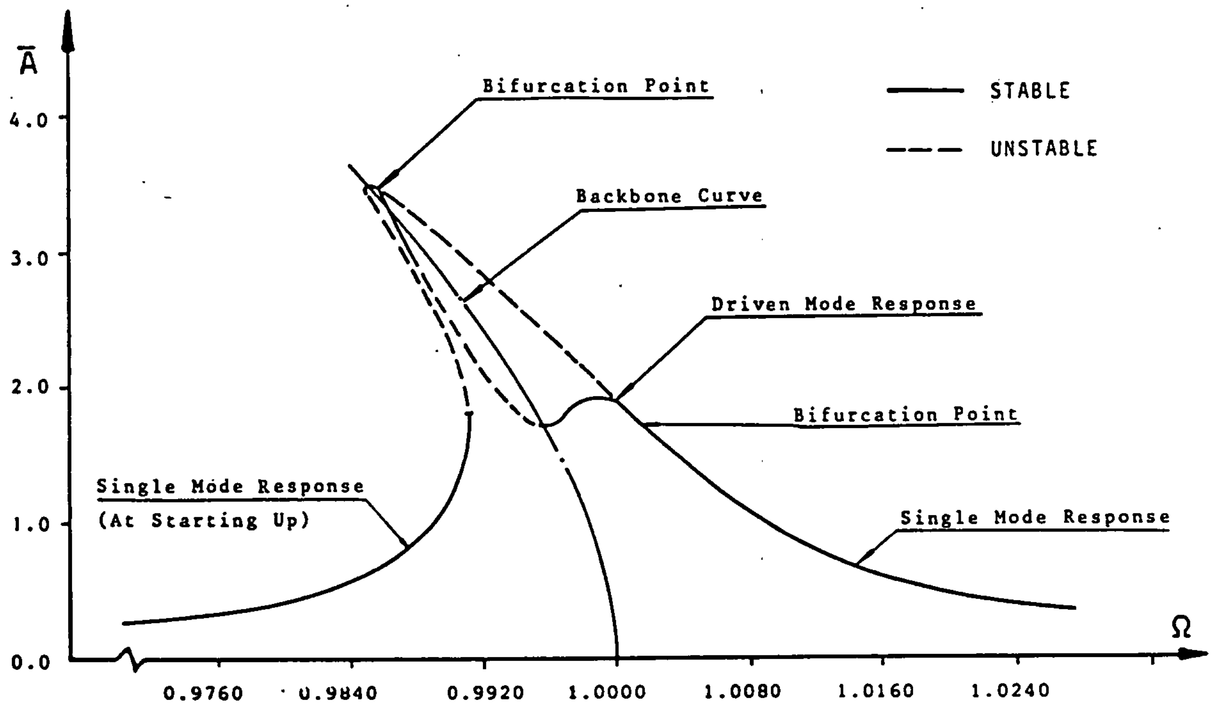

Fig. 2a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL.
DAMPING $\gamma = 1.35\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

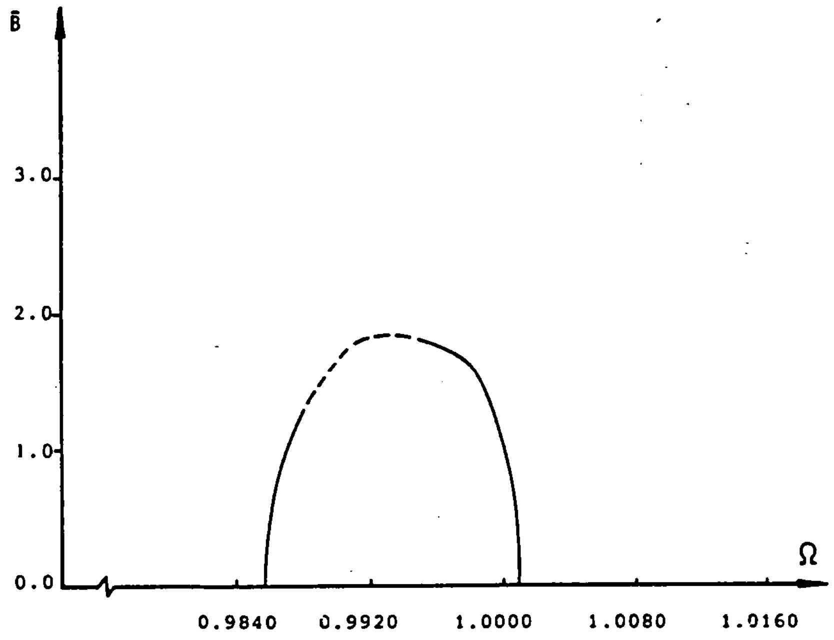

Fig. 2b COMPANION MODE RESPONSE.

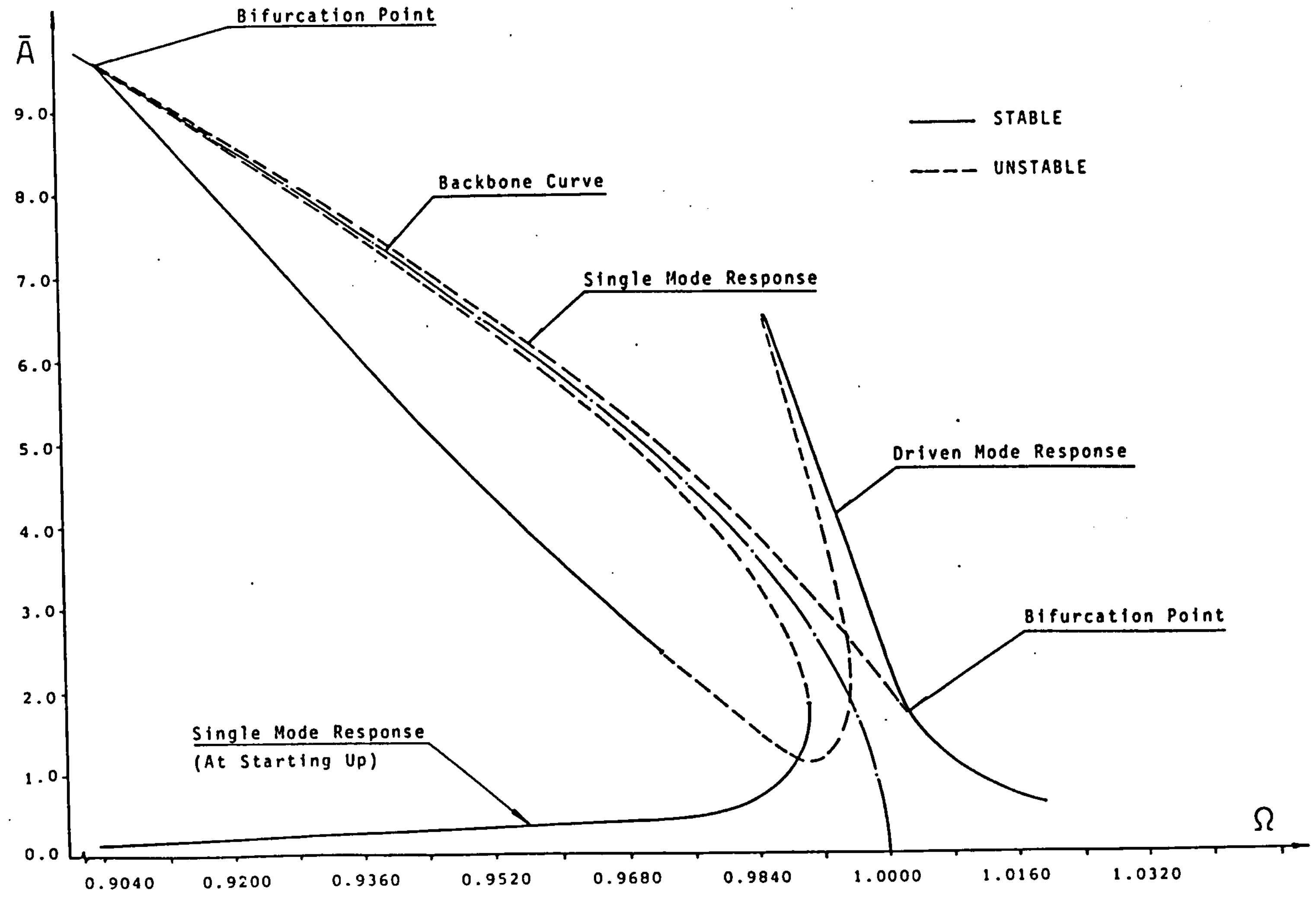

Fig. 3a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL. DAMPING $\gamma = 5\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

