Emergency Notes
CT 5316

These notes have been taken from a draft manuscript “Waves in oceanic and coastal waters” by L.H. Holthuijsen to be published separately (all rights reserved).
3. DESCRIPTION OF OCEAN WAVES

3.1 KEY CONCEPTS

- The short-term description of ocean waves requires stationary conditions. Time records of ocean waves (the sea surface elevation as a function of time at one location) need therefore to be as short as possible. However, characterising the waves with reliable averages requires the records to be as long as possible. The compromise at sea is: record lengths of 15 to 30 min. If the record is longer, it should be divided into such (nearly stationary, possibly overlapping) segments.
- The waves in each such stationary record can be characterized with average wave parameters such as the significant wave height and the significant wave period but these parameters do not provide a complete description of the waves.
- The significant wave height is fairly well correlated with the characteristic wave height as observed visually by experienced observers. This is not true for the wave period.
- To model a time record of ocean waves, the surface fluctuations are usually seen as the sum of a large number of statistically independent, harmonic waves (the wave components). This concept is called the random-phase/amplitude model.
- This model leads to the concept of the one-dimensional variance density spectrum, which shows how the variance of the sea surface elevation is distributed over the frequencies of the wave components that create the surface fluctuations.
- If the surface fluctuations are stationary (in a statistical sense) and Gaussian distributed, then the variance density spectrum provides a complete statistical description of the wave condition.
- The concept of the random-phase/amplitude model can be extended to the three-dimensional, moving sea surface, which is then seen as the sum of a large number of statistically independent, harmonic waves propagating in all directions across the sea surface. The corresponding two-dimensional variance density spectrum shows how the wave variance is distributed over the frequencies and directions of these harmonic wave components.
- The one-dimensional variance density spectrum can be obtained from the two-dimensional spectrum by integration over all directions.
- The variance density spectrum provides, in addition to a description in a statistical sense, also a description in a physical sense when multiplied with $\rho g$ ($\rho$ = specific density of water, $g$ = gravitational acceleration). The result is the energy density spectrum. It shows how the energy of the waves is distributed over the frequencies (and directions).

The spectral analysis of a time record of the sea surface elevation is treated in Appendix C.
### 3.5 THE WAVE SPECTRUM

#### 3.5.1 Introduction

The aim of describing ocean waves with a spectrum is not so much to describe one observation of the sea surface (i.e., one time record) in detail but rather to describe the sea surface as a stochastic process, i.e., to characterise all possible observations (time records) that could have been made, in the conditions of the actual observation. An observation is thus formally treated\(^1\) as one realisation of a stochastic process (see Appendix A). For ocean waves, this treatment is based on the random-phase/amplitude model, which leads to the wave spectrum, which is the most important form in which ocean waves are described\(^2\).

The basic concept of the wave spectrum is simple but its many aspects make it seem rather complicated. To distinguish the essence from these additional aspects, consider first a wave record (the surface elevation \(\eta(t)\) at one location as a function of time) with a duration \(D = 30\) min, obtained at sea with a wave buoy or a wave pole. We can then exactly reproduce the surface elevation in that record as the sum of a large number of harmonic wave components (a Fourier series):

\[
\eta(t) = \sum_{i=1}^{N} a_i \cos(2\pi f_i t + \alpha_i) \tag{3.5.1}
\]

where \(a_i\) and \(\alpha_i\) are the amplitude and phase respectively\(^3\) of each frequency \(f_i = i/D\). We can determine the values of these amplitude and phases for each frequency by Fourier analysing the wave record (see Appendix C). This would give us the amplitude and phase spectrum for this record (see Fig.3.5.1). By substituting the computed amplitudes and phases into Eq. (3.5.1), we can exactly reproduce the record.

\(^1\) The theory is taken from the description of noise (e.g., Tukey and Hamming, 1949) with some of the first applications to ocean waves by Barber and Ursell (1946) and Deacon (1949).

\(^2\) This eventually leads to a Fourier analysis of wave observations as treated in Appendix C.

\(^3\) The Greek alphabet:

<table>
<thead>
<tr>
<th>notation</th>
<th>name</th>
<th>notation</th>
<th>name</th>
<th>notation</th>
<th>name</th>
<th>notation</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>A α</td>
<td>alpha</td>
<td>H η</td>
<td>eta</td>
<td>N ν</td>
<td>nu</td>
<td>T τ</td>
<td>tau</td>
</tr>
<tr>
<td>B β</td>
<td>beta</td>
<td>Θ θ</td>
<td>theta</td>
<td>Ξ ζ</td>
<td>xi</td>
<td>Υ υ</td>
<td>epsilon</td>
</tr>
<tr>
<td>Γ γ</td>
<td>gamma</td>
<td>λ λ</td>
<td>lambda</td>
<td>Ω ω</td>
<td>omega</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Δ δ</td>
<td>delta</td>
<td>Κ κ</td>
<td>kappa</td>
<td>Π π</td>
<td>pi</td>
<td>Χ χ</td>
<td>chi</td>
</tr>
<tr>
<td>Ε ε</td>
<td>epsilon</td>
<td>Λ λ</td>
<td>lambda</td>
<td>Ρ ρ</td>
<td>rho</td>
<td>Ψ ψ</td>
<td>psi</td>
</tr>
<tr>
<td>Z ζ</td>
<td>zeta</td>
<td>Μ μ</td>
<td>mu</td>
<td>Σ σ</td>
<td>sigma</td>
<td>Ω ω</td>
<td>omega</td>
</tr>
</tbody>
</table>
For most wave records, the phases turn out to have any value between 0° and 360° without any preference for any one value. In a linear model of the waves (which is a basic assumption for the interpretation of this spectrum), this is always the case and we will ignore therefore ignore the phase spectrum (except that we keep in mind that the phases are uniformly distributed between 0° and 360° and apply that knowledge when called for). Then only the amplitude spectrum remains to characterise the wave record. If we were to repeat the experiment, i.e., measure the surface elevation again in identical conditions, the time record would be different and so would be the amplitude spectrum. To properly account for this sample character of the spectrum, we should repeat the experiment many times ($M$) and take the average over all these experiments to find the average amplitude spectrum:

$$
\tilde{a}_i = \frac{1}{M} \sum_{i=1}^{M} a_i \quad \text{for all frequencies } f_i
$$

(3.5.2)

For large values of $M$ the value of $\tilde{a}_i$ converges (tends to a constant value), thus solving the sampling problem. However, this amplitude spectrum is discrete, i.e., only the frequencies $f_i = i / D$ are present, whereas in fact all frequencies are present at sea. A first step to resolve this problem would be to distribute the average amplitude $\tilde{a}_i$ over the frequency interval $\Delta f$, giving an amplitude density $\tilde{a}_i / \Delta f$ at each frequency. All frequencies would thus be present because they have all been assigned an amplitude density. However, it is more meaningful to do this for the variance of each wave component $\frac{1}{2} \sigma_i^2$. There are two reasons for this. First, the variance is a more relevant statistical quantity than the amplitude. For instance, the sum of the variances of the wave components is equal to the variance of the sum of the wave components (= the random surface elevation; you may want to read this sentence again). In contrast to this, the sum of the amplitudes is not equal to the amplitude of the sum (there is no such thing as the amplitude of a random sea surface elevation). Second, the linear theory for surface gravity waves (see Chapter 5) shows that the energy of the waves is proportional to the variance. This implies that through the variance, a link is available to such physical properties as wave energy but also wave-induced particle velocity and pressure variations. Distributing the variance over the frequency interval gives the variance density: $\frac{1}{2} \sigma_i^2 \rightarrow \frac{1}{2} \sigma_i^2 / \Delta f$ (constant within the frequency band $\Delta f$). The variance is now distributed over all frequencies but the value of the variance density still jumps from one frequency band to the next. This is resolved by letting the frequency interval $\Delta f$ approach zero ($\Delta f \rightarrow 0$). The definition of the variance density spectrum thus becomes:

$$
E(f) = \lim_{\Delta f \to 0} \frac{1}{\Delta f} \frac{1}{2} \rho g \sigma_i^2
$$

(3.5.3)
Here, this (one-dimensional) frequency spectrum $E(f)$ has been introduced only very briefly and only to give the essence of the concept of the spectrum. It will be treated more extensively in the next Sections. The description of waves in geographic space (the surface elevation as a function of the horizontal coordinates, i.e., an spatial image of the waves) requires the introduction of a two-dimensional spectrum, which is totally analogous to the above frequency spectrum.

The great advantage of using the concept of the variance density spectrum rather than the actual surface elevation of the waves, is that (a) all statistical characteristics of the waves are available from the spectrum (e.g., simple, theoretical estimates can replace the statistical analysis of a wave record) and (b) the spectrum of the waves can be predicted with numerical models that are driven by (computed) winds (the surface elevation itself cannot be predicted) and (c) at scales longer than a few wave periods, the response of waves to wind, breaking etc. can only be represented by the spectrum.

### 3.5.2 Random-phase/amplitude model

The basic model for describing the moving surface elevation $\eta(t)$ is the random-phase/amplitude model in which the surface elevation is considered to be the sum of a large number of harmonic waves, each with a random amplitude and a random phase (constant in time but different for each realisation of the time record; for the concept of random variables and realisations, see Appendix A):

$$\eta(t) = \sum_{i=1}^{N} a_i \cos(2\pi f_i t + \alpha_i)$$  \hspace{1cm} (3.5.4)

where $N$ is a large number (of frequencies) and the underscores of amplitude $a_i$ and phase $\alpha_i$ indicate that these are now random variables, constant in time for each realization of $\eta(t)$, see Fig. 3.5.1. The specific values of the frequencies $f_i$ in this summation are not important as long as (a) the frequencies are densely distributed along a frequency axis (i.e., the difference between two sequential frequencies $f_i$ and $f_{i+1}$ should be small compared to some characteristic wave frequency and (b) they are in the correct range (typically between 0.05 Hz and 1.0 Hz for waves at sea).
The summation of many harmonic waves with random amplitudes and phases creates a random sea surface.

The random phase and amplitude, being random variables, require probability density functions to fully characterise them. In the random-phase/amplitude model, the phases at each frequency $f_i$, are uniformly distributed between $-\pi$ and $+\pi$ (see Fig. 3.5.2):

$$p(\alpha_i) = \frac{1}{2\pi} \quad \text{for} \quad -\pi < \alpha_i \leq \pi$$

and the amplitudes $a_i$ are Rayleigh\(^4\) distributed (see Fig. 3.5.2; with only one parameter $\mu_i$ varying over the frequencies $f_i$):

$$p(a_i) = \frac{a_i}{\mu_i^2} \exp\left(-\frac{a_i^2}{4\mu_i^2}\right) \quad \text{for} \quad a_i \geq 0$$

---

\(^4\) John William Strutt, 3rd Baron Rayleigh, (1842-1919), was an English physicist whose work on gases (the discovery of argon) earned him the Nobel prize in 1904. He also worked in the field of acoustics, optics and wave propagation in fluids.
where $\mu_i$ is the expected value of the amplitude $\mu_i = E\{a_i\}$ (see Appendix A for the notion a mean value as an expected value). Since $\mu_i = E\{a_i\}$ is the only parameter in Eq. (3.5.6), the statistical characteristics of $a_i$ are completely given by only this one parameter (per frequency). The function that shows this mean amplitude along the frequency axis is called the amplitude spectrum $E\{a_i\}$ (see Fig. 3.5.2).

To create one realization of $\eta(t)$ with Eq. (3.5.4), the values of the amplitudes $a_i$ and phases $\alpha_i$ are drawn from their respective probability density functions, at each frequency separately and independently. For each new realization of $\eta(t)$, the values of $a_i$ and $\alpha_i$ are again drawn randomly from these probability density functions. A wave record at sea can be seen as one such realization.

Regarding the applicability of the random-phase/amplitude model to real ocean waves, the following observations should be made:

**Fig. 3.5.2** The random-phase/amplitude model: at every frequency there is one uniform distribution of the random phase and one Rayleigh distribution of the random amplitude (characterised by the expected value $E\{a_i\}$). Top panels: for a series of frequencies, $f_i$, $i=1, 2, 3, 4, 5$ etc... Bottom panel: expected value of the amplitude as function of frequency = amplitude spectrum.
First, the random-phase/amplitude model generates a stationary, Gaussian process (see Appendix A for this concept). To use this approach for the conditions at sea, which are never really stationary, a wave record needs to be divided into segments that are each deemed to be approximately stationary (a duration of 15 to 30 min is commonly used for wave records obtained at sea). In addition, the wave components are not really mutually independent at sea (as in the random-phase/amplitude model) because they interact to some degree (see Chapters 6 and 8). However, these interactions are weak if the waves are not too steep and not in very shallow water and can be ignored, leaving the random-phase/amplitude model in place as the basic model to describe ocean waves.

Second, the random-phase/amplitude model is a summation of wave components at discrete frequencies \( f_i \) whereas in fact, a continuum of frequencies is present at sea. This aspect is the subject of the next section.

### 3.5.3 Variance density spectrum

The amplitude spectrum \( E\{a\} \) as a function of frequency \( f_i \), which was treated in the previous section, provides enough information to realistically describe the sea surface elevation as a stationary, Gaussian process. However, for several reasons it is more relevant to present the information in this spectrum in a different way: consider the variance \( E\{\frac{1}{2}a^2\} \) as a function of frequency \( f_i \) (see Fig. 3.5.3 and Note 3A) rather than the expectation of the amplitude \( E\{a\} \). In other words, consider the variance spectrum instead of the amplitude spectrum. The advantages of this have been indicated earlier in Section 3.5.1.

![Figure 3.5.3](image-url)  
*Fig. 3.5.3 The concepts of an amplitude spectrum and the corresponding variance spectrum of ocean waves.*

However, both the amplitude and variance density spectrum are based on discrete frequencies whereas nature does not select such discrete frequencies. All frequencies are present at sea. The random-phase/amplitude model needs therefore to be modified. This is done by distributing the variance \( E\{\frac{1}{2}a^2\} \) over the frequency interval around each frequency (the frequency band \( \Delta f_i \), equal to the interval between the frequencies and usually, but not necessarily, constant for all frequencies). The resulting variance density \( E^\prime(f_i) \) is then:

\[ E^\prime(f_i) = \frac{E\{\frac{1}{2}a^2\}}{\Delta f_i} \]

5 Note that the symbol for variance density \( E^\prime(\cdot) \) is different from the symbol for expected value \( E\{\cdot\} \).
\[ E'(f_i) = \frac{1}{\Delta f_i} E\{\gamma g_i^2\} \]  

(3.5.7)

\[ E(f) = \lim_{\Delta f_i \to 0} \frac{1}{\Delta f_i} E\{\gamma g_i^2\} \]  

(3.5.8)

Fig. 3.5.4 The transformation of the discrete amplitude spectrum of the random-phase/amplitude model to the continuous variance density spectrum.

The value of this variance density, although defined for all frequencies, still varies discontinuously from one frequency band to the next (see Fig. 3.5.4). A continuous version of the variance density spectrum is obtained by having the width of the frequency band \( \Delta f_i \) approach zero (see Fig. 3.5.4):

This function \( E(f) \) is called the variance density spectrum. It is the single most important concept in this book.
The variance density spectrum gives a complete description of the surface elevation of ocean waves in a statistical sense, provided that the surface elevation can be seen as a stationary, Gaussian process, i.e., all statistical characteristics can be expressed in terms of this spectrum.

It follows that the total variance \( \eta^2 \) (see Note 3A) of the sea surface elevation is the sum of the variances of all frequency bands \( \Delta f_i \):

\[
\eta^2 = \sum_i \frac{1}{\Delta f_i} E\{\eta_i^2\} \Delta f_i
\]

or, if \( \Delta f_i \to 0 \) as for the continuous variance density spectrum:

\[
\text{total variance} = \eta^2 = \int_0^\infty E(f) df
\]

For the interpretation of the variance density spectrum \( E(f) \) it is important to note that it shows how the variance of the sea surface elevation is distributed over the frequencies. It indicates how much each frequency band contributes to the total variance. However, it is rather difficult to form a mental picture of this: a statistical characteristic (variance) is distributed over the frequencies of the harmonic components that make up the process. It may help if we multiply the spectrum with \( \rho g \). We then obtain the energy density spectrum (by virtue of the fact that the wave energy is proportional to the variance; see Eq. 3.5.13). This spectrum shows how the wave energy is distributed over the frequencies (see Section 3.5.4), which seems to be easier to comprehend: it indicates how much each frequency band contributes to the total wave energy.

The dimension and S.I. unit of the variance density \( E(f) \) follow directly from the definition (Eq. 3.5.8): the dimension of the amplitude \( a_i \) is [length] and its S.I. unit is [m] while the dimension of the frequency band \( \Delta f_i \) is [time]^{-1} and its S.I. unit is [s^{-1}] or rather [Hz]. The dimension of \( E(f) \) is therefore [length^2/(1/time)] and its unit is either m^2 s or m^2/Hz (personally, I prefer the latter, because it shows better that frequencies are involved with dimension Hz, rather than some time interval with dimension s). Another way of finding the unit of \( E(f) \) is to note that the integral of \( E(f) \) (i.e., the surface area under \( E(f) \)), is equal to the total variance of the sea surface elevation, the S.I. unit of which is m^2. Since the unit of the horizontal axis (frequency \( f \)) is Hz, it follows that the unit of the vertical axis (variance density \( E(f) \)) should be m^2/Hz to arrive at unit m^2 for the integral.
NOTE 3A  

The variance of the sea surface elevation

The variance of the surface elevation \( \eta(t) \) is, by definition, the average of the surface elevation squared: \( \overline{\eta^2} \) (the overbar indicates time-averaging).

For a harmonic wave with amplitude \( a \), the variance is \( \overline{\eta^2} = \frac{1}{2}a^2 \).

For random ocean waves, a large number of such harmonic waves are added, as in the random-phase/amplitude model and the variance of this sum (i.e., of the random surface elevation \( \eta(t) \)) is equal to the sum of the individual variances ("the sum of the variances is the variance of the sum"):  
\[
\overline{\eta^2} = \overline{\eta^2} = \sum_{i=1}^{N} \overline{\eta_i^2} = \sum_{i=1}^{N} E\{\eta_i^2\}
\]

The square root of this variance is the standard deviation \( \sigma_\eta \) of the surface elevation, which can be seen as a vertical scale of the wave heights, for instance, the significant wave height \( H_s = 4\sigma_\eta \) (see Section 4.4.3).

