On boundary damping for elastic structures

Akkaya, Tugce

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ON BOUNDARY DAMPING
FOR ELASTIC STRUCTURES
On boundary damping for elastic structures

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
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voorzitter van het College voor Promoties,
in het openbaar te verdedigen
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doctor

Tugce AKKAYA

Master of Science in Applied Mathematics,
Celal Bayar University, Turkije,

geboren te Izmir, Turkije.
This dissertation has been approved by the
Promotor: Prof. dr. ir. A.W. Heemink
Copromotor: Dr. ir. W.T. van Horssen

Composition of the doctoral committee:

Rector Magnificus, Chairman
Prof. dr. ir. A.W. Heemink, Promotor, Delft University of Technology
Dr. ir. W.T. van Horssen, Copromotor, Delft University of Technology

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All rights reserved. No part of this publication may be reproduced in any form or by any means of electronic, mechanical, including photocopying, recording or otherwise, without the prior written permission from the author.
To my mother Mediha,
my father Ömer,
and my sister Gökçe
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Chapter 1

Introduction

1.1 Background

Mechanical vibrations stem from the oscillating response of elastic bodies to an internal or external force. Some mechanical vibrations are useful in life, for example, the “silent ring” mode for mobile phones, massage appliances, electric shavers, the motion of a tuning fork such as in musical instruments, watches, and medical uses. However, some vibrations can cause undesirable vibrations in mechanical systems, for instance, automobile vibrations leading to passenger discomfort, building vibrations during earthquakes, bridge vibrations due to strong winds.

All mechanical systems with mass and stiffness are subject to vibrations. Vibrations are induced when the mass is displaced from its equilibrium (resting) position due to an internal or external force. Following that, the mass accelerates and starts going back to the equilibrium position due to the restoring force. If there is no nonconservative force, such as friction, the system continues to oscillate around its equilibrium position. During the oscillation, there is a continual energy transformation from the kinetic energy to the potential energy, and vice versa. In real life, all mechanical systems have nonconservative forces, for example, damping, which cause the energy to dissipate in the system. Then, the energy transformation continues until all energy is dissipated by damping during vibration.

Mechanical vibrations can be categorised into three types: free vibration, forced vibration, and self-excited vibration. Free vibration occurs when the system is subject to no external force after an initial disturbance, such as initial displacement and initial velocity. A well-known example of free vibration is the motion of a guitar string after it is plucked. Forced vibration arises from having an external force after an initial disturbance, for example, the vibration of a building during an earthquake. Lastly, self-excited vibration is encountered when the system experiences a steady external force after initial disturbance, for instance, an army marching on a bridge. A famous example of self-excited vibration is the destruction of the Tacoma Narrows Bridge due to strong wind on November 7, 1940, in Washington State. For further information on types of vibrations the reader is referred to [53, 32].
CHAPTER 1. INTRODUCTION

In recent decades, research in the field of the vibrations of cables of cable stayed bridges has been one of the interesting research subjects due to external forces (wind or rain) induced oscillations of cables among both applied mathematicians and engineers. The main goal of these scientists is to understand and to suppress the undesired vibrations.

Inclined stay cables of bridges are usually attached to a pylon tower at one end and to the bridge deck at the other end (see, for example, Figure 1.1). As has been observed from engineering wind-tunnel experiments [31, 6], raindrops hitting the inclined stay cable cause the generation of one or more very small stream of water (rivulets) on the surface of the cable. The system mass changes when the rivulet is blown off. In [10, 57, 58], it has been shown that even a marginal change in mass can lead to instabilities in one-degree-of-freedom systems. The presence of rivulets on the cable changes the mass of the bridge system that can lead to instabilities, which are not fully understood.

Systems with a time-varying mass are found in physics, in engineering, and in fluid-structure interaction problems [33]. Oscillations of electric transmission lines and cables of cable-stayed bridges with water rivulets on the cable surface can also be considered to be time-varying dynamic systems [10]. When the rivulets are subjected to various mechanical or structural factors, they display interesting dynamical phenomena such as wave propagation, wave steeping, and the development of chaotic responses [31].

Due to low structural damping of a bridge, a wind-field containing raindrops may excite a galloping type of vibration. For instance, the Erasmus bridge in Rotterdam, started to swing under mild wind conditions shortly after it was opened to the traffic.
in 1996. To suppress the undesired oscillations of the bridge, dampers were installed as can be seen in Figure 1.2. Understanding the undesired oscillations of the bridge is important to prevent serious failures of the structures. In order to restrain the undesired vibrations of the mechanical structures different kinds of dampers such as tuned mass dampers and oil dampers can be used at the boundary.

![Figure 1.2: Used new dampers to the Erasmus bridge to prevent vibrations.](Photo:courtesy of TU Delft.)

The vibrations of the bridge cables with dampers can be described mathematically by string-like or beam-like problems. For string-like problems, Caswita [13] worked on the dynamics of inclined stretched strings which are attached to a fixed support at one end and a vibrating support at the other end. In order to stabilise the problem, boundary damping should be taken into account. To our knowledge, there is no literature on the use of boundary damping for a rain-wind induced oscillation of inclined cables. In order to understand how effective boundary damping is for rain-wind induced vibrations of inclined cables, string-like and beam-like problems with boundary damping should be first studied for a simple model. The effects of boundary damping on wind induced inclined string-like problems with time-varying mass due to rain not much is known up to now.

The main goal of this thesis is to model the vibrations of the cable in a simple, but still realistic way, that is, in a setting with infinitely many degrees of freedom, and to solve (with analytic and semi-analytic approaches) the initial-boundary value problems for the partial differential equations which will follow from the modelling procedure for the rain-wind induced vibrations of the cable. As mathematical tools to solve the problems considered in this thesis, the D’Alembert method, the Laplace Transform method and perturbation methods are used.
1.2 Outline of the thesis

The chapters of this thesis are based on a collection of several modified journal and conference papers, which are either published or are currently under review.

The vibrations of the bridge cables can be described mathematically by string vibrations, which are modeled by a second-order partial differential equation known as a wave equation. As bending stiffness is considered, the description of the vibration of the cables is represented by a fourth-order partial differential equation known as the Euler-Bernoulli beam equation. The equations of motion for string-like or beam-like problems can be derived by using Hamilton’s principle \[45\]. The chapters in this thesis are structured as follows.

In Chapter 2, the vibration of a semi-infinite string-like problem is modeled by an initial boundary value problem with (non)-classical boundary conditions. This string-like problem is considered as a simple model to study the reflection and damping properties for the systems with and without mass. The problem is formulated as

\[
\begin{align*}
    u_{tt} - \hat{c}^2 u_{xx} &= 0, \quad 0 < x < \infty, \quad t > 0, \\
    u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < \infty,
\end{align*}
\]

where the wave speed \( \hat{c}^2 = T/\rho \), \( T \) is the tension and \( \rho \) is the mass density of the string. \( u(x,t) \) is the vertical transversal displacement of the string, \( x \) is the position along the string, \( t \) is the time, \( f(x) \) represents the initial deflection, and \( g(x) \) the initial velocity. The string is attached to a mass-spring-dashpot system at \( x = 0 \). Therefore, the boundary condition for (1.1) is given by

\[
    mu_{tt}(0, t) = Tu_x(0, t) - ku(0, t) - \alpha u_t(0, t), \quad 0 \leq t < \infty.
\]

We assume that \( T \) (tension), \( m \) (mass), \( k \) (the stiffness of the spring) and \( \alpha \) (the damping coefficient of the dashpot) are all positive constants. The exact solutions of these initial-boundary value problems are obtained by using the D’Alembert formula. We also present the energy decay of the solution of the initial-boundary value problem and the boundedness of these solutions.

Next, in Chapter 3, the bending stiffness is considered and we examine a transversely vibrating homogeneous semi-infinite beam attached to a spring-dashpot system at \( x = 0 \).

\[
\begin{align*}
    u_{tt} + a^2 u_{xxxx} &= \frac{q}{\rho A}, \quad 0 < x < \infty, \quad t > 0, \\
    u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x < \infty,
\end{align*}
\]

where \( a^2 = (EI/\rho A) > 0 \). \( E \) is the Young’s modulus of elasticity, \( I \) is the moment of inertia of the cross-section, \( \rho \) is the density, \( A \) is the area of the cross-section, and \( q \) is an external load. For non-classical boundary damping, the boundary conditions for (1.4) are given by

\[
    EI u_{xx}(0, t) = 0, \quad EI u_{xxx}(0, t) = \alpha u_t(0, t), \quad 0 \leq t < \infty.
\]

We use the method of Laplace transforms to construct the (exact) solution and also derive closed form expressions for the Green’s functions for this problem. In addition,
it is shown how waves are damped and reflected, and how much energy is dissipated at the non-classical boundary.

Finally, in Chapter 4 the longitudinal and transversal in-plane vibrations of an inclined stretched beam with a time-varying mass, and in a uniform wind flow are studied. While one end of the string (at $x = 0$) is fixed, a sliding damper is applied at the other end of the beam (at $x = L$). The equations of motion describing the longitudinal and transversal displacements of the tensioned Euler Bernoulli Beam can be derived by using a variational principle [24]. The coupled system of partial differential equations to describe the in-plane displacements of the beam is reduced to a single partial differential equation by using Kirchhoff’s approach. We obtain

\[
\mu v_{xxxx} + v_{tt} - v_{xx} = \epsilon \left\{ \eta_1 + \sigma \frac{\Omega_1}{v_0} \cos(g_1 x - \Omega_1 t) \right\} v_t \\
- \sigma \sin(g_1 x - \Omega_1 t) v_{tt} + \eta_2 (1 - x) v_{xx} - \eta_2 v_x \\
+ \sigma \sin(g_1 x - \Omega_1 t) \eta_3 \right\}, \quad t > 0, \quad 0 < x < 1,
\]

with the boundary conditions

\[
v(0, t; \epsilon) = v_{xx}(0, t; \epsilon) = v_x(1, t; \epsilon) = 0, \quad (1.8)
\]

\[
\mu v_{xxx}(1, t; \epsilon) = v_x(1, t; \epsilon) + \epsilon [\lambda v_t(1, t; \epsilon) + \eta_2 v_x(1, t; \epsilon)]. \quad (1.9)
\]

The stability of solutions is studied in detail by using the multiple-timescales perturbation method.
Chapter 2

Reflection and Damping Properties for Semi-infinite String Equations with Non-classical Boundary Conditions

Abstract. In order to answer the main question as indicated in the previous chapter, we should start to examine boundary reflection and damping properties of the string-like problem as a simple model. In this chapter, initial-boundary-value problems for a linear wave (string) equation are considered. The main objective is to study boundary reflection and damping properties of waves in semi-infinite strings. This problem is of considerable practical interest in the context of vibration suppression at boundaries of elastic structures. Solutions of wave equations will be constructed for two different classes of boundary conditions. In the first class, a massless system consisting of a spring and damper will be considered at the boundary. In the second class, an additional mass will be added to the system at the boundary. The D’Alembert method will be used to construct explicit solutions of the boundary value problem for the one-dimensional wave equation on the semi-infinite domain. It will also be shown how waves are damped and reflected at these boundaries, and how much energy is dissipated at the boundary.

2.1 Introduction

Many researchers have paid considerable attention to the dynamics of mechanical structures due to rain, wind, earthquake, machines and traffic-induced vibrations. These vibrations in mechanical structures are of great importance because of their

---

Parts of this chapter have been published in [4] the Journal of Sound and Vibration 336 (2015) and as a contribution to the conference proceedings of ENOC 2014.
impact on our life. The motion of mechanical structures, such as for instance the vibrations of bridge cables and power transmission lines, can be described by mathematical models which are wave-like or string-like problems [56, 18]. In order to suppress the undesired oscillations of the mechanical structures, all kinds of dampers can be used at the boundary. Many dampers such as tuned mass dampers and oil dampers have historically been used to reduce the wind-induced vibrations of taut cables to safe levels, and so to prevent fatigue failures of the structures.

In the literature there are some fundamental examples of classical boundary conditions without mass such as fixed and free end conditions [27]. More complicated conditions for which the end point is connected to a spring and/or dashpot can be found in Graff [25] or in Morse and Feshbach [47]. In [25] the problem is solved by using the Fourier transform method. In addition, by using the method of characteristics for one-dimensional wave equations, Morse and Feshbach [47] solved the same problem. In all of these cases no mass was attached to the system at the boundary. Moreover, it seems that a mass-spring-dashpot system attached at the boundary has not been treated analytically before in the literature. The main goal of this chapter is to investigate a linear equation for a semi-infinite string with and without mass attached at the boundary. With our approach it is possible to compute directly for string-like elastic structures the exact damping properties when a boundary damper is added to the structure. Depending on the choices of the parameter values of the boundary damper (i.e. mass, stiffness and damping parameters) the effectiveness can be obtained exactly.

This chapter is organized as follows. In Section 2.2, we establish the governing equations of motion. In Section 2.3, we discuss variations on the relatively simple case without mass, which arise in various physical problems. In other words, the string is attached to a spring-dashpot system at \( x = 0 \) as shown in Figure 2.1(a). For the more complicated conditions, in Section 2.4, we turn our attention to a spring-dashpot system with mass as shown in Figure 2.1(b). For both cases, we present not only the energy decay of the solution of the initial-boundary value problem and the boundedness of these solutions, but also the reflection and damping properties of the system. Finally, in Section 2.5, we draw some conclusions.

### 2.2 The governing equations of motion

We will consider the perfectly flexible string on a semi-infinite interval. \( u(x,t) \) is the vertical transversal displacement of the string, where \( x \) is the position along the string, and \( t \) is the time. Let us assume that gravity and other external forces can be neglected. The equation of motion is for instance derived in reference [45], by using Hamilton’s principle:

\[
\ddot{u} - \hat{c}^2 u'' = 0, \quad 0 < x < \infty, \quad t > 0, \\
u(x,0) = f(x), \quad \dot{u}(x,0) = g(x), \quad x > 0,
\]

(2.1)

where the wave speed \( \hat{c}^2 = T/\rho \), \( T \) is the tension and \( \rho \) is the mass density of the string. Here, \( f(x) \) and \( g(x) \) represent the initial displacement and initial velocity of the string, respectively. Note that the overdot (\( \cdot \)) denotes the derivative with respect to time and the prime (\('\)) denotes the derivative with respect to the spatial variable.
2.2. THE GOVERNING EQUATIONS OF MOTION

The governing equations of motion are as follows:

(a) The spring-dashpot system without mass

\[ Tu'(0,t) = ku(0,t) + \alpha \dot{u}(0,t), \quad \text{if } m = 0, \quad (2.2) \]

and for the mass-spring-dashpot system, we will have

\[ m\ddot{u}(0,t) = Tu'(0,t) - ku(0,t) - \alpha \dot{u}(0,t), \quad \text{if } m \neq 0. \quad (2.3) \]

The wave travels between \( x = 0 \) and \( x = \infty \) as shown in Figure 2.1(a) and Figure 2.1(b). It is assumed that \( m \) (mass), \( k \) (the stiffness of the spring) and \( \alpha \) (the damping coefficient of the dashpot) are all positive constants. In order to put the equation in a non-dimensional form, the following dimensionless quantities are used:

\[ u^*(x^*, t^*) = \frac{u(x,t)}{L_*}, \quad x^* = \frac{x}{L_*}, \quad t^* = \frac{t}{T_*}, \quad f^*(x^*) = \frac{f(x)}{L_*}, \quad g^*(x^*) = \frac{g(x)}{L_*} \]

where \( L_* \) and \( T_* \) are some dimensional characteristic quantities for the length and the time respectively, and by inserting these non-dimensional quantities into Eq.(2.1), we obtain

\[ \ddot{u}(x,t) - \dddot{u}(x,t) = 0, \quad 0 < x < \infty, \quad t > 0, \quad (2.4) \]

with initial conditions

\[ u(x,0) = f(x), \quad \dot{u}(x,0) = g(x), \quad 0 \leq x < \infty, \quad (2.5) \]

and with boundary conditions

\[ u'(0,t) = \lambda u(0,t) + \beta \dot{u}(0,t), \quad t \geq 0 \quad \text{(if } m = 0), \quad (2.6) \]

or

\[ \ddot{u}(0,t) = \eta u'(0,t) + \mu \ddot{u}(0,t) + \psi \dot{u}(0,t), \quad t \geq 0 \quad \text{(if } m \neq 0), \quad (2.7) \]

\[ \text{Figure 2.1: Two different physical models for an viscoelastic string.} \]
where \( c^2 = L^2/T^2 \), \( \lambda = kL/T \), \( \beta = \alpha L/T \), \( \eta = TT^2/mL \), \( \mu = kT^2/m \) and \( \psi = \alpha T/m \). The asterisks indicating the dimensional quantities are omitted in Eq.(2.4) through (2.7) and henceforth for convenience.

In order to examine the reflection of waves, we will consider \( u(x,0) = f(x) \) and \( \dot{u}(x,0) = -f'(x) \) as initial conditions, which implies that we initially only have waves travelling to the left (i.e. travelling to the boundary at \( x = 0 \)).

### 2.3 Reflection at boundaries

#### 2.3.1 The spring-dashpot system \((m = 0)\)

In this section, we will consider the case of a string of semi-infinite length, extending in the positive direction from \( x = 0 \), where there is a support at \( x = 0 \) having transverse stiffness force, and resistance. The initial-boundary value problem for \( u(x,t) \) is given by

\[
\ddot{u}(x,t) - u''(x,t) = 0, \quad 0 < x < \infty, \quad t > 0, \tag{2.8}
\]

\[
u(x,0) = f(x), \quad \dot{u}(x,0) = g(x), \quad 0 \leq x < \infty, \tag{2.9}
\]

\[
u'(0,t) = \lambda u(0,t) + \beta \dot{u}(0,t), \quad 0 \leq t < \infty, \quad \lambda \geq 0, \quad \beta \geq 0, \tag{2.10}
\]

where \( f \in C^2 \), and \( g \in C^1 \). It is well-known that the general solution of the one-dimensional wave equation is given by

\[
u(x,t) = F(x-t) + G(x+t). \tag{2.11}
\]

Here \( F \) and \( G \) functions represent propagating disturbances, and by using the initial conditions, we obtain

\[
F(x) = \frac{1}{2} f(x) - \frac{1}{2} \int_0^x g(s) \, ds - \frac{K}{2}, \tag{2.12}
\]

\[
G(x) = \frac{1}{2} f(x) + \frac{1}{2} \int_0^x g(s) \, ds + \frac{K}{2}, \tag{2.13}
\]

where \( K \) is a constant of integration. Substitution of Eq.(2.12) and (2.13) into the general solution Eq.(2.11) gives the well-known D’Alembert formula for \( u(x,t) \)

\[
u(x,t) = \frac{1}{2} [f(x+t) + f(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds. \tag{2.14}
\]

For \( x-t < 0 \), \( f(x-t) \) is not yet defined in Eq.(2.14), and for \( x-t < s < 0 \), \( g(s) \) is not yet defined in Eq.(2.14). This “freedom” in \( f \) and in \( g \) will be used to satisfy the boundary condition Eq.(2.10). Substituting Eq.(2.14) into Eq.(2.10) yields:

\[
\frac{f'(-t)}{2} - \frac{g(-t)}{2} + \frac{f'(t)}{2} + \frac{g(t)}{2} = \lambda \left[ \frac{f(-t)}{2} + \frac{f(t)}{2} + \frac{1}{2} \int_{-t}^{t} g(s) \, ds \right]
\]

\[
+ \beta \left[ -\frac{f'(-t)}{2} + \frac{g(-t)}{2} + \frac{f'(t)}{2} + \frac{g(t)}{2} \right], \tag{2.15}
\]

\[
\]
where \( f \) and \( g \) can be chosen independently. If \( g \equiv 0 \), \( f \) has to satisfy
\[
\frac{f'(t)}{2} + \frac{f'(t)}{2} = \lambda \left[ \frac{f(t)}{2} + \frac{f(t)}{2} \right] + \beta \left[ -\frac{f'(t)}{2} + \frac{f'(t)}{2} \right],
\]
or equivalently
\[
f'(-t) - \left( \frac{\lambda}{1+\beta} \right) f(-t) = \left( \frac{\beta - 1}{1+\beta} \right) f(t) + \left( \frac{\lambda}{1+\beta} \right) f(t). \tag{2.16}
\]

This can be written as,
\[
f'(-t) - \kappa_0 f(-t) = \gamma_0 f'(t) + \kappa_0 f(t), \tag{2.17}
\]
where \( \kappa_0 = \left( \frac{\lambda}{1+\beta} \right) \) and \( \gamma_0 = \left( \frac{\beta - 1}{1+\beta} \right) \). When we substitute \( y(t) = f(-t) \) and \( y'(t) = -f'(t) \) into Eq.(2.17), we obtain a simple first order ordinary differential equation for \( y(t) = f(-t) \), which readily can be solved, yielding
\[
f(-t) = -\gamma_0 f(t) + f(0)e^{-\kappa_0 t}(\gamma_0 + 1) + \kappa_0(\gamma_0 - 1)e^{-\kappa_0 t} \int_0^t e^{\kappa_0 s} f(s) ds. \tag{2.18}
\]

Similarly, for \( f \equiv 0 \), we obtain
\[
g(-t) = -\gamma_0 g(t) + g(0)e^{-\kappa_0 t}(\gamma_0 + 1) + \kappa_0(\gamma_0 - 1)e^{-\kappa_0 t} \int_0^t e^{\kappa_0 s} g(s) ds. \tag{2.19}
\]
Replacing \( -t \) by \( x - t \) in Eq.(2.18), we obtain for \( x < t \)
\[
f(x - t) = -\gamma_0 f(t - x) + f(0)e^{\kappa_0(x-t)}(\gamma_0 + 1) + \kappa_0(\gamma_0 - 1)e^{\kappa_0(x-t)} \int_0^{t-x} e^{\kappa_0 s} f(s) ds. \tag{2.20}
\]

By substituting Eq.(2.19) and Eq.(2.20) into Eq.(2.14) for \( x - t < 0 \), yields
\[
\begin{align*}
u(x, t) & = \frac{1}{2} \left[ f(x + t) - \gamma_0 f(t - x) \right] + \frac{\gamma_0 + 1}{2} \left\{ f(0)e^{\kappa_0(x-t)} + \frac{g(0)}{\kappa_0} \left[ 1 - e^{\kappa_0(x-t)} \right] \right\} \\
& \quad + \frac{1}{2} \int_0^{x+t} g(s) ds - \frac{1}{2} \int_0^{t-x} g(s) ds + \kappa_0 \frac{\gamma_0 - 1}{2} e^{\kappa_0(x-t)} \int_0^{t-x} e^{\kappa_0 s} f(s) ds \\
& \quad - \frac{\gamma_0 - 1}{2} e^{\kappa_0(x-t)} \int_0^{t-x} e^{\kappa_0 s} g(s) ds. \tag{2.21}
\end{align*}
\]

When we consider \( u(x, 0) = f(x) \) and \( \dot{u}(x, 0) = -f'(x) \) as initial conditions, Eq.(2.14) becomes for \( x - t < 0 \):
\[
u(x, t) = -\gamma_0 f(t - x) + f(0)e^{\kappa_0(x-t)}(\gamma_0 + 1) + \kappa_0(\gamma_0 - 1)e^{\kappa_0(x-t)} \int_0^{t-x} e^{\kappa_0 s} f(s) ds, \tag{2.22}
\]
and for \( x - t > 0 \):
\[
u(x, t) = f(x - t). \tag{2.23}
\]
CHAPTER 2. REFLECTION AND DAMPING PROPERTIES FOR SEMI-INFINITE STRING EQUATIONS
WITH NON-CLASSICAL BOUNDARY CONDITIONS

Figure 2.2: The initial wave and its reflection.

Table 2.1 shows the extension of \( f(t) \) for negative arguments. As can be seen, the solution has an odd or an even extension when we take \( \beta = 0, \lambda \to \infty \) or \( \beta = 0, \lambda \to 0 \), respectively. Similarly, when we consider only damping at the boundary at \( x=0 \) (\( \lambda = 0 \)), the solution has again an odd or an even extension as \( \beta \to \infty \) or \( \beta \to 0 \), respectively. We have ideal damping as \( \beta = 1, \lambda = 0 \). Figure 2.2 also shows how a wave is reflected, that is, for \( f(x) = \sin^2(x) \) and \( g(x) = -f'(x) \) for \( \pi < x < 2\pi \), and \( f(x) = 0 \) elsewhere. Its reflection for \( x < 0 \) is plotted for different values of \( \lambda \) and \( \beta \).

Table 2.1: Extension of \( f(t) \) for negative arguments when \( m = 0 \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \beta )</th>
<th>extension of ( f(t) ) for the boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( &gt; 0 )</td>
<td>( &gt; 0 )</td>
<td>( -\gamma_0 f(0)e^{-\kappa_0 t}(\gamma_0 - 1) + \kappa_0(\gamma_0 - 1)e^{-\kappa_0 t}\int_0^t \lambda e^{\kappa_0 s}f(s)ds ), where ( \kappa_0 = \left( \frac{\lambda}{1+\beta} \right) ) and ( \gamma_0 = \left( \frac{\beta - 1}{1+\beta} \right) ).</td>
</tr>
<tr>
<td>( \geq 0 )</td>
<td>( 0 )</td>
<td>( -f(0) - \lambda \int_0^t e^{\lambda s}[-f'(s) + \lambda f(s)]ds = f(t) - 2\lambda e^{-\lambda t}\int_0^t e^{\lambda s}f(s)ds )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( \geq 0 )</td>
<td>( 1-\beta \frac{t}{1+\beta} f(t) + \frac{2\beta}{1+\beta} f(0) ).</td>
</tr>
</tbody>
</table>
2.3.2 The mass-spring-dashpot system \((m \neq 0)\)

In this section, a semi-infinite string with a mass added in the spring-dashpot system at \(x = 0\) is considered. We will determine the solution of

\[
\begin{align*}
\dddot{u}(x, t) - u''(x, t) &= 0, \quad 0 < x < \infty, \quad t > 0, \\
\dot{u}(x, 0) &= f(x), \quad \ddot{u}(x, 0) = g(x), \quad 0 \leq x < \infty, \\
\dddot{u}(0, t) &= \eta u'(0, t) - \mu \dot{u}(0, t) - \psi \ddot{u}(0, t), \quad 0 \leq t < \infty,
\end{align*}
\]

where \(\eta > 0, \mu \geq 0, \psi \geq 0\). We already know the general solution of the wave equation, and again by applying the boundary condition to Eq.(2.14), we obtain

\[
\begin{align*}
f''(t) + f''(-t) + g'(t) - g'(-t) &= \eta \left[ f'(-t) + f'(t) - g(-t) + g(t) \right] \\
- \mu \left[ f(-t) + f(t) + \int_t^1 g(s) \, ds \right] - \psi \left[ -f'(-t) + f'(t) + g(t) + g(-t) \right],
\end{align*}
\]

where \(f\) and \(g\) can be chosen independently. For \(g \equiv 0\),

\[
f''(-t) - (\eta + \psi)f'(-t) + \mu f(-t) = -f''(t) + (\eta - \psi)f'(t) - \mu f(t).
\]

We can rewrite Eq.(2.28) by putting the unknown function \(f(-t) = y(t)\), \(f'(-t) = -y'(t)\), and \(f''(-t) = y''(t)\). It then follows that

\[
y''(t) + \theta y'(t) + \mu y(t) = -f''(t) + (\eta - \psi)f'(t) - \mu f(t),
\]

where, \(\theta = (\eta + \psi)\). Similarly, \(g(-t)\) can be obtained when \(f \equiv 0\).

The characteristic equation corresponding to Eq.(2.28) is given by

\[
\lambda^2 + \theta \lambda + \mu = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-\theta \pm \sqrt{\Delta}}{2},
\]

where \(\Delta = \theta^2 - 4\mu\).

Solving the characteristic equation will give two roots, \(\lambda_1\) and \(\lambda_2\). The behaviour of the system depends on the relative values of the two fundamental parameters, \(\mu\) and \(\theta\). In particular, the qualitative behaviour of the system depends crucially on whether the quadratic equation for \(\lambda\) has two real solutions, one real solution, or two complex conjugate solutions.

