Discussions of Relative Magnitude of Governing Parameters in the Problem of Slender Ships in Waves

By Hajime MARUO
Department of Naval Architecture

Reprinted from
BULLETIN OF THE FACULTY OF ENGINEERING
YOKOHAMA NATIONAL UNIVERSITY
VOL. 23

YOKOHAMA, JAPAN

MARCH, 1974

Bulletin of the Faculty of Engineering, Yokohama National Univ., Vol. 23, March, 1974

Discussions of Relative Magnitude of Governing Parameters in the Problem of Slender Ships in Waves

By Hajime MARUO
Department of Naval Architecture

Abstract

The linearized theory of ship motion among waves with forward speed assumes smallness of two parameters, i.e. the wave steepness ratio and the ship slenderness ratio ε. There are other parameters, however, which govern the flow field. They are frequency parameter $\omega\sqrt{l/g}$ and Froude number $U/\sqrt{gl}$. As the relative order of magnitude between each parameter changes, the leading term of the inner solution for the fluid motion changes accordingly. Discussions are given concerning the following cases. Case 1. $\omega\sqrt{l/g}=O(1)$, $U/\sqrt{gl}=O(1)$ Case 2. $\omega\sqrt{l/g}=O(\varepsilon^{-1/2})$, $U/\sqrt{gl}=O(1)$ Case 3. $\omega\sqrt{l/g}=O(\varepsilon^{-1/2})$, $U/\sqrt{gl}=O(\varepsilon^{1/2})$ Case 4. $\omega\sqrt{l/g}=O(1)$, $U/\sqrt{gl}=O(\varepsilon^{1/2})$ Case 5. $\omega\sqrt{l/g}=O(1)\rightarrow O(\varepsilon^{-1/2})$ $U/\sqrt{gl}=O(\varepsilon^{1/2})$.

2: rel. hoge ω (korte golven)
3: idem, en lage snelheid,

1. Introduction

A mathematical analysis of the fluid motion around a ship moving at a free surface stands in need of the linearization of boundary conditions, in particular at the free surface. In the case of steady forward motion, the linearization can be attained simply by the systematic expansion of the velocity potential in terms of the beam length ratio or the slenderness ratio. Michell's thin ship theory[1] is a typical linearized theory which is consistent by itself, and the slender ship theory is another possibility. The fluid motion generated by an oscillating ship without forward speed can be linearized by means of the amplitude of the oscillation, and there is no need of the restriction for the ship from. If, however, once the forward speed is introduced to the oscillating ship, the consistent development of the linearized theory turns to be much intricate. The difficulty in finding out a rational solution which is not trivial was first demonstrated by Peters and Stoker.[2] They showed that the hydrodynamic reactions such as the added mass and damping do not appear in the order of approximation for a thin ship oscillating in the plane of symmetry. There were extensive works by Haskind[3] and Hanaoka[4] about thin ships in longitudinal oscillations in still water before that time. A full condemnation of these achievements by reason of inconsistency may not be fair, because the consistent structure of theory breaks down on account of just a single reason of the inclusion of the steady forward velocity, or more specifically, the velocity potential of the steady forward motion, and the thin ship theory is consistent at zero forward speed. Moreover it can give a right expression for the most important portion of the hydrodynamic reaction, the damping, as was shown by the rigorous argument of Newman.[5] The main difficulty in the oscillating thin ship with forward velocity lies in the fact that the disturbance generated by the periodical motion of the ship is weaker than the disturbance due to the forward motion, but the employment of two independent parameters, namely the motion amplitude and the beam length ratio, could remove the above trouble. More serious difficulty appears when the ship is moving in ambient waves. Cross products between the velocity of the incident waves and that due to the steady forward motion appear in the same order of magnitude as that of the fluid motion due to the oscillation. It gives the free surface condition much complication. If, on the other hand, the assumption of the slender ship is employed, the order of magnitude of the horizontal motion is higher than that of the vertical motion, with respect to the slenderness ratio. Therefore the effect of the steady forward motion does not appear in terms of the lowest order in the far field expansion. The effect of the steady potential may appear in the lowest order in the near field expansion, but the solution is not so formidable. It is rather strange that we cannot find any discussion so far, except one by the present writer,[6] on the systematic development of the slender ship theory when the ship is moving in ambient waves with finite forward speed. The first order solution can be found in a neat form, if the wave amplitude and the slenderness ratio are taken as independent parameters to make the systematic expansion. The trouble of the simple slender ship theory appears in another aspect. Some numerical computations[7] have revealed that the hydrodynamic forces calculated by the theory show much deviation from actual phenomena in most parts of the frequency range of practical interest, and the validity of the slender ship theory is limited in the case of very low frequency.

It is widely known that the calculation by the strip theory has given results which show a reasonable agreement with measurements.[8] It is known also that the strip theory is a rational approximation for slender ships oscillating in still water with high frequency but without forward speed. The high frequency means, on the other hand, that the ship is excited by the force of incident waves whose length is comparable with the beam of the ship. However the problem of practical importance does not concern such a short wave case, but discussions in the practical field are directed mostly toward the case that the wave length is nearly same as the length of the ship. Therefore we cannot be satisfied by the apparent agreement between the measured results and those by the strip theory. Another problem is the effect of the forward speed. The effort of Ogilive and Tuck[9] to find out a rational basis for the inclusion of forward velocity in the strip theory is rather painstaking.