3.5.4 Interpretation of the variance density spectrum

The variance density spectrum was introduced in the previous section by transforming the discrete amplitude spectrum into a continuous distribution of the variance over frequencies. This is a rather formal procedure showing which frequency band \( \Delta f \) contributes how much \( \Delta \text{var} \) to the total variance (see Fig. 3.5.5):

\[
\Delta \text{var} = \int_{\Delta f} E(f)df
\]

(3.5.11)

Fig. 3.5.5  The interpretation of the variance density spectrum as the distribution of the total variance of the sea surface elevation over frequencies.
Note that the contribution of one single frequency to the total variance is zero, because the spectral bandwidth of one single frequency is zero: \( \Delta f \to 0 \) and its contribution \( E(f) \Delta f \to 0 \).

The overall appearance of the waves can be inferred from the shape of the spectrum: the narrower the spectrum, the more regular the waves are. This is shown for three different wave conditions in Fig. 3.5.6. The narrowest spectrum corresponds to a wave record of a harmonic wave (the spectrum degenerates into a delta-function at one frequency; the variance density is infinite since the bandwidth of one single frequency is zero but its variance is finite). Distributing the variance over a slightly wider frequency band gives a slowly modulating harmonic wave. Distributing the wave variance over a still wider frequency band gives a rather chaotic wave field (irregular waves), because the components in the time record get out of phase with one another very quickly.

\[
E(f) \Delta f \to 0
\]

\[
\text{surface elevation } \eta(t) \quad \text{harmonic wave}
\]

\[
\text{surface elevation } \eta(t) \quad \text{modulated harmonic wave}
\]

\[
\text{surface elevation } \eta(t) \quad \text{irregular waves}
\]

Fig. 3.5.6  The (ir)regular character of the waves for three different widths of the spectrum.

As indicated above, the energy of the waves can be expressed in terms of the variance of the surface elevation because the energy of a harmonic wave (per unit horizontal surface area) is equal to the mean square elevation times the gravitational acceleration \( g \) and the density of water \( \rho \) (see the linear theory of surface gravity waves in Chapter 5), so that the total energy (i.e. summed over all components) is:

\[
E_{\text{total}} = \rho g \overline{\eta^2}
\]
We can therefore multiply the variance density spectrum \( E_{\text{variance}}(f) = E(f) \) with \( \rho g \) and obtain the energy density spectrum as:

\[
E_{\text{energy}}(f) = \rho g E_{\text{variance}}(f)
\]

(3.5.13)

Just as the variance density spectrum is used to describe the statistical aspects of the waves, so can the energy density spectrum be used to describe the physical aspects of waves (within the limitations of the stationary, Gaussian model and the linear theory of surface gravity waves). This close relationship leads to a rather inaccurate use of the word "spectrum". It refers to both the variance density spectrum and the energy density spectrum. Very often, the two terms are used indiscriminately with the context indicating which of the two is actually meant.

### 3.5.5 Alternative spectral formulations

The variance density spectrum can be defined in other ways than the one given above. The differences may relate to: (a) the spectral domain, (b) the formal definition.

#### Spectral domain

The variance density has been defined in Section 3.5.4 in terms of the frequency \( f = 1/T \) (where \( T \) is the period of the harmonic wave) but it can equally well be formulated in terms of the radian frequency \( \omega = 2\pi f T \). The corresponding spectrum \( E(\omega) \) is defined in the same manner as \( E(f) \), the only difference being that \( \cos(2\pi ft + \alpha) \) in Eq. (3.5.1) is replaced by \( \cos(\omega t + \alpha) \). These spectra are obviously related: spectrum \( E(\omega) \) can be readily expressed in terms of \( E(f) \) and vice versa but it must be born in mind that the total variance \( \overline{\eta^2} \) should be conserved in such transformations. In other words:

\[
\overline{\eta^2} = \int_0^\infty E(\omega) d\omega = \int_0^\infty E(f) df
\]

(3.5.14)

which is readily achieved by taking:

\[
E(\omega)d\omega = E(f)df
\]

(3.5.15)

or:

\[
E(\omega) = E(f) \frac{df}{d\omega} = E(f) J
\]

(3.5.16)

where \( J = \frac{df}{d\omega} \) is called the Jacobian\(^6\). In this case, of transforming \( E(f) \) into \( E(\omega) \), it has the value \( J = 1/(2\pi) \). Therefore, in the transformation, one must not only transform the frequencies \( f \) into \( \omega \), but also the density \( E(f) \) into the density \( E(\omega) \).

---

\(^6\) More generally, transforming one (density) function to another with the condition that the integral is conserved, can be achieved with: \( E(x) = E(y) J \) where \( y \) is a function of \( x \), i.e., \( y = f(x) \) and \( J = dy/\,dx \) is the (one-dimensional) Jacobian. This applies to all density functions of which the integral needs to be conserved in the transformation (e.g., probability density functions). The
Formal definition

The variance density spectrum has been defined in terms of the random-phase/amplitude model. An alternative definition of the spectrum, which has exactly the same interpretation, is based on the Fourier transform of the auto-covariance function of the sea surface elevation:

\[ E(f) = 2 \int_{-\infty}^{\infty} C(\tau) \cos(2\pi f \tau) d\tau \]  

(3.5.17)

where the auto-covariance function \( C(\tau) \) is defined as (see Appendix A):

\[ C(\tau) = E\{ \eta(t) \, \eta(t + \tau) \} \]  

(3.5.18)

For a stationary process, this auto-covariance function depends only on the time difference \( \tau \) (the interval between two arbitrarily chosen moments \( t_1 \) and \( t_2 \) in time: \( \tau = t_2 - t_1 \)). It contains all co-variances of the joint probability density functions of \( \eta(t) \) and \( \eta(t + \tau) \) for any \( t \) and \( \tau \). It therefore provides a complete description in a statistical sense, if the process is stationary and Gaussian. It has been noted in Section 3.5.3 (without proof) that the variance density spectrum too, provides such a complete description. This statement is based on the fact that a Fourier transform is reversible. As Eq. (3.5.17) shows that the variance density spectrum can be obtained by Fourier transforming the auto-covariance function, the auto-covariance function can be obtained by Fourier transforming the variance density spectrum. The one function can apparently be expressed in terms of the other without loss of information. This implies that the variance density spectrum contains the same information as the auto-covariance function. It therefore describes a stationary, Gaussian process as completely as the auto-covariance function does.

This definition of the variance density spectrum, based on the auto-covariance function, is not used very often because most computers calculate the amplitudes in Eq. (3.5.8) far more efficiently directly from a wave record (with a technique called the Fast Fourier Transform = FFT) than that they can calculate the auto-covariance function of Eq. (3.5.18) from the same record and its subsequent Fourier transform (Eq. 3.5.17).

3.5.6 Frequency, direction spectrum

The above one-dimensional variance density spectrum characterises the stationary, Gaussian surface elevation as a function of time (at one geographic location). To describe actual, three-dimensional, moving waves, the space dimension has to be added. To that end we expand the random-phase/amplitude model by considering a harmonic wave that propagates in \( x, y \)-space, in direction \( \theta \) relative to the positive \( x \)-axis:

\[ E(x_1, x_2) = E(y_1, y_2) J, \]  

where the transformation of a two-dimensional density function requires a two-dimensional Jacobian: 

\[ E(x_1, x_2) = E(y_1, y_2) J, \]  

where the Jacobian \( J \) is the determinant of the matrix

\[
\begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\
\frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2}
\end{bmatrix},
\]

which is \( J = \frac{\partial y_1}{\partial x_1} \frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2} \frac{\partial y_2}{\partial x_1} \).
\[ \eta(x, y, t) = a \cos(\omega t - kx \cos \theta - ky \sin \theta + \alpha) \]  

(3.5.19)

where radian frequency \( \omega = 2\pi / T \) (use \( \omega \) instead of \( f \) for the sake of brevity in the notation) and wave number \( k = 2\pi / L \) (where \( L \) is the wavelength of the harmonic wave). Analogous to the one-dimensional model, the corresponding three (!)-dimensional random-phase/amplitude model (in \( x \)-, \( y \)- and \( t \)-space) is then the sum of a large number of such propagating harmonic waves (Fig. 3.5.7):

\[ \eta(x, y, t) = \sum_{i=1}^{N} \sum_{j=1}^{M} \left[ a_{ij} \cos(\omega t - k_i x \cos \theta_j - k_j y \sin \theta_j + \alpha_{ij}) \right] \]  

(3.5.18)

Fig. 3.5.7 The random waves moving in time = sum of large number of harmonic wave components, travelling across the ocean surface with different periods, directions, amplitudes and phases (after Pierson et al., 1955a).

Adding two dimensions to the original one-dimensional random-phase/amplitude model (dimensions \( x \) and \( y \), added to time \( t \), or equivalently, wave number \( k \) and direction \( \theta \), added to frequency \( \omega \) ) would result in two more indices in the summation. However, the index for wave number \( k \) is equal to the index frequency \( \omega \) because frequency and wave number are related through the dispersion relationship of the linear theory for surface gravity waves (see Chapter 5):
$\omega^2 = gk\tanh(kd)$, where $d$ is the water depth, so that every wave number $k$ corresponds to one frequency $\omega$. The seemingly three-dimensional random-phase/amplitude model thus reduces to a two-dimensional model in terms of frequency (or wave number) and direction. Note that each wave component is thus indicated in Eq. (3.5.20) with two indices: $i$ (for the frequency or wave number) and $j$ (for the direction).

As in the one-dimensional model, every wave component in this two-dimensional model has a random amplitude $a_{i,j}$ (Rayleigh distributed) and a random phase $\alpha_{i,j}$ (uniformly distributed). And analogous to the definition of the one-dimensional spectrum, the exact values of the frequencies $\omega_i$ and the directions $\theta_j$ are not important as long as their interval is small (e.g. a small fraction of $1/T_{1/2}$ and a small fraction of $360^\circ$, respectively). This two-dimensional random-phase/amplitude model represents a Gaussian process that is stationary in time and homogeneous in $x,y$-space: a spatial pattern of chaotically moving surface elevations, seen as the sum of many wave components propagating with different amplitudes, phases and frequencies (or wave lengths) in different directions across the ocean surface. The effect is a realistic representation of random, short-crested waves (see Fig. 3.5.7).

By using the same techniques as before, the discrete two-dimensional amplitude spectrum can be transformed into a continuous two-dimensional variance density spectrum (see Figs. 3.5.8 and 3.5.9):

$$E(\omega,\theta) = \lim_{\Delta \omega \to 0, \Delta \theta \to 0} \frac{1}{\Delta \omega \Delta \theta} E\{\chi^2 a^2_{i,j}\}$$

(3.5.21)

or in terms of frequency $f$:

$$E(f,\theta) = \lim_{\Delta f \to 0, \Delta \theta \to 0} \frac{1}{\Delta f \Delta \theta} E\{\chi^2 a^2_{i,j}\}$$

(3.5.22)

**Fig. 3.5.8**  The two-dimensional spectrum of waves (shown in polar coordinates).
Obviously (using the proper Jacobian; see Section 3.5.5):

\[ E(\omega, \theta) = \frac{1}{2\pi} E(f, \theta) \tag{3.5.23} \]

The dimension and S.I. unit of \( E(f, \theta) \) follow directly from its definition: the dimension of the amplitude \( a_{ij} \) is [length] and its S.I. unit is [m]. The dimension of the frequency band \( \Delta f \) is [time]\(^{-1}\) and its S.I. unit is [s\(^{-1}\)] or [Hz]. The direction band \( \Delta \theta \) is dimensionless but its units are either radians or degrees. The dimension of \( E(f, \theta) \) is therefore, (from Eq. 3.5.22): [length\(^2\)/ (1/time)] and its unit is m\(^2\)/Hz/radian or m\(^2\)/Hz/degree.

\[ \Delta \text{var} = \int_{\Delta f} \int_{\Delta \theta} E(f, \theta) d\theta df \]

*Fig. 3.5.9* The contribution of a spectral bin \((\Delta f, \Delta \theta)\) to the total variance of the waves.

The two-dimensional spectrum \( E(f, \theta) \) shows, how the variance of \( \eta(x, y, t) \) is distributed over frequencies and directions just as the one-dimensional frequency spectrum shows how the variance is distributed over frequencies. The volume of \( E(f, \theta) \) is therefore equal to the total variance \( \overline{\eta^2} \) of the sea surface elevation\(^7\):

\[ \overline{\eta^2} = \int_{0}^{2\pi} \int_{0}^{\pi} E(f, \theta) d\theta df \tag{3.5.24} \]

---

\(^7\) Phillips, in his famous book Phillips (1977), defines a direction-frequency spectrum \( \Psi_0(f, \theta) \) such that \( \overline{\eta^2} = \int_{0}^{2\pi} \int_{0}^{\pi} \Psi_0(f, \theta) d\theta df \). This is unusual and confusing. Phillips (1985) explicitly points out this difference between \( \Psi_0(f, \theta) \) and \( E(f, \theta) \).
The contribution of the spectral bin \((\Delta f, \Delta \theta)\) to the total variance is (see Eq. 3.5.11 and Fig. 3.5.9):

\[
\Delta \text{var} = \int \int E(f, \theta) d\theta df
\]  

(3.5.25)

The one-dimensional frequency spectrum \(E(f)\), which does not contain any directional information, can be obtained from the frequency-direction spectrum \(E(f, \theta)\) by integration over all directions (per frequency):

\[
E(f) = \frac{2\pi}{\Omega} \int_0^\pi E(f, \theta) d\theta
\]

(3.5.26)

Fig. 3.5.10   An interpretation of the wave spectrum off the Dutch coast when a northerly swell, generated by a storm off the Norwegian coast meets a locally generated westerly wind sea.

3.5.7 The spectrum at sea

Suppose that a severe storm in the Norwegian Sea generates swell travelling south into the North Sea (see Fig. 3.5.10). That swell will arrive one or two days later off the Dutch coast where it may meet a young sea being generated by a local breeze from westerly directions. The spectrum there will then represent two wave systems: a swell from the north and a local wind sea from the west. Swell is generally of a much lower frequency than a local wind sea, so that the two wave systems are well separated, both in frequency and direction. Moreover, swell is rather regular and long-crested so that its spectrum is narrow (in both frequency and direction; see Section 6.4.2 for an explanation for this). In contrast to this, wind sea is generally irregular and short-crested and its spectrum is therefore
much broader. The spectrum off the Dutch coast will therefore be rather distinctive in this situation: a narrow, low-frequency spectrum oriented in southerly directions (direction of propagation) representing the swell and a much broader spectrum at higher frequencies oriented towards easterly directions, representing the locally generated wind sea. The one-dimensional spectrum, obtained by integrating the two-dimensional spectrum over the directions is equally distinctive because the swell and the local wind sea are well separated in frequency.

3.5.8 Wave number spectrum

In the previous sections, the surface elevation was considered as a function of time and space. It can also be considered as a function of space alone, i.e., at one moment in time (a "frozen" surface). The surface elevation can then be a function only of one spatial (horizontal) coordinate $x$, for instance in a photograph of the water surface through the glass side-wall of a wave flume (like Fig. A5 in Appendix A). It can also be a function of two spatial horizontal coordinates $(x, y)$, for instance in a pair of stereo-photographs (see Fig. 3.2.2) or from imaging radar (see Section 2.4.1.2). Some will say that these are two- or three-dimensional records because the surface is available in two or three dimensions; here we consider the independent variables $(x, y$ or $t$) when counting the number of dimensions.

3.5.8.1 One-dimensional wave number spectrum

The rationale and the definition of the one-dimensional wave number spectrum are identical to those of the one-dimensional frequency spectrum. The only differences are that time $t$ is replaced by the horizontal coordinate $x$ and that frequency $\omega = 2\pi/T$ is replaced by wave number $k = 2\pi/L$. The one-dimensional variance density spectrum in terms of wave number $E(k)$ is then defined as:

$$E(k) = \lim_{\Delta k \to 0} \frac{1}{\Delta k} E\left\{ \frac{1}{2} \bar{a}^2 \right\}$$

(3.5.27)

where $\Delta k$ is the wave number bandwidth. Since frequencies $\omega$ and wave numbers $k$ are related through the dispersion relationship of the linear theory for surface gravity waves (see Chapter 5), the wave number spectrum can be obtained from the frequency spectrum with:

$$E(k) = E(\omega) \frac{d\omega}{dk}$$

(3.5.28)

where $d\omega/dk$ is the Jacobian to transform from frequency to wave number domain (see Section 3.5.5; note that the value of this Jacobian happens to be equal to the group velocity as obtained from the dispersion relationship, see Section 5.4.3.2), so that:

$$E(k) = c_g E(\omega)$$

(3.5.29)

All characteristics of this one-dimensional wave number spectrum are identical or analogous to those of the frequency spectrum, e.g., the total wave variance is given by:

$$\bar{\eta}^2 = \int E(k)dk$$

(3.5.30)
3.5.8.2 Two-dimensional wave number spectrum

The harmonic wave components underlying the spectral description of the frozen sea surface, in two spatial dimensions \( x, y \) can be written as:

\[
\eta(x, y) = a_{i,j} \cos(k_{x,i}x + k_{y,j}y + \alpha_{i,j})
\]  

(3.5.31)

and the corresponding two-dimensional wave number spectrum \( E(k_x, k_y) \) is defined, analogous to the above definitions of the various spectra, as\(^8\):

\[
E(k_x, k_y) = \lim_{\Delta k_{x,i} \rightarrow 0} \lim_{\Delta k_{y,j} \rightarrow 0} \frac{1}{\Delta k_{x,i} \Delta k_{y,j}} E\left\{\gamma a^2_{i,j}\right\}
\]  

(3.5.32)

where \( \Delta k_{x,i} \) and \( \Delta k_{y,j} \) are the spectral bandwidths. Obviously, \( k = \sqrt{k_x^2 + k_y^2} \) and \( \theta = \arctan\left(\frac{k_y}{k_x}\right) \), so that another two-dimensional spectrum can be defined as:

\[
E(k, \theta) = \lim_{\Delta k \rightarrow 0} \lim_{\Delta \theta \rightarrow 0} \frac{1}{\Delta k \Delta \theta} E\left\{\gamma a^2_{i,j}\right\}
\]  

(3.5.33)

where \( \Delta k \) and \( \Delta \theta \) are the spectral bandwidths. The two spectra are related as:

\[
E(k, \theta) = E(k_x, k_y) J
\]  

(3.5.34)

where \( J = k \) is the Jacobian to transform the spectrum from the two-dimensional \( \vec{k} \)-domain to the two-dimensional \( k, \theta \) domain (see footnote in Section 3.5.5). The two-dimensional frequency-direction spectrum \( E(\omega, \theta) \) is obtained from \( E(k, \theta) \), simply with:

\[
E(\omega, \theta) = E(k, \theta) J
\]  

(3.5.35)

where the Jacobian is \( J = \frac{\partial k}{\partial \omega} = \frac{1}{c} \) (see above), and similarly, with \( c = \omega / k \):

\[
E(k_x, k_y) = \frac{cc}{\omega} E(\omega, \theta)
\]  

(3.5.36)

The one-dimensional wave number spectrum is readily obtained from \( E(k, \theta) \) by integrating over all directions:

---

\(^8\) The spectrum \( E(k_x, k_y) \) may also be written as \( E(\vec{k}) = E(k_x, k_y) \), where \( \vec{k} = (k_x, k_y) \) is called the wave number vector.
3.5.9 Spectrum acquisition

The techniques to acquire the one- or two-dimensional spectrum are essentially:

-- measure the sea surface elevation with in-situ or remote sensing techniques and analyse the record (see Appendix C) or,
-- predict it with numerical models using wind, tide and sea bottom topography information.