(i) **The case \(\Delta > 0\) or equivalently \(4\mu < \theta^2\)**. There are two different real roots. The homogeneous solution is given by

\[
y_h(t) = b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t},
\]

where \(b_1\) and \(b_2\) are constants. When we apply the method of variations of parameters [9], we obtain the general solution of Eq.(2.28)

\[
y(t) = b_1 y_1(t) + b_2 y_2(t) + Y(t),
\]

where \(y_1(t) = e^{\lambda_1 t}, \ y_2(t) = e^{\lambda_2 t}\), and \(Y(t)\) is given by

\[
Y(t) = -y_1(t) \int_0^t \frac{y_2(s) g(s)}{W(y_1, y_2)} \, ds + y_2(t) \int_0^t \frac{y_1(s) g(s)}{W(y_1, y_2)} \, ds,
\]

where \(W(y_1, y_2) = \int_0^t y_1(s) y_2(s) \, ds\).
in which the Wronskian \( W(y_1, y_2) \) is given by

\[
W(y_1, y_2)(t) = \begin{vmatrix}
  e^{\lambda_1 t} & e^{\lambda_2 t} \\
  \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t}
\end{vmatrix} = (\lambda_2 - \lambda_1)e^{(\lambda_1 + \lambda_2)t},
\]

and where \( g(s) = -f''(s) + (\eta - \psi)f'(s) - \mu f(s) \).

Then,

\[
Y(t) = -\frac{e^{\lambda_1 t}}{(\lambda_2 - \lambda_1)} \int_0^t e^{-\lambda_1 s}[-f''(s) + (\eta - \psi)f'(s) - \mu f(s)]ds \\
+ \frac{e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)} \int_0^t e^{-\lambda_2 s}[-f''(s) + (\eta - \psi)f'(s) - \mu f(s)]ds.
\]

By substituting Eq.(2.35) into the general solution Eq.(2.32) and by using integration by parts, we obtain

\[
y(t) = b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t} - f(t) + \frac{f(0)}{\lambda_2 - \lambda_1} [e^{\lambda_2 t}(\lambda_2 - \eta + \psi) - e^{\lambda_1 t}(\lambda_1 - \eta + \psi)] \\
+ \frac{f'(0)}{\lambda_2 - \lambda_1} (e^{\lambda_2 t} - e^{\lambda_1 t}) + \frac{e^{\lambda_1 t}}{\lambda_1 - \lambda_2} (\lambda_1^2 - \lambda_1(\eta - \psi) + \mu) \int_0^t e^{-\lambda_1 s} f(s)ds \\
- \frac{e^{\lambda_2 t}}{\lambda_2 - \lambda_2} (\lambda_2^2 - \lambda_2(\eta - \psi) + \mu) \int_0^t e^{-\lambda_2 s} f(s)ds.
\]

In order to determine the constants \( b_1 \) and \( b_2 \), we take \( t = 0 \) and then it follows from \( y(0) = f(0) \) and \( y'(0) = -f'(0) \) that

\[
b_1 = \frac{-f'(0) - \lambda_2 f(0)}{\lambda_1 - \lambda_2} \quad \text{and} \quad b_2 = \frac{-f'(0) - \lambda_1 f(0)}{\lambda_2 - \lambda_1}.
\]

Hence,

\[
f(-t) = -f(t) + f(0) \left[ (e^{\lambda_1 t} + e^{\lambda_2 t}) + \frac{(\eta - \psi)}{\lambda_2 - \lambda_1} (e^{\lambda_1 t} - e^{\lambda_2 t}) \right] \\
+ \frac{e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \left[ (\lambda_1^2 - \lambda_1(\eta - \psi) + \mu) \int_0^t e^{-\lambda_1 s} f(s)ds \right] \\
- \frac{e^{\lambda_2 t}}{\lambda_2 - \lambda_1} \left[ (\lambda_2^2 - \lambda_2(\eta - \psi) + \mu) \int_0^t e^{-\lambda_2 s} f(s)ds \right].
\]

(ii) **The case \( \Delta = 0 \) or equivalently \( 4\mu = \theta^2 \).** There is a double root \( \lambda \), which is real. In this case, with only one root \( \lambda \), the homogeneous solution of Eq.(2.29) is given by

\[
y_h(t) = (b_3 + b_4 t)e^{-\theta t/2},
\]

where \( b_3 \) and \( b_4 \) are constants. Completely similar to the previous case, \( y(t) = f(-t) \) can be computed, yielding

\[
f(-t) = -f(t) + f(0)e^{-\theta t/2}[2 + t(\psi - \eta)] + \left[ \frac{\theta(3\eta - \psi)}{4} + \mu \right] e^{-\theta t/2} \int_0^t s e^{\theta s/2} f(s)ds \\
+ \left[ 2\eta - t \left( \frac{\theta(3\eta - \psi)}{4} + \mu \right) \right] e^{-\theta t/2} \int_0^t e^{\theta s/2} f(s)ds.
\]
(iii) The case $\Delta < 0$ or equivalently $4\mu > \theta^2$. In this case $\lambda$ is complex valued, the homogeneous solution is given by

$$y_h(t) = e^{-\theta t/2} \left[ b_5 \cos \left( \frac{\sqrt{-\Delta t}}{2} \right) + b_6 \sin \left( \frac{\sqrt{-\Delta t}}{2} \right) \right], \quad (2.41)$$

where $b_5$ and $b_6$ are constants. The same method which is used for the case (i) to obtain the extension of $f(t)$ for negative arguments can also be applied for this case. Table 2.2 gives the extension of $f(t)$.

In order for these solutions to exist, $u$ and $f$ must be twice continuously differentiable and $g$ must be continuously differentiable, in addition

$$f''(0) = \eta f'(0) - \mu f(0) - \psi g(0), \quad \eta > 0, \quad \mu > 0, \quad \psi > 0. \quad (2.42)$$

We usually take $f$ and $g$ as being independent functions, then

$$\begin{cases} 
g(0) = 0, \text{and} \\
f''(0) = \eta f'(0) - \mu f(0). \end{cases} \quad (2.43)$$

Figure 2.3 demonstrates some reflected waves due to the mass-spring-dashpot system at $x = 0$ for the initial values with $f(x) = \sin^2(x)$ and $g(x) = -f'(x)$ for $\pi < x < 2\pi$, and $f(x) = 0$ elsewhere. For increasing values of the variables $\psi$ and $\mu$, the reflected waves become “odd” extensions of the original wave (see Figure 2.3 and Figure 2.4). However, it can be seen that the height of the reflected wave depends on the damping coefficient $\psi$, that is, the reflected wave for $\psi = 0$ is higher than that for $\psi \neq 0$.

For $\eta$ and $\psi$ fixed, and varying stiffness coefficient $\mu$, Figure 2.5 depicts some reflected waves in cases (i), (ii) and (iii). The reflected wave for $\eta = 1$, $\psi = 1$, $\mu = 1$ (case (ii)) is more or less in between the reflected waves for $\mu = 1/2$ and $\mu = 7/5$ indicating the case (i) and case (iii), respectively. Moreover, it is also shown in Figure 2.5(a) and Figure 2.5(b) how the damping coefficient $\psi$ influences the height of the reflected wave.

In addition to these, Figure 2.6 illustrates some reflected waves when the stiffness coefficient $\mu$ in the boundary condition at $x = 0$ is equal to zero. For increasing values of the variables $\eta$, the reflected waves become “even” extensions of the original wave as shown in Figure 2.6(a). Nevertheless, Figure 2.6(b) shows that the reflected waves become “odd” extensions of the incident wave due to increasing the damping coefficient $\psi$. Finally, Figure 2.6(c) displays that the width of the reflected wave depends on the tension coefficient $\eta$, that is, the width of the reflected wave becomes wider for decreasing value of the variable $\eta$. 

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Figure 2.3: Some reflected waves due to the mass-spring-dashpot system at $x = 0$. 

(a) $(\eta + \psi)^2 > 4\mu$

(b) $(\eta + \psi)^2 = 4\mu$

(c) $(\eta + \psi)^2 < 4\mu$
2.3. REFLECTION AT BOUNDARIES

\[ \eta^2 > 4\mu \]

\[ \eta^2 = 4\mu \]

\[ \eta^2 < 4\mu \]

Figure 2.4: Some reflected waves due to the mass-spring system at \( x = 0 \) with \( \psi = 0 \).
CHAPTER 2. REFLECTION AND DAMPING PROPERTIES FOR SEMI-INFINITE STRING EQUATIONS
WITH NON-CLASSICAL BOUNDARY CONDITIONS

2.4 The energy and its rate of change

2.4.1 The energy and boundedness of solutions in the case $m = 0$

The total energy $E(t)$ is the sum of the kinetic and the potential energies of the string and the potential energy of the spring, that is

$$E(t) = \frac{1}{2} \int_0^\infty (\dot{u}^2 + u'^2)dx + \frac{\lambda}{2} u^2(0,t).$$ (2.44)

Taking the time derivative of $E(t)$, we find

$$\dot{E}(t) = \int_0^\infty [\dot{u} \ddot{u} + u' \dot{u}']dx + \lambda u(0,t) \dot{u}(0,t),$$

by using integration by parts and by observing that there is no energy at $x = \infty$, we deduce that

$$\dot{E}(t) = -\beta \ddot{u}^2(0,t).$$ (2.45)

This implies that the energy $E(t)$ decreases in time. Then,

$$E(t) = E(0) - \beta \int_0^t \dot{u}^2(0,t)dt \leq E(0), \text{ for all } t \geq 0.$$ (2.46)

By using the Cauchy-Schwarz inequality, it then follows that

$$|u(x,t)| = \left| \int_0^x u_s(s,t)ds \right| \leq \sqrt{\int_0^\infty u_s^2(s,t)ds} \leq \sqrt{2E(t)} \leq \sqrt{2E(0)}.$$ (2.47)
2.4. THE ENERGY AND ITS RATE OF CHANGE

Figure 2.6: Some reflected waves when the stiffness coefficient $\mu$ in the boundary condition at $x = 0$ is equal to zero.
Table 2.2: Extension of $f(t)$ for negative arguments when $m \neq 0$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\mu$</th>
<th>$\psi$</th>
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<tbody>
<tr>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
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</table>

(i) $(\eta + \psi)^2 > 4\mu$: \[ f(t) + f(0) \left[ (e^{\lambda_1 t} + e^{\lambda_2 t}) + \frac{(\eta - \psi)}{2} (e^{\lambda_1 t} - e^{\lambda_2 t}) \right] + \frac{e^{\lambda_1 t}}{4} \left[ (\lambda_1 - \lambda_2) \eta + \mu \right] \int_0^t e^{-\lambda_1 s} f(s) ds \]

where $\lambda_1 = -\frac{\theta + \sqrt{\Delta}}{2}, \lambda_2 = -\frac{\theta - \sqrt{\Delta}}{2}, \Delta = \theta^2 - 4\mu$ and $\theta = \eta + \psi$.

(ii) $(\eta + \psi)^2 = 4\mu$: \[ -f(t) + f(0) e^{-\eta t/2} [2 + t(\psi - \eta)] + \left[ \frac{\theta(3\eta - \psi)}{4} + \mu \right] e^{-\eta t/2} \int_0^t e^{\eta s/2} f(s) ds \]

(iii) $(\eta + \psi)^2 < 4\mu$: \[ -f(t) + f(0) e^{-\eta t/2} \left[ 2 \eta + \frac{\theta^2}{4} \sin \left( \frac{\sqrt{-\Delta} t}{2} \right) + 2 \cos \left( \frac{\sqrt{-\Delta} t}{2} \right) \right] \]

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(i) $\eta^2 > 4\mu$: \[ f(t) + f(0) \left[ (e^{\lambda_1 t} + e^{\lambda_2 t}) + \frac{(\eta - \psi)}{2} (e^{\lambda_1 t} - e^{\lambda_2 t}) \right] + \frac{e^{\lambda_1 t}}{4} \left[ (\lambda_1^2 - \lambda_2^2) \eta + \mu \right] \int_0^t e^{-\lambda_1 s} f(s) ds \]

where $\lambda_1 = -\eta + \sqrt{\eta^2 - 4\mu}$ and $\lambda_2 = -\eta - \sqrt{\eta^2 - 4\mu}$.

(ii) $\eta^2 = 4\mu$: \[ -f(t) + f(0) e^{-\eta t/2} [2 - \eta t] + \left( \frac{\theta^2}{4} + \mu \right) e^{-\eta t/2} \int_0^t e^{\eta s/2} f(s) ds \]

\[ + \left[ 2 \eta - t \left( \frac{\eta^2}{4} + \mu \right) \right] e^{-\eta t/2} \int_0^t e^{\eta s/2} f(s) ds \]

(iii) $\eta^2 < 4\mu$: \[ -f(t) + f(0) e^{-\eta t/2} \left[ - \frac{2 \eta^2}{4} \sin \left( \frac{\sqrt{-\Delta} t}{2} \right) + 2 \cos \left( \frac{\sqrt{-\Delta} t}{2} \right) \right] \]

\[ + \left[ \frac{2 \eta^2}{\sqrt{-\Delta}} \cos \left( \frac{\sqrt{-\Delta} t}{2} \right) + 2 \eta \sin \left( \frac{\sqrt{-\Delta} t}{2} \right) \right] e^{-\eta t/2} \int_0^t e^{\eta s/2} f(s) ds \]

\[ + 2 \eta \cos \left( \frac{\sqrt{-\Delta} t}{2} \right) - \frac{2 \eta^2}{\sqrt{-\Delta}} \sin \left( \frac{\sqrt{-\Delta} t}{2} \right) \int_0^t e^{\eta s/2} f(s) ds - \frac{2 \eta^2}{\sqrt{-\Delta}} \sin \left( \frac{\sqrt{-\Delta} t}{2} \right) \int_0^t e^{\eta s/2} f(s) ds \]

where $\Delta^* = \eta^2 - 4\mu$. 

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</table>

\[ -f(t) + 2 f(0) \left[ 1 + \frac{\theta}{2} (e^{-\eta t} - 1) \right] + 2 \eta e^{-\eta t} \int_0^t e^{\eta s/2} f(s) ds \]

\[ -f(t) + 2 f(0) e^{-\eta t} + 2 \eta e^{-\eta t} \int_0^t e^{\eta s/2} f(s) ds \]
2.4. **THE ENERGY AND ITS RATE OF CHANGE**

2.4.1 **The energy and its rate of change**

The energy of the system is given by

\[
E(t) = \frac{1}{2} \int_{0}^{\infty} (\dot{u}^2 + u'^2) dx + \frac{1}{2\eta} \dot{u}^2(0,t) + \frac{\mu}{2\eta} u^2(0,t).
\]  

(2.48)

Taking the time derivative of \(E(t)\), it follows that

\[
\dot{E}(t) = -\frac{\psi}{\eta} \dot{u}^2(0,t).
\]

(2.49)

This implies that the energy decreases due to the mass-spring-dashpot system at \(x = 0\). Then,

\[
E(t) = E(0) - \frac{\psi}{\eta} \int_{0}^{t} \dot{u}^2(0,t) dt.
\]

(2.50)

Thus, \(u(x,t)\) is bounded if the initial energy is bounded [19]. In Figure 2.7(a), it is illustrated that the energy of the system is preserved as \(\beta = 0\) for the initial values \(u(x,0) = f(x)\), \(u_t(x,0) = -f'(x)\) with \(f(x) = \sin^2(x)\) for \(\pi < x < 2\pi\), and \(f(x) = 0\) elsewhere. However, as the damping parameter \(\beta\) becomes larger, the dashpot moves hardly and the energy dissipation decreases slowly. Moreover, when we fix the value of \(\beta\) and increase the value of \(\lambda\), it follows from Figure 2.7(b) that the energy dissipation is lower compared to keeping the value of \(\lambda\) fixed and increasing the value of \(\beta\).

2.4.2 **The energy and boundedness of solutions in the case \(m \neq 0\)**

The total energy \(E(t)\) is the sum of the kinetic and the potential energies of the string and the mass-spring system at \(x = 0\), that is,

\[
E(t) = \frac{1}{2} \int_{0}^{\infty} (\dot{u}^2 + u'^2) dx + \frac{1}{2\eta} \dot{u}^2(0,t) + \frac{\mu}{2\eta} u^2(0,t).
\]

(2.48)

Taking the time derivative of \(E(t)\), it follows that

\[
\dot{E}(t) = -\frac{\psi}{\eta} \dot{u}^2(0,t).
\]

(2.49)

This implies that the energy decreases due to the mass-spring-dashpot system at \(x = 0\). Then,

\[
E(t) = E(0) - \frac{\psi}{\eta} \int_{0}^{t} \dot{u}^2(0,t) dt.
\]

(2.50)
CHAPTER 2. REFLECTION AND DAMPING PROPERTIES FOR SEMI-INFINITE STRING EQUATIONS
WITH NON-CLASSICAL BOUNDARY CONDITIONS

(a) \( \eta = \psi = 1 \), and varying \( \mu \) in \( E(t) \)

(b) \( \eta = 1 \), and varying \( \psi, \mu \) in the case (ii)

(c) \( \psi = 0 \) or \( \psi = 1 \), and varying \( \eta \) with \( \mu = 0 \)

Figure 2.8: The energy as a function of time \( t \) due to (a-b) the mass-spring-damper system and (c) the mass-damper system at \( x = 0 \) for the initial values \( u(x,0) = f(x) \), \( u_t(x,0) = -f'(x) \) with \( f(x) = \sin^2(x) \) for \( \pi < x < 2\pi \), and zero elsewhere.
2.5. CONCLUSIONS

We already know from the previous section that \( u(x,t) \) is bounded if the initial energy is bounded.

\[
E(t) \leq E(0), \text{ for all } t \geq 0.
\]  
(2.51)

When the damping coefficient \( \psi > 0 \), it is obvious from Eq.(2.50) that energy of the system is dissipated. If \( \psi = 0 \), then \( E(t) = E(0) \), which expresses the conservation of energy.

Figure 2.8 shows the energy decay as a function of time \( t \) due to the mass-spring-dashpot system (see Figure 2.8(a) and Figure 2.8(b)) and the mass-dashpot system (see Figure 2.8(c)) at \( x = 0 \) for the initial values \( u(x,0) = f(x), \ u_t(x,0) = -f'(x) \) with \( f(x) = \sin^2(x) \) for \( \pi < x < 2\pi \), and \( f(x) = 0 \) elsewhere. In Figure 2.8(a), it can be seen that at \( t = 9.7 \) the incident wave hits the boundary, and from that time energy is dissipated. It follows that in case (iii) with \( \mu = 7/5 \) more energy is dissipated than in case (i) with \( \mu = 1/2 \) or in case (ii) with \( \mu = 1 \).

For \( \eta \) fixed, and varying \( \psi \) and \( \mu \), Figure 2.8(b) demonstrates energy decay due to the mass-spring-string system in the case (ii). It can be seen that energy of the system is conserved when the damping coefficient \( \psi \) is equal to zero (see Figure 2.8(b) and Figure 2.8(c)). Furthermore, the energy decay becomes larger for increasing values of the coefficients \( \psi \) and \( \mu \). However, energy dissipation starts to become less for larger values of the damping coefficient \( \psi \) (e.g. \( \psi = 2 \)). Similarly, it follows from Figure 2.8(c) that energy dissipation becomes less for parameter values of \( \eta \) larger than 1.

2.5 Conclusions

In this chapter, an initial-boundary value problem for a wave equation on a semi-infinite interval has been studied. We applied the D’Alembert formula to obtain the general solution for a one-dimensional wave equation, and examined the solution for various boundary conditions. This chapter provides an understanding of how waves are damped and reflected by these boundaries, and how much energy is dissipated at the boundary. It was also shown that the solution is bounded by using an energy integral.

The results as given in this chapter can be used in several applications. For instance, in [17, 59, 60] the authors used the reflected waves due to a mass-less spring-dashpot boundary to study the vibrations of violin strings. Also these reflected waves (due to a mass-less spring dashpot) were used in [60] as approximations for the reflected waves due to a mass-spring-dashpot boundary with nonzero mass. In this chapter we presented and computed the exact reflected waves. Part of the results presented in this chapter may be extended to axially moving strings, such as conveyor belts, elevator cables, and so on, as well as to beam equations to compute the reflections of waves by these boundaries, but some modifications are needed.
On Constructing a Green’s Function for a Semi-Infinite Beam with Boundary Damping

Abstract. In the previous chapter, the boundary reflection and damping properties of waves in semi-infinite strings were studied. The vibrations of the bridge cables can be described mathematically by string-like problem, which are modelled by a second-order partial differential equation known as wave equation. However, the mathematical model in Chapter 2 assumes that the bending stiffness is neglected and this may not be the case for real world physical problems. In this chapter, the description of the vibration of the cables is represented by a fourth-order partial differential equation known as the Euler-Bernoulli beam equation. The main aim is to contribute to the construction of Green’s functions for initial boundary value problems for the Euler-Bernoulli beam equations. We consider a transversely vibrating homogeneous semi-infinite beam with classical boundary conditions such as pinned, sliding, clamped or with non-classical boundary conditions such as dampers. This problem is of important interest in the context of the foundation of exact solutions for semi-infinite beams with boundary damping. The Green’s functions are explicitly given by using the method of Laplace transforms. The analytical results are validated by references and numerical methods. It is shown how the general solution for a semi-infinite beam equation with boundary damping can be constructed by the Green’s function method, and how damping properties can be obtained.

3.1 Introduction

In engineering, many problems describing mechanical vibrations in elastic structures, such as for instance the vibrations of power transmission lines [56] and bridge cables...
can be mathematically represented by initial-boundary-value problems for a wave or a beam equation. Understanding the transverse vibrations of beams is important to prevent serious failures of the structures. In order to suppress the undesired vibrations of the mechanical structures different kinds of dampers such as tuned mass dampers and oil dampers can be used at the boundary. Analysis of the transversally vibrating beam problems with boundary damping is still of great interest today, and has been examined for a long time by many researchers [52, 29, 62]. In order to obtain a general insight into the over-all behavior of a solution, having a closed form expression which represents a solution, can be very convenient. The Green’s function technique is one of the few approaches to obtain integral representations for the solution [27].

In many papers and books, the vibrations of elastic beams have been studied by using the Green’s function technique. A good overview can be found in e.g. [25, 26] and [55, 23, 27] for initial-value problems and for initial-boundary value problems, respectively. The initial-boundary value problem for a semi-infinite clamped bar has already been solved to obtain its Green’s function by using the method of Laplace transforms [50]. To our best knowledge, we have not found any literature on the explicit construction of a Green’s function for semi-infinite beam with boundary damping.

The outline of the present chapter is as follows. In Section 3.2, we establish the governing equations of motion. The aim of this chapter is to give explicit formula for the Green’s function for the following semi-infinite pinned, slided, clamped and damped vibrating beams as listed in Table 3.1. In Section 3.3, we use the method of Laplace transforms to construct the (exact) solution and also derive closed form expressions for the Green’s functions for these problems. In Section 3.4, three classical boundary conditions are considered and the Green’s functions for semi-infinite beams are represented by definite integrals. For pinned and sliding vibrating beams, it is shown how the exact solution can be written with respect to even and odd extensions of the Green’s function. In Section 3.5, we consider transversally vibrating elastic beams with non-classical boundary conditions such as dampers. The analytical results for semi-infinite beams in this case are compared with numerical results on a bounded domain [0, L] with L large. The damping properties are given by the roots of denominator part in the Laplace approach, or equivalently by the characteristic equation. Numerical and asymptotic approximations of the roots of a characteristic equation for the beam-like problem on a finite domain will be calculated. It will be shown how boundary damping can be effectively used to suppress the amplitudes of oscillation. In Section 3.6, the concept of local energy storage is described. Finally some conclusions will be drawn in Section 3.7.

### 3.2 The governing equations of motion

We will consider the transverse vibrations of a one-dimensional elastic Euler-Bernoulli beam which is infinitely long in one direction. The equations of motion can be derived by using Hamilton’s principle [45]. The function \(u(x, t)\) is the vertical deflection of the beam, where \(x\) is the position along the beam, and \(t\) is the time. Let us assume that gravity can be neglected. The equation describing the vertical displacement of
Table 3.1: Boundary conditions (BCs) for beams which are infinitely long in one direction

<table>
<thead>
<tr>
<th>Type of system</th>
<th>Left end condition</th>
<th>BCs at ( x = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pinned</td>
<td></td>
<td>( u = 0, \ E I u'' = 0 ).</td>
</tr>
<tr>
<td>Sliding</td>
<td></td>
<td>( u' = 0, \ E I u''' = 0 ).</td>
</tr>
<tr>
<td>Clamped</td>
<td></td>
<td>( u = 0, \ u' = 0 ).</td>
</tr>
<tr>
<td>Non-classical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damper</td>
<td></td>
<td>( E I u'' = 0, \ E I u''' = -\alpha \dot{u} ).</td>
</tr>
</tbody>
</table>

The beam is given by

\[
\ddot{u} + a^2 \ u''' = \frac{q}{\rho A}, \quad 0 < x < \infty, \quad t > 0, \tag{3.1}
\]

\[
\dot{u}(x,0) = f(x), \quad \dot{u}(x,0) = g(x), \quad 0 \leq x < \infty, \tag{3.2}
\]

where \( a^2 = (EI/\rho A) > 0 \). \( E \) is Young’s modulus of elasticity, \( I \) is the moment of inertia of the cross-section, \( \rho \) is the density, \( A \) is the area of the cross-section, and \( q \) is an external load. Here, \( f(x) \) represents the initial deflection and \( g(x) \) the initial velocity. Note that the overdot (\( \cdot \)) denotes the derivative with respect to time and the prime (\( ' \)) denotes the derivative with respect to the spatial variable \( x \).

In the book of Guenther and Lee [26], and Graff [25], the solution of the Euler-Bernoulli beam equation (3.1) with \( q = 0 \) on an infinite domain is obtained by using Fourier transforms, and is given by

\[
u(x,t) = \int_{-\infty}^{\infty} [K(\xi - x, t)f(\xi) + L(\xi - x, t)g(\xi)] \, d\xi, \tag{3.3}\]

where

\[
K(x,t) = \frac{1}{\sqrt{4\pi a t}} \sin \left( \frac{x^2}{4a t} + \frac{\pi}{4} \right), \tag{3.4}\]

and

\[
L(x,t) = \frac{x}{2a} \left[ S \left( \frac{x^2}{4a t} \right) - C \left( \frac{x^2}{4a t} \right) \right] + 2tK(x,t). \tag{3.5}\]
Here the functions $C(z)$ and $S(z)$ are the Fresnel integrals defined by

$$ C(z) = \int_0^z \frac{\cos(s)}{\sqrt{s}} \, ds, \quad \text{and} \quad S(z) = \int_0^z \frac{\sin(s)}{\sqrt{s}} \, ds. \quad (3.6) $$

In order to put the Eq. (3.1) and Eq. (3.2) in a non-dimensional form the following dimensionless quantities are used:

$$ u^*(x^*, t^*) = \frac{u(x, t)}{L_*}, \quad x^* = \frac{x}{L_*}, \quad t^* = \frac{\kappa t}{L_*}, \quad \kappa = \frac{1}{L_*} \sqrt{\frac{EI}{\rho A}}, $$

$$ f^*(x^*) = \frac{f(x)}{L_*}, \quad g^*(x^*) = \frac{g(x)}{\kappa}, \quad q^*(x^*, t^*) = \frac{q(x, t) \rho A \kappa^2}{L_*}, $$

where $L_*$ is the dimensional characteristic quantity for the length, and by inserting these non-dimensional quantities into Eq. (3.1)-(3.2), we obtain the following initial-boundary value problem:

$$ \dddot{u}(x, t) + u^{'''}(x, t) = q(x, t), \quad 0 < x < \infty, \quad t > 0, \quad (3.7) $$

$$ u(x, 0) = f(x), \quad \dot{u}(x, 0) = g(x), \quad 0 \leq x < \infty, \quad (3.8) $$

and the boundary conditions at $x = 0$ are given in Table 3.1. The asterisks indicating the dimensional quantities are omitted in Eq. (3.7) and Eq. (3.8), and henceforth for convenience.

In the coming sections, we will show how the Green’s functions for semi-infinite beams with boundary conditions given at $x = 0$, can be obtained in explicit form.

### 3.3 The Laplace transform method

In this section, Green’s functions will be constructed by using the Laplace transform method in order to obtain an exact solution for the initial-boundary value problem Eq. (3.7)-(3.8). Let us assume that the external force $q(x, t) = \delta(x - \xi) \otimes \delta(t)$ at the point $x = \xi$ at time $t = 0$, $\delta$ being Dirac’s function, and $f(x) = g(x) = 0$. The Green’s function $G_\xi(x, t), \xi > 0$, expresses the displacements along the semi-infinite beam.