One can turn to the three-dimensional slender ship theory and try to find out its possibility to give a right result in the case of moderate and high frequencies. To do this, the frequency must be regarded as a parameter of changing order of magnitude. The forward speed is another quantity of which the order of magnitude must be examined. It is not intended, in the present work, to find out any explicit formulae for the solution of the slender ship theory, but discussions will be focused on the possibility of consistent linearized theory when the frequency parameter $\omega\sqrt{l/g}$ and Froude number $U/\sqrt{gl}$ change their order of magnitude. The simple slender ship theory stands on the assumption that these parameters both are of the order of unity. It is known by the works of Vossers[10] and Joosen[11] that the simple strip theory is valid when $\omega\sqrt{l/g}=O(\varepsilon^{-1/2})$, $U/\sqrt{gl}=O(\varepsilon^{1/2})$, where ε is the slenderness ratio. Ogilive and Tuck discussed the case, $\omega\sqrt{l/g}=O(\varepsilon^{-1/2})$, $U/\sqrt{gl}=(1)$. A non-linear effect appears in the same order as the effect of forward speed. The latter two cases are short wave problem but the motion of the ambient wave was not given a due consideration.

The present paper intends to elucidate how the choice of the order of magnitude of these parameters affects the perturbation scheme and to what extent the linearization of the boundary condition is valid.

2. Expansion of boundary conditions by small parameters.

The problem which we are going to discuss is a slender body floating on a regular wave and moving with a uniform average speed U in the mean direction of its longitudinal axis. In the most general case, the direction of the forward velocity and the direction of wave propagation differ each other and the body or the ship makes oscillations of six-degrees of freedom around its mean position. To discuss the whole problem in a linearized scheme, the primary requisition is the small amplitude of oscillation. Since the ship's oscillation is excited by the action of the incident wave, the small amplitude of the wave is the basic premise, but the linearization of the fluid motion in regular waves needs a condition that the wave amplitude is much smaller than nothing but the wave length. Therefore the ratio of the wave amplitude to the wave length h/λ is the basic parameter. One can employ, instead of the above ratio, the maximum wave slope Kh, where K is the wave number $2\pi/\lambda$. Then let us define a small parameter $\delta=Kh$ for the use of the discussion of the order of magnitude. It must be noted that the order of magnitude of the wave amplitude changes as the order of the wave number changes. The definition of the long wave or short wave is based on the comparison with the length of the ship $2l$. Therefore the order of the wave number or frequency parameter is related to the dimensionless coefficient $Kl=2\pi/\lambda$ or $\omega\sqrt{l/g}$. One may put $l=1$, when there is no confusion, because l is regarded as the reference length in the whole system. Since the oscillation of the ship is excited by the action of the wave, the amplitude of the oscillation has the same order of magnitude as that of the wave amplitude. The velocity of the fluid motion generated by the oscillation of the ship is of the same order as the oscillatory part of the velocity of the ship's surface, which is of the same order as the velocity of the orbital motion of the wave. Therefore the whole system can be linearized by a single parater δ, if the steady forward motion does not exist, and no restriction is imposed on the shape of the ship. If the ship has a steady forward speed, on the other hand, a restriction must be imposed on the shape of the ship, in order to make the disturbance due to the forward motion small. The slender ship is assumed in the present discussion.

voor korte golven niet.

The ratio of the beam to the length, denoted by ε, is much smaller than unity and the ratio of the draft to the beam is of the order of 1.

The small parameters δ and ε are utterly independent of each other.

The perturbation method assumes the possibility of expansion of the velocity potential in series of ascending power of δ and ε. Since δ is independent of ε, the expansion can be done first with respect to δ.

The first term which is independent of δ gives the case when the ship moves with uniform velocity on still water. The linearized theory takes terms up to the first order with respect to δ. Then the term which is linear to δ is simple harmonic and gives the oscillatory part of the velocity potential. Next one may expand the above portions of the velocity potential, already linearized by δ, by the slenderness ratio ε. There is a term which is independent of ε in the oscillatory potential. It gives the incident waves which may be assumed as simple harmonic. The other part gives the disturbance by the ship.

Consider a relative motion with respect to the coordinates moving with the average forward velocity U, and take the axis of x in the direction opposite to the uniform velocity. Then the velocity potential can be written in the form like $Ux+\phi$, and ϕ satisfies the Laplace equation.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0\,, \tag{1}$$

in the space occupied by the fluid. The boundary conditions satisfied by the velocity potential are those on the ship's surface and on the free surface. If the depth of water is assumed infinite, the condition at the infinity is that the fluid velocity due to the disturbance by the ship vanishes there and the fluid motion is just the sum of the uniform flow and the incident wave. The radiation condition at a great distance should be considered too. The boundary condition at the ship's surface is

$$\frac{\partial f}{\partial t}+\left(U+\frac{\partial\phi}{\partial x}\right)\frac{\partial f}{\partial x}+\frac{\partial\phi}{\partial z}\frac{\partial f}{\partial z}-\frac{\partial\phi}{\partial y}=0\,. \tag{2}$$

at

$$y=f(x,z)\,. \tag{3}$$

The latter equation gives the expression for the ship's surface in reference to the coordinotes fixed in space. Now we assume, for simplicity, that the ship makes heaving and pitching in longitudinal waves. Designate the vertical position of the center of gravity by Z and the pitching angle ψ (positive for bow up), and take coordinates (x_0, y_0, z_0) fixed to the ship. Then the relation between coodinate systems (x, y, z) and (x_0, y_0, z_0) is

$$\left.\begin{aligned} x_0&=x\cos\psi-(z-Z)\sin\psi\\ y_0&=y\\ z_0&=x\sin\psi+(z-Z)\cos\psi \end{aligned}\right\} \tag{4}$$

