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Publication date
2015

Document Version
Final published version

Published in
Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems, AAMAS 2015

Now, Later, or Both: a Closed-Form Optimal Decision for a Risk-Averse Buyer

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ABSTRACT

Motivated by the energy domain, we examine a risk-averse buyer that has to purchase a fixed quantity of a continuous good. The buyer has two opportunities to buy: now or later. The buyer can spread the quantity over the two timeslots in any way, as long as the total quantity remains the same. The current price is known, but the future price is not. It is well known that risk neutral buyers purchase in whichever timeslot they expect to be the cheapest, regardless of the uncertainty of the future price. Research suggests, however, that most people may in fact be risk-averse. If the expected future price is lower than the current price, but very uncertain, then they may purchase in the present, or spread the quantity over both timeslots. We describe a formal model with a uniform price distribution and a two-segment piecewise linear risk aversion function. We provide a theorem that states the optimal decision as a closed-form expression.

Categories and Subject Descriptors

G.3 [**Mathematics of Computing**]: Probability and Statistics

Keywords

energy, optimization, price uncertainty, risk aversion, utility

1. INTRODUCTION

Sometimes a fixed amount of a continuous good can be bought in two different occasions or timeslots, where the price in one timeslot is known and the price in the other timeslot is unknown. An example of such a case is the trading of electricity in a day ahead market and a balancing market (e.g. [3]). The price in the day ahead market is more certain than the price in the balancing market. Another example is the charging of electric vehicles (EVs). Owners may have several options where to charge their EV. For example, an owner may be able to charge his vehicle at home or at a local charging station. The price at home is known to the owner, but the current price at the station is not.

2. PROBLEM DESCRIPTION

Appears in: *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2015), Bordini, Elkind, Weiss, Yolum (eds.), May, 4–8, 2015, Istanbul, Turkey.*

We examine a buyer that must purchase a fixed quantity Q of a continuous good. There are two timeslots to buy this good: in the present or in the future, respectively at unit price p_1 or p_2. Buyers have to decide how to spread the total quantity Q over the two timeslots. This decision is expressed as the quantity $Q_1 \in [0, Q]$ purchased in the present. The quantity $Q_2 = Q - Q_1$ to be purchased in the future is then simply the remaining quantity. The buyer can choose to buy the its demand in the present ($Q_1 = Q$), in the future ($Q_1 = 0$), or in both timeslots ($Q_1 \in (0, Q)$).

We assume the buyer does not prefer one timeslot over the other, except for the difference in cost. Thus, the most important variable to be evaluated is the total cost made in both timeslots. The following definition expresses the total cost made by the buyer in terms of the decision variable Q_1.

Definition 1. Let Q be the total quantity to be purchased, let p_1, p_2 respectively be the prices in the present and the future, and let $Q_1 \in [0, Q]$ be the quantity purchased in the present. Then, the **cost function** $Z(Q_1)$ is defined as

$$Z(Q_1) = p_1 Q_1 + p_2(Q - Q_1)$$

The cost function expresses a preference order over the set of possible decisions $[0, Q]$. If the buyer knew both prices p_1, p_2, then according to this preference order he would buy the entire quantity in whichever timeslot is the cheapest.

However, at the time of the decision, the buyer does not know the future price p_2. Fortunately, the buyer may have some information on what p_2 may be. To incorporate this knowledge, we model p_2 as a stochastic variable, of which the buyer knows the distribution. Consequently, the cost function yields a probability distribution over the cost, rather than a deterministic value. In our analysis, we assume p_2 has a uniform distribution $p_2 \sim U[a, b]$.

The optimal decision of the buyer depends on his attitude towards risk. In the literature, risk neutral buyers are very common; these minimize their expected total cost and are indifferent towards the uncertainty of the cost. For the setting above, risk neutral buyers purchase the entire quantity in whichever timeslot has the lowest expected price. Thus, this situation is the same as the deterministic case, except that p_2 is replaced by $\mathbb{E}p_2$. For risk neutral buyers, spreading the purchase over two timeslots is never strictly better than buying the entire quantity in the cheapest timeslot.

Though risk neutral buyers are common in the literature, there is evidence that buyers are in fact risk averse [4, 1]. Risk averse buyers do not only prefer low costs, but they also want to reduce the risk of bad outcomes. In case of two actions that yield equal expected costs, the buyer prefers the one with the least uncertain cost. Furthermore, a risk averse buyer may prefer an action that yields a higher expected cost, if this cost is less uncertain.

Risk aversion is usually modeled as maximization of the expected utility. The utility is a monotonic, concave transformation of the pay-off. Common examples of utility functions are exponential utility [4] and the piecewise linear utility [2]. Since our problem is formulated in terms of cost rather than pay-off, we model risk aversion as minimization of the expected *dis*utility. Disutility is a monotonic, convex transformation of the cost. More specifically, we use a two-segment piecewise linear disutility function.