We start by defining the Laplace operator as an integration with respect to the time variable $t$. The Laplace transform $g_\xi$ of $G_\xi$ with respect to $t$ is defined as

$$ g_\xi(x, p) = \mathcal{L}\{G_\xi(x, t)\} = \int_0^\infty e^{-pt} G_\xi(x, t) \, dt, \quad (3.9) $$

where $g_\xi$ is the Green’s function of the differential operator $L = (d^4/dx^4) + p^2$ on the interval $(0, \infty)$. The Green’s function $g_\xi$ satisfies the following properties [35]:

[G1] The Green’s function $g_\xi$ satisfies the fourth order ordinary differential equation in each of the two subintervals $0 < x < \xi$ and $\xi < x < \infty$, that is, $L g_\xi = 0$ except when $x = \xi$.

[G2] The Green’s function $g_\xi$ satisfies at $x = 0$ one of the homogeneous boundary conditions, as given in Table 3.1.
The Green’s function $g_\xi$ and its first and second order derivatives exist and are continuous at $x = \xi$.

The third order derivative of the Green’s function $g_\xi$ with respect to $x$ has a jump discontinuity which is defined as

$$\lim_{\epsilon \to 0} [g''_\xi(\xi + \epsilon) - g''_\xi(\xi - \epsilon)] = 1.$$  \hspace{1cm} (3.10)

The transverse displacement $u(x, t)$ of the beam can be represented in terms of the Green’s function as (see also [51]):

$$u(x, t) = -\int_0^\infty f(\xi) \dot{G}_\xi(x, t) \, d\xi + \int_0^\infty g(\xi) G_\xi(x, t) \, d\xi + \int_0^t \int_0^\infty q(\xi, \tau) G_\xi(x, t-\tau) \, d\xi \, d\tau.$$ \hspace{1cm} (3.11)

In the coming sections, we solve exactly the initial-boundary value problem for a beam on a semi-infinite interval for different types of boundary conditions.

Figure 3.1: The Green’s function $g(v, s)$ for a pinned end semi-infinite beam with the initial values $g(v, 0) = 0$, $g_s(v, 0) = 0$, and the external force $q(v, s) = \delta(v - 1) \otimes \delta(s)$. 

(a) initial phase of the wave

(b) fading-out wave
3.4 Classical boundary conditions

3.4.1 Pinned end: \( u(0, t) = u''(0, t) = 0 \)

In this section, we consider a semi-infinite beam equation, when the displacement and the bending moment are specified at \( x = 0 \), i.e. \( u(0, t) = u''(0, t) = 0 \), and when the beam has an infinite extension in the positive x-direction. By using the requirements [G1]-[G4], \( g_\xi \) is uniquely determined, and we obtain

\[
g_\xi = \frac{1}{8\beta^3} \left\{ e^{-\beta|x-\xi|} \left[ \cos\beta(x - \xi) + \sin\beta |x - \xi| \right] \right. \\
+ e^{-\beta(x+\xi)} \left[ -\cos\beta(x + \xi) - \sin\beta(x + \xi) \right] \}, \tag{3.12}
\]

where \( \beta^2 = p/2 \). In order to invert the Laplace transform, we use the formula (see [15], page 93)

\[
\mathcal{L}^{-1} \left[ (\sqrt{p^2})^{-1} \phi(\sqrt{p^2}) \right] = \int_0^t \mathcal{L}^{-1} \{ \phi(\tau) \} \, d\tau, \tag{3.13}
\]

and (see [48], page 279)

\[
\mathcal{L}^{-1} \left[ p^{-1/2} e^{-\sqrt{p\tau}} \cos(\sqrt{p\tau}) \right] = \frac{1}{\sqrt{\pi t}} \cos \left( \frac{z}{2t} \right), \tag{3.14}
\]

\[
\mathcal{L}^{-1} \left[ p^{-1/2} e^{-\sqrt{p\tau}} \sin(\sqrt{p\tau}) \right] = \frac{1}{\sqrt{\pi t}} \sin \left( \frac{z}{2t} \right), \tag{3.15}
\]
3.4. CLASSICAL BOUNDARY CONDITIONS

where \( z = \frac{|x + \xi|}{\sqrt{2}} \). The Green’s function yields

\[
G_\xi(x, t) = -\int_0^t [K(\xi - x, \tau) - K(\xi + x, \tau)] \, d\tau,
\]

(3.16)

where the kernel function is defined by

\[
K(x, \tau) = \frac{1}{\sqrt{4\pi\tau}} \sin \left( \frac{x^2}{4\tau} + \frac{\pi}{4} \right).
\]

(3.17)

When we assume for Eq. (3.7) and Eq. (3.8) that the external loading is absent \((q = 0)\), and that the initial displacement \(f(x)\) and the initial velocity \(g(x)\) are nonzero, one can find the solution of the pinned end semi-infinite beam in the form of Eq. (3.3) as

\[
u(x, t) = \int_0^\infty \left[ [K(\xi - x, t) - K(\xi + x, t)]f(\xi) + [L(\xi - x, t) - L(\xi + x, t)]g(\xi) \right] d\xi,
\]

(3.18)

where \( K \) and \( L \) are given by Eq. (3.4) and Eq. (3.5). It should be observed that Eq. (3.18) could have been obtained by using Eq. (3.3) and the boundary conditions \( u = u'' = 0 \) at \( x = 0 \). From which it simply follows that \( f \) and \( g \) should be extended as odd functions in their argument, and then by simplifying the so-obtained integral, one obtains Eq. (3.18).

On the other hand, when we consider that the external loading is nonzero, for example, \( q = \delta(x - \xi) \otimes \delta(t) \), and the initial disturbances are zero \((f = g = 0)\), the solution of pinned end semi-infinite beam can be written in a non-dimensional form. By substituting the following dimensionless quantities in Eq. (3.16)

\[
v = \frac{x}{\xi}, \quad s = \frac{t}{\xi^2}, \quad \sigma = \frac{t}{\tau}, \quad g(v, s) = \frac{G_\xi}{\xi}.
\]

(3.19)

We obtain

\[
g(v, s) = -\sqrt{s \over 4\pi} \int_1^\infty \left[ \sin \left( \frac{(\sigma(v - 1)^2}{4s} + \frac{\pi}{4} \right) - \sin \left( \frac{(\sigma(v + 1)^2}{4s} + \frac{\pi}{4} \right) \right] \frac{d\sigma}{\sigma^{3/2}}.
\]

(3.20)

Figure 3.1 shows the shape of the semi-infinite one-sided pinned beam during its oscillation. It can be observed how the amplitude of the impulse at \( x = \xi \) is increasing and how the deflection curves start to develop rapidly from the boundary at \( x = 0 \) as new time variable \( s \) is increasing, where \( s \) is given by Eq. (3.19).

3.4.2 Sliding end: \( u'(0, t) = u'''(0, t) = 0 \)

In this section, we consider a semi-infinite beam equation for \( x > 0 \), when the bending slope and the shear force are specified at \( x = 0 \), i.e. \( u'(0, t) = u'''(0, t) = 0 \). The same method which is used in Section 3.4.1 to obtain the Green’s function can also be applied for the sliding end semi-infinite beam. The Green’s function is given by

\[
G_\xi(x, t) = \int_0^t [K(\xi - x, \tau) + K(\xi + x, \tau)] \, d\tau,
\]

(3.21)
and the transverse displacement \( u(x, t) \) of the beam without an external loading is given by

\[
u(x, t) = \int_0^\infty \left[ K(\xi - x, t) + K(\xi + x, t) \right] f(\xi) + L(\xi - x, t) + L(\xi + x, t) \right] g(\xi) \, d\xi. \tag{3.22}
\]

Eq. (3.22) also could have been obtained by using Eq. (3.3) and the boundary conditions \( u' = u''' = 0 \) at \( x = 0 \). It follows that \( f \) and \( g \) should be extended as even functions in their argument, and then by simplifying the so-obtained integral, we obtain Eq. (3.22). By using the same dimensionless quantities as in Section 3.4.1, the non-dimensional form of the solution for the sliding end semi-infinite beam is given by:

\[
g(v, s) = -\sqrt{\frac{s}{4\pi}} \int_1^\infty \left[ \sin \left( \frac{\sigma(v - 1)^2}{4s} + \frac{\pi}{4} \right) + \sin \left( \frac{\sigma(v + 1)^2}{4s} + \frac{\pi}{4} \right) \right] \frac{d\sigma}{\sigma^{3/2}}, \tag{3.23}
\]

Similarly, Figure 3.2 demonstrates the shape of the semi-infinite one-sided sliding beam during its oscillation. It can be seen how the amplitude of the impulse at \( x = \xi \) is increasing and how the deflection curve is developing from the boundary at \( x = 0 \) as the new time variable \( s \) is increasing.

### 3.4.3 Clamped end: \( u(0, t) = u'(0, t) = 0 \)

In this section, we consider a semi-infinite beam equation for \( x > 0 \), when the deflection and the slope are specified at \( x = 0 \), i.e. \( u(0, t) = u'(0, t) = 0 \). The non-dimensional form for the Green’s function of the semi-infinite vibrating beam is now given by

\[
g(v, s) = -\sqrt{\frac{s}{4\pi}} \int_1^\infty \left[ \sin \left( \frac{\sigma(v - 1)^2}{4s} + \frac{\pi}{4} \right) - \sin \left( \frac{\sigma(v + 1)^2}{4s} + \frac{\pi}{4} \right) \right] \frac{d\sigma}{\sigma^{3/2}}.
- \sqrt{2}e^{-\sigma \nu/2s} \cos \left( \frac{\sigma(v - 1)^2}{4s} \right) \right] \frac{d\sigma}{\sigma^{3/2}}. \tag{3.24}
\]

Figure 3.3 depicts the fading-out waves for the elastic beam which is clamped at the boundary. Figure 3.4 demonstrates some initial phase of the “reflected” wave and the fading-out wave for the initial values with \( f(x) = \sin^2(x) \) and \( g(x) = 0 \) for \( \pi < x < 2\pi \), and \( f(x) = g(x) = 0 \) elsewhere. For more information on the Green’s function \( G(x, t) \) for a semi-infinite clamped beams the reader is referred to [50]. For the simple cases (i.e., for the pinned, sliding and clamped cases), we compared our results with some of the available, analytical results in the literature [49, 50]. Our results agreed completely with those results.
3.5 Non-classical boundary conditions

3.5.1 Damper end: $u''(0, t) = 0$, $u'''(0, t) = -\tilde{\lambda} \dot{u}(0, t)$

In this section, we consider a semi-infinite beam equation for $x > 0$, when the bending moment is zero and the shear force is proportional to the velocity (damper) at $x = 0$, i.e. $EIu'' = 0$, $EIu''' = -\alpha \ddot{u}$. After applying the dimensionless quantities $\tilde{\lambda} = \alpha L_s/\sqrt{EI\rho A}$ to the damper boundary conditions, it follows that $u'' = 0$, $u''' = -\tilde{\lambda} \dot{u}$. We obtain the Green’s function for the semi-infinite beam in a similar way as shown in the previous cases. By using the requirements $[G1]$-$[G4]$, $g_\xi$ is uniquely determined, and we obtain

$$g_\xi = \frac{1}{8\beta^3} \left\{ e^{-\beta|x-\xi|}[\cos\beta(x - \xi) + \sin\beta|x - \xi|] + e^{-\beta(x+\xi)}[-\cos\beta(x + \xi) - \sin\beta(x + \xi)]ight.$$  
$$+ \frac{4\beta^3 e^{-\beta(x+\xi)}}{2\beta^3 + \tilde{\lambda} p}[\cos\beta(x - \xi) + \cos\beta(x + \xi)]\right\}, \tag{3.25}$$
where $\beta^2 = p/2$. In order to invert the Laplace transform, we use the formula (see [15], page 93)

$$\mathcal{L}^{-1}\left[p^{-1} \phi(p)\right] = \int_0^t \mathcal{L}^{-1}\{\phi(\tau)\} \, d\tau.$$  \hfill (3.26)

\textbf{Figure 3.4:} Some reflected waves for a semi-infinite one-sided beam with $f(x) = \sin^2(x)$ and $g(x) = 0$ for $\pi < x < 2\pi$, and $f(x) = 0$ elsewhere.
3.5. NON-CLASSICAL BOUNDARY CONDITIONS

Figure 3.5: The Green’s function $g(v,s)$ for a semi-infinite beam with boundary damping ($\lambda = 1$) for the initial values $g(v,0) = 0$, $g_s(v,0) = 0$, and the external force $q(v,s) = \delta(v-1) \otimes \delta(s)$.

\[
\phi(p) = \frac{p^{-1/2}}{2\sqrt{2}} e^{-\sqrt{p}\eta}[\cos(\sqrt{p}\eta) + \sin(\sqrt{p}\eta)] - \frac{p^{-1/2}}{2\sqrt{2}} e^{-\sqrt{p}\mu}[\cos(\sqrt{p}\mu) + \sin(\sqrt{p}\mu)] \\
+ \frac{p^{-1/2}}{2\sqrt{2}} e^{-\sqrt{p}\mu} \frac{2p^{3/2}}{p^{3/2} + \sqrt{2}\lambda p}[\cos(\sqrt{p}\eta) + \cos(\sqrt{p}\mu)],
\]

(3.27)

where $\eta = \frac{(x+\xi)}{\sqrt{2}}$ and $\mu = \frac{|x-\xi|}{\sqrt{2}}$. In Eq. (3.27), we use Eq. (3.14)-(3.15) for the first two terms, and the following convolution theorem for the last term (see [15], page 92)

\[
\mathcal{L}^{-1} [\phi_1(p) \phi_2(p)] = f_1(t) * f_2(t) = \int_0^t f_1(r)f_2(t-r)dr,
\]

(3.28)

where

\[
\phi_1(p) = \frac{p^{-1/2}}{2\sqrt{2}}[\cos(\sqrt{p}\eta) + \cos(\sqrt{p}\mu)],
\]

(3.29)

\[
\phi_2(p) = e^{-\sqrt{p}\mu} \frac{2p^{3/2}}{p^{3/2} + \sqrt{2}\lambda p}.
\]

(3.30)
For the inverse Laplace transform of Eq. (3.29), we use the following formula (see [12], page 106)

\[
\mathcal{L}^{-1}\left[p^{-1/2}\cos(\sqrt{p}\eta)\right] = \frac{1}{\sqrt{\pi t}} \sin\left(\frac{\eta}{4t} + \frac{\pi}{4}\right),
\]

(3.31)

and for the inverse Laplace transform of Eq. (3.30), we use the following formulas (see [7], page 245-246)

\[
\mathcal{L}^{-1}\left[e^{-\sqrt{p} \mu}\right] = \sqrt{\frac{\mu}{\pi}} t^{-3/2} e^{-\mu/4t}, \text{Re}(\mu) > 0,
\]

(3.32)

\[
\mathcal{L}^{-1}\left[\frac{e^{-\mu \sqrt{p}}}{\sqrt{p} + \lambda \sqrt{2}}\right] = \frac{e^{-\mu^2/4t}}{\sqrt{\pi t}} - \frac{\sqrt{2} e^{\mu^2} \lambda^2 + 2 \lambda^2 t}{\sqrt{2\pi t}} \text{erfc}\left(\frac{\mu^2}{\sqrt{t}} + \lambda \sqrt{2t}\right),
\]

(3.33)

where the error function is defined as

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt.
\]

(3.34)

Then, the Green’s function is given by

\[g(v, s)\]

![Figure 3.6: The Green’s function $g(v, s)$ for a semi-infinite beam with different boundary damping parameters for the initial values $g(v, 0) = 0$, $g_s(v, 0) = 0$, and the external force $q(v, s) = \delta(v - 1) \otimes \delta(s)$ at $s = 0.8$.](image-url)
\[ G_\xi(x,t) = -\int_0^t \frac{1}{2\sqrt{\pi} \tau} \left[ \sin \left( \frac{(x - \xi)^2}{4\tau} + \frac{\pi}{4} \right) - \sin \left( \frac{(x + \xi)^2}{4\tau} + \frac{\pi}{4} \right) \right] d\tau \\
- \int_0^t \int_0^\tau \left[ \sin \left( \frac{(x - \xi)^2}{8(\tau - r)} + \frac{\pi}{4} \right) + \sin \left( \frac{(x + \xi)^2}{8(\tau - r)} + \frac{\pi}{4} \right) \right] \\
\left[ e^{-(x+\xi)^2/8\tau} \frac{(x + \xi - 4\tilde{\lambda}r)}{4\pi r \sqrt{r}(\tau - r)} + \frac{2\tilde{\lambda}^2}{\sqrt{2\pi}(\tau - r)} e^{\tilde{\lambda}(x+\xi)+2\tilde{\lambda}^2r} \\
erfc \left( \frac{(x + \xi + 4\tilde{\lambda}r)}{2\sqrt{2r}} \right) \right] dr d\tau. \tag{3.35} \]

When we assume that the external loading is nonzero, for example, \( q(x,t) = \delta(x - \xi) \otimes \delta(t) \), and the initial disturbances are zero (\( u(x,0) = f(x) = 0 \), \( \dot{u}(x,0) = g(x) = 0 \)), the solution for the semi-infinite beam with damping boundary can be written in a non-dimensional form by substituting the following dimensionless quantities in Eq. (3.5.1):

\[ v = \frac{x}{\xi}, \quad s = \frac{t}{\xi^2}, \quad \tilde{s} = \frac{\tau}{\xi^2}, \quad \sigma = \frac{r}{\tau}, \quad \varphi = \frac{\tau}{r}, \quad \tilde{\lambda} = \frac{\lambda}{\xi}, \quad g(v,s) = \frac{G_\xi}{\xi}, \]

we obtain

\[ g(v,s) = -\int_1^\infty \frac{\sqrt{s}}{2\sqrt{\pi} \sigma^3} \left[ \sin \left( \frac{(v - 1)^2\sigma}{4s} + \frac{\pi}{4} \right) - \sin \left( \frac{(v + 1)^2\sigma}{4s} + \frac{\pi}{4} \right) \right] d\sigma \\
- \int_1^\infty \int_1^\infty \left[ \sin \left( \frac{\sigma \varphi(v - 1)^2}{8(\varphi - 1)} + \frac{\pi}{4} \right) + \sin \left( \frac{\sigma \varphi(v + 1)^2}{8(\varphi - 1)} + \frac{\pi}{4} \right) \right] \\
\left[ e^{-\sigma \varphi(v+1)^2/8s} \frac{\sigma \varphi(v + 1) - 4\lambda s}{4\pi \sigma^2 \sqrt{\varphi - 1}} + \frac{\lambda^2 \sqrt{2s^3}}{\sqrt{\pi \lambda^5 \varphi^3} \varphi} e^{\lambda(v+1)+2\frac{s^2}{\sigma \varphi}} \\
erfc \left( \frac{\sigma \varphi(v + 1) + 4\lambda s}{2\sqrt{2s \sigma \varphi}} \right) \right] d\varphi d\sigma. \tag{3.36} \]

Figure 3.5 shows the shape for the semi-infinite beam with boundary damping during its oscillation. It is observed how the vibration is suppressed due to using a damper (\( \lambda = 1 \)) at the boundary \( x = 0 \). Figure 3.6 depicts the Green’s function of the semi-infinite beam for varying boundary damping parameters \( \lambda \) at \( s = 0.8 \). As can be seen, the damping boundary condition starts to behave like free and pinned boundary condition when we take \( \lambda \to 0 \) and \( \lambda \to \infty \), respectively. For the damping case, we compare our solution in the next section with a long bounded beam by applying the Laplace transform method for a certain value of \( \lambda \).
3.5.2 Damper-clamped end: \( u''(0, t) = -\lambda \dot{u}(0, t), \ u''(0, t) = 0, \ u(L, t) = 0, \ u'(L, t) = 0 \)

In this section, we compare our semi-infinite results with results for a bounded domain \([0, L]\) with \(L\) large. We can formulate the dimensionless initial boundary value problem describing the transverse vibrations of a damped horizontal beam which is attached to a damper at \( x = 0 \) as follows:

\[
\ddot{u}(x, t) + u'''(x, t) = q(x, t), \quad 0 < x < L, \quad t > 0, \tag{3.37}
\]

\[
u(x, 0) = f(x), \quad \dot{u}(x, 0) = g(x), \quad 0 \leq x < L, \tag{3.38}
\]

and boundary conditions,

\[
u''(0, t) = -\lambda \dot{u}(0, t), \quad \nu''(0, t) = 0, \quad t \geq 0, \tag{3.39}
\]

\[
u(L, t) = 0, \quad \nu'(L, t) = 0, \quad t \geq 0. \tag{3.40}
\]

We will also solve this problem by using the Laplace transform method which reduces the partial differential equation Eq. (3.68) to a non-homogeneous linear ordinary differential equation, which can be solved by using standard techniques [20, 28]. When we apply the Laplace transform method, which was defined in Eq. (3.9), to Eqs. (3.68)-(3.70), we obtain the following boundary value problem

\[
PDE: U''''(x, p) + p^2 U(x, p) = Q(x, p), \tag{3.40}
\]

\[
BCs: U''''(0, p) = \lambda \left[ p U(0, p) - f(0) \right], \tag{3.41}
\]

\[
U''(0, t) = U(L, p) = U'(L, p) = 0, \tag{3.42}
\]

where \( U(x, p) \) and \( Q(x, p) \) are the Laplace transforms of \( u(x, t) \) and \( q(x, t) \), and \( p \) is the transform variable. Here, \( Q(x, p) = \delta(x - \xi) + p u(x, 0) + \dot{u}(x, 0) \). We assume that the initial conditions are zero, that is \( u(x, 0) = f(x) = 0 \) and \( \ddot{u}(x, 0) = g(x) = 0 \).

The general solution of the homogeneous equation, that is, Eq. (3.71) with \( Q(x, p) = 0 \), is given by

\[
U(x, \beta) = C_1(\beta)\cos(\beta x) + C_2(\beta)\sin(\beta x) + C_3(\beta)\cosh(\beta x) + C_4(\beta)\sinh(\beta x), \tag{3.43}
\]

where \( C_j(\beta) \) are arbitrary functions for \( j = 1..4 \). For simplicity, we consider \( p^2 = -\beta^4 \), so that \( p = \pm i\beta^2 \). We consider only the case \( p = i\beta^2 \) for further calculations, because the case \( p = -i\beta^2 \) will also lead to the same \( p \). The particular solution of the non-homogeneous equation Eq. (3.71) can be defined by using the method of variation of parameters. We rewrite the general solution as follows:

\[
U(x, \beta) = K_1(\beta)\cos(\beta x) + K_2(\beta)\sin(\beta x) + K_3(\beta)\cosh(\beta x) + K_4(\beta)\sinh(\beta x)
\]

\[
+ \frac{1}{2\beta^2} \int_0^x Q(s, \beta)\left[\sinh(\beta(s-x)) - \sinh(\beta(s-x))\right] ds, \tag{3.44}
\]

where \( Q(s, \beta) = \delta(s - \xi) \). \( K_j(\beta) \) for \( j = 1..4 \) can be determined from the boundary conditions and the solution of Eq. (3.71) and Eq. (3.72) is given by

\[
U(x, \beta) = \int_0^L Q(s, \beta)H_1(s, \beta : x) ds + \int_0^x Q(s, \beta)H_2(s, \beta : x) ds, \tag{3.45}
\]

where \( H_1(s, \beta : x) \) and \( H_2(s, \beta : x) \) are the Heaviside functions of \( x \) at \( s \).
3.5. NON-CLASSICAL BOUNDARY CONDITIONS

(a) The first ten oscillation modes as approximation of the solution of $u(x, t)$ for a damper-clamped ended finite beam

(b) The first forty oscillation modes as approximation of the solution of $u(x, t)$ for a damper-clamped ended finite beam

(c) The Green’s function $g(v, s)$ for a one-sided damper ended semi-infinite beam

Figure 3.7: The comparison of the numerical and exact solutions of a damper-clamped ended finite beam ($L = 10$) and a damper ended semi-infinite beam with $\lambda = 1$ for the zero initial values and the external force $q(x, t) = \delta(x - 1) \otimes \delta(t)$ at times $t = 0.4$ and $t = 0.8$.

where

$$H_1(s, \beta : x) := \frac{1}{4\beta^3 h_{\lambda L}(\beta)} \left[ \Theta_1(x) \Upsilon_1(s, \beta) + \Theta_2(x) \Upsilon_2(s, \beta) + \Theta_3(x) \Upsilon_3(s, \beta) \right], \quad (3.45)$$
\[ \Theta_1(x) := \cos(\beta x) + \cosh(\beta x), \] (3.46)

\[ \Upsilon_1(s, \beta) := [\sin(\beta(L - s)) - \sinh(\beta(L - s))]\beta[\cos(\beta L) + \cosh(\beta L)] 
- [\cos(\beta(L - s)) - \cosh(\beta(L - s))]\beta[\sin(\beta L) + \sinh(\beta L)], \] (3.47)

\[ \Theta_2(x) := \sin(\beta x), \] (3.48)

\[ \Upsilon_2(s, \beta) := [\sin(\beta(L - s)) - \sinh(\beta(L - s))] [2\lambda \cosh(\beta L) 
+ \beta(\sin(\beta L) - \sinh(\beta L))] - [\cos(\beta(L - s)) - \cosh(\beta(L - s))] 
[2\lambda \sin(\beta L) - \beta(\cos(\beta L) + \cosh(\beta L))], \] (3.49)

\[ \Theta_3(x) := \sinh(\beta x), \] (3.50)

\[ \Upsilon_3(s, \beta) := [\sin(\beta(L - s)) - \sinh(\beta(L - s))] [2\lambda \cos(\beta L) 
- \beta(\sin(\beta L) - \sinh(\beta L))] - [\cos(\beta(L - s)) - \cosh(\beta(L - s))] 
[2\lambda \sin(\beta L) + \beta(\cos(\beta L) + \cosh(\beta L))], \] (3.51)

\[ h_{\lambda L}(\beta) := \beta[1 + \cos(\beta L)\cosh(\beta L)] + \lambda i[\cosh(\beta L)\sin(\beta L) - \sinh(\beta L)\cos(\beta L)], \] (3.52)

\[ H_2(s, \beta : x) := \frac{1}{2\beta^3} [\sin(\beta(s - x)) - \sinh(\beta(s - x))]. \] (3.53)

In order to obtain the solution of Eqs. (3.68)-(3.70), the inverse Laplace transform of \( U(x, p) \) will be applied by using Cauchy’s residue theorem, that is,

\[ u(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} U(x, p) dp = \sum_n \text{Res}(e^{pt} U(x, p), p = p_n), \] (3.54)

for \( \gamma > 0 \). Here \( \text{Res}(e^{pt} U(x, p), p = p_n) \) is the residue of \( e^{pt} U(x, p) \) at the isolated singularity at \( p = p_n \). The poles of \( U(x, p) \) are determined by the roots of the following characteristic equation

\[ h_{\lambda L}(\beta) := 0, \] (3.55)

which is a “transcendental equation” defined in Eq. (3.5.2). The zeros of \( h_{\lambda L}(\beta) \) for \( \lambda = 0 \), which reduces the problem to the clamped-free beam, have been considered in \cite{36}. By using Rouché’s theorem, it can be shown that the number of roots of \( h_{\lambda L}(\beta) := 0 (\lambda > 0) \) is equal to the same number of roots of \( h_L(\beta) := 0 (\lambda = 0) \). For the proof of Rouché’s theorem, the reader is refered to Ref. \cite{16}. Eq. (3.55) has infinitely many roots \cite{46}. By using the relation \( p = i\beta^2 \), we can determine the roots
of \( p \), which are defined in complex conjugate pairs, such that \( p_n = p_n^{\text{re}} \pm ip_n^{\text{im}} \), where \( n \in \mathbb{N} \) and \( p_n^{\text{re}}, \ p_n^{\text{im}} \in \mathbb{R} \). So, the damping rate and oscillation rate are given by \( p_n^{\text{re}} := -2\beta_n^{\text{re}}\beta_n^{\text{im}} \) and \( p_n^{\text{im}} := (\beta_n^{\text{re}})^2 - (\beta_n^{\text{im}})^2 \), respectively.