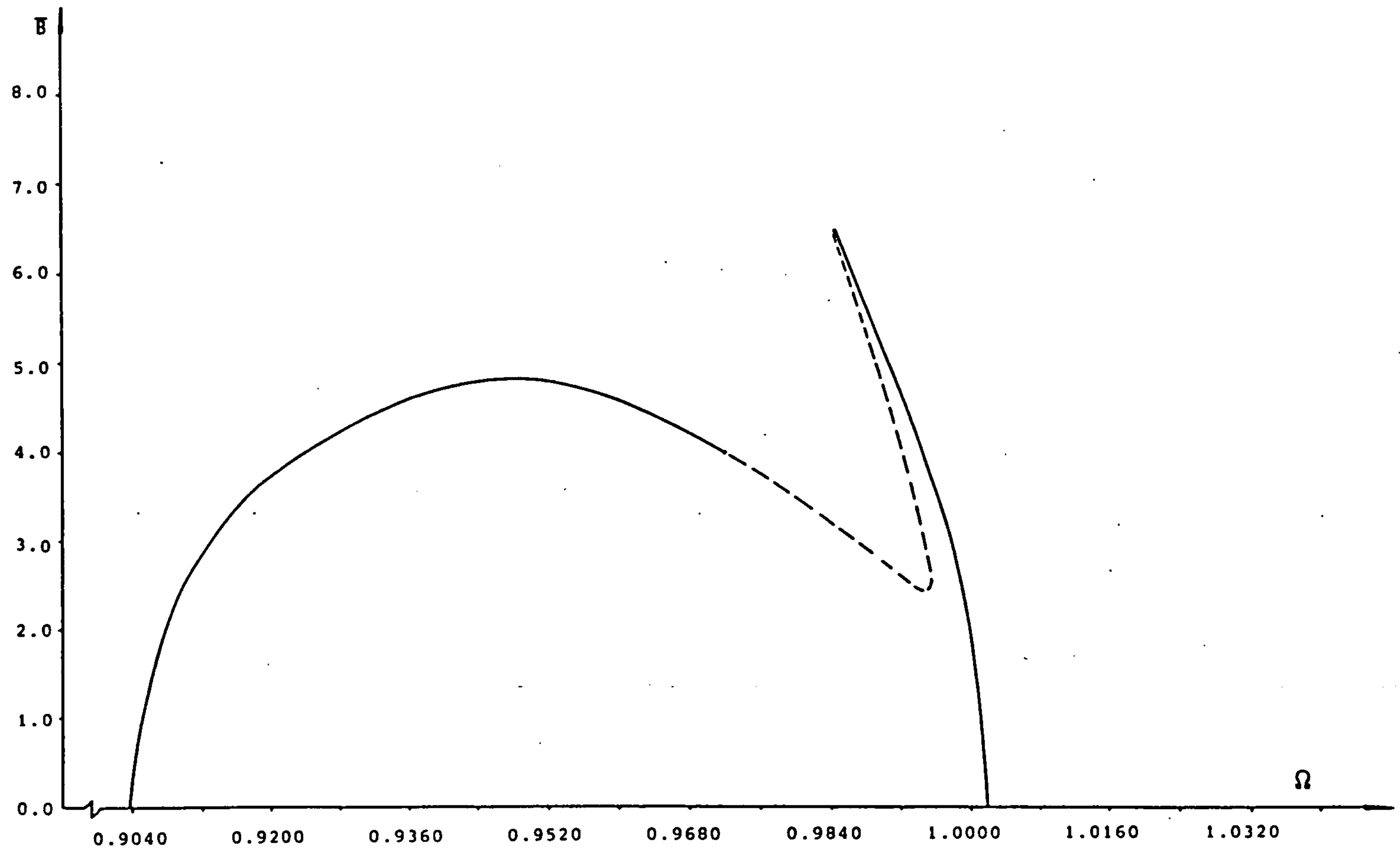

Fig. 3b COMPANION MODE RESPONSE.

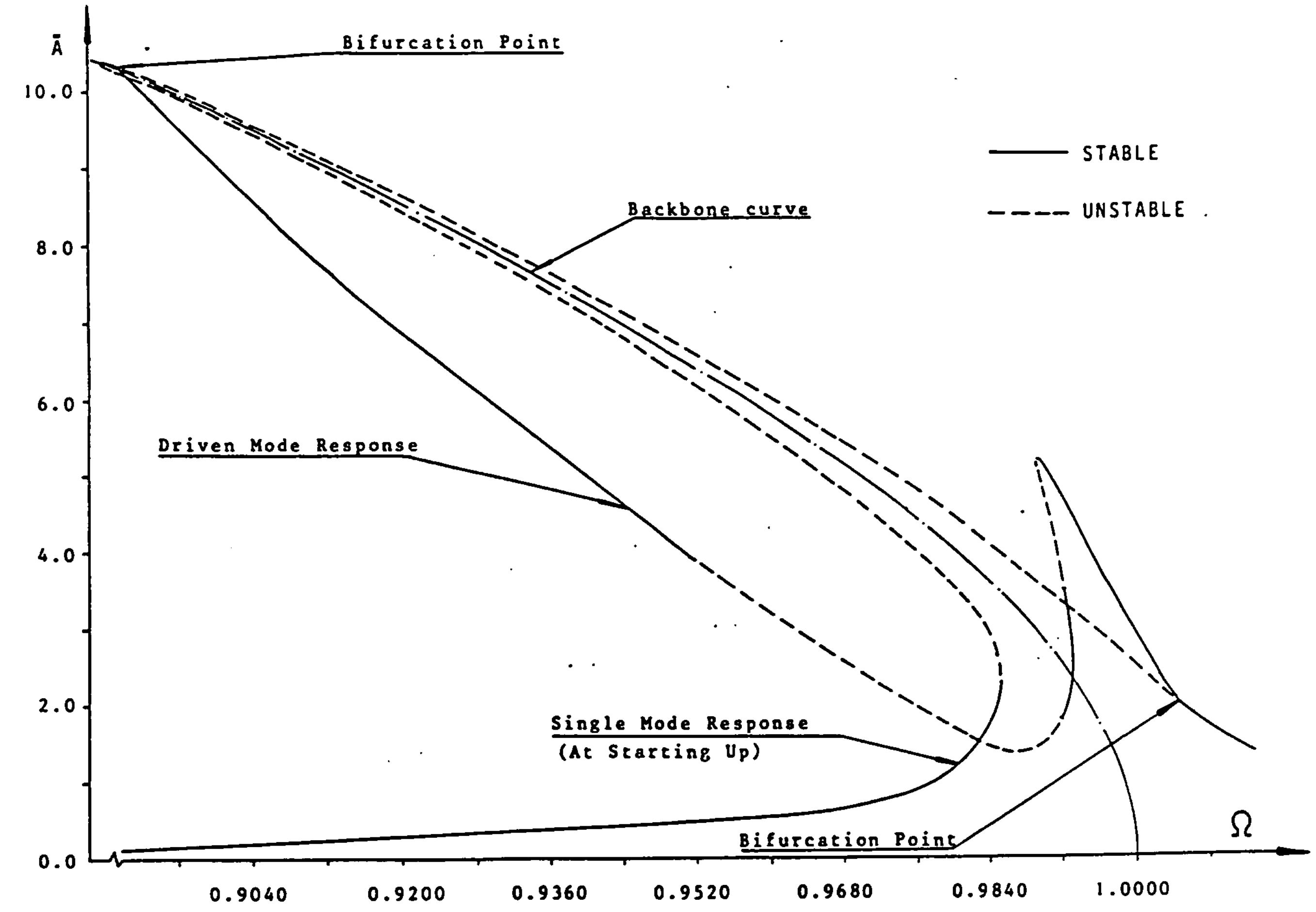

Fig. 4a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 8.50\times10^{-5}$.

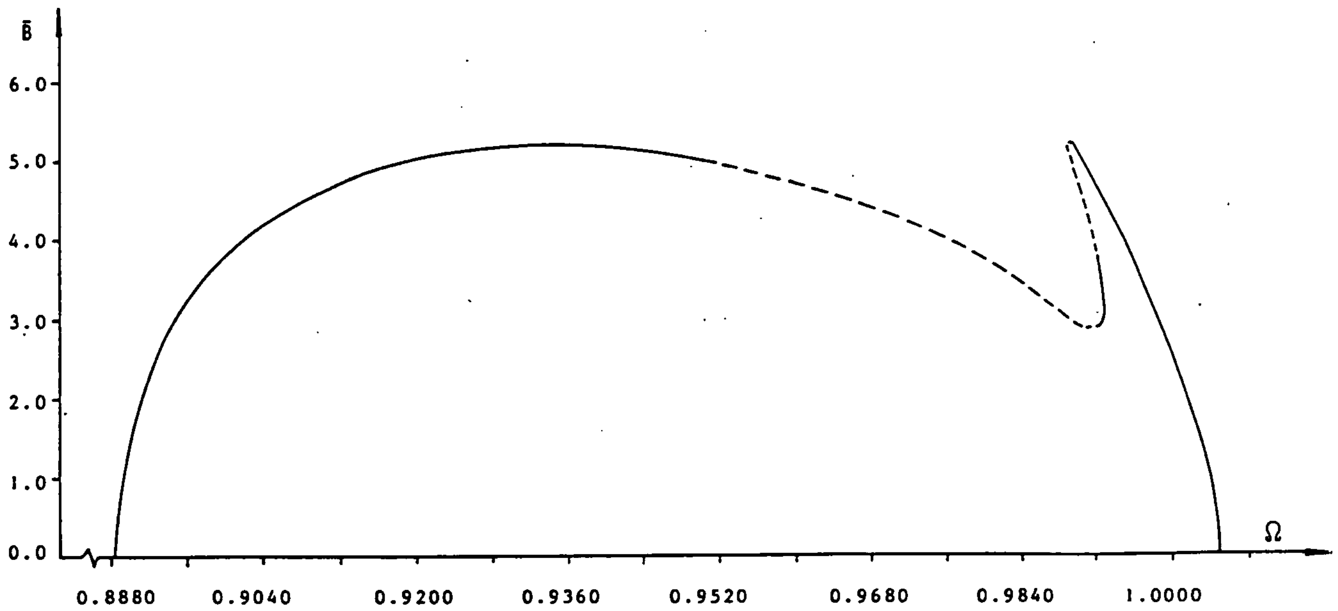

Fig. 4b COMPANION MODE RESPONSE.

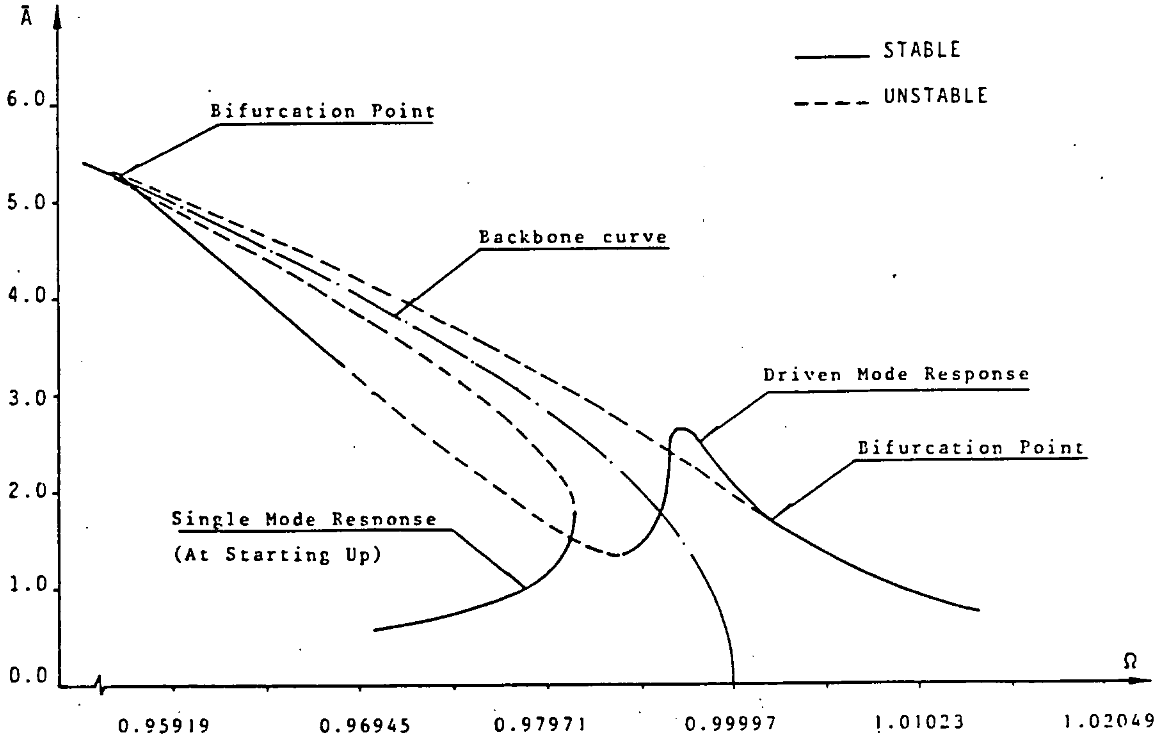

Fig. 5a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = 0.00$ AND $\delta_2 = 0.01$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 7$, $m = k = 1$, $n = 1 = 5$.

