DELFt UNIVERSITY OF TECHNOLOGY
FACULTY OF AEROspACE ENGINEERING

Memorandum M - 592

VIBRATION OF TWO RECTANGULAR PANELS
CONNECTED AT TWO POINTS AND "HOLD DOWN"
AT FOUR POINTS.

by

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The governing equations are derived using Lagrange's equations, whereby the constraint conditions are enforced via Lagrangian multipliers. The mode shapes and natural frequencies are determined via Kron's eigenvalue procedure. Calculation of the response to a random excitation in the 20 - 2000 Hz band.

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\[ [M] \text{ matrix see eq. 34} \]
\[ [N] \text{ matrix see eq. 34} \]
\[ Q \text{ fourier transform of } q(t) \]
\[ [R(t)] \text{ cross-correlation matrix} \]
\[ S(\omega) \text{ power spectral density} \]
\[ S_1, S_2 \text{ surface area plate 1 and 2} \]
\[ T \text{ kinetic energy} \]
\[ V \text{ potential or strain energy} \]
\[ W_1^1(x,y,t), W_2^2(x,y,t) \text{ momentary position of point (x,y) of plate 1 or 2 at time t} \]
\[ W_0 \text{ base offset} \]
\[ W_e^I(x,y), W_e^{II}(x,y) \text{ elastic deformation plate 1 & 2} \]
\[ \alpha \text{ matrix of the eigenmode vectors} \]
\[ \beta \text{ matrix see eq. 23 & 29} \]
\[ \gamma \text{ structural damping factor} \]
\[ \eta \frac{\rho L}{\rho H} \]
\[ \lambda_n \text{ Lagrangian multiplier} \]
\[ \nu \text{ poisson's ratio} \]
\[ \rho \text{ mass density} \]
\[ \tau \text{ time lag} \]
\[ \phi_i(x) \text{ polynomials of the Gram-Schmidt process for beam shape function in } x \]
\[ \psi_j(y) \text{ polynomials of the Gram-Schmidt process for beam shape function in } y \]
\[ \Omega^2 \frac{\rho w^2}{D} \]
\[ \omega \text{ angular frequency} \]
Summary

The normal modes and the response is derived using Lagrange's equations together with the Lagrangian multiplier method. The frequency expression is determined in terms of orthogonally generated polynomials. This leads to an eigenvalue expression together with a number of constraints equations. This is not a standard eigenvalue problem in the form of $[E - \lambda A]X = 0$

but an eigenvalue problem in the form of

$$
\begin{bmatrix}
E - \lambda A & -K \\
-K^T & 0
\end{bmatrix}
\begin{bmatrix}
X \\
\theta
\end{bmatrix} = 0
$$

which can be solved via a method developed by N.S. Sehmi [J.7] who is using a Kron eigenvalue procedure developed by W.H. Wittrick and F.W. Williams [J.2]. Then the response is calculated using the normal modes and the excitation given.
1. Introduction.

The assigned thesis topic, namely to find or develop a method to predict accurate localized response for structures randomly excited in the 100 - 500 Hz band, has evolved in the development of a 'new' method [lit. TU.1 & TU.2]. This 'new' method makes use of two existing methods. The Lagrange's equations are used to formulate the dynamic equations. The Lagrangian multipliers are used to enforce the constraint conditions. The normal mode method is used to calculate the localized response.

2. Problem description.

Space structures are often submitted to random vibrations in the 20 - 2000 Hz band. Until now the calculation of the response via finite element method does not give very accurate answers. Above 150 Hz for complicated structures (that consist of more than one type of element) the accuracy is rapidly decreasing.

In this report another method is used. The dynamical equations are derived via a minimalisation of the Lagrangian \( L = T - V \), using orthogonal polynomials for describing the elastic deformation.

There is a simplified construction of a solar array in folded position. The solar array consists of two sandwich panels and is held down at four points against the satellite. The yoke is being modeled by two discrete mass points. (see fig. 1.) The structure is excited via the two times four hold down points with a PSD(\( \tilde{u} \)) drawn in fig. 2.

![Solar array in folded position](image)

fig.1 Solar array in folded position.
3. a Theoretical model & dimensions.

In the following the equations used for plates are for isotropic plates, it is however also possible to write them in a similar manner for orthotropic plates.

![Diagram of a model with dimensions labeled](image)

- Dimensions: $a = 2000 \text{ mm}$
- $b = 1500 \text{ mm}$
- $c = 28 \text{ mm}$
- Plate density $\rho [\text{kg/mm}^3]$
- Plate thickness $h [\text{mm}]$
- Yoke mass $m = 1/2 \text{ kg}$
- $\rho \cdot h = 2 \times 10^{-6} [\text{kg/mm}^2]$

fig 3. The model used in the analysis.

The coordinates $x_i, y_i$ are for $i = 1, 2$ from the yoke mass

- $i = 3, \ldots, 10$ from the support points
- $i = 11, 12$ from the corner connections
3. b Derivation of the dynamic equations.

Let \( W^1 = \left\{ W_0 + W^I_{e}(x,y) \right\} e^{i\omega t} \)

\[ W^2 = \left\{ W_0 + W^{II}_{e}(x,y) \right\} e^{i\omega t} + c \] (1)

\( W^1 \) and \( W^2 \) are the momentary positions in the z-direction of the plate 1 and 2 the indices \( e \) stand for elastic deformation.

The kinetic energy (neglecting the mass of the two connecting rods and the four hold down point masses) is

\[
T = \frac{ph}{2} \int_{S_1} (\dot{W}^1)^2 dS_1 + \frac{ph}{2} \int_{S_2} (\dot{W}^2)^2 dS_2 + \frac{m}{2} \left\{ \left[ \dot{W}^1(x_1,y_1) \right]^2 + \left[ \dot{W}^1(x_2,y_2) \right]^2 \right\} 
\] (2)

The strain energy is

\[
V = \frac{1}{2} D^* \int_{S_1} \left[ \frac{\partial^2 W^1}{\partial x^2} \right]^2 + 2(1-\nu) \left( \frac{\partial^2 W^1}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 W^1}{\partial y^2} \right)^2 + 2v \frac{\partial^2 W^1}{\partial x^2} \frac{\partial^2 W^1}{\partial y^2} \right] dS_1 \\
+ \frac{1}{2} D^* \int_{S_2} \left[ \frac{\partial^2 W^2}{\partial x^2} \right]^2 + 2(1-\nu) \left( \frac{\partial^2 W^2}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 W^2}{\partial y^2} \right)^2 + 2v \frac{\partial^2 W^2}{\partial x^2} \frac{\partial^2 W^2}{\partial y^2} \right] dS_2 
\] (3)

In which \( D^* = D(1 + i\gamma) \) which is a structural damping factor.

\[
\dot{W}^1 = \left\{ W_0 + W^I_{e}(x,y) \right\} \omega \ e^{i\omega t} \\
(\dot{W}^1)^2 = - \left\{ W_0^2 + 2W_0W^I_{e}(x,y) + W^I_{e}(x,y) \right\} \omega^2 e^{2i\omega t} \\
W^2 = \left\{ W_0 + W^{II}_{e}(x,y) \right\} \omega \ e^{i\omega t} \\
(\dot{W}^2)^2 = - \left\{ W_0^2 + 2W_0W^{II}_{e}(x,y) + W^{II}_{e}(x,y) \right\} \omega^2 e^{2i\omega t} 
\] (4)

\[
\frac{\partial^2 W^1}{\partial x^2} = e^{i\omega t} \left\{ \frac{\partial^2 W^1}{\partial x^2} \right\} \\
\frac{\partial^2 W^1}{\partial y^2} = \left\{ \frac{\partial^2 W^1}{\partial y^2} \right\} \\
\frac{\partial^2 W^2}{\partial x^2} = e^{i\omega t} \left\{ \frac{\partial^2 W^2}{\partial x^2} \right\} \\
\frac{\partial^2 W^2}{\partial y^2} = \left\{ \frac{\partial^2 W^2}{\partial y^2} \right\} 
\] etc. (5)

Substitute the derivatives of eq. 1 into eq. 2 and 3.
Hence

\[ T = e^{2iwt} \left[ \frac{\rho \omega^2}{2} \int_{S_1} \left\{ W_0^2 + 2W_0W_e^I + W_e^I \right\} dS_1 + \right. \]
\[ \left. + \frac{\rho \omega^2}{2} \int_{S_2} \left\{ W_0^2 + 2W_0W_e^{II} + W_e^{II} \right\} dS_2 + \right. \]
\[ \left. + \frac{\mu \omega^2}{2} \left\{ W_0^2 + 2W_0W_e^I(x_1, y_1) + W_e^I(x_1, y_1) + \right. \right. \]
\[ \left. \left. + W_0^2 + 2W_0W_e^I(x_2, y_2) + W_e^I(x_2, y_2) \right\} \right] = \]
\[ e^{2iwt} \left[ \frac{\rho \omega^2}{2} \left( \int_{S_1} W_0^2 dS_1 + 2 \int_{S_1} W_0W_e^I dS_1 + \int_{S_1} W_e^I dS_1 + \right. \right. \]
\[ \left. \left. + \int_{S_2} W_0^2 dS_2 + 2 \int_{S_2} W_0W_e^{II} dS_2 + \int_{S_2} W_e^{II} dS_2 \right) + \right. \]
\[ \left. + \frac{\mu \omega^2}{2} \left( 2W_0W_e^I(x_1, y_1) + W_e^I(x_1, y_1) + W_0^2 + 2W_0W_e^I(x_2, y_2) + W_e^I(x_2, y_2) \right) \right) \] (6)

and

\[ V = e^{2iwt} \left[ \frac{D^*}{2} \int_{S_1} \left\{ \left( \frac{\partial W_e^I}{\partial x} \right)^2 + 2(1-\nu) \left( \frac{\partial W_e^I}{\partial x} \right)^2 \left( \frac{\partial W_e^I}{\partial y} \right)^2 + 2\nu \right. \right. \]
\[ \left. \left. + \frac{\partial^2 W_e^{II}}{\partial x^2} + 2(1-\nu) \left( \frac{\partial W_e^{II}}{\partial x} \right)^2 \left( \frac{\partial W_e^{II}}{\partial y} \right)^2 + 2\nu \right. \right. \]
\[ \left. \left. + \frac{\partial^2 W_e^{II}}{\partial y^2} \right\} dS_1 + \right. \]
\[ + \frac{D^*}{2} \left( \int_{S_2} \left( \frac{\partial W_e^I}{\partial x} \right)^2 dS_2 + 2(1-\nu) \left( \frac{\partial W_e^I}{\partial y} \right)^2 dS_2 + \right. \right. \]
\[ \left. \left. + 2\nu \int_{S_2} \frac{\partial^2 W_e^{II}}{\partial x^2} \right\} dS_2 + \int_{S_2} \left. \frac{\partial^2 W_e^{II}}{\partial y^2} \right\} dS_2 \right] \] (7)

\( W_e(x, y) \) is the elastic deformation of the plates and may be expressed as \( W_e(x, y) = \sum_{i} \int_{y} A_{1j} \varphi_{i} \psi_{j} \) in which \( \varphi_{i} \) is a function of \( x \) and \( \psi_{j} \) is a function of \( y \).

