EXPERIMENTS ON SHEAR TRANSFER IN CRACKS IN CONCRETE
PART II: ANALYSIS OF RESULTS

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Experiments on shear transfer in cracks in concrete

Part II: Analysis of results

Reportnr. 5-79-10

Researchnumber 3.1.7602

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Delft, November 1979

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NOTATIONS

All values used in this report are expressed in N, mm and N/mm².

\( a_x \) projection of a contact length in an \( xy \)-plane on the \( x \)-axis
\( a_{xt} \) sum of all values \( a_x \) over a length \( l \) of the crack
\( a_y \) projection of a contact length in an \( xy \)-plane on the \( y \)-axis
\( a_{yt} \) sum of all values \( a_y \) over a length \( l \) of the crack
\( f'_cc \) average cube crushing strength
\( f'_cp \) average prism crushing strength
\( f_R \) related rib area, characterizing the profiling of a reinforcing bar
\( p \) percentage of weight passing a sieve
\( p_k \) probability that an arbitrary point in the concrete lies within an aggregate particle
\( u \) embedment depth of a particle
\( u_{max} \) maximum embedment depth of a particle for which contact is obtained
\( w \) crack width
\( w_o \) initial crack width
\( A_c \) total area of the crack plane
\( A_x \) projection of the total contact area of a unit area of the crack plane on an \( x \)-plane
\( A_y \) projection of the total contact area of a unit area of the crack plane on an \( y \)-plane
\( D \) diameter of a spherical aggregate particle
\( D_{max} \) maximum diameter of the particles in a concrete mixture
\( E_c \) modulus of elasticity of concrete
\( E_s \) modulus of elasticity of steel
\( F_d \) dowel force
\( F_{d\theta} \) dowel force parallel to the crack plane for bars, inclined under an angle \( \theta \)

\( F_{iv} \) vertical component of aggregate interlock

\( F_{ih} \) horizontal component of aggregate interlock

\( F_s \) axial steel force

\( R \) radius of an aggregate particle

\( R_{\text{max}} \) maximum radius of aggregate particles in a concrete mixture

\( R_{\text{mc}} \) radius of the smallest aggregate particle providing "maximum contact"

\( \Delta \) shear displacement

\( \mu \) coefficient of friction between matrix and aggregate particles

\( \rho \) reinforcement ratio

\( \rho_o \) reinforcement percentage

\( \sigma \) normal stress

\( \sigma_{pu} \) normal stress at yielding of matrix during sliding of the crack faces

\( \sigma_{sx} \) steel stress at section \( x \)

\( \sigma_{cx} \) concrete stress at section \( x \)

\( \tau \) shear stress

\( \tau_{pu} \) shear stress at crushing of matrix during sliding of the crack faces

\( \phi \) diameter of a reinforcing bar
INTRODUCTION

In Part I a description has been given of the results of an experimental investigation into the behaviour of reinforced and unreinforced cracks subjected to shear loading. In Part II these results are analyzed. Section 2 of this report presents a systematical comparison of the experimental results, so that the influence of variation of parameters can be observed. Section 3 represents a theoretical analysis of the behaviour of cracks in plain concrete: a mathematical model has been developed, based on the actual material properties, which reflects the experimental tendencies with a fair degree of accuracy. This model has been used to explain results of other investigators and to reveal other features of the mechanism by an extended parameter study. The results of section 3 are used in section 4, studying the behaviour of reinforced cracks under shear loading. It is demonstrated how the individual components of aggregate interlock, dowel action and axial steel stress cooperate to ensure equilibrium in the crack and how the experimental results can be predicted by a calculation.
EXPERIMENTAL OBSERVATIONS

As a result of the systematic variation of variables in the test series the influence of a lot of parameters, which were expected to affect the behaviour of cracks under shear loading, could be visualized. These tendencies, emerging from comparisons of experimental results, are represented prior to further analysis.

2.1 The behaviour of cracks, crossed by embedded reinforcement

a. The influence of the accidental, global, roughness of the cracks

Comparing the behaviour of specimens with the same reinforcement and comparable concrete qualities, it is seen that repetition of tests does apparently not lead to significant differences (fig. 2.1 and 2.2). Both the crack opening path and the development of the shear stress as a function of the shear (and normal) displacement were approximately the same for the specimens compared. Hence the global structure of the crack planes does not lead to a significant scatter for the type of tests carried out.