In-situ techniques (e.g., a heave buoy; see Chapter 2), supplemented with the appropriate spectral analysis (see Appendix C) usually provide reasonable estimates of the one-dimensional frequency spectrum $E(f)$ or $E(\omega)$. Most of these techniques do not provide estimates of the two-dimensional spectrum $E(f,\theta)$ or $E(\omega,\theta)$, even if spatial characteristics of the waves are measured (e.g. the slope of the sea surface in two orthogonal directions with a pitch-and-roll buoy, see Section 2.3.1). But, they can provide some overall directional parameters; the mean direction and directional spreading (per frequency or averaged over frequencies; see Section 6.3.4 for the concept of the directional spreading). Remote-sensing techniques usually provide either estimates of the one-dimensional frequency spectrum $E(f)$ or $E(\omega)$ or estimates of the two-dimensional spectrum, usually the wave number spectrum $E(k_1,k_2)$. Numerical wave models (e.g., spectral for oceanic and coastal scales, see Chapters 6 and 8) predict the two-dimensional spectrum $E(f,\theta)$ or $E(\omega,\theta)$, from which the one-dimensional frequency spectrum $E(f)$ or $E(\omega)$ can be readily obtained (Section 3.5.6).

3.6 RESPONSE SPECTRA

The mechanical response of structures (or measurement instruments) to excitation by waves is often required in engineering practice (e.g., to compute the forces exerted by waves on the beams of an offshore platform). Many of such responses are non-linear. However, computations of such responses are often based on an assumed linear response, mostly because that greatly simplifies the computations and the interpretation of the results. If the response is linear (generally true for small forces and motions) and the system is constant in time and the surface elevation is stationary and Gaussian distributed, then the statistical characteristics of the response can be determined with readily available theoretical results for stationary, Gaussian processes (see Chapter 4). The word "linear" is rather crucial here. It means the following: consider a system which:

\[ a x(t) + b y(t) \rightarrow a X(t) + b Y(t) \]  

then the system is called linear, if a linear combination of these excitations gives the corresponding linear combination of the responses:

\[ E(k) = \int_{0}^{2\pi} E(k,\theta) d\theta \]  

(3.5.37)
It shows that a relative amplification $a$ or $b$, of an excitation $x(t)$ or $y(t)$ gives an equal relative amplification $a$ or $b$ of the response to that excitation $X(t)$ or $Y(t)$. It also shows that the responses to the excitations are independent: the response to excitation $x(t)$ is not affected by the response to excitation $y(t)$. One characteristic of a time-constant, linear system is that, when the excitation is harmonic with a given frequency, the response too is harmonic (with the same frequency):

$$\begin{align*}
x(t) &= \hat{x} \sin(\omega t + \alpha_x) \\
X(t) &= \hat{X} \sin(\omega t + \alpha_x) \quad \text{linear system}
\end{align*}$$

The response $X(t) = \hat{X} \sin(\omega t + \alpha_x)$ differs from the excitation $x(t) = \hat{x} \sin(\omega t + \alpha_x)$ only in amplitude and phase. In such a linear system the response can be described simply with the phase differences and the ratio of the amplitudes, which generally depend on the frequency of the excitation:

$$R_\alpha(\omega) = \alpha_x(\omega) - \alpha_y(\omega) \quad \text{phase response function} \quad (3.6.2)$$

and

$$\hat{R}(\omega) = \frac{\hat{X}(\omega)}{\hat{x}(\omega)} \quad \text{amplitude response function} \quad (3.6.3)$$

The response spectrum $E_y(\omega)$ is readily obtained from the combination of the excitation spectrum $E_x(\omega)$ and the square of the amplitude response function $\hat{R}(\omega)$ (the square is used because the spectral density is a measure of the square of the amplitude):

$$E_y(\omega) = \left[\hat{R}(\omega)\right]^2 E_x(\omega) \quad \text{response spectrum} \quad (3.6.4)$$

The response is then characterised by this response spectrum as much as the excitation is characterised by the excitation spectrum.

The two functions $\hat{R}_{ys}(\omega)$ and $R_\alpha(\omega)$ are called the frequency response functions of the system (amplitude and phase response function respectively). The response functions can sometimes be computed analytically for harmonic excitations for many different frequencies. They can often also be measured, in actual structures in the field or in laboratory conditions, for a wide range of frequencies. One technique is to excite the structure with one frequency after another, and measure the amplitude and phase response for all these frequencies. Another technique is to excite the structure with random waves and to divide the response spectrum by the excitation spectrum (see below). More advanced techniques account for unrelated effects in such measurements (also called noise) but these require a more advanced spectral analysis: a cross-spectral analysis, which falls outside the scope of this book. We can thus obtain the amplitude and phase response as a function of frequency in a variety of ways. Obvious examples are the (linearised) forces in or motions of a structure. Less obvious examples (because the above has been presented in terms of responses of structures)
are the motions of water particles or pressure in the water, induced by the waves. These can be estimated, at least to first order, with the linear wave theory (see Chapter 5) because these too are formulated in terms of ratios of amplitudes.

If the response is direction-sensitive, the above technique needs to be expanded to take wave directions into account. For that, a two-dimensional response function is required. It is defined in the same way as the frequency response function:

\[
\hat{R}(\omega, \theta) = \frac{\hat{X}(\omega, \theta)}{\hat{x}(\omega, \theta)}
\]  

(3.6.5)

where \( \hat{R}(\omega, \theta) \) is the two-dimensional frequency-direction response function. The amplitudes are the amplitude \( \hat{x}(\omega, \theta) \) of the exciting harmonic wave (travelling in direction \( \theta \)) and \( \hat{X}(\omega, \theta) \) of the response harmonic wave, respectively. The two-dimensional response spectrum is then readily calculated as:

\[
E_{r}(\omega, \theta) = \left[ \hat{R}(\omega, \theta) \right]^{2} E_{s}(\omega, \theta)
\]  

(3.6.6)

This seems obvious but that is not always the case. The exciting harmonic wave, being an ocean wave is readily seen as a wave with a direction of propagation. However, the response harmonic wave need not have a direction. For instance, the tension in a beam of an offshore platform has no direction. Still, the amplitude of such a (nondirectional) response can be defined. It is simply the amplitude of the response when the system is excited by an ocean wave from a certain direction, even if the response harmonic itself has no direction! In such cases the computation is carried out in two phases. First the above two-dimensional response spectrum \( E_{r}(\omega, \theta) \) is computed as an artefact of the procedure and then the one-dimensional frequency response spectrum is calculated as:

\[
E_{s}(\omega) = \int_{0}^{2\pi} E_{r}(\omega, \theta) \, d\theta
\]  

(3.6.7)
4. STATISTICS

4.1 KEY CONCEPTS

Short-term statistics
- The theoretical estimates of the short-term statistical characteristics of wind waves that are treated in this Chapter 4 are based on the assumption that the surface elevation is a stationary, Gaussian process.
- For such a process, Rice (1944, 1945) has given an analytical expression for the mean frequency of level crossing in terms of the variance density spectrum.
- With this expression it can be shown that for waves with a narrow spectrum, the crest height and the wave height are Rayleigh distributed with the zero-th order moment $m_0$ of the wave spectrum as the only parameter. Observations have shown that this is also true for waves with a broader spectrum.
- The significant wave height is readily estimated from the Rayleigh distribution as: $H_s = m_0 = 4\sqrt{m_0}$. This is typically 5% to 10% larger than the value $H_s = H_{1/3}$, estimated from measured time series.
- Observations show that for unimodal spectra the significant wave period is nearly equal to the peak period of the spectrum.
- The maximum individual wave height in a given duration (of stationary conditions) is a random variable, with a corresponding probability density function that can be determined from the wave spectrum. In most storms, this maximum individual wave height is about twice the significant wave height.
- The mean length of a wave group can be estimated from the width of the variance density spectrum.

Long-term statistics
- Long-term wave statistics (relating to durations of dozens of years or more) are obtained from observations or from computer simulations. A theoretical basis to analyze such observations and simulations is practically absent.
- Long-term extreme values (probability of exceedance or return periods) can be estimated with three approaches: the Initial Distribution approach, The Peak-Over-Threshold approach and the Annual Maximum approach.

4.2 INTRODUCTION

In this chapter some statistical characteristics of wind waves are considered on a short-term scale and a long-term scale. The short-term statistics relate to cumulative effects of the waves, such as the erosion of beaches or the fatigue of structures and to extreme values of the waves, such as occur in survival conditions that are used in the design of a structure. For the cumulative effects, some of the most relevant quantities are (a) the instantaneous surface elevation, (b) crossings of the surface elevation through certain levels, (c) the crest heights and (d) wave groups. For extreme values, relevant quantities are (e) the largest crest height or wave height in a certain duration (e.g., a storm). Long-term statistics (the "wave climate") relate primarily to the distribution of the significant wave height and the return period of extreme values of the significant wave height. These long-term statistics (at one geographic location) can be
estimated from (a) all available observations at that location, i.e., typically with a time interval of 3 hours (the Initial Distribution approach), (b) the maximum values of the significant wave height in storms only, at that location (above a certain threshold; Peak-Over-Threshold approach) and (c) the maxima values of the significant wave height per year only, at that location (the Annual Maximum approach). These three approaches will be briefly introduced.

Literature:

NOTE 4A
The moments of the wave spectrum

When the random sea surface elevation is treated as a stationary, Gaussian process, then all its statistical characteristics are determined by the variance density spectrum \( E(f) \). These characteristics will be expressed in terms of the moments of that spectrum, which are defined as:

\[
m_n = \int_0^\infty f^n E(f) \, df \quad \text{for} \quad n = 0, 1, 2, 3, \ldots
\]

The moment \( m_n \) is called the "n\(^{th}\)-order moment" of \( E(f) \). For example, the standard deviation \( \sigma_\eta \) of the random surface elevation (with zero-mean) is:

\[
\sigma_\eta = E(\eta^2)^{1/2} = \left[ \int E(f) \, df \right]^{1/2} = m_0^{1/2} \quad \text{if} \quad \mu_\eta = E(\eta) = 0
\]

4.3 INSTANTANEOUS SURFACE ELEVATION (deep water)

In the linear approximation of ocean waves (the random-phase/amplitude model of Section 3.5), the instantaneous sea surface elevation \( \eta(t) \), as it appears at one arbitrary moment \( t \) in time, is assumed to be Gaussian distributed. This means that, if the observation were to be repeated under exactly the same conditions, the elevation would have an unpredictable value, drawn from a Gaussian-distributed population. Assuming the mean to be zero, the Gaussian probability density function can be written as:

\[
p(\eta) = \frac{1}{(2\pi m_0)^{1/2}} \exp \left( -\frac{\eta^2}{2m_0} \right) \quad \text{for a zero-mean} \tag{4.3.1}
\]

where \( m_0^{1/2} \) is the standard deviation of the surface elevation (see Note 4A). An empirical confirmation that this is true would require obtaining a large set of wave records at sea (an ensemble), with identical statistical conditions (i.e., identical storms), which of course is impossible. Instead therefore, one considers the surface elevation as a function of time in one, stationary wave record (assuming that the process is ergodic, i.e., ensemble averages are equal to time averages, see Appendix A). Usually, the agreement between the observed and theoretical probability density functions is good, at least in open sea (deep water), but high elevations are observed slightly more frequently than according to the Gaussian model and low elevations are observed slightly less frequently. For steeper waves or in shallow water the discrepancies are more
pronounced because the waves are more non-linear. Two such observations with a fairly strong nonlinear character (one with a large wave steepness\(^9\) = 0.06, in deep water and one with a relatively large wave height in shallow water, \(H_{m_s} / \text{depth} = 0.44\)), are compared with the Gaussian model in Fig. 4.3.1.

\[ \text{Steepness is defined here as } 2\pi H_{m_s} / (gT^2) \text{ where the mean wave period is } T = m_j / m_1 \text{ (the zeroth- and first-order moment of the spectrum).} \]

![Fig. 4.3.1 Histograms of the observed surface elevation (deep and shallow water) and the corresponding Gaussian probability density functions: panel a, deep water = 70 m depth, significant wave height 2.70 m and mean period 5.3 s (data courtesy of FUGRO-OCEANOR, Trondheim, Norway), panels b, shallow water = 8.8 m depth and significant wave height 3.55 m (after Ochi and Wang, 1984; same observations in panels b1 and b2). The Gram-Charlier series provides a better fit for the strongly non-linear waves in shallow water.](image)

This character of the deviations is due to the fact that non-linear processes generally make the wave crests higher and sharper and the wave troughs shallower and flatter. The analogous differences between a harmonic wave and a non-linear wave (e.g., a Stokes wave, see Section 5.8.2) are shown in Fig. 4.3.2. It is obvious, because of the sharp and high crests in the non-linear wave, that the surface elevation remains longer at the higher levels for the non-linear wave than for the harmonic wave (near A in Fig. 4.3.2). Such a longer fraction of time corresponds to a larger probability that the surface elevation is located at these higher levels. This explains why the Gaussian model underestimates the observed high values of the random surface elevation.
Fig. 4.3.2 The analogy of the difference between a harmonic wave and a real (Stokes-type) wave (panel a) and the corresponding difference between a Gaussian distribution and an observed probability density function of the surface elevation at sea (panel b). Note that the axes in panel (b) are 90° rotated relative to the usual orientation (to show η vertically as in panel a).

In contrast to this, for the large, negative values, the relatively shallow and flat troughs in the non-linear wave (near B in Fig. 4.3.2) correspond to a smaller fraction of time at these negative values than in the harmonic wave. Near the mean of the record (η = 0; near C in Fig. 4.3.2), the relatively steep slope of the real surface reduces the fraction of time of the elevation around the value of zero and hence the probability of occurrence in this interval. The total effect of these deviations is a probability density function of the real surface elevation that is skewed to the negative values ("leaning" towards negative values; see Fig. 4.3.2, panel b) but its extremes, both positive and negative, are shifted upwards (along the η-axis). This skewed character is evident when visually inspecting a paper record of waves. It is immediately obvious (from the steeper crests and flatter troughs) whether the record is oriented upside-down or not. A probability density function that takes such skewness into account is the Edgeworth's form of the type A Gram-Charlier series (Longuet-Higgins, 1963; see Fig. 4.3.1 panel b2).

Literature:

4.4 WAVE HEIGHT AND PERIOD

The statistical characteristics of the wave height, probably the most important wave parameter for engineers, can be obtained theoretically from the wave spectrum. The derivation that is shown here is based on an expression due to Rice\(^\text{10}\) (1944, 1945 and 1954), for the average time interval between level crossings of the sea surface elevation (see Fig. 4.4.1). From this expression, the probability density function of the crest height and wave height can be derived.

Literature:

\(^{10}\) S. O. Rice, 1907-1986,
4.4.1 Wave period (deep water)

The integral of the Gaussian probability density function, gives the probability that \( \eta(t) \) is below a certain level \( \eta \) or, the fraction of time that \( \eta(t) \) is below that level. This fraction of time, by itself, does not give any information as to how often the surface elevation crosses that level or what the time interval is between such crossings (the time period \( T_\eta \) between level up- or down crossings; see Fig. 4.4.1).

![Graph showing surface elevation and level up-crossings](image)

**Fig. 4.4.1** The up-crossings of the sea surface elevation through level \( \eta \) and the corresponding time intervals \( T_\eta \).