In [Ref. J.11] the functions taken for \( \varphi_{i} \) and \( \psi_{j} \) are beam shape functions with the same boundary conditions the plate has.
Thus \( W_e^I = \sum_i \sum_j A_{ij}^I \varphi_i(x) \varphi_j(y) \) \hspace{1cm} (8)

\( W_e^{II} = \sum_i \sum_j A_{ij}^{II} \varphi_i(x) \varphi_j(y) \)

Substitution of eq. 8 into 6 and 7 we get

\[
T = e^{2i\omega t} \left[ \frac{-\omega^2}{2} \left( W_0^2 S_1 + 2W_0 \int \int_{S1T} \sum_i \sum_j A_{ij}^I \varphi_i \psi_j dS1 + \int \int_{S1T} \left( \sum_i \sum_j A_{ij}^I \varphi_i \psi_j \right)^2 dS1 + \right. \right.

\[+ W_0^2 S_2 + 2W_0 \left( \sum_i \sum_j A_{ij}^{II} \varphi_i \psi_j \right) dS2 + \int \int_{S2T} \left( \sum_i \sum_j A_{ij}^{II} \varphi_i \psi_j \right)^2 dS2 \left. \right)

\[+ \frac{\omega}{2} \left( 2W_0 \left[ W_0 + \sum_i \sum_j A_{ij}^I \left( \varphi_i(x1) \psi_j(y1) + \varphi_i(x2) \psi_j(y2) \right) \right] + \left\{ \sum_i \sum_j A_{ij}^I \varphi_i(x1) \psi_j(y1) \right\}^2 + \left\{ \sum_i \sum_j A_{ij}^I \varphi_i(x2) \psi_j(y2) \right\}^2 \right) \right)
\]

\[ (9) \]

\[
V = e^{2i\omega t} \left[ \frac{-D}{2} \left( \int \int_{S1T} \left( \sum_i \sum_j A_{ij}^I \varphi_i x \psi_j \right)^2 dS1 + \int \int_{S1T} \left( \sum_i \sum_j A_{ij}^I \varphi_i x \psi_j \right) \left( \sum_i \sum_j A_{ij}^I \varphi_i y \psi_j \right) dS1 + \right. \right.

\[+ \int \int_{S2T} \left( \sum_i \sum_j A_{ij}^{II} \varphi_i x \psi_j \right)^2 dS2 + \int \int_{S2T} \left( \sum_i \sum_j A_{ij}^{II} \varphi_i x \psi_j \right) \left( \sum_i \sum_j A_{ij}^{II} \varphi_i y \psi_j \right) dS2 + \right. \right.

\[+ \int \int_{S2T} \left( \sum_i \sum_j A_{ij}^{II} \varphi_i x \psi_j \right)^2 dS2 + \int \int_{S2T} \left( \sum_i \sum_j A_{ij}^{II} \varphi_i x \psi_j \right) \left( \sum_i \sum_j A_{ij}^{II} \varphi_i y \psi_j \right) dS2 \left. \right) \right] \right] \]

\[ (10) \]

The beam shape function in x and y direction are orthogonal polynomials generated with a Gram – Schmidt process [Ref.B.3].
\[ \varphi_{k+1}(x) = \begin{cases} f(x) - B_k \varphi_k(x) - C_k \varphi_{k-1}(x) & \text{for } k = 1, 2, 3, \ldots \\ \end{cases} \]

In which

\[ B_k = \frac{\int_0^b f(x)w(x)\varphi_k^2(x)dx}{\int_0^b w(x)\varphi_k^2(x)dx} \]

\[ C_k = \frac{\int_0^b w(x)\varphi_k^2(x)dx}{\int_0^b w(x)\varphi_{k-1}^2(x)dx} \]

Also \( \varphi_0 = 0 \), \( w(x) \) is the weighting function and is taken unity in this application because the plate thickness is uniform, \( f(x) \) is the generating function, chosen so that the higher order polynomials satisfy certain boundary conditions.

According to [Ref.J.11] \( \varphi_i = \text{const.} \cdot \sum_{i=1}^{5} R_i \left( \frac{x}{b} \right)^{1-1} \) and for a

Free - Free beam \( R_i = 0 \) for \( i = 2-5 \) and \( R_1 = 1 \) [Ref.J.11, table 1], hence \( \varphi_1 = \text{const.} \). These polynomials satisfy the orthogonality

condition,

\[ \int_0^b w(x) \varphi_k(x) \varphi_1(x)dx = \begin{cases} 0 & \text{if } k \neq 1 \\ B_{k1} & \text{if } k = 1 \end{cases} \]

\( B_{k1} \) depends on the normalization used. With the right normalization \( B_{k1} = 1 \) [Ref.J.10] which makes the solution easier.

The generating function \( f(x) = x/b \). Using the method of [Ref.J.10] the Gram - Schmidt process becomes

\[ B_k = \frac{\int_0^b f(x)w(x)\hat{\varphi}_k^2(x)dx}{\int_0^b w(x)\hat{\varphi}_k^2(x)dx} = \int_0^b f(x) w(x) \hat{\varphi}_k^2(x)dx \]

\[ C_k = \frac{\int_0^b w(x)\hat{\varphi}_k^2(x)dx}{\int_0^b w(x)\hat{\varphi}_{k-1}^2(x)dx} = 1 \]

(12)

in which \( \hat{\varphi}_k(x) \) is the normalized polynomial \( \hat{\varphi}_k(x) = \frac{1}{\sqrt{B_{k1}}} \varphi_k(x) \).
Hence eq. 11 is now

$$\varphi_{k+1}(x) = \left\{ f(x) - \int_{0}^{b} f(x) \, w(x) \, \hat{\varphi}_{k}^{2}(x) \, dx \right\} \hat{\varphi}_{k}(x) - \hat{\varphi}_{k-1}(x) \quad k = 1, 2, 3, \ldots$$

$$= \left\{ \frac{x}{b} - \int_{0}^{b} \frac{\hat{\varphi}_{k}^{2}(x)}{b} \, dx \right\} \hat{\varphi}_{k}(x) - \hat{\varphi}_{k-1}(x)$$

(13)

The functions for the y direction (thus \( \psi_{j} \)) are generated in a like manner to \( \varphi_{i}(x) \), simply by replacing \( x \) by \( y \) and \( b \) by \( a \). For a free edge, the plates natural edge condition of zero normal bending and zero Kirchoff shear force cannot be satisfied term by term when using separation of variables solution, due to the Poisson effect.

The generation of \( \varphi_{i} \) and \( \psi_{j} \) gives

$$\hat{\varphi}_{0} = 0 \quad ; \quad \hat{\psi}_{0} = 0$$

$$\hat{\varphi}_{1} = \sqrt{\frac{1}{b}} \quad ; \quad \hat{\psi}_{1} = \sqrt{\frac{1}{a}}$$

$$\hat{\varphi}_{2} = \sqrt{\frac{12}{b}} \left[ \frac{x}{b} - \frac{1}{2} \right] \quad ; \quad \hat{\psi}_{2} = \sqrt{\frac{12}{a}} \left[ \frac{y}{a} - \frac{1}{2} \right]$$

$$\hat{\varphi}_{3} = \sqrt{\frac{180}{b}} \left[ \left( \frac{x}{b} \right)^{2} - \left( \frac{x}{b} \right) + \frac{1}{6} \right] \quad ; \quad \hat{\psi}_{3} = \sqrt{\frac{180}{a}} \left[ \left( \frac{y}{a} \right)^{2} + \left( \frac{y}{a} \right) + \frac{1}{6} \right]$$

$$\hat{\varphi}_{4} = \quad ; \quad \hat{\psi}_{4} = \quad \text{etc.} \quad (14)$$

The generation of higher order polynomials can be done using the computer program REDUCE (from MIT). [ref. B.6]
The Lagrangian $L = T - V$ is of the following form

$$
L = e^{2i\omega t} \left[ \frac{\rho \omega^2}{2} \left( w_0^2 S_1 + 2w_0 \sum_j A^I_{i,j} \int_{S_1} \varphi_i \varphi_j dS_1 + \int_{S_1} \left( \sum_j A^I_{i,j} \varphi_i \varphi_j \right)^2 dS_1 \right)
+ w_0^2 S_2 + 2w_0 \sum_j A^I_{i,j} \int_{S_2} \varphi_i \varphi_j dS_2 + \int_{S_2} \left( \sum_j A^I_{i,j} \varphi_i \varphi_j \right)^2 dS_2 \right) + \\
+ \frac{m \omega^2}{2} \left[ 2w_0 \left( w_0 + \sum_j A^I_{i,j} \left\{ \left( \varphi_i(x_1) \varphi_j(x_1) + \varphi_i(x_2) \varphi_j(x_2) \right) \right\} \right)
+ \left( \sum_j A^I_{i,j} \varphi_i(x_1) \varphi_j(x_1) \right)^2 \\
+ \left( \sum_j A^I_{i,j} \varphi_i(x_2) \varphi_j(x_2) \right)^2 \right] + \\
- \frac{D^*}{2} \left( \sum_j \sum_j A^I_{i,j} \varphi_i \varphi_j \right)^2 dS_1 + 2(1-\nu) \sum_j \sum_j \sum_j A^I_{i,j} \varphi_i \varphi_j \varphi_{i,j} dS_1 + \\
+ \sum_j \sum_j \sum_j A^I_{i,j} \varphi_i \varphi_j \varphi_{i,j} \varphi_{i,j} dS_1 + 2\nu \sum_j \sum_j \sum_j A^I_{i,j} \varphi_i \varphi_j \varphi_{i,j} \varphi_{i,j} dS_1 + \\
+ \sum_j \sum_j A^{II}_{i,j} \varphi_i \varphi_j \varphi_{i,j} \varphi_{i,j} dS_2 + 2(1-\nu) \sum_j \sum_j \sum_j A^{II}_{i,j} \varphi_i \varphi_j \varphi_{i,j} \varphi_{i,j} dS_2 + \\
+ \sum_j \sum_j A^{II}_{i,j} \varphi_i \varphi_j \varphi_{i,j} \varphi_{i,j} dS_2 + 2\nu \sum_j \sum_j \sum_j A^{II}_{i,j} \varphi_i \varphi_j \varphi_{i,j} \varphi_{i,j} dS_2 \right) \right) (15)
$$

