Elementarity and dimension

Non impeditus ab utta scientia

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Outline

1. Dimensions
2. Elementarity
3. Proofs using elementarity
   - Formulas
   - Bases
   - To work
4. Sources
Covering dimension

Caveat: all spaces are (at least) normal
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**Definition**

$$\dim X \leq n$$ if every finite open cover has a (finite) open refinement of order at most $$n + 1$$
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**Definition**

\[ \dim X \leq n \] if every finite open cover has a (finite) open refinement of order at most \( n + 1 \) (i.e., every \( n + 2 \)-element subfamily has an empty intersection).
There is a convenient characterization.
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Theorem (Hemmingsen)
\[ \dim X \leq n \iff \text{every } n+2\text{-element open cover has a shrinking with an empty intersection.} \]
Large inductive dimension

**Definition**

\[ \text{Ind } X \leq n \text{ if between every two disjoint closed sets } A \text{ and } B \text{ there is a partition } L \text{ that satisfies } \text{Ind } L \leq n - 1. \]
Large inductive dimension

Definition

Ind $X \leq n$ if between every two disjoint closed sets $A$ and $B$ there is a partition $L$ that satisfies Ind $L \leq n - 1$.

The starting point: Ind $X \leq -1$ iff $X = \emptyset$. 
Large inductive dimension

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\( L \) is a partition between \( A \) and \( B \) means:
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Definition

\( \text{Ind } X \leq n \) if between every two disjoint closed sets \( A \) and \( B \) there is a partition \( L \) that satisfies \( \text{Ind } L \leq n - 1 \).

The starting point: \( \text{Ind } X \leq -1 \) iff \( X = \emptyset \).

\( L \) is a partition between \( A \) and \( B \) means: there are closed sets \( F \) and \( G \) that cover \( X \) and satisfy: \( F \cap B = \emptyset, G \cap A = \emptyset \) and \( F \cap G = L \).
Definition

\[ D_g X \leq n \] between every two disjoint closed sets \( A \) and \( B \) there is a cut \( C \) that satisfies \( D_g L \leq n - 1 \).
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Dg $X \leq n$ between every two disjoint closed sets $A$ and $B$ there is a cut $C$ that satisfies $Dg L \leq n - 1$.
The starting point: $Dg X \leq -1$ iff $X = \emptyset$. 
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\( \text{Dg} \, X \leq n \) between every two disjoint closed sets \( A \) and \( B \) there is a cut \( C \) that satisfies \( \text{Dg} \, L \leq n - 1 \).

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The starting point: \( \text{Dg} \ X \leq -1 \) iff \( X = \emptyset \).

\( C \) is a cut between \( A \) and \( B \) means: \( C \cap K \neq \emptyset \) whenever \( K \) is a subcontinuum of \( X \) that meets both \( A \) and \( B \).
(In)equalities

- For $\sigma$-compact metric $X$: $\dim X$
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(In)equalities

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- The second is fairly recent (1999).
For $\sigma$-compact metric $X$: $\dim X = \text{Ind } X = \text{Dg } X$

The first equality is classical and holds for all metric $X$

the second is fairly recent (1999).

There is for each $n$ a locally connected Polish $X_n$ with $\text{Dg } X = 1$ and $\dim X_n = n$ (Fedorchuk, van Mill)
More inequalities

- \( \text{Dg } X \leq \text{Ind } X \) (each partition is a cut)
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- $\text{dim} \, X \leq \text{Ind} \, X$ (Vedenissof)
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More inequalities

- $\text{Dg } X \leq \text{Ind } X$ (each partition is a cut)
- $\dim X \leq \text{Ind } X$ (Vedenissof)
- $\dim X \leq \text{Dg } X$ (Fedorchuk)

We will reprove the last two inequalities.
A structure (group, field, lattice) $A$ is an **elementary** substructure of a similar structure $B$ if
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Definition

A structure (group, field, lattice) $A$ is an elementary substructure of a similar structure $B$ if every equation with parameters from $A$ that has a solution in $B$ already has a solution in $A$.