The average of that interval, \( \overline{T}_\eta \), can be readily expressed in terms of the spectrum Rice (1944, 1945 and 1954):

\[
\overline{T}_\eta = \sqrt{m_0/m_2} \exp \left( -\frac{\eta^2}{2m_0} \right)
\]  

(4.4.1)

where \( m_0 \) and \( m_2 \) are the zero-th and second-order moment (see Note 4A) respectively of the variance density spectrum \( E(f) \). The mean frequency of these level-crossings, \( f_\eta = \overline{T}_\eta^{-1} \) is correspondingly:

\[
f_\eta = \sqrt{m_2/m_0} \exp \left( -\frac{\eta^2}{2m_0} \right)
\]  

(4.4.2)

A special case is the mean zero-crossing period \( \overline{T}_0 \) (see Section 3.3.3), which can be obtained from Eq. (4.4.1) with \( \eta = 0 \):

\[
\overline{T}_0 = \sqrt{m_0/m_2}
\]  

(4.4.3)
This estimate of the mean zero-crossing period is sometimes denoted as $\bar{T}_0 = T_{m^2}$. The reciprocal of this, the mean zero-crossing frequency $\bar{f}_o = \bar{T}_0^{-1}$ is obviously:

$$\bar{f}_o = \frac{m_2}{m_0}$$ (4.4.4)

Unfortunately, the value of $m_2$ (and therefore also the estimate of $\bar{T}_0$ with Eq. 4.4.3) is sensitive to small errors or variations in the measurement technique or in the analysis technique. For instance, the integration interval to estimate $m_0$ and $m_2$ from the spectrum should strictly range from $f = 0$ to $f = \infty$, whereas in actual practise it is from 0 to some practical upper limit (e.g., the Nyquist frequency; see Appendix C). Moreover, the definition of the moments of the spectrum shows that, in general, the values of higher-order moments are rather sensitive to noise in the high-frequency range of the spectrum. The reason is that in the computation of higher-order moments, the spectral levels of higher frequencies, where noise is usually relatively large, are enhanced far more than the spectral levels of the lower frequencies. Similarly, if the wave record itself is used to directly estimate $\bar{T}_0$ (instead of using the moments of the spectrum), the definition says that all wave heights in the wave record should be included in the averaging procedure. In actual practise a (low) threshold value for the waves is used (typically a few centimetres), to avoid including non-physical variations near the zero-level of the wave record (e.g., related to instrument noise). These considerations suggest that $\bar{T}_0$ is not always the most reliably estimated characteristic wave period. Another mean period is therefore sometimes used, which is less dependent on high-frequency noise. It is defined as the inverse of the mean frequency of the wave spectrum:

$$T_{m^2} = f_{\text{mean}} = \left(\frac{m_2}{m_0}\right)^{-1}$$ (4.4.5)

Still another characteristic wave period is the significant wave period $T_{1/3}$ (see Section 3.3.3). Like the mean period $T_{m^2}$ it is less dependent on high-frequency noise as it depends only on the higher waves. A theoretical expression in terms of the spectrum is available but it is rather complicated and it will not be treated here (see Kitano et al., 2001). The following relationships are empirical (based on observations or computer simulations, e.g., Goda, 1988). For swell (or more precisely: waves with a narrow spectrum), the zero-crossing period is nearly the same for all waves. The mean zero-crossing period $\bar{T}_0$ is then practically equal to $T_{1/3}$ which, in this case, is almost equal to the peak period of the spectrum (the inverse of the frequency with maximum energy density):

$$\bar{T}_0 = T_{1/3} = T_{\text{peak}}$$ (4.4.6)

For wind sea (with a wider spectrum), this is not the case, but it has been found empirically (Goda, 1978) that, if the spectrum is unimodal, the average period of the higher waves is approximately equal to 0.9 times the inverse of the peak frequency $f_{\text{peak}}$. To illustrate this, each line in Fig. 4.4.2 shows how $\bar{T}_H$ (the mean zero-crossing period for waves with wave height $H$) varies with wave height $H$. For all lines in this Fig. 4.4.2, i.e., for all spectra considered, this wave period is approximately 0.9 times the peak period for wave heights larger than $1.5\bar{H}$ (i.e., for the 17% highest waves):
\[ T_{\text{H}} = 0.9 T_{\text{peak}} \]

for unimodal wind sea spectra and \( H \geq 1.5 \bar{H} \)

Since the significant wave period \( T_{1/3} \) is taken from these higher waves:

\[ T_{1/3} = \overline{T_H} = 0.9 T_{\text{peak}} \]

for unimodal wind sea spectra

\[ \frac{\overline{T_H}}{T_{\text{peak}}} \]

\[ \begin{align*}
0 & \quad 0.5 & \quad 1.0 \\
0.0 & \quad 1.0 & \quad 2.0 & \quad 3.0
\end{align*} \]

\[ H / \bar{H} \]

\[ \text{spectrum narrower} \]

\[ \text{Fig. 4.4.2 The mean zero-crossing wave period for a given wave height (normalised with the peak period) as a function of wave height (normalised with the mean wave height) for spectra with different spectral width (after Goda, 1978).} \]

Literature:
Kitano et al. (2001)

4.4.2 Crest height (deep water)

One would expect the value of a crest height \( \eta_{\text{crest}} \), defined as the maximum elevation in a wave to be positive (Fig. 4.4.3). It is after all a maximum. If the spectrum is narrow, this is certainly the case. However, if spectrum is wide, i.e., the waves are irregular, the crest height thus defined may well be negative. This difference shows that the width of the spectrum affects the shape of the probability density function of the crest heights: a narrow spectrum corresponds to positive crest heights only; a wide spectrum corresponds to positive and negative crest heights.

For waves with a narrow spectrum, the derivation of the statistical characteristics of the crest height is relatively simple. For such waves, the total number of crests is equal to the number of up-crossings through the zero-level (see Fig. 4.4.4). In addition, the total number of crests above a positive level \( \eta \) is equal to the number of up-crossings through that level. The fraction of crests above a certain level \( \eta_{\text{crest}} > \eta \) can then be estimated from the number of crests in duration \( D \) as:

\[ \frac{\text{number of crests with } \left( \eta_{\text{crest}} > \eta \right) \text{ in duration } D}{\text{total number of crests in duration } D} = \frac{D / \overline{T_{\eta}}}{D / T_{\text{peak}}} = \frac{f_{\eta}}{f_0} \]  \hspace{1cm} (4.4.9)
Fig. 4.4.3 The exclusively positive crest heights in wave conditions with a narrow spectrum (regular appearance of the sea surface) and the positive and negative crest heights in wave conditions with a wide spectrum (irregular appearance of the sea surface).

Interpreting this fraction as the probability of $\eta_{\text{crest}}$ exceeding the level $\eta$ and substituting the expressions for $\bar{f}_\eta$ and $\bar{f}_0$, Eqs. (4.4.2) and (4.4.4) in the right-hand-side of Eq. (4.4.9) gives:

$$\Pr\{ \eta_{\text{crest}} > \eta \} = \frac{\bar{f}_\eta}{\bar{f}_0} = \frac{m_2 \exp\left(-\frac{\eta^2}{2m_0}\right)}{m_0} = \exp\left(-\frac{\eta^2}{2m_0}\right)$$  \hspace{1cm} (4.4.10)

From which the cumulative distribution function $\Pr\{ \eta_{\text{crest}} < \eta \} = 1 - \Pr\{ \eta_{\text{crest}} > \eta \}$ is obtained as:

$$p_{2,\text{crest}}(\eta) = \Pr\{ \eta_{\text{crest}} \leq \eta \} = 1 - \exp\left(-\frac{\eta^2}{2m_0}\right)$$  \hspace{1cm} (4.4.11)
Fig. 4.4.4  The number of crests in wave conditions with a sufficiently narrow spectrum (regular appearance of the waves) is equal to the number of level up-crossings.

The probability density function of $\eta_{\text{crest}}$ is then obtained as the derivative of $P_{\eta_{\text{crest}}} (\eta)$:

$$p_{\eta_{\text{crest}}} (\eta) = \frac{\eta}{m_0} \exp \left( -\frac{\eta^2}{2m_0} \right)$$

which is shown in Fig. 4.4.5 (see Note 4B about the notation in Eqs. 4.4.11 and 4.4.12). These functions are of the Rayleigh-type (i.e., the independent variable $\eta$ in the cumulative distribution function occurs to the second power in the exponent). A Rayleigh distribution has only one parameter, which in this case is the zero-th order moment $m_0$ of the variance density spectrum (and not the zero-th order moment of the Rayleigh distribution or any other function!). Since all statistical characteristics of $\eta_{\text{crest}}$ are determined by this function, they can all be expressed in terms of this moment $m_0$ alone (provided that the spectrum is narrow). For instance, the mean and standard deviation of the crest height for waves with a narrow spectrum are:

$$\mu_{\text{crest}} = E \{ \eta_{\text{crest}} \} = \frac{\pi}{\sqrt{2}} \sqrt{m_0}$$

$$\sigma_{\eta_{\text{crest}}} = \sqrt{E \{ \eta_{\text{crest}}^2 \} - E^2 \{ \eta_{\text{crest}} \} } = \sqrt{2 - \frac{\pi}{2} \sqrt{m_0}}$$
For waves with a wide spectrum, i.e., with an irregular appearance of the sea surface elevation, the probability density function is not readily derived (see Note 4C for quantifying the width of the spectrum). However, if we consider the maximum crest height per wave $\hat{\eta}_{\text{crest}}$ (i.e., between two consecutive zero up-crossings; see Fig. 4.4.6; note the $\hat{}$ in the notation), then observations have shown that the distribution function of this maximum is practically a Rayleigh distribution (at least for values of $\hat{\eta}_{\text{crest}}$ that are not too low, e.g., $\hat{\eta}_{\text{crest}} \geq \sqrt{m_0}$).

Fig. 4.4.5 The Rayleigh probability density function of the crest height $\eta_{\text{crest}}$ for wave conditions with a narrow spectrum.

Fig. 4.4.6 There is only one maximum crest height between zero up-crossings, also for waves with a wide spectrum.
**NOTE 4B**  
Notation in probability functions

The probability functions that we consider, relate to certain stochastic variables e.g., the cumulative distribution function of the surface elevation $\eta$. This distribution function gives the probability that the surface elevation $\eta$ is lower than some given level $\eta$:

$$P(\eta) = \Pr\{\eta < \eta\}.$$  

Note that the stochastic variable $\eta$ is underscored whereas the level $\eta$ is not (the level under consideration is not a stochastic variable). Strictly speaking we should make that distinction in the notation. The cumulative distribution function of $\eta$ should therefore be written as $P_\eta(\eta)$: the stochastic variable as a subscript and the level as the argument. However, when the stochastic variable and the level are indicated with the same symbol, the subscript is ignored: $P_\eta(\eta) = P(\eta)$. But, sometimes the stochastic variable is indicated with another symbol than the level under consideration. For instance, the crest height, $\eta_{crest}$, may or may not exceed a certain level $\eta$. In such cases, we must make the distinction, hence $P_{\eta_{crest}}(\eta)$ for the cumulative distribution function of $\eta_{crest}$.

**NOTE 4C**  
Spectral width parameters

For arbitrarily shaped spectra, the probability density functions of the crest height and the wave height depend on the spectral width, which can be quantified with a parameter $\varepsilon$ defined by Cartwright and Longuet-Higgins (1956) as:

$$\varepsilon = \left(1 - \frac{m_2^4}{m_4m_1}\right)^{1/2}$$

For $\varepsilon \to 0$ (a very narrow spectrum), $P_{\eta_{crest}}(\eta)$ and $P(H)$ approach the Rayleigh distribution, whereas for $\varepsilon \to 1$ (a very wide spectrum), the distribution approaches a Gaussian distribution (a very irregular appearance of the waves with as many positive as negative crest heights). Note that this is a theoretical result for a spectrum with an arbitrary shape.

The spectrum of ocean waves very often has a tail with a shape given by $\alpha g^2 f^{-5}$ (where $\alpha$ is a constant and $g$ is gravitational acceleration; see Section 6.3.3). The value of the fourth-order moment $m_4$, which is required to estimate $\varepsilon$, is then dominated by the upper limit of the integration (usually the Nyquist frequency, see Appendix C). Moreover, $m_4$ is a fairly high-order moment of the spectrum and its estimation is therefore rather sensitive to noise in the spectrum at high frequencies (or the presence of nonlinear effects, the higher-harmonics of which appear as secondary, high-frequency peaks in the spectrum of the waves). In actual practise of ocean waves, the value of $\varepsilon$ does therefore not depend only on the shape of the spectrum, but also, and to a high degree, on the cut-off frequency and the errors in the high-frequency part of the spectrum (which are usually relatively large). The parameter $\varepsilon$ should therefore be used with great care. An alternative spectral width parameter $\nu$ is given by Longuet-Higgins (1975):

$$\nu = \left(\frac{m_4m_2^4}{m_1^4} - 1\right)^{1/2}$$

Which suffers to a lesser degree from the same problem. Another spectral width parameter, denoted as $\kappa$ and due to Battjes and Van Vledder (1984), is defined as:
\[ \kappa^2 = \frac{1}{m_0^2} \left[ \int_{0}^{\infty} E(f) \cos(2\pi f \tau) df \right]^2 + \left[ \int_{0}^{\infty} E(f) \sin(2\pi f \tau) df \right]^2 \] with \( \tau = \bar{\tau}_0 = \sqrt{m_0 / m_2} \).

These spectral width parameters are also control the groupiness character of the waves (see Section 4.5). In this role, \( \kappa \) is superior in several respects to \( \varepsilon \) and \( \nu \) (see Van Vledder, 1992).

**Literature:**
Forristall (2000)

### 4.4.3 Wave height (deep water)

For waves with a narrow spectrum (the appearance of the sea surface elevation is regular), the height of the wave is practically equal to twice the height of the crest: \( H = 2 \eta_{crest} \) (see Fig. 4.4.7). The probability density function of \( H \) can then be readily determined from the probability density function of \( \eta_{crest} \) with a simple transformation, using a Jacobian as explained in Section 3.5.5:

\[ p(H) = p_{\eta_{crest}}(\eta) \frac{d\eta_{crest}}{dH} \] (4.4.14)

so that with Eq. (4.4.12), the probability density function of the wave height \( H \) is:

\[ p(H) = \frac{\eta_{crest}}{m_0} \exp \left( -\frac{\eta_{crest}^2}{2m_0} \right) \frac{d\eta_{crest}}{dH} \] (4.4.15)

so that with \( \eta_{crest} = \sqrt{H} \):

\[ p(H) = \frac{H}{4m_0} \exp \left( -\frac{H^2}{8m_0} \right) \quad \text{for } H \geq 0 \] (4.4.16)

which is also a Rayleigh distribution. All statistical characteristics of \( H \) follow from this Rayleigh distribution, e.g., the mean and the root-mean-square (rms) value of the wave height are:

\[ \bar{H} = E\{H\} = \sqrt{2\pi m_0} \] (4.4.17)

and

\[ H_{\text{rms}} = \left[ E\{H^2\} \right]^{\frac{1}{2}} = \sqrt{8m_0} \] (4.4.18)
The cumulative distribution function of $H$ can be obtained by integrating the probability density function, but it can also, and more directly, be obtained from the cumulative distribution function of $\eta_{\text{crest}}$ (which is not a density function and the transformation does not require using a Jacobian). Just substituting $\eta_{\text{crest}} = H/2$ in this cumulative distribution function gives the cumulative distribution function for the individual wave height $H$:

$$\text{Pr}\{ H \leq H \} = 1 - \exp\left( -\frac{H^2}{8m_0} \right)$$  \hspace{1cm} (4.4.19)

Which, of course, can also be obtained by integrating the probability density function $p(H)$ of Eq. (4.4.16).

![Fig. 4.4.7 The wave height $H$ for wave conditions with a narrow spectrum.](image)

As indicated in Section 3.3.2, the significant wave height is defined as the mean value of the 1/3-highest wave heights. This fraction of the waves can be identified in the Rayleigh distribution, so that the significant wave height can be determined from that distribution. We will denote this estimate of the significant wave height not as $H_s$ but as $H_{s0}$ (analogous to the notation of the mean zero-crossing period $T_{m0}$), to distinguish it from the estimate obtained visually $H_s = H_{sv}$, or from the wave record directly $H_s = H_{1/3}$. The wave heights that are involved in the definition of the significant wave height are located in the 1/3 highest part of the Rayleigh distribution, where $H > H^*$, with $H^*$ defined by (see Fig. 4.4.8)\textsuperscript{11}:

$$\frac{1}{3} \int_{H^*}^{\infty} p(H) \, dH = \frac{1}{3}$$  \hspace{1cm} (4.4.20)

\textsuperscript{11} Since $H = \sqrt{\ln(Q^{-1})}H_{m0}$ (where $Q = \text{Pr}\{ H > H \}$, from Eq. 4.4.19), it follows that $H_s = \sqrt{\ln(3)}H_{m0} = 1.048H_{m0}$. 
The significant wave height in the Rayleigh probability density function.

The mean value of these 1/3-highest wave heights is, by definition, the significant wave height. It can be determined as an expected value, i.e., with the zero-th and first order moments of the 1/3 highest part of the distribution:

\[
H_{m_0} = E\{H\}_{H \geq H'} = \frac{\int_{H'}^\infty H \cdot p(H) dH}{\int_{H'}^\infty p(H) dH}
\]  
(4.4.21)

Substituting the analytical expression for the Rayleigh distribution in this Eq. (4.4.21) gives the following result:

\[
H_{m_0} = 4.005... \sqrt{m_0}
\]  
(4.4.22)

or, for all practical purposes:

\[
H_{m_0} = 4\sqrt{m_0}
\]  
(4.4.23)

Substituting this expression back into the Rayleigh distribution shows that 13.5% of the wave heights exceed this value. For the Rayleigh distribution, the ratio between \(H_{m_0}\) and other characteristic wave heights is fixed, for instance:

\[
\bar{H} = E\{H\} = \sqrt{\frac{\pi}{8}} H_{m_0}
\]  
(4.4.24)

and

\[
H_{rms} = \frac{1}{\sqrt{2}} H_{m_0}
\]  
(4.4.25)

Observations have shown that wave heights in deep water are indeed almost Rayleigh distributed (if the waves are not too steep). This is illustrated in Fig. 4.4.9 with a cumulative distribution of observed wave heights from 5 hurricanes in
the Gulf of Mexico, plotted on Rayleigh paper\textsuperscript{12}. The clustering of the observations around a straight line shows that these observations are indeed close to Rayleigh distributed (i.e., the shape of the observed distribution is close to the shape of the Rayleigh distribution).

Although these and other observations confirm the applicability of the shape of the Rayleigh distribution, they show a somewhat smaller significant wave height than predicted from the zero-the order moment of the spectrum $m_0$ with $H_{m_0} = 4\sqrt{m_0}$. There are several reasons for this. One is, that in the above theoretical derivation, a narrow spectrum was assumed with no correlation between one wave height and the next. In actual sea conditions, the spectrum has a finite width and consecutive wave heights are correlated. In addition, the surface elevation is not perfectly Gaussian distributed due to nonlinear processes such as wave breaking and nonlinear wave-wave interactions. The consequence is that the significant wave height estimated from a zero-crossing analysis, $H_{\chi}$ may be 5\% to 10\% lower than the significant wave height as estimated from the spectrum with $H_{m_0} = 4\sqrt{m_0}$). Longuet-Higgins (1980) suggests $H_{1/3} = 0.925H_{m_0}$ for the observations of Fig. (4.4.9). This is in close agreement with the results for other observations (see Fig. 4.4.10). Numerical simulations based on a linear superposition of independent wave components, give slightly higher values than these empirical findings from the field ($H_{1/3} = 0.95H_{m_0}$; Goda, 1988a).