Then there are the constraint equations

$$
W^1(x_1,y_1,t) = W_0 \quad i = 3 - 6 \quad \forall \ t \quad \text{or} \quad W^I_e(x_1,y_1) = 0 \quad i = 3 - 6
$$

$$
W^2(x_1,y_1,t) = W_0 + c \quad i = 7 - 10 \quad \forall \ t \quad \text{or} \quad W^{II}_e(x_1,y_1) = 0 \quad i = 7 - 10
$$

$$
W^2(x_1,y_1,t) - W^1(x_1,y_1,t) = c \quad i = 11, 12 \quad \forall \ t
$$

or $W^{II}_e(x_1,y_1) - W^I_e(x_1,y_1) = 0 \quad i = 11, 12$ (16)
Using the eq. 8 and substitute these into eq. 16 we get

\[ \chi_k (A^I_{ij}) = \sum_1^4 \sum_j A^I_{ij} \varphi_i (x_{k+2}) \psi_j (y_{k+2}) = 0 \quad \text{for } k = 1 - 4 \]

\[ \chi_k (A^{II}_{ij}) = \sum_1^4 \sum_j A^{II}_{ij} \varphi_i (x_{k+2}) \psi_j (y_{k+2}) = 0 \quad \text{for } k = 5 - 8 \]

\[ \chi_k (A^I_{ij}, A^{II}_{ij}) = \left\{ \sum_1^4 \left( A^{II}_{ij} - A^I_{ij} \right) \varphi_i (x_{k+2}) \psi_j (y_{k+2}) \right\} = 0 \quad \text{for } k = 9, 10 \]

(17)

The Lagrangian equations are

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}_r} \right) - \frac{\partial L}{\partial x_s} = \sum_k \lambda_k \frac{\partial \chi_k}{\partial A^\ell_{rs}} \]

(18)

Because \( A^\ell_{rs} \) is no function of the time, thus \( \frac{\partial L}{\partial \dot{x}_r} = 0 \)

Hence eq. 18 can be rewritten into

\[ \frac{\partial L}{\partial \dot{x}_r} + \sum_k \lambda_k \frac{\partial \chi_k}{\partial A^\ell_{rs}} = 0 \quad r,s = 1,2,3,..; \ell = I,II \]

(19)

[Ref.B.1; eq. 6.115]

Substituting eqs. 15 and 16 into eq. 18 yields after some regrouping

\[ \ell = 1 \quad \frac{\partial \omega^2}{2} \left\{ 2w_0 \left[ \sum_{s1} \varphi_r \psi_s \varphi_r \psi_s ds_1 + 2 \sum_{s1} \sum_j A^I_{ij} \varphi_i \psi_j \varphi_r \psi_s ds_1 \right] + \right. \]

\[ + \frac{m \omega^2}{2} \left[ 2w_0 \left( \varphi_r (x1) \psi_s (y1) + \varphi_r (x2) \psi_s (y2) \right) + \right. \]

\[ + 2 \sum_{j} \sum_j A^I_{ij} \varphi_i (x1) \psi_j (y1) \varphi_r (x1) \psi_s (y1) + \]

\[ + \left. \sum_j \sum_j A^I_{ij} \varphi_i (x2) \psi_j (y2) \varphi_r (x2) \psi_s (y2) \right\} + \]

\[ + \]
\[
- \frac{D^*}{2} \left\{ 2 \int_{S_1} \sum_j \sum_j A^I_{ij} \phi_{xx} \psi_{r_r} \psi_s dS_1 + \\
+ 4(1-\nu) \int_{S_1} \sum_j \sum_j A^I_{ij} \phi_{r_r} \psi_{r_r} \psi_s dS_1 + \\
+ 2 \int_{S_1} \sum_j \sum_j A^I_{ij} \phi_{yy} \psi_{r_r} dS_1 + \\
+ 2\nu \left\{ \int_{S_1} \sum_j \sum_j A^I_{ij} \phi_{xx} \psi_{r_r} \psi_s dS_1 + \\
+ \int_{S_1} \sum_j \sum_j A^I_{ij} \phi_{yy} \psi_{r_r} \psi_s dS_1 \right\} \right\} + \\
+ \sum_{k=5}^{10} \lambda_k \phi_r(x_{k+2}) \psi(y_{k+2}) - \sum_{k=5}^{10} \lambda_k \phi_r(x_{k+2}) \psi(y_{k+2}) = 0 \quad a
\]

\[
\tau = 2 - \frac{\rho h^2}{2} \left\{ 2w_0 \int_{S_2} \phi \psi dS_2 + 2 \int_{S_2} \sum_j \sum_j A^{II}_{ij} \phi_{r_r} \psi_s dS_2 + \\
+ 4(1-\nu) \int_{S_2} \sum_j \sum_j A^{II}_{ij} \phi_{yy} \psi_s dS_2 + \\
+ 2 \int_{S_2} \sum_j \sum_j A^{II}_{ij} \phi_{xx} \psi_s dS_2 + \\
+ 2\nu \left\{ \int_{S_2} \sum_j \sum_j A^{II}_{ij} \phi_{xx} \psi_{r_r} \psi_s dS_2 + \\
+ \int_{S_2} \sum_j \sum_j A^{II}_{ij} \phi_{yy} \psi_{r_r} \psi_s dS_2 \right\} \right\} + \\
+ \sum_{k=5}^{10} \lambda_k \phi_r(x_{k+2}) \psi(y_{k+2}) = 0 \quad b \quad (20)
\]
Let \( \frac{\rho \omega^2}{D} = \Omega^2 \); \( \frac{m}{\rho h} = \eta \)

\[
\int_0^b \frac{\partial}{\partial x} \varphi_k \frac{\partial}{\partial x} \varphi_r \, dx = E_{ir}^{(k,1)}
\]

\[
\int_0^a \frac{\partial}{\partial y} \psi_x \frac{\partial}{\partial y} \psi_s \, dy = F_{js}^{(m,n)}
\]

\[
\varphi_i(x_k) \psi_j(y_k) = K_{ij}(x_k, y_k) \quad ; \quad \iint_S \varphi_i \psi_j \, dS = G_{ij}
\] (21)

Substituting these into the eq. 20a-b.

Hence,

for \( \ell = 1 \)

\[
\Omega^2 \left\{ W_0 G_{rs} \right. + \sum_{j} \sum_{j} A_{ij}^{(0,0)} E_{ir}^{(0,0)} F_{js}^{(0,0)} + \eta W_0 \left[ K_{rs}(x_1, y_1) + K_{rs}(x_2, y_2) \right] + \eta \sum_{j} \sum_{j} A_{ij}^{(1,1)} \left[ K_{ij}(x_1, y_1) K_{rs}(x_1, y_1) + K_{ij}(x_2, y_2) K_{rs}(x_2, y_2) \right] + \n \end{array}
\]

\[
- (1 + \eta \gamma) \sum_{j} \sum_{j} A_{ij}^{(2,2)} E_{ir}^{(2,2)} F_{js}^{(2,2)} + 2(1 - \nu) E_{ir}^{(1,1)} F_{js}^{(1,1)} + E_{ir}^{(0,0)} F_{js}^{(2,2)} + \nu \left[ E_{ir}^{(2,0)} F_{js}^{(2,0)} + E_{ir}^{(0,2)} F_{js}^{(0,2)} \right] \}
\]

\[
+ \frac{\lambda}{D} \sum_{k=1}^{4} K_{rs}(x_{k+2}, y_{k+2}) - \frac{10\lambda}{D} K_{rs}(x_{k+2}, y_{k+2}) = 0
\]

for \( \ell = 2 \)

\[
\Omega^2 \left\{ W_0 G_{rs} \right. + \sum_{j} \sum_{j} A_{ij}^{(1,1)} E_{ir}^{(0,0)} F_{js}^{(0,0)} \right. \quad (1 + \eta \gamma) \sum_{j} \sum_{j} A_{ij}^{(2,2)} E_{ir}^{(2,2)} F_{js}^{(0,0)} + \n \end{array}
\]

\[
+ 2(1 - \nu) E_{ir}^{(1,1)} F_{js}^{(1,1)} + E_{ir}^{(0,0)} F_{js}^{(2,2)} + \nu \left[ E_{ir}^{(2,0)} F_{js}^{(2,0)} + E_{ir}^{(0,2)} F_{js}^{(0,2)} \right] \}
\]

\[
+ \sum_{k=5}^{10} \frac{\lambda}{D} K_{rs}(x_{k+2}, y_{k+2}) = 0
\]
Also let \( Z_{i,j} = E_{i,r} F_{j,s} \) + \( 2(1-\nu)E_{i,r} F_{j,s} \) + \( \nu \left[ E_{i,r} F_{j,s} + E_{i,r} F_{j,s} \right] \) \( (23) \)

and using the orthogonality of the shape functions

\[
E_{i,r}^{(0,0)} = \begin{cases} 0 & i \neq r \\ B_{i,r} & i = r \\ j \neq s \\ B_{i,s} & j = s \end{cases} \quad F_{j,s}^{(0,0)} = \begin{cases} 0 & j \neq s \\ B_{j,s} & j = s \end{cases}
\]

and \( B_{i,r} = B_{j,s} = 1 \) because the polynomials are normalized to unity.