In order to construct asymptotic approximations of the roots of \( h_{\lambda L}(\beta) \), we first multiply Eq. (3.55) by \( L \), and define \( \tilde{\beta} = \beta L \) and \( \tilde{\lambda} = \lambda L \). Hence, we obtain

\[
h_{\tilde{\lambda}}(\tilde{\beta}) \equiv \tilde{\beta}[1 + \cos(\tilde{\beta})\cosh(\tilde{\beta})] + \tilde{\beta}i[\cosh(\tilde{\beta})\sin(\tilde{\beta}) - \sinh(\tilde{\beta})\cos(\tilde{\beta})] = 0. \tag{3.56}
\]

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<th>( n )</th>
<th>( \beta_{\text{num},n} )</th>
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Table 3.2: Numerical approximations of the solutions \( \beta_n \) and \( p_n \) of the characteristic equation Eq. (3.55) for the case \( L = 10 \) and \( \lambda = 1 \).
Next, multiplying $h_{\tilde{\lambda}}(\tilde{\beta})$ by $(2)/(\tilde{\beta}e^{\tilde{\beta}})$, the characteristic equation yields
\[
\cos(\tilde{\beta}) = \mathcal{O}(|\tilde{\beta}|^{-2}) + i \left( \frac{\tilde{\lambda}}{\tilde{\beta}}[\cos(\tilde{\beta}) - \sin(\tilde{\beta})] + \mathcal{O}(|\tilde{\beta}|^{-3}) \right),
\]
(3.57)
or
\[
\cos(\tilde{\beta}) = \mathcal{O}(|\tilde{\beta}|^{-1}),
\]
(3.58)
which is valid in a small neighbourhood of $k = (n - \frac{1}{2})$ for all $n > 0$. After applying Rouché’s theorem (see [22]), the following asymptotic solutions for $\beta_n$ and $p_n$ are obtained
\[
\beta_n = \mp \frac{1}{L} \left[ k\pi + \mathcal{O}(|n|^{-2}) + i \left( \frac{\lambda L}{k\pi} + \mathcal{O}(|n|^{-2}) \right) \right],
\]
(3.59)
and
\[
p_n = -\frac{2\lambda}{L} + \mathcal{O}(|n|^{-1}) + i \left( \frac{(k\pi)^4 - (\lambda L)^2}{(kL\pi)^2} + \mathcal{O}(|n|^{-1}) \right),
\]
(3.60)
which are valid and represent the asymptotic approximations of the damping rates of the eigenvalues for sufficiently large $n \in \mathbb{N}$.

The first twenty roots $\beta_{\text{num},n}$ and $p_{\text{num},n}$, which are computed numerically by using Maple, and the first twenty asymptotic approximations of the roots of the Eq. (3.55) are listed in Table 3.2. For higher modes, it is found that the asymptotic and numerical approximations of the damping rates are very close to each other, and the numerical damping rates, which are the real part of $p_{\text{num},n}$, converges to $-0.2$.

The characteristic equation Eq. (3.55) has three unique real-valued roots; $p = 0$ is one of these roots. Note that $p = 0$ is not a pole of $U(x,p)$. That is why, the only contribution to the inverse Laplace transform is the first integral of Eq. (3.5.2). The implicit solution of the problem Eqs. (3.68)-(3.70) is given by
\[
u(x,t) = e^{p_{\text{re}}t}H(x,p_{\text{re}}) + e^{p_{\text{im}}t}H(x,p_{\text{im}})
+ \sum_{n=1}^{N} e^{p_{\text{re}}^n t} \left( [H(x,p_n) + H(x,p_n^*)] \cos(p_{\text{re}}^n t)
+ i[H(x,p_n) - H(x,p_n^*)] \sin(p_{\text{re}}^n t) \right),
\]
(3.61)
where $H(x,p_n^*)$ is the complex conjugate of $H(x,p_n)$, and $H(x,p_n)$ is given by
\[
H(x,p_n) := \frac{R(x,p_n)}{\partial_p(\Omega(p_n))|_{p=p_n}},
\]
(3.62)
where
\[
R(x,p_n) := \Theta_1(x)\Upsilon_1(s,\beta_n) + \Theta_2(x)\Upsilon_2(s,\beta_n) + \Theta_3(x)\Upsilon_3(s,\beta_n),
\]
(3.63)
and
\[
\partial_p(\Omega(p_n))|_{p=p_n} := \left( \frac{\partial\Omega(\beta_n)}{\partial\beta_n} \frac{\partial\beta_n}{\partial p_n} \right),
\]
(3.64)
3.6. THE ENERGY IN THE DAMPED CASE

\( \Omega(\beta_n) := 4\beta_n^3 h_{\lambda L}(\beta_n). \)  

(3.65)

By using the relation \( p_n := i\beta_n^2, \beta_n := \beta_n^re + i\beta_n^im \) is defined by

\[ \beta_n^{re} := \frac{\sqrt{p_n^{im} + \sqrt{(p_n^{re})^2 + (p_n^{im})^2}}}{\sqrt{2}}, \]

(3.66)

and

\[ \beta_n^{im} := \frac{-p_n^{re}\sqrt{2}}{2\sqrt{p_n^{im} + \sqrt{(p_n^{re})^2 + (p_n^{im})^2}}}. \]

(3.67)

The numerical approximations of the roots which are listed in Table 3.2 can be substituted into Eq. (3.5.2) to obtain explicit approximations of the problem Eqs. (3.68)-(3.70). Figure 3.7 shows the comparison of the numerical and exact solutions of a damper-clamped ended finite beam \((L = 10)\) and a damper ended semi-infinite beam with \(\lambda = 1\) for the zero initial values and the external force \(q(x, t) = \delta(x - 1) \otimes \delta(t)\) at times \(t = 0.4\) and \(t = 0.8\). It can be seen that the numerical results in Figure 3.7(a) and Figure 3.7(b) are similar to the analytical (exact) results in Figure 3.7(c) when the number of modes become sufficiently large.

3.6 The energy in the damped case

In this section, we derive the energy of the transversally free vibrating homogeneous semi-infinite beam \((q = 0)\)

\[ \ddot{u}(x, t) + u'''(x, t) = 0, \quad 0 < x < \infty, \quad t > 0, \]  

(3.68)

subject to the boundary conditions \(u''(0, t) = 0,\) and \(u'''(0, t) = -\tilde{\lambda}\dot{u}(0, t).\) By multiplying Eq. (3.68) with \(\dot{u},\) we obtain the following expression

\[ \{\frac{1}{2}(\dot{u}^2 + u'')\}_t + \{\dot{u}\dot{u}''' + \dot{u}'u''\}_x = 0, \]

(3.69)

By integrating Eq. (3.69) with respect to \(x\) from \(x = 0\) to \(x = \infty\) and with respect to \(t\) from \(t = 0\) to \(t = t,\) respectively, we obtain the total mechanical energy \(E(t)\) in the interval \((0, \infty).\) This energy \(E(t)\) is the sum of the kinetic and the potential energy of the beam, that is,

\[ E(t) = \frac{1}{2} \int_0^\infty (\dot{u}^2 + u''^2)dx. \]

(3.70)

The time derivative of the energy \(E(t)\) is given by

\[ \dot{E}(t) = -\tilde{\lambda}\dot{u}^2(0, t), \]

(3.71)

where \(\tilde{\lambda}\) is the boundary damping parameter. And so, it follows from Eq. (3.71) that:

\[ E(t) = E(0) - \tilde{\lambda} \int_0^t u_s^2(0, s)ds. \]

(3.72)

When the damping parameter \(\tilde{\lambda} > 0,\) it follows from Eqn. (3.72) that energy of the system is dissipated. If \(\tilde{\lambda} = 0,\) then \(E(t) = E(0),\) which represents conservation of energy.
3.7 Conclusions

In this chapter, an initial-boundary value problem for a beam equation on a semi-infinite interval has been studied. We applied the method of Laplace transforms to obtain the Green’s function for a transversally vibrating homogeneous semi-infinite beam, and examined the solution for various boundary conditions. In order to validate our analytical results, explicit numerical approximations of the damping and oscillating rates were constructed by using the Laplace transform method to finite domain. It has been shown that the numerical results approach the exact results for sufficiently large domain length and for sufficiently many number of modes. The total mechanical energy and its time-rate of change can also be derived.

This chapter provides an understanding of how the Green’s function for a semi-infinite beam can be calculated analytically for (non)-classical boundary conditions. The method as given in this chapter can be used for other boundary conditions as well.
Chapter 4

On boundary damping to reduce the rain-wind oscillations of an inclined cable

Abstract. In the previous chapter, the boundary reflection and damping properties of waves in semi-infinite beams were studied. In order to model the vibration of stretched bridge cables in a more realistic way, we need to consider the effect of gravity, of the external wind forces and of the time-varying mass due to rain. In this chapter a model will be derived to describe the rain-wind induced oscillations of an inclined cable. Water-rivulets running along the cable and aerodynamic forces acting on the cable are taken into account to describe these oscillations. A boundary damper is assumed to be present near the lower endpoint of the cable. For a linearly formulated initial-boundary value problem for a tensioned beam equation describing the in-plane transversal oscillations of the cable, the effectiveness of this damper is determined by using a two timescales perturbation method. It is shown how mode-interactions play an important role in the dynamic behaviour of the cable system. Some resonant and non-resonant cases have been studied in detail.

4.1 Introduction

The study on how to damp vibrations in stay cables, which are attached to a pylon at one end and to a bridge deck at the other end, is of great importance not only in structural engineering but also in applied mathematics. The combined effect of rain and of wind can change the aerodynamic properties of the cable stayed bridge and can lead to relatively large amplitude vibrations of the cables. For example one can refer to the Erasmus bridge in Rotterdam, which started to vibrate heavily under mild wind-rain conditions shortly after its opening in 1996. This bridge was temporarily closed to the traffic as a safety precaution. As a temporarily measure, polypropylene

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Parts of this chapter have been submitted to the Nonlinear Dynamics, 2017.
ropes were installed between the cables and the bridge desk. Later these ropes were replaced by hydraulic dampers as a permanent measure and by conforming to the aesthetic of the bridge designed by the architects. Much research on this problem has been done both numerically and experimentally to understand the mechanisms of rain-wind induced vibration of inclined stay cables [11, 21].

As has been observed from wind-tunnel experiments, raindrops hitting the inclined stay cable cause the generation of one or more rivulets on the surface of the cable. The presence of flowing water on the cable changes the mass and the aerodynamic properties of the bridge system that can lead to instabilities. These water rivulets on the cable surface can be considered as a time-varying mass in the system [2, 1]. In order to suppress undesirable vibrations in bridge structures, different kinds of dampers such as tuned mass dampers and oil dampers can be installed between the cables and the bridge deck. Numerous work has been done from the theoretical and the experimental point of view to predict the optimal damper location and type. Jacquot [34] calculated the optimal value and location of the viscous damper which is located in a randomly forced horizontal cantilever beam.

These physical problems of rain-wind induced vibration of inclined cables can be modelled mathematically by initial-boundary-value problems for string-like or beam-like problems. For string-like problems, the static state due to gravity and the dynamic state due to a parametrical and a transversal excitation at one of the ends of the inclined string were studied in [14]. For the inclined cable subjected to wind
with a moving rivulet on its surface, the nonlinear dynamic model is investigated by considering the equilibrium position of the rivulet [39]. The effect of the static condensation of the longitudinal displacement due to the cable inclination and the cable total tension is investigated by using numerical and analytical techniques in [54]. In addition, the interaction among the three excitation sources: self excitation, which is caused by a mean wind flow, and external and parametric excitations due to vertical motion of the ground support, on the nonlinear dynamics of the inclined cable related to a cable-stayed bridge were studied in [40]. In many papers the rain-wind-induced vibrations of inclined taut cables have been studied, but much less work has been done on the rain-wind-induced vibrations of an inclined cable subjected to wind with time-varying rivulets on its surface.

In this chapter we consider the linear (and nonlinear) dynamic response due to a damper near one of the support ends of the inclined cable. On the cable a time-varying (rain) mass rivulet is assumed to be present (see also Figure 4.1). The stationary windflow will lead to the phenomenon of self excitation of the cable. The damper location in the cable-stayed bridge is quite close to the anchorage of the cable. For more information on the effect of the bending stiffness of a tensioned cable for varying damper locations, the reader is referred to [30, 42, 41].

The aim of this chapter is to provide an understanding of how effective boundary damping is for inclined stretched beams with a small bending rigidity. These problems for strings or beams are considered to be basic models for oscillations of cables from a practical viewpoint. The outline of this chapter is as follows. In Section 4.2, we apply a variational method in order to derive the governing equations of motion of the tensioned Euler-Bernoulli beam, and obtain a system of three coupled PDEs. By using Kirchhoff’s approach, the number of PDEs in the system is reduced to two PDEs (one for the in-plane motion, and one for out-of-plane motion) or to a single PDE (only for the in-plane motion). The main aim of this section is to give a general model to describe the dynamics of rain-wind-induced vibrations of an inclined beam, where the gravity effect in schematically shown in Figure 4.1. In Section 4.3, we only consider the in-plane motion of the inclined beam with only one rivulet on the cable. In Section 4.4, the two-timescales perturbation method is used to solve the problem and some (non)resonance frequencies are determined. In this chapter we will consider the pure resonance case and the non-resonance case. All other resonance cases can be investigated similarly. Finally, the conclusions are presented in Section 4.5.

### 4.2 Equations of motion

We consider an inclined, perfectly flexible, elastic unstretched beam with a small bending rigidity on a finite interval $x \in [0, L]$, which is attached to a dashpot $\lambda$ at $x = L$, and to a vertical rigid bar at $x = 0$ (see Figure 4.1). $u$, $w$ and $v$ are the displacements in $x$-direction, $y$-direction and $z$-direction, respectively. From Newton’s second law, the equations of motion can be stated as follows: the time derivative of the linear momentum of the system is equal to the sum of external forces which are elastic forces, gravity, drag- and lift-forces due to the uniform wind flow $v_0$, blowing under a yaw angle $\beta$ as can be seen in Figure 4.1. We assume that the tension due to stretching in the beam is large enough, such that the small sag of the beam, that
is, the displacements in $x$ and $z$-direction due to gravity, can be neglected. Similarly, due to this tension, we neglect the wind-force along the cable in $x$-direction. Hence, the total tension in the beam at $x = x_0$ can be written by

$$ T(x_0) = T_0 + M(x_0, t) g A \sin(\alpha) x_0, \quad (4.1) $$

where $T_0$ is the pretension in the beam, $g$ is the acceleration due to gravity, $A$ is the cross-sectional area of the beam, $\alpha$ is the angle between the beam and the horizontal plane, and $M(x, t)$ is mass of the beam including the time-varying mass of the water rivulet.

Let the coordinates $(x, y, z)$ of a material point of the unstretched beam be $(x, 0, 0)$ with $x \in [0, L]$, where the $x$-axis, $y$-axis and $z$-axis are defined in Figure 4.1. The dynamic displacement of this material point is denoted by $u(x, t) i$, $v(x, t) k$ and $w(x, t) j$, where $i$, $k$ and $j$ are the unit vectors along the $x$-axis, $y$-axis and $z$-axis [14]. The vector position $R(x, t)$ of this material point in the dynamic state is obtained by

$$ R(x, t) = \left[ x + \frac{T_0}{AE} x + \frac{M(x, t) g \sin(\alpha) x^2}{2E} + u(x, t) \right] i + v(x, t) j + w(x, t) k, \quad (4.2) $$

where $E$ is Young’s modulus. The relative strain per unit length of the stretched beam is written by

$$ \epsilon_{x0}(x, t) = \left| \frac{\partial}{\partial x} R(x, t) \right| - 1 $$

$$ = \sqrt{1 + \frac{T_0}{AE} + \frac{M(x, t) g \sin(\alpha) x}{E} + u_x^2} + v_x^2 + w_x^2 - 1, \quad (4.3) $$

where $u_x$, $v_x$ and $w_x$ represent the derivative with respect to $x$ of $u(x, t)$, $v(x, t)$ and $w(x, t)$, respectively. By assuming that $u_x^2$ is small with respect to $u_x$ and $v_x^2 + w_x^2$, and by expanding the square-root in a Taylor series, we have approximately

$$ \epsilon_{x0} \approx \frac{T_0}{AE} + \frac{M(x, t) g \sin(\alpha) x}{E} + u_x + \frac{v_x^2}{2} + \frac{w_x^2}{2}. \quad (4.4) $$

For the curvature of the beam in the $(x, y, z)$-space (out-of-plane) or for the curvature of the beam in the $(x, y)$-plane (in-plane), the reader is referred to [8, 43]. The axial strain of a generic point of the beam located at distances $y$ and $z$ (in the $y$ and $z$ directions) from the centerline of the beam is defined by

$$ \epsilon_{xx} \approx -y v_{xx} - z w_{xx}. \quad (4.5) $$

Hence, the total strain of a line-element of the beam is approximately given by

$$ \epsilon_x = \epsilon_{x0} + \epsilon_{xx} \approx \frac{T_0}{AE} + \frac{M(x, t) g \sin(\alpha) x}{E} + u_x + \frac{v_x^2}{2} + \frac{w_x^2}{2} - y v_{xx} - z w_{xx}. \quad (4.6) $$

The equations of motion describing the vibrations of the beam will be obtained by using Hamilton’s principle [24]. In order to apply this principle, the kinetic energy
and potential energy for the beam should be calculated. The potential energy density of the beam is given by (using Hooke’s Law [37]):

$$P = \frac{EL}{2} \epsilon_x^2 - M(x, t) g A L [u \sin(\alpha) + v \sin(\alpha)]$$  \hfill (4.7)

$$= \frac{E A L}{2} \left[ \frac{T_0}{AE} + \frac{M(x, t) g \sin(\alpha) x}{E} + u + \frac{v_x^2}{2} + \frac{w_x^2}{2} \right]^2 + \frac{EL}{2} (I_y v_{xx}^2 + I_z w_{xx}^2) - M(x, t) g A L [u \sin(\alpha) + v \cos(\alpha)],$$

where $I_y$ and $I_z$ represent the axial moments of area about the y- and z-axes, respectively. Hence, the potential energy of the beam is given by

$$E_P = \frac{E A L}{2} \int_0^L \left[ \frac{T_0}{AE} + \frac{M(x, t) g \sin(\alpha) x}{E} + u + \frac{v_x^2}{2} + \frac{w_x^2}{2} \right]^2 dx$$  \hfill (4.8)

$$+ \frac{E}{2} \int_0^L (I_y v_{xx}^2 + I_z w_{xx}^2) dx - A g \int_0^L M(x, t) [u \sin(\alpha) + v \cos(\alpha)] dx.$$

In addition, the kinetic energy density of the beam is given by

$$K = \frac{A L}{2} M(x, t) [u_t^2 + v_t^2 + w_t^2],$$  \hfill (4.9)

and the kinetic energy of the beam is given by

$$E_K = \frac{A}{2} \int_0^L M(x, t) [u_t^2 + v_t^2 + w_t^2] dx.$$  \hfill (4.10)

The Hamiltonian integral is $\mathcal{F} = \mathcal{F}(t_2) - \mathcal{F}(t_1) = \int_{t_1}^{t_2} (E_K - E_P) dt$. If we denote the integrand with $f(u, u_x, u_t, v, v_x, v_{xx}, v_t, w, w_x, w_{xx}, w_t)$, then the three Euler-Lagrange equations of the variational problem $\delta \mathcal{F} = 0$, which we have to solve according to the Hamiltonian principle, are as follows:

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial u_x} \right) + \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial u_t} \right) - \frac{\partial f}{\partial u} = 0,$$

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial v_x} \right) - \frac{\partial}{\partial x^2} \left( \frac{\partial f}{\partial v_{xx}} \right) + \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial v_t} \right) - \frac{\partial f}{\partial v} = F_y,$$

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial w_x} \right) - \frac{\partial}{\partial x^2} \left( \frac{\partial f}{\partial w_{xx}} \right) + \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial w_t} \right) = F_z.$$
or equivalently, the equations of motion are given by

\[
\frac{\partial}{\partial t} \left[ M(x,t) u_t \right] - E \frac{\partial}{\partial x} \left[ \frac{T_0}{AE} + \frac{M(x,t)g\sin(\alpha)x}{E} + u_x + \frac{v_x^2}{2} + \frac{w_x^2}{2} \right] - M(x,t)g\sin(\alpha) = 0, \tag{4.11}
\]

\[
\frac{\partial}{\partial t} \left[ M(x,t) v_t \right] - E \frac{\partial}{\partial x} \left[ v_x \left( \frac{T_0}{AE} + \frac{M(x,t)g\sin(\alpha)x}{E} + u_x + \frac{v_x^2}{2} + \frac{w_x^2}{2} \right) \right] + \frac{\partial^2}{\partial x^2} \left( \frac{EI_y}{A} v_{xx} \right) - M(x,t)g\cos(\alpha) = \frac{F_y}{A}, \tag{4.12}
\]

\[
\frac{\partial}{\partial t} \left[ M(x,t) w_t \right] - E \frac{\partial}{\partial x} \left[ w_x \left( \frac{T_0}{AE} + \frac{M(x,t)g\sin(\alpha)x}{E} + u_x + \frac{v_x^2}{2} + \frac{w_x^2}{2} \right) \right] + \frac{\partial^2}{\partial x^2} \left( \frac{EI_z}{A} w_{xx} \right) = \frac{F_z}{A}, \tag{4.13}
\]

where \( F_y \) and \( F_z \) are the aerodynamic forces in the in-plane and in the out-of-plane, respectively, and are defined by

\[
F_y = -[D \sin(\phi) + L \cos(\phi)], \tag{4.14}
\]

\[
F_z = [D \cos(\phi) + L \sin(\phi)]. \tag{4.15}
\]

Here \( D \) and \( L \) are the magnitudes of the drag and lift forces, respectively, which may be given by

\[
D = \frac{1}{2} \rho_a d L c_D(x,t;\phi_i^*) v_s^2, \tag{4.16}
\]

\[
L = \frac{1}{2} \rho_a d L c_L(x,t;\phi_i^*) v_s^2, \tag{4.17}
\]

where \( \rho_a \) is the air-density, \( d \) is the diameter of the cross-section of the circular part of the beam cable, \( L \) is the length of beam, \( v_s \) is the virtual wind velocity which is given by

\[
v_s^2 = (v_\infty \cos(\gamma) - w_i)^2 + (v_\infty \sin(\gamma) + v_i)^2. \tag{4.18}
\]

Figure 4.2 shows the center of a cross-section of the cable with rivulets. \( \psi_1 \) and \( \psi_2 \) are the displacements of the upper and the lower rivulet, respectively, and \( \phi_i^* \) is the wind attack angle, which is defined by \( \phi_i^* = \phi - \psi_i, \ i = 1, 2 \). The angle between the virtual wind velocity \( v_s \) and the horizontal axis is defined by \( \phi \), which can be
expressed as

\[
\phi(t) = \arctan \left( \frac{v_\infty \sin(\gamma) + v_t}{v_\infty \cos(\gamma) - w_t} \right), \quad (4.19)
\]

\[
\approx \arctan(\tan(\gamma)) + \frac{v_t}{v_\infty} \cos(\gamma) + \frac{w_t}{v_\infty} \sin(\gamma) - \frac{v^2_t}{v^2_\infty} \sin(\gamma) \cos(\gamma)
\]

\[
+ \frac{v_t w_t}{v^2_\infty} \left( 2 \cos^2(\gamma) - 1 \right) + \frac{w^2_t}{v^2_\infty} \sin(\gamma) \cos(\gamma) + \frac{v^3_t}{v^3_\infty} \left( \cos(\gamma) - \frac{4}{3} \cos^3(\gamma) \right)
\]

\[
+ \frac{v_t w_t}{v^3_\infty} \left( \sin(\gamma) - 4 \cos^2(\gamma) \sin(\gamma) \right) + \frac{v_t^2 w_t}{v^3_\infty} \left( 4 \cos^3(\gamma) - 3 \cos(\gamma) \right)
\]

\[
+ \frac{w^3_t}{v^3_\infty} \left( \frac{4}{3} \cos^2(\gamma) \sin(\gamma) - \frac{1}{3} \sin(\gamma) \right) + \ldots,
\]

where \(v_\infty = v_0 \sqrt{\cos^2(\beta) + \sin^2(\alpha) \sin^2(\beta)}\) is the effective wind flow, and \(\gamma\) is the angle of attack, which depends on the inclination angle \(\alpha\) and the yaw angle \(\beta\) defined by [11]

\[
\gamma(\alpha, \beta) = \arcsin \left( \frac{\sin(\alpha) \sin(\beta)}{\sqrt{\cos^2(\beta) + \sin^2(\alpha) \sin^2(\beta)}} \right). \quad (4.20)
\]
As can be seen in Figure 4.2, the position of the upper and lower rivulet are determined by \( \psi_1 \) and \( \psi_2 \), respectively. From experimental data [6], we may assume that the total time-varying mass due to these rivulets on the cable changes periodically and may be defined to have the form [2, 3]

\[
m_1(x, t) = M_1 \mu_1(x, t) = M_1 \left(1 + A_1 \sin(\gamma_1 x - \Omega_1 t)\right),
\]

\[
m_2(x, t) = M_2 \mu_2(x, t) = M_2 \left(1 + A_2 \sin(\gamma_2 x - \Omega_2 t)\right),
\]

where \( M_1 > 0 \) and \( M_2 > 0 \) are the constant mass of the upper and lower rivulets, respectively. \( A_1 > 0 \) and \( A_2 > 0 \) are small parameters, and \( \gamma_1 > 0 \), \( \gamma_2 > 0 \), \( \Omega_1 > 0 \), \( \Omega_2 > 0 \). If \( M_0 > 0 \) is the constant mass of the beam cable, the total mass can be given by

\[
M(x, t) = M \mu(x, t) = M \left(1 + \bar{A}_1 \sin(\gamma_1 x - \Omega_1 t) + \bar{A}_2 \sin(\gamma_2 x - \Omega_2 t)\right)
\]

where \( M = M_0 + M_1 + M_2 > 0 \), \( \bar{A}_1 = M_1 A_1 / M \), and \( \bar{A}_2 = M_2 A_2 / M \).

The quasi-steady drag \( c_D(x, t; \phi^*_i) \) and lift \( c_L(x, t; \phi^*_i) \) coefficients may be obtained from wind-tunnel measurements. Yamaguchi shows the experimental results for the drag, lift and moment coefficients at various angles of wind attack with the ratio of diameters of rivulet and cable in [61]. These results show that the drag, lift and moment coefficients for different ratios of diameters of rivulet and cable are similar to each other. Due to these results, we define that the drag and lift coefficients may be written in periodic form in a small \( \alpha_0, \beta_0 \) and \( \gamma_0 \) neighbourhood of fixed “rivulet” profiles as follows:

\[
c_D(x, t; \phi^*_i) = c_D \left[ \kappa_1 \mu_1(x, t) + \kappa_2 \mu_2(x, t) \right],
\]

\[
c_L(x, t; \phi^*_i) = c_L \left[ \left( \phi^*_1(t) - \alpha_1 \right) \bar{k}_1 \mu_1(x, t) + \left( \phi^*_2(t) - \alpha_2 \right) \bar{k}_2 \mu_2(x, t) \right]
\]

\[
+ c_L \left[ \left( \phi^*_1(t) - \alpha_1 \right)^2 \bar{k}_1 \mu_1(x, t) + \left( \phi^*_2(t) - \alpha_2 \right)^2 \bar{k}_2 \mu_2(x, t) \right].
\]

Here \( \kappa_i, \bar{k}_i \) and \( \bar{k}_i \) are constants for \( i = 1, 2 \) indicating the effect of the mass change of the rivulets on the drag and lift coefficients [38], and

\[
c_D = c_D \left(1 + \alpha_0 \sin(\gamma_0 x - \Omega_0 t)\right),
\]

\[
c_L_1 = c_L_1 \left(1 + \beta_0 \sin(\gamma_0 x - \Omega_0 t)\right),
\]

\[
c_L_3 = c_L_3 \left(1 + \sigma_0 \sin(\gamma_0 x - \Omega_0 t)\right),
\]

where \( \alpha_0, \beta_0 \) and \( \sigma_0 \) are small parameters, and \( \gamma_0 > 0 \), \( \Omega_0 > 0 \).

