The domain of attraction of the $\alpha$-sun operator for type II and type III distributions

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Let $(Y_n)$ be a sequence of independent random variables with common distribution $F$ and define the iteration:

$$X_0 = x_0, X_n := X_{n-1} \vee (\alpha X_{n-1} + Y_n), \quad \alpha \in [0, 1).$$

We denote by $\mathcal{D}(\Phi_\alpha)$ the domain of maximal attraction of $\Phi_\alpha$, the extreme value distribution of the first type. Greenwood and Hooghiemstra showed in 1991 that for $F \in \mathcal{D}(\Phi_\alpha)$ there exist norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $a_n^{-1}\{X_n - b_n/(1 - \alpha)\}$ has a non-degenerate (distributional) limit. In this paper we show that the same is true for $F \in \mathcal{D}(\Psi_\alpha) \cup \mathcal{D}(\Lambda)$, the type II and type III domains. The method of proof is entirely different from the method in the aforementioned paper. After a proof of tightness of the involved sequences we apply (modify) a result of Donnelly, concerning weak convergence of Markov chains with an entrance boundary.

Keywords: extremal limits; self-similar Markov processes; weak convergence

1. Introduction

Let $(Y_n)_{n \geq 1}$ be a sequence of independent random variables with common distribution function $F$ and define the iteration

$$X_0 = x_0, X_n := X_{n-1} \vee (\alpha X_{n-1} + Y_n), \quad n \geq 1, \quad \alpha \in [0, 1).$$

We denote by $\mathcal{D}(G)$ the domain of maximal attraction of the distribution $G$, where $G$ is one of the extreme value distributions. For $F \in \mathcal{D}(G)$ and $a_n > 0$, $b_n \in \mathbb{R}$ such that $F^n(a_n x + b_n) \to G(x)$, for all $x$, we define, for $n \geq 1$,

$$Y_{n,j} := \frac{Y_j - b_n}{a_n}, \quad j = 1, 2, \ldots.$$

For $\alpha \in [0, 1)$ and $x_0 \in \mathbb{R}$, the random element $X_n(\cdot) \in D[0, \infty)$ (the space of cadlag functions, equipped with the Skorohod topology) is defined by

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Observe that the relation between the sequence of random variables $X_n$ given by (1) and the sequence of processes $X_n(\cdot)$ is

$$X_n\left(\frac{j}{n}\right) = a_n^{-1} \left( X_j - \frac{b_n}{1 - \alpha} \right).$$

The motivation for studying recursive sequences such as (1) comes from a stochastic solar energy model (cf. Haslett 1980). Note that for $\alpha = 0$ the sequence $X_n$ is the sequence of partial maxima:

$$X_n = x_0 \lor Y_1 \lor \cdots \lor Y_n,$$

whereas for $\alpha = 1$ (this value is not included in the definition (1)) we obtain

$$X_n = x_0 + [Y_1]^+ + \cdots + [Y_n]^+ \quad ([x]^+ = x \lor 0, x \in \mathbb{R}).$$

Hence the sequence $X_n$ defined by (1) is between maxima and sums of independent random variables, and from that viewpoint of theoretical interest.

Greenwood and Hooghiemstra (1991) showed that for $F \in \mathcal{D}(\Phi_\gamma)$, where

$$\Phi_\gamma(x) := \exp(-x^{-\gamma})1_{[0,\infty)}(x),$$

the process $X_n(\cdot)$ converges weakly in $D[0, \infty)$ to a self-similar Markov process $Z(\cdot)$. Furthermore the distribution of $Z(1)$ admits a density $h_\alpha$ on $(0, \infty)$, given as the unique density solution of the equation

$$h_\alpha(x) = \gamma \int_0^x (x - \alpha u)^{-\gamma} h_\alpha(u) \, du, \quad x > 0.$$ 

In this case $X_n(0) = a_n^{-1}\{x_0 - b_n/(1 - \alpha)\} \to 0$, and the proof proceeds by showing that the functional induced by (2) on the point process $\sum \delta\{j/n, X_n(j/n)\}$ is continuous.