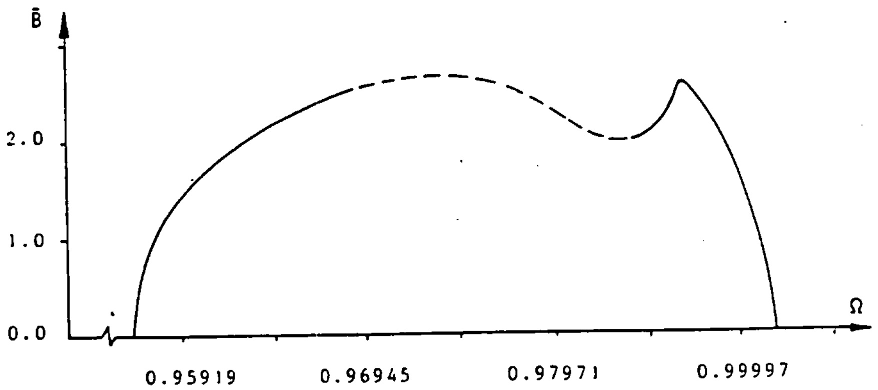

Fig.5b COMPANION MODE RESPONSE.

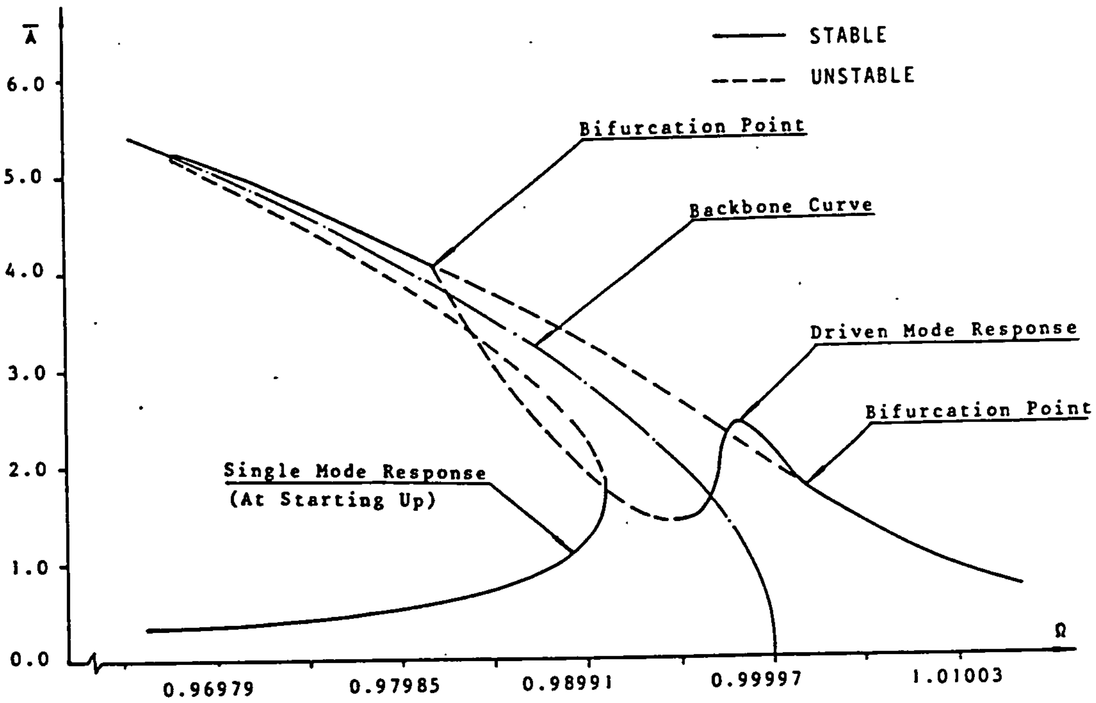

Fig. 6a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = 0.00$ AND $\delta_2 = 0.05$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 7$, $m = k = 1$, $n = 1 = 5$.

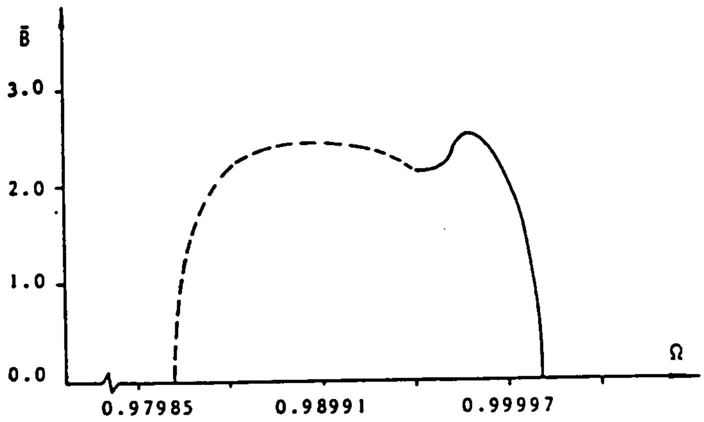

Fig.6b COMPANION MODE RESPONSE.

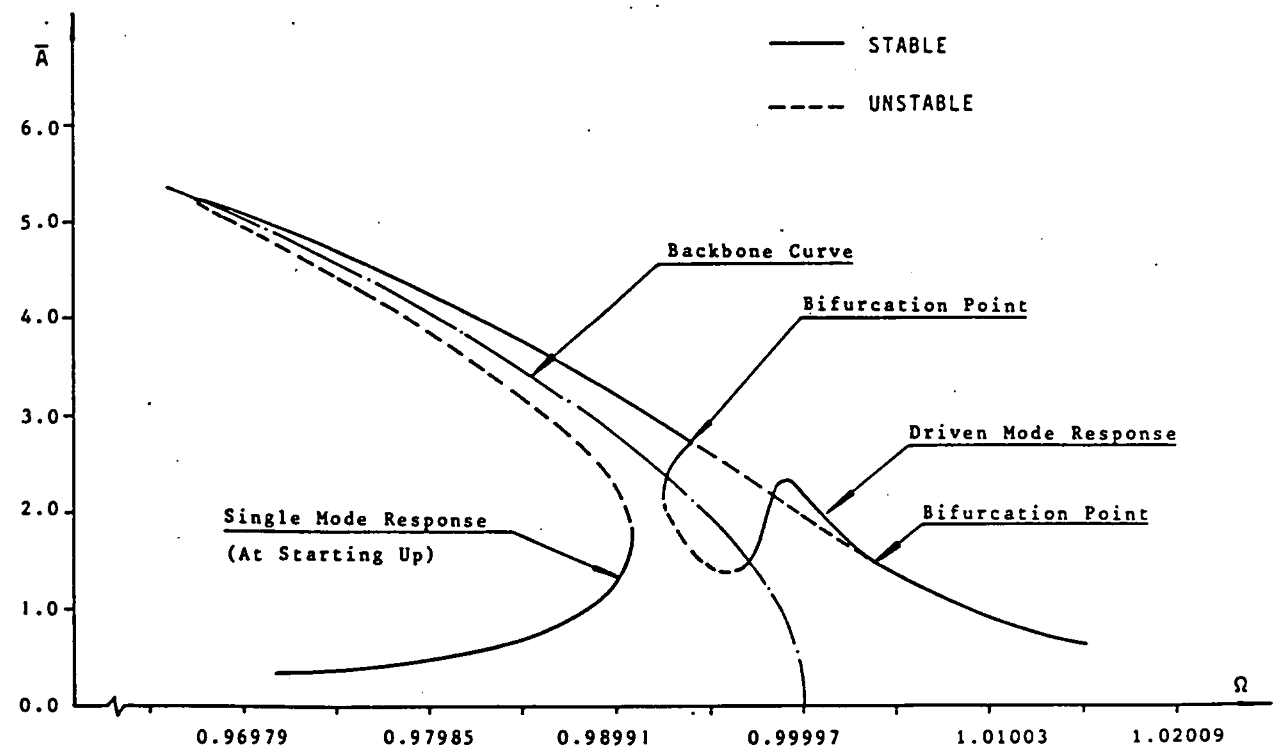

Fig. 7a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = 0.00$ AND $\delta_2 = 0.07$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 7$, $m = k = 1$, $n = 1 = 5$.

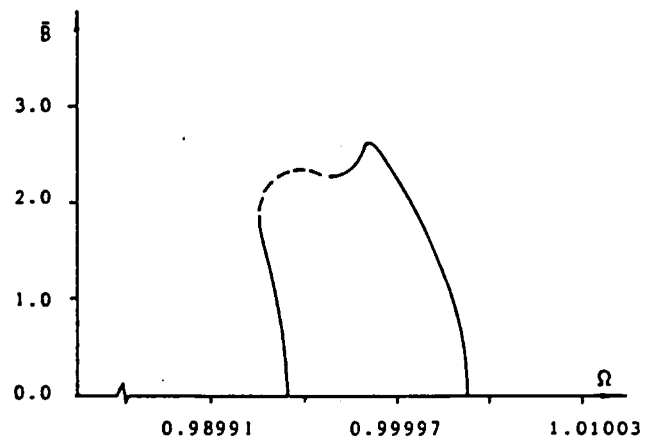

Fig. 7b COMPANION MODE RESPONSE.