By neglecting terms of degree four and higher terms, we obtain the in-plane wind force as follows:

\[
F_y \approx - \frac{\rho_a}{2} d L v_\infty^2 \left\{ A_{00} + \frac{v_t}{v_\infty} A_{10} + \frac{w_t}{v_\infty} A_{01} + \frac{v_t^2}{v_\infty^2} A_{20} + \frac{v_t w_t}{v_\infty^2} A_{11} + \frac{w_t^2}{v_\infty^2} A_{02} + \frac{v_t^3}{v_\infty^3} A_{30} + \frac{v_t^2 w_t}{v_\infty^3} A_{21} + \frac{v_t w_t^2}{v_\infty^3} A_{12} + \frac{w_t^3}{v_\infty^3} A_{03} \right\},
\]
and similarly, we compute the out-of-plane wind force as follows:

\[ F_z \approx \frac{\rho a^2 dL v_\infty^2}{2} \left\{ B_{00} + \frac{v_t}{v_\infty} B_{10} + \frac{w_t}{v_\infty} B_{01} + \frac{w_t^2}{v_\infty^2} B_{20} + \frac{v_t w_t}{v_\infty} B_{11} + \frac{w_t^2}{v_\infty} B_{02} + \frac{v_t^2}{v_\infty^2} B_{21} + \frac{v_t w_t^2}{v_\infty^3} B_{12} + \frac{w_t^3}{v_\infty^3} B_{03} \right\} \quad (4.30) \]

where \( A_{ij} \) and \( B_{ij} \) for \( i, j = 0, 1, 2, 3 \) as follows:

\[
A_{ij} = c_D \left[ \kappa_1 \mu_1(x, t) + \kappa_2 \mu_2(x, t) \right] a_{ij0} + c_{L_1} \left[ \kappa_1 \mu_1(x, t) a_{ij1} + \kappa_2 \mu_2(x, t) a_{ij2} \right] + c_{L_3} \left[ \kappa_1 \mu_1(x, t) a_{ij3} + \kappa_2 \mu_2(x, t) a_{ij4} \right],
\]

and

\[
B_{ij} = c_D \left[ \kappa_1 \mu_1(x, t) + \kappa_2 \mu_2(x, t) \right] b_{ij0} + c_{L_1} \left[ \kappa_1 \mu_1(x, t) b_{ij1} + \kappa_2 \mu_2(x, t) b_{ij2} \right] + c_{L_3} \left[ \kappa_1 \mu_1(x, t) b_{ij3} + \kappa_2 \mu_2(x, t) b_{ij4} \right].
\]

The detailed expressions of the \( a_{ijk} \)- and \( b_{ijk} \)-coefficients for \( k = 1, 2, 3, 4 \) can be found in Appendix A. When we assume only in-plane horizontal cable motion with only the upper rivulet present, that is, \( \gamma(\alpha, \beta) = 0, \mu_2(x, t) = 0, \) and \( \kappa_1 = \bar{\kappa}_1 = \bar{\bar{\kappa}}_1 = \mu_1(x, t) = 1, \) the same coefficients of \( c_D, c_{L_1} \) and \( c_{L_3} \) are obtained as in [56].

### 4.3 Further Simplifications

For further calculations, we only consider in-plane motion \( (w = 0) \) of the inclined beam, and we assume that there is only one rivulet on the cable, that is, \( \gamma_1 \approx \gamma_2, \Omega_1 \approx \Omega_2. \) Thus, we define \( M(x, t) = \tilde{M}(1 + \tilde{A} \sin(\gamma_1 x - \Omega_1 t)), \) where \( \tilde{A} \) is a small parameter. We substitute Eq. (4.29) and Eq. (4.30), into Eq. (4.12)-(4.13) and rewrite in order to obtain the equations of motion for the beam with a time-varying mass, yielding

\[
\ddot{u} - \frac{E}{M} \frac{\partial}{\partial x} \left( u_x + \frac{v_x^2}{2} \right) = \left[ \tilde{A} \Omega_1 \cos(\gamma_1 x - \Omega_1 t) \right] u_t - \left[ \tilde{A} \sin(\gamma_1 x - \Omega_1 t) \right] \dddot{u},
\]

\[
+ \left[ 2 + 2 \tilde{A} \cos(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha) x
\]

\[ + \left[ 2 + 2 \tilde{A} \sin(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha),
\]

\[ (4.33)\]
where \( A_{ij} \) for \( i, j = 0, 1, 2, 3 \) are defined in Eq. (4.31). Eq. (4.33)-(4.34) represent the in-plane motion of the inclined beam cable system in the longitudinal and transversal direction, that is, in \( x \)-, and \( y \)-direction. We introduce the new variables \( \bar{u}(x,t) \) and \( \bar{v}(x,t) \) as defined by:

\[
\begin{align*}
\bar{u}(x,t) &= \bar{u}(x,t) + \hat{u}(x), \\
\bar{v}(x,t) &= \bar{v}(x,t) + \hat{v}(x),
\end{align*}
\]

where \( \hat{u}(x) \) and \( \hat{v}(x) \) are the ”stationary” solutions by neglecting small time perturbation (see Appendix B.). When we substitute the new variables Eq. (4.35)-(4.36) into the Eq. (4.33)-(4.34), we obtain

\[
\begin{align*}
\bar{u}_{tt} - \frac{E}{M} \frac{\partial}{\partial x} \left( \bar{u}_x + \bar{v}_x \hat{v}_x + \frac{\bar{v}_x^2}{2} \right) &= \left[ \bar{A} \sum_1 \cos(\gamma_1 x - \Omega_1 t) \right] \bar{u}_t \\
&\quad - \left[ \bar{A} \sin(\gamma_1 x - \Omega_1 t) \right] \bar{u}_{tt} \\
&\quad + \left[ \bar{A} \gamma_1 \cos(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha) x \\
&\quad + \left[ 2 \bar{A} \sin(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha),
\end{align*}
\]
4.3. FURTHER SIMPLIFICATIONS

\[
\frac{EI_y}{AM} \dddot{v}_{xxxx} - \frac{T_0}{AM} \dddot{v}_{xx} + \dddot{v}_t - \frac{E}{M} \frac{\partial}{\partial x} \left[ (\dddot{v}_x + \ddot{v}_x) \left( \dddot{v}_x + \dddot{v}_x \dddot{v}_x + \frac{\dddot{v}_x^2}{2} \right) \right]
\]

(4.38)

\[\dddot{v}_x \left( \dddot{u}_x + \dddot{v}_x \right) \right] = \left[ \frac{\dddot{v}_x}{\dddot{v}_x} \left( \dddot{v}_x + \dddot{v}_x \dddot{v}_x + \frac{\dddot{v}_x^2}{2} \right) \right]
\]

(4.39)

These coupled partial differential equations can be reduced to a single partial differential equation by applying Kirchhoff’s approximation. It will be assumed that \( \dddot{u} \) and \( \dddot{u} \) are \( \mathcal{O}(\epsilon) \), \( \bar{v} \) and \( \bar{v} \) are \( \mathcal{O}(\epsilon) \), \( \bar{A} \) is \( \mathcal{O}(\epsilon) \), \( g \sin(\alpha) = P_0 \) is \( \mathcal{O}(1) \), and \( \frac{E}{M} = P_1^* \) is \( \mathcal{O}(1/\epsilon) \), where \( \epsilon \) is a small parameter with \( 0 < \epsilon \ll 1 \). Then, by using these assumptions, Eq. (4.37) up to order \( \epsilon \) becomes

\[-P_1^* \frac{\partial}{\partial x} \left( \dddot{u}_x + \dddot{v}_x \dddot{v}_x + \frac{\dddot{v}_x^2}{2} \right) = P_0^* \left[ 2\bar{A} \sin(\gamma_1 x - \Omega_1 t) + \bar{A} \gamma_1 x \cos(\gamma_1 x - \Omega_1 t) \right] \]

(4.40)

First we integrate Eq. (4.39) with respect to \( x \) from 0 to \( x \), yielding

\[-P_1^* \left( \dddot{u}_x + \dddot{v}_x \dddot{v}_x + \frac{\dddot{v}_x^2}{2} \right) = -h(t) + \int_0^x 2P_0^* \bar{A} \sin(\gamma_1 \bar{x} - \Omega_1 t) d\bar{x} \]

(4.41)

and then from 0 to \( L \), obtaining

\[-P_1^* \left[ \dddot{u}(L, t) - \dddot{u}(0, t) + \int_0^L \left( \dddot{u}_x \ddot{v}_x + \frac{\dddot{v}_x^2}{2} \right) dx \right] = - \int_0^L h(t) dx \]

(4.42)

Hence,

\[h(t) = \frac{P_1^*}{L} \left[ \dddot{u}(L, t) - \dddot{u}(0, t) + \int_0^L \left( \dddot{v}_x \dddot{v}_x + \frac{\dddot{v}_x^2}{2} \right) dx \right] + \frac{\bar{A} P_0^*}{\gamma_1} \left( \cos(\Omega_1 t) - \cos(\gamma_1 L - \Omega_1 t) \right) .\]
When we substitute Eq. (4.42) into Eq. (4.40), we obtain

\[
\left( \ddot{u}_x + \dddot{v}_x \hat{v}_x + \frac{\dddot{v}_x^2}{2} \right) = \frac{1}{L} \left[ \ddot{u}(L,t) - \ddot{u}(0,t) + \int_0^L \left( v_x \hat{v}_x + \frac{\dddot{v}_x^2}{2} \right) \, dx \right] + \frac{\bar{A}P_1^*}{P_1^* \gamma_1} \left[ - \cos(\gamma_1 L - \Omega_1 t) + \cos(\gamma_1 x - \Omega_1 t) \right] \\
- \gamma_1 x \sin(\gamma_1 x - \Omega_1 t) \right].
\]

Similarly the equation for \( v \) in Eq. (4.38) can be rewritten in

\[
P_2^* \dddot{v}_{xxx} - P_3^* \ddot{v}_{xx} + \dddot{v}_t - P_1^* \frac{\partial}{\partial x} \left[ \dddot{v}_x + \dddot{v}_x \right] \left( \dddot{u}_x + \dddot{v}_x \hat{v}_x + \frac{\dddot{v}_x^2}{2} \right) + \dddot{v}_x \left( \dddot{u}_x + \frac{\dddot{v}_x^2}{2} \right) = \frac{\bar{A} \Omega_1}{A} \cos(\gamma_1 x - \Omega_1 t) \dddot{v}_t - \frac{\bar{A} \sin(\gamma_1 x - \Omega_1 t)}{P_1^*} \dddot{v}_t \\
+ \frac{\bar{A} \gamma_1}{P_1^*} \cos(\gamma_1 x - \Omega_1 t) \dddot{v}_x + \dddot{v}_x + P_0^* x \dddot{v}_{xx} + \dddot{v}_x + \dddot{v}_x + P_0^* \dddot{v}_x \\
+ \frac{\bar{A} \sin(\gamma_1 x - \Omega_1 t)}{P_1^*} \dddot{v}_x + \dddot{v}_x + P_0^* \dddot{v}_x \\
+ \frac{\bar{A} \sin(\gamma_1 x - \Omega_1 t)}{P_1^*} \dddot{v}_x + \dddot{v}_x + P_0^* \dddot{v}_x \\
- \frac{\rho_a}{2AM} dL v_\infty^2 \left\{ \dddot{v}_t v_\infty + \frac{\dddot{v}_t^2}{v_\infty^2} A_{10} + \frac{\dddot{v}_t^3}{v_\infty^3} A_{20} + \frac{\dddot{v}_t^3}{v_\infty^3} A_{30} \right\},
\]

where \( P_2^* = \frac{E I_y}{A L} \) is \( O(1/\epsilon) \), \( P_3^* = \frac{L_0}{A M} \) is \( O(1/\epsilon) \), and \( P_4^* = g \cos(\alpha) \) is \( O(1) \). Substituting \( \left( \dddot{u}_x + \dddot{v}_x \hat{v}_x + \frac{\dddot{v}_x^2}{2} \right) \) from Eq. (4.43) and \( \left( \dddot{u}_x + \frac{\dddot{v}_x^2}{2} \right) \) from Eq. (B.6) as given in Appendix B. into Eq. (4.44) we obtain

\[
P_2^* \dddot{v}_{xxx} + \dddot{v}_t - P_3^* \dddot{v}_{xx} = \frac{P_1^*}{L} \left( \dddot{v}_x + \dddot{v}_x \hat{v}_x \right) \left[ \dddot{u}(L,t) - \dddot{u}(0,t) \right] \\
+ \int_0^L \left( \dddot{v}_x \hat{v}_x + \frac{\dddot{v}_x^2}{2} \right) \, dx + \left[ \bar{A} \frac{\Omega_1}{A} \cos(\gamma_1 x - \Omega_1 t) \right] \dddot{v}_t \\
- \left[ \bar{A} \sin(\gamma_1 x - \Omega_1 t) \right] \dddot{v}_t + \left[ \bar{A} \sin(\gamma_1 x - \Omega_1 t) \right] P_0^* \dddot{v}_x + \dddot{v}_x + \dddot{v}_x + P_0^* \dddot{v}_x \\
+ \left[ \bar{A} \cos(\gamma_1 x - \Omega_1 t) \right] - \bar{A} \cos(\gamma_1 x - \Omega_1 t) P_0^* \dddot{v}_x + \dddot{v}_x + P_0^* \dddot{v}_x \\
+ \left[ P_0^* (L - x) + \frac{P_1^*}{2L} \int_0^L \dddot{v}_x \, dx \right] \dddot{v}_{xx} + \left[ \bar{A} \sin(\gamma_1 x - \Omega_1 t) \right] P_4^* \\
- \frac{\rho_a}{2AM} dL v_\infty^2 \left\{ \dddot{v}_t v_\infty + \frac{\dddot{v}_t^2}{v_\infty^2} A_{10} + \frac{\dddot{v}_t^3}{v_\infty^3} A_{20} + \frac{\dddot{v}_t^3}{v_\infty^3} A_{30} \right\}.
\]

As can be seen in Figure 4.1, the inclined cable is attached to a sliding damper at \( x = L \) and to a pylon at \( x = 0 \). The boundary conditions are given up to order \( \epsilon^2 \) by

\[
\dddot{u}(0,t) = EI_y \dddot{v}_{xx}(0,t) = \dddot{v}(0,t) = \dddot{u}(L,t) = EI_y \dddot{v}_x(L,t) = 0,
\]
and
\[ EI_y \ddot{v}_{xx}(L, t) = [T_0 + AMLg\sin(\gamma)] \ddot{v}_x(L, t) + \lambda \tilde{v}_t(L, t), \] (4.47)
and the initial conditions are
\[ \begin{align*}
\tilde{u}(x, 0) &= u_0(x), \quad \tilde{u}_t(x, 0) = u_1(x), \\
\tilde{v}(x, 0) &= v_0(x), \quad \tilde{v}_t(x, 0) = v_1(x).
\end{align*} \] (4.48) (4.49)

In order to put the equations in a non-dimensional form the following dimensionless quantities are used:
\[ \begin{align*}
x^* &= x/L, \quad t^* = t/L \sqrt{T_0/AM}, \quad \tilde{v}^*(x^*, t^*) = \frac{\tilde{v}(x, t)}{L}, \quad \tilde{v}_t^*(x^*) = \frac{\tilde{v}(x)}{L}, \\
\gamma_1^* &= \frac{\gamma_1 L}{T_0}, \quad v_1^*(x^*) = \sqrt{AM/T_0} v_1(x), \quad \Omega_1^* = \frac{\Omega_1 L}{\sqrt{AM/T_0}}, \quad \mu = \frac{EI_y}{T_0 L^2}, \quad \lambda^* = \frac{\lambda}{\sqrt{AMT_0}}.
\end{align*} \]

Then Eq. (4.45) in a non-dimensional form becomes
\[ \begin{align*}
\mu \ddot{v}_{xxx} + \ddot{v}_t - \dddot{v}_{xx} &= \eta_1 (\ddot{v}_{xx} + \ddot{v}_{xx}) \left[ \int_0^1 \left( \dddot{v}_x \dddot{v}_x + \frac{\dddot{v}_x^2}{2} \right) dx \right] \\
&+ \left[ A \Omega_1 \cos(\gamma_1 x - \Omega_1 t) \right] \dddot{v}_t + \eta_{10} \dddot{v}_t \\
&- \left[ A \sin(\gamma_1 x - \Omega_1 t) \right] \dddot{v}_t + \eta_{20} \dddot{v}_t^2 + \eta_{30} \dddot{v}_t^3 \\
&- \left[ A \sin(\gamma_1 x - \Omega_1 t) \right] \eta_2 (\dddot{v}_x + \dddot{v}_x) - \eta_2 \dddot{v}_x \\
&+ \left[ A \cos(\gamma_1 x - \Omega_1 t) - \tilde{A} \cos(\gamma_1 L - \Omega_1 t) \right] \frac{\eta_2}{\gamma_1} (\dddot{v}_{xx} + \dddot{v}_{xx}) \\
&+ \left[ \eta_2 (1 - x) + \eta_4 \int_0^1 \dddot{v}_x^2 dx \right] \dddot{v}_{xx} + \left[ A \sin(\gamma_1 x - \Omega_1 t) \right] \eta_3,
\end{align*} \] (4.50)

with the boundary conditions
\[ \begin{align*}
\tilde{v}(0, t) &= \ddot{v}_{xx}(0, t) = \ddot{v}_x(1, t) = 0, \\
\mu \ddot{v}_{xx}(1, t) &= \dddot{v}_x(1, t) + \epsilon \left[ \tilde{A} \dddot{v}_t(1, t) + \eta_2 \dddot{v}_x(1, t) \right],
\end{align*} \] (4.51) (4.52)

and the initial conditions
\[ \begin{align*}
\ddot{v}(x, 0) &= v_0(x), \quad \dddot{v}_t(x, 0) = v_1(x),
\end{align*} \] (4.53)

where \( \tilde{v}_1 = \frac{EA}{T_0} \) is \( O(1) \), \( \tilde{v}_{10} = -\frac{1}{2\sqrt{AMv_\infty}} \rho_a d L^2 v_\infty A_{10} > 0 \) is \( O(\epsilon) \), \( \tilde{v}_{20} = -\frac{1}{2AM} \rho_a d L^2 A_{20} \) is \( O(\epsilon) \), \( \tilde{v}_{30} = -\frac{1}{2AMv_\infty} \sqrt{\frac{T_0}{AM}} \rho_a d L^2 A_{30} \) is \( O(\epsilon) \), \( \tilde{v}_2 = \frac{AMLg \sin(\alpha)}{T_0} \) is \( O(\epsilon) \), \( \tilde{v}_3 = \frac{AM}{T_0} g \cos(\alpha) \) is \( O(\epsilon) \), and \( \tilde{v}_4 = \frac{E}{2T_0 L} \) is \( O(1) \). The damping coefficient \( \lambda \) is assumed to be of \( O(\epsilon) \), that is, \( \lambda = \epsilon \tilde{\lambda} \). We also assume that \( \ddot{v}(x, t) = \epsilon v(x, t) \), \( \tilde{v}_{10} = \epsilon \tilde{v}_{10} \), \( \tilde{v}_2 = \epsilon \tilde{v}_2 \), \( \tilde{v}_3 = \epsilon \tilde{v}_3 \), and \( \tilde{A} = \epsilon \tilde{\sigma} \). The asterisks indicating the dimensionless quantities.
are omitted in Eq. (4.50) through Eq. (4.53), and henceforth for convenience. Then, by using these assumptions, Eq. (4.50) up to order $\epsilon^2$ becomes

$$\mu v_{xxxx} + v_{tt} - v_{xx} = \epsilon \left[ \eta_{10} + \sigma \Omega_1 \cos(\gamma_1 x - \Omega_1 t) \right] v_t$$

$$- \sigma \sin(\gamma_1 x - \Omega_1 t) v_{tt} + \eta_2 (1 - x) v_{xx}$$

$$- \eta_2 v_x + \sigma \sin(\gamma_1 x - \Omega_1 t) \eta_3 \right], \ t > 0, \ 0 < x < 1,$$

with the boundary conditions

$$v(0, t; \epsilon) = v_{xx}(0, t; \epsilon) = v_x(1, t; \epsilon) = 0,$$

$$\mu v_{xx}(1, t; \epsilon) = v_x(1, t; \epsilon) + \epsilon [\tilde{\lambda} v_t(1, t; \epsilon) + \eta_2 v_x(1, t; \epsilon)],$$

and the initial conditions

$$v(x, 0; \epsilon) = v_0(x), \ v_t(x, 0; \epsilon) = v_1(x).$$

In the following section the initial-boundary value problem Eq. (4.54)-(4.57) will be studied further.

### 4.4 Application of the two-timescales perturbation method

In this section, the initial-boundary value problem Eq. (4.54)-(4.57) will be studied and an approximation of the solution of the initial-boundary value problem up to order $\epsilon$ will be constructed by using a two-timescales perturbation method. We assume that $v(x, t)$ can be expanded in a formal power series in $\epsilon$, that is,

$$v(x, t; \epsilon) = v_0(x, t) + \epsilon v_1(x, t) + \epsilon^2 v_2(x, t) + \cdots,$$

where all $v_i(x, t)$ and their derivatives for $i = 0, 1, 2, \cdots$ are $O(1)$ on a time-scale of order $\epsilon^{-1}$. The approximation of the solution may have secular terms which are unbounded terms in time. In order to avoid these secular terms, we will apply the two-timescales perturbation method by introducing a slow time-scale $\tau = \epsilon t$, and

$$v(x, t; \epsilon) = y(x, t, \tau; \epsilon).$$

The following transformations are needed for the time derivatives

$$v_t = y_t + \epsilon y_{\tau},$$

$$v_{tt} = y_{tt} + 2\epsilon y_{t\tau} + \epsilon^2 y_{\tau\tau}.$$  

Substitution of Eq. (4.59)-(4.61) into Eq. (4.54) yields:

$$\mu y_{xxxx} + y_{tt} - y_{xx} = \epsilon \left[ -2 y_{t\tau} + \left[ \eta_{10} + \sigma \Omega_1 \cos(\gamma_1 x - \Omega_1 t) \right] y_t$$

$$- \sigma \sin(\gamma_1 x - \Omega_1 t) y_{tt} + \eta_2 (1 - x) y_{xx} - \eta_2 y_x$$

$$+ \sigma \sin(\gamma_1 x - \Omega_1 t) \eta_3 \right] + O(\epsilon^2),$$

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with the boundary conditions
\[ y(0, t, \tau; \epsilon) = y_{xx}(0, t, \tau; \epsilon) = y_x(1, t, \tau; \epsilon) = 0, \quad (4.63) \]
\[ \mu y_{xxx}(1, t, \tau; \epsilon) = y_x(1, t, \tau; \epsilon) + \epsilon [\lambda(y_t(1, t, \tau; \epsilon)
+ \epsilon y_\tau(1, t, \tau; \epsilon)] + \eta_2 y_x(1, t, \tau; \epsilon)], \quad (4.64) \]
and the initial conditions
\[ y(x, 0, 0; \epsilon) = v_0(x), \quad y_t(x, 0, 0; \epsilon) + \epsilon y_\tau(x, 0, 0; \epsilon) = v_1(x). \quad (4.65) \]

Assuming that
\[ y(x, t, \tau; \epsilon) = y_0(x, t, \tau; \epsilon) + \epsilon y_1(x, t, \tau; \epsilon) + \epsilon^2 y_2(x, t, \tau; \epsilon) + \cdots, \quad (4.66) \]
then by collecting terms of equal powers in \( \epsilon \), it follows from the problem for \( y(x, t, \tau; \epsilon) \) that the \( \mathcal{O}(1) \)-problem is
\[ \mu y_{0,xxx} + y_{0,t} - y_{0,xx} = 0, \quad (4.67) \]
\[ y_0(0, t, \tau) = y_{0,xx}(0, t, \tau) = 0, \quad (4.68) \]
\[ y_0(x, 1, \tau) = 0, \quad (4.69) \]
\[ \mu y_{0,xxx}(1, t, \tau) - y_{0,x}(1, t, \tau) = 0, \quad (4.70) \]
\[ y_0(x, 0, 0) = v_0(x), \quad \text{and} \quad y_{0,t}(x, 0, 0) = v_1(x), \quad (4.71) \]
and that the \( \mathcal{O}(\epsilon) \)-problem is
\[ \mu y_{1,xxx} + y_{1,t} - y_{1,xx} = -2y_{0,t} + \left[ \eta_1 + \sigma \Omega_1 \cos(\gamma_1 x - \Omega_1 t) \right] y_{0,t} \]
\[ - \sigma \sin(\gamma_1 x - \Omega_1 t) y_{0,tt} + \eta_2 (1 - x) y_{0,xx} \]
\[ - \eta_2 y_{0,x} + \sigma \sin(\gamma_1 x - \Omega_1 t) \eta_3, \quad (4.72) \]
\[ y_1(0, t, \tau) = y_{1,xx}(0, t, \tau) = 0, \quad (4.73) \]
\[ y_1(x, 1, \tau) = 0, \quad (4.74) \]
\[ \mu y_{1,xxx}(1, t, \tau) - y_{1,x}(1, t, \tau) = \lambda y_{0,t}(1, t, \tau) + \eta_2 y_{0,x}(1, t, \tau), \quad (4.75) \]
\[ y_1(x, 0, 0) = 0, \quad \text{and} \quad y_{1,t}(x, 0, 0) + y_{0,t}(x, 0, 0) = 0. \quad (4.76) \]

The method of separation of variables will be applied to solve the problem Eq. (4.67)-(4.71). The solution of the \( \mathcal{O}(1) \)-problem may be given in a special form
\[ y_0(x, t, \tau) = T(t, \tau) \phi(x). \quad (4.77) \]

By substitution of Eq. (4.77) into Eq. (4.67) and by dividing the so-obtained equation by \( T(t, \tau) \phi(x) \) yields:
\[ \frac{T_{tt}(t, \tau)}{T(t, \tau)} = \frac{\phi_{xx}(x)}{\phi(x)} - \mu \frac{\phi^{(iv)}(x)}{\phi(x)} = -\omega. \quad (4.78) \]

A separation constant is defined \(-\omega\) so that the time-dependent part of the product solution oscillates for real and positive eigenvalues (for the proof we refer the reader to [52]). We obtain a time-dependent part
\[ T_{tt}(t, \tau) + \omega T(t, \tau) = 0, \quad (4.79) \]
and the general solution of the time-dependent part is a linear combination of sines and cosines in \( t \),
\[
T(t, \tau) = \sigma_1(\tau)\cos(\sqrt{\omega}t) + \sigma_2(\tau)\sin(\sqrt{\omega}t),
\]
where \( \sigma_1 \) and \( \sigma_2 \) are arbitrary function in \( \tau \). In addition, we obtain a space-dependent part
\[
\phi_{xxx}(x) - \frac{1}{\mu} \phi_{xx}(x) - \frac{\omega}{\mu} \phi(x) = 0,
\]
and the boundary conditions Eq. (4.68)-(4.70) yield
\[
\phi(0) = \phi_{xx}(0) = \phi_x(1) = \mu \phi_{xxx}(1) - \phi_x(1) = 0.
\]

The characteristic equation for Eq. (4.81) is given by
\[
m^4 - \frac{m^2}{\mu} - \frac{\omega}{\mu} = 0,
\]
and the solutions of Eq. (4.81) are given by
\[
\phi(x) = c_1 \sinh(ax) + c_2 \cosh(ax) + c_3 \sin(bx) + c_4 \cos(bx),
\]
where \( c_i \) for \( i = 1, 2, 3, 4 \) are constants, and
\[
a = \sqrt{1 + \sqrt{1 + 4\mu^2 \omega^2}}, \quad b = \sqrt{-1 + \sqrt{1 + 4\mu^2 \omega^2}}.
\]
The non-trivial solutions are found by using the boundary conditions Eq. (4.82), leading to the characteristic equation
\[
f_\mu(\omega) = -\mu ab \cos(b) \cosh(a)(a^2 + b^2) - a \cosh(a) \sin(b) + b \cos(b) \sinh(a) = 0.
\]

It follows from Eq. (4.85) that the eigenvalues \( \omega_n = \mu b_n^4 + b_n^2 \) can be numerically computed for given values of \( \mu \). The first ten eigenvalues \( \omega_n \) are listed in Table 1. The eigenfunctions belonging to different eigenvalues are orthogonal with respect to the inner product; for details the reader is referred to [52],
\[
< \phi_m(x), \phi_n(x) > = \int_0^1 \phi_m(x) \phi_n(x) \, dx,
\]
and the eigenfunctions of the problem Eq. (4.81)-(4.82) can be determined and are given by
\[
\phi_n(x) = \theta_n \sinh(a_n x) + \sin(b_n x),
\]
where \( \theta_n = -\frac{b_n \cos(b_n)}{a_n \cosh(a_n)}, \quad a_n = \sqrt{1 + \sqrt{1 + 4\mu \omega_n^2}}, \quad b_n = \sqrt{-1 + \sqrt{1 + 4\mu \omega_n^2}}. \)

Hence, infinitely many non-trivial solutions of the initial-boundary problem Eq. (4.67)-(4.71) have been determined. By using the superposition principle, the solution is obtained
\[
y_0(x, t, \tau) = \sum_{n=1}^{\infty} \left[ A_n(\tau) \cos(\sqrt{\omega_n}t) + B_n(\tau) \sin(\sqrt{\omega_n}t) \right] \phi_n(x).
\]
where \( \phi_n(x) \) is given by Eq. (4.87), and where \( A_n \) and \( B_n \) are arbitrary functions in \( \tau \) which can be used to avoid secular terms in \( y_1(x, t, \tau) \). By using the initial conditions Eq. (4.71) we can determine \( A_n(0) \) and \( B_n(0) \), which are given by
\[
A_n(0) = \frac{1}{\zeta_n} \int_0^1 v_0(x) \phi_n(x) \, dx,
\]
and
\[
B_n(0) = \frac{1}{\zeta_n} \int_0^1 v_0(x) \phi_n(x) \, dx.
\]
and
\[
\sqrt{\omega_n} B_n(0) = \frac{1}{\zeta_n} \int_0^1 v_1(x) \phi_n(x) \, dx,
\]
where
\[
\zeta_n = \int_0^1 \phi_n^2(x) \, dx.
\]

\[\text{(4.90)}\]

\[\text{(4.91)}\]