![Graph](image-url)

Fig. 2.1 Comparison of the behaviour of two similar specimens

110208 : Mix 1, $f'_{cc} = 30.7 \text{ N/mm}^2$, 2 stirrups $\phi$ 8 mm, $w_o = 0.02 \text{ mm}$

110208t: Mix 1, $f'_{cc} = 35.9 \text{ N/mm}^2$, 2 stirrups $\phi$ 8 mm, $w_o = 0.03 \text{ mm}$
Fig. 2.2 Comparison of the behaviour of two similar specimens

110808 : Mix 1, $f'_c = 30.7 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm}$, $w_o = 0.02 \text{ mm}$

110808h : Mix 1, $f'_c = 29.4 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm}$, $w_o = 0.01 \text{ mm}$

b. The influence of the initial crack width

Some specimens were precracked in such a way that the yield stress of the rebars was exceeded and the crack didn't close as far as in the other cases (0.07 and 0.09 mm against 0.01-0.03 mm normally). In fig. 2.3 the identical specimens 110208 and 110208t ($f'_c = 30.7-35.9 \text{ N/mm}^2$ and $w_o = 0.02-0.03 \text{ mm}$) are compared with a third specimen, also with the same properties but with a greater initial crack width ($w_o = 0.09 \text{ mm}$). It is seen that both for the crack opening path and for the $\tau$-$\Delta$ relation the lines of the specimen with the greater initial crack width tend to join the lines of the other specimens. This was also observed comparing on the one hand the specimens 110808 and 110808h ($f'_c = 29.4-30.7 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm}$, $w_o = 0.01-0.02$) and on the other hand the specimen 110808hg ($f'_c = 29.4 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm}$, but $w_o = 0.07 \text{ mm}$) (fig. 2.4).
Fig. 2.3 Influence of the initial crack width $w_o$ on the shear stress - shear displacement relation (a) and on the crack opening path (b)

110208 : Mix 1, $f'_c = 30.7 \text{ N/mm}^2$, 2 stirrups $\phi 8 \text{ mm}, w_o = 0.02 \text{ mm}$
110208t: Mix 1, $f'_c = 35.9 \text{ N/mm}^2$, 2 stirrups $\phi 8 \text{ mm}, w_o = 0.03 \text{ mm}$
110208g: Mix 1, $f'_c = 29.4 \text{ N/mm}^2$, 2 stirrups $\phi 8 \text{ mm}, w_o = 0.09 \text{ mm}$

Fig. 2.4 Influence of initial crack width $w_o$ on the shear stress - shear displacement relation (a) and on the crack opening path (b)

110808 : Mix 1, $f'_c = 30.7 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm}, w_o = 0.02 \text{ mm}$
110808h : Mix 1, $f'_c = 29.4 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm}, w_o = 0.01 \text{ mm}$
110808hg: Mix 1, $f'_c = 29.4 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm}, w_o = 0.07 \text{ mm}$
c. **The behaviour at repeated loading**

A lot of specimens were unloaded and subsequently reloaded. During unloading the crack opening path was generally approximately followed, in backward direction (fig. 2.5 and 2.6). After reloading the crack opening path was followed without discontinuity. In the $\tau$-$\Delta$ diagram the unloading and reloading procedure was characterized by a considerable amount of hysteresis (fig. 2.5a). However, also here the original $\tau$-$\Delta$ line was continued after reloading, as if no influence of load history was felt. This type of behaviour was observed in many cases (Part I, fig. 2.14-2.19).

Some specimens were again reloaded and unloaded after a period of 5 months. The influence of the increase of the concrete strength appeared in the $\tau$-$\Delta$ diagram by a higher value of $\tau$ (fig. 2.6a). However, the crack opening path seemed not to be influenced by the strength of the concrete. Also this type of behaviour was observed in several cases (Part I, fig. 2.19, 25, 26, 29).

![Graph showing shear stress-displacement relation](image)

**Fig. 2.5** The effect of unloading and reloading on the shear stress - shear displacement relation and on the crack opening path of specimen 110208t (mix 1, $f'_{cc} = 35.9$ N/mm$^2$, 2 stirrups $\phi$ 8 mm, $w_o = 0.3$ mm)
Fig. 2.6 The effect of unloading and reloading after short time and after long time (5 months) on the shear stress - shear displacement relation and on the crack opening path of specimen 110408 (mix 1, $f'_{cc} = 30.7 \text{N/mm}^2$, 4 stirrups $8 \text{mm}$ and $w_o = 0.03 \text{mm}$)

**d. The influence of a variation of the number of rebars and their diameter at constant reinforcement ratio**

A number of tests were carried out on specimens with equal reinforcing ratios but a different number of bars and, consequently, different diameters. For the mixes 1 and 2 ($f'_{cc} \approx 30 \text{N/mm}^2$) a specimen with 7 stirrups $6 \text{mm}$ ($\rho_o = 1.10\%$) could be compared with a specimen with 4 stirrups $8 \text{mm}$ ($\rho_o = 1.12\%$). Next for mix 2 a specimen with 8 stirrups $8 \text{mm}$ ($\rho_o = 2.23\%$) could be compared with a specimen with 2 stirrups $16 \text{mm}$ ($\rho_o = 2.23\%$). The results of these comparisons are represented in the figures 2.7-2.9.
Fig. 2.7 The influence of the variation of bar size at constant reinforcement ratio on the shear stress - shear displacement relation (a) and on the crack opening path (b), for:

110408: Mix 1, $f'_{cc} = 29.2 \text{ N/mm}^2$, 4 stirrups $\phi 8 \text{ mm (} \rho_o = 1.12\% )$ and $w_o = 0.03 \text{ mm}$