These are (apparently) very rich substructures
Examples

- the field $\mathbb{Q}$ is not an elementary substructure of the field $\mathbb{R}$
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- the field \( \mathbb{Q} \) is not an elementary substructure of the field \( \mathbb{R} \); consider \( x \cdot x = 2 \)
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- The ordered set $\mathbb{Z}$ is not an elementary substructure of the ordered set $\mathbb{Q}$
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Examples

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- The ordered set $\mathbb{Q}$ is an elementary substructure of the ordered set $\mathbb{R}$
There are plenty of elementary substructures.
How to make them

There are plenty of elementary substructures.

**Theorem (Löwenheim-Skolem)**

*Assume our language of discourse is countable.*
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Assume our language of discourse is countable. Let $B$ be a structure suitable for that language and let $X \subseteq B$ such that there is an elementary substructure $A$ of $B$ such that $X \subseteq A$ and $|A| \leq |X| + \aleph_0$. 
There are plenty of elementary substructures.

**Theorem (Łoewenheim-Skolem)**

Assume our language of discourse is countable. Let $B$ be a structure suitable for that language and let $X \subseteq B$ then there is an elementary substructure $A$ of $B$ such that $X \subseteq A$ and $|A| \leq |X| + \aleph_0$. 
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Here is Hemmingsen’s characterization of $\dim X \leq n$
Covering dimension

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\[
(\forall x_1)(\forall x_2) \cdots (\forall x_{n+2})(\exists y_1)(\exists y_2) \cdots (\exists y_{n+2}) \left[ (x_1 \cap x_2 \cap \cdots \cap x_{n+2} = \emptyset) \rightarrow ((x_1 \leq y_1) \land (x_2 \leq y_2) \land \cdots \land (x_{n+2} \leq y_{n+2}) \land (y_1 \cap y_2 \cap \cdots \cap y_{n+2} = \emptyset) \land (y_1 \cup y_2 \cup \cdots \cup y_{n+2} = 1)) \right].
\]
Large inductive dimension

We can express $\text{Ind } X \leq n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)
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We can express $\text{Ind } X \leq n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

$$(\forall x)(\forall y)(\exists u)$$

$$[(((x \leq a) \land (y \leq a) \land (x \cap y = 0)) \rightarrow (\text{partn}(u, x, y, a) \land I_{n-1}(u)))]$$
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We can express $\text{Ind } X \leq n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

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where $\text{partn}(u, x, y, a)$ says that $u$ is a partition between $x$ and $y$ in the (sub)space $a$: 
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where $\text{partn}(u, x, y, a)$ says that $u$ is a partition between $x$ and $y$ in the (sub)space $a$:

$$(\exists f)(\exists g) \left( (x \cap f = 0) \land (y \cap g = 0) \land (f \cup g = a) \land (f \cap g = u) \right).$$
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We can express $\text{Ind } X \leq n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

\[(\forall x)(\forall y)(\exists u) \left[ ((x \leq a) \land (y \leq a) \land (x \cap y = \emptyset)) \rightarrow (\text{partn}(u, x, y, a) \land I_{n-1}(u)) \right]\]

where $\text{partn}(u, x, y, a)$ says that $u$ is a partition between $x$ and $y$ in the (sub)space $a$:

\[(\exists f)(\exists g) ((x \cap f = \emptyset) \land (y \cap g = \emptyset) \land (f \cup g = a) \land (f \cap g = u))\].