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \mu = 0.001 )</th>
<th>( \mu = 0.01 )</th>
<th>( \mu = 0.1 )</th>
<th>( \mu = 1 )</th>
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<tbody>
<tr>
<td>( \omega_1 )</td>
<td>4.16497</td>
<td>4.30777</td>
<td>5.00501</td>
<td>10.54607</td>
</tr>
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<td>( \omega_2 )</td>
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<td>29.20039</td>
<td>73.56205</td>
<td>517.34633</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>67.51921</td>
<td>101.78930</td>
<td>444.20353</td>
<td>3868.72929</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>137.55593</td>
<td>269.11062</td>
<td>17807.12342</td>
<td>174300.30645</td>
</tr>
<tr>
<td>( \omega_5 )</td>
<td>241.83773</td>
<td>601.31819</td>
<td>4196.24404</td>
<td>40145.67525</td>
</tr>
<tr>
<td>( \omega_6 )</td>
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<td>1191.92269</td>
<td>9214.09788</td>
<td>89435.96197</td>
</tr>
<tr>
<td>( \omega_7 )</td>
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<td>2157.81266</td>
<td>17807.12342</td>
<td>174300.30645</td>
</tr>
<tr>
<td>( \omega_8 )</td>
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<td>3639.25581</td>
<td>31378.01114</td>
<td>308765.61727</td>
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<tr>
<td>( \omega_9 )</td>
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<td>5799.89869</td>
<td>51563.23375</td>
<td>509196.62255</td>
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<tr>
<td>( \omega_{10} )</td>
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<td>8826.76635</td>
<td>80233.04556</td>
<td>794295.86667</td>
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</table>

\[\text{Table 4.1: Some eigenvalues } \omega_i \text{ which are roots of Eq. (4.85) with } \omega_i \text{ the } i\text{-th root.}\]

Now we solve the \( O(\epsilon) \)-problem Eq. (4.72)-(4.76). Due to having an inhomogeneous boundary condition Eq. (4.75), we use the following transformation to convert the problem into a problem with homogeneous boundary conditions
\[
y_1(x, t, \tau) = V(x, t, \tau) + (x^4 - 2x^3 \frac{12\mu + 2}{12\mu + 2}) h(t, \tau),
\]
where
\[
h(t, \tau) = \tilde{\lambda} y_{0t}(1, t, \tau) + \eta_2 y_{0x}(1, t, \tau).
\]

\[\text{(4.92)}\]

\[\text{(4.93)}\]

Substituting Eq. (4.92) into Eq. (4.72)-(4.76), we obtain
\[
\mu V_{xxxx} + V_{tt} - V_{xx} = -2y_{0tt} + \left[ \eta_{10} + \sigma \Omega_1 \cos(\gamma_1 x - \Omega_1 t) \right] y_{0t} - \eta_2 y_{0x} \]
\[
- \sigma \sin(\gamma_1 x - \Omega_1 t) y_{0tt} + \eta_2 (1 - x) y_{0xx} + \sigma \sin(\gamma_1 x - \Omega_1 t) \eta_3
\]
\[
+ \left( \frac{12x^2 - 12x - 24\mu}{12\mu + 2} \right) h(t, \tau) - \left( \frac{x^4 - 2x^3}{12\mu + 2} \right) h_{tt}(t, \tau),
\]
\[
V(0, t, \tau) = V_{xx}(0, t, \tau) = V_x(1, t, \tau) - \left( \frac{2}{12\mu + 2} \right) h(t, \tau) = 0,
\]
\[
\mu V_{xxx}(1, t, \tau) - V_x(1, t, \tau) = 0,
\]
\[\text{(4.94)}\]

\[\text{(4.95)}\]

\[\text{(4.96)}\]
\[ V(x, 0, 0) = -\left(\frac{x^4 - 2x^3}{12\mu + 2}\right) h(0, 0), \]  
\[ V_t(x, 0, 0) = -y_0 \tau(x, 0, 0) - \left(\frac{x^4 - 2x^3}{12\mu + 2}\right) h_t(0, 0), \]  
where \( h(t, \tau) \) is given by Eq. (4.93), and where \( h(0, 0) \) and \( h_t(0, 0) \) are given by

\[ h(0, 0) = \tilde{\lambda} v_1(1) + \eta_2 v_0 x(1), \]
\[ h_t(0, 0) = \tilde{\lambda} \left[ v_0 xx(1) - \mu v_0 xxxx(1) \right] + \eta_2 v_1 x(1). \]

In order to solve Eq. (4.94)-(4.98), \( V(x, t, \tau) \) is written in the following eigenfunction expansion

\[ V(x, t, \tau) = \sum_{m=1}^{\infty} V_m(t, \tau) \phi_m(x), \]  
and by substituting Eq. (4.101) into the partial differential equation Eq. (4.94), we obtain

\[ \sum_{m=1}^{\infty} [V_{m,t,t}(t, \tau) + \omega_m V_m(t, \tau)] \phi_m(x) = -2y_0 \tau \]
\[ + \left[ \eta_{10} + \sigma \Omega_1 \cos(\gamma_1 x - \Omega_1 t) \right] y_0 t \]
\[ - \eta_2 y_0 x - \sigma \sin(\gamma_1 x - \Omega_1 t) y_0 tt + \eta_2 (1 - x) y_0 xx \]
\[ + \sigma \sin(\gamma_1 x - \Omega_1 t) \eta_3 + \left(\frac{12x^2 - 12x - 24\mu}{12\mu + 2}\right) h(t, \tau) \]
\[ - \left(\frac{x^4 - 2x^3}{12\mu + 2}\right) h_{tt}(t, \tau). \]

We expand \( \frac{x^4 - 2x^3}{12\mu + 2} \) and \( \frac{12x^2 - 12x - 24\mu}{12\mu + 2} \) into a series of eigenfunctions \( \phi_m(x) \), we obtain

\[ \frac{x^4 - 2x^3}{12\mu + 2} = \sum_{m=1}^{\infty} c_m \phi_m(x), \]
\[ \frac{12x^2 - 12x - 24\mu}{12\mu + 2} = \sum_{m=1}^{\infty} d_m \phi_m(x), \]
where

\[ c_m = \frac{1}{\zeta_m} \int_0^1 \frac{x^4 - 2x^3}{12\mu + 2} \phi_m(x) dx, \]
\[ d_m = \frac{1}{\zeta_m} \int_0^1 \frac{12x^2 - 12x - 24\mu}{12\mu + 2} \phi_m(x) dx, \]
and where \( \zeta_m \) is given by Eq. (4.91). By multiplying both sides of Eq. (4.102) by \( \phi_n(x) \), and then by integrating from \( x = 0 \) to \( x = 1 \), and by using the orthogonality properties of the eigenfunctions, we obtain
\[
\left[ V_{\tau\tau}(t, \tau) + \omega_n V_n(t, \tau) \right] = -c_n h_{\tau\tau}(t, \tau) + d_n h(t, \tau) - 2T_{n\tau}(t, \tau) \quad (4.107)
\]
\[
+ \frac{\eta_2}{\zeta_n} T_n(t, \tau)(\Phi_{nn} - \Phi_{nn}) \\
+ \frac{\eta_2}{\zeta_n} \sum_{m=1, m \neq n}^{\infty} T_m(t, \tau)(\Phi_{mn} - \Phi_{mn}) \\
+ \frac{\sigma \eta_3}{\zeta_n} \left( \cos(\Omega_1 t) \hat{\Upsilon}_n - \sin(\Omega_1 t) \Upsilon_n \right) + \eta_{10} T_{n\tau}(t, \tau) \\
+ \frac{\sigma \Omega_1}{\zeta_n} T_{n\tau}(t, \tau) \left( \cos(\Omega_1 t) \Psi_{nn} + \sin(\Omega_1 t) \hat{\Psi}_{nn} \right) \\
+ \frac{\sigma}{\zeta_n} \sum_{m=1, m \neq n}^{\infty} \left[ \Omega_1 T_{m\tau}(t, \tau) \left( \cos(\Omega_1 t) \Psi_{mn} + \sin(\Omega_1 t) \hat{\Psi}_{mn} \right) \\
+ T_{m\tau}(t, \tau) \left( \sin(\Omega_1 t) \Psi_{mn} - \cos(\Omega_1 t) \hat{\Psi}_{mn} \right) \right],
\]

where

\[
T_m(t, \tau) = A_m(\tau) \cos(\sqrt{\omega_m} t) + B_m(\tau) \sin(\sqrt{\omega_m} t), \quad (4.108)
\]

and

\[
\Phi_{mn} = \int_0^1 \frac{d\phi_m(x)}{dx} \phi_n(x) dx, \quad (4.109)
\]
\[
\hat{\Phi}_{mn} = \int_0^1 (1 - x) \frac{d^2\phi_m(x)}{dx^2} \phi_n(x) dx, \quad (4.110)
\]
\[
\Psi_{mn} = \int_0^1 \cos(\gamma_1 x) \phi_m(x) \phi_n(x) dx, \quad (4.111)
\]
\[
\hat{\Psi}_{mn} = \int_0^1 \sin(\gamma_1 x) \phi_m(x) \phi_n(x) dx, \quad (4.112)
\]
\[
\Upsilon_m = \int_0^1 \cos(\gamma_1 x) \phi_m(x) dx, \quad (4.113)
\]
\[
\hat{\Upsilon}_m = \int_0^1 \sin(\gamma_1 x) \phi_m(x) dx. \quad (4.114)
\]

It follows from Eq. (4.93) and Eq. (4.88) that \( h(t, \tau) \) and \( h_{\tau\tau}(t, \tau) \) can be written as

\[
h(t, \tau) = \tilde{\lambda} \sum_{m=1}^{\infty} T_{m\tau}(t, \tau) \phi_m(1) + \eta_2 \sum_{m=1}^{\infty} T_m(t, \tau) \frac{d\phi_m(1)}{dx}, \quad (4.115)
\]
\[
h_{\tau\tau}(t, \tau) = \tilde{\lambda} \sum_{m=1}^{\infty} T_{m\tau\tau}(t, \tau) \phi_m(1) + \eta_2 \sum_{m=1}^{\infty} T_{m\tau\tau}(t, \tau) \frac{d\phi_m(1)}{dx}. \quad (4.116)
\]
Hence, by using Eq. (4.115) and Eq. (4.116) it follows that Eq. (4.107) can be rewritten as

\[
\left[ V_{nt}(t, \tau) + \omega_n V_n(t, \tau) \right] =
\]

\[
c_n \sum_{m=1}^{\infty} \left\{ \sin(\sqrt{\omega_m}t) \left[ -\tilde{\lambda}\phi_m(1)A_m(\tau)\sqrt{\omega_m^2} + \eta_2 \frac{d\phi_m(1)}{dx}B_m(\tau)\omega_m \right] \
+ \cos(\sqrt{\omega_m}t) \left[ \tilde{\lambda}\phi_m(1)B_m(\tau)\sqrt{\omega_m^2} + \eta_2 \frac{d\phi_m(1)}{dx}A_m(\tau)\omega_m \right] \right\} \\
+ d_n \sum_{m=1}^{\infty} \left\{ \sin(\sqrt{\omega_m}t) \left[ -\tilde{\lambda}\phi_m(1)A_m(\tau)\sqrt{\omega_m} + \eta_2 \frac{d\phi_m(1)}{dx}B_m(\tau) \right] \
+ \cos(\sqrt{\omega_m}t) \left[ \tilde{\lambda}\phi_m(1)B_m(\tau)\sqrt{\omega_m^2} + \eta_2 \frac{d\phi_m(1)}{dx}A_m(\tau) \right] \right\} \\
+ \frac{\eta_2}{\zeta_n} \sum_{m=1, m \neq n}^{\infty} (\Phi_{mn} - \Phi_{nn}) \left[ A_m(\tau)\cos(\sqrt{\omega_m}t) + B_m(\tau)\sin(\sqrt{\omega_m}t) \right] \\
+ \sin(\sqrt{\omega_n}t) \left\{ 2\sqrt{\omega_n} \frac{dA_n(\tau)}{d\tau} - A_n(\tau) \left[ \eta_10 \sqrt{\omega_n} + \tilde{\lambda}\phi_n(1)\sqrt{\omega_n}(\zeta_n \omega_n + d_n) \right] \
+ B_n(\tau) \left[ \frac{\eta_2}{\zeta_n} (\Phi_{nn} - \Phi_{nn}) + \eta_2 \frac{d\phi_n(1)}{dx}(\zeta_n \omega_n + d_n) \right] \right\} \\
+ \frac{\eta_2}{\zeta_n} \left( \tilde{\lambda}\phi_n(1)\sqrt{\omega_n}(\zeta_n \omega_n + d_n) \right) \\
+ \frac{\sigma}{\zeta_n} \left( \cos(\Omega_1 t) \tilde{\gamma}_n - \sin(\Omega_1 t) \gamma_n \right) \\
+ \sin(\Omega_1 t + \sqrt{\omega_n}t) \frac{\sigma}{2\zeta_n} (\Omega_1 \sqrt{\omega_n} + \omega_n) \left[ -A_n(\tau)\Psi_{nn} + B_n(\tau)\hat{\Psi}_{nn} \right] \\
+ \sin(\Omega_1 t - \sqrt{\omega_n}t) \frac{\sigma}{2\zeta_n} (\Omega_1 \sqrt{\omega_n} - \omega_n) \left[ A_n(\tau)\Psi_{nn} + B_n(\tau)\hat{\Psi}_{nn} \right] \\
+ \cos(\Omega_1 t + \sqrt{\omega_n}t) \frac{\sigma}{2\zeta_n} (\Omega_1 \sqrt{\omega_n} + \omega_n) \left[ A_n(\tau)\hat{\Psi}_{nn} + B_n(\tau)\Psi_{nn} \right] \\
+ \cos(\Omega_1 t - \sqrt{\omega_n}t) \frac{\sigma}{2\zeta_n} (\Omega_1 \sqrt{\omega_n} - \omega_n) \left[ -A_n(\tau)\hat{\Psi}_{nn} + B_n(\tau)\Psi_{nn} \right] \\
+ \frac{\sigma}{2\zeta_n} \sum_{m=1, m \neq n}^{\infty} \left\{ \sin(\Omega_1 t + \sqrt{\omega_m}t)(\Omega_1 \sqrt{\omega_m} + \omega_m) \left[ -A_m(\tau)\Psi_{mn} + B_m(\tau)\hat{\Psi}_{mn} \right] \
+ \sin(\Omega_1 t - \sqrt{\omega_m}t)(\Omega_1 \sqrt{\omega_m} - \omega_m) \left[ A_m(\tau)\Psi_{mn} + B_m(\tau)\hat{\Psi}_{mn} \right] \
+ \cos(\Omega_1 t + \sqrt{\omega_m}t)(\Omega_1 \sqrt{\omega_m} + \omega_m) \left[ A_m(\tau)\hat{\Psi}_{mn} + B_m(\tau)\Psi_{mn} \right] \
+ \cos(\Omega_1 t - \sqrt{\omega_m}t)(\Omega_1 \sqrt{\omega_m} - \omega_m) \left[ -A_m(\tau)\hat{\Psi}_{mn} + B_m(\tau)\Psi_{mn} \right] \right\} \right].
4.4. APPLICATION OF THE TWO-TIMESCALES PERTURBATION METHOD

It can be easily seen from Eq. (4.117) that there are infinitely many values of \( \Omega_1 \), which can cause internal resonances. The possible resonance cases are as follow:

(i) The non-resonant case: if \( \Omega_1 \) is not within an order \( \epsilon \)-neighbourhood of the frequencies \( \sqrt{\omega_n} \pm \sqrt{\omega_m} \) for all \( m \) and \( n \).

(ii) The near resonance case (Resonance detuning): if \( \Omega_1 \) is within an order \( \epsilon \)-neighbourhood of \( \sqrt{\omega_n} \pm \sqrt{\omega_m} \) for certain fixed \( m \) and \( n \).

(iii) The pure resonance case: if \( \Omega_1 = \sqrt{\omega_n} + \sqrt{\omega_m} \) for certain fixed \( m \) and \( n \). This is the sum type resonance case. If \( \Omega_1 = \sqrt{\omega_n} - \sqrt{\omega_m} \) for certain fixed \( m \) and \( n \), it is the difference type resonance case.

In the following subsections, we will study the non-resonant case, the sum type resonance case, the difference type resonance case and by considering only some of the first few modes and omitting the higher order modes.

For simplicity we will now assume that \( \Omega_1 = \sqrt{\omega_1} + \sqrt{\omega_2} \) (or \( \Omega_1 = \sqrt{\omega_2} - \sqrt{\omega_1} \)). Resonance can occur when \( \Omega_1 = \pm \sqrt{\omega_n} \pm \sqrt{\omega_m} \) for some \( n \) and \( m \). We have to consider the following three cases:

(i) \( \Omega_1 = -\sqrt{\omega_n} - \sqrt{\omega_m} \), which has no solution since the right-hand side is negative while the left-hand side is positive.

(ii) \( \Omega_1 = \sqrt{\omega_n} + \sqrt{\omega_m} \), which has only the trivial solution \( m = 2, n = 1 \) or \( m = 1, n = 2 \).

(iii) \( \Omega_1 = \sqrt{\omega_n} - \sqrt{\omega_m} \) (or equivalently \( \Omega_1 = \sqrt{\omega_m} - \sqrt{\omega_n} \)), which has only solutions by looking at the Table 4.1 for certain values of \( \mu \). There are possibilities that three modes are interacting. For example, \( \sqrt{\omega_1} + \sqrt{\omega_2} \cong \sqrt{\omega_4} - \sqrt{\omega_2} \) for \( \mu = 0.001 \).

<table>
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<th>( \mu = 0.01 )</th>
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<td>( \tilde{\rho}_n )</td>
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CHAPTER 4. ON BOUNDARY DAMPING TO REDUCE THE RAIN-WIND OSCILLATIONS OF AN INCLINED CABLE

Figure 4.3: Transverse displacements $y_0$ at (a) $x = 0.5$ and at (b) $x = 1$ against time $t$ with the initial displacement $v_0(x) = 0.1\sin(\pi x)$ and the initial velocity $v_1(x) = 0.05\sin(\pi x)$ for $\mu = 0.001$, $\eta_{10} = \eta_2 = \epsilon = 0.1$, $\lambda = 0.5$, and $N = 10$.

Figure 4.4: Transverse displacements $y_0$ at (a) $x = 0.5$ and at (b) $x = 1$ against time $t$ with the initial displacement $v_0(x) = 0.1\sin(\pi x)$ and the initial velocity $v_1(x) = 0.05\sin(\pi x)$ for $\mu = 0.001$, $\eta_{10} = \eta_2 = \epsilon = 0.1$, $\lambda = 0.05$, and $N = 10$. 
4.4. APPLICATION OF THE TWO-TIMESCALES PERTURBATION METHOD

Figure 4.5: Transverse displacements $y_0$ at (a) $x = 0.5$ and at (b) $x = 1$ against time $t$ with the initial displacement $v_0(x) = 0.1\sin(\pi x)$ and the initial velocity $v_1(x) = 0.05\sin(\pi x)$ for $\mu = 1$, $\eta_{10} = \eta_{2} = \epsilon = 0.1$, $\lambda = 0.5$, and $N = 10$.

Figure 4.6: Transverse displacements $y_0$ at (a) $x = 0.5$ and at (b) $x = 1$ against time $t$ with the initial displacement $v_0(x) = 0.1\sin(\pi x)$ and the initial velocity $v_1(x) = 0.05\sin(\pi x)$ for $\mu = 1$, $\eta_{10} = \eta_{2} = \epsilon = 0.1$, $\lambda = 0.05$, and $N = 10$. 
4.4.1 The non-resonant case

When $\Omega_1 \neq \sqrt{\omega_n} \pm \sqrt{\omega_m} + \mathcal{O}(\epsilon)$ for all $m$ and $n$ only resonances occur due to the fourth and fifth term in the right hand side of Eq. (4.117), that is, Eq. (4.117) can be rewritten as

$$\left[V_{nt}(t,\tau)+\omega_n V_n(t,\tau)\right] = \sin(\sqrt{\omega_n}t)\left\{2\sqrt{\omega_n} \frac{dA_n(\tau)}{d\tau} - A_n(\tau)\left[\eta_{10}\sqrt{\omega_n} + \bar{\lambda}\phi_n(1)\sqrt{\omega_n}(c_n\omega_n + d_n)\right] + B_n(\tau)\left[\eta_2\Phi_{nn} - \Phi_{nn} + \eta_2\frac{d\phi_n(1)}{dx}(c_n\omega_n + d_n)\right]\right\}$$

$$+ \cos(\sqrt{\omega_n}t)\left\{-2\sqrt{\omega_n} \frac{dB_n(\tau)}{d\tau} + B_n(\tau)\left[\eta_{10}\sqrt{\omega_n} + \bar{\lambda}\phi_n(1)\sqrt{\omega_n}(c_n\omega_n + d_n)\right]ight\} + "NST"$$

where “NST” stands for terms that lead to nonsecular terms in $V_n$. In order to remove secular terms, it follows from Eq. (4.118) that $A_n(\tau)$ and $B_n(\tau)$ have to satisfy

$$\frac{dA_n(\tau)}{d\tau} - A_n(\tau)X_n + B_n(\tau)Y_n = 0,$$

$$\frac{dB_n(\tau)}{d\tau} - B_n(\tau)X_n - A_n(\tau)Y_n = 0,$$

where $X_n$ and $Y_n$ are defined by

$$X_n = \left[\frac{\eta_{10}}{2} + \frac{\bar{\lambda}}{2}\phi_n(1)(c_n\omega_n + d_n)\right],$$

$$Y_n = \left[\frac{\eta_2}{2\sqrt{\omega_n}}(\Phi_{nn} - \Phi_{nn}) + \frac{\eta_2}{2\sqrt{\omega_n}}\frac{d\phi_n(1)}{dx}(c_n\omega_n + d_n)\right].$$

The solution of Eq. (4.119) and Eq. (4.120) is given by

$$A_n(\tau) = \left[A_n(0)\cos(Y_n \tau) - B_n(0)\sin(Y_n \tau)\right]e^{X_n \tau},$$

$$B_n(\tau) = \left[A_n(0)\sin(Y_n \tau) + B_n(0)\cos(Y_n \tau)\right]e^{X_n \tau},$$

where $A_n(0)$ and $B_n(0)$ are given by Eq. (4.89) and Eq. (4.90), respectively. By using Eq. (4.105) and Eq. (4.106) with Eq. (4.81) and Eq. (4.82) it can easily be shown that $\frac{1}{2}(c_n\omega_n + d_n) = \frac{\mu}{\sqrt{c_n(12\mu+2)}} \frac{d^2\phi_n(1)}{dx^2} - \frac{1}{2}\bar{\lambda}\phi_n(1)$. Hence Eq. (4.108) with Eq. (4.123) and Eq. (4.124) can be rewritten as

$$T_m(t,\tau) = e^{X_n \tau}\left[A_m(0)\cos(\sqrt{\omega_m}t - Y_n \tau) + B_m(0)\sin(\sqrt{\omega_m}t - Y_n \tau)\right].$$
Now, from Eq. (4.118) with Eq. (4.119)-(4.125), we can obtain $V_n(t, \tau)$ straightforwardly. Obviously, $V_n(t, \tau)$ will be bounded on a time-scale of order $1/\epsilon$, and so will be $V_n(x, t, \tau)$.

In Table 4.2 numerical approximations of $\omega_n$ and the damping parameter $\tilde{p}_n$ are given for different values of $\mu$. For different bending stiffness, we can choose the damping parameter in such a way that all modes are damped by taking $\tilde{\lambda}$ appropriately compared to the $\eta_{10}$ coefficient. It is also clear to see from Table 4.2 that for smaller values of $\mu$ we should take larger $\tilde{\lambda}$ to have damping for all modes. In Table 4.3, it can be seen as expected that the bending stiffness $\mu$ and the damping parameter $\tilde{\lambda}$ influence the stability.

For $\mu = 0.001$, Figure. 4.3 and Figure. 4.4 demonstrate the vibration response at the middle $x = 0.5$ and at the end $x = 1$ points of the beam for different values of the damping factor $\tilde{\lambda}$, Figure. 4.3 is plotted for $\tilde{\lambda} = 0.5$ and Figures. 4.4 is plotted for $\tilde{\lambda} = 2$; the initial conditions are specified as $v_0(x) = 0.1\sin(\pi x)$ and $v_1(x) = 0.05\sin(\pi x)$. For $\mu = 1$, Figure. 4.5 and Figure. 4.6 illustrate similar behaviour. These figures show that the amplitudes of the transverse vibrations decrease faster for increasing of $\mu$ and $\tilde{\lambda}$. Similar results for the string-like problem have been observed in [20].

### 4.4.2 The sum type resonance case: $\Omega_1 = \sqrt{\omega_2} + \sqrt{\omega_1}$

We will consider now only the first two modes and omit the higher order modes. That is, we will assume that for the given $\mu$ values only these two modes might occur in a resonance interaction. Eq. (4.117) can be rewritten as

$$
\begin{align*}
[V_{1tt}(t, \tau) + \omega_1 V_1(t, \tau)] &= \sin(\sqrt{\omega_1}t) \left\{ 2\sqrt{\omega_1} \frac{dA_1(\tau)}{d\tau} ight. \\
&\quad - A_1(\tau) \left[ \eta_{10} \sqrt{\omega_1} + \tilde{\lambda} \phi_1(1) \sqrt{\omega_1} (c_1 \omega_1 + d_1) \right] \\
&\quad + B_1(\tau) \left[ \frac{\eta_2}{\zeta_1} (\dot{\Phi}_{11} - \Phi_{11}) + \eta_2 \frac{d\phi_1(1)}{dx} (c_1 \omega_1 + d_1) \right] \} \\
&\quad + \cos(\sqrt{\omega_1}t) \left\{ - 2\sqrt{\omega_1} \frac{dB_1(\tau)}{d\tau} ight. \\
&\quad + B_1(\tau) \left[ \eta_{10} \sqrt{\omega_1} + \tilde{\lambda} \phi_1(1) \sqrt{\omega_1} (c_1 \omega_1 + d_1) \right] \\
&\quad + A_1(\tau) \left[ \frac{\eta_2}{\zeta_1} (\dot{\Phi}_{11} - \Phi_{11}) + \eta_2 \frac{d\phi_1(1)}{dx} (c_1 \omega_1 + d_1) \right] \} \\
&\quad + \sin(\Omega_1 t - \sqrt{\omega_2} t) \frac{\sigma}{2\zeta_1} (\Omega_1 \sqrt{\omega_2} - \omega_2) \left[ A_2(\tau) \Psi_{21} + B_2(\tau) \dot{\Psi}_{21} \right] \\
&\quad + \cos(\Omega_1 t - \sqrt{\omega_2} t) \frac{\sigma}{2\zeta_1} (\Omega_1 \sqrt{\omega_2} - \omega_2) \left[ - A_2(\tau) \dot{\Psi}_{21} + B_2(\tau) \Psi_{21} \right] \\
&\quad + “\text{NST}”,
\end{align*}
$$

and a similar equation for $V_2(t, \tau)$ can be obtained from Eq. (4.126). In order to avoid secular terms, it follows from the equations for $V_1(t, \tau)$ and $V_2(t, \tau)$ that $A_1(\tau)$, $B_1(\tau)$, $A_2(\tau)$ and $B_2(\tau)$ have to satisfy
parameters are determined by the physics, which are taken from [2, 1] in Table 4.4.