Rewriting eq. 22 using eq. 23 & 24

Hence

for \( \ell = 1 \)

\[
A_{rs}^I \Omega^2 + \sum_I \sum_J A_{i,j}^I \left( (1 + i\gamma) Z_{i,j} + \eta \Omega^2 \left[ K_{ij}(x_1,y_1) K_{rs}(x_1,y_1) + K_{ij}(x_2,y_2) K_{rs}(x_2,y_2) \right] \right) + \sum_{k=1}^4 \frac{\lambda_k}{D} K_{rs}(x_{k+2},y_{k+2}) - \sum_{k=5}^{10} \frac{\lambda_k}{D} K_{rs}(x_{k+1},y_{k+1}) = -\Omega^2 \left[ \omega_0 G_{rs} + \eta \left( K_{rs}(x_1,y_1) + K_{rs}(x_2,y_2) \right) \right] \]

for \( \ell = 2 \)

\[
A_{rs}^{II} \Omega^2 + \sum_I \sum_J A_{i,j}^{II} \left( (1 + i\gamma) Z_{i,j} + \frac{10\lambda_k}{D} K_{rs}(x_{k+2},y_{k+2}) \right) = -\Omega^2 \omega_0 G_{rs}
\]

Now let \( \alpha_{rsIJ} = \frac{\eta \left[ K_{ij}(x_1,y_1) K_{rs}(x_1,y_1) + K_{ij}(x_2,y_2) K_{rs}(x_2,y_2) \right]}{Z_{i,j}^{rs}} \)

Then eq. 25 becomes

\[
A_{rs}^I \Omega^2 + \sum_I \sum_J A_{i,j}^I Z_{i,j}^{rs} (1 + i\gamma - \alpha_{rsIJ} \sum_I \sum_J A_{i,j}^{II} Z_{i,j}^{rs}) + \sum_{k=1}^4 K_{rs}(x_{k+2},y_{k+2}) + \sum_{k=5}^{10} \frac{\lambda_k}{D} K_{rs}(x_{k+1},y_{k+1}) = -\Omega^2 \left[ \omega_0 G_{rs} + \eta \left( K_{rs}(x_1,y_1) + K_{rs}(x_2,y_2) \right) \right] \]

\( (27^a) \)
the Eqs. 17 become using eq. 21

\[ x_k = \sum_i \sum_j A_{ij}^I K_{ij}(x_{k+2}^*, y_{k+2}^*) = 0 \quad \text{for } k = 1, -1, 4 \]

\[ x_k = \sum_i \sum_j A_{ij}^{II} K_{ij}(x_{k+2}^*, y_{k+2}^*) = 0 \quad \text{for } k = 5, -3, 8 \]

\[ x_k = \sum_i \sum_j \left( A_{ij}^{II} - A_{ij}^I \right) K_{ij}(x_{k+2}^*, y_{k+2}^*) = 0 \quad \text{for } k = 9, 10 \]  \hspace{1cm} (28)

The eqs. 28b, 27a & 28 form a system with an equal number of equations and unknowns.

Let \( [Z] \) and \( [\alpha] \) be

\[
[Z] = \begin{bmatrix}
Z_{1111} & Z_{1211} & Z_{1311} & Z_{1411} & \cdots & Z_{1j11} & \cdots & \cdots & \cdots & Z_{1n11} \\
Z_{1112} & & & & & & & & & \\
Z_{11rs} & Z_{12rs} & Z_{13rs} & Z_{14rs} & \cdots & Z_{1jrs} & \cdots & \cdots & \cdots & Z_{1nnrs} \\
Z_{11nn} & Z_{12nn} & Z_{13nn} & Z_{14nn} & \cdots & Z_{1jnn} & \cdots & \cdots & \cdots & Z_{1nnnn} \\
\end{bmatrix}
\]

(29)

\[ [\alpha] = \begin{bmatrix}
\alpha_{1111} & \alpha_{1211} & \alpha_{1311} & \cdots & \alpha_{1j11} & \cdots & \cdots & \cdots & \cdots & \alpha_{1n11} \\
\alpha_{1112} & & & & & & & & & \\
\alpha_{11rs} & \alpha_{12rs} & \alpha_{13rs} & \cdots & \alpha_{1jrs} & \cdots & \cdots & \cdots & \cdots & \alpha_{1nnrs} \\
\alpha_{11nn} & \alpha_{12nn} & \alpha_{13nn} & \cdots & \alpha_{1jnn} & \cdots & \cdots & \cdots & \cdots & \alpha_{1nnnn} \\
\end{bmatrix}
\]

And let \([C] = [Z] [\alpha] + [I]\)

Also \( \left\{ A^I \right\} = \left\{ A_{11}^I, A_{12}^I, A_{13}^I, A_{rs}^I, A_{nn}^I, A_{11}^{II}, A_{12}^{II}, A_{13}^{II}, A_{rs}^{II}, A_{nn}^{II} \right\} \) \hspace{1cm} (30)

19
\[
\left\{ \lambda^1 \right\} = \left\{ \lambda_1 / D, \lambda_2 / D, \ldots, \lambda_i / D, \ldots, \lambda_{10} / D \right\}
\]

and let \([K]\) be

\[
[K] = \frac{1}{D} \begin{bmatrix}
K_{11}(x3y3) & -K_{11}(x1y1) & K_{11}(x1y11) & K_{11}(x1y12) \\
-\varnothing & K_{rs}(x3y3) & -K_{rs}(x3y3) & K_{rs}(x6y6) \\
K_{nn}(x3y3) & -K_{nn}(x3y3) & K_{nn}(x6y6) & -K_{nn}(x6y6) \\
\end{bmatrix}
\]

The excitation vector \(\{E\}\) is

\[
\begin{bmatrix}
G_{11} + \eta[K_{11}(x1,y1) + K_{11}(x2,y2)] \\
G_{12} + \eta[K_{12}(x1,y1) + K_{12}(x2,y2)] \\
G_{rs} + \eta[K_{rs}(x1,y1) + K_{rs}(x2,y2)] \\
G_{nn} + \eta[K_{nn}(x1,y1) + K_{nn}(x2,y2)] \\
\end{bmatrix} = \Omega^2 \mathcal{W}_0 \left[
\begin{array}{c}
G_{11} \\
G_{12} \\
G_{rs} \\
G_{nn}
\end{array}
\right],
\]

(32)
Then the system of equations is as follows

\[
\begin{bmatrix}
Z(1+i\gamma)-\Omega^2 C \\
Z(1+i\gamma)-\Omega^2 I
\end{bmatrix}
\begin{bmatrix}
A \\
K
\end{bmatrix}
= 
\begin{bmatrix}
E \\
\lambda
\end{bmatrix}
\]

(33)

First we have to solve the normal mode problem thus no damping and no excitation. \(D^* = D\) or \(\gamma = 0\) and no excitation thus \(\{E\} = 0\).

Then the system of eq. 33 becomes

\[
\begin{bmatrix}
M \\
-K^T
\end{bmatrix}
\begin{bmatrix}
N \\
\varnothing
\end{bmatrix}
\begin{bmatrix}
A \\
\lambda
\end{bmatrix}
= \varnothing
\]

(34)

where the following matrices have been used

\[
M = \begin{bmatrix}
Z & \varnothing \\
\varnothing & Z
\end{bmatrix}
\quad\text{and}\quad
N = \begin{bmatrix}
C & \varnothing \\
\varnothing & I
\end{bmatrix}
\]

Notice that \([M]\) is a real symmetric and non-negative definite matrix, and \([N]\) is real symmetric positive definite matrix.

This system is solved via a Newtonian procedure developed by N.S. Sehmi [Ref.J.7], which is described in appendix A, using an algorithm by Wittrick, Williams and Simpson described in appendix B [Ref.J.1, J2, J3].
3.6 Calculation of the response.

Now we know the normal modes of the structure and the consecutive eigenfrequencies. Thus with the excitation given in the problem description the response can be calculated.

The dynamic equations are eq. 33

\[
\begin{bmatrix}
M(1+i\gamma) - \Omega^2 N
\end{bmatrix} \mathbf{A} = \mathbf{F}
\]

For the calculation of the response \( \mathbf{A} \) and \( \mathbf{F} \) are depending on \( t \) again.

Thus

\[
\begin{bmatrix}
M(1+i\gamma) - \Omega^2 N
\end{bmatrix} \mathbf{A}(t) = \mathbf{F}(t)
\]

(35)

now let \( \mathbf{A}(t) = [X] \mathbf{q}(t) \) \( \quad (36) \)

In which \( [X] \) is the modal matrix containing the normal modes. In eq. 35 we don’t need to use the boundary constraints because eq. 36 makes that all displacements are functions of normal modes which satisfy the boundary conditions.

\( [\Lambda] \) is the diag. matrix containing \( \{\omega_1^2, \omega_2^2, \ldots, \omega_i^2, \ldots, \omega_q^2\} \).

substitution of eq. 36 into eq. 35 yields

\[
[M][X]\mathbf{q}(t) + i\gamma[M][X]\mathbf{q}(t) - \frac{\rho h \omega^2}{D} [N][X]\mathbf{q}(t) = \mathbf{F}(t)
\]

(37)

Premultiply with \( [X]^T \) we obtain

\[
[X]^T[M][X](1+i\gamma)\mathbf{q}(t) - \frac{\rho h \omega^2}{D} [X]^T[N][X]\mathbf{q}(t) = [X]^T\mathbf{F}(t)
\]

and

\[
[X]^T\mathbf{F}(t) = [\Lambda][\Lambda]^{-1}[X]^T\mathbf{F}(t) = [\Lambda]\mathbf{f}(t)
\]

(38)

In which

\[
\mathbf{f}(t) = [\Lambda]^{-1}[X]^T\mathbf{F}(t)
\]

(39)

Now let \( Q(\omega) = \int_{-\infty}^{\infty} \mathbf{q}(t) e^{-i\omega t} \, dt \)

\[
E(\omega) = \int_{-\infty}^{\infty} \mathbf{f}(t) e^{-i\omega t} \, dt
\]

(40)

Taking the Fourier transform of eq. 38 one gets
\[ Q(\omega) \left[ [X]^T \left\{ [M](1 + i\gamma) - \frac{\rho h u^2}{D} [N] \right\} [X] \right] = [A][E(\omega) \right] \quad (41) \]

Now let \( Q(\omega) = \begin{bmatrix} H(\omega) \end{bmatrix} \quad \mathcal{E}(\omega) \)

thus \[ \left[ H(\omega) \right] = \left[ [X]^T \left\{ [M](1 + i\gamma) - \frac{\rho h u^2}{D} [N] \right\} [X] \right]^{-1} \quad (42) \]

\( [H(\omega)] \) is the complex frequency matrix.