We start with $I_{-1}(a)$, which denotes $a = \emptyset$. 
Here we have the recursive definition of a formula $\Delta_n(a)$:
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$\forall x \forall y \exists u$

$[ ( \{ x \leq a \} \wedge \{ y \leq a \} \wedge \{ x \cap y = 0 \} ) \rightarrow ( \text{cut}(u, x, y, a) \wedge \Delta_{n-1}(u) ) ]$,
Here we have the recursive definition of a formula $\Delta_n(a)$:

$$(\forall x)(\forall y)(\exists u)
\left[\left((x \leq a) \land (y \leq a) \land (x \cap y = 0)\right) \rightarrow \left(\text{cut}(u, x, y, a) \land \Delta_{n-1}(u)\right)\right],$$

and $\Delta_{-1}(a)$ denotes $a = 0$. 
The formula \( \text{cut}(u, x, y, a) \) expresses that \( u \) is a cut between \( x \) and \( y \) in \( a \):
The formula \( \text{cut}(u, x, y, a) \) expresses that \( u \) is a cut between \( x \) and \( y \) in \( a \):

\[
(\forall v)[((v \leq a) \land \text{conn}(v) \land (v \cap x \neq \emptyset) \land (v \cap y \neq \emptyset)) \rightarrow (v \cap u \neq \emptyset)],
\]
The formula $\text{cut}(u, x, y, a)$ expresses that $u$ is a cut between $x$ and $y$ in $a$:

$$(\forall v)[((v \leq a) \land \text{conn}(v) \land (v \cap x \neq o) \land (v \cap y \neq o)) \rightarrow (v \cap u \neq o)],$$

and $\text{conn}(a)$ says that $a$ is connected:
The formula \( \text{cut}(u, x, y, a) \) expresses that \( u \) is a cut between \( x \) and \( y \) in \( a \):

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\]

and \( \text{conn}(a) \) says that \( a \) is connected:

\[
(\forall x)(\forall y)[((x \cap y = o) \land (x \cup y = a)) \rightarrow ((x = o) \lor (x = a))],
\]
Wherefore formulas?

Romeo and Juliet, Act 2, scene 2 (alternate)
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O Formulas, Formulas! — Wherefore useth thou Formulas?
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- \( \dim X \leq n \) iff \( 2^X \) satisfies \( \delta_n \)
- \( \text{Ind } X \leq n \) iff \( 2^X \) satisfies \( I_n(X) \)
Romeo and Juliet, Act 2, scene 2 (alternate):
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- $\dim X \leq n \iff 2^X$ satisfies $\delta_n$
- $\text{Ind } X \leq n \iff 2^X$ satisfies $I_n(X)$
- $\text{Dg } X \leq n \iff 2^X$ satisfies $\Delta_n(X)$
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Theorem

Let $X$ be compact. Then $\dim X \leq n$ iff some (every) lattice-base for its closed sets satisfies $\delta_n$. 

Theorem

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Proof.

Both directions use swelling and shrinking to replace the finite families by combinatorially equivalent subfamilies of the base.
Large inductive dimension

Theorem

Let $X$ be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\text{Ind } X \leq n$. 
Large inductive dimension

**Theorem**

Let $X$ be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\text{Ind } X \leq n$.

**Proof.**

Induction: given $A$ and $B$ expand them to $A', B' \in \mathcal{B}$. Then find $L \in \mathcal{B}$, between $A'$ and $B'$, such that $\mathcal{B}_L = \{D \in \mathcal{B} : D \subseteq L\}$ satisfies $I_{n-1}(L)$. As $\mathcal{B}_L$ is a base for the closed sets of $L$ we know, by inductive assumption, that $\text{Ind } L \leq n - 1$. 

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Elementarity and dimension
Large inductive dimension

**Theorem**

Let $X$ be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\text{Ind } X \leq n$.

**Proof.**

Induction: given $A$ and $B$ expand them to $A', B' \in \mathcal{B}$. Then find $L \in \mathcal{B}$, between $A'$ and $B'$, such that $\mathcal{B}_L = \{D \in \mathcal{B} : D \subseteq L\}$ satisfies $I_{n-1}(L)$. As $\mathcal{B}_L$ is a base for the closed sets of $L$ we know, by inductive assumption, that $\text{Ind } L \leq n - 1$.