110706: Mix 1, $f'_{cc} = 31.7 \text{ N/mm}^2$, 7 stirrups $\phi 6 \text{ mm (} \rho_o = 1.10\% )$ and $w_o = 0.02 \text{ mm}$

It appeared that in all the cases slightly higher $\tau$-values were developed by the specimens containing the bars with the smallest diameters. To explain this phenomenon two arguments can be advanced. Firstly in [23, pp. 44-45] it was shown that, for equal reinforcement ratios, a slightly higher dowel resistance is developed. Theoretically this difference is expressed by the ratio $(\phi_2/\phi_1)^{0.25}$ in which $\phi_2$ is the greater and $\phi_1$ the smaller diameter. Consequently, 7 stirrups $\phi 6 \text{ mm}$ give a dowel resistance which is 7% greater than that provided by 4 stirrups $\phi 8 \text{ mm}$, and 8 stirrups $\phi 8 \text{ mm}$ give a 19% greater dowel resistance than 2 stirrups $\phi 16 \text{ mm}$.
Fig. 2.8 The influence of the variation of bar size at constant reinforcement ratio on the shear stress - shear displacement relation (a) and on the crack opening path (b), for:

120408: Mix 2, $f'_{cc} = 29.5 \text{ N/mm}^2$, 4 stirrups $\phi 8 \text{ mm} \left( \rho_o = 1.12\% \right)$ and $w_o = 0.04 \text{ mm}$

120706: Mix 2, $f'_{cc} = 29.2 \text{ N/mm}^2$, 7 stirrups $\phi 6 \text{ mm} \left( \rho_o = 1.10\% \right)$ and $w_o = 0.02 \text{ mm}$

In [23, fig. 3.44] a comparison was made between the contributions of aggregate interlock (based on [17]), and dowel action (based on [16]). In [6] it was shown that an axial force in the reinforcing bars reduces the dowel action. As a rough estimation, for a crack width of 0.25 mm, the dowel action amounts 15% of the shear resistance if the reinforcement ratio is 2.23% and 8% if the reinforcement ratio is 1.12%. Taking these values into account the theoretical difference in total shear stress could only amount 1% in favour of 7 $\phi 6$ and 3% in favour of 8 $\phi 8$ mm.
Fig. 2.9 The influence of the variation of bar size at constant reinforcement ratio on the shear stress - shear displacement relation (a) and on the crack opening path (b), for:

120808: Mix 2, $f'_{cc} = 29.5 \text{ N/mm}^2$, 8 stirrups $\phi 8 \text{ mm} (\rho_o = 2.23\%), w_o = 0.02 \text{ mm}$

120216: Mix 2, $f'_{cc} = 29.2 \text{ N/mm}^2$, 2 stirrups $\phi 16 \text{ mm} (\rho_o = 2.23\%), w_o = 0.03 \text{ mm}$

A second, more probable, explanation is found in the differences in restraint stiffness of the reinforcing bars, since bars with greater diameters exhibit a less favourable bond behaviour. The relation between the pull out forces and the bond slip at the crack have been calculated, based on a method developed by REHM [18] and MARTIN [11], (section 4.2.1). Since the pull out forces can be "translated" to normal compressive stresses on the crack plane and the bond slip at the crack is equal to half the crack width, the relation of the restraint stresses and the crack width can be constructed (section 4.2.1, fig. 4.5). In Fig. 4.5b, c it is seen that, due to less favourable bond behaviour, the restraint...
stresses develop slightly slower in the specimens with 4 stirrups \( \phi 8 \text{ mm} \) and 2 stirrups \( \phi 16 \text{ mm} \) than in the comparable specimens with 7 stirrups \( \phi 6 \text{ mm} \) and 8 stirrups \( \phi 8 \text{ mm} \). Below, in e, it will be shown that there is a significant relation between the shear stresses developed and the restraint stiffness.

Contrary to the shear stress-displacement relation the crack opening path is apparently not influenced by changes in the number of bars and their diameter at constant reinforcing ratio (fig. 2.7-2.9b).

e. The influence of the number of stirrups and the concrete strength \( f'_{cc} \)

The influence of the number of bars and the concrete strength was systematically investigated. Five series of specimens with different concrete qualities were tested, all series consisting of four specimens reinforced with 2, 4, 6 and 8 stirrups \( \phi 8 \text{ mm} \). In this way the influence of the concrete quality at equal reinforcement could be studied in 4 series and the influence of the amount of reinforcement at constant concrete quality in 5 series. This is represented schematically in Table 2.I.