The equation of the ship's surface in reference to the coordinates (x_0, y_0, z_0) can be written as

$$y_0=f_0(x,z_0)\,, \tag{5}$$

Assuming Z and ψ as quantities of the order of δ, and omitting terms of higher order, we obtain

$$\frac{\partial f}{\partial t}=-\frac{\partial f_0}{\partial x_0}z\dot{\varphi}-\frac{\partial f_0}{\partial z_0}(\dot{Z}-x\dot{\varphi}),$$

$$\frac{\partial f}{\partial x}=\frac{\partial f_0}{\partial x_0}+\frac{\partial f_0}{\partial z_0}\varphi,$$

$$\frac{\partial f}{\partial z}=\frac{\partial f_0}{\partial z_0}-\frac{\partial f_0}{\partial x_0}\varphi,$$

where dot means the time-derivative. Then eq. (2) becomes

$$-\frac{\partial f_0}{\partial x_0}z\dot{\varphi}-\frac{\partial f_0}{\partial z_0}(\dot{Z}-x\dot{\varphi})+\left(U+\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial f_0}{\partial x_0}+\frac{\partial f_0}{\partial z_0}\varphi\right)$$
$$+\frac{\partial\phi}{\partial z}\left(\frac{\partial f_0}{\partial z_0}-\frac{\partial f_0}{\partial x_0}\varphi\right)-\frac{\partial\phi}{\partial y}=0, \tag{6}$$

If we pick up terms which are independent of time, we obtain the boundary condition for the steady potential, denoted by $U\phi_0$

$$\left(1+\frac{\partial\phi_0}{\partial x}\right)\frac{\partial f_0}{\partial x_0}+\frac{\partial\phi_0}{\partial z}\frac{\partial f_0}{\partial z_0}-\frac{\partial\phi_0}{\partial y}=0. \tag{7}$$

Take length n along the outward normal to the ship's surface, and denote the direction cosines of the normal n_x, n_y, n_z. Then the above equation can be written in the form

$$\frac{\partial\phi_0}{\partial n}+n_x=0. \tag{8}$$

Now let us examine the order of magnitude when the ship is regarded very slender. n_x means the slope of the ship's surface, so that its order of magnitude is the slenderness ratio ε. Since the disturbance velocity potential is singular along the x-axis, the differentiation of the velocity potential along the normal changes the order of magnitude by ε^{-1}. This fact can be shown materially by the employment of so called 'strained coordinates' which measure lengthwise direction and lateral direction by different scales, such as x/l, y/b, z/b, where l is the half length and b is the half breadth of the ship. This procedure is well known and we shall not repeat it here. Therefore the relation between the order of magnitude of ϕ_0 and that of $\partial\phi_0/\partial n$ is

$$\partial\phi_0/\partial n=\varepsilon^{-1}O(\phi_0),$$

or

$$\phi_0=\varepsilon O(\partial\phi_0/\partial n).$$

Since $n_x=O(\varepsilon)$, the above relation results

$$\phi_0=O(\varepsilon^2).$$

The oscillatory part is composed by the incident wave potential ϕ_w, which is independent of ε, and the oscillatory disturbance ϕ_1. The former can be regarded as a given function. Now let us examine the order of magnitude of each term of eq. (6). It is easily understood that $f_0=O(\varepsilon)$, $z=O(\varepsilon)$, $\partial f_0/\partial x_0=O(\varepsilon)$, $\partial f_0/\partial z_0=\varepsilon^{-1}O(\varepsilon)=O(1)$ $\partial\phi_0/\partial x=O(\varepsilon^2)$, $\partial\phi_0/\partial y=\varepsilon^{-1}O(\phi_0)=O(\varepsilon)$, $\partial\phi_0/\partial z=O(\varepsilon)$ $\partial\phi_1/\partial x=O(\phi_1)$, $\partial\phi_1/\partial y=\varepsilon^{-1}O(\phi_1)$, $\partial\phi_1/\partial z=\varepsilon^{-1}O(\phi_1)$.

Some consideration is needed as to the derivative of ϕ_w.

We can express the regular wave by the velocity potential

$$\phi_w=h\sqrt{g/K}\exp\left[Kz-iK(x\cos\alpha+y\sin\alpha)-i\omega t\right]. \tag{9}$$

This potential means a regular wave of amplitude h and wave length $\lambda=2\pi/K$, propagating in the direction, making an angle α with the x-axis. Since the coordinates are moving with the ship, ω means the circular frequency of encounter, and the absolute frequency is given by

$$\omega_0=\omega-UK. \tag{10}$$

There is a relation between the wave number and the frequency as follows.

$$K=\omega_0^2/g=(\omega-UK)^2/g \tag{11}$$

The order of magnitude of ω or K may not be unity, so that one needs to include these quantities in the argument of the order. It can be assumed that the amplitude of ship's oscillation is of the same order of magnitude of the wave amplitude, but the frequency of the oscillation is not the frequency of the wave ω_0 but the frequency of encounter ω. The wave amplitude h is the order of δ/K, so that the velocity of the oscillatory motion of the ship has the order $\omega\delta K^{-1}$. The fluid velocity of the incident wave has, on the other hand, has the order $\delta K^{-1/2}$. They are not necessarily of the same order. Keeping these facts in mind, we examine the order of magnitude of the oscillatory part of eq. (6). Taking the first order terms with respect to δ, we obtain