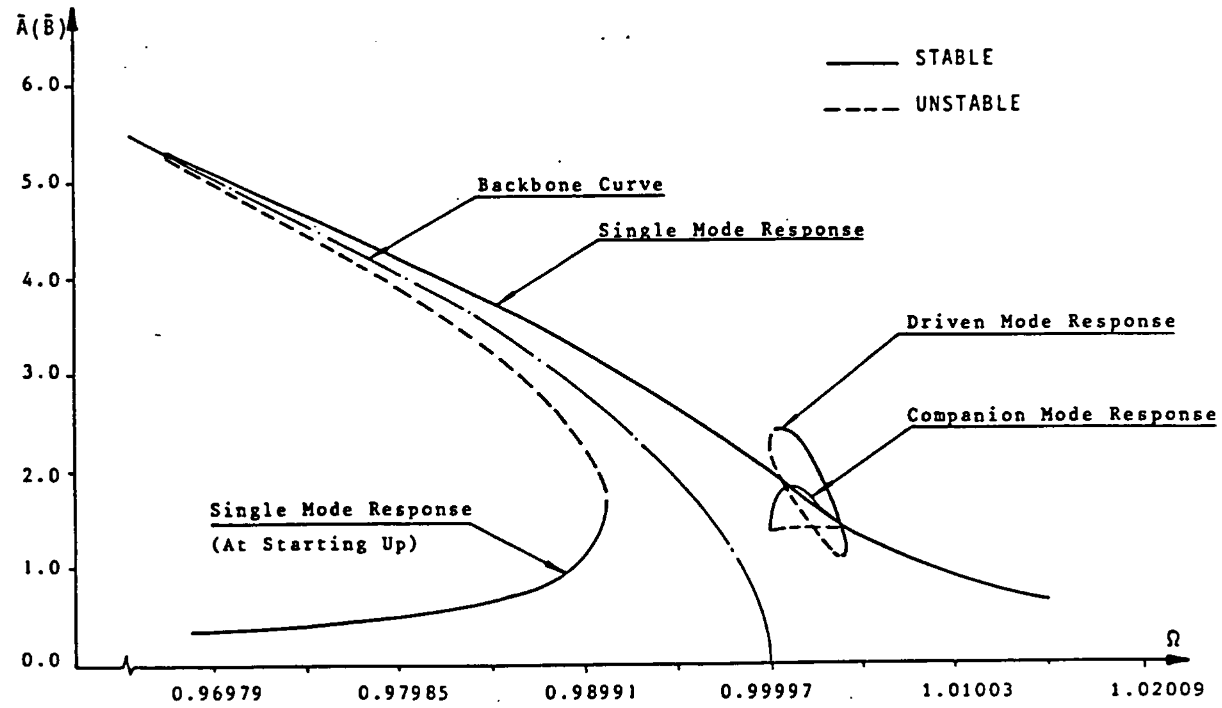

Fig. 8 AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = 0.00$ AND $\delta_2 = 0.10$. DAMPING $\gamma = 9\times10^{-5}$; EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 7$, $m = k = 1$, $n = 1 = 5$.

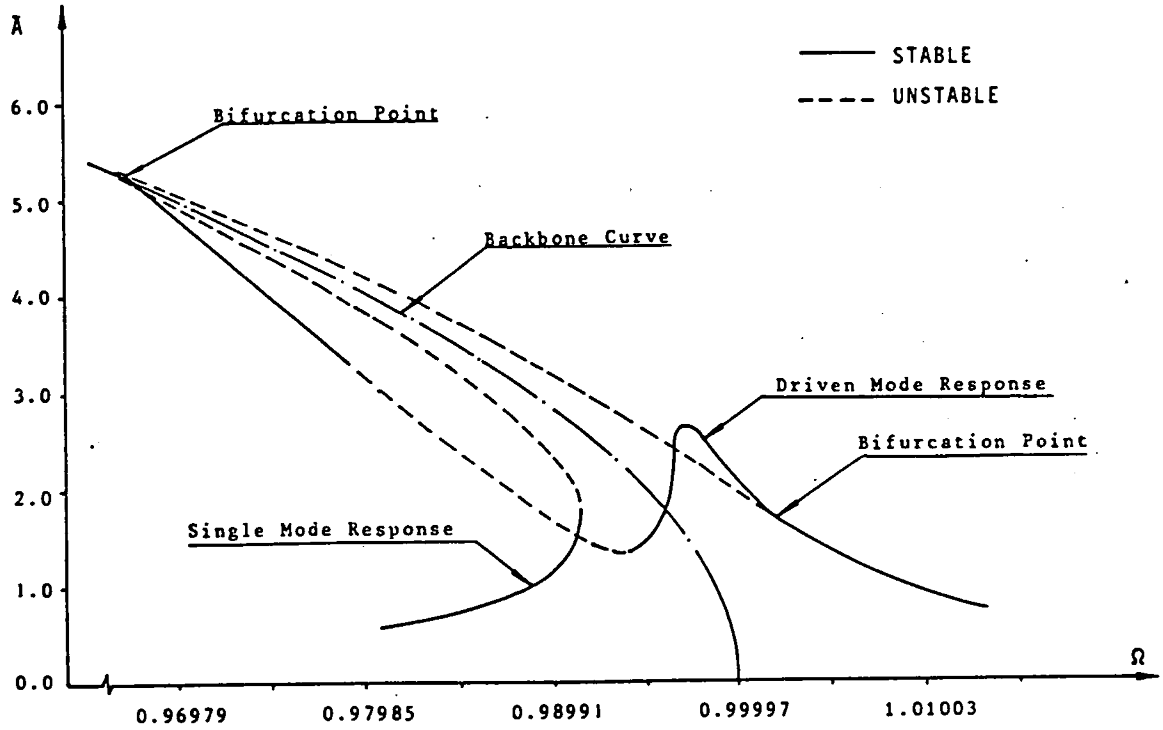

Fig. 9a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = 0.00$ NAD $\delta_2 = 0.05$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS i = 7, m = k = 1, n =10, 1 = 5.

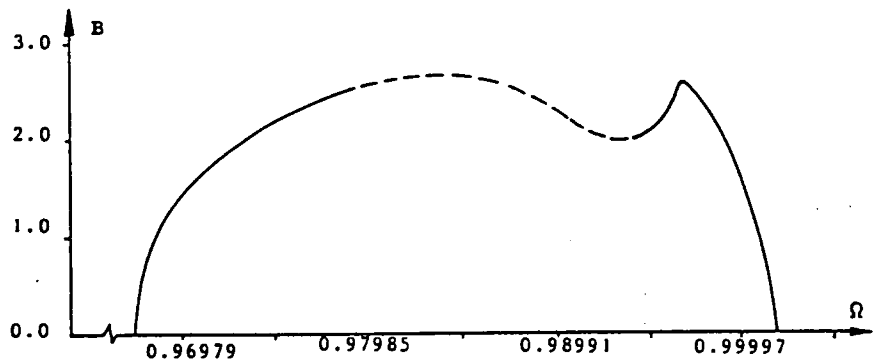

Fig. 9b COMPANION MODE RESPONSE.

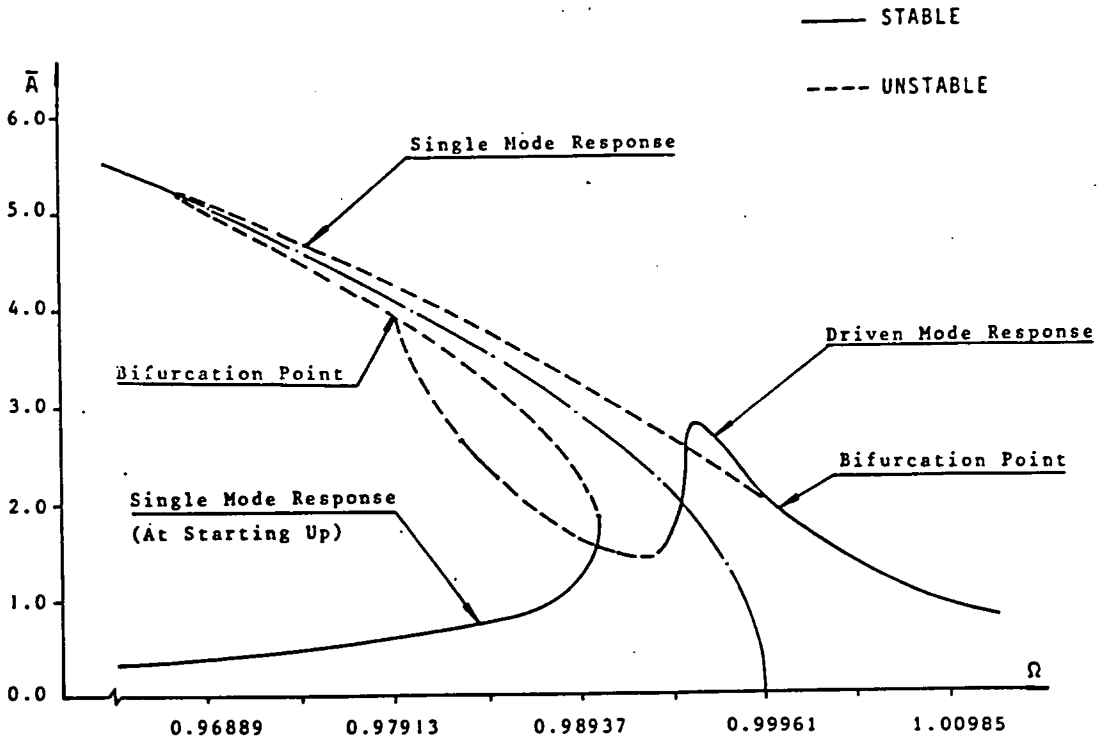

Fig.10a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = -0.02$ AND $\delta_2 = 0.00$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 2$, $m = k = 1$, $n = 10$, $l = 5$.

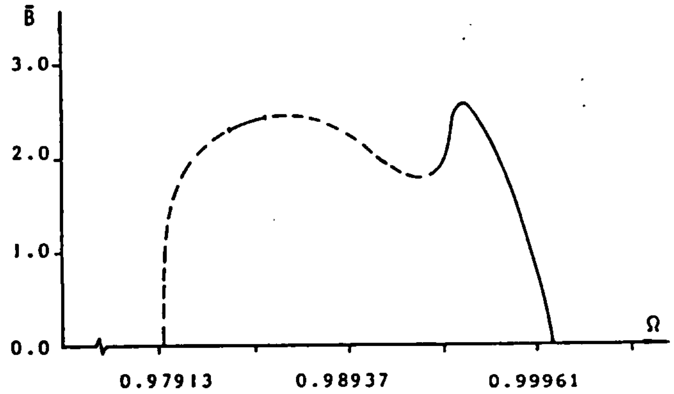

Fig.10b COMPANION MODE RESPONSE.