\textsuperscript{12} Plot $[-\ln(1 - P(x))]^{1/2}$ against $x$. 

\textbf{Fig. 4.4.9} The cumulative distribution of observed individual wave heights $H$, normalized with the standard deviation of the surface elevation $\sqrt{m_0}$, from 5 hurricanes in the Gulf of Mexico (data from Forristall, 1978), on Rayleigh paper and the suggestion of Longuet-Higgins (1980) to scale the Rayleigh distribution for these observations with a factor 0.925.
The significant wave height $H_{1/3}$ estimated directly from the time record of the waves, compared to the theoretical estimate $H_m = 4\sqrt{m_0}$. Data from location K13 in the southern North Sea during December 2003, courtesy of Royal Netherlands Meteorological Institute. The best-fit linear approximation (least-squares fit) is close to the suggestion of Longuet-Higgins (1980) for the hurricane data of Forristall (1978).

A formulation of the Rayleigh distribution that is independent of this discrepancy would be the following self-scaling formulation (substitute Eq. 4.4.23 into Eq. 4.4.19):

$$\text{Pr}\left\{ H < H \right\} = 1 - \exp\left\{ -2\left( \frac{H}{H_{1/3}} \right)^2 \right\} \quad \text{for } H \geq 0$$  \hspace{1cm} (4.4.26)

The corresponding probability density function can be readily obtained by differentiating this cumulative distribution (or by substituting Eq. 4.4.23 into Eq. 4.4.16):

$$p(H) = \frac{4H}{H_{1/3}} \exp\left\{ -2\left( \frac{H}{H_{1/3}} \right)^2 \right\} \quad \text{for } H \geq 0$$ \hspace{1cm} (4.4.27)

Since the shape of the distribution function is not altered by this substitution, all relationships between characteristic values of the wave height $H$ are unaffected and therefore still valid for real ocean waves (to the extent that the Rayleigh distribution applies).

**Literature:**
4.4.4 Wave height (shallow water)

The wave height distribution in shallow water deviates from the distribution in deep water due to the effects of non-linear phenomena of which the most extreme example is wave breaking, e.g. in the surf zone. Generally accepted theoretical derivations, such as those that lead to the Rayleigh distribution in deep water, are not available for shallow water. It appears that, in spite of this, the Rayleigh distribution fits the observations of waves in shallow water reasonably well. However, a closer inspection reveals that the distribution is affected at the higher values of the wave heights. This has lead Battjes and Groenendijk (2000) to replace the tail of the Rayleigh distribution with the tail of the more general Weibull distribution (see Fig. 4.4.1; the Rayleigh distribution is a special case of the Weibull distribution, see below):

\[
\Pr\{H < H\} = 1 - \exp\left\{-2\left(\frac{H}{H_t}\right)^k\right\}
\]

all individual waves in shallow water \hspace{1cm} (4.4.28)

where \(i = 1\) for \(H \leq H_v\), and \(i = 2\) for \(H > H_v\), where \(H_v\) is a transition wave height. The values for the coefficients, suggested by Battjes and Groenendijk (2000) are \(k_1 = 2\) (which makes the expression a Rayleigh distribution for \(H \leq H_v\)) and \(k_2 = 3.6\). The values of \(H_1\) and \(H_2\) both follow from the values of \(H_{rms}\) and the transition wave height \(H_v\), which depend on the zero-th order moment of the wave spectrum, the local water depth and the local bottom slope (Battjes and Groenendijk, 2000).

Some situations require that the probability density function is known for breaking waves, for instance to estimate the dissipation rate of wave energy in the surf zone (see Section 8.4.3.2). Thornton and Guza (1983) fitted a Rayleigh distribution to their observation of breaking waves (see Fig. 4.4.12) but found that a Rayleigh distribution weighted with a weight function \(W(H)\) fitted these observations better:

\[
p_{H_v}^*(H) = \frac{2H}{H_{rms}^2} \exp\left\{-\left(\frac{H}{H_{rms}}\right)^2\right\} W(H)
\]

for individual breaking waves in shallow water \hspace{1cm} (4.4.29)

with:

\[
W(H) = \left\{\begin{array}{ll}
1 & \text{if } H < H_{rms} \\
0 & \text{if } H > H_{rms}
\end{array}\right.
\]
\[ W(H) = \left( \frac{H_{rms}}{\gamma d} \right)^n \left[ 1 - \exp \left( - \frac{H}{\gamma d} \right) \right] \]

where \( p^*_{H_0}(H) \) is a density function, the surface area of which is the probability of breaking (and not unity; it should therefore not be properly called a probability density function; hence the asterisk \(^*\) in the notation). Thornton and Guza (1983) suggested \( \gamma \approx 0.42 \) and \( n = 4 \) for their observations.

\[ \int p^*(H/H_0) \, d(H/H_0) = \text{probability of breaking (after Thornton and Guza, 1983).} \]

**Fig. 4.4.12** The distribution of the observed normalised wave height \( H/H_0 \) of individual breaking waves \( p^*(H/H_0) \) and of all waves \( p(H/H_0) \), in surf zone conditions (\( H_0 \) is the offshore rms-wave height). The function \( p^*(H/H_0) \) is scaled such that \( \int p^*(H/H_0) \, d(H/H_0) = \text{probability of breaking} \).

**Literature:**

## 4.5 WAVE GROUPS

For some engineering problems, the arrival of a series of high waves (a wave group) is of great importance. For instance, the stability of a rubble-mound breakwater and the overtopping of a dike by waves depend on the number of waves in such a group. A wave group can be defined more precisely as an uninterrupted sequence of waves with wave heights higher than an arbitrarily chosen but usually high, threshold value \( H \). The length of such a wave group is by definition, the number of waves (\( N \)) in the group.

To derive the probability that the length of an arbitrarily chosen wave group is larger than a value \( N \), imagine a long, (statistically) stationary time series of wave heights:
A wave group \textit{starts} when a wave height is larger than the threshold value \(H\) and the preceding wave height is smaller than \(H\), that is, the wave group starts at \(H\), if \(H > H\) and \(H_{i+1} < H\). A wave group \textit{ends} when a wave height is higher than \(H\) and the following wave height is smaller than \(H\), that is, the wave group ends at \(H\), if \(H > H\) and \(H_{j+1} < H\).

If \(j = i\), that is, the first wave of the group is also the last wave of the group, the length of the group is obviously \(N = 1\). This is rather trivial but by definition it is a group. The probability of this to occur is given by (remember that, from a formal point of view, \(H\) is the first wave of the group):

\[
\Pr\{N = 1\} = \Pr\{H_i \geq H \text{ and } H_{i+1} < H\} \tag{4.5.1}
\]

If individual wave heights are independent, then this \textit{joint} probability \(\Pr\{H_i \geq H \text{ and } H_{i+1} < H\}\) is equal to the product of the \textit{individual} probabilities (properly called marginal probabilities, i.e. the probability of one variable, independent of the other variable):

\[
\Pr\{N = 1\} = \Pr\{H_i \geq H\} \cdot \Pr\{H_{i+1} < H\} \tag{4.5.2}
\]

We selected the group such that always (i.e. for every group) \(H_i > H\), in other words, \(\Pr\{H_i \geq H\} = 1\) for every group, so that, with Eq. (4.5.2):

\[
\Pr\{N = 1\} = \Pr\{H_{i+1} < H\} = 1 - Q_H \tag{4.5.3}
\]

where \(Q_H = \Pr\{H_{i+1} \geq H\}\).

If the second wave in the group is also high, i.e., \(H_{i+1} > H\), then the length of the group is 2 or more. When the next wave height is lower than the threshold value, \(H_{i+2} < H\), the length of the wave group is \(N = 2\) and the probability that \(N = 2\) is then given as (if the wave heights are statistically independent):

\[
\Pr\{N = 2\} = \Pr\{H_i \geq H \text{ and } H_{i+1} \geq H \text{ and } H_{i+2} < H\}

= \Pr\{H_i \geq H\} \cdot \Pr\{H_{i+1} \geq H\} \cdot \Pr\{H_{i+2} < H\} = 1 \cdot Q_H \cdot (1 - Q_H) = Q_H (1 - Q_H) \tag{4.5.4}
\]

In the same way, the probability that the group length \(N = 3\) is given as:

\[
\Pr\{N = 3\} = \Pr\{H_i \geq H \text{ and } H_{i+1} \geq H \text{ and } H_{i+2} \geq H \text{ and } H_{i+3} < H\}.
\]
The probability $\bar{N} = N$ is similarly given as:

$$
\Pr\{ N = N \} = Q_H^{N-1} (1 - Q_H)
$$

(4.5.6)

These estimates of probabilities are based on the assumption that the wave heights are mutually independent. However, wave heights correlate to some degree (i.e., they are not independent). A high wave is generally followed by another high wave (and a low wave is generally followed by another low wave). The probability $Q_H$ must therefore be replaced with a probability that involves the effect of the preceding wave height. This is a conditional probability, which is the probability that a wave height $H_{i+1} > H$, under the condition that the preceding wave height $H_i > H$. The notation of this conditional probability is $\Pr\{ H_{i+1} > H \mid H_i > H \} = R_H$. The probability of the group length being equal to $\bar{N} = N$, would then be, with the same rationale as above, but now with this dependency between wave heights taken into account (compare with Eq. 4.5.6):

$$
\Pr\{ N = N \} = R_H^{N-1} (1 - R_H)
$$

(4.5.7)

The mean length of a wave group, expressed in terms of the number of waves in the group is then given by (see Kimura, 1980):

$$
\bar{T}_{\text{high waves}} = (1 - R_H)^{-1}
$$

(4.5.8)

It is likewise possible to define a group of low waves (i.e., a sequence of wave heights lower than a certain threshold $H$). The average length of a group of such low waves, is:

$$
\bar{T}_{\text{low waves}} = (1 - R_H^*)^{-1}
$$

(4.5.9)

where:

$$
R_H^* = \Pr\{ H_{i+1} < H \mid H_i < H \}
$$

(4.5.10)

It is remarkable that for typical wind sea conditions (with a JONSWAP spectrum, see Section 6.3.3) the mean distance between two consecutive groups of high waves, i.e., the combined length of a group of high waves plus the length of a group of low waves: $\bar{T} = \bar{T}_{\text{high waves}} + \bar{T}_{\text{low waves}}$ is approximately 7 (for a threshold value $H$ between the mean wave height $\bar{H}$ and the significant wave height $H_{1/3}$). In spite of the scepticism of some wave researchers, this corresponds nicely with the old rule-of-thumb: "every seventh wave is the highest wave" (e.g., Rudyard Kipling, in "The first jungle book: the white seal" or Henri Charrière in "Pappilon").
Even a casual wave observation shows that in conditions with relatively regular waves (conditions with a narrow spectrum), the wave groups are relatively long, whereas in conditions with irregular waves (wide spectrum), the wave groups are relatively short. The groupiness of waves and the probability $R_d$ depend therefore on the width of the spectrum. In fact, the statistical distribution of the group length can be expressed in terms of the spectrum with the spectral width $\kappa$ (see Note 4C) as one of the parameters.

Literature:

4.6 EXTREME VALUES (stationary conditions)

For many engineering problems it is important to understand the statistical characteristics of extreme wave heights, in particular of the maximum surface elevation $\eta_{\text{max}}$ in a certain duration $D$ (see Fig. 4.6.1) or the maximum wave height $H_{\text{max}}$ in that duration (remember that we are considering stationary conditions).

The statistical characteristics of these maxima are fully described by the cumulative distribution functions for the maximum elevation $P_{\eta_{\text{max}}} (\eta) = \Pr\{\eta_{\text{max}} < \eta\}_D$ or for the maximum wave height $P_{H_{\text{max}}} (H) = \Pr\{H_{\text{max}} < H\}_D$. To determine the maximum elevation in the duration $D$ it is relevant to note this maximum elevation is equal to the maximum crest height in that duration, so that:

$$P_{\eta_{\text{max}}} (\eta) = \Pr\{\eta_{\text{max}} < \eta\}_D = \Pr\{\eta_{\text{max,crest}} < \eta\}_D$$

(4.6.1)

The question of finding the cumulative distribution function of the maximum elevation is thus transformed into the question of finding the cumulative distribution function of the maximum crest height $\Pr\{\eta_{\text{max,crest}} < \eta\}_D$. 

![Fig. 4.6.1 The maximum crest height in a duration D.](image)
4.6.1 Extreme elevations

The rationale for arriving at cumulative distribution function of the maximum crest height or maximum elevation, \( \Pr\{\eta_{\text{max,crest}} < \eta\} \) is the following. The probability that an arbitrarily chosen crest height in a given sea state (i.e., one arbitrarily chosen crest height in a record with a given spectrum of a stationary wave condition) exceeds a level \( \eta \) is given by \( \Pr\{\eta_{\text{crest}} > \eta\} \). For the sake of brevity, we will denote this probability as \( Q_{\text{crest}} = \Pr\{\eta_{\text{crest}} > \eta\} \), which normally is taken to be a Rayleigh distribution (see Section 4.4.2). The probability that this arbitrarily chosen crest height does not exceed the level \( \eta \) is then \( 1 - Q_{\text{crest}} \).

The probability that two arbitrarily chosen crest heights in the wave record do not exceed the level \( \eta \) is given by \( (1 - Q_{\text{crest}})^2 \) (if the crest heights are statistically independent, which is not entirely true, see Section 4.5, but the errors involved are acceptable, see below). For the same reason and under the same conditions, is the probability that all crest heights in the duration \( D \) do not exceed the level \( \eta \), given by:

\[
\Pr\{\text{all } \eta_{\text{crest}} \leq \eta\}_D = (1 - Q_{\text{crest}})^N
\]  

(4.6.2)

where \( N \) is the total number of crests in duration \( D \). If all crest heights are lower than the level \( \eta \), then all elevations (and the maximum crest height) are lower than this level, so that:

\[
\Pr\{\text{all } \eta < \eta\}_D = \Pr\{\eta_{\text{max}} < \eta\}_D = \Pr\{\eta_{\text{max,crest}} < \eta\}_D = (1 - Q_{\text{crest}})^N
\]  

(4.6.3)

The probability that the crest height will be larger than the level \( \eta \) (one or more times) in duration \( D \), is equal to the probability that not all crests heights are lower than level \( \eta \):

\[
\Pr\{\eta_{\text{max,crest}} > \eta\}_D = 1 - (1 - Q_{\text{crest}})^N
\]  

(4.6.4)

(see Note 4D for an approximation).

**NOTE 4D**

An approximation for \( (1 - Q_{\text{crest}})^N \)

If the number of crests in any duration of reasonable length (a storm) is large (\( N \gg 1 \)) and the probability of exceedance \( Q_{\text{crest}} \) is small (order \( 1/N \)), then:

\[
(1 - Q_{\text{crest}})^N = \exp(-NQ_{\text{crest}})
\]

All that remains to be determined to compute this probability is the total number of crests \( N \) in duration \( D \). This can be done by noting that for a narrow spectrum the number of crests is equal to the number of upward or downward zero-crossings, which is determined by the mean zero-crossing frequency and the duration:
WAVES IN OCEANIC AND COASTAL WATERS – STATISTICS

\[ N = \bar{f}_0 D = \frac{m^2}{m_0} D \]  

(4.6.5)

The corresponding probability density function \( p_{\eta_{\text{max}}} (\eta) \) of the maximum elevation in duration \( D \) is the derivative of

\[ \Pr\{\eta_{\text{max}} < \eta\} = \Pr\{\eta_{\text{max,crest}} < \eta\} : \]

\[ \Pr\{\eta_{\text{max}} < \eta\} = \frac{d(1 - Q_{\text{crest}})^N}{d\eta} \]

(4.6.6)

It is shown in Fig. 4.6.2. The maximum value of this probability density function is located at the mode of \( \eta_{\text{max}} \) (interpreted as the most probable value). It can be shown that this most probable value of the maximum elevation \( \eta_{\text{max}} \) is:

\[ \text{mod}(\eta_{\text{max}}) = \sqrt{2\ln N} \sqrt{m_0} \]

(4.6.7)

The expected value (i.e., the mean) of the maximum elevation \( \eta_{\text{max}} \) is:

\[ E\{\eta_{\text{max}}\} = \left(1 + \frac{0.29 \ln N}{\ln N}\right) \sqrt{2\ln N} \sqrt{m_0} \]

(4.6.8)

Obviously, with more waves in a storm, the values of \( \text{mod}(\eta_{\text{max}}) \) and \( E\{\eta_{\text{max}}\} \) increase with increasing values of \( N \) but Eqs. (4.6.7) and (4.6.8) show that the increase is only very slow because of the logarithm and the square root in these expressions. In other words, these values are rather insensitive to the value of \( N \). An error in estimating the value of \( N \) has therefore usually no serious consequences for the estimates of \( \text{mod}(\eta_{\text{max}}) \) and \( E\{\eta_{\text{max}}\} \) (see Table 4.1). This illustrates that these results are also rather insensitive to the earlier assumption that wave crests would be statistically independent.