In this paper we prove weak convergence of $X_n(\cdot)$ for $F \in \mathcal{D}(\Psi_\gamma, \Phi_\gamma)$, where

$$\Psi_\gamma(x) := \exp\left[-(-x)^\gamma\right]1_{(-\infty,0]}(x) + 1_{(0,\infty)}(x),$$

$$\Lambda(x) := \exp(-e^{-x}).$$

For $F \in \mathcal{D}(\Psi_\gamma, \Phi_\gamma)$ we have $X_n(0) = a_n^{-1}\{x_0 - b_n/(1 - \alpha)\} \to -\infty$. In these cases the method of proof is entirely different from that in the work of Greenwood and Hooghiemstra (1991). It is based on the weak convergence of Markov processes to a limiting Markov process with entrance boundary. The proof uses monotonicity of the relevant Markov process and tightness of the sequence $X_n(t)$ for fixed positive $t$. In Sections 2 and 3 we prove weak convergence, aside from the tightness of $X_n(t)$, which we postpone to Section 4.
2. The convergence result for type II distributions

Let $F \in \mathcal{D}(\Psi_x)$; then $r := \sup \{x: F(x) < 1\} < \infty$, and $1 - F(r - x^{-1}) = x^{-\gamma} L(x)$, with $L$ slowly varying at infinity. Set $b_n = r$ and $a_n := r - \inf \{y: 1 - F(y) \leq n^{-1}\}$. The points $(j/n, Y_{nj})$, $n \geq 1$, $j = 1, 2, \ldots$ are contained in $E := (0, \infty) \times (-\infty, 0)$. To prepare for the formulation of the convergence result we first specify what will be the limiting Markov process. Denote by $N$ a Poisson point process on $E$ with intensity measure the product of Lebesgue measure $dt$ and the measure $d\mu$, where

$$\mu(y, 0) = |y|^{\gamma}, \quad y < 0.$$ 

For $x < 0$ we denote by $N_x$ the points of $N$ in the strip $(0, \infty) \times [x, 0)$. We order the points of $N_x$ according to the first coordinate and denote them by $(t_1, j_1), (t_2, j_2), \ldots$, where $0 < t_1 < t_2 < \cdots$ and $j_k \in [x, 0)$. The continuous-time Markov process $Z_x(\cdot)$ with state space $[x, 0)$ is defined by

$$Z_x(t) := \begin{cases} x, & 0 \leq t < t_1, \\ Z_x(t_{k-1}) \lor \{\alpha Z_x(t_{k-1}) + j_k\}, & t_k \leq t < t_{k+1}. \end{cases}$$

We shall show that, for $x \to -\infty$, the process $Z_x(\cdot)$ converges almost surely to a process $Z(\cdot)$ with $Z(0) = -\infty$, almost surely, whereas, for any $t > 0$, we have $-\infty < Z(t) < 0$, almost surely, and where the conditional distribution of $(Z(s)|Z(t) = x)$ is given by the distribution of $Z_x(s - t)$, $s > t$. This final statement is clear from the definition of $Z_x$. The process $Z(\cdot)$ will be the limit of $X_n(\cdot)$ on $D(0, \infty)$. Here is a proof of the statements concerning $Z(\cdot)$.

Since we have, for $x < y$ and each $t \geq 0$,

$$Z_x(t) \leq Z_y(t) \leq 0,$$

the almost sure convergence of $Z_x(t)$ to a value $Z(t)$, possibly $-\infty$, follows. As for each $x$ the process $Z_x(\cdot)$ is non-decreasing we obtain that $Z(\cdot)$ is non-decreasing and we hence conclude that $Z_x(\cdot)$ converges almost surely to a non-decreasing random function $Z(\cdot)$, as $x \to -\infty$. If we show that for arbitrary $t > 0$ the collection $\mathcal{II} := \{Z_x(t), x < 0\}$ is uniformly tight, then $-\infty < Z(t) \leq 0$, $t > 0$. The tightness of $\mathcal{II}$ is a consequence of the three lemmas below, the first of which goes back to Rényi and is well known.

**Lemma 1.** Fix $x < 0$. Let $\sigma_j$, $j = 1, 2, \ldots$ be the points of a Poisson process on $\mathbb{R}^+$ with intensity $|x|^{\gamma}$. Independent of this Poisson process we define an independent, identically distributed sequence $\beta_1, \beta_2, \ldots$ with distribution

$$P(\beta_1 \leq y) = 1 - \left|\frac{y}{x}\right|^\gamma, \quad x \leq y \leq 0.$$ 

Then the point process $N'_x := \sum_j \delta_{(\sigma_j, \beta_j)}$ is equal in distribution to $N_x$.