No equivalence, see later.
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Let $X$ be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then
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Let $X$ be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can’t say anything about $Dg X$. 
Theorem

Let $X$ be compact. If some lattice lattice-base, $B$, for its closed sets satisfies $\Delta_n(X)$ then we can’t say anything about $Dg X$.

Proof.

Let $X = [0, 1]$ and let $B$ be the lattice-base generated by the family of sets of the form $[0, q] \cup \{q + 2^{-n} : n \in \omega\}$ (q rational) and $[p, 1] \cup \{p - 2^{-n} : n \in \omega\}$ (p irrational).
**Theorem**

Let $X$ be compact. If some lattice lattice-base, $B$, for its closed sets satisfies $\Delta_n(X)$ then we can’t say anything about $Dg\ X$.

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Let $X = [0, 1]$ and let $B$ be the lattice-base generated by the family of sets of the form $[0, q] \cup \{q + 2^{-n} : n \in \omega\}$ ($q$ rational) and $[\rho, 1] \cup \{\rho - 2^{-n} : n \in \omega\}$ ($\rho$ irrational).

$B$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously.
Theorem

Let $X$ be compact. If some lattice lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can’t say anything about $\text{Dg} X$.

Proof.

Let $X = [0, 1]$ and let $\mathcal{B}$ be the lattice-base generated by the family of sets of the form $[0, q] \cup \{q + 2^{-n} : n \in \omega\}$ ($q$ rational) and $[p, 1] \cup \{p - 2^{-n} : n \in \omega\}$ ($p$ irrational). $\mathcal{B}$ has no connected elements, hence it satisfies $\Delta_0(X)$ vacuously but $\text{Dg}[0, 1] = 1$. 
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Let $X$ be a compact Hausdorff space and let $L$ be an elementary sublattice of the lattice $2^X$ of all closed subsets of $X$. 
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Consider the Wallman space $wL$ of $L$. 
Let $X$ be a compact Hausdorff space and let $L$ be an elementary sublattice of the lattice $2^X$ of all closed subsets of $X$.

Consider the Wallman space $wL$ of $L$.

What can we say about $\dim wL$, $\Ind wL$ and $\Dg wL$?
Theorem

\[ \dim wL = \dim X \]
Covering dimension

**Theorem**

\[ \dim wL = \dim X \]

**Proof.**

Notice that \( \delta_n \) and its negation state that certain systems of equations have solutions.
Covering dimension

Theorem
\[ \dim wL = \dim X \]

Proof.
Notice that \( \delta_n \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( \delta_n \) iff \( L \) satisfies \( \delta_n \).
Covering dimension

Theorem
\[ \dim wL = \dim X \]

Proof.
Notice that \( \delta_n \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( \delta_n \) iff \( L \) satisfies \( \delta_n \). Previous theorem: \( L \) satisfies \( \delta_n \) iff \( 2^{wL} \) does.
Covering dimension

Theorem
\[ \dim wL = \dim X \]

Proof.
Notice that \( \delta_n \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( \delta_n \) iff \( L \) satisfies \( \delta_n \). Previous theorem: \( L \) satisfies \( \delta_n \) iff \( 2^{wL} \) does. It follows that \( \dim X \leq n \) iff \( \dim wL \leq n \) for all \( n \).
Large inductive dimension

Theorem

\[ \text{Ind } wL \leq \text{Ind } X \]
# Large inductive dimension

## Theorem

\[ \text{Ind } wL \leq \text{Ind } X \]

## Proof.

Notice that \( I_n(a) \) and its negation state that certain systems of equations have solutions.
Theorem

\[ \text{Ind } wL \leq \text{Ind } X \]

Proof.

Notice that \( I_n(a) \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( I_n(X) \) iff \( L \) does.
Theorem

\[ \text{Ind } wL \leq \text{Ind } X \]

Proof.