Table 2.I. Scheme of comparable specimens to study the influence of the concrete quality and the amount of reinforcement

<table>
<thead>
<tr>
<th>stirrups</th>
<th>( 2 \phi 8 \text{ mm} )</th>
<th>( 4 \phi 8 \text{ mm} )</th>
<th>( 6 \phi 8 \text{ mm} )</th>
<th>( 8 \phi 8 \text{ mm} )</th>
</tr>
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<tr>
<td>Mix 1: ( f'<em>{cc} = 29.5 \text{ N/mm}^2 ) ( D</em>{\text{max}} = 16 \text{ mm} )</td>
<td></td>
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<td></td>
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<tr>
<td>Mix 2: ( f'<em>{cc} = 29.2 \text{ N/mm}^2 ) ( D</em>{\text{max}} = 16 \text{ mm} )</td>
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<tr>
<td>Mix 3: ( f'<em>{cc} = 56.1 \text{ N/mm}^2 ) ( D</em>{\text{max}} = 16 \text{ mm} )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mix 4: ( f'<em>{cc} = 19.9 \text{ N/mm}^2 ) ( D</em>{\text{max}} = 16 \text{ mm} )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mix 5: ( f'<em>{cc} = 38.2 \text{ N/mm}^2 ) ( D</em>{\text{max}} = 32 \text{ mm} )</td>
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The influence of the amount of reinforcement at constant concrete quality can be observed in the figures 2.10-2.14, representing the \( \tau - \Delta \) relations and the crack opening pathes in the range \( \Delta \) and \( w < 0.5 \text{ mm} \) for the five different concrete mixes. It is not quite clear if the crack opening path is influenced by the amount of reinforcement. For the concrete with the lowest strength (mix 4, \( f'_{cc} = 19.9 \text{ N/mm}^2 \), fig. 2.10b) an increase of \( \Delta \) seems to occur for increasing reinforcement. This tendency is also observed for the series with intermediate strength (mixes 1 and 2, \( f'_{cc} = 29.5 \text{ N/mm}^2 \), fig. 2.11b and 2.13b). For the larger strengths (mixes 5 with \( f'_{cc} = 38.2 \text{ N/mm}^2 \) and 3 with \( f'_{cc} = 56.1 \text{ N/mm}^2 \), fig. 2.12b and 2.14b)

Fig. 2.10 The influence of the number of stirrups on the shear stress - shear displacement relation (a) and on the crack opening path (b) for mix 4, with \( f'_{cc} = 19.9 \text{ N/mm}^2 \), \( D_{max} = 16 \text{ mm} \)

240208: 2 stirrups \( \phi \) 8 mm
240408: 4 stirrups \( \phi \) 8 mm
240608: 6 stirrups \( \phi \) 8 mm
240808: 8 stirrups \( \phi \) 8 mm
no influence is observed. Furtheron it is obvious that an increase of the amount of reinforcement has an important positive influence on the shear strength. Comparing the figures 2.10-2.14 it can be concluded that this influence is greater for greater concrete strengths.

Fig. 2.11 The influence of the number of bars on the shear stress - shear displacement relation (a) and on the crack opening path (b), for mix 1, with $f_{cc}' = 30.7$ N/mm$^2$, $D_{max} = 16$ mm

110208: 2 stirrups $\phi$ 8 mm
110408: 4 stirrups $\phi$ 8 mm
110608: 6 stirrups $\phi$ 8 mm
110808: 8 stirrups $\phi$ 8 mm
Fig. 2.12 The influence of the number of bars on the shear stress - shear displacement relation (a) and on the crack opening path (b), for mix 3, with $f'_cc = 56.1 \text{ N/mm}^2$ and $D_{max} = 16 \text{ mm}$
130208: 2 stirrups $\phi 8 \text{ mm}$; 130408: 4 stirrups $\phi 8 \text{ mm}$;
130608: 6 stirrups $\phi 8 \text{ mm}$; 130808: 8 stirrups $\phi 8 \text{ mm}$

Fig. 2.13 The influence of the number of stirrups on the shear stress - shear displacement relation (a) and on the crack opening path (b), for mix 2 (disc.grad.), with $f'_cc = 29.2 \text{ N/mm}^2$, $D_{max} = 16 \text{ mm}$
120208: 2 stirrups $\phi 8 \text{ mm}$; 120408: 4 stirrups $\phi 8 \text{ mm}$;
120608: 6 stirrups $\phi 8 \text{ mm}$; 120808: 8 stirrups $\phi 8 \text{ mm}$
Fig. 2.14 The influence of the number of stirrups on the shear stress - shear displacement relation (a) and on the crack opening path (b), for mix 5, with $f'_c = 38.2 \, \text{N/mm}^2$, $D_{\text{max}} = 32 \, \text{mm}$

- $250208$: 2 stirrups $\phi 8 \, \text{mm}$
- $250408$: 4 stirrups $\phi 8 \, \text{mm}$
- $250608$: 6 stirrups $\phi 8 \, \text{mm}$
- $250808$: 8 stirrups $\phi 8 \, \text{mm}$

Fig. 2.15 The influence of the number of stirrups on the shear stress - shear displacement relation (a) and the crack opening path (b) for mix 4 ($f'_c = 29.2 \, \text{N/mm}^2$, $D_{\text{max}} = 16 \, \text{mm}$), represented for a great range of displacements.