**Lemma 2.** Let $(X_n)$ be defined by (1) with initial value $X_1 = -1$, and with $(Y_n)$ an independent, identically distributed sequence with distribution

$$F(y) = 1 - |y|^\gamma, \quad -1 \leq y \leq 0.$$ (5)
Then
\[
\sup_{n \geq 1} n^{1/\gamma} EX_n \geq A,
\]
where \( A < 0 \) is given by \( |A| := \{(1 + \gamma)/\gamma\}(1 - \alpha)^{-1 - \gamma} \).

**Remark 1.** Note that \( F \) given in (5) belongs to \( D(\Psi_\gamma) \) and that for this specific distribution the norming constants are given by \( b_n = 0 \) and \( a_n = n^{-1/\gamma} \). The proof below is equal to the tightness proof of Theorem 3 in Section 4 for \( F \) given in (5). Because of the smoothness of \( F \) the proof of Lemma 2 is easier than that of Theorem 3.

**Proof.** The conditional expectation \( E(X_{n+1}|X_n) = X_n + \int_0^1 \{1 - F(y)\} \, dy \); so by taking double expectations and using the Jensen inequality
\[
EX_{n+1} = E g(X_n) \geq g(EX_n),
\]
where \( g(u) := u + \{(1 - \alpha)|u|\}^{1+\gamma}/(1 + \gamma), \) \(-1 \leq u \leq 0.\) Put \( u_n := EX_n \) and \( v_n := An^{-1/\gamma}.\)
We shall prove by induction that \( u_n \geq v_n \) for all \( n \geq 1.\) For \( n = 1, u_1 = -1 \) and \( v_1 = A < -1.\) Assume that \( u_n \geq v_n \) for some \( n.\) By (6) and the monotonicity of \( g,\)
\[
u_{n+1} \geq g(u_n) \geq g(v_n).
\]
The inequality \( g(v_n) \geq v_{n+1} \) follows because \( n\{1 - \{n/(n + 1)\}^{1/\gamma}\} \leq 1/\gamma, \) for all \( n \geq 1 \) and \( \gamma > 0.\)

**Lemma 3.** For any \( t > 0,\)
\[
\lim_{M \to \infty} \lim_{x \to -\infty} P(Z_x(t) \geq -M) = 1.
\]

**Proof.** By monotonicity it is sufficient to show (7) for a sequence \( x_n \to -\infty.\) Let
\[
\tau_n := \inf \{s > 0: \# \text{ points of } N \text{ contained in the set } (0, s] \times [-n^{1/\gamma}, 0) = n\}.
\]
Observe from Lemma 1 that, for \( x_n = -n^{1/\gamma}, \) there holds \( Z_{x_n}(\tau_n) \overset{d}{=} n^{1/\gamma} X_n, \) if \( X_1 := -1 \) and \( F \) given in (5). Because \( N \) is a Poisson process with intensity \( dt \times d\mu \) the random variable \( \tau_n \) is the sum of \( n \) independent and exponentially distributed random variables each with parameter \( n.\) It is straightforward that \( \tau_n \to 1, \) a.s. Hence it follows from Lemma 2 and the monotonicity of \( Z_{\cdot}(\cdot) \) that for each \( t > 1 \) the statement (7) holds. The result for \( 0 < t \leq 1 \) is easily obtained by noting that for any subsequence \( n_k \) we have, with \( m_k = [nk] \),
\[
\lim_{k \to \infty} n_k^{1/\gamma} X_{[nk]} = t^{-1/\gamma} \lim_{k \to \infty} m_k^{1/\gamma} X_{m_k}.
\]
We now formulate and prove our main result for \( F \in D(\Psi_\gamma).\)