$$\begin{aligned}&-\frac{\partial f_0}{\partial x_0}z\dot{\psi}-\frac{\partial f_0}{\partial z_0}(\dot{Z}-x\dot{\psi})+U\frac{\partial f_0}{\partial z_0}\psi+U\frac{\partial\phi_0}{\partial x}\frac{\partial f_0}{\partial z_0}\psi\\&+\frac{\partial f_0}{\partial x}\frac{\partial}{\partial x}(\phi_1+\phi_w)+\frac{\partial f_0}{\partial z_0}\frac{\partial}{\partial z}(\phi_1+\phi_w)-U\frac{\partial\phi_0}{\partial z}\psi\\&-\frac{\partial}{\partial y}(\phi_0+\phi_w)-U(Z-x\psi)\left(\frac{\partial f_0}{\partial z_0}\frac{\partial^2\phi_0}{\partial z^2}-\frac{\partial^2\phi_0}{\partial y\partial z}\right)=0.\end{aligned} \tag{12}$$

where the last term comes from the steady potential at time-varying position. If we assume $U=O(1)$ and $\omega=O(1)$, it will be found that $\phi_1=O(\delta\varepsilon)$. Then taking terms up to the order of $\delta\varepsilon$, we obtain

$$\begin{aligned}&\frac{\partial f_0}{\partial z_0}(\dot{Z}-x\dot{\psi})+U\frac{\partial f_0}{\partial z_0}\psi+\frac{\partial f_0}{\partial z_0}\frac{\partial}{\partial z}(\phi_1+\phi_w)-\frac{\partial}{\partial y}(\phi_1+\phi_w)\\&+\frac{\partial f_0}{\partial x_0}\frac{\partial\phi_w}{\partial x}-U(Z-x\psi)\left(\frac{\partial f_0}{\partial z_0}\frac{\partial^2\phi_0}{\partial z^2}-\frac{\partial^2\phi_0}{\partial y\partial z}\right)=0.\end{aligned} \tag{13}$$

The term of the lowest order has the order of ε. Consider the outward normal ν to the sectional form of the hull in the plane perpendicular to the x-axis and designate its direction cosines by ν_y, ν_z. If ε^2 is omitted, the boundary condition can be written in the form like

$$(\dot{Z}-x\dot{\psi}+U\psi)\nu_z+\frac{\partial\phi_1}{\partial\nu}-U(Z-x\psi)\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_0}{\partial z}\right)+\frac{\partial\phi_w}{\partial\nu}=0\,. \tag{14}$$

Assuming that ϕ_0 and ϕ_w are given functions and shifting terms, we have

$$\frac{\partial\phi_1}{\partial\nu}=-(\dot{Z}-x\dot{\psi}+U\psi)\nu_z+U(Z-x\psi)\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_0}{\partial z}\right)-\frac{\partial\phi_w}{\partial\nu}\,, \tag{15}$$

This is the boundary condition to be satisfied by ϕ_1 on the hull surface. All therms are of the order of δ, so that the order of magnitude of ϕ_1 must be $\delta\varepsilon$. One can divide the periodical potential ϕ_1 into a part determined by the oscillation of the ship and a part originated by the diffraction of the ambient wave. The former one is the radiation potential ϕ_R, for which the boundary condition becomes

$$\frac{\partial\phi_R}{\partial\nu}=-(\dot{Z}-x\dot{\psi}+U\psi)\nu_z+U(Z-x\psi)\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_0}{\partial z}\right)\,, \tag{16}$$

while the latter is the diffraction potential ϕ_D, which has to satisfy the boundary condition

$$\frac{\partial\phi_D}{\partial_0\nu}=-\frac{\partial\phi_w}{\partial\nu}\,. \tag{17}$$

Next let us consider the boundary condition of the free surface. If the form of the free surface is given by the equation

$$z=\zeta(x,y,t)\,, \tag{18}$$

the kinematical condition there is

$$\frac{\partial\zeta}{\partial t}+\left(U+\frac{\partial\phi}{\partial x}\right)\frac{\partial\zeta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}-\frac{\partial\phi}{\partial z}=0\,. \tag{19}$$

There is another condition that the pressure is constant on the free surface. Then Bernouilli's theorem gives

$$\frac{\partial\phi}{\partial t}+U\frac{\partial\phi}{\partial x}+\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}+g\zeta=0\,. \tag{20}$$

These equations hold on the curved surface $z=\zeta$. Putting

$$F(x,y,z,t)=\frac{\partial\phi}{\partial t}+U\frac{\partial\phi}{\partial x}+\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}\,, \tag{21}$$

we can write

$$\zeta=-\frac{1}{g}F(x,y,\zeta,t)\,. \tag{22}$$

Therefore eq. (19) can be written in the form

$$\frac{\partial F}{\partial t}+\left(U+\frac{\partial\phi}{\partial x}\right)\frac{\partial F}{\partial x}+\frac{\partial\phi}{\partial y}\,\frac{\partial F}{\partial y}+\frac{\partial\phi}{\partial z}\,\frac{\partial F}{\partial z}+g\frac{\partial\phi}{\partial z}=0\,. \tag{23}$$

This is the exact non-linear form of the free surface condition. Let us examine the order of magnitude in the near field, bearing in mind the fact that the differentiation with respect to y or z changes the order by ε^{-1}, owing to the assumption of the slender ship. We have written the velocity potential in the form like

$$\phi=U\phi_0+\phi_1+\phi_w\,,$$

Inserting in (23) the above decomposition of the potential, and picking up the time-independent part, we obtain

$$\frac{\partial^2\phi_0}{\partial x^2}+2\frac{\partial^2\phi_0}{\partial x\partial y}\,\frac{\partial\phi_0}{\partial y^2}+\frac{3}{2}\,\frac{\partial^2\phi_0}{\partial y^2}\left(\frac{\partial\phi_0}{\partial y}\right)^2+\frac{\partial\phi_0}{\partial x}\,\frac{\partial^2\phi_0}{\partial y^2}+\frac{g}{U^2}\,\frac{\partial\phi_0}{\partial z}+O(\varepsilon^3)=0\,, \tag{24}$$