We obtain from the system Eq. (4.127)-(4.130),

\[
\begin{align*}
\frac{dA_1(\tau)}{d\tau} &= A_1(\tau)X_1 - B_1(\tau)Y_1 - A_2(\tau)Z_2 - B_2(\tau)C_2, \\
\frac{dB_1(\tau)}{d\tau} &= A_1(\tau)Y_1 + B_1(\tau)X_1 - A_2(\tau)C_2 + B_2(\tau)Z_2, \\
\frac{dA_2(\tau)}{d\tau} &= -A_1(\tau)Z_1 - B_1(\tau)C_1 + A_2(\tau)X_2 - B_2(\tau)Y_2, \\
\frac{dB_2(\tau)}{d\tau} &= -A_1(\tau)C_1 + B_1(\tau)Z_1 + A_2(\tau)Y_2 + B_2(\tau)X_2,
\end{align*}
\]

where \(X_1, X_2, Y_1, Y_2, Z_1, Z_2, C_1\) and \(C_2\) are defined by

\[
\begin{align*}
X_1 &= \left[ \frac{\eta_0}{2} + \frac{\lambda}{2} \phi_1(1)(c_1\omega_1 + d_1) \right], \\
X_2 &= \left[ \frac{\eta_0}{2} + \frac{\lambda}{2} \phi_2(1)(c_2\omega_2 + d_2) \right], \\
Y_1 &= \left[ \frac{\eta_2}{2\zeta_1\sqrt{\omega_1}}(\Phi_{11} - \Phi_{11}) + \frac{\eta_2}{2\sqrt{\omega_1}} \frac{d\phi_1(1)}{dx}(c_1\omega_1 + d_1) \right], \\
Y_2 &= \left[ \frac{\eta_2}{2\zeta_2\sqrt{\omega_2}}(\Phi_{22} - \Phi_{22}) + \frac{\eta_2}{2\sqrt{\omega_2}} \frac{d\phi_2(1)}{dx}(c_2\omega_2 + d_2) \right], \\
Z_1 &= \frac{\sigma\Psi_{12}}{4\zeta_2\sqrt{\omega_2}}(\Omega_1\sqrt{\omega_1} - \omega_1), \\
Z_2 &= \frac{\sigma\Psi_{21}}{4\zeta_1\sqrt{\omega_1}}(\Omega_1\sqrt{\omega_2} - \omega_2), \\
C_1 &= \frac{\sigma\hat{\Phi}_{12}}{4\zeta_2\sqrt{\omega_2}}(\Omega_1\sqrt{\omega_1} - \omega_1), \\
C_2 &= \frac{\sigma\hat{\Phi}_{21}}{4\zeta_1\sqrt{\omega_1}}(\Omega_1\sqrt{\omega_2} - \omega_2).
\end{align*}
\]

We obtain from the system Eq. (4.127)-(4.130),

\[\dot{X} = AX\]

where

\[
X = \begin{pmatrix} A_1(\tau) \\ B_1(\tau) \\ A_2(\tau) \\ B_2(\tau) \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} X_1 & -Y_1 & -Z_2 & -C_2 \\ Y_1 & X_1 & -C_2 & Z_2 \\ -Z_1 & -C_1 & X_2 & -Y_2 \\ -C_1 & Z_1 & Y_2 & X_2 \end{pmatrix}.
\]

and where \(\dot{X}\) represents the derivative of \(X\) with respect to \(\tau\). The matrix \(A\) for a given configuration is only depending on the damping parameter \(\lambda\). The other parameters are determined by the physics, which are taken from [2, 1] in Table 4.4.
In order to stabilise the system we should determine the damping parameter $\lambda$ in such a way that all the real parts of the eigenvalues of the matrix $A$ are negative. As can be seen in Table 4.4, if we assume $\lambda = 0$, there is an instability in the system due to the wind force $\eta_0$. While for increasing values of $\lambda$, we observe that the cable system becomes stable, which also depends on the value of the bending stiffness $\mu$.

### 4.4.3 The difference type resonance case: $\Omega_1 = \sqrt{\omega_2} - \sqrt{\omega_1}$

A similar analysis as given for the sum type resonance case, can also be applied in the difference type resonance case. We will consider $\Omega_1 = \sqrt{\omega_2} - \sqrt{\omega_1}$, which is assumed to have only solution $m = 2, n = 1$ (or $m = 1, n = 2$). Then, Eq. (4.117) can be rewritten as

$$
\left[ V_{1t}(t, \tau) + \omega_1 V_1(t, \tau) \right] = \sin(\sqrt{\omega_1}t) \left\{ 2\sqrt{\omega_1} \frac{dA_1(\tau)}{d\tau} - A_1(\tau) \left[ \eta_{10} \sqrt{\omega_1} + \lambda \phi_1(1) \sqrt{\omega_1}(c_1 \omega_1 + d_1) \right] + B_1(\tau) \left[ \eta_2 (\Phi_{11} - \Phi_{11}) + \eta_2 \frac{d\phi_1(1)}{dx} (c_1 \omega_1 + d_1) \right] \right\} + \cos(\sqrt{\omega_1}t) \left\{ -2\sqrt{\omega_1} \frac{dB_1(\tau)}{d\tau} + B_1(\tau) \left[ \eta_{10} \sqrt{\omega_1} + \lambda \phi_1(1) \sqrt{\omega_1}(c_1 \omega_1 + d_1) \right] + A_1(\tau) \left[ \eta_2 (\Phi_{11} - \Phi_{11}) + \eta_2 \frac{d\phi_1(1)}{dx} (c_1 \omega_1 + d_1) \right] \right\} + \sin(\Omega_1 t + \sqrt{\omega_2} t) \frac{\sigma}{2 \zeta_1} (\Omega_1 \sqrt{\omega_2} + \omega_2) \left[ -A_2(\tau) \Psi_{21} + B_2(\tau) \Phi_{21} \right] + \cos(\Omega_1 t + \sqrt{\omega_2} t) \frac{\sigma}{2 \zeta_1} (\Omega_1 \sqrt{\omega_2} + \omega_2) \left[ A_2(\tau) \Phi_{21} + B_2(\tau) \Psi_{21} \right] + \text{“NST”},
$$

and a similar equation for $V_2(t, \tau)$ can be obtained from Eq. (4.131). In order to avoid secular terms, it follows from the equations for $V_1(t, \tau)$ and $V_2(t, \tau)$ that $A_1(\tau), B_1(\tau), A_2(\tau)$ and $B_2(\tau)$ have to satisfy

$$
\frac{dA_1(\tau)}{d\tau} = A_1(\tau) X_1 - B_1(\tau) Y_1 + A_2(\tau) \tilde{Z}_2 - B_2(\tau) \tilde{C}_2, \quad (4.132)
$$

$$
\frac{dB_1(\tau)}{d\tau} = A_1(\tau) Y_1 + B_1(\tau) X_1 + A_2(\tau) \tilde{C}_2 + B_2(\tau) \tilde{Z}_2, \quad (4.133)
$$

$$
\frac{dA_2(\tau)}{d\tau} = A_1(\tau) \tilde{Z}_1 - B_1(\tau) \tilde{C}_1 + A_2(\tau) X_2 - B_2(\tau) Y_2, \quad (4.134)
$$

$$
\frac{dB_2(\tau)}{d\tau} = A_1(\tau) \tilde{C}_1 + B_1(\tau) \tilde{Z}_1 + A_2(\tau) Y_2 + B_2(\tau) X_2, \quad (4.135)
$$

where $X_1, X_2, Y_1, Y_2, \tilde{Z}_1, \tilde{Z}_2, \tilde{C}_1$ and $\tilde{C}_2$ are defined by
CHAPTER 4. ON BOUNDARY DAMPING TO REDUCE THE RAIN-WIND OSCILLATIONS OF AN
INCLINED CABLE

\[ X_1 = \left[ \frac{\eta_{10}}{2} + \frac{\lambda}{2} \phi_1(1)(c_1 \omega_1 + d_1) \right], \]

\[ X_2 = \left[ \frac{\eta_{10}}{2} + \frac{\lambda}{2} \phi_2(1)(c_2 \omega_2 + d_2) \right], \]

\[ Y_1 = \left[ \frac{\eta_2}{2 \zeta_1 \sqrt{\omega_1}} (\Phi_{11} - \Phi_{11}) + \frac{\eta_2}{2 \sqrt{\omega_1}} \frac{d\phi_1(1)}{dx} (c_1 \omega_1 + d_1) \right], \]

\[ Y_2 = \left[ \frac{\eta_2}{2 \zeta_2 \sqrt{\omega_2}} (\Phi_{22} - \Phi_{22}) + \frac{\eta_2}{2 \sqrt{\omega_2}} \frac{d\phi_2(1)}{dx} (c_2 \omega_2 + d_2) \right], \]

\[ \tilde{Z}_1 = \frac{\sigma \Psi_{12}}{4 \zeta_2 \sqrt{\omega_2}} (\Omega_1 \sqrt{\omega_1} + \omega_1), \]

\[ \tilde{Z}_2 = \frac{\sigma \Psi_{21}}{4 \zeta_1 \sqrt{\omega_1}} (\Omega_1 \sqrt{\omega_2} + \omega_2), \]

\[ \tilde{C}_1 = \frac{\sigma \hat{\Psi}_{12}}{4 \zeta_2 \sqrt{\omega_2}} (\Omega_1 \sqrt{\omega_1} + \omega_1), \]

\[ \tilde{C}_2 = \frac{\sigma \hat{\Psi}_{21}}{4 \zeta_1 \sqrt{\omega_1}} (\Omega_1 \sqrt{\omega_2} + \omega_2). \]

We obtain from the system Eq. (4.132)-(4.135),

\[ \dot{X} = AX \]

where

\[ X = \begin{pmatrix} A_1(\tau) \\ B_1(\tau) \\ A_2(\tau) \\ B_2(\tau) \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} X_1 & -Y_1 & \tilde{Z}_2 & -\tilde{C}_2 \\ Y_1 & X_1 & \tilde{C}_2 & \tilde{Z}_2 \\ \tilde{Z}_1 & -\tilde{C}_1 & X_2 & -Y_2 \\ \tilde{C}_1 & \tilde{Z}_1 & Y_2 & X_2 \end{pmatrix}. \]

and where \( \dot{X} \) represents the derivative of \( X \) with respect to \( \tau \). As in the sum type resonance case, this matrix \( A \) for a given configuration is only depending on the damping parameter \( \lambda \). In Table 4.5, it can easily be seen that there is a change from instability to stability, which is around \( \bar{\lambda} = 0.05 \).
### Table 4.3: Numerical approximations of $\tilde{\omega}_n$ in the non-resonance case for $\eta_0 = \eta_2 = \gamma_1 = 0.1$, $\sigma = 2.8$, and different values of $\mu$.

<table>
<thead>
<tr>
<th>$\omega_n$</th>
<th>$\tilde{\omega}_1$</th>
<th>$\lambda = 0$</th>
<th>$\lambda = 0.05$</th>
<th>$\lambda = 0.1$</th>
<th>$\lambda = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 0.001$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\tilde{\omega}_1$</td>
<td>0.05000-0.05043i</td>
<td>0.01553-0.05043i</td>
<td>-0.01893-0.05043i</td>
<td>-0.29467+0.05043i</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$\tilde{\omega}_2$</td>
<td>0.05000+0.05043i</td>
<td>0.01553+0.05043i</td>
<td>-0.01893+0.05043i</td>
<td>-0.29467+0.05043i</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>$\tilde{\omega}_1$</td>
<td>0.05000+0.12096i</td>
<td>0.00138+0.12096i</td>
<td>-0.04724+0.12096i</td>
<td>-0.43620+0.12096i</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>$\tilde{\omega}_2$</td>
<td>0.05000+0.12096i</td>
<td>0.00138+0.12096i</td>
<td>-0.04724+0.12096i</td>
<td>-0.43620+0.12096i</td>
</tr>
<tr>
<td>$\mu = 0.1$</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\tilde{\omega}_1$</td>
<td>0.05000+0.04832i</td>
<td>0.01277+0.04832i</td>
<td>-0.02445+0.04832i</td>
<td>-0.32227+0.04832i</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$\tilde{\omega}_2$</td>
<td>0.05000+0.10884i</td>
<td>-0.01009+0.10884i</td>
<td>-0.07018+0.10884i</td>
<td>-0.55092+0.10884i</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>$\tilde{\omega}_1$</td>
<td>0.05000+0.15541i</td>
<td>-0.02986+0.15541i</td>
<td>-0.10972+0.15541i</td>
<td>-0.74859+0.15541i</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>$\tilde{\omega}_2$</td>
<td>0.05000+0.15541i</td>
<td>-0.02986+0.15541i</td>
<td>-0.10972+0.15541i</td>
<td>-0.74859+0.15541i</td>
</tr>
<tr>
<td>$\mu = 1$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\tilde{\omega}_1$</td>
<td>0.05000+0.02687i</td>
<td>-0.01575+0.02687i</td>
<td>-0.08150+0.02687i</td>
<td>-0.60751+0.02687i</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$\tilde{\omega}_2$</td>
<td>0.05000+0.02687i</td>
<td>-0.01575+0.02687i</td>
<td>-0.08150+0.02687i</td>
<td>-0.60751+0.02687i</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>$\tilde{\omega}_1$</td>
<td>0.05000+0.02510i</td>
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<td>-1.77718+0.02510i</td>
<td>-9.08590+0.02510i</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>$\tilde{\omega}_2$</td>
<td>0.05000+0.02510i</td>
<td>-0.08630+0.02510i</td>
<td>-1.77718+0.02510i</td>
<td>-9.08590+0.02510i</td>
</tr>
</tbody>
</table>
Table 4.4: Numerical approximations of $\tilde{\omega}_n$ in the sum type resonance case for $\eta_1 = \eta_2 = \gamma_1 = 0.1$, $\sigma = 2.8$, and different values of $\mu$.

<table>
<thead>
<tr>
<th>$\tilde{\omega}$</th>
<th>$\tilde{\lambda} = 0$</th>
<th>$\tilde{\lambda} = 0.05$</th>
<th>$\tilde{\lambda} = 0.1$</th>
<th>$\tilde{\lambda} = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 0.001$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\omega}_1$</td>
<td>0.04845+0.11079i</td>
<td>0.01800+0.04043i</td>
<td>-0.05054+0.11145i</td>
<td>-0.44189+0.11611i</td>
</tr>
<tr>
<td>$\tilde{\omega}_2$</td>
<td>0.04845-0.11079i</td>
<td>0.01800-0.04043i</td>
<td>-0.01564-0.04085i</td>
<td>-0.28898+0.04557i</td>
</tr>
<tr>
<td>$\tilde{\omega}_3$</td>
<td>0.05155+0.04017i</td>
<td>-0.00109+0.11104i</td>
<td>-0.05054-0.11145i</td>
<td>-0.28898-0.04557i</td>
</tr>
<tr>
<td>$\tilde{\omega}_4$</td>
<td>0.05155-0.04017i</td>
<td>-0.00109-0.11104i</td>
<td>-0.01564+0.04085i</td>
<td>-0.05054+0.11145i</td>
</tr>
<tr>
<td>$\mu = 0.01$</td>
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</tr>
<tr>
<td>$\tilde{\omega}_1$</td>
<td>0.04786+0.09626i</td>
<td>0.01689+0.03638i</td>
<td>-0.07576+0.09834i</td>
<td>-0.31627+0.04557i</td>
</tr>
<tr>
<td>$\tilde{\omega}_2$</td>
<td>0.04786-0.09626i</td>
<td>0.01689-0.03638i</td>
<td>-0.07576-0.09834i</td>
<td>-0.31627-0.04557i</td>
</tr>
<tr>
<td>$\tilde{\omega}_3$</td>
<td>0.05214+0.03554i</td>
<td>-0.01421+0.09705i</td>
<td>-0.01887+0.03772i</td>
<td>-0.55691+0.10610i</td>
</tr>
<tr>
<td>$\tilde{\omega}_4$</td>
<td>0.05214-0.03554i</td>
<td>-0.01421-0.09705i</td>
<td>-0.01887-0.03772i</td>
<td>-0.55691-0.10610i</td>
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<tr>
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<tr>
<td>$\tilde{\omega}_1$</td>
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<td>0.01955+0.02646i</td>
<td>-0.03344+0.03425i</td>
<td>-0.43828+0.04090i</td>
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<tr>
<td>$\tilde{\omega}_2$</td>
<td>0.03514-0.02827i</td>
<td>0.01955-0.02646i</td>
<td>-0.03344-0.03425i</td>
<td>-0.43828-0.04090i</td>
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<tr>
<td>$\tilde{\omega}_3$</td>
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<td>-0.08868+0.05346i</td>
<td>-0.20482+0.06108i</td>
<td>-1.15302+0.06767i</td>
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<td>-0.08868-0.05346i</td>
<td>-0.20482-0.06108i</td>
<td>-1.15302-0.06767i</td>
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<td>$\mu = 1$</td>
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<tr>
<td>$\tilde{\omega}_1$</td>
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<td>0.03971+0.02213i</td>
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</tr>
<tr>
<td>$\tilde{\omega}_2$</td>
<td>0.15268-0.00330i</td>
<td>0.03971-0.02213i</td>
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<tr>
<td>$\tilde{\omega}_3$</td>
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<td>-0.21419+0.02066i</td>
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<td>-2.04522+0.02662i</td>
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<td>-0.40248-0.02553i</td>
<td>-2.04522-0.02662i</td>
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</table>
4.5 Conclusions

In this chapter, initial-boundary value problems for a tensioned beam equation are studied. The model is derived to describe the rain-wind induced oscillations of an inclined cable. We applied a multiple-timescales perturbation method in order to observe whether or not mode interactions between vibration modes occur for certain values of the bending stiffness and the damping parameter. The results show that the system in both the pure resonance case and the non-resonance case can be stabilised by a boundary damper. Some of these cases are studied in Section 4.4. Mode interactions

<table>
<thead>
<tr>
<th>( \tilde{\omega} )</th>
<th>( \tilde{\lambda} = 0 )</th>
<th>( \tilde{\lambda} = 0.05 )</th>
<th>( \tilde{\lambda} = 0.1 )</th>
<th>( \tilde{\lambda} = 0.5 )</th>
</tr>
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<td></td>
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<tr>
<td>( \tilde{\omega}_1 )</td>
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<td>0.03561-0.06786i</td>
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<td>-0.44619-0.11951i</td>
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<td>-0.01870-0.10353i</td>
<td>-0.00034-0.06328i</td>
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<tr>
<td>( \tilde{\omega}_2 )</td>
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<td>0.03746-0.06118i</td>
<td>-0.09085-0.10097i</td>
<td>-0.31460-0.04791i</td>
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<td>-0.03478-0.09599i</td>
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<td>-0.55859-0.10926i</td>
</tr>
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<tr>
<td>( \tilde{\omega}_1 )</td>
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<td>0.03048-0.03872i</td>
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</tr>
<tr>
<td>( \tilde{\omega}_4 )</td>
<td>-0.05598-0.04590i</td>
<td>-0.21457-0.04221i</td>
<td>-0.40173-0.03742i</td>
<td>-2.04492+0.02840i</td>
</tr>
</tbody>
</table>

Table 4.5: Numerical approximations of \( \tilde{\omega}_n \) in the difference type resonance case for \( \eta_0 = \eta_2 = \gamma_1 = 0.1, \sigma = 2.8, \) and different values of \( \mu. \)
between two and more modes depending on the bending stiffness $\mu$ are possible. More complicated resonance cases can be $\Omega_1 = \sqrt{\omega_n} \pm \sqrt{\omega_m}$ or $\Omega_1 = 2\sqrt{\omega_n}$, when $\Omega_1 = \sqrt{\omega_N} + \sqrt{\omega_M}$ (or $\Omega_1 = \sqrt{\omega_N} - \sqrt{\omega_M}$) for some fixed $N$ and $M$. These cases still have to be studied, and can be studied by using the techniques as shown in Section 4.4 of this chapter.

This chapter provides an understanding of how effective boundary damping can be for the in-plane transversal oscillations of the cable. The same approach can be used for in-plane and out-of-plane transversal oscillations of elastic structures.
Aerodynamic parameters

\[ b := [-(\psi_1 + \alpha_1) + \arctan(\tan(\gamma))], \quad (A.1) \]
\[ c := [-(\psi_2 + \alpha_2) + \arctan(\tan(\gamma))], \quad (A.2) \]
\[ a_{000} := \sin(\gamma), \quad (A.3) \]
\[ a_{001} := b \cos(\gamma), \quad (A.4) \]
\[ a_{002} := c \cos(\gamma), \quad (A.5) \]
\[ a_{003} := b^3 \cos(\gamma), \quad (A.6) \]
\[ a_{004} := c^3 \cos(\gamma), \quad (A.7) \]
\[ a_{100} := 2 - \cos^2(\gamma), \quad (A.8) \]
\[ a_{101} := \cos(\gamma)[\cos(\gamma) - b \sin(\gamma)], \quad (A.9) \]
\[ a_{102} := \cos(\gamma)[\cos(\gamma) - c \sin(\gamma)], \quad (A.10) \]
\[ a_{103} := b^2 \cos(\gamma)[3 \cos(\gamma) - b \sin(\gamma)], \quad (A.11) \]
\[ a_{104} := c^2 \cos(\gamma)[3 \cos(\gamma) - c \sin(\gamma)], \quad (A.12) \]
\[ a_{010} := -\sin(\gamma) \cos(\gamma), \quad (A.13) \]
\[ a_{011} := \sin(\gamma)[\cos(\gamma) - b \sin(\gamma)], \quad (A.14) \]
\[ a_{012} := \sin(\gamma)[\cos(\gamma) - c \sin(\gamma)], \quad (A.15) \]
\[ a_{013} := b^2 \sin(\gamma)[3 \cos(\gamma) - b \sin(\gamma)], \quad (A.16) \]
\[ a_{014} := c^2 \sin(\gamma)[3 \cos(\gamma) - c \sin(\gamma)], \quad (A.17) \]
\[ a_{200} := \frac{\sin(\gamma)}{2}[2 + \cos^2(\gamma)], \quad (A.18) \]
\[ a_{201} := \frac{\cos(\gamma)}{2}[2b - 3b \cos^2(\gamma) - 2 \sin(2\gamma)], \quad (A.19) \]
\[ a_{202} := \frac{\cos(\gamma)}{2}[2c - 3c \cos^2(\gamma) - 2 \sin(2\gamma)], \quad (A.20) \]
\[ a_{203} := \frac{b \cos(\gamma)}{2} [2b^2 + \cos^2(\gamma)(6 - 3b^2) - 6b \sin(2\gamma)], \quad (A.21) \]

\[ a_{204} := \frac{c \cos(\gamma)}{2} [2c^2 + \cos^2(\gamma)(6 - 3c^2) - 6c \sin(2\gamma)], \quad (A.22) \]

\[ a_{110} := \cos^3(\gamma), \quad (A.23) \]

\[ a_{111} := 4\cos^3(\gamma) - 3\cos(\gamma) + b[\sin(\gamma) - 3\sin(\gamma)\cos^2(\gamma)], \quad (A.24) \]

\[ a_{112} := 4\cos^3(\gamma) - 3\cos(\gamma) + c[\sin(\gamma) - 3\sin(\gamma)\cos^2(\gamma)], \quad (A.25) \]

\[ a_{113} := 6b\sin(\gamma)\cos^2(\gamma) + b^2[12\cos^3(\gamma) - 9\cos(\gamma)] + b^3[-3\sin(\gamma)\cos^2(\gamma) + \sin(\gamma)], \quad (A.26) \]

\[ a_{114} := 6c\sin(\gamma)\cos^2(\gamma) + c^2[12\cos^3(\gamma) - 9\cos(\gamma)] + c^3[-3\sin(\gamma)\cos^2(\gamma) + \sin(\gamma)], \quad (A.27) \]

\[ a_{020} := -\frac{\sin(\gamma)}{2}[3 + \cos^2(\gamma)], \quad (A.28) \]

\[ a_{021} := 2\sin(\gamma)\cos^2(\gamma) - \sin(\gamma) - \frac{3b}{2}\sin^2(\gamma)\cos(\gamma), \quad (A.29) \]

\[ a_{022} := 2\sin(\gamma)\cos^2(\gamma) - \sin(\gamma) - \frac{3c}{2}\sin^2(\gamma)\cos(\gamma), \quad (A.30) \]

\[ a_{023} := 3b\sin^2(\gamma)\cos(\gamma) + b^2[6\sin(\gamma)\cos^2(\gamma) - 3\sin(\gamma)] - \frac{3b^3}{2}\sin^2(\gamma)\cos(\gamma), \quad (A.31) \]

\[ a_{024} := 3c\sin^2(\gamma)\cos(\gamma) + c^2[6\sin(\gamma)\cos^2(\gamma) - 3\sin(\gamma)] - \frac{3c^3}{2}\sin^2(\gamma)\cos(\gamma), \quad (A.32) \]

\[ a_{300} := \frac{\cos^2(\gamma)}{2} [5 \cos^2(\gamma) - 4], \quad (A.33) \]

\[ a_{301} := -\frac{\cos(\gamma)}{6}[-15b \sin(\gamma)\cos^2(\gamma) + 23 \cos^3(\gamma) + 6b \sin(\gamma) - 18 \cos(\gamma)], \quad (A.34) \]

\[ a_{302} := -\frac{\cos(\gamma)}{6}[-15c \sin(\gamma)\cos^2(\gamma) + 23 \cos^3(\gamma) + 6c \sin(\gamma) - 18 \cos(\gamma)], \quad (A.35) \]

\[ a_{303} := \left(1 - \frac{23b^2}{2}\right)\cos^4(\gamma) + \left(\frac{5b^3}{2} - 9b\right)\sin(\gamma)\cos^3(\gamma) + 9b^2 \cos^2(\gamma) - b^3 \sin(\gamma)\cos(\gamma), \quad (A.36) \]

\[ a_{304} := \left(1 - \frac{23c^2}{2}\right)\cos^4(\gamma) + \left(\frac{5c^3}{2} - 9c\right)\sin(\gamma)\cos^3(\gamma) + 9c^2 \cos^2(\gamma) - c^3 \sin(\gamma)\cos(\gamma), \quad (A.37) \]

\[ a_{210} := \frac{3}{2}\sin(\gamma)\cos^3(\gamma), \quad (A.38) \]

\[ a_{211} := b \left[\frac{15}{2}\cos^2(\gamma) - \frac{15}{2}\cos^4(\gamma) - 1\right] + 5\sin(\gamma)\cos(\gamma) - \frac{23}{2}\sin(\gamma)\cos^3(\gamma), \quad (A.39) \]

\[ a_{212} := c \left[\frac{15}{2}\cos^2(\gamma) - \frac{15}{2}\cos^4(\gamma) - 1\right] + 5\sin(\gamma)\cos(\gamma) - \frac{23}{2}\sin(\gamma)\cos^3(\gamma), \quad (A.40) \]

\[ a_{213} := \left(27b - \frac{15b^3}{2}\right)\cos^4(\gamma) + \left(3 - \frac{69b^2}{2}\right)\sin(\gamma)\cos^3(\gamma) \]

\[ + \left(\frac{15b^3}{2} - 21b\right)\cos^2(\gamma) + 15b^2\sin(\gamma)\cos(\gamma) - b^3, \quad (A.41) \]
\[ a_{214} := \left(27c - \frac{15c^3}{2}\right)\cos^4(\gamma) + \left(3 - \frac{69c^2}{2}\right)\sin(\gamma)\cos^3(\gamma) \]  
(A.42)

\[ + \left(\frac{15c^3}{2} - 21c\right)\cos^2(\gamma) + 15c^2\sin(\gamma)\cos(\gamma) - c^3, \]

\[ a_{120} := \frac{3}{2}\sin^2(\gamma)\cos^2(\gamma), \]  
(A.43)

\[ a_{121} := b\left(- \frac{15}{2}\sin(\gamma)\cos^3(\gamma) + \frac{9}{2}\sin(\gamma)\cos(\gamma)\right) + \frac{17}{2}\cos^4(\gamma) - \frac{19}{2}\cos^2(\gamma) + 2, \]  
(A.44)

\[ a_{122} := c\left(- \frac{15}{2}\sin(\gamma)\cos^3(\gamma) + \frac{9}{2}\sin(\gamma)\cos(\gamma)\right) + \frac{17}{2}\cos^4(\gamma) - \frac{19}{2}\cos^2(\gamma) + 2, \]  
(A.45)

\[ a_{123} := \left(\frac{51b^2}{2} - 3\right)\cos^4(\gamma) + \left(27b - \frac{15b^3}{2}\right)\sin(\gamma)\cos^3(\gamma) \]  
(A.46)

\[ + \left(3 - \frac{57b^2}{2}\right)\cos^2(\gamma) + \left(\frac{9b^3}{2} - 15b\right)\sin(\gamma)\cos(\gamma) + 6b^2, \]

\[ a_{124} := \left(\frac{51c^2}{2} - 3\right)\cos^4(\gamma) + \left(27c - \frac{15c^3}{2}\right)\sin(\gamma)\cos^3(\gamma) \]  
(A.47)