**Theorem 1.** Let \( F \in D(\Psi_\gamma) \) and \( x_0 < r/(1 - \alpha). \) On \( D(0, \infty) \) we have
\[
X_n(\cdot) \overset{d}{\to} Z(\cdot),
\]
where \( Z(\cdot) \) is the Markov process with entrance boundary introduced above.
Proof. The coordinate projection $X_n(t)$ at time $t > 0$ is uniformly tight as a consequence of Theorem 3 in Section 4, because

$$\lim_{n \to \infty} \frac{a_{[nt]}}{a_n} = t^{-1/\gamma},$$

and

$$X_n(t) = a_n^{-1} \left( X_{[nt]} - \frac{r}{1 - \alpha} \right) = \frac{a_{[nt]}}{a_n} a_{[nt]}^{-1} \left( X_{[nt]} - \frac{r}{1 - \alpha} \right).$$

Next we check that the sequence $X_n(\cdot)$ is tight in $D[a, b]$, the space of cadlag functions with $t \in [a, b]$ for each pair $a$, $b$ with $0 < a < b < \infty$. Given that $X_n(a) = x \in [-M, 0]$, the process $X_n(t)$, $t \geq a$, is non-decreasing and converges weakly to $Z_a(t - a)$, $t \geq a$, because of convergence of the underlying point processes and continuity of the map $(x, y) \to x \vee (ax + y)$. Hence, if $n_k$ is a subsequence for which $X_{n_k}(a)$ converges weakly on $\mathbb{R}$, then $X_{n_k}(\cdot)$ converges weakly on $D[a, b]$. Consequently the sequence $X_n$ is relatively compact on $D[a, b]$ (and hence tight by Prohorov’s theorem).

Take a particular weakly convergent subsequence of $X_n(\cdot)$ and denote its limit by $\hat{Z}(\cdot) \in D(0, \infty)$ (for convenience we shall also index the subsequence by $n$). For $t > 0$ we denote by $\mathcal{C}_t$ the set of continuity points of the distribution of $\hat{Z}(t)$. We shall show that the process $\hat{Z}(\cdot)$ satisfies the following.

(i) For each $M \geq 0$, $\lim_{h \downarrow 0} P(\hat{Z}(h) \leq -M) = 1$.

(ii) For $0 < s < t$, $x \in \mathcal{C}_s$ and $y \in \mathcal{C}_t$,

$$P(\hat{Z}(s) \leq x, \hat{Z}(t) \leq y) = \int_{-\infty}^{\hat{Z}(t) - s} P(\hat{Z}(s) \in du) P(Z_u(t - s) \leq y).$$

(iii) The finite-dimensional distributions of $\hat{Z}(\cdot)$ coincide with those of $Z(\cdot)$.

From (iii) the theorem follows, because the finite-dimensional distributions form a determining class.

If $-M \in \mathcal{C}_h$, then

$$P(\hat{Z}(h) \leq -M) = \lim_{n \to \infty} P(X_n(h) \leq -M)$$

$$\geq \lim_{n \to \infty} P(\sup_{1 \leq j \leq [nh]} Y_{n,j} \leq -M(1 - \alpha))$$

$$= \exp \{ -hM^\gamma(1 - \alpha)^\gamma \} \to 1, \ h \downarrow 0.$$
Since for each \( u \) we have \( P(X_n(t) \leq y | X_n(s) = u) = P(Z_u(t - s) \leq y) \) and, since the map \( u \rightarrow P(Z_u(t) \leq x) \) is bounded and continuous, we obtain (ii) from the definition of weak convergence.

In order to prove (iii) for the one-dimensional distributions write, for \( 0 < h < t \) and \( x \in \mathbb{R} \),

\[
P(\hat{Z}(t) \leq x) = \int_{-\infty}^{0} P(\hat{Z}(h) \in d\mu)P(Z_u(t) - h \leq x)\\ 
\geq \int_{-\infty}^{-M} P(\hat{Z}(h) \in d\mu)P(Z_u(t) - h \leq x)\\ 
\geq P(Z_{-M}(t - h) \leq x)P(\hat{Z}(h) \leq -M) \rightarrow P(Z(t) \leq x),
\]

by letting first \( h \downarrow 0 \) and then \( M \rightarrow \infty \). On the other hand

\[
P(\hat{Z}(t) \leq x) = \int_{-\infty}^{0} P(\hat{Z}(h) \in d\mu)P(Z_u(t) - h \leq x) \\
\leq P(Z(t) - h \leq x) \rightarrow P(Z(t) \leq x).
\]

Hence the distribution of \( \hat{Z}(t) \) coincides with that of \( Z(t) \). Statement (iii) for two-dimensional distributions and also for arbitrary finite-dimensional distributions is now an easy consequence of (ii) and the equality of the one-dimensional distributions at each positive time \( t \).