- $240208$: 2 stirrups $\phi 8 \, \text{mm}$
- $240408$: 4 stirrups $\phi 8 \, \text{mm}$
- $240608$: 6 stirrups $\phi 8 \, \text{mm}$
- $240808$: 8 stirrups $\phi 8 \, \text{mm}$
Fig. 2.16 The influence of the amount of reinforcing steel on the shear stress - shear displacement relation (a) and the crack opening path (b) for mix 1 (fcfc = 29.4–36.6 N/mm², Dmax = 16 mm), represented for a great range of displacements. 110208/110208t/110208g = 2 stirrups ø 8 mm, 210608 = 6 stirrups ø 8 mm, 210216 = 2 stirrups ø 16 mm, 210316 = 3 stirrups ø 16 mm, 110706 = 7 stirrups ø 6 mm and 210204 = 2 stirrups ø 4 mm.

Fig. 2.17 The influence of the amount of reinforcing steel on the shear stress - shear displacement relation (a) and the crack opening path (b) for mix 3 (fc'cc = 56.1 N/mm², Dmax = 16 mm) represented for Δ < 2 mm and w < 1 mm. 230208 = 2 stirrups ø 8 mm, 230408 = 4 stirrups ø 8 mm, 230608 = 6 stirrups ø 8 mm, 230808 = 8 stirrups ø 8 mm.
Fig. 2.18 The influence of the amount of reinforcing steel on the shear stress-shear displacement relation (a) and the crack opening path (b) for mix 2 \( (f'_{cc} = 29.2 \text{ N/mm}^2, D_{\text{max}} = 16 \text{ mm}) \), represented for the range \( w < 1 \text{ mm and } \Delta < 2 \text{ mm} \). 120208 = 2 stirrups \( \phi 8 \text{ mm} \), 120408 = 4 stirrups \( \phi 8 \text{ mm} \), 120608 = 6 stirrups \( \phi 8 \text{ mm} \), 120808 = 8 stirrups \( \phi 8 \text{ mm} \), 120706 = 7 stirrups \( \phi 6 \text{ mm} \).

Fig. 2.19 The influence of the amount of reinforcing steel on the shear stress-shear displacement relation (a) and the crack opening path (b) for mix 5 \( (f'_{cc} = 38.2 \text{ N/mm}^2, D_{\text{max}} = 32 \text{ mm}) \), for values \( \Delta < 2 \text{ mm} \) and \( w < 1 \text{ mm} \). 250208 = 2 stirrups \( \phi 8 \text{ mm} \), 250408 = 4 stirrups \( \phi 8 \text{ mm} \), 250608 = 6 stirrups \( \phi 8 \text{ mm} \), 250808 = 8 stirrups \( \phi 8 \text{ mm} \).
The figures 2.10-2.14 have been repeated in 2.15-2.19 for greater displacement intervals of \( w \) (up to 1 mm) and \( \Delta \) (up to 2 mm), so that also the behaviour after yielding of the reinforcing bars could be studied.

In these figures it can be observed that the slight influence of the reinforcement ratio on the crack opening path for small displacements, if existing, disappears for greater displacements. Apparently all mixes have an own, characteristic crack opening path, independent of the reinforcement ratio. This is most clearly revealed in fig. 2.16 representing the crack opening paths for 12 specimens with reinforcement ratios between \( \rho_o = 0.14\% \) and \( \rho_o = 3.35\% \). Only specimen 210204 (with \( \rho_o = 0.14\% \)) shows a deviating result: all other specimens, with \( 0.56\% < \rho_o < 3.35\% \) follow the same crack opening line. No discontinuity occurs after reaching the top of the \( \tau-\Delta \) diagram. The average displacement lines for the various mixes are compared in fig. 2.20.

![Diagram](image)

**Fig. 2.20** Average crack opening paths for the various mixes
Small differences are observed between the lines of mixes 1, 2, 4 and 5. Mix 3 ($f'_{cc} = 56.1 \text{ N/mm}^2$) exhibits an increase of the shear displacement with regard to the other mixes. This can be attributed to the fact that for this high concrete strength the crack intersects a part of the aggregate particles, resulting in a reduced shear resistance of the crack plane for comparable $w$, $\Delta$ values. This effect is more pronounced in the case of lightweight concrete, in which the crack runs through all lightweight particles.

![Graph showing the influence of differences in concrete mix and strength at constant reinforcement (2 stirrups $\phi$ 8 mm, $\rho_o = 0.56\%$) on the shear strength - shear displacement relation (a) and on the crack opening path (b).](image)

The influence of the concrete type and strength at equal reinforcement (Table 2.1., horizontal lines) is shown in the figures 2.21-2.24 by combining the results of the specimens with equal reinforcement from different series.
Fig. 2.22 The influence of differences in concrete mix and strength at constant reinforcement (4 stirrups $\phi$ 8 mm, $\rho_o = 1.12\%$) on the shear strength - shear displacement relation (a) and on the crack opening path (b).