<table>
<thead>
<tr>
<th>( N )</th>
<th>((2 \ln N)^{1/2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000 (real value)</td>
<td>3.90 (real value)</td>
</tr>
<tr>
<td>2500 (error relative to 2000 = 25%)</td>
<td>3.96 (error relative to 3.90 = 1.5%)</td>
</tr>
<tr>
<td>4000 (error relative to 2000 = 100%)</td>
<td>4.07 (error relative to 3.90 = 4.3%)</td>
</tr>
</tbody>
</table>

Table 4.1 The sensitivity of \( \text{mod}(\eta_{\text{max}}) \) and \( E\{\eta_{\text{max}}\} \) for errors in estimating the number of crests \( N \).
It is generally not wise to use the value of \( \eta_{\text{max}} \) itself to base a design on, because the probability that the actually occurring maximum elevation exceeds \( \eta_{\text{max}} \) is considerable. This probability is equal to (from Eq. 4.6.6; see Note 4D):

\[
\Pr\{\eta_{\text{max}} > \eta\} = 1 - \exp(-\eta Q_{\text{crest}}) \quad \text{for } N \gg 1 \text{ and } Q_{\text{crest}} \ll 1
\] (4.6.9)
and since \( N Q_{\text{crest}} = 1 \) for \( \eta_{\text{max}} = \text{mod}(\eta_{\text{max}}) \):

\[
\Pr[\eta_{\text{max}} > \eta] = 1 - \exp(-1) = 0.63
\]

Which implies that there is a 0.63 probability that the actual maximum surface elevation in a storm will exceed its most probable value (but not by much, as the probability density function is rather narrow, see Fig. 4.6.2).

The agreement between observed values of the maximum elevation in a given duration and the above theoretical estimate is generally good, as shown in Fig. 4.6.3 where the theoretical estimate (Eq. 4.6.8) is compared with measurements as a function of the number of waves per record (mean values from many wave records).

![Graph showing measured and theoretically estimated values of the mean maximum elevation.](image)

**Fig. 4.6.3** Measured and theoretically estimated values of the mean maximum elevation \( E\{\eta_{\text{max}}\} \) in records with \( N \) waves (normalised with the standard deviation \( \sqrt{m_0} \); after Cartwright, 1958). These maxima were obtained from very long sequences, which were constructed by concatenating, normalised time series.

**Literature:**
Cartwright and Longuet-Higgins (1956)

### 4.6.2 Extreme wave heights

The cumulative distribution function of the maximum individual wave height in a duration \( D \) can be derived with the same rationale as the above with the following results:

\[
\Pr[H_{\text{max}} \leq H] = (1 - Q_H)^N = \exp(-NQ_H)
\]  

(4.6.11)

The mode of \( H_{\text{max}} \), i.e., the most probable value of \( H_{\text{max}} \), is:

\[
\text{mod}(H_{\text{max}}) = \sqrt[1/2]{\ln N \cdot m_0}
\]

(4.6.12)

The probability density function of \( H_{\text{max}} \) is just as narrow as the probability density function of \( \eta_{\text{max}} \) (see Fig. 4.6.2), so that for a given wave record \( H_{\text{max}} = \text{mod}H_{\text{max}} \) (see Note 4E). However, it is generally not wise to use \( \text{mod}H_{\text{max}} \) as the design wave height in engineering practise because the actually occurring maximum wave height has a probability of 0.63
to exceed $H_{\text{max}}$ (see above for $\text{mod} \left( \eta_{\text{max}} \right)$). Sometimes, an extremely high wave occurs that seems to be inconsistent with the above theory. These waves are called "freak" waves or "rogue" waves (see Note 4E).

It is important to note that the above is based on statistical arguments, assuming that the wave heights are all taken from one distribution. That may not always be the case. For instance, in shelf seas the limited water depth may affect very high waves and not the lower waves, which implies that the high waves are taken from another distribution than the low waves. The estimation of the statistical character of very high waves should therefore not be based exclusively on the observations of low waves if the physical regime changes with wave height.

**Literature:**

### NOTE 4E

**The maximum wave height in a storm and freak (or rogue) waves**

The fact that the probability density function of $H_{\text{max}}$ is narrow is sometimes used to estimate $H_{\text{max}}$ very rapidly from the value of $H_s = H_{m_0}$ (which, in operational situations, is often the wave height predicted by a meteorological centre). If the duration of a storm is 6 hours and the average zero-crossing wave period is about 10 s, then $N = 2160$ and it follows then from Eq. (4.6.12), that:

$$H_{\text{max}} = 1.96 \approx 2H_{m_0}$$

Since this theoretical estimate of $H_{\text{max}}$ is just as insensitive to the value of $N$ as the theoretical estimate of $\text{mod} \left( \eta_{\text{max}} \right)$, this simple relationship between $H_{\text{max}}$ and $H_{m_0}$ is often inversed by engineers to quickly estimate the significant wave height $H_s$ from a wave record:

$$H_s = \frac{1}{2}H_{\text{max}}$$

They only need to look up the maximum wave height in the wave record and *presto*, they have estimated $H_{m_0}$! However, remember that $H_{1/3} = 0.90H_{m_0}$ to 0.95 $H_{m_0}$ (see Section 4.4.3).

The above estimate of the maximum wave height in a storm of two times the significant wave height is only an estimate of the most likely maximum. In actual storms, the value will be somewhat higher or lower but not much because the probability density function of the maximum wave height is rather narrow. The occurrence of a very large wave height (larger than 2.5 times the significant wave height, say) is therefore exceptional. If such an exceptionally high wave occurs, one would expect a certain built up towards such an event: the wave would likely be preceded by one or two or perhaps even three other high waves. However, that is not always the case. Sometimes, an extremely high wave occurs without any such warning. It appears out of nowhere, seemingly without any relation with the prevailing wave conditions. For long time, tales of the sea about such monster waves crashing against ships were regarded as sailor fantasy. However, extensive measurements at sea now available have revealed that at least part of these tales reflect actual facts. An example of a measurement of such a wave, with a crest height of 18.5 m when the significant wave height was “only” 12 m, is given below.
A freak wave observed at the Draupner platform in the central North Sea on January 1, 1995. The crest height of this wave was about 18.5 m whereas the significant wave height was about 12 m (after Haver and Andersen, 2000).

Such waves are called "freak" waves or "rogue" waves. The preceding trough is sometimes referred to as "a hole in the sea". There is no consensus as to what exactly a freak wave is. I favour the description of survivors who have seen one and lived to tell the tale: an exceptionally high, steep breaking wave with an unusually long crest with an almost vertical front preceded by a deep trough. It seems to have a fairly stable form and it suddenly appears out of nowhere. This wave definitely stands out against its background. Breaking of a freak wave seems to be mostly of a spilling nature, but plunging may occur. Here is an eyewitness account of such a wave: a single wave with a crest height of about 7 m in a situation in which the significant wave height was only 4 m. It is from Luigi Cavaleri, who operates an oceanographic observation tower in the Adriatic Sea just south of Venice (Italy; and who helped me write this book, see the Introduction of this book):

"We were on the tower during the night in the middle of a storm with about 4 m significant wave height. I was fixing an instrument at about 7 metres above the sea level, on a long horizontal extension of the platform. Suddenly I heard something like a train coming, and, looking in the dark, I could spot the whitish crest of a wave running against the tower at my height. There was nothing I could do. The crest passed barely below my feet before exploding against the structure. When I turned towards the tower, for a couple of seconds I could see only water. My colleagues, who were watching from an upper deck, were soaked. I was dry."

Focusing of wave energy by a meandering strong current (by current refraction, see Section 7.3.5) has been suggested to explain accidents involving large waves in certain areas. A classical example is the Agulhas current, off the southern tip of South Africa. However, energy focusing may lead to exceptionally high sea conditions in a certain area but not to a single extremely high wave. A more promising and solid approach is based on theories that explain the instability of the sea surface in certain conditions. In a random sea we find sequences of larger and smaller waves. It may happen that in a sequence of steep and fairly regular waves (a local inhomogeneity), a single large wave begins to extract energy from its neighbours, growing at their expense. As a matter of fact, one of the characteristics of a freak wave is that smaller waves and a deep trough precede the freak wave. This makes them even more dangerous as it surprises the crew of the unhappy ship that encounters such an event. A crew is generally not able to respond fast enough to avoid considerable damage, to the point of complete disaster (loss of the ship without any trace). After a while, the wave dies down and returns its energy to its surroundings. Such a development of a single wave can be modelled analytically or numerically in great detail. However, it should be stressed that, the occurrence of a freak wave is governed by statistics. There is no way we can anticipate where and when it will happen. At most we can estimate the probability of its appearance.
4.7 LONG-TERM STATISTICS (WAVE CLIMATE)

4.7.1 Introduction

In the previous sections, the statistical characteristics of the waves were considered for short-term, stationary conditions, usually for the duration of a wave record (typically 15-30 min) or a storm (typically 6 to 12 hours, which obviously stretches the condition of stationarity). For long-term statistics, e.g., statistics over durations of a few dozen years or more, the conditions are not stationary and the problem of describing waves needs to be approached in a different way. For these time scales it is not feasible to present the waves as a time series of the surface elevation itself (with a sample interval of one second or less). Instead, each stationary condition (e.g., a time record of 15-30 min) is replaced by a value of the significant wave height, period and mean wave direction. This gives a long-term sequence of these values with a time interval (typically 3 hours) that is much smaller than the duration of the sequence (see Fig. 4.7.1).

![Graph showing significant wave height over a 10-year period (1980-1989; NODC buoy 46005, position 131° W, 46° N, i.e., in the northern Pacific Ocean, data from the American National Oceanographic Data Center, downloaded from http://www.nodc.noaa.gov/BUOY/46005.html). Note the unusually high value in early 1988 and the gaps in other years.]

Fig. 4.7.1 The significant wave height over a 10-year period (1980-1989; NODC buoy 46005, position 131° W, 46° N, i.e., in the northern Pacific Ocean, data from the American National Oceanographic Data Center, downloaded from http://www.nodc.noaa.gov/BUOY/46005.html). Note the unusually high value in early 1988 and the gaps in other years.

The long-term time sequence thus obtained can be analyzed to estimate statistical characteristics of the waves, for instance to obtain design conditions for maritime structures. Usually the analysis is limited to long-term statistical distributions of the significant wave height but other parameters as such as the duration of storms and calms (persistence statistics) or return period of severe conditions are also considered in engineering practise.
A remarkable difference with the short-term statistics of waves is that there is no basic theoretical model underlying the long-term statistics. However, some results of the Extreme Value theory can be used to analyse and interpret long-term observations if a number of fundamental conditions is complied with. The most important of these are that (a) the values in the time sequence must be statistically independent and (b) they must be identically distributed. This is called *independent and identically distributed* (abbreviated to i.i.d.), i.e., they should be random samples from a population with a given statistical distribution. Unfortunately, these conditions can pose serious problems for real ocean waves because consecutive values in the time series of observations are usually not independent (they are correlated, i.e., a large value of the significant wave height is usually preceded and followed by another high value, at least when the time interval between the observations is less than one day or so).

To achieve statistical independence one should therefore consider only observed values that are sufficiently far separated in time. However, this problem is often ignored, notably in the Initial Distribution approach, to be described next, but it can be solved by selecting values at large intervals (e.g., one value per storm, as in the Peak-Over-Threshold approach or one value per year, as in the Annual Maximum approach, both to be described next). In addition, the observed values are often not identically distributed, because waves can be generated by different types of storms, which suggests a variety of statistical distributions. This problem is not so easily solved. For instance, swell is generated in distant storms and wind sea is generated by local winds (see Section 6.4.2), suggesting different statistical distributions for these two wave conditions. For many oceanic locations therefore the wave climate should be separated in a swell climate and a wind sea climate. Such distinction (which is often ignored) may not be sufficient: each of these climates may have to be split again into two or more climates because swell may originate from different parts of the ocean, each with its own swell-generating weather patterns. For instance, swell off the Californian coast is generated in the northern hemisphere but also in the southern hemisphere with obviously different weather climates. Wind sea may be generated by hurricanes in an area where the daily weather is dominated by trade winds, requiring different climate descriptions for the common, daily conditions and the extreme conditions of the hurricanes. In coastal waters, the situation is even more complicated, since the physical mechanisms that affect the waves may change as the significant wave height and period increase due to the effect of the limited water depth (possibly imposing a maximum on the significant wave height due to depth-induced wave breaking). Such physically imposed upper limit of the wave heights may not be noticeable in the observed wave heights because these are too low but such an unobserved, presumed upper limit would be very relevant when extrapolating the observations to extreme conditions.

*Literature:*

### 4.7.2 Initial Distribution approach

Often, the first step in analysing the long-term time series of the significant wave height, mean wave period and mean wave direction, is to estimate the joint probability density function of \( p(H, \bar{T}_0, \bar{\theta}) \), usually by ordering the observed values and presenting the results in two-dimensional histograms of \( H \) and \( \bar{T}_0 \) per directional sector. The (actual or relative) number of observations is then presented (instead of the probability density) in bins of size \( \Delta H, \Delta \bar{T}_0 \) (per directional sector of size \( \Delta \bar{\theta} = \) typically 30° or 45°). In some atlases, these histograms are given per season or per month. By adding the numbers over the directional sectors one obtains the histogram for the significant wave height and period irrespective of direction, representing the joint distribution \( p(H, \bar{T}_0) \). Such a joint
distribution is illustrated in Fig. 4.7.2, which also shows that the observed wave steepness \( s = \frac{2\pi H_s}{\sqrt{gT_s^2}} \) (see Section 4.4.1 for definitions) is limited to approximately \( s \leq 1:15 \) (this is a universal, physical limitation in deep water, imposed by wave breaking), while on average \( s = 1:30 \) (depending on the mix of swell and wind sea in the area of the observations). By adding the numbers for the period for a given significant wave height or the numbers for the significant wave height for a given mean wave period, one obtains the histogram for either the significant wave height or the mean period separately, representing the probability density functions \( p(H_s) \) and \( p(T_0) \) respectively (see Fig. 4.7.3).

Literature:
Repko et al. (2000)
For many applications, the histograms are adequate because only the statistics of the sorted values would be needed within the range of observed values, e.g., to analyse fatigue effects in a structure. Extreme conditions usually fall outside the range of observed values and to estimate these conditions one needs to extrapolate the observations (typically only for the significant wave height). This is usually done by fitting some curve through the histogram and extrapolating that curve to the desired low probability of occurrence. In the absence of any theory, the curve fitting is entirely empirical: a number of candidate distributions (analytical expressions) with a number of free parameters is chosen and the values of these parameters are estimated by fitting the distributions to the data. The distribution that fits the data best is then used for the extrapolation. To facilitate judging such a fit by eye, it is convenient to use the cumulative distribution function \( P(H_s) = \Pr\{H_s \leq H_s\} \) rather than the probability density function \( p(H_s) \), because, when plotted on paper with the proper scale, the cumulative distribution function will appear as a straight line around which the data should cluster (if the candidate distribution fits the data, for instance, Fig. 4.7.4). Objective goodness-of-fit tests are also available (e.g., the Student-\( t \)-test, the \( \chi^2 \)-test and the Kolmogorov-Smirnov test).

The choice of the candidate distributions to be fitted to the data is rather arbitrary but past experience helps to limit the choice to only a few. A two-parameter distribution is the most convenient for a fit-by-eye, because a straight line on paper has only two free parameters (e.g., an intercept and a slope). However, it is obvious that a distribution with more free parameters would generally provide a better fit because it has more degrees of freedom. It is therefore advisable to also consider distributions with three free parameters. For an objective fit (see below), any number of parameters is permitted (within reason, and certainly considerably less than the number of data points). Two distributions that are widely used for the long-term distribution of the significant wave height are given in Note 4F: the log-normal distribution and the Weibull distribution.

**NOTE 4F**

Long-term distributions for the significant wave height \( H_s \)

**Log-normal distribution:**

\[
\Pr\{H_s \leq H_s\} = \frac{1}{2\pi} \int_{-\infty}^{\ln H_s - A} \exp\left(-\frac{1}{2} x^2\right) dx
\]

**Weibull distribution\( ^* \):**

\[
\Pr\{H_s \leq H_s\} = 1 - \exp\left(-\left(\frac{H_s - A}{B}\right)^C\right) \quad \text{for } H_s > A \text{ and } C > 0
\]

\[
= 0 \quad \text{for } H_s \leq A
\]

The parameter \( A \) is a location parameter (the position of the distribution on the \( H_s \)-axis). In the Weibull distribution this parameter also represents a lower limit of the significant wave height (a permanent minimum background sea). The parameter \( B \) provides a normalisation (scaling), which determines the width of the distributions. The parameter \( C \) in the Weibull distribution is a shape parameter (e.g. for \( C = 1 \), the 2-parameter Weibull distribution reduces to an exponential distribution and for \( C = 2 \) to a Rayleigh distribution).

\( ^* \) This distribution is called the Weibull distribution for minima (although it is not used for the minima here).
To fit the candidate distributions to the observations requires that a probability of non-exceedance be assigned to each observed value. There are two procedures to determine this probability of non-exceedance: (a) when the number of observations is large, one "bins" the observations (i.e., determine the number of observations falling in certain intervals, i.e., the "bins" of a histogram, see Fig. 4.7.3) or (b), when the number of observations is small, one assigns a probability to each observation individually. In the bin option, the probability of $H_{i,j}$ not exceeding the value $H_{i,j}$ (the lower limit of bin number $j$) is:

$$\Pr\{H_{i,j} < H_{i,j}\} = n_i / N$$  \hspace{1cm} (4.7.1)

where $n_i$ is the number of observations lower than $H_{i,j}$ and $N$ is the total number of observations. When the number of observations is small, the intuitive estimate of the probability per individual observation would be

$$\Pr\{H_{i,j} < H_{i,j}\} = 1 - i / N$$ \hspace{1cm} (where $i$ is the ranking number of the observation; ranking the highest observation as $i = 1$; this is called the "plotting position"). Actually, statisticians tell us that, due to effects of sample variability of the observed values (slightly different results when the analysis would be repeated with other samples from the same population), the plotting position depends on the distribution from which the observation is assumed to be taken, e.g., for the Weibull distribution, Goda (1988b, based on Petrukas and Aagaard, 1970) recommends:

$$\Pr\{H_{i,j} < H_{i,j}\} = 1 - \frac{i - \alpha}{N + \beta}$$ \hspace{1cm} \text{with} \hspace{1cm} \alpha = 0.20 + 0.27 / \sqrt{C} \hspace{1cm} \text{and} \hspace{1cm} \beta = 0.20 + 0.23 / \sqrt{C}$$  \hspace{1cm} (4.7.2)

where $C$ is the shape parameter of the distribution (see Note 4F).