\( [H^*(\omega)] \) is the complex conjugate of \( [H(\omega)] \).

Also we know \[ W^1_e(x, y, t) = \sum_j \sum_j A_{ij}^1(t) \varphi(x) \psi(y) \quad (43) \]

and \[ W^2_e(x, y, t) = \sum_j \sum_j A_{ij}^2(t) \varphi(x) \psi(y) \]

or in matrix notation \[ \begin{bmatrix} W^1_e(x, y, t) \\ W^2_e(x, y, t) \end{bmatrix} = \begin{bmatrix} K(x, y) & 0 \\ 0 & K(x, y) \end{bmatrix}^T A(t) \quad (44) \]

(using notations eq. 21)

The cross-correlation of the response is \[ R_{w_e}(x, y, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} \begin{bmatrix} W^1_e(x, y, t) \\ W^2_e(x, y, t) \end{bmatrix} \begin{bmatrix} W^1_e(x, y, t + \tau), W^2_e(x, y, t + \tau) \end{bmatrix} dt \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} \begin{bmatrix} K(x, y) & 0 \\ 0 & K(x, y) \end{bmatrix}^T \{A(t)\} \{A(t + \tau)\} \begin{bmatrix} K(x, y) & 0 \\ 0 & K(x, y) \end{bmatrix} dt \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}^T \{X|q(t)\} \{q(t + \tau)\}^T \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} dt \]

\[ = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}^T [X] \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} \{q(t)\} \{q(t + \tau)\}^T dt [X]^T \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \]

\[ = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}^T [X] \begin{bmatrix} R_{q(\tau)}(\tau) \end{bmatrix} [X]^T \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \quad (45) \]
In which
\[
\begin{bmatrix}
R_{\alpha}(\tau)
\end{bmatrix} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \{g(t)\} \{g(t+\tau)\} \, dt
\]  
(46)

Also
\[
\begin{bmatrix}
R_{\alpha}(t)
\end{bmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ H^*(\omega) \right] \left[ S_{f}(\omega) \right] \left[ H(\omega) \right] e^{i\omega t} \, d\omega
\]  
(47)

where
\[
\begin{bmatrix}
S_{f}(\omega)
\end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix}
R_{f}(\tau)
\end{bmatrix} e^{-i\omega \tau} \, d\tau
\]  
(48)

and
\[
\begin{bmatrix}
R_{f}(\tau)
\end{bmatrix} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \{f(t)\} \{f(t+\tau)\}^T \, dt
\]  
(49)

In eq. 39 we see that \(\{f(t)\} = [\Lambda]^{-1} \{X\}^T \{F(t)\}\) thus
\[
\{f(t+\tau)\}^T = \{F(t+\tau)\}^T [X][\Lambda]^{-1}
\]  
(50)

Substitution of eq. 50 into eq. 49 hence
\[
\begin{bmatrix}
R_{f}(\tau)
\end{bmatrix} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[ \Lambda \right]^{-1} \{X\}^T \{F(t)\} \{F(t+\tau)\}^T [X][\Lambda]^{-1} \, dt
\]  
(51)

In which
\[
\begin{bmatrix}
R_{E}(\tau)
\end{bmatrix} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \{E(t)\} \{E(t+\tau)\}^T \, dt
\]  
(52)

Introducing eq. 51 into eq. 48 yields
\[
\begin{bmatrix}
S_{f}(\omega)
\end{bmatrix} = \int_{-\infty}^{\infty} \left[ \Lambda \right]^{-1} \{X\}^T \begin{bmatrix}
R_{E}(\tau)
\end{bmatrix} [X][\Lambda]^{-1} e^{-i\omega \tau} \, d\tau
\]  
(53)

\[
= \left[ \Lambda \right]^{-1} \{X\}^T \left[ \int_{-\infty}^{\infty} \begin{bmatrix}
R_{E}(\tau)
\end{bmatrix} e^{-i\omega \tau} \, d\tau \right] [X][\Lambda]^{-1}
\]  
(54)

\[
= \left[ \Lambda \right]^{-1} \{X\}^T \begin{bmatrix}
S_{E}(\omega)
\end{bmatrix} [X][\Lambda]^{-1}
\]  
(55)
where 
\[ S_E(\omega) = \int_{-\infty}^{\infty} R_{E}(\tau) e^{-i\omega\tau} d\tau \]  
(54)

Substituting eq. 53 into eq. 47 we obtain

\[ R_{Q}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ H^*(\omega) \right][\Lambda]^{-1}[X]^T \left[ S_E(\omega) \right][X][\Lambda]^{-1} \left[ H(\omega) \right] e^{i\omega\tau} d\omega \]
(55)

and the result into eq. 44

\[ R_{w_e}(x, y, \tau) = \frac{1}{2\pi} \begin{bmatrix} K & Q \end{bmatrix}^T \left[ X \right] \int_{-\infty}^{\infty} \left[ H^*(\omega) \right][\Lambda]^{-1}[X]^T \left[ S_E(\omega) \right][X][\Lambda]^{-1} \left[ H(\omega) \right] e^{i\omega\tau} d\omega [X]^T \begin{bmatrix} K & Q \end{bmatrix} \]
(56)

In eq. 32 the excitation vector is given

\[ F = \Omega^2 W_0 \begin{bmatrix} G_{11} + \eta[K_{11}(x_1, y_1) + K_{11}(x_2, y_2)] \\ G_{12} + \eta[K_{12}(x_1, y_1) + K_{12}(x_2, y_2)] \\ \vdots \\ G_{nn} + \eta[K_{nn}(x_1, y_1) + K_{nn}(x_2, y_2)] \end{bmatrix} = \rho \omega^2 \begin{bmatrix} G_{11} + \eta[K(x_1, y_1) + K(x_2, y_2)] \\ G_{12} \\ \vdots \\ G_{nn} \end{bmatrix} \]

\[ = \rho \omega^2 \begin{bmatrix} G_{11} + \eta[K(x_1, y_1) + K(x_2, y_2)] \\ G_{12} \\ \vdots \\ G_{nn} \end{bmatrix} \]

in which \( G = \begin{bmatrix} G_{11} & G_{12} & \vdots & G_{1n} \\ G_{21} & G_{22} & \vdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{bmatrix} \) and \( \eta K = \begin{bmatrix} K_{11}(x_1, y_1) + K_{11}(x_2, y_2) \\ K_{12}(x_1, y_1) + K_{12}(x_2, y_2) \\ \vdots \\ K_{nn}(x_1, y_1) + K_{nn}(x_2, y_2) \end{bmatrix} \)
(57)
Substitution of eq. 57 and eq. 32 into eq. 54 using eq. 52 yields

\[
\begin{align*}
[S_E(\omega)] &= \int_{-\infty}^{\infty} [R_E(\tau)] e^{-i\omega \tau} d\tau = \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \{E(t)\} \{E(t+\tau)\}_T^T \cdot d\tau e^{-i\omega \tau} d\tau = \\
&= \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{\phi \omega^2}{D} W_0(t) \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T \cdot e^{-i\omega \tau} \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T \cdot W_0(t+\tau) \frac{\phi \omega^2}{D} dt e^{-i\omega \tau} d\tau = \\
&= \frac{\rho h \omega^4}{D^2} \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T \cdot \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} W_0(t) W_0(t+\tau) dt e^{-i\omega \tau} \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T = \\
&= \frac{\rho h \omega^4}{D^2} \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T \cdot \int_{-\infty}^{\infty} \left[ R_{W_0}(\tau) \right] e^{-i\omega \tau} \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T = \\
&= \frac{\rho h \omega^4}{D^2} \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T \cdot \left[ S_{W_0}(\omega) \right] \left[ G + \eta K \right]_T = (58)
\end{align*}
\]

In the problem description and Ref. TU1 the power spectral density of the acceleration is given. Now is the relation between the PSD of the displacement and the acceleration equal to

\[
[S_{W_0}(\omega)] = \omega^4 \left[ S_{W_0}(\omega) \right] (59)
\]

Hence

\[
[S_E(\omega)] = \frac{\rho h}{D^2} \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right]_T \cdot \left[ S_{W_0}(\omega) \right] \left[ G + \eta K \right]_T = (60)
\]

Now the cross-correlation of the displacement can be calculated.

\[
[R_{W_e}(x,y,\tau)] = \frac{\rho h}{2\pi D^2} \left[ \begin{array}{c} \Delta K(x,y) \ 0 \\ 0 \ K(x,y) \end{array} \right]_T [X] * \\
\cdot \int_{-\infty}^{\infty} \left[ \begin{array}{c} H(\omega) \ \Delta^{-1}[X]_T \left[ \begin{array}{c} G + \eta K \\ G \end{array} \right] \left[ S_{W_0}(\omega) \right] \left[ G + \eta K \right]_T [X] \Delta^{-1}[H(\omega)] e^{i\omega \tau} dw \right. \\
\cdot \left. [X]^T \left[ \begin{array}{c} K(x,y) \ 0 \\ 0 \ K(x,y) \end{array} \right] \right] (61)
\]

The cross-correlation of the acceleration which is needed for
calculating the load levels can be derived in a same manner.