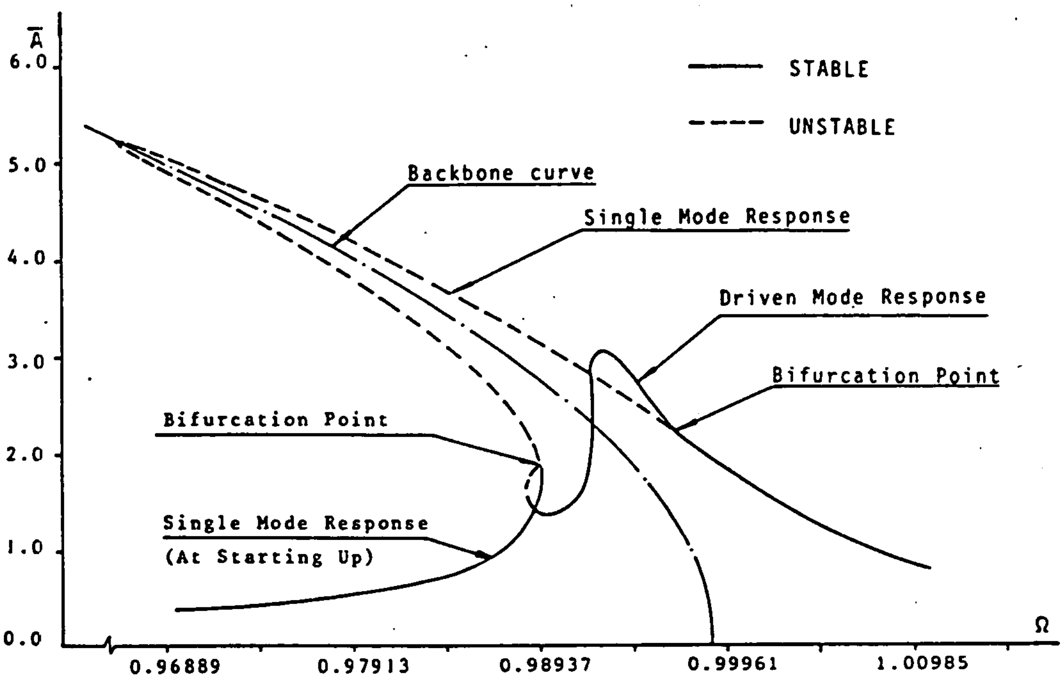

Fig. 11a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = -0.06$ AND $\delta_2 = 0.00$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 2$, $m = k = 1$, $n = 10$, $l = 5$.

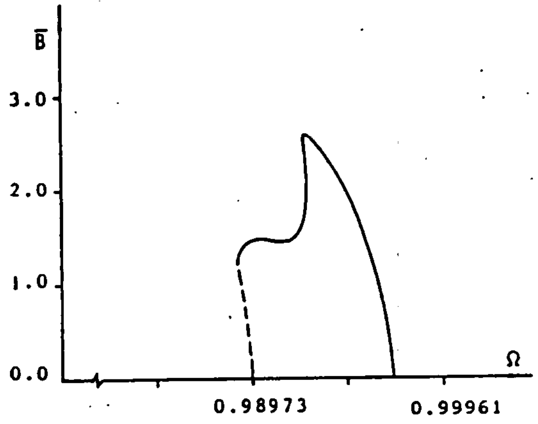

Fig. 11b COMPANION MODE RESPONSE.

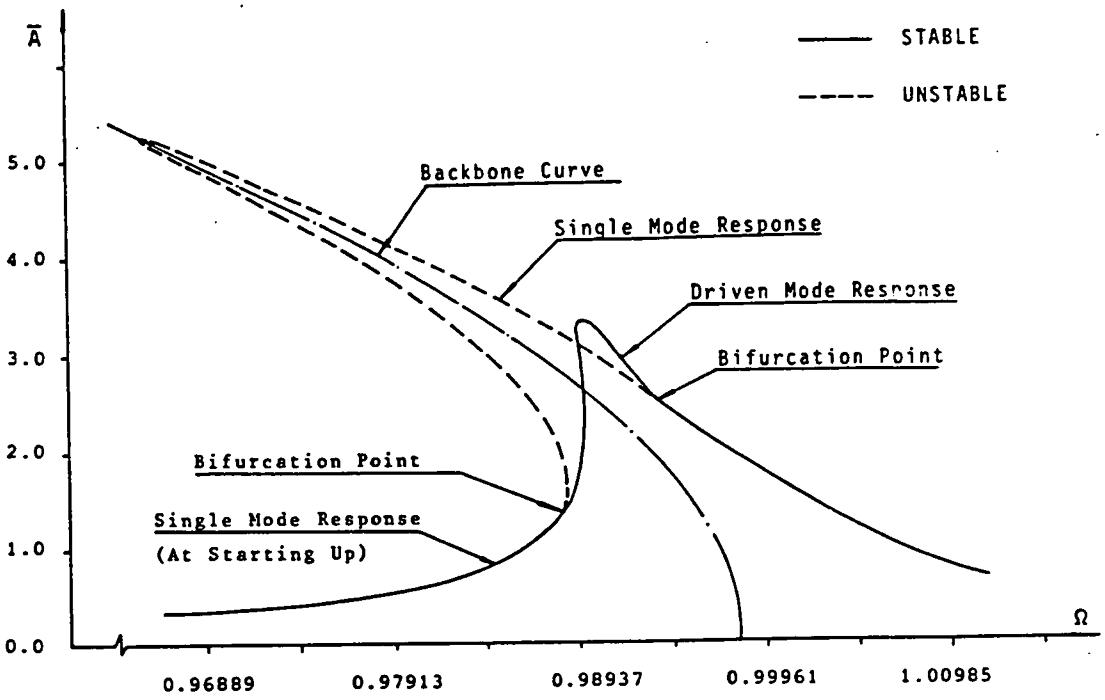

Fig.12a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = -0.10$ AND $\delta_2 = 0.00$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 2$, $m = k = 1$, $n = 10$, $l = 5$.

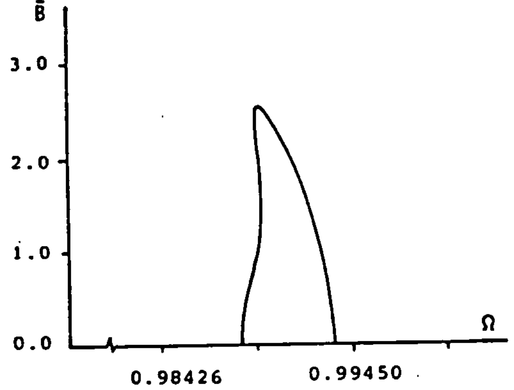

Fig.12b COMPANION MODE RESPONSE.

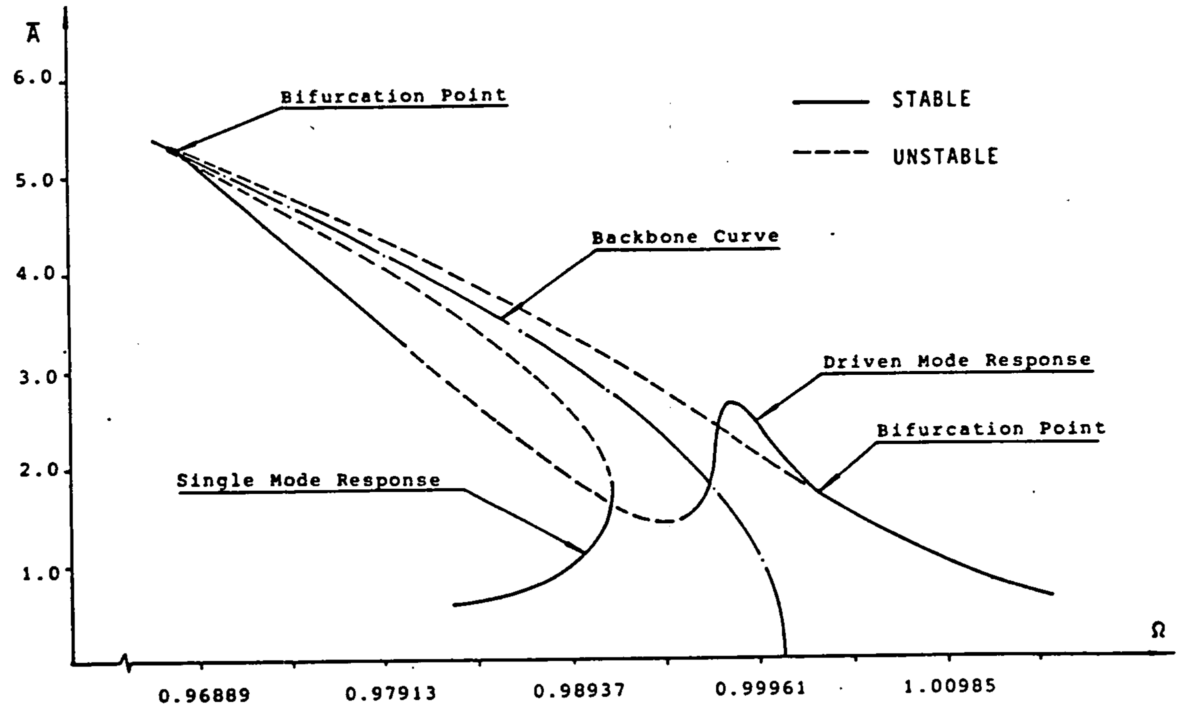

Fig.13a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = -0.06$ AND $\delta_2 = 0.00$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 7$, $m = k = 1$, $n = 1 = 5$.