Having thus established the probability values of the observations, one can estimate the parameters of the various candidate distributions by fitting these distributions, subjectively by eye or objectively with a formal procedure, e.g., with a least-squares technique (that minimise the sum of the squared differences between the observations and the candidate distribution), or a maximum-likelihood technique (that maximise the probability that the observations are taken from the candidate distribution), or a moments technique (that chose the parameters of the candidate distribution such that the lower-order moments of the observed distribution and of the candidate distribution are equal). The most primitive procedure seems to be the fit-by-eye: plot the values on paper with proper scales along the axes and draw a straight line through the data. If the data belong to the distribution that corresponds to the scales, they should arrange themselves along a straight line. For the distributions considered in the long-term statistics of this Chapter 4, such scales are readily constructed (except for the log-normal distribution) as log or double-log scales, depending on the distribution considered, e.g., plotting

$$y = \ln\left(\frac{1 - \Pr\{H_{i,j} > H_{i,j}\}}{\Pr\{H_{i,j} < H_{i,j}\}}\right)$$ \hspace{1cm} \text{against} \hspace{1cm} x = C \ln \left(\frac{(H_{i,j} - A)}{B}\right)$$, gives a straight line$^{13}$ for the Weibull distribution (see Fig. 4.7.4; this particular data set is also well approximated by the log-normal distribution when the seasonal variation in

---

$^{13}$ There are several methods to represent the data as straight lines. Statisticians and mathematicians tend to plot the observed values of $H_i$ against the predicted value of $H_i$ (at a given value of non-exceedence of the fitted candidate distribution; for a perfect fit, this would give a straight line). These plots are called quantile plots (see Note 4G). The traditional technique in engineering is to plot, in one plot, both the observed distribution and the candidate distribution, i.e., $\Pr\{H_{i,j} < H_{i,j}\}$ against $H_i$ using log- or double-log scales (see Fig. 4.7.4). The candidate distribution always appears as a straight line (due to the scaling along the axes) whereas the observed distribution will appear as a straight line only in case of a perfect fit. Such a plot can be used for estimating by eye the parameters of a two-parameter distribution (from the slope and intercept of a straight line that is fitted through the observed distribution).
significant wave height is removed, except for the high values of $H_s$ where the fit remains poor, see Note 4G). The advantage of a fit by eye is that the engineer is able to favour the higher values in the data set at his professional discretion. The alternative, i.e., fitting the distribution with an objective procedure, is not as objective as it may sound because without a theoretical basis, the choice of the technique is still subjective. In any case, an objective fit should always be inspected by eye to verify that the fit is reasonable, in particular for the high observed values in which engineers are usually most interested (results may unintentionally be seriously biased to the low values of the data of which there are many of). Assigning more importance to higher values in the observations can also be achieved in objective fit procedures by properly weighing these higher values or by ignoring the lower values (which is called "censoring"; see Fig. 4.7.4 where the low values $H_s < 1$ m were removed).

$$\Pr\{H_s \leq H_s\} = \ln\left[-\ln\left(1 - \Pr\{H_s \leq H_s\}\right)\right]$$

**Fig. 4.7.4**  The long-term Weibull distribution (for minima) of the significant wave height, for the years 1980-2003 of NODC buoy 46005 of Fig. 4.7.1 (all values below 1.00 m removed). The straight line represents the best-fit analytical distribution (maximum likelihood). The position of the exceptional storm in early 1988 is not obvious (see Figs. 4.77 and 4.79).
WAVES IN OCEANIC AND COASTAL WATERS – STATISTICS

NOTE 4G  
Seasonal variation removed from the long-term distribution of the significant wave height  
(Initial Distribution approach)

When the seasonal variation of the significant wave height is removed from the time series of \( H_s \) of Fig. 4.7.4 (by scaling the values of \( H_s \) with a harmonic component with period of 365 days), then the fit between the observations and the log-normal distribution improves markedly for most of the observations, but the fit is still poor for the highest values:

The long-term, log-normal distribution of the significant wave height (quantile plots), for the years 1980-2003 of NODC buoy 46005 of Fig. 4.7.1). Left-hand panel: no seasonal scaling; right-hand panel: seasonally scaled values. The straight lines represent the best-fit analytical distributions (maximum likelihood).

The extrapolation of the long-term distribution provides the probability that an (unobserved) high value of the significant wave height is exceeded. It does not indicate when such an event will actually happen. That of course is unpredictable (in the long-term) but with some extra information it is possible to determine how often it will happen. In many engineering design procedures this is expressed in terms of a return period, i.e., the average time interval between occurrences of the extreme significant wave height, or, better stated: the average time interval between successive up-crossings of the significant wave height through a chosen level (see Fig. 4.7.5). This return period can be estimated from the long-term cumulative distribution function \( P(H_s) = \Pr\{H_s \leq H_s^+\} \) if also the average duration of exceedance per event is known (an event = a series of consecutive values of \( H_s \) that are all above the chosen level, preceded and followed by lower values; the duration of an event is also known as "persistence"). To obtain this estimate of the return period, consider a long period of \( N \) years in which the significant wave height crosses the chosen level \( n \) times (in upward direction only; see Fig. 4.7.5).
The average time interval between these up-crossings is, by definition, the return period \( RP_{H_d>H_s} \) (approximately \( RP_{H_d>H_s} = N/n \) years, in this example). For estimating the number of these up-crossings \( n \), we need to interpret the probability of exceedance \( \Pr\{H_s > H_d\} \) as the fraction of time during which \( H_s > H_d \). For instance, if the probability of exceedance of the level \( 10 \) m is \( 0.00183 \), then the total duration \( D \) during which the significant wave height exceeds this level of \( 10 \) m, is \( D = 16 \) hour per year on average (i.e., averaged over many years). If the average duration per event \( (\bar{d}_{H_d>H_s}) \) is \( 8 \) hours, then obviously, with this total duration of \( 16 \) hours per year, the up-crossings through the level of \( 10 \) m occur twice per year (on average). This frequency of occurrence (number of occurrences per year) is apparently determined from \( 0.00183 \times 24 \times 365/8 \) (year\(^{-1}\)), when \( \bar{d}_{H_d>H_s} \) is expressed in hours. The return period is the inverse of this frequency, \( RP_{H_d>H_s} = 8 / (0.00183 \times 24 \times 365) = 4371 \) hours = \( ½ \) year. Expressed analytically:

\[
RP_{H_d>H_s} = \frac{\bar{d}_{H_d>H_s}}{\Pr\{H_s > H_d\} \times 24 \times 365} \text{ (year) when } \bar{d}_{H_d>H_s} \text{ in hours, Initial Distribution approach (4.7.3)}
\]

or

\[
RP_{H_d>H_s} = \frac{\bar{d}_{H_d>H_s}}{\Pr\{H_s > H_d\}} = \frac{\bar{d}_{H_d>H_s}}{1 - P(H_s)} \text{ (unit of } \bar{d}_{H_d>H_s}, \text{ Initial Distribution approach (4.7.4)}
\]

where \( P(H_s) \) is the cumulative distribution function: \( P(H_s) = \Pr\{H_s < H_d\} = 1 - \Pr\{H_s > H_d\} \). The estimation of the return period in this manner requires information about the average duration of events during which \( H_s > H_d \), which can only be obtained from observed or simulated time sequences of \( H_s \). Strangely enough, the return period is sometimes estimated as \( RP_{H_d>H_s} = \Delta t_{H_s} / \{1 - P(H_s)\} \), where \( \Delta t_{H_s} \) is the time interval between the observations of \( H_s \), typically 3
hours, or even $RP_{H > H_i} = 1/\left\{1 - P(H_i)\right\}$, which implies that $\tilde{d}_{H > H_i}$ would be unity (the dimension and unit of the return period $RP_{H > H_i}$ being mysterious because it has the same dimension and unit as $\tilde{d}_{H > H_i}$ which has been replaced by unity). Needless to say that the return period thus estimated is seriously wrong or even nonsensical.

The return period of calms, i.e., the periods that $H_i < H_i$, which is important for activities such as towing an offshore platform to its location of operation, is estimated similarly:

$$RP_{H_i < H_i} = \frac{\tilde{d}_{H_i < H_i}}{Pr[H_i < H_i]} = \frac{\tilde{d}_{H_i < H_i}}{P(H_i)} \quad \text{ (unit of } \tilde{d}_{H > H_i} \text{ ) calms in Initial Distribution approach} \quad (4.7.5)$$

**Literature:**

### 4.7.3 Peak-Over-Threshold approach

The statistics of extreme values of the significant wave height can also be estimated with another approach than the above. In the peak-over-threshold approach considered here, only the maximum value of $H_s$ in each of a large number of storms is considered (see Fig. 4.7.6). A storm is defined here as an uninterrupted sequence of values of $H_s$ all exceeding a certain, fairly high value (threshold value $H_s,\text{threshold}$), preceded and followed by a lower value\(^{14}\). The value to be chosen for this threshold depends very much on the local conditions. For severe climates, a threshold value of 5 m may be needed, whereas for calm climates, a value of 1 m may be better suited. The criterion is, that a sufficient number of storms (preferably several dozen) can be identified in the long-term time record. For each such storm the maximum significant wave height is then identified as the highest (= peak) value in that storm: $H_s,\text{peak}$ (the peak-over-threshold).

\[ H_s (m) \]

**Fig. 4.7.6** Storm with successive crossings of the significant wave height through a threshold level.

\(^{14}\) Sometimes small gaps between such storms are ignored, to avoid breaking-up a phenomenon that obviously is one storm (seen from a meteorological point of view).
The Extreme Value Theory (e.g., Castillo, 1988; Coles, 2001) tells us that the distribution of the maximum in such a sequence of values is the Generalized Pareto distribution (GP distribution see Note 4G)\(^{15}\). In other words, the maximum significant wave height in a storm is Pareto-distributed (under certain conditions, e.g., the values must be independent and identically distributed and the threshold value must be relatively high; see Fig. 4.7.7).

This approach has two important advantages over the Initial Distribution approach: (a) if the wave climate contains more than one distribution (due to different physical regimes, see Section 4.7.1), the selection of only the high values of the significant wave height tends to concentrate the analysis on the regime that dominates the extremes and (b) the storms are statistically independent events, which simplifies the interpretation of the results of the analysis (e.g., estimating the sampling errors involved).

\[ \Pr\{H_{\text{peak, peak}} \leq H_{\text{peak}}\} = -\ln(1 - \Pr\{H_{\text{peak, peak}} \leq H_{\text{peak}}\} ) \]

Fig. 4.7.7  The long-term Generalized Pareto distribution of the peak-over-threshold significant wave height for threshold value \( A = H_{\text{threshold}} = 5 \text{ m} \) (resulting in 24 storms per year on average), for the years 1980-2003 of NODC buoy 46005 of Fig. 4.7.1. The straight line represents the best-fit analytical distribution (maximum likelihood; \( C = 0 \) would correspond to an exponential distribution; the indicated value of \( C = -0.174 \) is so far from zero (the estimated standard deviation of \( C = 0.036 \)), that the observed values are probably not taken from an exponential distribution). The position of the exceptional storm in early 1988 is obvious.

\(^{15}\) The convergence theorem is that the distribution of the maximum of a large set of independent and identically distributed (i.i.d.) random variables larger than some threshold value, is a Generalized Pareto Distribution: \( \max(x_1, \ldots, x_n \mid x_i > x_{\text{threshold}}) \rightarrow GPD \) for \( n \rightarrow \infty \).
Once the parameters of the distribution of $H_{s,\text{peak}}$ have been determined by fitting the distribution to the data, an estimate of the return period $RP_{H_{s,\text{peak}}} > H_{s,\text{peak}}$ of $H_{s,\text{peak}}$ can be made. This return period is defined in analogy with the return period in the Initial Distribution approach: it is the average time interval between the storms in which $H_{s,\text{peak}} \geq H_{s,\text{peak}}$. To introduce the estimation of this return period, consider all storms in the long-term time series for which $H_{s,\text{threshold}} = 5 \text{ m}$ and suppose that the statistical analysis gives a $0.005$ probability of exceedance of the $9 \text{ m}$ level, by the peak value in the storms: $Pr\{H_{s,\text{peak}} > 9 \text{m}\} = 0.005$. It then follows that $1$ out of every $200$ storms has such a peak value (a severe storm). Suppose that the average interval between all storms (defined by $H_s > 5 \text{ m}$), is $\Delta T_{\text{storm}} = 18 \text{ weeks}$ (to be estimated from the long-term time record), then the average interval between severe storms with $H_{s,\text{peak}} > 9 \text{ m}$ is $200 \times 18 \text{ weeks} \approx 70 \text{ years}$.

Expressed analytically:

\[
RP_{H_{s,\text{peak}}} > H_{s,\text{peak}} = \frac{\Delta T_{\text{storm}}}{1 - P\{H_{s,\text{peak}}\}} \quad \text{(units of $\Delta T_{\text{storm}}$)} \quad \text{Peak-Over-Threshold approach} \quad (4.7.6)
\]

One advantage of this Peak-Over-Threshold approach is that it has some theoretical basis (the Extreme Value theory) and that only storms in which the significant wave height exceeds some value need to be considered (thus reducing any hindcast effort to only these storms). This approach seems therefore to be an attractive compromise between, on the one hand, the Initial Distribution approach, in which a very large number of values of the significant wave height is used but which has no theoretical basis, and on the other, the Annual Maximum approach (to be treated next) which also has theoretical support of the Extreme Value theory but for which usually only a small number of observations is available (equal to the number of years in the long-term time record).

**NOTE 4G**

Long-term distribution for the maximum significant wave height
(Peak-Over-Threshold approach)

Generalised Pareto distribution:

\[
Pr\{H_{s,\text{peak}} \leq H_{s,\text{peak}}\} = 1 - \left\{1 + C \left(\frac{H_{s,\text{peak}} - A}{B}\right)^{-1/C}\right\}^{-1/C} \quad \text{for} \quad H_{s,\text{peak}} > A
\]

The parameter $A$ is a threshold value $A = H_{s,\text{threshold}}$. The parameter $B$ provides a normalisation (scaling) and the parameter $C$ is a shape parameter. For $C \rightarrow 0$, the distribution reduces to the exponential distribution:

\[
Pr\{H_{s,\text{peak}} \leq H_{s,\text{peak}}\} = 1 - \exp\left(-\frac{H_{s,\text{peak}} - A}{B}\right) \quad \text{for} \quad H_{s,\text{peak}} > A
\]

**Literature:**

4.7.4 Annual Maximum approach

Occasionally, another approach, the annual maximum approach, is used. Consider a population of random values from which a large set of samples is arbitrarily drawn (creating the parent distribution). The Extreme Value theory tells us that under fairly general conditions, the distribution of the maximum value of that set is the Generalized Extreme Value (GEV) distribution\(^\text{16}\) and that, if the parent distribution is a log-normal distribution or a Weibull distribution, this GEV distribution reduces to a Gumbel distribution (e.g., Castillo, 1988; Coles, 2001; see Note 4H). To use these theoretical findings in a wave climate analysis, consider one year of values of the significant wave height and call the maximum value in this set \(H_{\text{s,AM}}\). A time series of \(N\) years thus gives \(N\) values of \(H_{\text{s,AM}}\) (see Fig. 4.7.8). Since the initial distribution of the significant wave height (the parent distribution) is often close to a Weibull distribution or a log-normal distribution (see Section 4.7.2), it follows that \(H_{\text{s,AM}}\) should be (nearly) Gumbel distributed (see Fig. 4.7.9).

\[ \text{Fig. 4.7.8 The annual maxima in a long-term time record of the significant wave height (observations as in Fig. 4.7.1, supplemented with an artist impression of hindcast results).} \]

The parameters of this GEV distribution can be estimated from the observed values of \(H_{\text{s,AM}}\) with any of the methods mentioned earlier in Section 4.7.2. To introduce the estimation of the return period in this approach, consider a situation in which the probability of \(H_{\text{s,AM}}\) exceeding the level of 7.5 m is 0.02. This exceedance then occurs (on average) twice every hundred samples, or once in every fifty years. The return period of \(H_{\text{s,max \ year}}\) can apparently be estimated with:

\[ \text{return period } (H_{\text{s,AM}} > H_{\text{s,AM}}) = RP_{H_{\text{s,AM}} > H_{\text{s,AM}}} = \frac{1 \text{ year}}{\Pr \left( H_{\text{s,AM}} > H_{\text{s,AM}} \right)} \]  

or:

\[ (4.7.7) \]

\(^{16}\) The convergence theorem is that the distribution of the maximum of a large set of independent and identically distributed (i.i.d.) random variables, is a Generalized Extreme Value Distribution: \(\max(x_1, \ldots, x_n) \rightarrow \text{GEV for } n \rightarrow \infty\).
\[ \frac{1}{1 - P(H_{s,AM})} \] (year)

\[ P_H(H_{s,AM}) \]

\[ \Pr[H_{s,AM} < H_{s,AM}] \]

\[ \Pr[H_{s,AM} < H_{s,AM}] \]

\[ A = 9.74 \text{ m} \]
\[ B = 1.28 \]
\[ C = -0.17 \]

\[ \ln \ln \Pr[H_{s,AM} < H_{s,AM}] \]

\[ \ln \ln \Pr[H_{s,AM} < H_{s,AM}] \]

\[ H_{s,AM} \]

\[ \frac{1}{C} \ln \left[ 1 + C(H_{s,AM} - A)/B \right] \]

\[ \frac{1}{C} \ln \left[ 1 + C(H_{s,AM} - A)/B \right] \]

\[ 100 \text{ year} \]
\[ 75 \text{ year} \]
\[ 50 \text{ year} \]
\[ 20 \text{ year} \]
\[ 10 \text{ year} \]
\[ 5 \text{ year} \]

\[ \text{storm of early 1988} \]

\[ \text{Generalised Extreme Value distribution} \]

\[ \text{all yearly maximum significant wave heights} \]

\[ \ln \ln \Pr[H_{s,AM} < H_{s,AM}] \]

\[ \ln \ln \Pr[H_{s,AM} < H_{s,AM}] \]

\[ H_{s,AM} \]

\[ 100 \text{ year} \]
\[ 75 \text{ year} \]
\[ 50 \text{ year} \]
\[ 20 \text{ year} \]
\[ 10 \text{ year} \]
\[ 5 \text{ year} \]

Fig. 4.7.9 The long-term Generalised Extreme Value distribution of the annual-maximum significant wave height, for the years 1980-2003 of NODC buoy 46005 of Fig. 4.7.1. The straight line represents the best-fit analytical distribution (maximum likelihood; \( C = 0 \) would correspond to a Gumbel distribution; the indicated value of \( C = -0.17 \), is close enough to zero, with an estimated standard deviation of \( C \) of 0.123, that at least in this case, the annual-maximum significant wave height is probably Gumbel distributed). The position of the exceptional storm in early 1988 is obvious: the fitted distribution in the range \( H_{s,AM} \geq 12 \text{ m} \), suggests that the value of 13.6 m in this exceptional storm would occur only once in 70 years (on average), as shown by the dashed line.)
4.7.5 Wave atlases

The data underlying the long-term statistics are often based on wave observations carried out in the context of routine observation programs of government agencies or the offshore industry, using buoys or wave poles (the most common instruments), recently supplemented with ocean-looking satellites. Some of these observations are used only for operational purposes and are never stored or only for a short period of time. Other observation programs are specifically aimed at acquiring and storing data to provide a basis for estimating long-term wave statistics. An alternative to these long-term observation programs are computer simulations. Such simulations (called "hindcasts"; see Chapters 6 and 8) are based on archived wind fields that are available for many parts of the world at national or international meteorological institutes. These hindcasts can often provide wave information over much longer periods than observations because meteorological (wind) archives are generally much older than wave archives. Moreover, such simulation studies can be carried out within a few months or one or two years whereas an equivalent observation program would require dozens of years.