\[
\begin{bmatrix}
R_{w_e}^*(x, y, \tau)
\end{bmatrix} = \frac{\rho^2 h^2}{2\pi D^2} \begin{bmatrix}
K(x, y) & 0 \\
0 & K(x, y)
\end{bmatrix}^T [X] \times \\
\times \int_{-\infty}^{\infty} \begin{bmatrix}
H(\omega)
\end{bmatrix} [A]^{-1} [X]^T \begin{bmatrix}
G + \eta K \\
G
\end{bmatrix} S_{\tilde{W}_0}(\omega) \begin{bmatrix}
G + \eta K
\end{bmatrix}^T [X] [A]^{-1} \begin{bmatrix}
H(\omega)
\end{bmatrix} \omega^4 e^{i\omega \tau} d\omega \\
\times [X]^T \begin{bmatrix}
K(x, y) & 0 \\
0 & K(x, y)
\end{bmatrix}
\]

(62)

For \( \tau = 0 \),

\[
\begin{bmatrix}
R_{w_e}(x, y, 0)
\end{bmatrix} = \begin{bmatrix}
E[W_{w_e}(x, y, t)^2] & E[W_{w_e}(x, y, t)\tilde{W}_{w_e}(x, y, t)] \\
E[W_{w_e}(x, y, t)\tilde{W}_{w_e}(x, y, t)] & E[\tilde{W}_{w_e}(x, y, t)^2]
\end{bmatrix}
\]

(63)

or \[
\begin{bmatrix}
R_{w_e}(x, y, 0)
\end{bmatrix} = \begin{bmatrix}
E[W_{w_e}(x, y, t)^2] & E[W_{w_e}(x, y, t)\tilde{W}_{w_e}(x, y, t)] \\
E[W_{w_e}(x, y, t)\tilde{W}_{w_e}(x, y, t)] & E[\tilde{W}_{w_e}(x, y, t)^2]
\end{bmatrix}
\]

(64)

in which \( E[f(x, y, t)] \) is the mean

\[
E[f(x, y, t)] = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} f(x, y, t) dt
\]

(65)

The integral in eq. 61 & 62 has to be evaluated numerically, the most easy (and with the smallest inaccuracy) is to sum with a step of \( k\pi \).

\[
\begin{bmatrix}
H(\omega)
\end{bmatrix} [A]^{-1} [X]^T \begin{bmatrix}
G + \eta K
\end{bmatrix} S_{\tilde{W}_0}(\omega) \begin{bmatrix}
G + \eta K
\end{bmatrix}^T [X] [A]^{-1} \begin{bmatrix}
H(\omega)
\end{bmatrix} d\omega = \\
4000 \frac{\pi}{k}
\]

\[
\sum_{j=-\frac{4000}{k}}^{\frac{4000}{k}} S_{\tilde{W}_0}(j, k\pi) \begin{bmatrix}
H(j, k\pi)
\end{bmatrix} [A]^{-1} [X]^T \begin{bmatrix}
G + \eta K
\end{bmatrix} \begin{bmatrix}
G + \eta K
\end{bmatrix}^T [X] [A]^{-1} \begin{bmatrix}
H(j, k\pi)
\end{bmatrix} k\pi
\]

(66)

equally the integral of eq. 62
\[
\int_{-4000\pi}^{4000\pi} \left[ H^*(\omega) \right] (\Lambda)^{-1} [X]^T \left\{ \begin{bmatrix} G + \eta K \\ G \end{bmatrix} \right\} S_{W_0}(\omega) \left\{ \begin{bmatrix} G + \eta K \\ G \end{bmatrix} \right\}^T [X] (\Lambda)^{-1} \left[ H(\omega) \right] \omega^4 d\omega = \\
\sum_{k}^{4000} k \left[ S_{W_0}(j.k\pi) \right] \left[ H^*(j.k\pi) \right] (\Lambda)^{-1} [X]^T \left\{ \begin{bmatrix} G + \eta K \\ G \end{bmatrix} \right\} \left\{ \begin{bmatrix} G + \eta K \\ G \end{bmatrix} \right\}^T [X] (\Lambda)^{-1} \left[ H(j.k\pi) \right] (j.k\pi)^4 k \pi
\] (67)

This evaluation takes rather a lot of calculations because it needs to calculate \left[ H^*(j.k\pi) \right] and \left[ H(j.k\pi) \right] a couple of hundred or thousand times.

The dimensions of \left[ H(\omega) \right] are q x q which is equal to 2 * n^2 n equal the order of the polynomial thus for example n = 3 \Rightarrow q = 18, n = 4 \Rightarrow q = 32, or n = 5 \Rightarrow q = 50. According to [Ref. J11] n = 5 or 6.


4.1 Conclusions.

- This method can be used to calculate the localized response (see eq. 61 & 62).
- This method will (probably) give accurate answers if one can find the appropriate shape functions.
- For higher modes many beam shape functions \varphi_i(x) and \psi_j(x) are necessary to give accurate description of the normal mode shapes hence the computing time will grow fast with the complexity of the problem.

4.2 Recommendations.

- Further research is needed in the calculation of the complex frequency matrix (find out if it can be diagonalized).
- Further research is needed in finding better orthonormal mode shape functions.
Appendix A.

A Newtonian procedure to solve a Kron eigenvalue problem via an algorithm developed by N.S. Sehmi. [Ref.J.7].

The problem of eq. 34 is alike of eq. 11 of [Ref.J.7].

\[
\begin{bmatrix}
E - \lambda A & -K^T \\
-K & \varnothing
\end{bmatrix}
\begin{bmatrix}
x \\
\beta
\end{bmatrix} = 0
\]

In which first the standard eigenvalue problem

\[
\begin{bmatrix}
E - \lambda A
\end{bmatrix} x = 0
\]

has to be solved. \[A.1^a\]

\(E\) is a real symmetric non-negative definite matrix. \(A\) is a real symmetric positive definite matrix. Let \(\begin{bmatrix} M \end{bmatrix}\) be the matrix consisting the eigenvectors \(x_i\) of eq. \(A.1^a\) and \(\begin{bmatrix} \Lambda \end{bmatrix}\) be the matrix with the eigenvalues. Then \(\begin{bmatrix} M^T \end{bmatrix} \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} \Lambda \end{bmatrix}\) and

\(\begin{bmatrix} M^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}\) and \(\begin{bmatrix} I \end{bmatrix}\) is the identity matrix.

To solve \(\begin{bmatrix} E - \lambda A \end{bmatrix} x = 0\) \[A.2\]

we first decompose \(A\) into \(\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} L^T \end{bmatrix}\) \[A.3\]

in which \(L\) is a lower triangular matrix formed via a Cholesky decomposition of \(A\). [Ref.B.5;§5.8], also let \(x = L^{-T} y\) in which

\(L^{-T} = \begin{bmatrix} L^{-1} \end{bmatrix} = \begin{bmatrix} L^T \end{bmatrix}^{-1}\) \[A.4\]

[Ref.B.5:p.43]. Substitution into eq. \(A.2\) we get

\[
\begin{bmatrix}
E L^{-T} - \lambda L L^{-T}
\end{bmatrix} y = 0
\]

premultiplied with \(L^{-1}\) we get

\[
\begin{bmatrix}
L^{-1} E L^{-T} - \lambda^{-1} L L^{-1} L^{-T}
\end{bmatrix} y = 0
\] or

\[
\begin{bmatrix}
L^{-1} E L^{-T} - \lambda I
\end{bmatrix} y = 0
\]

being a standard eigenvalue problem to be solved via an appropriate method e.g. the power method with hotelling deflation or by the
Jacobi method [Ref. B.5: §5.5, 5.6 or §5.9].
Then calculate the \( x_i = L y_i \) and normalize the \( x_i \) to \( x_i = 1 \).

Now \( \begin{bmatrix} M \\ \Lambda \end{bmatrix} \) and \( \begin{bmatrix} \Lambda \\ \Lambda \end{bmatrix} \) are known. Back to the original problem eq. A.1, via the

transformation \( \mathbf{x} = \begin{bmatrix} M \\ \Lambda \end{bmatrix} \mathbf{q} \) the eq. A.1 gives,

\[
\begin{bmatrix}
\Lambda - \lambda I & -\Gamma \\
-\Gamma^T & \varnothing
\end{bmatrix}
\begin{bmatrix}
\mathbf{q} \\
\mathbf{p}
\end{bmatrix} = \mathbf{0}
\]  

A.7

and

\( \Gamma = \begin{bmatrix} K & M \end{bmatrix}^T \)  

A.8

The first equation of A.7 gives for \( \mathbf{q} \)

\[
\mathbf{q} = \begin{bmatrix} \Lambda - \lambda I \end{bmatrix}^{-1} \Gamma \mathbf{p}
\]  

A.9

eliminating \( \mathbf{q} \) from the second equation

\( \Gamma^T \mathbf{q} = \Gamma^T \begin{bmatrix} \Lambda - \lambda I \end{bmatrix}^{-1} \Gamma \mathbf{p} = \mathbf{0} \) or \( R(\lambda) \mathbf{p} = \mathbf{0} \)  

A.10

with \( R = \Gamma^T \begin{bmatrix} \Lambda - \lambda I \end{bmatrix}^{-1} \Gamma = \Gamma^T \Omega \Gamma \) with

\[
D = \begin{bmatrix} \Lambda - \lambda I \end{bmatrix}^{-1}
\]  

A.11

\( \begin{bmatrix} D \end{bmatrix} \) is a diagonal matrix of the following form

\[
D = \begin{bmatrix}
1/\lambda_1 - \lambda & 1/\lambda_2 - \lambda \\
1/\lambda_1 - \lambda & 1/\lambda_2 - \lambda \\
1/\lambda_3 - \lambda & 1/\lambda_4 - \lambda
\end{bmatrix}
\]  

which is easily constructed.

\( \begin{bmatrix} R \end{bmatrix} \) is a symmetric matrix of order \( n \), \( n \) equal to the number of constraint relations.

The characteristic values of \( \lambda \) arise from the solution of

\[
\begin{bmatrix} R(\lambda) \end{bmatrix} \mathbf{p} = \mathbf{0} \)  Two solutions are possible
case a. is most common while case b. is rare; it implies that motion can occur with all constraint forces zero (thus the mode has nodes at all points of constraints). Not every time \( |R(\lambda) | = \omega \) is the \( \beta \) vector equal to zero. For every \( \lambda = \lambda \_i \) (where the \( \lambda \_i \) are the diagonal terms of the \( \Lambda \) matrix) \( |R(\lambda) | = \omega \).