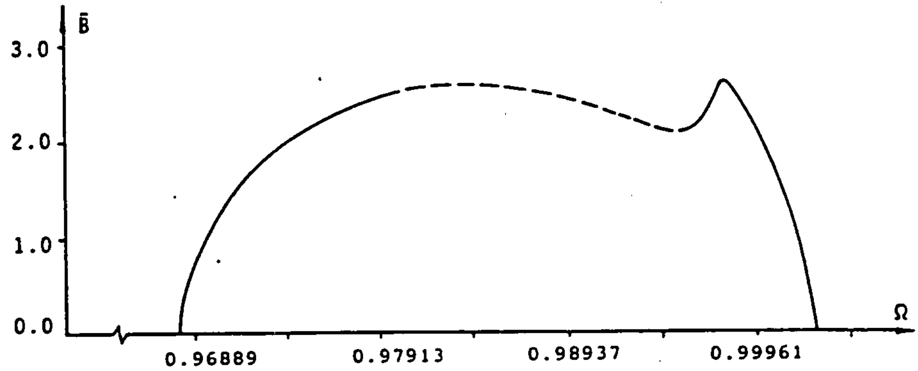

Fig.13b COMPANION MODE RESPONSE.

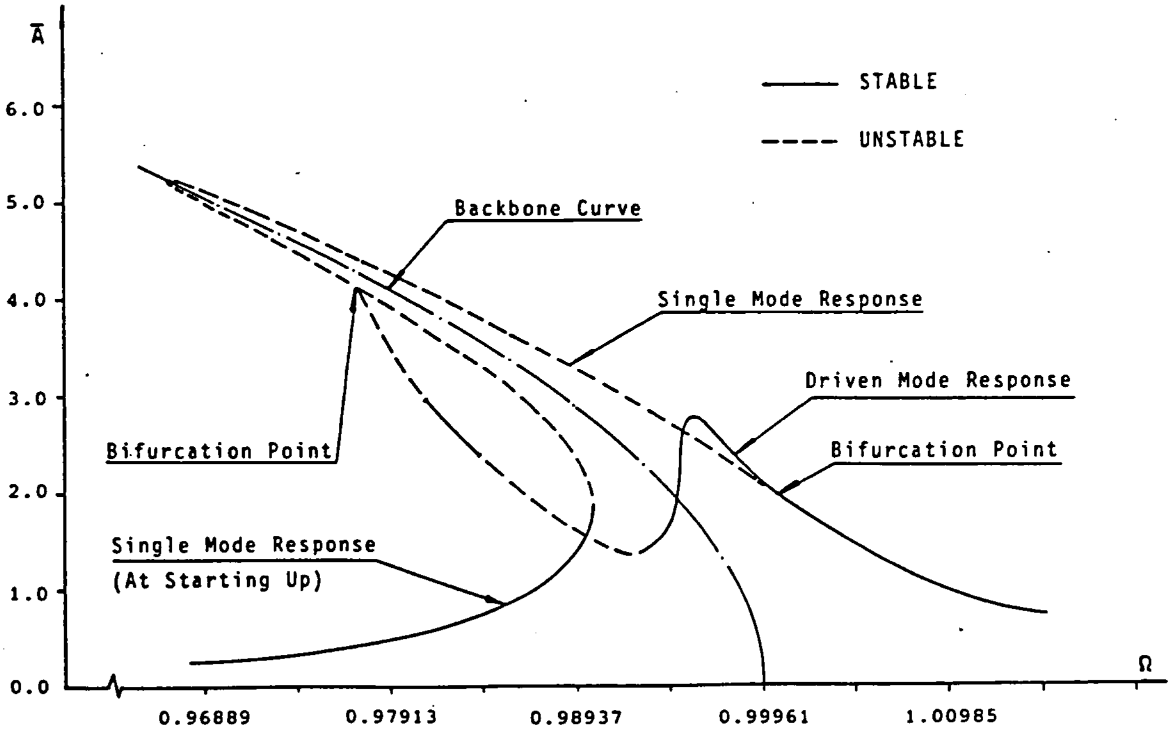

Fig.14a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = -0.02$ AND $\delta_2 = 0.02$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 2$, $m = k = 1$, $n = l = 5$.

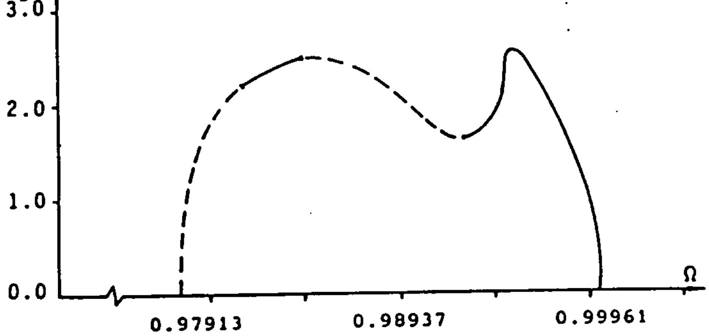

Fig.14b COMPANION MODE RESPONSE.

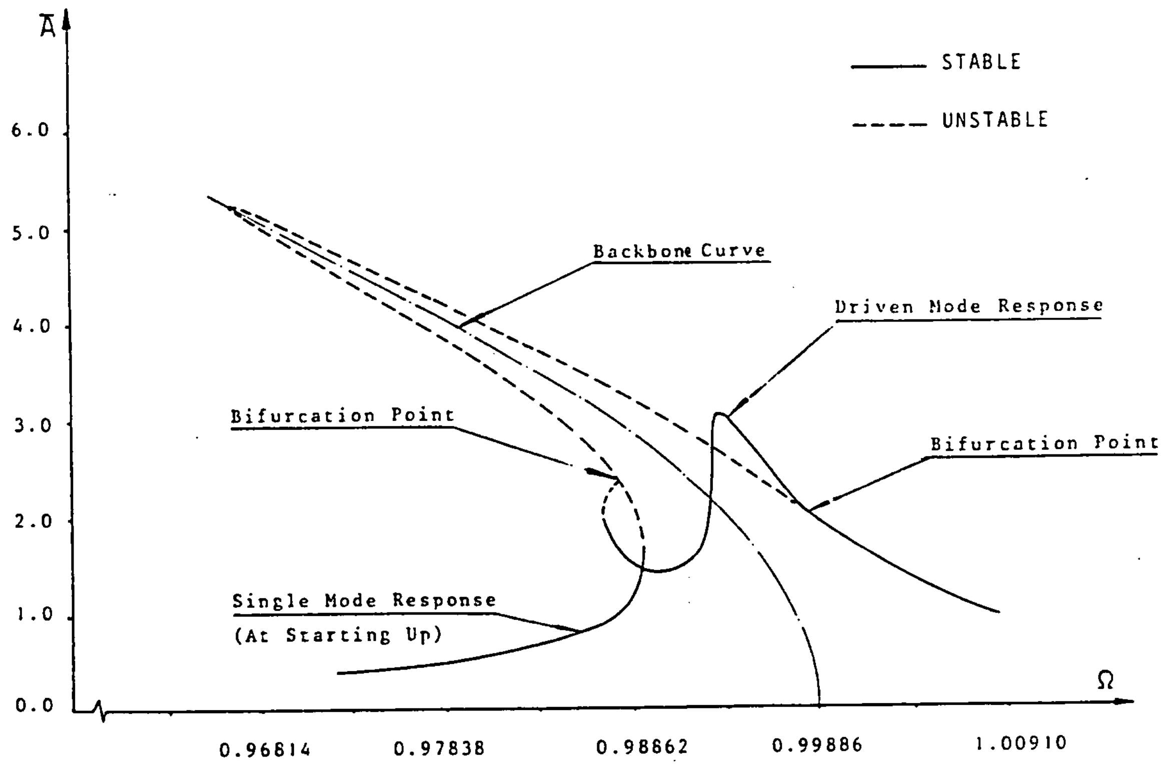

Fig.15a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = -0.06$ AND $\delta_2 = 0.05$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 2$, $m = k = 1$, $n = l = 5$.

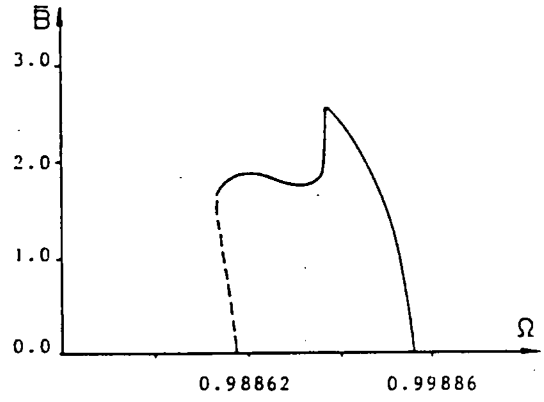

Fig.15b COMPANION MODE RESONSE.

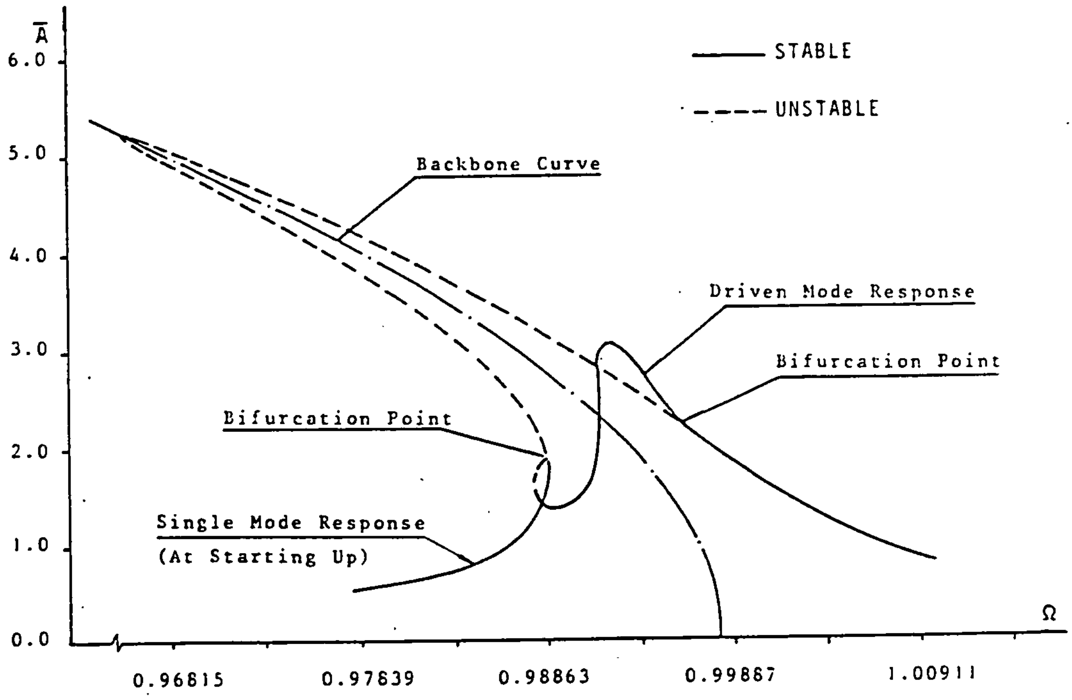

Fig.16a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH $\delta_1 = -0.06$ AND $\delta_2 = 0.05$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS $i = 2$, $m = k = 1$, $n = 10$, $l = 5$.