*Remark 2.* The above proof is an adaptation of the proof of Theorem 1 of Donnelly (1991). One of the differences is that in the present paper the state space of the Markov process is a subset of \( \mathbb{R} \), whereas Donnelly treats countable state spaces; also the way we prove tightness on \( D(0, \infty) \) differs from Donnelly’s approach.

*Corollary 1.* For \( F \in \mathcal{D}(\Psi, \gamma) \) and \( (X_n) \), with \( x_0 < r/(1 - \alpha) \), the sequence defined in (1), we have

\[
a_n^{-1} \left( X_n - \frac{b_n}{1 - \alpha} \right) \overset{d}{\rightarrow} X,
\]

where the limit \( X \) has density \( h_{\alpha} \) on \((-\infty, 0)\), given by the unique density solution of the functional equation

\[
h_{\alpha}(x) = \gamma |x| \int_{x/\alpha}^{\infty} |x - \alpha u|^{\gamma} h_{\alpha}(u) \, du, \quad x < 0. \tag{9}
\]

*Proof.* For \( x < 0 \), an elementary argument using the definition of \( Z(\cdot) \) gives, for \( h \rightarrow 0 \),

\[
P(Z(t + h) > x) - P(Z(t) > x) = h \int_{x/\alpha}^{\infty} |x - \alpha u|^{\gamma} P(Z(t) \in du) + o(h).
\]
This equation can be rewritten, using the self-similarity of \( Z(\cdot) \),

\[
P(Z(1) > x(t + h)^{1/\gamma}) - P(Z(1) > xt^{1/\gamma}) = h \int_{x/a}^{x} |x - au|^\gamma P(Z(1) \in t^{1/\gamma} \, du) + o(h).
\]

The functional equation (9) now follows by standard arguments and by using the equality \( X = Z(1) \). That (9) has a unique density solution can be seen by calculating the moments

\[
\mu_k := \int_{-\infty}^{0} |x|^{k\gamma} h_\alpha(x) \, dx, \quad k = 0, 1, \ldots
\]

It follows from (9) that

\[
\mu_k = \mu_{k+1} \int_{\alpha}^{1} y^{k\gamma-1}(y - \alpha)^{\gamma} \, dy,
\]

and hence by a theorem of Carleman (cf. Feller 1971, p. 227), the moments \( \mu_0 = 1, \mu_1, \ldots \) uniquely determine the density \( h_\alpha \).

\[ \square \]

### 3. The convergence result for type III distributions

In this section we treat the case where \( F \in \mathcal{D}(\Lambda) \). In order to define the limit process of \( X_n(\cdot) \) for this case let \( \mathcal{N} \) be the Poisson process on \((0, \infty) \times \mathbb{R}\) with intensity measure \( dt \times d\mu \), where \( \mu(x, \infty) = e^{-x}, x \in \mathbb{R} \). The point process \( \mathcal{N}_x \) is the restriction of \( \mathcal{N} \) to \((0, 1) \times (x, \infty)\). On the points \((t_1, j_1), (t_2, j_2), \ldots\), of \( \mathcal{N}_x \), we define \( Z_x(\cdot) \) by (4). Further we denote by \( Z(\cdot) \) the almost sure limit of \( Z_x(\cdot) \), as \( x \to -\infty \). Along the lines of Section 2 we have the following.

**Theorem 2.** Let \( F \in \mathcal{D}(\Lambda) \) and \( x_0 < r/(1 - \alpha) \). On \( D(0, \infty) \) we have

\[
X_n(\cdot) \overset{d}{\to} Z(\cdot).
\]