When the number of constraints increase the possibility of solutions of case b. decrease.

In appendix B an algorithm developed by Simpson [Ref. J.3] is shown, which states that for a given trial \( \lambda \), say \( \lambda \_i \), the exact number of natural frequencies of the structure which exceed \( \lambda \_i \) equals

\[
K(\lambda \_i) = s [D(\lambda \_i)] - [R(\lambda \_i)]
\]

Where \( K(\lambda \_i) \) represents the exact number of eigenfrequencies exceeding \( \lambda \_i \) and \( s[D(\lambda \_i)] \) and \( s[R(\lambda \_i)] \) are the sign counts of the matrix \( D \) or \( R \) respectively.

In this form \( R \) is not in a convenient form for scanning \( \lambda \). A way for scanning \( R \) is reducing it to a scalar upon which a quadratically convergent scheme to locate the characteristic values may be implemented.

Thus let

\[
\begin{bmatrix}
\beta \\
\end{bmatrix} = \begin{bmatrix}
R_1 & R_2 \\
R_2^T & R_4 \\
\end{bmatrix} \begin{bmatrix}
\bar{\beta} \\
\beta_n \\
\end{bmatrix} = 0
\]

A.15

and \( \beta_n \) is the last element of \( \beta \). Expanding the first equation of A.15 we get

\[
\bar{\beta} = -\begin{bmatrix}
R_1 \\
\end{bmatrix}^{-1} \begin{bmatrix}
R_2 \\
\end{bmatrix} \beta_n
\]

A.16

Then eliminating \( \bar{\beta} \) from the second equation

\[
\begin{bmatrix}
R_4 - R_2^T R_1 R_2 \\
\end{bmatrix} \beta_n = r_4 \cdot \beta_n
\]

A.17

Eq. A.17 suggests an inversion of \( R_1 \) for every trial frequency, this is not necessary because \( r_4 \) is formed during the sign count process, i.e.

\[
r_4 = R_{nn}(\lambda \_i)
\]

A.18
in which $R_{nn}(\lambda^*)$ is the last diagonal element of the upper
triangulated matrix $R_{nn}(\lambda^*)$.

For the non trivial solutions $r_4 = 0$; $\beta_n = 0$ and
$r_4 = \infty$; $\beta_n = 0$.

a Newton - Raphson process is applied to equation (A.17).

Hence $\lambda_{i+1} = \lambda_i - \frac{r_4(\lambda_i)}{(dr_4/d\lambda)}$ A.19

To get the derivative of $r_4$ in terms of the vector $\beta$ first assume
$|R_1| \neq 0$ and that there are no multiple roots, then for the non
trivial case $r_4 = 0$; $\beta_n = 1$ and with eq. A.16 we can write

$$\beta^T = \left\{ \begin{bmatrix} -R_1^{-1}R_2 \end{bmatrix}^T, 1 \right\}$$ A.20

Then

$$r_4 = R_4 - R_2^T R_1^{-1} R_2 = \beta^T R \beta$$ A.21

Differentiation with respect to $\lambda$ gives

$$\frac{dr_4}{d\lambda} = 2 \frac{d\beta^T}{d\lambda} R \beta + \beta^T \left[ \frac{dR}{d\lambda} \right] \beta$$ A.22

in which

$$\frac{d\beta^T}{d\lambda} = \left[ -\frac{d}{d\lambda} \left( R_2^T R_1^{-1} \right), 0 \right]$$ A.23

and

$$R \beta = \left\{ 0, r_4 \right\}^T$$

thus

$$\frac{dr_4}{d\lambda} = \beta^T \frac{dR}{d\lambda} \beta$$ A.24

Eq. A.11 says $R = \Gamma^T D \Gamma$ with $D = \left[ \Lambda - \lambda I \right]^{-1}$

thus

$$\frac{dR}{d\lambda} = \Gamma^T \frac{dD}{d\lambda} \Gamma = \Gamma^T D^2 \Gamma$$ A.25

Now

$$\frac{dr_4}{d\lambda} = \beta^T \Gamma^T D \Gamma \beta = \left( DR \beta \right)^T DR \beta$$ A.26

From eq. A.26 we can see that $\frac{dr_4}{d\lambda}$ can never be negative.

Since the poles of $r_4$ are given by $|R_1| = 0$ (eq. A.17) it follows
that there can be only one zero between two consecutive of $r_4$. If
multiple roots exist $\frac{dr_4}{d\lambda} = 0$.

The number of poles $k_p^*$ exceeded by $\lambda^*$ is given by an expression
like eq. A.14

$$k_p^*(\lambda^*) = s \left[ D(\lambda^*) \right] - s \left[ R_1(\lambda^*) \right]$$ A.27

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The pole count, at each estimate of $\lambda_1$, eq. A.19 implies that $m - 1 < K(\lambda_1) < m + 1$ when $K(\lambda_1) = m$. 

Hence eq. A.19 now becomes $\lambda_{i+1} = \lambda_i - r_4(\lambda_i)\left((D\Gamma\beta)^T(D\Gamma\beta)\right)^{-1}((D\Gamma\beta)^T(d\Gamma\beta))$. 

The assumption made that $\beta_n \neq 0$ can lead to a problem, when $\beta_n$ is very small it follows that poles and required $\lambda$'s are very close near each other. This can lead to very sharp changes in the gradient of $r_4$ which can lead to "pole jumping" or overshooting the pole bounds.

Fig. A1: When $\beta_n$ is small, poles and zeros are pushed together resulting in "pole jumping".

The value of $\lambda_{i+1}$ of eq. A.29 would have to be discarded and a new $\lambda_{i+1}$ has to be chosen closer to required $\lambda$. This leads to slow convergence and extra Newton steps.

The process is deemed to have converged when

$$\left| \lambda_{i+1} - \lambda_i \right| / \lambda_i < \varepsilon$$

in which $\varepsilon$ is the error level.

When convergence is reached the vector $\beta$ can be determined via sequential back substitution in

$$[R] \beta = 0$$

$\beta$ can be obtained by setting $\beta_n = 1$ and then calculating $\beta_{n-1}, \beta_{n-2}$ etc. [Ref. B.5:p-112].

Using eq. A.9 $q = D\Gamma\beta$ can be determined. For more information about this method see [J.1, J.2, J.3, J.5, J.6, J.7].
Appendix B

A "sign count" algorithm for [D] and [R] developed by W.H. Wittrick, F.W. Williams [J.2] and A.Simpson [J.3],

Consider any real symmetric matrix which is non singular. The number of positive eigenvalues + the number of negative eigenvalues are the rank of the matrix.

Thus let \([A]\) be \(n \times n\), nonsingular, then \(\Pi(A) + \nu(A) = n\) \(\quad (B.1)\)
in which \(\Pi(A)\) are the number of positive eigenvalues and \(\nu(A)\) are the number of negative eigenvalues. [Ref.B.6:p-187].

The signature of \([A]\) is \(\Pi(A) - \nu(A) = \text{signature}(A)\) \(\quad (B.2)\)
The index of \([A]\) is \(\Pi(A) = \text{index}(A)\) \(\quad (B.3)\)

In [B.6] there was no name for \(\nu(A)\) there for Wittrick and Williams proposed to name it sign count. In order to compute the sign count, the simplest and fastest way involves the reduction of \([A]\) to upper triangular form by a Gaussian elimination procedure. In [J.2] by Williams and Wittrick the Gaussian elimination procedure should be made without row interchanges, the sign count \(\Pi(A)\) is the number of negative elements on the diagonal of \([A]^\nu\), the upper triangular form of \([A]\). In [J.7] by Sehmi the Gaussian elimination procedure should be done with optional diagonal pivoting. This also leads to \([A]^\nu\) which is merely reordered. The sign count is also equal the number of negative elements on the diagonal of \([A]^\nu\). This is done because the "pole jumping" mentioned in appendix A is then minimized.
Appendix C.

An example done with the theory of $\text{3}^a$, $\text{3}^b$ & $\text{3}^c$.

Fig. C1, Beam with mass at it's end, connected to a vertical strut which is exited at its base.

With dimensions,

- $E$ is Young's modulus
- $I$ is moment of inertia
- $K$ is spring stiffness of strut $EA/L$
- $m$ mass of beam per unit length
- $M$ concentrated mass at beam end
- $l$ length of beam
- $L$ length of strut

- $E = 70.00 \times 10^9$ [N/m$^2$]
- $I = 106.7 \times 10^9$ [m$^4$]
- $K = 1.400 \times 10^7$ [N/m]
- $m = 3.375 \times 10^0$ [Kg/m]
- $M = 25.00 \times 10^0$ [kg]
- $l = 1.000 \times 10^0$ [m]
- $L = 0.500 \times 10^0$ [m]

The kinetic energy is,

$$ T = \frac{m}{2} \int_0^l \left\{ \frac{\ddot{W}}{} \right\}^2 dx + \frac{M}{2} \left\{ \frac{\ddot{W}(y=L)}{L} \right\}^2 $$

The potential or strain energy is,

$$ V = \frac{EI^*}{2} \int_0^l \left\{ \frac{\partial^2 W_e^1}{\partial x^2} \right\}^2 dx + \frac{K^*}{2} \left\{ \frac{W_e^2(y=L)}{L} \right\}^2 $$

(C.1)
Like in chapter 2 the * means that $EI^* = EI(1 + i\gamma)$ and also $K^* = K(1 + i\gamma)$ in which $\gamma$ is the structural damping factor. (In this calculation .02 is used).

