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Stochastic renewal process models for estimation of damage cost over the life-cycle of a structure

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ABSTRACT

In the life-cycle cost analysis of a structure, the total cost of damage caused by external hazards like earthquakes, wind storms and flood is an important but highly uncertain component. In the literature, the expected damage cost is typically analyzed under the assumption of either the homogeneous Poisson process or the renewal process in an infinite time horizon (i.e., asymptotic solution). The paper reformulates the damage cost estimation problem as a compound renewal process and derives general solutions for the mean and variance of total cost, with and without discounting, over the life cycle of the structure. The paper highlights a fundamental property of the renewal process, referred to as renewal decomposition, which is a key to solving a wide range of life cycle analysis problems. The proposed formulation generalizes the results given in the literature, and it can be used to optimize the design and life cycle performance of structures.

1. Introduction

1.1. Background

The life-cycle cost analysis involves many elements, such as cost of construction, operation, maintenance, decommissioning, and many other activities, over a specified time horizon or service life of the structure. In the reliability-based optimization, Rosenblueth and Mendoza [1] pointed out the three most important components of the life cycle cost, namely, initial construction cost, benefits derived from the system and losses due to failures. The term *damage cost* is used in this paper to denoted the total losses due to failures that incur due to loss of services, damage to contents and cost of repairing and restoring the damaged structure.

In the life cycle analysis, one of the most uncertain elements is the damage cost that might result due to exposure to external hazards, such as earthquakes, wind storms and floods. Uncertainty in the estimation of damage cost arises from intrinsic uncertainties associated with the occurrence frequency and intensity of a given type of hazard, as well as the structural response to the hazard.

In recent times, research interests in the life cycle analysis has peaked, as it has become a focus of the performance-based design as well as optimization of decisions related to maintenance planning and retrofitting of structures.

In structural engineering, the homogeneous Poisson process (HPP) model for occurrences of a hazard has been traditionally used to estimate the expected life cycle damage cost, such as in the seismic risk analysis [2]. Although the HPP model greatly simplifies the analytical formulation, this model is not likely to represent the stochastic nature of a wide ranging hazards and threats. Therefore, the expected cost analysis performed under the HPP assumption cannot be considered a generic analysis of the problem.

The main aim of this paper is to provide a clear and comprehensive exposition of key ideas of the theory of stochastic renewal processes in a way to generalize the life-cycle analysis of the damage cost. In particular, derivations of the expected value and the variance of the cost, with and without discounting, are presented in a coherent manner. Explicit analytical results are derived for the HPP and Erlang processes, which are especial cases of general results presented in the paper. A practical example of seismic retrofitting is presented. An ulterior motive of this study is to help new generation of engineers understand the key concepts of stochastic process models for life-cycle cost analysis.

Since the paper is primarily concerned with damage cost resulting from external hazards, the effect of internal degradation (e.g., corrosion and fatigue) on the life cycle cost is not considered here.

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The inspection and maintenance costs to prevent failures resulting from internal degradation are also ignored. The topic of life cycle cost analysis considering a stochastic degradation process and a condition-based maintenance policy are already presented in separate studies by Cheng and Pandey [3] and Pandey et al. [4].

1.2. Literature

With the advent of probabilistic models for risk analysis in 1970s, there was a great deal of interest in using the total risk as a basis for optimizing the structural design codes. Whitman and Cornell [5] presented a comprehensive approach to evaluate the total seismic risk associated with a design that is expected to face multiple seismic events during its service life. The next comprehensive study on this topic was presented by Rosenblueth [6], who introduced the stochastic renewal process for estimating the expected present value of losses caused by infrequent hazards, such as earthquakes, strong winds and tsunamis. In this study, the expected discounted cost of structural failures and repairs was derived using the method of the Laplace transform. The problem of optimum design of structures under dead, live and seismic loads was considered in Rosenblueth [7]. The optimization involved minimization of expected discounted value of costs and losses over the life cycle of the structure. In earthquake engineering, calculation of lifetime seismic damage cost continued to be an active area of research [2], though the stochastic analysis is almost exclusively based on the homogeneous Poisson process (HPP) model. Porter et al. [8] presented computation of the variance of discounted seismic risk under the assumption of HPP model, perhaps the first time in the seismic literature. The derivation was based on the order statistics property of the Poisson process, which cannot be extended to a renewal process model.

Takahashi et al. [9] pointed out that the occurrence of large magnitude earthquakes, referred to as the 'characteristic earthquake' depends on the previous history of earthquake activity at the source. Therefore, a non-Poisson, non-stationary stochastic model must be used to describe their occurrences, whereas HPP model is more suitable for smaller earthquakes occurring more or less randomly. They adopted a renewal process model based on the Brownian Passage Time distribution and approximately evaluated the expected discounted cost of seismic damage. A detailed evaluation of structural damage and cost given a seismic event has also been an active area of research [10,11].

The interest in the renewal process model for life cycle cost optimization was rekindled by Rackwitz [12], in which Rosenblueth's model was extended to combine it with the Life Quality Index framework proposed by Pandey et al. [13]. In a series of papers, Rackwitz and his co-workers applied the renewal process model to a more general class of problems in which the effect of degradation and maintenance was also included in life cycle cost analysis [14–16]. Most of this work was concerned with the evaluation of expected discounted cost and losses. Goda and Hong [17] applied the Monte Carlo simulation method to evaluate the mean, standard deviation and probability distribution of the seismic life cycle cost. An application of the utility theory to life cycle analysis was presented by Cha and Ellingwood [18].

1.3. Limitations of existing literature

Although there is a fairly substantial body of the literature on stochastic modeling of life cycle cost analysis, the following limitations in the analytical formulation are noted:

- In the stochastic life cycle analysis, the homogeneous Poisson process model is omnipresent [8,12,2]. The HPP model leads to considerable analytical simplifications and avoids dealing with intricacies of the theory of the renewal process.
  The analysis is mostly limited to the expected cost and expected discounted cost. The computation of the variance is largely non-existent, with an exception of Porter et al. [8], who derived variance of the cost.
- Although the stochastic renewal process models were employed hitherto, their success has been mostly limited to the computation of expected cost in an asymptotic sense. In fact, a clear formulation for the expected discounted cost in a finite time horizon is not available.
  The asymptotic analysis is based on the elementary renewal theorem which says that the cost rate asymptotically converges to a ratio of the expected cost in a single renewal cycle to the expected cycle length. This asymptotically solution is so simple to use that it completely bypasses a formal stochastic formulation of the problem. For this reason, the literature is replete with the use of the asymptotic solution, even in cases where it is not consistent with a short and finite time planning horizon, required for financial planning and capital budgeting [3].
- The evaluation of variance of the life cycle cost and its discounted value in a stochastic renewal model has not been discussed at all in the life cycle analysis literature.
  A main reason for lack of generalities in renewal process based models is the method of the Laplace Transform that was used by most researchers to solve the problem [12,6]. Although this method allows to write a compact expression for the Laplace transform of the expected costs, its inverse in not easy to find for a general distribution of the inter-occurrence time. Therefore, this approach is mostly limited to a few special cases like the exponential distribution (i.e., HPP model) and the Erlang distribution.

1.4. Objectives and organization

The central objective of this paper is to present a clear and comprehensive formulation to compute expected value and variance of the damage cost, with and without discounting, that may incur over the life cycle of a structure due to exposure to external hazards like earthquake, wind, snow and flood. To achieve this objective, a general formulation based on the theory of stochastic renewal process is presented, which overcomes the limitations of the existing literature as stated in Section 1.3. The mean and variance of discounted cost can now be computed in a finite time horizon for a general renewal process.

The information about the mean and variance of life cycle cost can be used to improve decision making regarding the design alternatives and options of retrofitting of a structure within a "mean–variance" based utility framework. For example, a utility function given as the sum of mean and some multiple of standard deviation of cost can be maximized as a part of the decision making process.

In this paper, analytical results are also derived for a especial case of the Erlang renewal process. An interesting finding of the paper is that there is large variability associated with the estimate of the damage cost, as marked by a large coefficient of variation (COV ≈ 1). It means that an exclusive reliance on the expected cost in optimization would not yield desired result in practice due to potentially large variability in the actual outcome.

The paper is organized as follows. Section 2 presents the basic terminology and concepts of the stochastic renewal process model. The renewal decomposition, a fundamental concept used extensively in this paper, is clearly described. The lack of understanding of this key concept led many researches to adopt the Laplace Transform approach. Section 3 derives the expected cost and variance of the damage cost, and this formulation is extended to discounted cost analysis in Section 4. Analytical results for HPP and the Erlang renewal process are derived in Section 5. A practical example related to seismic retrofitting of a wooden house is presented in Section 6. The last Section 7 summarizes key findings of this study. Additional analytical derivations are presented in Appendix A.