One of the results of the statistical analysis, the histograms that represent long-term probability distribution functions (see Fig. 4.7.2), are sometimes published as wave atlases. Some cover all the worlds’ oceans while others cover only selected regions. Examples of such atlases are:

Based on (visual) observations:
- Ocean Wave Statistics (Hogben and Lumb, 1967)
- Global Wave Statistics (Hogben et al., 1985)
- Wind and wave climate in the Netherlands sector of the North Sea between 53° and 54° North latitude (Bouws, 1978)

Based on satellite observations:

\[ \text{Pr}\{H_{s,AM} \leq H_{s,AM}\} = \exp\left\{- \left[ 1 + C \left( \frac{H_{s,AM} - A}{B} \right) \right]^{-1/C} \right\} \quad \text{for} \quad 1 + C \frac{H_{s,AM} - A}{B} > 0 \quad \text{and} \quad B > 0 \]

For \( C < 0 \), this distribution is also known as the Weibull distribution (for maxima), which is usually written as:

\[ \text{Pr}\{H_s \leq H_s\} = \exp\left\{- \left( \frac{H_s - A}{B'} \right)^C \right\} \quad \text{with} \quad B' = B/C \quad \text{and} \quad C' = -1/C . \]

For \( C > 0 \) this distribution is also known as the Fréchet distribution or Fisher-Tippett III distribution. For \( C \to 0 \), the distribution reduces to the Gumbel distribution or Fisher-Tippett I distribution:

\[ \text{Pr}\{H_{s,AM} \leq H_{s,AM}\} = \exp\left\{- \exp\left\{- \frac{H_{s,AM} - A}{B} \right\} \right\} \quad \text{for} \quad B > 0 \]

The parameter \( A \) is a location parameter (the position of the distribution on the \( H_s \)-axis. The parameter \( B \) provides a normalisation (scaling) and the parameter \( C \) is a shape parameter.
Based on computer simulations:
- Waves of the Adjacent Seas of Japan (Japan Oceanographic Data Center?),
- European Wave Energy Atlas (Weratlas, internet),
- Statistica delle Onde Estreme Mare Tirreno, Tosi et al. (1984)

Based on mixed sources:
- Wind and Wave Climate Atlas of Canada (MacLaren Plansearch Limited, 1991)
- World Wave Atlas (Barstow, 1996)

Occasionally the actual time series are published on the Internet (see Fig. 7.4.1 for an example).
APPENDIX A  RANDOM VARIABLES

1 ONE STOCHASTIC VARIABLE

1.a Characterisation

The surface elevation in the presence of waves, at one location and at any one moment \( t \), in time, \( \eta(t) \) will be treated as a stochastic variable (a variable whose exact value cannot be predicted). For instance, consider a wave flume in which a wind machine generates waves (Fig. A.1). Somewhere in the flume, at location A, a wave gauge measures the surface elevation as a function of time and at some point in time \( t \), the surface elevation would have a value \( \eta(t) \).

If the experiment would be repeated, this value (same time and location) would be \( \eta(t) \) (note that the superscript is used to indicate the experiment number). If the experiment were repeated again, the surface elevation would be \( \eta(t) \), and so on and so forth. This value of the surface elevation at time \( t \) can obviously not be predicted. It is therefore a stochastic variable, denoted as \( \eta(t) \) (underscored, to show that it is a stochastic variable). Of course, the surface elevations at other times \( \eta(t) \), \( \eta(t) \) etc. are equally unpredictable and therefore also stochastic variables.
In this Appendix A, an unspecified stochastic variable is denoted as \( x \). It is fully characterized by its probability density function \( p(x) \), which is defined such that the probability of \( x \) attaining a value between \( x \) and \( x + dx \) is given by (see Fig. A.2):

\[
\Pr\{x < x \leq x + dx\} = \int_x^{x+dx} p(x)dx = p(x)dx
\]

(A.1)

\[\int_x^{x+dx} p(x)dx \approx p(x)dx\]

\( p(x) \) is called the cumulative distribution function \( P(x) \) of a stochastic variable \( x \).

It follows that the probability of \( x \) being less than or equal to \( x \) is (see Fig. A.3):

\[
\Pr\{x \leq x\} = \int_{-\infty}^{x} p(x)dx = P(x)
\]

(A.2)

\( P(x) \) is called the cumulative distribution function of \( x \) (sometimes the word "cumulative" is ignored). Obviously, the probability that the value of any stochastic variable is less than \( \infty \) is 1. This implies that the maximum value of the cumulative distribution function \( P(x) \) is 1 and that the surface area of a probability density function \( p(x) \) is always 1.0:

\[
\Pr\{x \leq \infty\} = \int_{-\infty}^{\infty} p(x)dx = P(\infty) = 1
\]

(A.3)
Note that statisticians express probabilities in terms of fractions rather than in terms of percentages, as you would do when talking to a friend. The average or mean value of $x$ can be determined from the probability density function $p(x)$. Such an average is called the "expected value" or "expectation", denoted as $E[.]$ and it is determined from the first-order moment \(^{17}\), divided by the zero-th order moment of $p(x)$:

$$E[x] = E[x] = \int_{-\infty}^{\infty} x p(x) dx \bigg/ \int_{-\infty}^{\infty} p(x) dx$$

(A.4)

Since the zero\(^{th}\)-order moment of a probability density function is always $\int_{-\infty}^{\infty} p(x) dx = 1$, it follows that:

$$E[x] = \int_{-\infty}^{\infty} x p(x) dx$$

(A.5)

Averages of functions of $x$ can also be defined as expected values. For instance, the variance $\sigma_x^2$ of $x$ is the average of the function $f(x) = (x - \mu_x)^2$, where $\mu_x$ is the expected value $E[x]$, so that:

$$\sigma_x^2 = E[(x - \mu_x)^2] = \int_{-\infty}^{\infty} (x - \mu_x)^2 p(x) dx$$

(A.6)

$\sigma_x$ is called the standard deviation of $x$ (it is a measure of the width of the probability density function).

More in general, the expected value of a function $f(x)$ is defined as:

---

\(^{17}\) The n-th order moment of a function $h(x)$ is by definition $m_n = \int_{-\infty}^{\infty} x^n h(x) dx$.
\[ E\{ f(x) \} = \int_{-\infty}^{\infty} f(x)p(x)dx \]  
(A.7)

**1b Gaussian probability density function**

Many processes in nature behave in such a way that the well-known *Gaussian* probability density function applies:

\[ p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left( -\frac{(x-\mu)^2}{2\sigma^2} \right) \]  
(A.8)

A theoretical explanation of this almost universal applicability is provided by the *Central Limit Theorem*, which simply formulated, says: the sum of a large number of independent stochastic variables (without one being dominant) is Gaussian distributed. Since many natural phenomena are the result of a large number of causes, it is reasonable to find the Gaussian probability density function to apply so often. The Gaussian probability density function is also called the *normal* probability density function. However, the Gaussian distribution is by no means the only one that applies to natural phenomena. Many others may also apply, e.g., the Rayleigh probability density function, the exponential probability density function or the Weibull probability density function, to name only a few. Note that the definitions of the mean \( \mu \) and the standard deviation \( \sigma \) are defined independently of the specific probability density function and therefore independent of the applicability of the Gaussian probability density function.

**1c Estimation**

An average is often estimated (e.g., from observations in an experiment), not from the probability density function \( p(x) \) but from a set of sample values of \( x \) (also called *realizations*). Such a set of sample values is called an *ensemble*, and the average is called an ensemble average, denoted as \( <> \). For instance:

\[ \text{mean} = \mu = <x> = \frac{1}{N} \sum_{i=1}^{N} x_i \]  
(A.9)

\[ \text{variance} = \sigma^2 = <(x-<x>)^2> = \frac{1}{N} \sum_{i=1}^{N} (x_i - <x>)^2 \]  
(A.10)

where \( N \) is the number of samples.

**2 TWO STOCHASTIC VARIABLES**

**2.a Characterisation**
A set of two stochastic variables \((x, y)\) is fully characterized by the joint probability density function of \(x\) and \(y\): \(p(x, y)\). This two-dimensional probability density function is defined, in analogy with the above, such that the probability of \(x\) attaining a value between \(x\) and \(x + dx\) and of \(y\) (simultaneously) attaining a value between \(y\) and \(y + dy\) is given by:

\[
\Pr\{x < x \leq x + dx \text{ and } y < y \leq y + dy\} = \int_x^{x+dx} \int_y^{y+dy} p(x, y) \, dx \, dy = p(x, y) \, dx \, dy
\]

(A.11)

The two stochastic variables may be related to one another. One variable is then said be dependent on the other. When they are linearly related (are grouped around a straight line, when one is plotted against the other), they are said to be correlated. Note that two stochastic variables can be related but uncorrelated (see Fig. A.4).

The degree of correlation (the degree to which the pairs of \((x, y)\) cluster around a straight line) is quantified with the correlation coefficient \(\gamma_{x,y}\), which is defined as the normalised co-variance \(C_{x,y}\) of the two variables:

\[
\gamma_{x,y} = C_{x,y} / (\sigma_x \sigma_y)
\]

(A.12)

where the co-variance is:

\[
C_{x,y} = E\{ (x - \mu_x)(y - \mu_y) \}
\]

(A.13)

2.b Gaussian probability density function
The two-dimensional *Gaussian* probability density function for the two stochastic variables \( x \) and \( y \) is:

\[
p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y (1 - \gamma_{x,y}^2)^{1/2}} \exp\left(-\frac{1}{1 - \gamma_{x,y}^2} \left(\frac{(x - \mu_x)^2}{2\sigma_x^2} - \gamma_{x,y} \frac{(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y} + \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)\right)
\]

(A.14)

### 3. AN ORDERED SET OF STOCHASTIC VARIABLES (= STOCHASTIC PROCESS)

#### 3.a Characterisation

Stochastic variables may not only be related or correlated. They may also be *ordered* in some sense, i.e., the variables exist in some kind of sequence, particularly when (many) more than two variables are considered. For instance, bottles produced by a machine appear one by one from that machine. Their length (which is a stochastic variable) is therefore ordered in the sequence of appearance from the machine. In this example, the ordering is one-dimensional but that need not always be the case. For instance, the height of students (a stochastic variable!) in a lecture room is ordered by the (two-dimensional, horizontal) position of the students in that room. The weight of leaves on a tree is ordered in three-dimensional space. Such an *ordered set of stochastic variables* is called a *stochastic process*. The (very large) set of surface elevations \( \eta \), observed as a function of time in the wind flume of Section 1.a, is an example of an ordering of very many stochastic variables in *time*. (Note that in a time sequence \( \eta(t_1) \), the stochastic variable \( \eta \) at time \( t_1 \) is another stochastic variable than \( \eta \) at time \( t_2 \) which is another stochastic variable than \( \eta \) at time \( t_3 \), etc.). Obviously, when the surface elevation at one moment in time is large, then a fraction of a second later, the surface elevation will also be large: the surface elevations at short time intervals are related and even correlated. Only after some lapse of time will the relation between the surface elevations be lost.

This stochastic *process* can be visualized with a mental experiment. Consider again the wind-generated waves in the flume of Section 1.a. The measurement starts at \( t = 0 \) when the wind starts to blow over still water. Initially the waves are small but as the wind continues to blow, the waves will grow larger and longer (upper panel in Fig. A.5 = realization 1). Such an experiment is one *realization* of the stochastic *process* \( \eta(t_1), \eta(t_2), \eta(t_3), \ldots, \eta(t_r), \ldots \), where \( \eta(t_i) \) is the surface elevation at point A at moment \( t_i \) in the experiment. The experiment can be repeated at will: the wind machine is turned off and after the water surface has returned to its still flat level, the wind machine is turned on again (at \( t = 0 \)), which starts the next experiment. Obviously, there are as many realizations of \( \eta(t_i) \) as there are experiments.
time records

observations at location A

= samples (realizations) of a stochastic process

Fig. A.5  A set of four realizations of the surface elevation as a function of time at one location in the laboratory flume of Fig. A.1 (essentially identical experiments, i.e., same wind speed etc.). The waves grow as time increases until some degree of stationarity is attained.

Like any stochastic variable, \( \eta(t) \) is fully characterized by its probability density function. This implies that one probability density function is required for each moment in time \( t_i \) to statistically characterise the surface elevation at that time. To characterise the surface elevations as a process, we need additionally all
joint probability density functions, i.e., at each moment in time \( t_i \) the joint probability density functions \( p(\eta(t_i), \eta(t_j)) \) for all \( t_j, t_i \), which in this case is an infinite number of such functions.

### 3b Stationary processes

If the joint probability density functions are constant in time all statistical characteristics of the waves are independent of time and the process is said to be stationary (although the statistics still depend on time intervals \( t_j - t_i \)). The stationarity of a process greatly simplifies the description, as only the statistical characteristics at one moment in time are required (including all relationships with the stochastic variables at other time intervals!). The analogous condition for variables that are ordered in space is called homogeneous. If only the averages and the variances of the variables (the main statistical characteristics) are constant in time or space, the process is called weakly stationary or weakly homogeneous.

### 3c Gaussian processes

If all joint probability density functions of a process (stationary or not) are Gaussian, the process is called a Gaussian process. A Gaussian process is relatively simple to describe, as only the averages of each pair of variables and their co-variance are required. In general, the covariance \( C_{x,y} \) is a function of the two moments in time at which the stochastic variables \( x = x(t_i) \) and \( y = x(t_j) \) are considered:

\[ C_{x,y} = C_{x,y}(t_i, t_j). \]

Writing \( t_i = t \) and \( t_j = t + \tau \), this may also be written as

\[ C_{x,y}(t_i, t_j) = E\left\{ (x(t) - \mu_x(t)) (x(t + \tau) - \mu_x(t + \tau)) \right\} = C(t, \tau). \]

The co-variance may therefore be said to be a function of time \( t \) and time interval \( \tau \).

### 3d Stationary, Gaussian process

A process with the combination of the above two characteristics, is a stationary Gaussian process. This type of process is even simpler to describe: only the mean and the co-variances at one moment in time are required (because they are identical at all other times). The co-variance is then (only) a function of the time interval \( \tau \) and if the average of the variable is taken to be zero (as usual for the surface elevation of waves), it can be written as:

\[ C(\tau) = E\left\{ x(t) x(t + \tau) \right\} \]

(A.15)

### 3e Ergodic process

If averaging over time (or space) gives the same results as averaging over the ensemble, then the process is called ergodic, for instance, the mean and variance of a zero-mean ergodic process are:

\[ \mu_i = \langle x(t_i) \rangle = \frac{1}{D} \int_D x(t) dt = 0 \]

(A.16)
the variance, \[ \sigma^2_x = \langle [x(t)]^2 \rangle = \frac{1}{D} \int [x(t)]^2 \, dt \]  
(A.17)

and the auto-covariance is:

\[ C(\tau) = \langle x(t)x(t+\tau) \rangle = \frac{1}{D} \int [x(t)x(t+\tau)] \, dt \]  
(A.18)

where the brackets \(<.>\) denote ensemble averaging and \(D\) is the length of the time interval (duration) over which the time average is taken. More in general, for the average of a function \(f[x(t)]\) of \(x(t)\):

\[ < f[x(t)] > = \frac{1}{D} \int f[x(t)] \, dt \]  
(A.19)

As the surface elevation in the presence of waves in stationary conditions happens to be ergodic\(^{18}\), all averages that are needed to describe waves can be estimated as time averages, even if formally speaking, ensemble averages are required. This is fortunate as we would not be able to obtain an ensemble in real nature (it would involve repeating over and over again, the same conditions at sea that generated the waves).

4. THE SEA SURFACE ELEVATION

The sea surface elevation in the presence of waves as a function of time is often treated as a Gaussian process. This is supported by measurements, which have shown this to be very nearly true in many cases (see Section 4.3). But there are also theoretical grounds for this: the surface elevation at any one moment in time \(t\) is seen as the sum of the elevations at that time of a large number of harmonic wave components which are generated and which have travelled across the sea surface independently from each other. The Central Limit Theorem (see Section 1.b of this Appendix A) shows that therefore the sea surface elevation should be Gaussian distributed. But not always. Wave components that create steep waves (or high waves in shallow water), do interact and are therefore not independent. Deviations from the Gaussian model therefore occur, particularly in the surf zone where the interactions between the wave components are strong.

\(^{18}\) Not all stationary processes are ergodic, for instance, switching on an electric circuit that produces a current that is constant in time but at an unpredictable level (which may well be Gaussian distributed), gives a non-ergodic process: the ensemble is a series of functions in time (each consisting of an unpredictable constant) that is described by statistical characteristics that are constant in time but the average in each experiment (realisation) is different from the average in another realisation. The process is stationary and may even be Gaussian but since the time average is not equal to the ensemble average, it is not ergodic.