From fig C1 we can see that

$$w^1(x,t) = \dot{w}_0^1 + w_e^1(x,t) = \dot{w}_0^1 + w_e^1(x)e^{i\omega t} \quad \text{a}$$

$$w^2(y,t) = \dot{w}_0^2(t) + w_e^2(y,t) = \left\{ \dot{w}_0^{II} + w_e^{II}(y) \right\} e^{i\omega t} + y \quad \text{b}$$

(C.2)

Now let $W_e^I(x) = \sum a_i \hat{\phi}_i(x) \quad i = 1, 2 \quad \text{a}$

and $W_e^{II}(x) = \sum b_j \hat{\phi}_j(y) \quad j = 1 \quad \text{b}$

(C.3)

In which $\hat{\phi}_i$ are orthonormal beam shape functions like in [ref.J.11].

$$\hat{\phi}_1(x) = \frac{x}{\sqrt{2.31111111}} \cdot \left\{ 6 \cdot \frac{x}{1} - 4 \cdot \left( \frac{x}{1} \right)^2 + \left( \frac{x}{1} \right)^3 \right\}$$

$$\hat{\phi}_2(x) = 4.165651243 \ast x \ast \left\{ \frac{x}{1} - 0.802197802 \right\} \ast \left\{ 6 \cdot \frac{x}{1} - 4 \cdot \left( \frac{x}{1} \right)^2 + \left( \frac{x}{1} \right)^3 \right\}$$

and $\hat{\phi}_i(y) = \sqrt{3} \ast (\frac{y}{L})$.  \hspace{1cm} (C.4)

![Diagram](image)

Fig. C2 The used orthonormal beam shape modes for $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\phi}_1$.  

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Substitute eq. C.3 into eq. C.2 and the result into eq. C.1, hence

\[ T = e^{2i\omega t} \left\{ \frac{m\omega^2}{2} \int_0^1 \left( \sum a_i \hat{\phi}_i(x) \right)^2 \, dx + \frac{M\omega^2}{2} \left[ (W_0^2)^2 + 2W_0^2b_1\hat{\phi}_1(y=L) + b_1^2\hat{\phi}_1^2(y=L) \right] \right\} \]

\[ V = e^{2i\omega t} \frac{EI^*}{2} \int_0^1 \left( \sum a_i \hat{\phi}_1(x) \right)^2 \, dx + \frac{K^*}{2} b_1^2\hat{\phi}_1^2(y=L) \]

(C.5)

The constraint equation \( \chi_1 = W^1(x=1,t) - W^2(y=L,t) = 0 \) (C.6)

substitution of eq. C.3 and C.2 into eq. C.6 yields,

\[ \chi_1 = \sum a_i \hat{\phi}_i(x=1) - W_0^2 - b_1\hat{\phi}_1(y=L) = 0 \]

(C.7)

Using the Lagrangian equations like eq. 19 in chapter 2 we get

\[ \frac{\partial L}{\partial a_j} + \lambda \frac{\partial x_1}{\partial a_j} = 0 \quad j = 1, 2 \]

(C.8)

\[ \frac{\partial L}{\partial b_1} + \lambda \frac{\partial x_1}{\partial b_1} = 0 \]

hence,

\[ m\omega^2 \int_0^1 \left\{ \sum a_i \hat{\phi}_i(x) \right\} \hat{\phi}_j(x) \, dx - EI^* \int_0^1 \left\{ \sum a_i \hat{\phi}_i(x) \right\} \hat{\phi}_j(x) \, dx + \lambda \hat{\phi}_j(x=1) = 0 \]

(C.9)

\[ M\omega^2 \left\{ W_0^2 \hat{\phi}_1(y=L) + b_1\hat{\phi}_1(y=L) \right\} - K^* b_1^2\hat{\phi}_1^2(y=L) - \lambda \hat{\phi}_1(y=L) = 0 \]

(C.9)
Using, \( \frac{m\omega^2}{EI} = \Omega^2 \),
\[
\begin{align*}
\frac{M}{m} = \eta^2 \\
\frac{K}{EI} = \nu \quad \text{and} \quad \nu/\eta^2 = \frac{K}{EI} \frac{m}{M}
\end{align*}
\]
and
\[
\int_0^1 \hat{\Phi}_i(x) \hat{\Phi}_j(x) \, dx = \begin{cases} 
0 & i \neq j \\
1 & i = j
\end{cases}
\]  \( (C.10) \)

\[
\int_0^1 \hat{\Phi}_i(x) \hat{\Phi}_j(x) \, dx = B_{1j} = B_{ji}
\]

also \( \frac{\lambda}{EI} = \Gamma \)

Thus the eq. C.9 will become using the eq. C.10

\[
\begin{cases}
\Omega^2 a_1 - (1 + i\gamma) \left( a_1 B_{11} + a_2 B_{21} \right) + \Gamma \hat{\Phi}_1(x=1) = 0 \\
\Omega^2 a_2 - (1 + i\gamma) \left( a_1 B_{12} + a_2 B_{22} \right) + \Gamma \hat{\Phi}_2(x=1) = 0
\end{cases}
\]  \( (C.11) \)

\[
\Omega^2 \eta^2 \hat{\Phi}_1^2(y=L) b_1 - \nu \hat{\Phi}_1^2(y=L) b_1 - \Gamma \hat{\Phi}_1(y=L) = -\Omega^2 \eta^2 \hat{\Phi}_1(y=L) W_0^2
\]

or in matrix notation,

\[
\begin{bmatrix}
B_{11}(1 + i\gamma) - \Omega^2 & B_{12}(1 + i\gamma) & 0 & -\hat{\Phi}_1(x=1) \\
B_{12}(1 + i\gamma) & B_{22}(1 + i\gamma) - \Omega^2 & 0 & -\hat{\Phi}_2(x=1) \\
0 & 0 & (\nu - \eta^2 \Omega^2) & \hat{\Phi}_1(y=L) \\
0 & 0 & 0 & \hat{\Phi}_1(y=L)
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
b_1 \\
b_1
\end{bmatrix} = \begin{bmatrix}
F \\
\nu F \\
\Omega^2 W_0^2 F
\end{bmatrix}
\]  \( (C.12) \)

Together with the constraint equation we have a system of four equations and four unknowns.
Thus we have

\[
\begin{bmatrix}
(1+i\gamma)[M] - \Omega^2[N] & -[K] \\
-[K]^T & 0
\end{bmatrix} \begin{bmatrix}
a_1 \\
a_2 \\
b_1 \\
\Gamma_1
\end{bmatrix} = \begin{bmatrix}
F \\
W^2 \\
W_0
\end{bmatrix}
\]  

(C.13)

This system of equations is similar to eq. 33 of chapter 2. Now the eigenvalue procedure described in appendix A has to be performed, with \( \gamma \) equal zero. Yielding the eigenvector matrix \([X]\) and the eigenvalue matrix \([\Lambda]\).

\[
[X] = \begin{bmatrix}
-7.36079E-03 & -2.28736E-03 & 0.00000E+00 \\
6.17604E-03 & 4.02186E-03 & 0.00000E+00 \\
8.55707E-04 & -4.06416E-05 & 0.00000E+00 \\
1.00000E+00 & 1.00000E+00 & 1.00000E+00
\end{bmatrix}
\]  

(C.14)

\[
[\Lambda] = \begin{bmatrix}
2.85850E+02 & 0.00000E+00 & 0.00000E+00 \\
0.00000E+00 & 9.98440E+02 & 0.00000E+00 \\
0.00000E+00 & 0.00000E+00 & 1.00000E+37
\end{bmatrix}
\]  

(C.15)

The third eigenvalue is infinity.

Fig.C3 The characteristic equation of \( r_4 \) plotted between \( \Omega^2 = 0 \) and \( \Omega^2 = 750 \).
Fig.C4  The two fundamental modes of the structure.

For the response calculation we go back to the eq. C.13 and use only the next part of it

$$
\left( (1+i\gamma)[M] - \Omega^2[N] \right) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \{ F \} 
$$

(C.18)

Now calculate the response with the use of eq 61 of chapter 2.

$$
\begin{bmatrix} R_{\hat{w}_e}(x,y,\tau) \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} \mathcal{K}(x,y) & 0 \\ 0 & \mathcal{K}(x,y) \end{bmatrix}^T [X] * 
\begin{bmatrix}
\int_{-\infty}^{\infty} \left[ \mathcal{H}^*(\Omega) [\Lambda]^{-1}[X] \right]^T \begin{bmatrix}
\begin{bmatrix} \mathcal{G}^+ & \eta_0 \mathcal{K} \\ \mathcal{G} \end{bmatrix} & \begin{bmatrix} \mathcal{S}_0 & \mathcal{G} \end{bmatrix} \end{bmatrix} [X] [\Lambda]^{-1} \begin{bmatrix}
\mathcal{H}(\Omega) \end{bmatrix} \Omega^4 e^{i\Omega \tau} \text{d}\Omega
\end{bmatrix}
\end{bmatrix}
$$

(C.17)

in which

$$
\begin{bmatrix} R_{\hat{w}_e}(x,y,0) \end{bmatrix} = \begin{bmatrix}
E[\hat{w}_e^1(x,y,t)^2] & E[\hat{w}_e^1(x,y,t)\hat{w}_e^2(x,y,t)] \\
E[\hat{w}_e^1(x,y,t)\hat{w}_e^2(x,y,t)] & E[\hat{w}_e^2(x,y,t)^2]
\end{bmatrix}
$$

(C.18)

$$
E[f(x,y,t)] = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} f(x,y,t) \text{d}t
$$

(C.19)
and

\[
\begin{bmatrix}
K(x,y) & 0 \\
0 & K(x,y)
\end{bmatrix} =
\begin{bmatrix}
\hat{\phi}_1(x) & \hat{\phi}_2(x) & 0 \\
0 & 0 & \hat{\phi}_1(y)
\end{bmatrix}
\]  \hspace{1cm} (C.20)

\[
H(\Omega) = \left[(X)^{T}[M][X](1 + i\gamma) - \omega^2[X]^{T}[N][X]\right]^{-1}
\]  \hspace{1cm} (C.21)

\[
\begin{bmatrix}
G + \eta K \\
G
\end{bmatrix} =
\begin{bmatrix}
0 \\
0^2 \hat{\phi}_1(L)
\end{bmatrix}
\]  \hspace{1cm} (C.22)

The PSD of the excitation \[\hat{W}_0(\Omega)\] is in this case

\[
S_W(\Omega) = \frac{1}{10} \cdot \frac{1}{4\pi} \cdot \sqrt{\frac{m}{EI}} = 1.082788 \times 10^{-4}
\]  \hspace{1cm} (C.23)

The results for the mean square acceleration along the x-axis is drawn in Fig.C5.

**Fig.C5** The mean square acceleration of the beam along the x-axis.
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**TU.2** B.H. Bulder

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