2. Stochastic renewal process: basic concepts

In the context of the life cycle analysis, a stochastic renewal process can be used to describe repeated occurrences of an event at random times. In Fig. 1, an event is shown to recur at times $S_1, S_2, \dots, S_n$, which is randomly distributed in an interval $(0, t]$. The recurring event can be an external hazard or a renewal of the structure or any other event depending on the problem. In this Section, key mathematical concepts related to the renewal process are described in a self-contained manner.

2.1. Point process and counting process

Mathematically, a (simple) point process is a random and strictly increasing sequence of real numbers, $S_0 = 0 < S_1 < S_2 < \cdots$, on the set of positive real numbers $\mathbf{R}_+$ without a finite limit point, i.e., as $i \to \infty, \lim S_i \to \infty$. The origin of the process is denoted as $S_0 = 0$. The cumulative distribution function of S_i is denoted as $F_{S_i}(x) = \mathbb{P}[S_i \leqslant x]$. A point process can be equivalently represented by a sequence of *random* inter-arrival times, $T_1, T_2, \dots$, with $T_n = S_n - S_{n-1}$. The arrival time, S_i, can thus be written as a partial sum of inter-occurrence times, i.e., $S_i = T_1 + T_2 \cdots + T_i$.

The number of events in the time interval $(0, t]$, denoted as $N(t)$, is formally defined as

$$N(t) = \max\{i, S_i \leqslant t\}, \quad (t \geqslant 0). \tag{1}$$

The process $\{N(t); t \geqslant 0\}$ is referred to as the *counting process* associated with the partial sums $S_i, i \geqslant 1$. Since the events $\{N(t) = i\}$ and $\{S_i \leqslant t < S_{i+1}\}$ are equal, the marginal probability distribution of $N(t)$ can be written as

$$\mathbb{P}[N(t) = i] = \mathbb{P}[S_i \leqslant t < S_{i+1}] = F_{S_i}(t) - F_{S_{i+1}}(t), \quad i = 0, 1, \dots \tag{2}$$

To derive this probability term, the following relations are used:

$$\begin{aligned} &\mathbb{P}[S_i \leqslant t] = \mathbb{P}[S_i \leqslant t, S_{i+1} \leqslant t] + \mathbb{P}[S_i \leqslant t, S_{i+1} > t] \\ &\iff \mathbb{P}[S_i \leqslant t < S_{i+1}] = \mathbb{P}[S_i \leqslant t] - \mathbb{P}[S_{i+1} \leqslant t] = F_{S_i}(t) - F_{S_{i+1}}(t) \end{aligned}$$

Note that $\mathbb{P}[S_i \leqslant t, S_{i+1} \leqslant t] = \mathbb{P}[S_{i+1} \leqslant t]$, since the first event, $(S_i \leqslant t)$, is a subset of the second event, $(S_{i+1} \leqslant t)$. Furthermore, $F_{S_0}(t) = \mathbb{P}[S_0 \leqslant t] = 1$ for any $t \geq 0$, since $S_0 = 0$. As $i \to \infty, \lim S_i \to \infty$, such that $F_{S_i}(t) \to 0$ for any finite value of $t \geq 0$. With these conditions, it can be shown using Eq. (2) that $\sum_{i=1}^{\infty} \mathbb{P}[N(t) = i] = 1$.

A joint distribution for $0 < t_1 < t_2 < \cdots < t_k$, and $0 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$, can be written as

$$\begin{aligned} &\mathbb{P}[N(t_1) = n_1, N(t_2) = n_2, \dots, N(t_k) = n_k] \\ &\quad = \mathbb{P}\left[S_{n_1} \leqslant t_1 < S_{n_1+1}, \dots, S_{n_k} \leqslant t_k < S_{n_k+1}\right] \end{aligned}$$

In summary, the finite-dimensional distributions of the counting process $N(t)$ is completely determined by the joint distributions of the random vectors $(S_1, \dots, S_k), k \geqslant 1$.

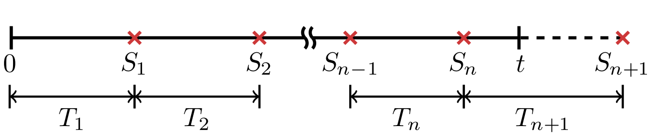

Fig. 1. A schematic of the renewal process.

Proposition 2.1. *If the two sequences,* $0 < S_1 < S_2 < \cdots$ *and* $0 < \tilde{S_1} < \tilde{S_2} < \cdots$, *have an identical distribution, i.e.,*

$$(S_1, \dots, S_k) \stackrel{d}{=} (\tilde{S_1}, \dots, \tilde{S_k}), \quad \text{for all} \quad k \geqslant 1,$$

then the associated counting processes, $N(t)$ *and* $\widetilde{N}(t)$, *also have the same distribution.*

2.2. Renewal processes

A point process is called an ordinary *renewal process* if the inter-occurrence times $T_1, T_2, \dots$, form a sequence of non-negative, independent and identically distributed (iid) random variables with a distribution $F_T(t)$. The word "renewal" implies that the process is reset after each occurrence of the event of interest. The homogeneous Poisson process is a well known example of a renewal process in which T follows an exponential distribution.

For a renewal process, the probability distribution of S_i is an i-fold convolution $F_T^{(i)}(t)$ defined as

$$F_{S_i}(t) = \mathbb{P}[T_1 + T_2 \cdots + T_i \leqslant t] = F_T^{(i)}(t), \tag{3}$$

which can be evaluated in a sequential manner as

$$F_T^{(i)}(t) = \int_0^t F_T^{(i-1)}(t - y)\, \mathrm{d}F_T(y), \quad (i \geq 2) \tag{4}$$

Note that $F_T^{(1)}(t) = F_T(t)$ and $\mathrm{d}F_T(t) = f_T(t)\mathrm{d}t$ when the probability density of T exists. The convolution, $F_T^{(i)}(t)$, is not easy to evaluate in a general setting because of numerical difficulties associated with the computation of higher order convolution integrals.

2.3. Renewal function

The renewal function, $\Lambda(t)$, is defined as the expected number of renewals in a time interval $(0, t]$.

A binary indicator function is introduced which makes it easier to write concise mathematical statements. The indicator function tests a logical condition in the following way:

$$\mathbf{1}_A = \begin{cases} 1 & \text{only if } A \text{ is true} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

From basic probability theory, the expected value of an indicator function is equal to the probability of occurrence of the condition being tested, i.e., $\mathbb{E}[\mathbf{1}_{\{A\}}] = \mathbb{P}[A]$.

To derive the renewal function, the number of renewals is written in terms of an indicator function as,

$$N(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{S_i \leqslant t\}}. \tag{6}$$

such its expected value can be evaluated as

$$\mathbb{E}[N(t)] = \Lambda(t) = \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbf{1}_{\{S_i \leqslant t\}}\right] = \sum_{i=1}^{\infty} \mathbb{P}[S_i \leqslant t] = \sum_{i=1}^{\infty} F_T^{(i)}(t). \tag{7}$$

This expression is not useful in computation, as it involves an infinite series of convolutions. To circumvent this difficulty, an integral equation for the renewal function is derived in the following manner.

Rearranging Eq. (7) and substituting from Eq. (4) leads to

$$\begin{aligned} \Lambda(t) &= F_T^{(1)}(t) + \sum_{i=2}^{\infty} F_T^{(i)}(t) = F_T(t) + \sum_{i=1}^{\infty} F_T^{(i+1)}(t) \\ &= F_T(t) + \sum_{i=1}^{\infty} \int_0^t F_T^{(i)}(t - y)\, \mathrm{d}F_T(y). \end{aligned}$$

Interchanging the sum and integral and using Eq. (7) leads to

$$\sum_{i=1}^{\infty}\int_0^t F_T^{(i)}(t-y)\,\mathrm{d}F_T(y) = \int_0^t \left(\sum_{i=1}^{\infty}F_T^{(i)}(t-y)\right)\mathrm{d}F_T(y)$$
$$= \int_0^t \Lambda(t-y)\,\mathrm{d}F_T(y),$$

The final result is the following integral equation for the renewal function:

$$\Lambda(t) = F_T(t) + \int_0^t \Lambda(t-y)\,\mathrm{d}F_T(y). \tag{8}$$

The renewal rate is defined as the expected number of renewals per unit time given by the time derivative ([19]):

$$\lambda(t) = \frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} \tag{9}$$

If the random variable has probability density function, $f_T(t)$, then the following integral equation can be written for the renewal rate:

$$\lambda(t) = f_T(t) + \int_0^t \lambda(t-y) f_T(y)\,\mathrm{d}y \tag{10}$$

2.3.1. Solution of the renewal equation

A classical approach to solve the renewal integral equation is based the Laplace transform method, which is briefly described here without any mathematical formalities. The Laplace transform (LT) of a function, such as the probability density, $f_T(t)$, is defined as

$$f_T^*(s) = \int_0^{\infty} e^{-st} f_T(t)\,\mathrm{d}t$$

The LT of the cumulative distribution function is given as $F_T^*(s) = \int_0^{\infty} e^{-st} F_T(t)\,\mathrm{d}t = f_T^*(s)/s$. Using a basic result that the LT of a convolution of two functions is the product of their LTs, the renewal Eq. (8) can be solved by taking LT of both sides as

$$\Lambda^*(s) = F_T^*(s) + \Lambda^*(s) f_T^*(s),$$

which leads to the final solution:

$$\Lambda^*(s) = \frac{f_T^*(s)}{s(1 - f_T^*(s))} \tag{11}$$

Thus, given the LT of the PDF of the inter-occurrence time, $f_T^*(s)$, the LT of the renewal function can be easily obtained. However, the inversion of $\Lambda^*(s)$ to obtain the renewal function, $\Lambda(t)$, in the original time domain requires more complex numerical methods and algorithms.