Fig. 2.23 The influence of differences in concrete mix and strength at constant reinforcement (6 stirrups $\phi$ 8 mm, $\rho_o = 1.68\%$) on the shear strength - shear displacement relation (a) and on the crack opening path (b).
The relation between the shear strength and the concrete strength is obvious (fig. 2.21a-2.24a). In the figures 2.21b-2.24b it is confirmed that all concrete types have their own, particular crack opening path (compare with fig. 2.15-2.19b).

f. The influence of variations in minor roughness of the crack faces

Experimental results, described in [12], indicated that the sand fractions of concrete mixtures could play a dominant role in the behaviour of reinforced cracks, subjected to shear forces. In order to investigate whether variations in the distribution of the sand fractions would work out on the behaviour, a special, gap graded concrete was designed. The aggregate sieve line of this mixture was discontinuous in this respect, that all particles with sizes
Fig. 2.25 Comparison of the influence of the elimination of the sand fraction between 0.25-1.00 mm on the behaviour of a reinforced crack subjected to a shear force (both mixes: $f' = 29-30$ N/mm$^2$)

a. Comparison of sieve lines

b. Comparison of shear stress - shear displacement relations for various amounts of reinforcement (2, 4, 6 and 8 stirrups $\phi$ 8 mm)

c. Comparison of crack opening pathes
between 0.25 and 1.00 mm were excluded, while quartz powder was added to obtain a feasible mixture. The cube crushing strength of the specimens, made with this type of concrete (type 2), was 29.5 N/mm², so that they could immediately be compared with the specimens made of a concrete with a continuous sieve line (type 1), and a cube crushing strength of 30.7 N/mm². The sieve lines of the aggregate are compared in fig. 2.25a. The shear stress - shear displacement curves and the crack opening pathes of both series are compared in fig. 2.25b and c. As can be seen, no significant differences can be observed.

g. The influence of the scale of the aggregate

Due to variations in concrete strength only two specimens were available with comparable concrete strength, the same reinforcement and different $D_{\text{max}}$. For these specimens (210608 with 6 stirrups $\phi$ 8 mm, $f'_{\text{cc}} = 36.6$ N/mm², $D_{\text{max}} = 16$ mm and 250608 with 6 stirrups $\phi$ 8 mm, $f'_{\text{cc}} = 38.2$ N/mm², $D_{\text{max}} = 32$ mm) the shear stress - shear displacement curves and the crack opening pathes are compared in fig. 2.26a and b. Apparently, the scale of the aggregate has hardly any influence for the range considered.

Fig. 2.26 The influence of the scale of the aggregate on the shear stress - shear displacement relation (a) and on the crack opening path (b) for two specimens with the same amount of reinforcement and comparable concrete strength. 210608 = 6 stirrups $\phi$ 8 mm, $f'_{\text{cc}} = 36.6$ N/mm², $D_{\text{max}} = 16$ mm, 250608 = 6 stirrups $\phi$ 8 mm, $f'_{\text{cc}} = 38.2$ N/mm², $D_{\text{max}} = 32$ mm
Fig. 2.27 The influence of the inclination of the reinforcement (2 stirrups $\phi$ 8 mm) on the shear stress - shear displacement relation (a) and on the crack opening path (b) at constant concrete quality (mixture 1)

21.0.45 : $f'_{cc} = 34.2 \text{ N/mm}^2$, $\theta = 45^0$
21.068 : $f'_{cc} = 34.2 \text{ N/mm}^2$, $\theta = 68^0$
11.090 \text{1)} : $f'_{cc} = 32.0 \text{ N/mm}^2$, $\theta = 90^0$
21.112 : $f'_{cc} = 34.2 \text{ N/mm}^2$, $\theta = 112^0$
21.135 : $f'_{cc} = 34.2 \text{ N/mm}^2$, $\theta = 135^0$

\text{1)} average of tests 110208, 110208t and 110208g
h. The influence of the inclination of the reinforcement to the crack plane

To study the effect of an inclination of the reinforcement to the crack plane a series was designed in which 8 specimens contained all 2 stirrups ø 8 mm, arranged at varying inclinations. Unfortunately in 4 tests deviations may have occurred by malfunction of the top hinge of the loading equipment. The remaining 4 tests have been combined with the results of other tests on specimens with the same concrete quality but reinforcement perpendicular to the crack plane. Therefore the results of the tests on specimens 110208, 110208t and 110208g have been combined to average lines, denoted with the code of a fictitious specimen 11.090. Fig. 2.27a shows the shear stress - shear displacement relations for the various inclinations. Fig. 2.27b shows the crack opening pathes. It is seen that the crack opening pathes are the same for all specimens except the one with $\theta = 135^\circ$. In fig. 2.27a it is seen that the effectivity of the reinforcement increases with decreasing value of $\theta$.