Notice that \( I_n(a) \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( I_n(X) \) iff \( L \) does. By previous theorem we know \( \text{Ind } wL \leq n \), whenever \( L \) satisfies \( I_n(wL) \).
Theorem

\[ \text{Ind } wL \leq \text{Ind } X \]

Proof.

Notice that \( I_n(a) \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( I_n(X) \) iff \( L \) does. By previous theorem we know \( \text{Ind } wL \leq n \), whenever \( L \) satisfies \( I_n(wL) \).

Thus: \( \text{Ind } X \leq n \) implies \( \text{Ind } wL \leq n \).
Theorem

\[ Dg \, wL \leq Dg \, X \]
Theorem

\[ D_g wL \leq D_g X \]

Nonproof

Notice that \( \Delta_n(a) \) and its negation state that certain systems of equations have solutions.
### Theorem

\[ Dg \omega L \leq Dg X \]

### Nonproof

Notice that \( \Delta_n(a) \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( \Delta_n(X) \) iff \( L \) does.
### Theorem

\[ Dg \, wL \leq Dg \, X \]

### Non-proof

Notice that \( \Delta_n(a) \) and its negation state that certain systems of equations have solutions. By elementarity we see that \( 2^X \) satisfies \( \Delta_n(X) \) iff \( L \) does.

By previous theorem we know nothing yet about \( Dg \, wL \).
Proof.

Let $A$ and $B$ be closed and disjoint in $\mathcal{W}_L$. Wlog, $A, B \in \mathcal{L}$.

There is $C \in \mathcal{L}$ that is a cut between $A$ and $B$ in $X$ and that satisfies $\Delta_{n-1}(C) \leq n-1$.

Inductive assumption: $D_{wL}C \leq n-1$ in $\mathcal{W}_L$, because $\mathcal{M} = \{D \in \mathcal{L} : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and $C$-in-$\mathcal{W}_L$ is $\mathcal{W}_M$.

Still to show: $C$-in-$\mathcal{W}_L$ is a cut between $A$ and $B$ in $\mathcal{W}_L$.
### Theorem

\[ \text{Dg } wL \leq \text{Dg } X \]

### Proof.

Let \( A \) and \( B \) be closed and disjoint in \( wL \). Wlog: \( A, B \in L \).
Theorem

\[ \text{Dg } wL \subseteq \text{Dg } X \]

Proof.

Let \( A \) and \( B \) be closed and disjoint in \( wL \). Wlog: \( A, B \in L \).
There \( C \in L \) that is a cut between \( A \) and \( B \) in \( X \) and that satisfies
\[ \Delta_{n-1}(C) \leq n - 1. \]
Theorem

\[ \text{Dg } wL \leq \text{Dg } X \]

Proof.

Let \( A \) and \( B \) be closed and disjoint in \( wL \). Wlog: \( A, B \in L \).
There \( C \in L \) that is a cut between \( A \) and \( B \) in \( X \) and that satisfies \( \Delta_{n-1}(C) \leq n - 1 \).
Inductive assumption: \( \text{Dg } C \leq n - 1 \) in \( wL \)
Theorem

\[ \text{Dg } wL \leq \text{Dg } X \]

Proof.

Let \( A \) and \( B \) be closed and disjoint in \( wL \). Wlog: \( A, B \in L \).
There \( C \in L \) that is a cut between \( A \) and \( B \) in \( X \) and that satisfies \( \Delta_{n-1}(C) \leq n-1 \).
Inductive assumption: \( \text{Dg } C \leq n-1 \) in \( wL \), because \( M = \{ D \in L : D \subseteq C \} \) is an elementary sublattice of \( \{ D \in 2^X : D \subseteq C \} \) and \( C \)-in-\( wL \) is \( wM \).
Theorem

\[ \operatorname{Dg} wL \leq \operatorname{Dg} X \]

Proof.