It has been found that a direct numerical solution of the renewal integral equation method by a trapezoidal integral rule is fairly simple, practical and accurate method [20]. In this paper, a modified numerical algorithm of Tijms [19] is used to solve the integral equation.

2.4. Concept of the renewal decomposition

Although the renewal equation can be derived from elementary concepts of probability theory, an underlying important concept is the regenerating property of the renewal process. Because this property is not well understood in clear mathematical terms, the engineering applications have been mostly limited to the evaluation of expected value in fairly simple settings. The concept of the regenerative property, also referred to as the renewal decomposition, allows to solve more involved problems.

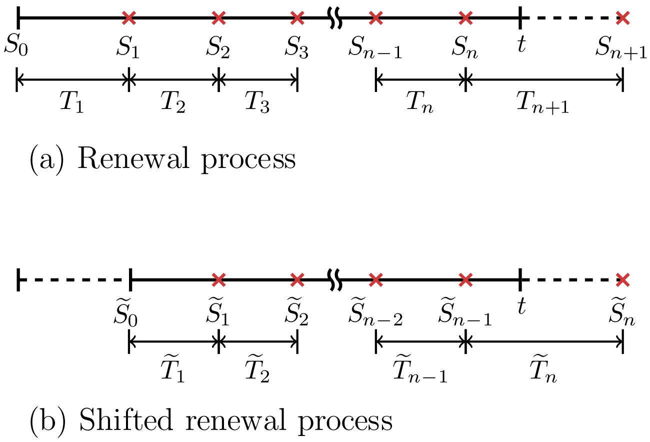

Fig. 2. An illustration of the renewal decomposition argument.

The renewal decomposition refers to a basic property of the renewal process that after every renewal a (probabilistic) replica of the original process starts again. In a practical engineering context, this property is easy to understand. For example, after a failure of a machine, when it is replaced by an identical new machine, the process of machine operation restarts afresh. Similarly, after a seismic event, the repair of a structure to restore its condition to the original (new) state is another example of renewal process. The probabilistic implications of this intuitive property can be formalized as follows.

Fig. 2(a) shows a renewal process in an interval $(0, t]$ with the number of renewals $N(t)$. Suppose this process is observed after the first event that occurred at time $S_1 = T_1$, as shown in Fig. 2 (b). Thus, the shifted renewal process, observed in the time interval, $(S_1, S_1 + t]$, is associated with the sequence of inter-arrival times, $T_2, T_3, \ldots$, or alternatively denoted as $\widetilde{T_1}, \widetilde{T_2}, \ldots$ with $\widetilde{T}_i = T_{i+1}$. The corresponding partial sum is denoted as $\widetilde{S}_i$, such that $\widetilde{S}_i = S_{i+1} - T_1$.

The number of renewals in the shifted process is given as

$$\widetilde{N}(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{\widetilde{S}_i \leqslant t\}}. \tag{12}$$

Proposition 2.2. *The renewal decomposition property means that*

1. *The counting process $N(t)$ has the same distribution as $\widetilde{N}(t)$.*
2. *The shifted process is independent of the time of shift, i.e., $\widetilde{N}(t)$ and T_1 are independent.*

2.4.1. Application to the derivation of renewal equation

Recall the definition of the number of renewals in the original process

$$N(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{S_i \leqslant t\}} = \mathbf{1}_{\{S_1 \leqslant t\}} + \sum_{i=1}^{\infty} \mathbf{1}_{\{S_{i+1} \leqslant t\}}.$$

Since $\widetilde{S}_i = S_{i+1} - T_1$, the sum in the righthand side can also be written as

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{S_{i+1} \leqslant t\}} = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_1 + \widetilde{S}_i \leqslant t\}} = \sum_{i=1}^{\infty} \mathbf{1}_{\{\widetilde{S}_i \leqslant t - T_1\}} = \widetilde{N}(t - T_1),$$

Thus, the final decomposition of the original process is obtained as

$$N(t) = \mathbf{1}_{\{T_1 \leqslant t\}} + \widetilde{N}(t - T_1). \tag{13}$$

To derive the renewal function, take the expectation of both sides of Eq. (13)

$$\mathbb{E}[N(t)] = \mathbb{E}\left[\mathbf{1}_{\{T_1 \leqslant t\}}\right] + \mathbb{E}\left[\widetilde{N}(t - T_1)\right]$$

Note that $\widetilde{N}(t - T_1) = 0$ for $t < T_1$. From the independence of T_1 and $\widetilde{N}(t)$ (see Proposition 2.2-2)

$$\mathbb{E}\left[\widetilde{N}(t - T_1)\right] = \int_0^t \mathbb{E}\left[\widetilde{N}(t - x)\right] \mathrm{d}F_T(x).$$

Since $N(t)$ and $\widetilde{N}(t)$ have identical distribution (see Proposition 2.2-1), the following equality holds:

$$\mathbb{E}\left[\widetilde{N}(t - x)\right] = \mathbb{E}[N(t - x)] = \Lambda(t - x)$$

Substituting these two results into Eq. (8) leads to the renewal integral equation:

$$\Lambda(t) = F_T(t) + \int_0^t \Lambda(t - x)\, \mathrm{d}F_T(x). \quad (14)$$

The renewal function can be easily computed using an algorithm given by Tijms [19], which is based on the trapezoidal integration method.

The renewal decomposition idea has been successfully applied to solve a more complex problem of the unavailability analysis of nuclear safety systems [21].

In many elementary textbooks the derivation of the renewal Eq. (8) is explained as "conditioning on the time first renewal". Despite its intuitive appeal, this approach does not go far enough to formulate a solution of complex problems. We believe that the concept of renewal decomposition, Eq. (13), is mathematically rigorous, and technically correct argument that is applicable to a larger class of problems, as shown in this paper.

Proposition 2.3. *Consider an integral equation of the form*

$$z(t) = \phi(t) + \int_0^t z(t - x)\, \mathrm{d}F_T(x),$$

where F_T is a cumulative distribution function, $F_T(0) = 0$, and $\phi(t)$ is a known, bounded function. The solution of this integral equation can be written in terms of the renewal function $\Lambda(t)$ associated with T as [19]

$$z(t) = \phi(t) + \int_0^t \phi(t - x)\, \mathrm{d}\Lambda(x).$$

2.5. Marked and compound renewal processes

In addition to the inter-occurrence time (T), the severity (or intensity) of a hazard tends to be highly uncertain, and it can also be modelled by another random variable, X. Thus, a recurring hazard can be modelled as a sequence of random vectors $(T_i, X_i), i = 1, 2, \ldots$, which are assumed to be independent and identically distributed. This sequence is called a *marked point process*, as shown in Fig. 3. It must be emphasized that the joint distribution of (T_i, X_i) is independent of that of (T_j, X_j), $i \neq j$, but a dependence between T_i and X_i is permitted.

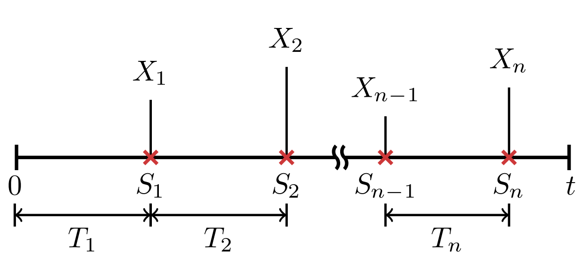

Fig. 3. An example of a marked renewal process.

The compound process refers to the cumulative effect of a marked renewal process. For example, if each occurrence of a hazard results in the structural damage cost of C \$, a random variable, then the total (or cumulative) cost in an interval $(0, t)$ is given as a random sum:

$$K(t) = \sum_{i=1}^{N(t)} C_i \quad (15)$$

The total cost, $K(t)$, is mathematically referred to as a compound renewal process. The mean and variance of the compound process are useful in the life cycle cost analysis, as shown later in the paper.