Definition 2. The piecewise linear disutility function $u(Z)$ is defined as

$$u(Z) = \begin{cases} Z & \text{if } Z \leq \alpha \\ \alpha + \beta(Z - \alpha) & \text{if } Z \geq \alpha \end{cases}$$

with parameters $\alpha > 0$ and $\beta > 1$.

For any decision Q_1, the disutility is obtained by applying the disutility function $u(Z)$ to the total cost $Z(Q_1)$. Due to the convexity of the disutility function, minimization of the expected disutility $\mathbb{E}[u(Z(Q_1))]$ results in risk-averse behavior. Thus, the optimal decision Q_1^* is defined as follows.

Definition 3. The **optimal consumption** Q_1^* is given by

$$Q_1^* = \operatorname*{argmin}_{Q_1 \in [0,Q]} \mathbb{E}[u(Z(Q_1))].$$

A possible application of a piecewise linear disutility function is to describe the preferences of a buyer with a certain budget, who has to pay interest over the portion of the cost that exceeds this budget. All outcomes up to threshold α yield a disutility equal to the total cost, while all outcomes beyond α yield a disutility higher than the cost. Hence, outcomes exceeding α are penalized more than they would have been if the buyer were risk neutral.

3. THE SOLUTION

In this section we give a closed-form expression for the optimal action Q_1^*. If the current price p_1 is lower than the expected future price $\mathbb{E}p_2$, then buying the entire quantity Q in the present minimizes the expected cost. Moreover, since p_1 is known while p_2 is uncertain, this also minimizes the expected disutility. Thus, if $p_1 < \mathbb{E}p_2$, then a risk-averse buyer buys Q in the present, as there are no advantages to delaying. On the other hand, if the current price p_1 is higher than the expected future price $\mathbb{E}p_2$, then delaying the entire quantity Q minimizes the expected cost. However, since p_1 is known while p_2 is uncertain, this also yields a higher uncertainty than buying in the present. If p_1 is sufficiently high, though, then the benefit of a lower expected cost outweighs the drawback of a higher uncertainty. More specifically, this is the case if $p_1 > \sigma$, where σ is defined as follows:

Definition 4. The **delay threshold price**, denoted σ, is

$$\sigma = \frac{a + \sqrt{\beta} b}{1 + \sqrt{\beta}}.$$

If $p_1 \in [\mathbb{E}p_2, \sigma]$, then a risk averse buyer may spread the purchase over both timeslots. This is a trade-off between expected cost reduction and uncertainty reduction. The following theorem states the optimal action of a risk-averse buyer in all situations described above.

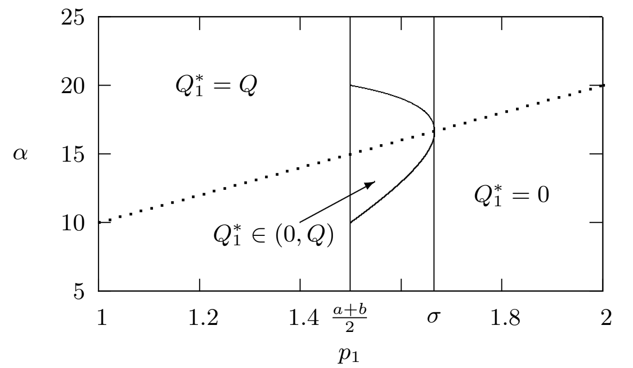

Figure 1: Optimal quantity to buy in the first timeslot. The dotted line shows $\alpha = p_1 Q$. For points on this line within interval $p_1 \in [\frac{a+b}{2}, \sigma)$, it holds that $Q_1^* = Q$. For $\alpha = p_1 Q$ and $p_1 = \sigma$ all Q_1 are optimal.

THEOREM 1. *The optimal quantity to buy now is*

$$Q_1^* = \begin{cases} Q & p_1 < \frac{a+b}{2} \\ 0 & p_1 > \sigma \\ \max(0, Q - \frac{|\alpha - p_1 Q|}{\sqrt{\rho}}) & \frac{a+b}{2} < p_1 < \sigma \end{cases}$$

where

$$\rho = \frac{\beta(b - p_1)^2 - (p_1 - a)^2}{\beta - 1}.$$

Some special cases are excluded in this equation. If $p_1 = \frac{a+b}{2}$, then all Q_1 such that there is no uncertainty regarding $Z(Q_1) \leq \alpha$ are optimal. If $p_1 = \sigma$ and $p_1 Q \neq \alpha$, then $Q_1^* = 0$. If $p_1 = \sigma$ and $p_1 Q = \alpha$, then any $Q_1 \in [0, Q]$ is optimal.

4. CONCLUSION

We have given a closed-form expression for the optimal behavior of a risk-averse buyer with a two-segment piecewise linear utility function and a uniformly distributed future price distribution (Theorem 1). This gives insight into risk averse buying with two timeslots. Moreover, our solution may open further research and development in finding closed-form solutions for other classes of this problem.

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