Then term of the lowest order is $(g/U^2)\partial\phi_0/\partial z$ which has the order of ε. Therefore the free surface condition for the steady potential of the lowest order is

$$\frac{\partial\phi_0}{\partial z}=0 \quad \text{at } z=0\,, \tag{25}$$

that is identical with the condition of rigid wall. We cannot proceed to the next order without handling the non-linear terms in the free surface condition. Let us examine next, the periodical part of the free surface condition, that can be written

$$\begin{aligned}&\frac{\partial^2\phi_1}{\partial t^2}+2U\frac{\partial^2\phi_1}{\partial t\partial x}+U^2\frac{\partial\phi_1}{\partial x^2}+g\frac{\partial\phi_1}{\partial z}+2U\frac{\partial\phi_0}{\partial y}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial\phi_1}{\partial y}\\&+2U^2\frac{\partial^2\phi_0}{\partial x\partial y}\,\frac{\partial\phi_1}{\partial y}+U^2\frac{\partial}{\partial y}\left(\frac{\partial\phi_0}{\partial y}\right)^2\frac{\partial\phi_1}{\partial y}+U^2\left(\frac{\partial\phi_1}{\partial y}\right)^2\frac{\partial^2\phi_1}{\partial y^2}\\&-U\frac{\partial^2\phi_0}{\partial z^2}\left(\frac{\partial\phi_1}{\partial t}+U\frac{\partial\phi_1}{\partial x}\right)+U^2\left\{\frac{\partial\phi_0}{\partial x}+\frac{1}{2}\left(\frac{\partial\phi_0}{\partial y}\right)^2\right\}\frac{\partial^2\phi_1}{\partial z^2}\\&-U\frac{\partial^2\phi_0}{\partial z^2}\left(\frac{\partial\phi_w}{\partial t}+U\frac{\partial\phi_w}{\partial x}\right)+O(\delta\varepsilon^2)=0\,.\end{aligned} \tag{26}$$

The lowest order term is of the order of δ and the next one is of the order of $\delta\varepsilon$. Though these are linear with respect to ϕ_1, if ϕ_0 and ϕ_w are assumed to be known functions, the non-linear effect appears in the form of product terms of ϕ_1 and ϕ_0, which make the boundary value problem unpractical. If only the term of the lowest order is taken, the free surface condition for the radiation potential becomes

$$\frac{\partial\phi_R}{\partial z}=0\,, \tag{27}$$

and we have the rigid wall condition again.

The free surface condition in the far field takes a different form from the above, because the differentiation with respect to y or z does not affect the order of magnitude. Therefore the leading term of the equation gives simply the linearized boundary condition as follows,

$$\frac{\partial^2\phi}{\partial t^2}+2U\frac{\partial^2\phi}{\partial t\partial x}+U^2\frac{\partial x^2}{\partial^2\phi}+g\frac{\partial\phi}{\partial z}=0\,, \quad \text{at } z=0\,. \tag{28}$$

For the periodical motion with circular frequency ω, we can write $\phi_1=e^{i\omega t}\Phi$ and the boundary condition becomes

$$-\omega^2\phi_1+2iU\omega\frac{\partial\phi_1}{\partial x}+U^2\frac{\partial^2\phi_1}{\partial x^2}+g\frac{\partial\phi_1}{\partial z}=0\,. \tag{29}$$

The Green function for this boundary condition is known and is expressed by

$$\begin{aligned}
G(x,y,z\,;\,x',y',z')&=\frac{1}{r_1}-\frac{1}{r_2}\\
&+\frac{2}{\pi}\int_{-\infty}^{\infty}dm\int_0^{\infty}dn\exp\left[-|y-y'|\sqrt{m^2+n^2}+im(x-x')\right]\\
&\times\{\cos(nz+\varepsilon)\cos(nz'+\varepsilon)-\cos nz\cos nz'\}/\sqrt{m^2+n^2}\\
&+\frac{2}{g}\int_{-\infty}^{\infty}dm\exp\left[(z+z')(mU+\omega)^2/g-|y-y'|\sqrt{m^2-(mU+\omega)^4/g^2}\right.\\
&\left.+im(x-x')\right](mU+\omega)^2/\sqrt{m^2-(mU+\omega)^4/g^2}\,,
\end{aligned} \tag{30}$$

where

$$r_1=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\,,$$

$$r_2=\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}\,,$$

$$\varepsilon=-(mU+\omega)/gn\,,$$

and the radical of imaginary in the last integral takes an appropriate sign in accordance with the radiation condition. The outer solution is expressed by means of this Green function.

3. Some cases when order of magnitude of the frequency parameter and/or the Froude number changes.

We discussed in the preceding section the boundary condition when the frequency and the forward speed are assumed to be of the order of unity. As mentioned before, however, the order of magnitude of these quantities can be different from unity. Then the boundary conditions will take another form.

Here we will discuss five cases.