3. Damage cost analysis (DCA)

3.1. Basic concepts

Suppose single occurrence of a hazard results in the damage cost of C \$, which is modelled as a random variable to account for uncertainties arising from random intensity of hazard and other design features. The damage cost per event (C) has a mean μ_C and standard deviation σ_C. As mentioned in Section 2.5, the total cost, $K(t)$, is a compound renewal process defined by an iid sequence of random vectors, $(T_1, C_1), (T_2, C_2), \ldots, (T, C)$, with non-negative random variables T and C. The joint distribution of (T_i, C_i) is independent of that of (T_j, C_j) for any $i \neq j$. However, the renewal cycle cost, C_i, and the duration, T_i, can be dependent. Therefore, the total cost $K(t)$ over the time interval $(0, t]$ is given as

$$K(t) = \sum_{i=1}^{N(t)} C_i = \sum_{i=1}^{\infty} C_i \mathbf{1}_{\{S_i \leqslant t\}}, \quad (16)$$

where $N(t)$ is the counting process associated with the *iid* sequence, $T_1, T_2, \ldots$. It is interesting to point out that the renewal function, $\Lambda(t)$, associated with this process is a key input to the evaluation of moments of $K(t)$, with and without discounting, as shown in the remainder of the paper.

Here, integral equations for the first two moments of the damage cost are derived in a general setting where the cost C and the inter-occurrence time T are dependent random variables, with a joint distribution, $F_{C,T}(c, t)$. A especial case of C being independent of T is tackled in Appendix A.

3.2. Expected cost

The derivation of the expected damage cost relies on the idea of renewal decomposition, as explained in Fig. 2 and Section 2.4.

Let $K(t)$ be an original compound renewal process (see Fig. 3) as defined by Eq. (16). Let $\widetilde{K}(t)$ be the shifted process in the time interval $(S_1, S_1 + t]$, which starts after the first event occurring at time $S_1 = T_1$. So, $\widetilde{K}(t)$ can be interpreted as the cost over $(0, t]$ associated with the shifted iid sequence $(T_2, C_2), (T_3, C_3), \ldots$, which is given as

$$\widetilde{K}(t) = \sum_{i=1}^{\infty} \widetilde{C}_i \mathbf{1}_{\{\widetilde{S}_i \leqslant t\}} = \sum_{i=1}^{\infty} C_{i+1} \mathbf{1}_{\{T_2 + \cdots + T_{i+1} \leqslant t\}} \quad (17)$$

Proposition 3.1. *The renewal decomposition of a compound process implies that*

1. *The original process $K = \{K(t); t \geqslant 0\}$ and the shifted process $\widetilde{K} = \{\widetilde{K}(t); t \geqslant 0\}$ are identically distributed, and*

2. *The shifted compound process $\widetilde{K}(t)$ is independent of the random vector (T_1, C_1).*

Using these properties, a decomposition formula for the damage cost can be derived by splitting the sum in Eq. (16) at the time of first renewal, $S_1 = T_1$. Thus,

$$\begin{aligned} K(t) &= C_1 \mathbf{1}_{\{S_1 \leqslant t\}} + \sum\nolimits_{i=1}^{\infty} C_{i+1} \mathbf{1}_{\{S_{i+1} \leqslant t\}} \\ &= C_1 \mathbf{1}_{\{S_1 \leqslant t\}} + \sum\nolimits_{i=1}^{\infty} C_{i+1} \mathbf{1}_{\{T_1 + (T_2 + \cdots + T_{i+1}) \leqslant t\}} \\ &= C_1 \mathbf{1}_{\{S_1 \leqslant t\}} + \sum\nolimits_{i=1}^{\infty} C_{i+1} \mathbf{1}_{\{(T_2 + \cdots + T_{i+1}) \leqslant t - T_1\}} \end{aligned}$$

It follows from Eq. (17) that

$$K(t) = C_1 \mathbf{1}_{\{S_1 \leqslant t\}} + \widetilde{K}(t - T_1), \tag{18}$$

where $\widetilde{K}(t - T_1) = 0$ for $t < T_1$. Taking the expectation of both sides of Eq. (18) leads to

$$\mathbb{E}[K(t)] = \mathbb{E}\big[C_1 \mathbf{1}_{\{T_1 \leqslant t\}}\big] + \mathbb{E}\Big[\widetilde{K}(t - T_1)\Big].$$

from Proposition 3.1,

$$\mathbb{E}\Big[\widetilde{K}(t - T_1)\Big] = \int_0^t \mathbb{E}\Big[\widetilde{K}(t - y)\Big]\, \mathrm{d}F_T(y) = \int_0^t \mathbb{E}[K(t - y)]\, \mathrm{d}F_T(y).$$

Defining the function

$$\phi(t) = \mathbb{E}\big[C \mathbf{1}_{\{T \leqslant t\}}\big], \tag{19}$$

which is an increasing function and bounded by $\mathbb{E}[C]$, which is a finite value of the mean of C. Thus, $\mathbb{E}[K(t)]$ satisfies the following renewal integral equation

$$\mathbb{E}[K(t)] = \phi(t) + \int_0^t \mathbb{E}[K(t - y)]\, \mathrm{d}F_T(y). \tag{20}$$

It follows from Proposition 2.3 on page 11 that the above integral equation has a unique solution:

$$\mathbb{E}[K(t)] = \phi(t) + \int_0^t \phi(t - y)\, \mathrm{d}\Lambda(y). \tag{21}$$

where $\Lambda(t)$ is the renewal function associated with the inter-occurrence time T. This solution is fairly general and it allows to consider a dependence between C and T.

As before, the Laplace transform (LT) method can be used in principle to solve the integral equation of the expected cost. The LT of the expected cost can be directly written as

$$K^*(s) = \phi^*(s) + \phi^*(s)\lambda^*(s) = \frac{\phi^*(s)}{(1 - f_T^*(s))} \tag{22}$$

This solution involves LTs of $\phi(t)$and $f_T(t)$ and its inversion is not easy except, in some elementary cases. Therefore, this approach is not discussed any further in the paper.

3.3. Asymptotic solution

The stochastic process of the life cycle cost, $K(t)$, has a remarkable asymptotic property. The total cost per unit time or cost rate, $K(t)/t$, as well as the expected cost per unit time has an asymptotic limit given as

$$k^\infty = \lim_{t \to \infty} \frac{K(t)}{t} = \lim_{t \to \infty} \frac{\mathbb{E}[K(t)]}{t} = \frac{\mathbb{E}[C]}{\mathbb{E}[T]} \tag{23}$$

For a detailed mathematical exposition of this topic, the readers are referred to Gallager [22].

In simple terms, the asymptotic limit of the expected cost per unit time is a ratio of the expected cost and length of a single renewal cycle. Using this result, the expected cost in a time interval $(0, t]$ can be approximately estimated as $\mathbb{E}[K(t)] \approx k^\infty t$.

It is also clear that by adopting the asymptotic result, a formal stochastic analysis of the total cost estimation problem can be completely avoided.

3.4. Second moment of the damage cost

To evaluate the variance of the damage cost, the second moment (or mean square) of the cost is needed, for which the starting point is the basic definition of mean square applied to the decomposition formula given by Eq. (18):

$$\mathbb{E}\Big[K^2(t)\Big] = \mathbb{E}\bigg[\Big(C_1 \mathbf{1}_{\{S_1 \leqslant t\}} + \widetilde{K}(t - T_1)\Big)^2\bigg] \tag{24}$$

Defining a function $\psi(t)$ as

$$\psi(t) = \mathbb{E}\Big[C^2 \mathbf{1}_{\{T \leqslant t\}}\Big] + 2\mathbb{E}\Big[C\widetilde{K}(t - T)\mathbf{1}_{\{T \leqslant t\}}\Big], \tag{25}$$

the function $\mathbb{E}\Big[K^2(t)\Big]$ satisfies a renewal equation:

$$\begin{aligned} \mathbb{E}\Big[K^2(t)\Big] &= \psi(t) + \mathbb{E}\Big[\widetilde{K}^2(t - T_1))\Big] \\ &= \psi(t) + \int_0^t \mathbb{E}\Big[K^2(t - y)\Big]\, \mathrm{d}F_T(y) \end{aligned}$$

The final solution for the mean-square of life cycle damage cost is

$$\mathbb{E}\Big[K^2(t)\Big] = \psi(t) + \int_0^t \psi(t - y)\, \mathrm{d}\Lambda(y) \tag{26}$$

4. Discounted cost analysis

The expected value of the discounted damage cost $K_D(t)$ can also be elegantly derived using the renewal decomposition described in Proposition 3.1.