Let \( A \) and \( B \) be closed and disjoint in \( wL \). Wlog: \( A, B \in L \).

There \( C \in L \) that is a cut between \( A \) and \( B \) in \( X \) and that satisfies

\[ \Delta_{n-1}(C) \leq n - 1. \]

Inductive assumption: \( \operatorname{Dg} C \leq n - 1 \) in \( wL \), because

\( M = \{ D \in L : D \subseteq C \} \) is an elementary sublattice of

\( \{ D \in 2^X : D \subseteq C \} \) and \( C \)-in-\( wL \) is \( wM \).

Still to show: \( C \)-in-\( wL \) is a cut between \( A \) and \( B \) in \( wL \).
Let $F$ be a closed set in $wL$ that meets $A$ and $B$ but not $C$. We show $F$ is not connected.

Find $H$ in $L$ around $F$, disjoint from $C$.

Back in $X$ no component of $H$ meets $C$, hence it does not meet both $A$ and $B$.

By well-known topology and elementarity there are disjoint elements $H_A$ and $H_B$ of $L$ such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.

Down in $wL$ we have exactly the same relations, so $H_A$ and $H_B$ show $F$ is not connected.
Proof (continued)

Let $F$ be a closed set in $wL$ that meets $A$ and $B$ but not $C$. We show $F$ is not connected.
Proof (continued)

Let $F$ be a closed set in $wL$ that meets $A$ and $B$ but not $C$. We show $F$ is not connected. Find $H$ in $L$ around $F$, disjoint from $C$. 

By well-known topology and elementarity there are disjoint elements $H_A$ and $H_B$ of $L$ such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$. Down in $wL$ we have exactly the same relations, so $H_A$ and $H_B$ show $F$ is not connected.
Proof (continued)

Let $F$ be a closed set in $wL$ that meets $A$ and $B$ but not $C$. We show $F$ is not connected.

Find $H$ in $L$ around $F$, disjoint from $C$.

Back in $X$ no component of $H$ meets $C$, hence it does not meet both $A$ and $B$. 

Proof (continued)

Let $F$ be a closed set in $wL$ that meets $A$ and $B$ but not $C$. We show $F$ is not connected.
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Back in $X$ no component of $H$ meets $C$, hence it does not meet both $A$ and $B$.
By well-known topology and elementarity there are disjoint elements $H_A$ and $H_B$ of $L$ such that
Proof (continued)

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Find $H$ in $L$ around $F$, disjoint from $C$. Back in $X$ no component of $H$ meets $C$, hence it does not meet both $A$ and $B$.

By well-known topology and elementarity there are disjoint elements $H_A$ and $H_B$ of $L$ such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.

Down in $wL$ we have exactly the same relations, so $H_A$ and $H_B$ show $F$ is not connected.
Let $X$ be compact Hausdorff and let $L$ be a *countable* elementary sublattice of $2^X$. Then
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**Vedenissof:** $\dim X = \dim wL = \text{Ind } wL \leq \text{Ind } X$
Let \( X \) be compact Hausdorff and let \( L \) be a countable elementary sublattice of \( 2^X \). Then

**Vedenissof:** \( \dim X = \dim wL = \text{Ind} wL \leq \text{Ind} X \)

**Fedorchuk:** \( \dim X = \dim wL = \text{Dg} wL \leq \text{Dg} X \)
Let $X$ be compact Hausdorff and let $L$ be a countable elementary sublattice of $2^X$. Then

**Vedenissof:** $\dim X = \dim wL = \Ind wL \leq \Ind X$

**Fedorchuk:** $\dim X = \dim wL = \Dg wL \leq \Dg X$

There are $X$ with $\dim X < \Dg X$, so $\Dg wL < \Dg X$ and $\Ind wL < \Ind X$ are possible.
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*On the Brouwer dimension of compact spaces*, Mathematical Notes* 73* (2003), 271–279,

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