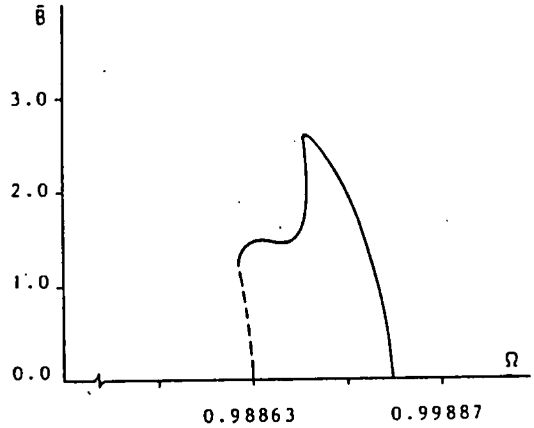

Fig.16b COMPANION MODE RESPONSE.

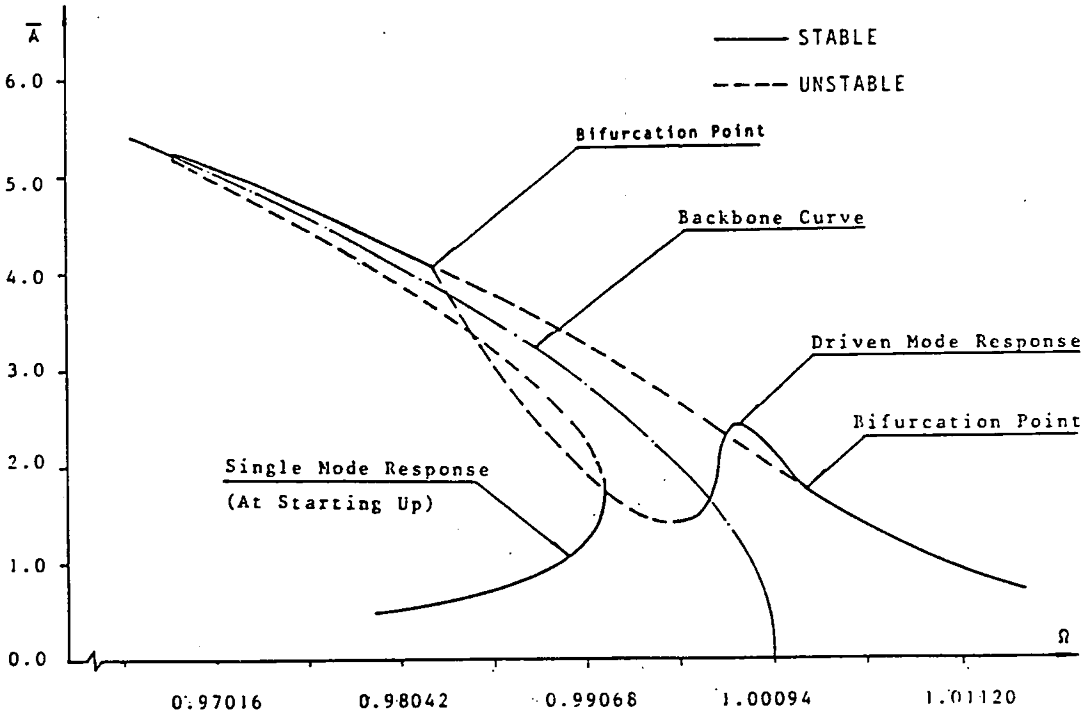

Fig.17a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF IMPERFECT SHELL WITH δ_1 = -0.06 AND δ_2 = 0.05. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$. COUPLING CONDITIONS i = 7, m = k = 1, n = l = 5.

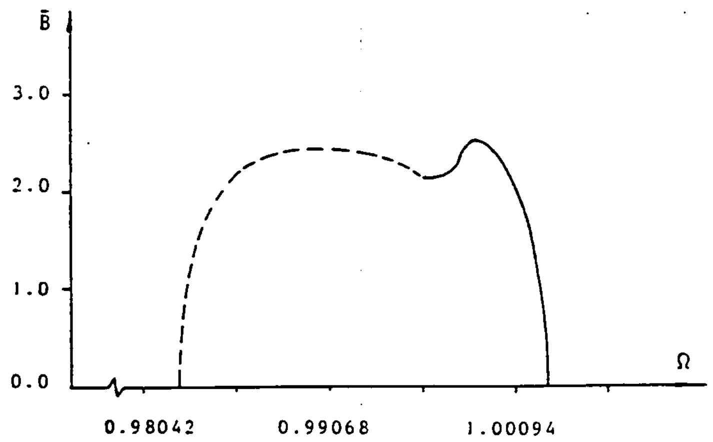

Fig.17b COMPANION MODE RESPONSE.

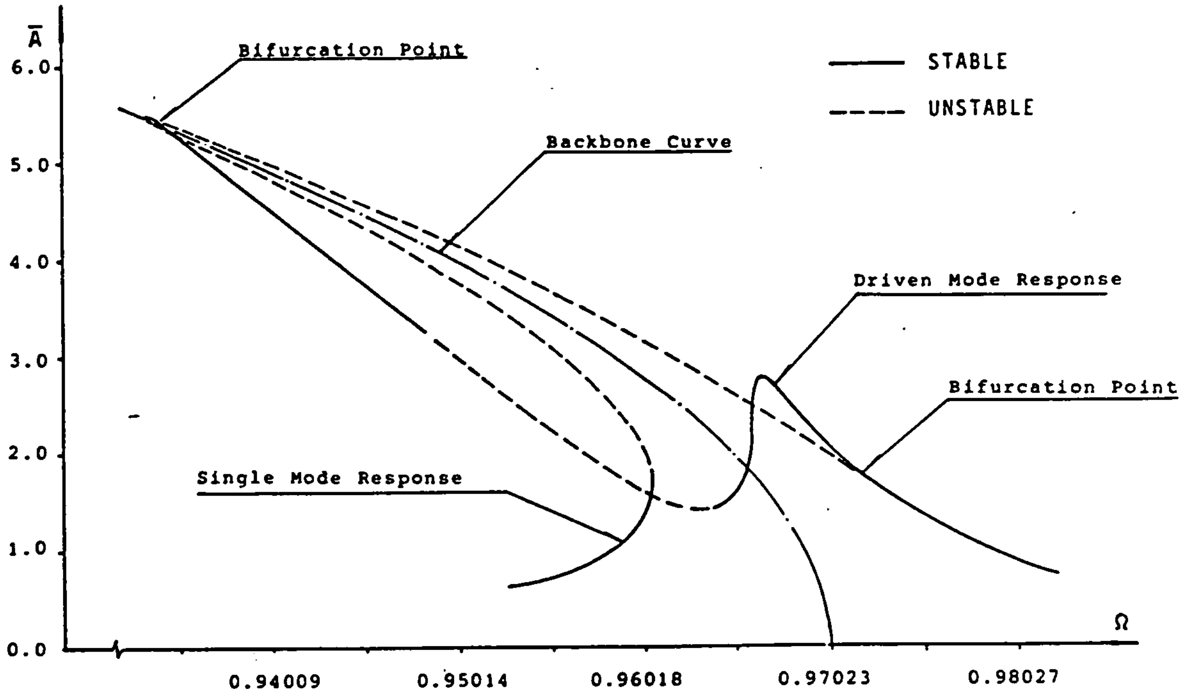

Fig.18a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL. AXIAL COMPRESSIVE LOAD $\lambda = 0.1$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

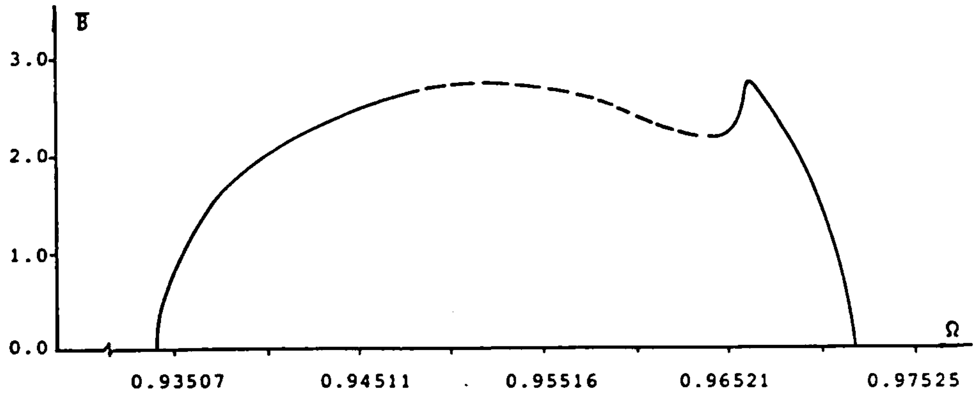

Fig.18b COMPANION MODE RESPONSE.