The cost C_i incurring at time S_i is discounted back to present time, $S_0 = 0$, as $C_i e^{-\rho S_i}$, where $\rho > 0$ is the discount rate. Thus,the total discounted cost can be written as

$$K_D(t) = \sum_{i=1}^{\infty} C_i e^{-\rho S_i} \mathbf{1}_{\{S_i \leqslant t\}}, \quad (\rho > 0). \tag{27}$$

4.1. Expected discounted cost

The discounted cost over the time interval $(S_1, S_1 + t]$ is given by

$$\sum_{i=1}^{\infty} C_{i+1} e^{-\rho S_{i+1}} \mathbf{1}_{\{S_{i+1} \leqslant S_1 + t\}} = e^{-\rho S_1} \widetilde{K}_D(t),$$

where

$$\widetilde{K}_D(t) = \sum_{i=1}^{\infty} C_{i+1} e^{-\rho(T_2 + \cdots + T_{i+1})} \mathbf{1}_{\{T_2 + \cdots + T_{i+1} \leqslant t\}}.$$

Clearly, $\widetilde{K}_D(t)$ is the discounted cost associated with the shifted sequence $(T_2, C_2), (T_3, C_3), \ldots$. The renewal decomposition property implies that the processes $K_D = \{K_D(t); t \geqslant 0\}$ and $\widetilde{K}_D = \{\widetilde{K}_D(t); t \geqslant 0\}$ are identically distributed and the first cycle (T_1, C_1) and the process $\widetilde{K}_D$ are independent.

To derive the renewal equation, the above sum (27), as before, is split into a first renewal cycle, $T_1 < t$, and the rest of the sum as

$$K_D(t) = C_1 e^{-\rho T_1} \mathbf{1}_{\{T_1 \leqslant t\}} + e^{-\rho T_1} \widetilde{K}_D(t - T_1). \tag{28}$$

Define the function

$$\phi_D(t) = \mathbb{E}\Big[C e^{-\rho T} \mathbf{1}_{\{T \leqslant t\}}\Big]. \tag{29}$$

and taking the expectation of Eq. (28) and further simplifications lead to the following integral equation:

$$\mathbb{E}[K_D(t)] = \phi_D(t) + \int_0^t e^{-\rho s}\mathbb{E}[K_D(t-y)]\,\mathrm{d}F_T(s) \tag{30}$$

To reduce the above renewal equation to a standard format, both sides are multiplied with $e^{\rho t}$. Denoting $K'_D(t) = e^{\rho t}\mathbb{E}[K_D(t)]$ leads to

$$K'_D(t) = \phi_D(t)e^{\rho t} + \int_0^t K'_D(t-s)\,\mathrm{d}F_T(s)$$

As before, the solution of the above integral equation is as follows:

$$K'_D(t) = \phi_D(t)e^{\rho t} + \int_0^t \phi_D(t-s)e^{\rho(t-s)}\,\mathrm{d}\Lambda(s)$$

It can be reverted to original notations as

$$\mathbb{E}[K_D(t)] = \phi_D(t) + \int_0^t \phi_D(t-s)e^{-\rho s}\,\mathrm{d}\Lambda(s), \quad (\rho > 0) \tag{31}$$

4.1.1. Asymptotic solution

The derivation of the asymptotic limit of the expected discounted cost as $t \to \infty$ begins with Eq. (27), which can be rewritten as

$$k_D^\infty = \lim_{t\to\infty}\mathbb{E}[K_D(t)] = \mathbb{E}\left[\sum_{i=1}^{\infty} C_i e^{-\rho S_i}\right] = \sum_{i=1}^{\infty}\mathbb{E}\left[C_i e^{-\rho T_1}\cdots e^{-\rho T_i}\right]$$

The last step uses the fact that $S_i = T_1 + T_2 + \cdots T_i$. Since $T_1, \ldots, T_i$ are *iid*, the above equation can be simplified as

$$k_D^\infty = \sum_{i=1}^{\infty}\mathbb{E}\left[Ce^{-\rho T}\right]\left(\mathbb{E}\left[e^{-\rho T}\right]\right)^{i-1} = \mathbb{E}\left[Ce^{-\rho T}\right]\sum_{i=0}^{\infty}\left(\mathbb{E}\left[e^{-\rho T}\right]\right)^{i}$$

Substituting the standard formula for the sum of the geometric series in above equation leads to the final result:

$$k_D^\infty = \frac{\mathbb{E}\left[Ce^{-\rho T}\right]}{1-\mathbb{E}\left[e^{-\rho T}\right]} \tag{32}$$

For a more comprehensive asymptotic analysis of moments of discounted cost, the readers are referred to van der Weide et al. [23]. The asymptotic discounted cost was commonly used in earlier studies, such as Rosenblueth [6] and Joannni and Rackwitz [14].

4.2. Second moment of the discounted cost

The renewal equation for the second moment of the discounted cost is derived by squaring both sides of Eq. (28)taking the expectation, which leads to

$$\mathbb{E}\left[K_D^2(t)\right] = \psi_D(t) + \int_0^t e^{-2\rho s}\mathbb{E}\left[K_D^2(t-s)\right]\mathrm{d}F_T(s)$$

where

$$\psi_D(t) = \mathbb{E}\left[\left(C_1^2 + 2C_1\widetilde{K}_D(t-T_1)\right)e^{-2\rho T_1}\mathbf{1}_{\{T_1\leqslant t\}}\right]. \tag{33}$$

Note that this derivation also uses the renewal decomposition properties as in previous cases. For sake of brevity, intermediate steps are not shown here, and the final result is presented:

$$\mathbb{E}\left[K_D^2(t)\right] = \psi_D(t) + \int_0^t e^{-2\rho s}\psi_D(t-s)\,\mathrm{d}\Lambda(s). \tag{34}$$

The solutions presented in Sections 3 and 4 show that the proposed approach based on the renewal decomposition is quite versatile method for deriving the moments of a compound renewal process.

5. Special cases: analytical results

5.1. C independent of T

In many instances, the damage cost per event (C) can be independent of the occurrence time between events (T). This assumption is commonly used in the seismic risks analysis. In this case, the evaluation of moments of the damage cost can be greatly simplified, as shown by derivations given in Appendix A. The reason for simplification is that the solution approach does not involve an integral equation. Rather, formulas are directly derived using the basic definition of a random sum, The final analytical solutions are presented in Table 1. To calculate all the solutions given in Table 1, only a single integral equation for the renewal function, $\Lambda(t)$, needs to be solved.

5.2. Homogeneous poisson process (HPP)

The HPP is the simplest and most widely used renewal process in which the time between events is an exponentially distributed random variable with the distribution $F_T(x) = 1 - e^{-\lambda x}$ and the mean, $\mu_T = 1/\lambda$. The distribution of $N(t)$ is explicitly given by the Poisson probability mass function,

$$\mathbb{P}[N(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots \tag{35}$$

The renewal function of HPP is a linear function of time

$$\Lambda(t) = \lambda t, \tag{36}$$

and the renewal rate, λ, is a constant. When C and T are assumed to be independent, the expected cost can be easily obtained from Eq. (A.8) as (also given in Table (1)):

$$\mathbb{E}[K(t)] = \mu_C\lambda t \tag{37}$$

The second moment of the cost can be obtained from Eq. (A.9)

$$\mathbb{E}\left[K^2(t)\right] = \mu_{2C}\lambda t + (\mu_C\lambda t)^2, \tag{38}$$

which leads to the variance of cost as

$$\sigma^2_{K(t)} = \mu_{2C}\lambda t. \tag{39}$$

The expected discounted cost can be obtained from Eq. (A.3) as

$$\mathbb{E}[K_D(t)] = \mu_C\int_0^t e^{-\rho x}\lambda\mathrm{d}x = \frac{\mu_C\lambda}{\rho}(1-e^{-\rho t}). \tag{40}$$

This standard formula is most commonly used in seismic risk analysis [2,8].

The second moment of the discounted cost can be obtained from Eq. (A.5) as

Table 1
Moments of the damage cost when C is independent of T

Case	Moment	Expression	Eq, No.
With discounting	Mean	$\mathbb{E}[K_D(t)] = \mu_C\int_0^t e^{-\rho x}\mathrm{d}\Lambda(x)$	(A.3)
	Mean square	$\mathbb{E}\left[K_D^2(t)\right] = \mu_{2C}\int_0^t e^{-2\rho x}\mathrm{d}\Lambda(x) + 2\mu_C\int_0^t e^{-2\rho x}\mathbb{E}[K_D(t-x)]\mathrm{d}\Lambda(x)$	(A.7)
No discounting	Mean	$\mathbb{E}[K(t)] = \mu_C\Lambda(t)$	(A.8)
-	Mean square	$\mathbb{E}\left[K^2(t)\right] = \mu_{2C}\Lambda(t) + 2(\mu_C)^2 \times \int_0^t \Lambda(t-x)\mathrm{d}\Lambda(x)$	(A.9)

$$\mathbb{E}\left[K_D^2(t)\right] = \mu_{2C}\int_0^t e^{-2\rho x}\lambda \mathrm{d}x + 2(\mu_C)^2\int_0^t e^{-2\rho x}\left[\int_0^{t-x} e^{-\rho y}\lambda \mathrm{d}y\right]\lambda \mathrm{d}x \tag{41}$$

The algebraic simplification of the above equation leads to the following result:

$$\mathbb{E}\left[K_D^2(t)\right] = \mu_{2C}(\frac{\lambda}{2\rho}(1-e^{-2\rho t})) + 2(\mu_C)^2\left(\frac{\lambda^2}{2\rho^2}(1-e^{-\rho t})^2\right)$$

since the second term is the square of the mean discounted cost (see Eq. (40), the variance of the discounted cost can be written as

$$\sigma^2_{K_D(t)} = \frac{\mu_{2C}\lambda}{2\rho}(1-e^{-2\rho t}) \tag{42}$$

This is result is the same as that reported by Porter et al. [8]. Analytical results for HPP model are summarized in Table 2.