Case 1. $\omega\sqrt{l/g}=O(1)$, $U/\sqrt{gl}=O(1)$

This is the case discussed in the previous section, and the result by this assumption may be called the simple slender ship theory. The free surface condition in the near field is

$$g\frac{\partial\phi_1}{\partial z}-U\frac{\partial^2\phi_0}{\partial z^2}\left(\frac{\partial\phi_w}{\partial t}+U\frac{\partial\phi_w}{\partial x}\right)=0\,,\quad \text{at } z=0\,. \tag{31}$$

If we divide the periodical potential ϕ_1 in the radiation potential ϕ_R and the diffraction potential ϕ_D, we have again the rigid wall condition for ϕ_R as eq. (27).

Because of the relation for the surface elevation namely

$$\zeta_w=-\frac{1}{g}\left(\frac{\partial\phi_w}{\partial t}+U\frac{\partial\phi_w}{\partial x}\right),$$

the condition for the diffraction potential can be written as

$$\frac{\partial\phi_w}{\partial z}=-U\zeta_w\frac{\partial^2\phi_D}{\partial z^2}\,,\quad \text{at } z=0\,. \tag{32}$$

This boundary condition at the plane $z=0$ can be transformed to the boundary condition on the hull surface. If we put

$$\phi_2=\phi_1+U\zeta_w\frac{\partial\phi_D}{\partial z}\,, \tag{33}$$

it satisfies the rigid wall condition

$$\frac{\partial\phi_2}{\partial z}=0\,,\quad \text{at } z=0\,. \tag{34}$$

The boundary condition for ϕ_2 on the hull surface becomes

$$\frac{\partial\phi_2}{\partial\nu}=-(\dot{Z}-x\dot{\varphi}+U\varphi)\nu_z+U(Z-x\varphi)\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_0}{\partial z}\right)$$
$$-\frac{\partial\phi_w}{\partial\nu}+U\zeta_w\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_0}{\partial z}\right). \tag{35}$$

In the case of a slender ship on longitudinal waves, one can write

$$\frac{\partial\phi_w}{\partial\nu}=\nu_z\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\zeta_w\,.$$

Therefore the bouudary condition for ϕ_2 on the hull surface can be written

$$\frac{\partial\phi_2}{\partial\nu}=\nu_z\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)(Z-x\varphi-\zeta_w)-U\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_0}{\partial z}\right)(Z-x\varphi-\zeta_w)\,. \tag{36}$$

It is interesting to observe that the above equation means the condition as if each section makes a movement of vertical displacement $Z-x\varphi-\zeta_w$ which is the deviation from the wave surface.

The boundary value problem with boundary conditons (36) and (34) can be solved without difficulty in the two-dimensional motion. If we designate the solution by $\phi_2^{(2D)}$, the velocity potential ϕ_1 is given by

$$\phi_1=\phi_2^{(2D)}-U\zeta_x\frac{\partial\phi_0}{\partial z}+g(x)\,, \tag{37}$$

where $g(x)$ is a function of x only, which can be determined by means of matching with the outer solution expressed by the Greeen function (30). We cannot take account of the term of next order in the free surface condition, without handling the non-linear effect.

Case 2. $\omega\sqrt{l/g}=O(\varepsilon^{-1/2})$, $U/\sqrt{gl}=O(1)$

This is the case of high frequency and high speed, which was discussed by Ogilvie and Tuck. The relation between the wave length and the ship's length is $\lambda/2l=\pi/kl=O(\varepsilon^{1/2})$. Therefore the wave length is not very short in comparison with the ship's length. The wave height is of the order of $\delta\varepsilon^{1/2}$ and the velocity of the orbital motion is δ, so that the order of ϕ_1 is $\delta\varepsilon$.

If we take up to the order of $\delta\varepsilon^{1/2}$ in the free surface condition, the condition for the radiation potential ϕ_R becomes

$$\frac{\partial t^2}{\partial^2\phi_R}+g\frac{\partial\phi_R}{\partial z}+2U\frac{\partial^2\phi_R}{\partial t\partial x}+2\frac{\partial\phi_0}{\partial y}\frac{\partial^2\phi_R}{\partial t\partial y}-\frac{\partial\phi_0}{\partial z^2}\frac{\partial\phi_R}{\partial t}=0\,, \tag{38}$$

The first two terms are of the order of δ and the others are of the order of $\delta\varepsilon^{1/2}$. If only the terms of the lowest order are taken, the linearized free surface condition

$$\frac{\partial^2\phi_R}{\partial t^2}+g\frac{\partial\phi_R}{\partial z}=0\,. \tag{39}$$

is obtained. The effect of the forward speed cannot be taken into consideration, unless quadratic terms which are the product of the steady potential and the radiation potential can be handled. The outer solution is expressed by means of the Green function defined by (30) such as

$$\phi_R\approx-\frac{1}{4\pi}e^{i\omega t}\int dx'\cdot m(x')G(x,y,z\,;\,x',0,0)\,. \tag{40}$$

Taking an asymptotic expansion for large ω, the double integral of the third term on the right hand side of (30) becomes $2/r_2$, while the last term becomes

$$-\frac{2i}{g}\int_{-\infty}^{\infty}\exp\left[(z+z')(mU+\omega)^2/g-i|y-y'|(mU+\omega)^2/g+im(x-x')\right]dm$$

$$\approx-\frac{2i}{g}e^{(z+z'-i|y-y'|)\omega^2/g}\int_{-\infty}^{\infty}\left\{1+(z+z'-i|y-y'|)\frac{2\omega Um}{g}\right\}e^{im(x-x')}dm\,.$$

Therefore Fouries integral theorem gives

$$\phi_R\approx-\frac{2i}{g}e^{i\omega t+(z-i|y|)\omega^2/g}\left\{m(x)+i(z+i|y|)\frac{2\omega U}{g}m'(x)\right\}. \tag{41}$$

The first term in the parenthesis means a plane wave propagating in the y-

direction and gives the two-dimensional motion in the plane perpendicular to the x-axis, but $m'(x)=dm(x)/dx$ in the second terms means that the interference exists between different sections. Therefore the boundary value problem cannot be formulated by the two-dimensional form. If only the term of the lowest order is taken, the boundary condition takes a simple form for which the two-dimensional solution is valid. The effect of the forward speed appears in the higher order, and can be expressed as a correction term which will be added to the result of the strip theory. Ogilvie and Tuck were able to give compact expressions for the correction terms to hydrodynamic forces and moments. According to them, the effect of forward speed does not appear in the added mass and damping but appears only in the coupling terms between different modes of oscillations.