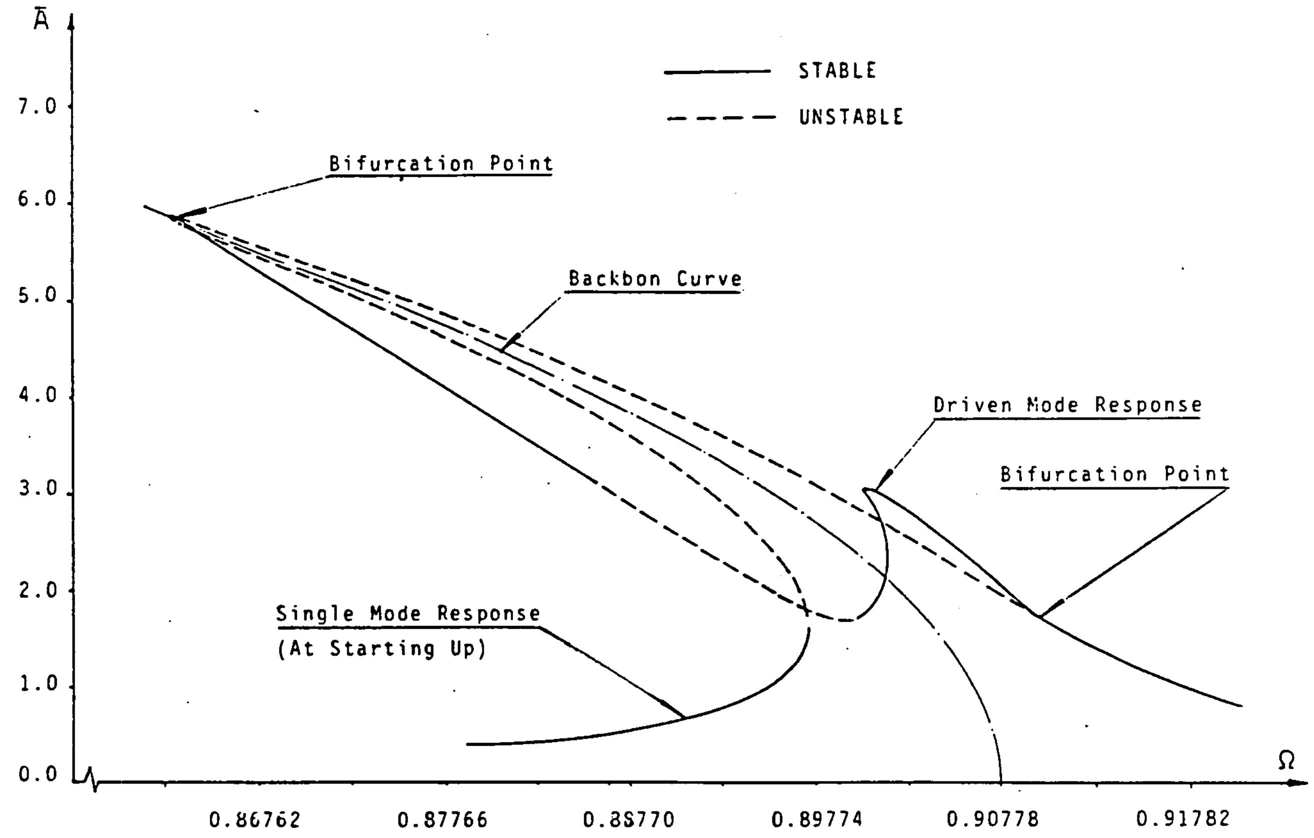

Fig.19a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL. AXIAL COMPRESSIVE LOAD $\lambda = 0.3$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

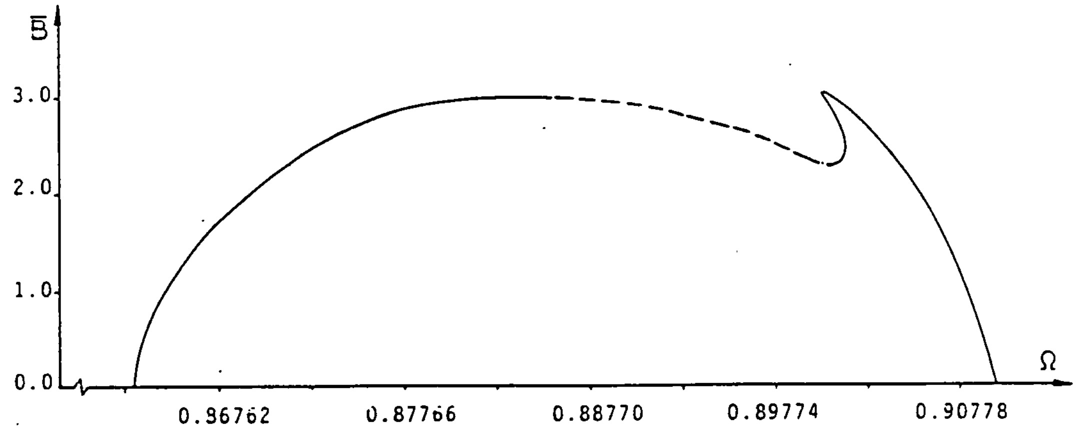

Fig.19b COMPANION MODE RESPONSE.

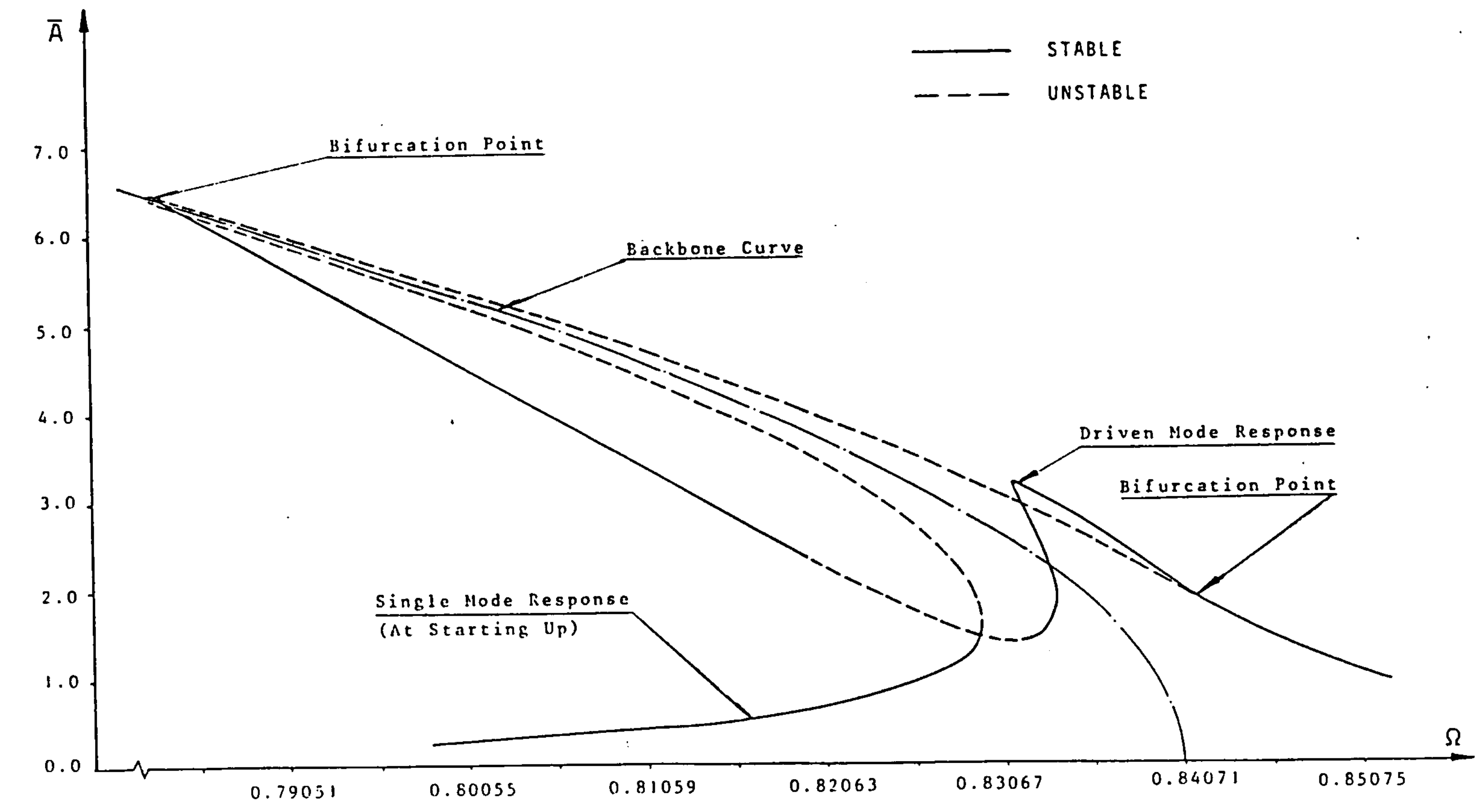

Fig.20a AMPLITUDE-FREQUENCY RELATIONSHIP OF DAMPED VIBRATION OF PERFECT SHELL. AXIAL COMPRESSIVE LOAD $\lambda = 0.5$. DAMPING $\gamma = 9\times10^{-5}$, EXCITATION $\bar{F}_D = 4.25\times10^{-5}$.

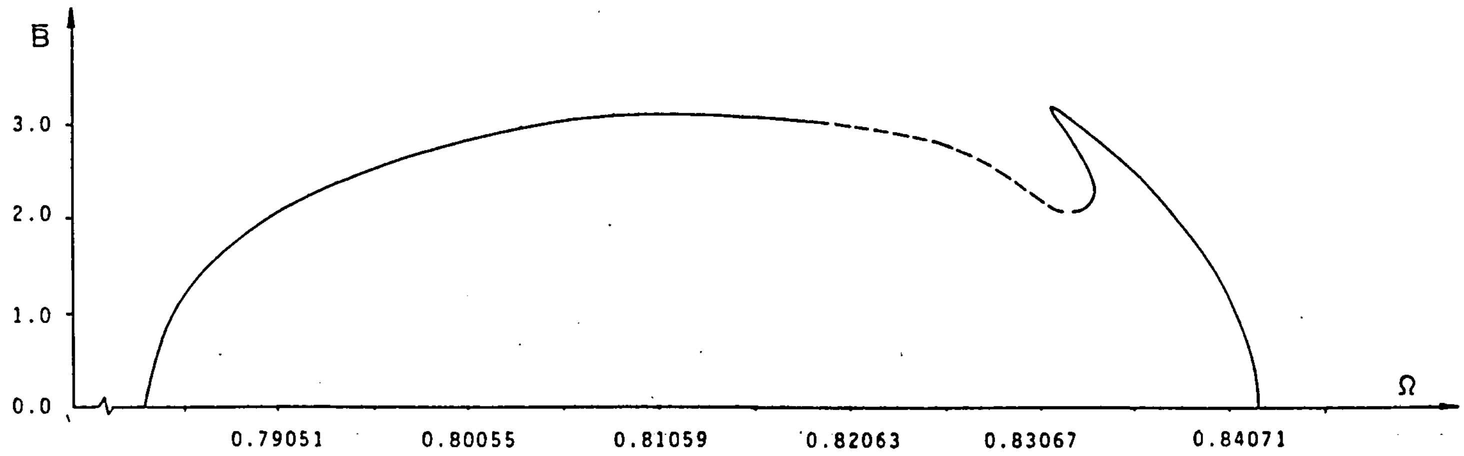

Fig.20b COMPANION MODE RESPONSE.

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