**Corollary 2.** For \( F \in \mathcal{D}(\Lambda) \) and \((X_n)\), with \( x_0 < r/(1 - \alpha) \), the sequence defined in (1), we have

\[
a_n^{-1}\left(X_n - \frac{b_n}{1 - \alpha}\right) \overset{d}{\to} X,
\]

where the limit \( X \) has density \( h_\alpha \) on \( \mathbb{R} \) given by

\[
h_\alpha(x) := (1 - \alpha)\left[\Gamma((1 - \alpha)^{-1})\right]^{-1} \exp\{-x - e^{-x(1-\alpha)}\}, x \in \mathbb{R},
\]

and where \( \Gamma(t) := \int_{0}^{\infty} x^{t-1}e^{-x} \, dx, t > 0 \).
Proof. For \( x \in \mathbb{R} \) and \( h \to 0 \),
\[
P(Z(t + h) > x) - P(Z(t) > x) = h \int_{-\infty}^{x} \exp \{-(x - au)\} P(Z(t) \in du) + o(h).
\]
From (11) the density of \( X \overset{d}{=} Z(1) \) can be obtained, using the self-similarity of \( \exp \{-Z(t)\} \).

Remark 3. Note that the density in (10) has the form
\[
h_\alpha(x)\,dx = c\exp(-\alpha x)\,d\Lambda\{x(1 - \alpha)\}, \quad \alpha \in [0, 1).
\]
However, for \( \alpha \neq 0 \) this density is not of the Gumbel type, i.e., there are no constants \( a \) and \( b \) such that
\[
h_\alpha(x)\,dx = d\Lambda(ax + b).
\]

4. Tightness of sequences

In this section we prove tightness for the sequence
\[
a_n^{-1}\left(X_n - \frac{b_n}{1 - \alpha}\right),
\]
with \((X_n)\) the sequence given by (1).

Theorem 3. For \( F \in \mathcal{D}(\Psi_f) \) and \( x_0 < r/(1 - \alpha) \), there exist norming constants \( a_n > 0 \) and \( b_n \in \mathbb{R} \) such that the sequence \( \{X_n - b_n/(1 - \alpha)\}/a_n \) is tight on \((-\infty, 0)\). A possible choice of \((a_n)\) and \((b_n)\) is
\[
b_n \equiv r, \quad a_n := r - \inf \{x: 1 - F(x) \leq n^{-1}\}.
\]

Proof. Note by induction that \( X_n \leq x_0 \lor M_n/(1 - \alpha) \), where \( M_n = Y_1 \lor Y_2 \cdots \lor Y_n \), however, it is not possible to obtain a lower bound for \( X_n \) in terms of \( M_n \). From the well known extreme value limit for \((M_n - b_n)/a_n\) we obtain 0 as a distributional upper bound for \( \{X_n - b_n/(1 - \alpha)\}/a_n \).

Choose a sequence \( \theta_n \) of positive real numbers with \( a_n/\theta_n \to 1 \), and satisfying
\[
\lim_{n \to \infty} n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right) = \gamma^{-1}.
\]
(12)
This is possible since \( a_n = a(n) \), where
\[
a(y) := r - \inf \{x: 1 - F(x) \leq y^{-1}\}, \quad y \geq 1,
\]
and \( a \) is regularly varying; for details see Galambos and Seneta (1973) and de Bruijn (1959). Our goal is to prove that there exists a constant \( A_0 > 0 \) and an integer \( n_0 \) such that
\[
\mathbb{E}\frac{X_n - r/(1 - \alpha)}{\theta_n} \geq -A_0, \quad n \geq n_0.
\]
This inequality, together with the upper bound \( X_n \leq x_0 \vee M_n/(1 - \alpha) \), implies tightness of \( \{X_n - b_n/(1 - \alpha)\}/\theta_n \) and hence of \( \{X_n - b_n/(1 - \alpha)\}/a_n \), since \( a_n/\theta_n \to 1 \). So all we need to prove is inequality (13).

Choose \( A_1 > 0 \) with \( A_1^\gamma = \frac{3}{2}[(\gamma + 1)/\gamma](1 - \alpha)^{-1-\gamma} \), and put \( \eta = (4\gamma)^{-1} \). Since \( n\{1 - F(r - \theta_n z)\} \) converges uniformly to \( z^\gamma \) on compacta, we can find \( n_1 \) such that, for \( n \geq n_1 \),

\[
\frac{1}{A_1} \int_0^{A_1(1-\alpha)} n\{1 - F(r - \theta_n z)\} \, dz \geq \frac{1}{A_1} \int_0^{A_1(1-\alpha)} (z^\gamma - \eta) \, dz
\]

\[
= \frac{1}{\gamma + 1} A_1^\gamma (1 - \alpha)^{\gamma+1} - (1 - \alpha)\eta
\]

\[
\geq \frac{3}{2\gamma} - \eta
\]

\[
= \frac{5}{4\gamma}.
\]