Case 3. $\omega\sqrt{l/g}=O(\varepsilon^{-1/2})$, $U/\sqrt{gl}=O(\varepsilon^{1/2})$

This is the case of short wave such that the wave length is of the same order as that of the breadth of the ship. In this case the order of magnitude of ϕ_1 is $\delta\varepsilon^{3/2}$. The free surface condition for the radiationpotential becomes

$$\frac{\partial^2\phi_R}{\partial t^2}+g\frac{\partial\phi_R}{\partial z}=0\,, \quad \text{at } z=0\,, \tag{42}$$

up to the order of $\delta\varepsilon^{1/2}$. The next term is of the order of $\delta\varepsilon^{3/2}$, in which the effect of the forward speed appears with the quadratic terms. Therefore the forward speed cannot be taken into account by the linearized theory. The asymptotic form for the outer solution takes the form

$$\phi_R\approx-\frac{2i}{g}e^{(z-i|y|)\omega^2/g}m(x)\,, \tag{43}$$

that means out-going plane waves, so that the motion is purely two-dimensional and the strip theory without forward speed is valid. However the strip theory is not applicable to the diffraction potential because of the short wave length. ! klopt.

Case 4. $\omega\sqrt{l/g}=O(1)$, $U/\sqrt{gl}=O(\varepsilon^{1/2})$

The wave length in this case is comparable with the ship's length, and the forward speed is not so high. Since the Froude number of ordinary ships is not much higher than the order of 0.3, the speed parameter U^2/gl is of the order of 10^{-1}. Therefore the present case corresponds to the ordinary ships in waves of moderate length.

The order of magnitude of the radiation potential is $\delta\varepsilon$ but the effect of forward speed appears in the term of the order of $\delta\varepsilon^{3/2}$. The lowest order term in the free surface condition has the order of δ and the next order is $\delta\varepsilon$. If we take up to the order of $\delta\varepsilon$, the free surface condition in the near field becomes

$$\frac{\partial^2\phi_1}{\partial t^2}+g\frac{\partial\phi_1}{\partial z}+U\zeta_{x}\frac{\partial^2\phi_0}{\partial z^2}=0\,, \quad \text{at } z=0\,. \tag{44}$$

The quadratic terms appears in the higher order of $\delta\varepsilon^{3/2}$, so that the radiation potential satisfies the linearized free surface condition without forward speed, eq. (42), and the free surface condition for the diffraction potential can be transformed to the boundary condition on the hull surface as was shown before. Because of the assumption of the slender ship, the velocity potential of the lowest order in the near field takes the form

$$\phi_1 = \phi_1^{(2D)} + g(x)\,, \tag{45}$$

where $\phi_1^{(2D)}$ is the two-dimensional potential function. Therefore the boundary condition on the hull surface does not relate to the three-dimensional part so long as the lowest order term is taken. However the three-dimensional part has an effect in the term of the order higher than the lowest one by ε. The free surface condition (44) involves the terms of this order. Therefore the consistent approximation must involve the three-dimensional effect. The effect of the forward speed in the three-dimensional portion is of still higher order, so that it need not be taken into account. The three-dimensional part of the inner solution is determined by matching with the outer solution, and the resulting expression takes the form[12]

$$\begin{aligned}\phi_1 = \phi_1^{(2D)} - e^{i\omega t}\Big[& m(x)(1+Kz)(\gamma+\pi i) + \frac{1}{2}(1+Kz)\int m'(x')sgn(x-x') \\ & \times \ln(2K|x-x'|)dx' - \frac{\pi}{4}K(1+Kz)\int m(x') \\ & \times \{H_0(K|x-x'|) + Y_0(K|x-x'|) + 2iJ_0(K|x-x'|)\}dx'\Big]\,,\end{aligned} \tag{46}$$

where γ is Euler's constant, H_0 is the Struve function, Y_0 and J_0 are Bessel functions, and $m(x)$ is the strength of the source term in the two-dimensional potential. The free surface condition for the radiation potential is

$$\frac{\partial \phi_R}{\partial z} - K\phi_R = 0\,, \quad \text{at } z=0\,, \tag{47}$$

In order to deal with the free surface condition for the diffraction potential, the auxiliary function ϕ_2 defined by (33) must be employed. It satisfies the free surface condition similar to (47) and the effect of the forward speed is transformed to the boundary condition on the hull surface, such as the last term of (35). The method of solution of the boundary value problem is similar to what is described in the literature (12).