5.3. Renewal process – Erlang(2) distribution

In this Section, analytical results are derived for a renewal process in which the inter-occurrence time, T, follows the Erlang distribution with the shape parameter 2. The PDF and CDF of this distribution are given as

$$f_T(t) = \lambda^2\, t e^{-\lambda t}, \quad F_T(t) = 1-(1+\lambda t)e^{-\lambda t} \tag{43}$$

The mean and standard deviation of this distribution are $2/\lambda$ and $\sqrt{2}/\lambda$, respectively. The renewal function of the Erlang-2 renewal process was presented in Tijms [19] as

$$\Lambda_T(t) = \frac{1}{2}\,\lambda\, t - \frac{1}{4}(1-e^{-2\lambda t}) \tag{44}$$

Using the above renewal function, the expected damage cost can be easily calculated from Eq. (A.8). To evaluate the mean-square of the life cycle cost using Eq. (A.9), the following result is needed:

$$\int_0^t \Lambda(t-x)\mathrm{d}\Lambda(x) = \frac{\lambda^2 t^2}{8} + \frac{3}{16}(1-e^{-2\lambda t}) - \frac{\lambda t}{8}(2+e^{-2\lambda t})$$

The expected value of discounted cost can be derived from Eq. (A.3) as

$$\frac{1}{\mu_C}\quad \mathbb{E}[K_D(t)] = \int_0^t e^{-\rho x}\,\mathrm{d}\Lambda(x)$$
$$= \frac{\lambda}{2\rho}(1-e^{-\rho t}) - \frac{\lambda}{2(2\lambda+\rho)}(1-e^{-(2\lambda+\rho)t})$$

To evaluate the mean square of the discounted cost, the following key integral is derived as

$$\frac{1}{c_1}\int_0^t e^{-2\rho x}\mathbb{E}[K_D(t-x)]\mathrm{d}\Lambda(x) = \frac{\lambda}{\rho(\lambda+\rho)} - \frac{2}{\rho}e^{-\rho t} + \frac{(2\lambda+\rho)}{\rho(2\lambda-\rho)}e^{-2\rho t} - \frac{2}{(2\lambda-\rho)}e^{-(2\lambda+\rho)t} + \frac{1}{(\lambda+\rho)}e^{-2(\lambda+\rho)t}$$

Table 2
Moments of the damage cost in the HPP model.

Case	Quantity	Expression
With discounting	Mean	$(\mu_C\lambda/\rho)(1-e^{-\rho t})$
	Variance	$(\mu_{2C}\,\lambda)/(2\rho))(1-e^{-2\rho t})$
No discounting	Mean	$\mu_C\lambda t$
	Variance	$\mu_{2C}\lambda t$

Note that the constant c_1 is defined as

$$c_1 = \frac{\mu_C\lambda^3}{4\rho(2\lambda+\rho)}$$

These analytical results are quite useful in verifying the numerical solution of the above problem through the use of the renewal function.

5.4. Numerical example

Analytical results derived in this Section are quite useful to illustrate the variation of mean and variance of the life-cycle damage cost. In both HPP and Erlang(2) models, the mean inter-occurrence time has an identical value of 25 years with PDFs shown in Fig. 4. The cost of damage per event has mean of μ_C = 100 thousand$ and COV of 0.1. The discount rate is taken as $\rho = 0.05$ per year. The planning horizon is varied from 5 to 60 years, and in each case the mean and standard deviation of the life cycle damage cost were calculated. Results for HPP model shown in Fig. 5 present an interesting observation that the standard deviation of damage cost exceeds far more than the mean cost in a short time horizon (< 20 years). It means that any optimization based on the expected cost would be rendered meaningless

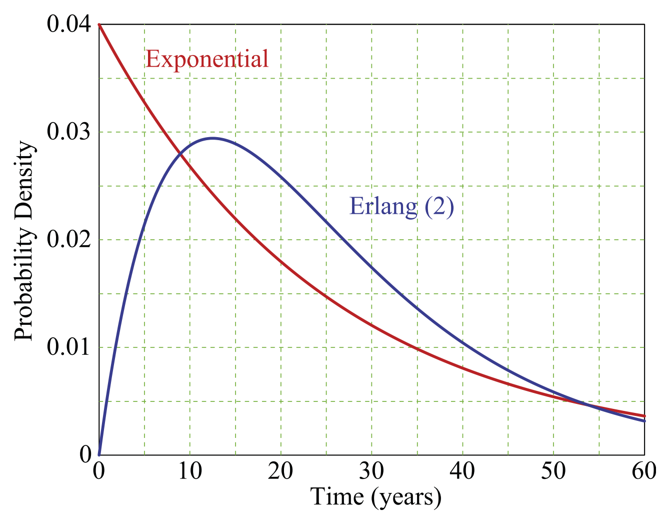

Fig. 4. Probability density functions of the exponential and the Erlang distributions.

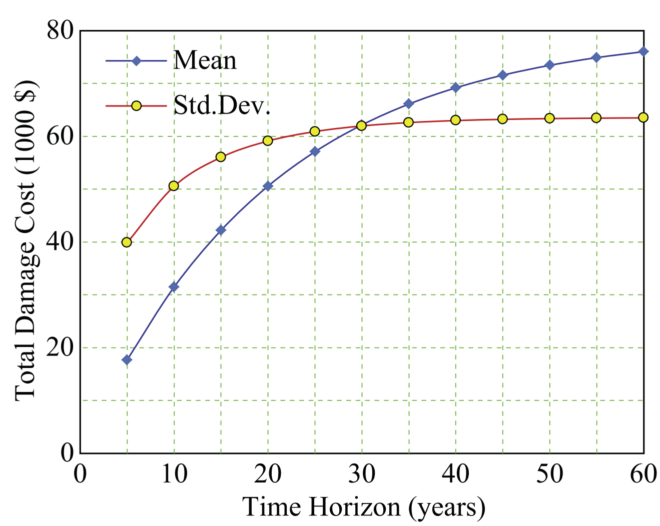

Fig. 5. Discounted damage cost: results for the HPP model.

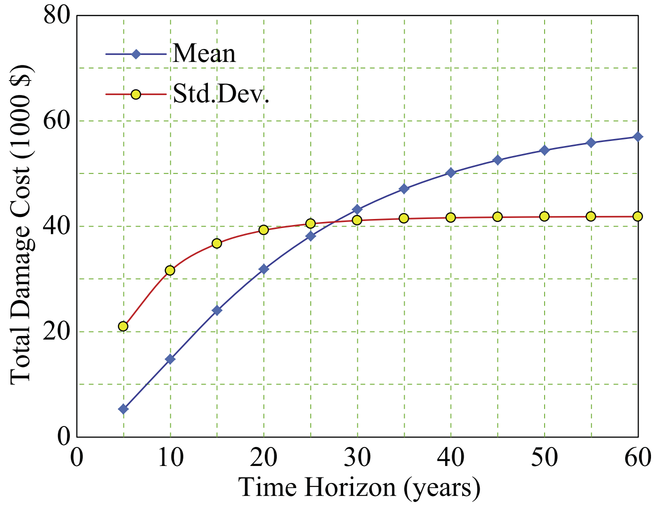

Fig. 6. Discounted damage cost: results for the Erlang process model.