2.2 The behaviour of the specimens with external restraint bars

The results of the tests on specimens with external restraint bars have been represented in Part I, fig. 3.3-3.8. The most striking difference with the results of the tests on the specimens with embedded bars was found in the dependency of the crack opening path on the external restraint stiffness. Whereas in the case of embedded bars great variation of restraint stiffness was not found to have a significant influence on the crack opening path, for specimens with external restraint bars even small variations in restraint stiffness were felt. Fig. 2.28 shows the influence of the stiffness of the external restraint bars (solid lines - fig. 2.28c) on the crack opening path (solid lines - fig. 2.28a and b). These lines are compared with those of the specimens with embedded bars (dotted lines). These lines have been calculated with a method, presented in section 4.2.1 and Appendix III. Even for much higher restraint stiffness values the crack opening path for specimens with embedded bars is less steep than that of the specimens with external restraint bars.
Fig. 2.28 The influence of the restraint stiffness of the external reinforcing bars (solid lines in c) on the crack opening path (solid lines in b and c). Comparison with the results of specimens with embedded bars (2, 4 and 6 stirrups $\phi$ 8 mm - dotted lines). All specimens: mix 1 ($f'_{cc}$ = 30-36 N/mm$^2$).

Code of specimens with external bars: mix no./initial crack width (mm)/restraint stress at $w = 0.6$ mm. The restraint stiffness of the embedded bars was calculated according to a method, presented in section 4.2.1.

It seems improbable that this is due to dowel action: although dowel action could increase the resistance against external shear forces it does not level the differences in restraint stiffness normal to the crack plane (see also MEHLHORN [13], and [23] pp. 45-47).
So, in spite of dowel action these differences ought to cause different values of $A$ at constant $w$, or, different crack opening pathes. Another mechanism must be responsible for this phenomenon.

2.3 The behaviour of the specimens with embedded reinforcing bars, provided with soft sleeves over a short length besides the crack area

One additional series was made, consisting of 4 specimens with 2, 4, 6 and 8 stirrups $\& 8$ mm, mix 1, $f'_c = 36.1$ N/mm$^2$. These specimens differed from the remainder of the specimens with embedded bars in that soft sleeves 40 mm long and about 3 mm thick were secured around the legs of the stirrups where they crossed the shear plane. In this way two effects were eliminated:

- dowel action
- the local deterioration of the concrete, where the bars cross the crack. This local deterioration is caused by the effects of the extraction of the bars, attended with the creation of micro-cracks in the vicinity of the crack plane, and in a later stage by the compression stresses under the bars due to dowel action.

The crack opening pathes of the specimens of this series are represented in fig. 2.29a (dotted lines). It is seen that now an increase of restraint stiffness results in a steeper crack opening path, which is comparable with the behaviour of specimens with external restraint rods. A comparison is made with two specimens, the external restraint stiffness of which is of the same order, and which have the same concrete type and strength (solid lines in fig. 2.29a-c). The restraint stiffness of both groups of specimens is compared in fig. 2.29. (The dotted lines in fig. 2.29c are calculated in section 4.3, fig. 4.13). For values of the crack width $w < 0.5$ mm, $1/0/7.8$ approximates $310608$, and $1/0/3.6$ lies (for $w > 0.1$ mm) between $310208$ and $310408$. Comparing the diagrams of fig. 2.29a and b, it can be concluded that the arrangement of the curves is in good agreement with what could be expected on basis of fig. 2.29c.

Hence, it is probable that the difference in behaviour between specimens with external restraint bars and embedded reinforcing bars is caused by an additional mechanism of shear transfer, located at the places where the bars cross the crack. This will be discussed more in detail in section 4.3.
Fig. 2.29  Comparison between the results of the tests on specimens with reinforcing bars covered with sleeves over a short length, and two specimens with external restraint rods:

a. crack opening path

b. shear stress - shear displacement diagram

c. restraint stiffness

310208 : Mix 1, $f'_{cc} = 36.1 \text{ N/mm}^2$, 2 stirrups $\phi 8$ mm, sleeves
310408 : Mix 1, $f'_{cc} = 36.1 \text{ N/mm}^2$, 4 stirrups $\phi 8$ mm, sleeves
310608 : Mix 1, $f'_{cc} = 36.1 \text{ N/mm}^2$, 6 stirrups $\phi 8$ mm, sleeves
310808 : Mix 1, $f'_{cc} = 36.1 \text{ N/mm}^2$, 8 stirrups $\phi 8$ mm, sleeves
1/0/3.6: Mix 1, $f'_{cc} = 36.7 \text{ N/mm}^2$, restraint stiffness $\sigma_w = 0.6 \text{ mm} = 3.6 \text{ N/mm}^2$
1/0/7.6: Mix 1, $f'_{cc} = 38.5 \text{ N/mm}^2$, restraint stiffness $\sigma_w = 0.6 \text{ mm} = 7.6 \text{ N/mm}^2$
3 ANALYSIS OF SHEAR TRANSFER IN CRACKS IN PLAIN CONCRETE

3.1 Basic assumptions

To explain certain properties of concrete, this material is sometimes represented as a two-phase system; in a matrix (phase I) a collection of aggregate particles (phase II) are embedded. Commonly not only the hardened cement paste, but also the fine aggregate particles are considered to form part of the matrix: anyhow the limit, up to which the smaller particles are attributed to the matrix, for instance 0.25 mm, is rather arbitrary. The properties of the matrix are however not very much influenced by the choice of this value, but are mainly governed by the properties of the hardened cement paste [25].