According to (12) we can find \( n_2 \) such that, for \( n \geq n_2 \),

\[
n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right) \leq \gamma^{-1} + \eta = \frac{5}{4\gamma}.
\]

Hence for \( n \geq n_0 = n_1 \vee n_2 \),

\[
\frac{1}{A_1} \int_0^{A_1(1-\alpha)} n\{1 - F(r - \theta_n z)\} \, dz \geq \frac{5}{4\gamma} \geq n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right).
\]

(14)

We are now ready to show (13). Note that

\[
E(X_{n+1}|X_n) = X_n + \int_{(1-\alpha)X_n}^r \{1 - F(y)\} \, dy;
\]

so by taking double expectations and using the Jensen inequality

\[
EX_{n+1} = Eg(X_n) \geq g(EX_n),
\]

(15)

where \( g(u) := u + \int_{(1-\alpha)u}^r \{1 - F(y)\} \, dy, \) \( u < r/(1 - \alpha) \). Put \( u_n := EX_n \) and \( v_n := r/(1 - \alpha) - A_0 \theta_n \), where \( A_0 > A_1 \) is taken large enough to satisfy

\[
u_{n_0} \geq v_{n_0}.
\]

We shall prove by induction that

\[
u_n \geq v_n
\]

(16)

for all \( n \geq n_0 \). Assuming that (16) holds for some \( n \geq n_0 \) it follows from the monotonicity of \( g \) on \((-\infty, r/(1 - \alpha))\) and (15) that

\[
u_{n+1} \geq g(u_{n+1}) \geq g(v_{n+1}).
\]

Hence we shall obtain \( u_{n+1} \geq v_{n+1} \) if we show that

\[
g(v_n) \geq v_{n+1}, \quad \forall n \geq n_0.
\]

(17)
The inequality (17) is equivalent to
\[
v_n + \int_{(1-\alpha)v_n}^{r} (1 - F(y)) \, dy \geq v_{n+1}, \quad \forall n \geq n_0,
\]
or, after setting \( y = r - A_\theta n z \),
\[
\int_0^{1-\alpha} n \{ 1 - F(r - A_\theta n z) \} \, dz \geq n \left( 1 - \frac{\theta_{n+1}}{\theta_n} \right), \quad \forall n \geq n_0.
\] (18)

Inequality (18), and hence (17), follows from
\[
\int_0^{1-\alpha} n \{ 1 - F(r - A_\theta n z) \} \, dz \geq \int_0^{1-\alpha} n \{ 1 - F(r - A_1 \theta z) \} \, dz
\]
\[
= \frac{1}{A_1} \int_0^{(1-\alpha)A_1} n \{ 1 - F(r - \theta z) \} \, dz
\]
\[
\geq n \left( 1 - \frac{\theta_{n+1}}{\theta_n} \right),
\]
for all \( n \geq n_0 \), according to (14).

The proof of tightness of the sequence \( a_n^{-1} \{ X_n - b_n/(1-\alpha) \} \), in case \( F \in D(\Lambda) \), can be given in a similar way; therefore we omit this proof.

**Theorem 4.** For \( F \in D(\Lambda) \) and \( x_0 < r/(1-\alpha) \), \( (a_n) \) and \( (b_n) \) such that \( F^n(a_n x + b_n) \to \Lambda(x) \) we have that \( \{ X_n - b_n/(1-\alpha) \}/a_n \) is tight on \( \mathbb{R} \).

5. Concluding remarks

(i) Together with the paper of Greenwood and Hooghiemstra (1991) this paper gives sufficient conditions on \( F \) to ensure that \( \{ X_n - b_n/(1-\alpha) \}/a_n \) has a distributional limit. It is known that for \( \alpha = 0 \) these conditions are also necessary. Whether this is also the case for \( 0 < \alpha < 1 \) we do not know.

(ii) The recursion (1) can be written as
\[
X_n = X_{n-1} + [Y_n - (1-\alpha)X_n]^+.
\]

A description of what kind of results can be expected if we let \( \alpha \) depend on \( n \) such that \( \alpha_n \to 1 \) is given in the work of den Hollander et al. (1991).
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References


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