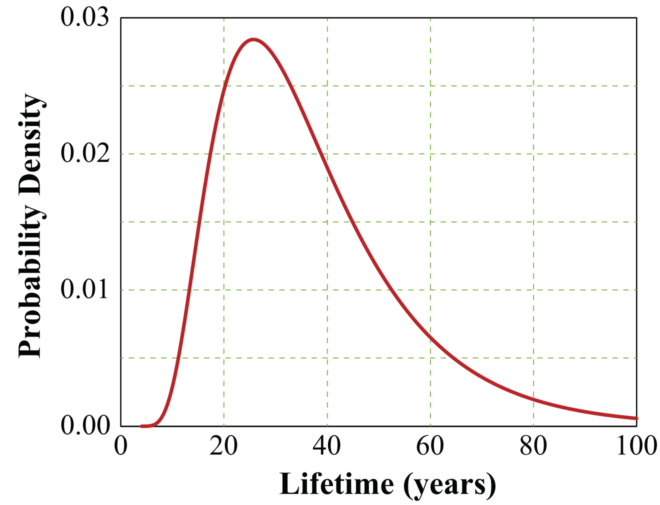

Fig. 7. Probability density the BPT distribution of earthquake occurrence interval.

by such a large volatility in the damage cost. Even in a 60 year time horizon, COV of the damage cost (=0.83) is quite high. The reason for this observation is that the standard deviation of cost is a function of the mean square of C (or 2nd moment, μ_{2C}), as shown by Eq. (42). Asymptotic values of the mean and standard deviation of the damage cost are calculated as 80 and 63.56 thousand$, respectively.

Results for the mean and standard deviation of the discounted damage cost, $K_D(t)$, in Erlang(2) model, as shown in Fig. 6, are qualitatively the same as those for the HPP case. Large volatility marked by large standard deviation is also present in this case. In quantitative terms, both mean and standard deviation are smaller than those calculated for HPP case. The COV of cost in a 60 year time horizon is 0.73, which is slightly smaller than that for HPP case. Asymptotic values of the mean and standard deviation of the damage cost are calculated as 60.95 and 41.86 thousand$, respectively.

6. Practical example: seismic risk analysis

6.1. Retrofitting of a wooden house

This example is inspired by the life cycle cost analysis of retrofitting of a two-storey wooden house by a base isolation system [24]. The house was located in Japan in a region vulnerable to high intensity earthquakes. Initial construction cost of the house was estimated as $300,000 and the value of contents in the house as $160,000. The seismic resistance of the house can be strengthened by installing a base isolation at a cost of $26,000. The question is about the cost effectiveness of the base isolation system in comparison to the risk of seismic damage that the house faces in absence of the base isolation system.

The seismic hazard at the site is posed by a characteristic earthquake of magnitude 7.5. The inter-occurrence time is assumed to follow the Brownian Passage Time (BPT) distribution with a mean of $\mu = 37.1$ years, a COV of $\alpha = 0.5$ [24]. The probability density of BPT distribution is given as (see Fig. 7):

$$f_T(t) = \sqrt{\frac{\mu}{2\pi\alpha^2 t^3}} \exp\left[-\frac{(t-\mu)^2}{2\mu t\alpha^2}\right] \quad (45)$$

An elaborate simulation based method was developed to estimate the cost of damage caused by a characteristic earthquake Takahashi et al. [9]. The simulation model considered uncertainties in the fault rupture, wave propagation, surface soil amplification and the dynamic response of building. The damage cost was estimated as a function of the nonlinear response of the building to simulated ground motions. Based on 100 simulations of the seismic response, the expected cost of damage to the house per earthquake event was estimated as $\mu_C = \$75,000$. COV of the damage cost is assumed in this paper as 0.1, since the original study did not give any specific value. By installing a base isolation system, simulations showed that the expected damage cost can be significantly reduced to $5,000. The COV of C is still assumed to be unchanged from 0.1.

The key objective of the life cycle analysis is to examine if it is worth installing the base isolation system for a 50 year service life of the house. Takahashi et al. [9] proposed the expected discounted cost as a basis for decision making. However, their analysis was considerably simplified by assuming that only one seismic event could occur in the service life of the structure. This assumption nullifies the need for a renewal process model, and the problem can be analyzed as a “first failure” problem.

The present analysis begins with the computation of the renewal function associated with the BPT distribution for the inter-occurrence time of characteristic earthquakes. The expected occurrence (or renewal) rate of occurrence of earthquakes is shown in Fig. 8. This was computed using the renewal Eq. (8). Moments of

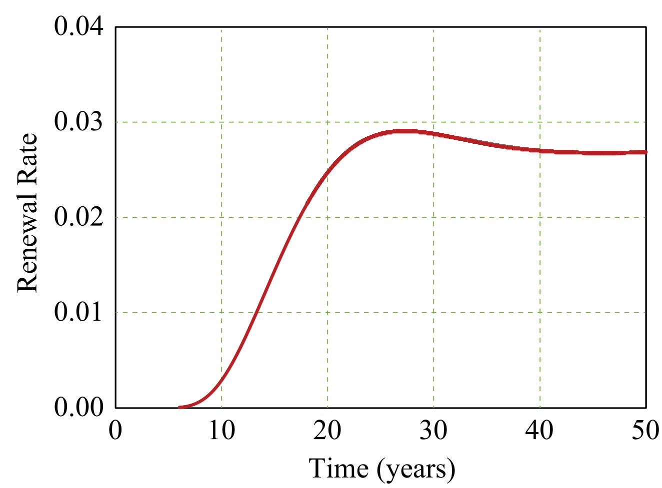

Fig. 8. The rate of earthquake occurrences: BPT distribution.

Table 3
Life-cycle damage cost estimates for a wood frame house.

Case	Without discounting Mean ($)	SD ($)	With Mean ($)	Discounting SD ($)
	Renewal process-BPT model			
Without base isolation	72,974	46,304	16,804	13,559
With base isolation	4865	3086	1120	904
Expected cost reduction	68,109	–	15,684	
Expected net benefit	+42,109	–	−10,316	
	HPP Model			
Without base isolation	101,078	87,502	37,112	39,000
With base isolation	6738	5833	2474	2600
Expected cost reduction	94,340	–	34,638	
Expected net benefit	+68,340	–	+8638	

the life cycle damage cost were calculated using the formulas given in Table 1.

Numerical results for life cycle damage cost, with and without discounting, are presented in Table 3. First consider the non-discounted cost of seismic damage over a 50 year period. The expected cost without the base isolation is calculated as $72,974 with a standard deviation of $46,304. With the base isolation system, the expected cost damage cost is reduced to $4865 with a standard deviation of $3,086. The expected reduction in damage cost is thus $68,109. The net benefit, after deducting the cost of base isolation system of $26,000, turns out to be $42,109. Based on expected cost analysis, it is beneficial to instal the base isolation system in the house. It is however important to recognize that large variability associated with the damage cost may preclude the realization of the projected benefit.

Based on an annual interest rate of 5%, the expected discounted cost was calculated as $16,804 when the house has no base isolation system. With base isolation, the expected cost reduced to $1120. Since the installation cost of $26,000 exceeds the benefit, i.e., a reduction in the damage cost ($ 15,684), the base isolation is not a cost-effective solution.

If the damage cost were calculated assuming that earthquake occurrences follow the homogeneous Poisson process model, the results turn out to be significantly different from those obtained using BPT renewal process model (see Table 3).. The expected cost of damage and the net benefit of base isolation are estimated as $101,100 and $+68,300, respectively. Using the discount rate of 5%, the expected cost and the net benefit turn out to be $37,100 and +$8,600. Thus, the HPP model leads to a conclusion that the proposed base isolation is a beneficial solution.

7. Conclusions

In the life cycle cost analysis, the total cost of damage caused by external hazards over the life cycle of a structure is a fairly uncertain element due to random nature of the time of occurrence and intensity of hazards. In structural engineering, the homogeneous Poisson process is widely used to model an external hazard, such as earthquakes. Under this model, simple analytical expressions for the mean and variance of damage cost can be derived. However, these results are not useful in cases where the inter-occurrence time deviates from the exponential distribution.

The paper presents a systematic development of a more general stochastic process model of the damage cost analysis in which occurrences of a hazard and its cost consequences are conceptually modelled as a marked renewal process. In this approach, the life cycle damage cost turns out to be a compound stochastic renewal process. Based on this model, the paper derives formulas for the mean and variance of discounted and non-discounted cost in a specified service life of a structure or any other system.

The proposed solutions bring the following new elements in the life cycle cost analysis::

- In the engineering literature, the expected discounted cost is typically calculated using an asymptotic solution, i.e., when the service life approaches infinity. This is unrealistic in cases where the financial planning is done for a finite service life of the structure. This paper presents all the solutions for finite service life which can also be extended to obtain the asymptotic solutions.
- The variance of the discounted cost in a renewal process model is presented in the paper, which is not available in the literature.
- In cases where the damage cost per event (C) depends on the inter-occurrence time (T), the mean and mean-squares of the life cycle damage cost can be computed only through renewal-type integral equations. These integral equations are derived in the paper, which are also unavailable in the current literature. Note that a dependence between C and T is introduced when a preventive maintenance policy is introduced in the analysis.
- The proposed solution approach based on a concept of the renewal decomposition is more versatile than the Laplace transform approach, which is traditionally used to write the solution of an integral equation in spite of the fact that inversion of the Laplace transform in a general setting is impractical.