\[ + \left(3 - \frac{57c^2}{2}\right)\cos^2(\gamma) + \left(\frac{9c^3}{2} - 15c\right)\sin(\gamma)\cos(\gamma) + 6c^2, \]

\[ a_{030} := \frac{1}{2}\sin^3(\gamma)\cos(\gamma), \]  
(A.48)

\[ a_{031} := \frac{\sin(\gamma)}{6}[23\cos^3(\gamma) - 17\cos(\gamma)] + \frac{b\sin^2(\gamma)}{2}[1 - 5\cos^2(\gamma)], \]  
(A.49)

\[ a_{032} := \frac{\sin(\gamma)}{6}[23\cos^3(\gamma) - 17\cos(\gamma)] + \frac{c\sin^2(\gamma)}{2}[1 - 5\cos^2(\gamma)], \]  
(A.50)

\[ a_{033} := \left(\frac{23b^2}{2} - 1\right)\cos^3(\gamma)\sin(\gamma) - \left(\frac{5b^3}{2} - 9b\right)\sin^2(\gamma)\cos^2(\gamma) \]  
(A.51)

\[ + \left(1 - \frac{17b^2}{2}\right)\sin(\gamma)\cos(\gamma) + \left(\frac{b^3}{2} - 3b\right)\sin^2(\gamma), \]

\[ a_{034} := \left(\frac{23c^2}{2} - 1\right)\cos^3(\gamma)\sin(\gamma) - \left(\frac{5c^3}{2} - 9c\right)\sin^2(\gamma)\cos^2(\gamma) \]  
(A.52)

\[ + \left(1 - \frac{17c^2}{2}\right)\sin(\gamma)\cos(\gamma) + \left(\frac{c^3}{2} - 3c\right)\sin^2(\gamma), \]

\[ b_{000} := \cos(\gamma), \]  
(A.53)

\[ b_{001} := b\sin(\gamma), \]  
(A.54)

\[ b_{002} := c\sin(\gamma), \]  
(A.55)

\[ b_{003} := b^3\sin(\gamma), \]  
(A.56)

\[ b_{004} := c^3\sin(\gamma), \]  
(A.57)
\[ b_{100} := \sin(\gamma)\cos(\gamma), \quad (A.58) \]
\[ b_{101} := \cos(\gamma)[b \cos(\gamma) + \sin(\gamma)], \quad (A.59) \]
\[ b_{102} := \cos(\gamma)[c \cos(\gamma) + \sin(\gamma)], \quad (A.60) \]
\[ b_{103} := b^2 \cos(\gamma)[3\sin(\gamma) + b \cos(\gamma)], \quad (A.61) \]
\[ b_{104} := c^2 \cos(\gamma)[3\sin(\gamma) + c \cos(\gamma)], \quad (A.62) \]
\[ b_{010} := -1 - \cos^2(\gamma), \quad (A.63) \]
\[ b_{011} := \sin(\gamma)[b \cos(\gamma) + \sin(\gamma)], \quad (A.64) \]
\[ b_{012} := \sin(\gamma)[c \cos(\gamma) + \sin(\gamma)], \quad (A.65) \]
\[ b_{013} := b^2 \sin(\gamma)[3\sin(\gamma) + b \cos(\gamma)], \quad (A.66) \]
\[ b_{014} := c^2 \sin(\gamma)[3\sin(\gamma) + c \cos(\gamma)], \quad (A.67) \]
\[ b_{200} := \frac{\cos^3(\gamma)}{2}, \quad (A.68) \]
\[ b_{201} := -\frac{\cos(\gamma)}{2}[2 - 4 \cos^2(\gamma) + 3b \sin(\gamma)\cos(\gamma)], \quad (A.69) \]
\[ b_{202} := -\frac{\cos(\gamma)}{2}[2 - 4 \cos^2(\gamma) + 3c \sin(\gamma)\cos(\gamma)], \quad (A.70) \]
\[ b_{203} := -\frac{3b}{2} \cos(\gamma)[b^2 \sin(\gamma)\cos(\gamma) - 4b \cos^2(\gamma) - 2\sin(\gamma)\cos(\gamma) + 2b], \quad (A.71) \]
\[ b_{204} := -\frac{3c}{2} \cos(\gamma)[c^2 \sin(\gamma)\cos(\gamma) - 4c \cos^2(\gamma) - 2\sin(\gamma)\cos(\gamma) + 2c], \quad (A.72) \]
\[ b_{110} := -\sin^3(\gamma), \quad (A.73) \]
\[ b_{111} := b \left[3\cos^3(\gamma) - 2\cos(\gamma)\right] + 4\sin(\gamma)\cos^2(\gamma) - \sin(\gamma), \quad (A.74) \]
\[ b_{112} := c \left[3\cos^3(\gamma) - 2\cos(\gamma)\right] + 4\sin(\gamma)\cos^2(\gamma) - \sin(\gamma), \quad (A.75) \]
\[ b_{113} := b^3 \left[3\cos^3(\gamma) - 2\cos(\gamma)\right] + 3b^2 \left[4\cos^2(\gamma)\sin(\gamma) - \sin(\gamma)\right] - 6b\cos^3(\gamma) - \cos(\gamma), \quad (A.76) \]
\[ b_{114} := c^3 \left[3\cos^3(\gamma) - 2\cos(\gamma)\right] + 3c^2 \left[4\cos^2(\gamma)\sin(\gamma) - \sin(\gamma)\right] - 6c\cos^3(\gamma) - \cos(\gamma), \quad (A.77) \]
\[ b_{020} := -\frac{\cos(\gamma)}{2}[\cos^2(\gamma) - 3], \quad (A.78) \]
\[ b_{021} := \frac{\sin(\gamma)}{2}[3b \cos^2(\gamma) + 4\sin(\gamma)\cos(\gamma) - b], \quad (A.79) \]
\[ b_{022} := \frac{\sin(\gamma)}{2}[3c \cos^2(\gamma) + 4\sin(\gamma)\cos(\gamma) - c], \quad (A.80) \]
\[ b_{023} := \frac{b}{2} \sin(\gamma)[3b^2 \cos^2(\gamma) + 12b \sin(\gamma)\cos(\gamma) - 6\cos^2(\gamma) - b^2 + 6], \quad (A.81) \]
\[ b_{024} := \frac{c}{2} \sin(\gamma)[3c^2 \cos^2(\gamma) + 12c \sin(\gamma)\cos(\gamma) - 6\cos^2(\gamma) - c^2 + 6], \quad (A.82) \]
\[ b_{300} := -\frac{1}{2}\sin(\gamma)\cos^3(\gamma), \quad (A.83) \]
\( b_{301} := -\frac{\cos(\gamma)}{6}[3b \cos^3(\gamma) + 23 \sin(\gamma)\cos^2(\gamma) - 6\sin(\gamma)], \quad \text{(A.84)} \)

\( b_{302} := -\frac{\cos(\gamma)}{6}[3c \cos^3(\gamma) + 23 \sin(\gamma)\cos^2(\gamma) - 6\sin(\gamma)], \quad \text{(A.85)} \)

\( b_{303} := \left(9b - \frac{b^3}{2}\right)\cos^4(\gamma) + \left(1 - \frac{23b^2}{2}\right)\sin(\gamma)\cos^3(\gamma) - 6b \cos^2(\gamma) + 3b^2 \sin(\gamma)\cos(\gamma), \quad \text{(A.86)} \)

\( b_{304} := \left(9c - \frac{c^3}{2}\right)\cos^4(\gamma) + \left(1 - \frac{23c^2}{2}\right)\sin(\gamma)\cos^3(\gamma) - 6c \cos^2(\gamma) + 3c^2 \sin(\gamma)\cos(\gamma), \quad \text{(A.87)} \)

\( b_{210} := -\frac{3}{2}\sin^2(\gamma)\cos^2(\gamma), \quad \text{(A.88)} \)

\( b_{211} := b[-\frac{15}{2}\cos^3(\gamma)\sin(\gamma) + 3\sin(\gamma)\cos(\gamma)] + \frac{19}{2}\cos^4(\gamma) - \frac{17}{2}\cos^2(\gamma) + 1, \quad \text{(A.89)} \)

\( b_{212} := c[-\frac{15}{2}\cos^3(\gamma)\sin(\gamma) + 3\sin(\gamma)\cos(\gamma)] + \frac{19}{2}\cos^4(\gamma) - \frac{17}{2}\cos^2(\gamma) + 1, \quad \text{(A.90)} \)

\( b_{213} := \left(\frac{69b^2}{2} - 3\right)\cos^4(\gamma) + \left(27b - \frac{15b^3}{2}\right)\cos^3(\gamma)\sin(\gamma) \quad \text{(A.91)} \)

\[ + \left(3 - \frac{63b^2}{2}\right)\cos^2(\gamma) + \left(3b^3 - 12b\right)\cos(\gamma)\sin(\gamma) + 3b^2, \]

\( b_{214} := \left(\frac{69c^2}{2} - 3\right)\cos^4(\gamma) + \left(27c - \frac{15c^3}{2}\right)\cos^3(\gamma)\sin(\gamma) \quad \text{(A.92)} \)

\[ + \left(3 - \frac{63c^2}{2}\right)\cos^2(\gamma) + \left(3c^3 - 12c\right)\cos(\gamma)\sin(\gamma) + 3c^2, \]

\( b_{120} := -\frac{3}{2}\cos(\gamma)\sin^3(\gamma), \quad \text{(A.93)} \)

\( b_{121} := b[\frac{15}{2}\cos^4(\gamma) - \frac{15}{2}\cos^2(\gamma) + 1] + \frac{23}{2}\cos^3(\gamma)\sin(\gamma) - \frac{13}{2}\cos(\gamma)\sin(\gamma), \quad \text{(A.94)} \)

\( b_{122} := c[\frac{15}{2}\cos^4(\gamma) - \frac{15}{2}\cos^2(\gamma) + 1] + \frac{23}{2}\cos^3(\gamma)\sin(\gamma) - \frac{13}{2}\cos(\gamma)\sin(\gamma), \quad \text{(A.95)} \)

\( b_{123} := \left(\frac{15b^3}{2} - 27b\right)\cos^4(\gamma) + \left(\frac{69b^2}{2} - 3\right)\cos^3(\gamma)\sin(\gamma) \quad \text{(A.96)} \)

\[ + \left(33b - \frac{15b^3}{2}\right)\cos^2(\gamma) + \left(3 - \frac{39b^2}{2}\right)\cos(\gamma)\sin(\gamma) + b^3 - 6b, \]

\( b_{124} := \left(\frac{15c^3}{2} - 27c\right)\cos^4(\gamma) + \left(\frac{69c^2}{2} - 3\right)\cos^3(\gamma)\sin(\gamma) \quad \text{(A.97)} \)

\[ + \left(33c - \frac{15c^3}{2}\right)\cos^2(\gamma) + \left(3 - \frac{39c^2}{2}\right)\cos(\gamma)\sin(\gamma) + c^3 - 6c, \]

\( b_{030} := -\frac{\sin^2(\gamma)}{2}[\cos^2(\gamma) - 2\cos(\gamma) + 1], \quad \text{(A.98)} \)
\[ b_{031} := \frac{\sin(\gamma)}{6} \left[ 15b \cos^2(\gamma) + 23 \cos^2(\gamma) \sin(\gamma) - 9b \cos(\gamma) - 5\sin(\gamma) \right], \quad (A.99) \]

\[ b_{032} := \frac{\sin(\gamma)}{6} \left[ 15c \cos^2(\gamma) + 23 \cos^2(\gamma) \sin(\gamma) - 9c \cos(\gamma) - 5\sin(\gamma) \right], \quad (A.100) \]

\[ b_{033} := \left(\frac{5b^3}{2} - 9b\right) \cos^3(\gamma) \sin(\gamma) + \left(\frac{23b^2}{2} - 1\right) \cos^2(\gamma) \sin^2(\gamma) + \left(9b - \frac{3b^3}{2}\right) \sin(\gamma) \cos(\gamma) + \left(1 - \frac{5b^2}{2}\right) \sin^2(\gamma), \quad (A.101) \]

\[ b_{034} := \left(\frac{5c^3}{2} - 9c\right) \cos^3(\gamma) \sin(\gamma) + \left(\frac{23c^2}{2} - 1\right) \cos^2(\gamma) \sin^2(\gamma) + \left(9c - \frac{3c^3}{2}\right) \sin(\gamma) \cos(\gamma) + \left(1 - \frac{5c^2}{2}\right) \sin^2(\gamma). \quad (A.102) \]
Stationary Solution $\hat{u}_x$ and $\hat{v}_x$

The stationary solution follows from Eq. (4.33) and (4.34) by considering no time-dependence, yielding

$$-\frac{E}{M} \frac{\partial}{\partial x} \left( \hat{u}_x + \frac{\hat{v}_x^2}{2} \right) = \left[ \tilde{A} \gamma_1 \cos(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha) x$$

$$+ \left[ 2 + 2 \tilde{A} \sin(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha), \tag{B.1}$$

$$\frac{EI_y}{AM} \hat{v}_xxxx - \frac{T_0}{AM} \hat{v}_{xx} - \frac{E}{M} \frac{\partial}{\partial x} \left[ \hat{v}_x (\hat{u}_x + \frac{\hat{v}_x^2}{2}) \right] = \left[ \tilde{A} \gamma_1 \cos(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha) x \hat{v}_x$$

$$+ \left[ 1 + \tilde{A} \sin(\gamma_1 x - \Omega_1 t) \right] g \sin(\alpha) \hat{v}_{xx}$$

$$+ \left[ 1 + \tilde{A} \sin(\gamma_1 x - \Omega_1 t) \right] g \cos(\alpha)$$

$$- \frac{\rho_a}{2AM} dLv_\infty^2 A_{00}. \tag{B.2}$$

It will be assumed that $\hat{u}$ is $\mathcal{O}(\epsilon^2)$, $\hat{v}$ is $\mathcal{O}(\epsilon)$, $\tilde{A}$ is $\mathcal{O}(\epsilon)$, $g\sin(\alpha) = P_0^* \mathcal{O}(1)$, $\frac{E}{M} = P_1^*$ is $\mathcal{O}(1/\epsilon)$, $\frac{EI_y}{AM} = P_2^*$ is $\mathcal{O}(1/\epsilon)$, and $\frac{T_0}{AM} = P_3^*$ is $\mathcal{O}(1/\epsilon)$, where $\epsilon$ is a small parameter with $0 < \epsilon \ll 1$. Then, by using these assumptions, Eq. (B.1) and (B.2) becomes

$$-P_1^* \frac{\partial}{\partial x} \left( \hat{u}_x + \frac{\hat{v}_x^2}{2} \right) = 2g \sin(\alpha) + \mathcal{O}(\epsilon), \tag{B.3}$$

$$P_2^* \hat{v}_{xxxx} - P_3^* \hat{v}_{xx} - P_1^* \frac{\partial}{\partial x} \left[ \hat{v}_x (\hat{u}_x + \frac{\hat{v}_x^2}{2}) \right] = \frac{\partial}{\partial x} \left[ P_0^* x \hat{v}_x \right] + g \cos(\alpha) - \frac{\rho_a}{2AM} dLv_\infty^2 A_{00} + \mathcal{O}(\epsilon). \tag{B.4}$$
with the boundary conditions

\[ \hat{u}(L) = \hat{u}(0) = 0, \quad (B.5) \]

\[ \hat{\nu}(0) = \hat{\nu}_{xx}(0) = \hat{\nu}_x(L) = 0, \]

\[ EIy \hat{\nu}_{xxx}(L) = [T_0 + AMLg \sin(\alpha)] \hat{\nu}_x(L). \]

When Eq. (B.3) and (B.4) are integrated with respect to \( x \), we may rewrite

\[ \left( \hat{u}_x + \frac{\hat{u}_x^2}{2} \right) = - \frac{2g \sin(\alpha) x}{P_1^*} + k_1, \quad (B.6) \]

\[ \hat{\nu}_{xxx} - \frac{P_3^*}{P_2^*} \hat{\nu}_{xx} - \frac{P_1^*}{P_2^*} \frac{\partial}{\partial x} \left[ \hat{\nu}_x \left( \hat{u}_x + \frac{\hat{u}_x^2}{2} \right) \right] = \frac{P_0^*}{P_2^*} x \hat{v}_x + \frac{g \cos(\alpha) x}{P_2^*} - \frac{\rho a}{2AMP_2^*} dL v_0^2 A_{00} x + k_2, \quad (B.7) \]

where \( k_1 \) and \( k_2 \) are constants of integration. Substitute Eq. (B.6) into Eq. (B.7), we obtain

\[ \hat{\nu}_{xxx} - \hat{\nu}_x (k_3 - xk_4) = k_2 + xk_5, \quad (B.8) \]

where \( k_3 = \frac{T_0}{EIy} + k_1 \frac{A}{y} \), \( k_4 = \frac{P_3^*}{P_2^*} \), and \( k_5 = \frac{P_0^*}{P_2^*} - \frac{\rho a}{2AMP_2^*} dL v_0^2 A_{00} \). In order to solve the non-homogeneous PDE Eq. (B.8), we will apply the method of variation of parameters and obtain

\[ \hat{\nu}(x) = \int_0^x C_1 A_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) d\bar{x} + \int_0^x C_2 B_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) d\bar{x} \]

\[ - \int_0^x A_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) \left\{ \int_0^x \left( k_2 + sk_5 \right) \frac{\cos(\alpha) x}{W_r(s)} B_i \left( \frac{k_3 - sk_4}{k_4^{2/3}} \right) ds \right\} d\bar{x} \]

\[ + \int_0^x B_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) \left\{ \int_0^x \left( k_2 + sk_5 \right) \frac{\cos(\alpha) x}{W_r(s)} A_i \left( \frac{k_3 - sk_4}{k_4^{2/3}} \right) ds \right\} d\bar{x} + C_3. \]

Here \( A_i \) and \( B_i \) are Airy functions, and \( C_1, C_2 \) and \( C_3 \) are constants of integration, and

\[ W_r(x) = A_i \left( \frac{k_3 - xk_4}{k_4^{2/3}} \right) \frac{d}{dx} \left[ B_i \left( \frac{k_3 - xk_4}{k_4^{2/3}} \right) d \right] - B_i \left( \frac{k_3 - xk_4}{k_4^{2/3}} \right) \frac{d}{dx} \left[ A_i \left( \frac{k_3 - xk_4}{k_4^{2/3}} \right) \right]. \]

Substituting Eq. (B.9) into Eq. (B.6) and integrated with respect to \( x \), we obtain

\[ \hat{u}(x) = - \int_0^x \left( \frac{2g}{P_1^*} \sin(\alpha) \bar{x} + k_1 \right) d\bar{x} + C_4 \]

\[ - \frac{1}{2} \int_0^x \left\{ C_1 A_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) + C_2 B_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) \right\} \]

\[ - A_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) \left\{ \int_0^x \left( k_2 + sk_5 \right) \frac{\cos(\alpha) x}{W_r(s)} B_i \left( \frac{k_3 - sk_4}{k_4^{2/3}} \right) ds \right\} \]

\[ + B_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) \left\{ \int_0^x \left( k_2 + sk_5 \right) \frac{\cos(\alpha) x}{W_r(s)} A_i \left( \frac{k_3 - sk_4}{k_4^{2/3}} \right) ds \right\}^2 d\bar{x}. \]
where $C_4$ is a constant of integration. By satisfying boundary conditions Eq. (B.5) for stationary solution, we obtain

$$k_1 := rac{g \sin(\alpha)L}{P_1} + \frac{1}{2L} \int_0^L \hat{v}_x^2 \, d\bar{x}, \quad (B.12)$$

$$k_2 := \frac{N_3 F_1 - F_3 N_1}{N_1 F_2 + F_1 N_2}, \quad (B.13)$$

$$C_1 := -C_2 \frac{B_i \left(1, \frac{k_3}{k_4^{2/3}}\right)}{A_i \left(1, \frac{k_3}{k_4^{2/3}}\right)}, \quad (B.14)$$

$$C_2 := \frac{N_3 F_2 + F_3 N_2}{N_1 F_2 + F_1 N_2}, \quad (B.15)$$

$$C_3 := C_4 = 0, \quad (B.16)$$

where

$$N_1 := \left\{ -A_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) B_i \left(1, \frac{k_3}{k_4^{2/3}}\right) + B_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) \right\}, \quad (B.17)$$

$$N_2 := \left\{ -A_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) \int_0^L \frac{1}{W r(\bar{x})} B_i \left(\frac{k_3 - \bar{x}k_4}{k_4^{2/3}}\right) d\bar{x} \right. + \left. B_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) \int_0^L \frac{1}{W r(\bar{x})} A_i \left(\frac{k_3 - \bar{x}k_4}{k_4^{2/3}}\right) d\bar{x} \right\}, \quad (B.18)$$

$$N_3 := \left\{ A_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) \int_0^L \frac{\bar{x}k_5}{W r(\bar{x})} B_i \left(\frac{k_3 - \bar{x}k_4}{k_4^{2/3}}\right) d\bar{x} \right. - \left. B_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) \int_0^L \frac{\bar{x}k_5}{W r(\bar{x})} A_i \left(\frac{k_3 - \bar{x}k_4}{k_4^{2/3}}\right) d\bar{x} \right\}, \quad (B.19)$$

$$F_1 := \frac{EI_y}{T_0 + AM L \sin(\alpha)} (k_3 - Lk_4) \left\{ B_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) - A_i \left(\frac{k_3 - Lk_4}{k_4^{2/3}}\right) \frac{B_i \left(1, \frac{k_3}{k_4^{2/3}}\right)}{A_i \left(1, \frac{k_3}{k_4^{2/3}}\right)} \right\}, \quad (B.20)$$
\[ F_2 := \frac{EI_y}{[T_0 + AMLg \sin(\alpha)]} (k_3 - Lk_4) \left\{ A_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \int_0^L \frac{1}{W(r)} B_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) d\bar{x} \right\} \]

\[ - B_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \int_0^L \frac{1}{W(r)} A_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) d\bar{x} \right\} \]

\[ + \frac{EI_y}{T_0 k_4^{1/3}} \left\{ A_i (1, \frac{k_3 - Lk_4}{k_4^{2/3}}) \frac{1}{W(r)} B_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \right\} \]

\[ - B_i (1, \frac{k_3 - Lk_4}{k_4^{2/3}}) \frac{1}{W(r)} A_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \right\} \]

\[ F_3 := \frac{EI_y}{[T_0 + AMLg \sin(\alpha)]} (k_3 - Lk_4) \left\{ A_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \int_0^L \frac{\bar{x}k_5}{W(r)} B_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) d\bar{x} \right\} \]

\[ - B_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \int_0^L \frac{\bar{x}k_5}{W(r)} A_i \left( \frac{k_3 - \bar{x}k_4}{k_4^{2/3}} \right) d\bar{x} \right\} \]

\[ + \frac{EI_y}{T_0 k_4^{1/3}} \left\{ - A_i (1, \frac{k_3 - Lk_4}{k_4^{2/3}}) \frac{Lk_5}{W(r)} B_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \right\} \]

\[ + B_i (1, \frac{k_3 - Lk_4}{k_4^{2/3}}) \frac{Lk_5}{W(r)} A_i \left( \frac{k_3 - Lk_4}{k_4^{2/3}} \right) \right\} . \]
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Summary

Many mathematical models, which describe oscillations in elastic structures such as suspension bridges, conveyor belts and elevator cables, can be formulated as initial-boundary value problems for string (wave) equations, or for beam equations. In order to build more durable, elegant and lighter mechanical structures, the undesired vibrations can be suppressed by using dampers.

In this thesis, the effect of boundary damping on elastic structures is studied. In Chapter 2, as a simple model of oscillations of a cable, a semi-infinite string-like problem is modelled by an initial boundary value problem with (non)-classical boundary conditions. We apply the classical method of D’Alembert to obtain the exact solution which provides information about the efficiency of the damper at the boundary.

In Chapter 3, initial-boundary value problems for a beam equation on a semi-infinite interval and on a finite interval have been studied. The method of Laplace transforms is applied to obtain the Greens function for a transversally vibrating homogeneous semi-infinite beam, and the exact solution for various boundary conditions are examined. The analytical results confirm earlier obtained results, and are validated by explicit numerical approximations of the damping and oscillating rates. The study shows that the numerical results approximate the exact results for sufficiently large domain lengths and for a sufficiently high number of modes. Moreover, the study provides an understanding of how the Greens functions for a semi-infinite beam can be computed analytically for (non)-classical boundary conditions.

Finally, in Chapter 4 the studies as presented in Chapter 2 and in Chapter 3 are extended to inclined structures. A model is derived to describe the rain-wind induced oscillations of an inclined cable. For a linearly formulated initial-boundary value problem for a tensioned beam equation describing the in-plane transversal oscillations of the cable, the effectiveness of a boundary damper is determined by using a two timescales perturbation method. Not only the influence of boundary damping but also the influence of the bending stiffness on the stability properties of the solution have been studied.
Samenvatting

Veel wiskundige modellen die trillingen in elastische constructies, zoals hangbruggen, lopende banden en liftkabels, beschrijven kunnen worden geformuleerd als initieel randvoorwaarde problemen voor golfvergelijkingen of balkvergelijkingen. Om duurzamere, slankere en lichtere constructies te bouwen moeten ongewenste trillingen worden onderdrukt met behulp van dempers.

In dit proefschrift bestuderen we het effect van demping in de oplegpunten van elastische constructies. In hoofdstuk 2 introduceren we een eenvoudig model dat trillingen in een kabel beschrijft als een initieel-randvoorwaarde probleem voor een half-oneindige kabel met klassieke en niet-klassieke randvoorwaarden. We passen de klassieke methode van D’Alembert toe om de exacte oplossing te verkrijgen. Dit geeft inzicht geeft in de effectiviteit van de demper.

In hoofdstuk 3 worden initieel-randvoorwaarde problemen voor een balkvergelijking op een half-oneindig domein en op een eindig domein bestudeerd. Laplace transformaties worden toegepast om de Greense functie te verkrijgen voor een homogene half-oneindige balk die dwars op de golfrichting vibreert. Daarnaast wordt de analytische oplossing bestudeerd voor verscheidene randvoorwaarden. De analytische resultaten bevestigen de eerder gevonden resultaten en worden gevalideerd met behulp van numerieke benaderingen van de mate van demping en trilling. Deze studie laat zien dat de numerieke resultaten de exacte resultaten benaderen voor voldoende lange domeinen en voor een voldoende aantal trillingscomponenten. Ook geeft deze studie meer inzicht in de manier waarop Greense functies analytisch kunnen worden bepaald voor een half-oneindige balk met klassieke en niet-klassieke randvoorwaarden.

Ten slotte worden in hoofdstuk 4 de studies zoals beschreven in hoofdstuk 2 en 3 uitgebreid naar hellende constructies. Er wordt een model afgeleid om trillingen te beschrijven die veroorzaakt worden door een combinatie van regen en wind op een hellende kabel. We bepalen de effectiviteit van dempers in de oplegpunten voor een lineair geformuleerd initieel-randvoorwaarde probleem voor een gespannen kabel, dat trillingen beschrijft in n dwarsrichting. Dit wordt gedaan met behulp van een twee-tijdschalen perturbatiemethode. Naast de invloed van demping wordt ook de invloed van buigstijfheid op de stabiliteitseigenschappen van de oplossing bestudeerd.
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List of publications and presentations

Publications in Refereed Journals


Publications in Refereed Proceedings


International Conferences and Workshops - oral/poster presentations


- 5th Women in Mathematics Summer School on Mathematical Theories towards Environmental Models, 27 May-1 June, 2013, International Centre for Theoretical Physics (ICTP), Trieste, Italy.

Tuğçe Akkaya was born on July 9, 1987, in Izmir, Turkey. In 2004 she completed her secondary education at Karşıyaka Gazi Lisesi in Izmir. In the same year she started her studies in Mathematics at the Faculty of Arts and Sciences, Celal Bayar University, Manisa, Turkey. From the same faculty, she received her Master of Science degree in Applied Mathematics in 2011.

In September 2012, she came to the Netherlands and joined the Mathematical Physics group in the Delft Institute of Applied Mathematics at Delft University of Technology (TU Delft) as a PhD researcher, and worked on the project “Rain-wind induced oscillations of cables” under the supervision of Dr. W.T. van Horssen. During her PhD, she presented her research in national and international conferences/workshops such places as Vienna (ENOC2014), Frankfurt (IUTAM2015), Houston (ASME-IMECE2015). She received an excellent poster paper award at the 8th European Nonlinear Dynamics Conference (ENOC). In addition to her research, she was a member of the EWI Faculty PhD council at TU Delft.

Since November 2017, Tuğçe is working as a Lecturer at the University of Twente.