Case 5. $\omega\sqrt{l/g} = O(1) \to O(\varepsilon^{-1/2})$, $U/\sqrt{gl} = O(\varepsilon^{1/2})$

We have examined so far the cases in which ω and U have definite order of magnitude. Since however, these parameters have wider variation in general, the discussion which holds only in a limited range of the parameters is not convenient for practical purposes. The Froude number of conventional merchantile vessels is not much higher than 0.3, so that the speed parameter U^2/gl is of the order of 10^{-1} as in the previous case. However the wave length has a wide variation from the same order as the ship's beam to the order of the ship's length, and an approximation which can cover such a wide range of frequency is desirable. Let us consider here a valid approximation in the frequency range from $O(1)$ to $O(\varepsilon^{-1/2})$. The case $\omega = O(1)$ is identical with Case 4. If we change the order of ω to $\varepsilon^{-1/2}$, the order of ωU becomes unity and non-linear terms appear in the order higher by ε in the free surface condition. Since this is the order of terms which were taken into consideration in Case 4, the linearized solution will become inconsistent when ω is increased to $O(\varepsilon^{-1/2})$. If we take only the term of the lowest order, the radiation potential is of the order of $\delta\varepsilon$, and the free surface condition for it is of the order of δ, namely

$$\frac{\partial \phi_R}{\partial z}=0\,. \tag{48}$$

The next term which has been omitted is of the order of $\delta\varepsilon$. The effect of the forward speed in the radiation potential appears in the order of $\delta\varepsilon^{3/2}$ which still satisfies the boundary condition (48). In Case 3, on the other hand, the term of the lowest order in the radiation potential is of the order of $\delta\varepsilon^{3/2}$, which satisfies the free surface condition

$$g\frac{\partial \phi_R}{\partial z}+\frac{\partial^2 \phi_R}{\partial t^2}=0\,. \tag{49}$$

If we employ the above equation as the free surface condition, it means that we are taking account of terms up to the order of $\delta\varepsilon^{3/2}$ in the radiation potential. If, on the other hand, the order of ω is changed towards unity, the second term of (49) becomes the higher order. However it will not result in any harmful effect, even if the second term in (49) is retained. The effect of the forward speed, which comes from the boundary condition at the hull surface is involved with the order of approximation when $\omega=O(1)$, but will shift to the higher order term of $O(\delta\varepsilon^{3/2})$ in the Case of $\omega=O(\varepsilon^{-1/2})$. The effect of the three-dimensional motion appears in the order higher than the lowest order by ε, so that it is outside the present order of approximation.

Thus the valid approximation in the present case is the two-dimensional motion with the free surface condition given by (49), and the boundary value problem can be solved at each section independently, that means the strip theory is valid. The boundary condition on the hull surface is expressed by eq. (36) which involves the effect of forward speed. Though the second term on the right hand side may make some trouble, it is not the case in the calculation of hydrodynamic forces and moments as long as the radiation potential is concerned, owing to the theorem found by Tuck.[9]

The above argument will provide justification of the strip theory by which the effect of forward speed is taken into account. The strip theory may be applicable to the diffraction potential too when the wave length is not very short, but further examination is needed, because it does not hold in the case of short wave, as was mentioned in Case 3.

Conclusion

The principal aim of the present argument is to examine to what extent the usual linearized form of the free surface condition of the form

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}=0\,,\quad \text{at } z=0\,,$$

is valid, and the ordinary strip theory is applicable.

It has been found that the possibility of the strip theory as a rational and self-consistent approximation is rather limited. If the frequency parameter and the Froude number have definite order of magnitude, the strip theory can become a consistent approximation only when a ship is oscillating in still water with high frequency but with a forward speed which is not higher than the order of $\varepsilon^{1/2}$. The effect of the forward speed appears in the higher order term which can not be included in the linearized theory. If we need, however, a practical method which is applicable to the range of wider variation of the frequency parameter, the strip theory which involves the effect of forward speed can be regarded as a lower approximation for the computation of hydrodynamic forces and moments on an oscillating ship in still water. Though this method is a weaker solution, it still has a uniform validity in a wider range of varying frequency. This fact may provide a justification for the good agreement between the measured result and the computation by means of the strip theory.

References

1) J. H. Michell, The wave resistance of a ship. Phil. Mag. (5) 45 (1898).
2) A. S. Peters, J. J. Stoker, The motion of a ship as a floating rigid body in a seaway. Comm. Pure & Appl. Math. 10 (1957).
3) M. D. Haskind, The oscillation of a ship in still water. Izv. Akad. Nauk SSSR Tekhn. Nauk (1956).
4) T. Hanaoka, The motion of a ship among waves and the theory of wave-resistance. J.S.N.A. 98 (1956).
5) J. N. Newman, A linearized theory for the motion of a thin ship in regular waves. J.S.R. 5-1 (1961).
6) H. Maruo, Application of the slender body theory to the longitudinal motion of a ship among waves. Bull. Faculty Eng. Yokohama Nat. Univ. 16 (1967).
7) W. P. A. Joosen, Oscillating slender ships at forward speed. Pub. 268 NSMB.
8) J. Gerritsma, W. Beukelman, Comparison of calculated and measured having and pitching motions of a Series 60, C_b=0.70 ship model in regular longitudinal waves. Proc. 11th ITTC (1966).
9) T. F. Ogilvie, E. O. Tuck, A rational strip theory of ship motions. Part 1. Dep. Nav. Arch & Marine Eng. Univ. Michigan 013 (1969).
10) G. Vossers, Some applications of the slender body theory in ship hydrodynamics. Wageningen (1962).
11) W. P. A. Joosen, Slender body theory for an oscillating ship at forward speed. 5th Symp. Naval Hydro. Bergen (1964).
12) H. Maruo, An improvement of the slender body theory for oscillating ships with zero forward speed. Bull. Faculty Eng. Yokohama Nat. Univ. 19 (1970).