The examples presented in the paper highlight the importance of considering the standard deviation of life cycle cost in the decision making, because its value tends to be of the same order of magnitude as the mean. Therefore, a decision solely based on expected cost is likely to be unrealistic in practical cases. This result motivates the application of more advanced concepts of decision theory.

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Appendix A. Analytical derivations: *C* independent of *T*

In this Section, formulas for the mean and mean-squares of the life cycle damage cost, $K(t)$, are derived when the damage cost per event, C, is independent of the inter-occurrence time, T.

The Appendix first derives the first two moments of the discounted cost. From these results, formulas for the moments of non-discounted cost are obtained as a special case with unit discount function.

A.1. Discounted cost

A.1.1. Expected cost

The life cycle damage cost, as given by Eq. (27), is rewritten in terms of a more general discounting function $h(S_i)$ as

$$K_D(t) = \sum_{i=1}^{\infty} C_i h(S_i) \mathbf{1}_{\{S_i \leqslant t\}} \tag{A.1}$$

Note that in case of exponential discounting, $h(S_i) = e^{-\rho S_i}$, $\rho > 0$. Since C and T are independent, the expected cost can be directly written as

$$\mathbb{E}[K_D(t)] = \mathbb{E}[C_i] \sum_{i=1}^{\infty} \mathbb{E}\left[h(S_i)\mathbf{1}_{\{S_i \leqslant t\}}\right] = \mu_C \sum_{i=1}^{\infty} \int_0^t h(x)\,\mathrm{d}F_{S_i}(x) \tag{A.2}$$

The following partial integration

$$\int_0^t h(x)\,\mathrm{d}F_{S_i}(x) = h(t)F_{S_i}(t) - \int_0^t F_{S_i}(x)\,\mathrm{d}h(x),$$

leads to

$$\sum_{i=1}^{\infty} \int_0^t h(x)\,\mathrm{d}F_{S_i}(x) = h(t)\Lambda(t) - \int_0^t \Lambda(x)\,\mathrm{d}h(x) = \int_0^t h(x)\,\mathrm{d}\Lambda(x),$$

This is result is obtained by interchanging the sum with the integration and using the definition of $\Lambda(t)$, as given by Eq. (7). Thus, the final result is obtained as

$$\mathbb{E}[K_D(t)] = \mu_C \int_0^t h(x)\,\mathrm{d}\Lambda(x) \tag{A.3}$$

It is interesting to note that even for a general discount function the expected discounted cost has a fairly simple analytical form.

A.1.2. Second moment of the discounted cost

To obtain the second moment of the discounted cost, both sides of Eq. (A.1) are squared and written as

$$\begin{aligned} K_D^2(t) &= \left(\sum_{i=1}^{\infty} C_i h(S_i)\mathbf{1}_{\{S_i \leqslant t\}}\right)^2 \\ &= \sum_{i=1}^{\infty} C_i^2 h^2(S_i)\mathbf{1}_{\{S_i \leqslant t\}} + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} C_i C_j h(S_i)\mathbf{1}_{\{S_i \leqslant t\}} h(S_j)\mathbf{1}_{\{S_j \leqslant t\}} \end{aligned} \tag{A.4}$$

Following the steps given in the previous Section, the expected value of the first term in the righthand side of (A.4) can be evaluated in a straightforward manner as

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{\infty} C_i^2 h^2(S_i)\mathbf{1}_{\{S_i \leqslant t\}}\right] &= \mathbb{E}\left[C^2\right]\mathbb{E}\left[\sum_{i=1}^{\infty} h^2(S_i)\mathbf{1}_{\{S_i \leqslant t\}}\right] \\ &= \mu_{2C}\int_0^t h^2(x)\,\mathrm{d}\Lambda(x) \end{aligned}$$

The expectation of the second term requires additional efforts as shown by the following steps. For $j > i$

$$\begin{aligned} &\mathbb{E}\left[C_i C_j h(S_i)\mathbf{1}_{\{S_i \leqslant t\}} h(S_j)\mathbf{1}_{\{S_j \leqslant t\}}\right] \\ &= \mu_C^2 \mathbb{E}\left[h(S_i)\mathbf{1}_{\{S_i \leqslant t\}} h(S_i + T_{i+1} + \cdots + T_j)\mathbf{1}_{\{S_i + T_{i+1} + \cdots + T_j \leqslant t\}}\right] \\ &= \mu_C^2 \int_0^t h(x)\mathbb{E}\left[h(x + T_{i+1} + \cdots + T_j)\mathbf{1}_{\{x + T_{i+1} + \cdots + T_j \leqslant t\}}\right]\mathrm{d}F_{S_i}(x), \end{aligned}$$

Since $T_{i+1} + \cdots + T_j \overset{d}{=} S_{j-i}$, it follows that for $x < t$

$$\mathbb{E}\left[h(x + T_{i+1} + \cdots + T_j)\mathbf{1}_{\{x + T_{i+1} + \cdots + T_j \leqslant t\}}\right] = \int_0^{t-x} h(x+y)\,\mathrm{d}F_{S_{j-i}}(y),$$

so

$$\begin{aligned} &\mathbb{E}\left[\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} C_i C_j h(S_i)\mathbf{1}_{\{S_i \leqslant t\}} h(S_j)\mathbf{1}_{\{S_j \leqslant t\}}\right] \\ &= \mu_C^2 \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} \int_0^t h(x)\int_0^{t-x} h(x+y)\,\mathrm{d}F_{S_{j-i}}(y)\,\mathrm{d}F_{S_i}(x). \end{aligned}$$

Since

$$\begin{aligned} \sum_{j=i+1}^{\infty} \int_0^{t-x} h(x+y)\,\mathrm{d}F_{S_{j-i}}(y) &= \sum_{k=1}^{\infty} \int_0^{t-x} h(x+y)\,\mathrm{d}F_{S_k}(y) \\ &= \int_0^{t-x} h(x+y)\,\mathrm{d}\Lambda(y), \end{aligned}$$

it follows that

$$\begin{aligned} &\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} \int_0^x h(x)\int_0^{t-x} h(x+y)\,\mathrm{d}F_{S_{j-i}}(y)\,\mathrm{d}F_{S_i}(x) \\ &= \sum_{i=1}^{\infty} \int_0^x h(x)\left[\int_0^{t-x} h(x+y)\,\mathrm{d}\Lambda(y)\right]\mathrm{d}F_{S_i}(x) \\ &= \int_0^t h(x)\left[\int_0^{t-x} h(x+y)\,\mathrm{d}\Lambda(y)\right]\mathrm{d}\Lambda(x). \end{aligned}$$

Summing up, the final result is obtained as

$$\mathbb{E}\left[K_D(t)^2\right] = \mu_{2C}\int_0^t h^2(x)\,\mathrm{d}\Lambda(x) + 2\mu_C^2 \int_0^t h(x)\left[\int_0^{t-x} h(x+y)\,\mathrm{d}\Lambda(y)\right]\mathrm{d}\Lambda(x). \tag{A.5}$$

In case of the exponential discounting function, the above formula can be simplified to

$$\mathbb{E}\left[K_D^2(t)\right] = \mu_{2C}\int_0^t e^{-2\rho x}\mathrm{d}\Lambda(x) + 2(\mu_C)^2 \int_0^t e^{-2\rho x}\left[\int_0^{t-x} e^{-\rho y}\mathrm{d}\Lambda(y)\right]\mathrm{d}\Lambda(x). \tag{A.6}$$

Using Eq. (A.3), the above expression can be rewritten in terms of the expected cost as

$$\mathbb{E}\left[K_D^2(t)\right] = \mu_{2C}\int_0^t e^{-2\rho x}\mathrm{d}\Lambda(x) + 2\mu_C \int_0^t e^{-2\rho x}\mathbb{E}[K_D(t-x)]\mathrm{d}\Lambda(x) \tag{A.7}$$

A.2. Non-discounted cost

Formulas for non-discounted cost can be obtained from the formulas for discounted cost by simply setting $h(x) = 1$. Thus, the expected cost can be directly obtained from Eq. (A.3) as

$$\mathbb{E}[K(t)] = \mu_C \Lambda(t). \tag{A.8}$$

Eq. (A.8) is a standard result for the mean of a compound renewal process ([22]). The result for the second moment of the cost can be directly written using Eq. (A.5) as

$$\begin{aligned} \mathbb{E}\left[K^2(t)\right] &= \mu_{2C}\Lambda(t) + 2(\mu_C)^2 \int_0^t \mathrm{d}\Lambda(x)\int_0^{t-x} \mathrm{d}\Lambda(y) \\ &= \mu_{2C}\Lambda(t) + 2(\mu_C)^2 \int_0^t \Lambda(t-x)\mathrm{d}\Lambda(x) \end{aligned} \tag{A.9}$$

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