Matrix and aggregate particles differ generally in strength and stiffness. The contact area between both materials, the bond zone, is the weakest link of the system. Hence cracking occurs commonly through the matrix, but along the circumference of the aggregate particles. Only in the case of high-strength concrete (with high matrix strength) and lightweight concrete (with low particle strength) cracks are observed running both through the matrix and the particles. Generally crack faces are encountered with a structure as indicated in fig. 3.1.

![Fig. 3.1 Structure of a crack plane in concrete for intermediate or low strength](image)

Natural aggregate particles have an irregular shape. It is assumed that these particles are randomly orientated, so that no directions of preference exist. For the mathematical model the particles are simplified to spheres, for which it is supposed that they can be
intersected by the crack plane on all depths with the same probability. Furtheron the crack plane is supposed to be a flat plane. This last assumption seems to be admissible since repetition of tests showed that the accidental global crack plane has no significant influence on the behaviour.

If there is a certain crack width and the crack interfaces are subjected to a shear displacement, the faces will get in touch with each other at a number of points, which will develop to contact areas due to yielding of the matrix (fig. 3.2).

![Fig. 3.2 Development of a contact area between matrix and aggregate](image)

If the shear load on the plane of cracking is increased and crack opening provokes a reaction normal to the plane, for instance due to reinforcement, the following mechanism will occur. The contact areas initially tend to slide; as a result of this sliding the contact area is reduced which results in too high contact stresses: hence new yielding occurs, until equilibrium of forces is obtained in parallel and normal direction.

Hardened cement paste is a visco-elastic material: the plastic deformations dominate the elastic deformations. Therefore the stress-strain relation of the matrix material is assumed to be rigid-plastic, as represented in fig. 3.3.

![Fig. 3.3 Rigid-plastic stress-strain relation of the matrix material](image)
The stress at which yielding occurs is denoted $\sigma_{pu}$. In multiaxial stress situations this value depends on the stresses in other directions and only combinations of stresses may be considered when failure is discussed. However, the contact areas are about to slide which implicates that always the maximum amount of friction is developed so that a constant maximum shear stress and a constant $\sigma_{pu}$ value will be present. Therefore the equilibrium conditions are formulated based on a uniform critical stress combination $(\sigma_{pu}, \tau_{pu})$ with $\tau_{pu} = \mu \sigma_{pu}$.

The equilibrium at the contact area can be described using fig. 3.4.

![Fig. 3.4a, b Equilibrium conditions at the contact area](image)

The radial stresses $\sigma_{pu}$ are composed to a force $F_{\sigma} = 2 \sigma_{pu} R \sin \theta$.

The vertical component of this force is $F_{\sigma v} = 2 \sigma_{pu} R \sin \theta \sin \alpha$ and the horizontal component is $F_{\sigma h} = 2 \sigma_{pu} R \sin \theta \cos \alpha$. The stresses $\tau_{pu}$ are composed to a force $F_{\tau} = 2 \tau_{pu} R \sin \theta$. The vertical component of this force is $F_{\tau v} = 2 \tau_{pu} R \sin \theta \cos \alpha$ and the horizontal component is $F_{\tau h} = 2 \tau_{pu} R \sin \theta \sin \alpha$.

The horizontal projection of the contact area is equal to $a_x = 2 R \sin \theta \sin \alpha$ and the vertical projection is $a_y = 2 R \sin \theta \cos \alpha$.

The total resulting vertical reaction can then be formulated as $F_v = \sigma_{pu} a_x - \tau_{pu} a_y$ and the total horizontal reaction as $F_h = \sigma_{pu} a_y + \tau_{pu} a_x$ (fig. 3.4b).
Since \( \tau_{pu} = \mu \cdot \sigma_{pu} \) this is simplified to the final relations

\[
F_v = \sigma_{pu} (a_x - \mu a_y)
\]

\[
F_h = \sigma_{pu} (a_y + \mu a_x)
\]

The total resistance of the crack plane is the sum of all particle contributions:

\[
\Sigma F_v = \sigma_{pu} (\Sigma a_x - \mu \Sigma a_y)
\]

\[
\Sigma F_h = \sigma_{pu} (\Sigma a_y + \mu \Sigma a_x)
\]

The values \( \sigma_{pu} \) and \( \mu \) are material constants, whereas the values \( a_x \) and \( a_y \) have to be calculated. Aspects that have to be taken into account into this calculation are the variation of particle diameters, depending on the mix-composition, and the possibility that every particle can be intersected by the crack plane at various levels.

Due to the structure of the crack plane the phenomenon of shear transfer in a crack is actually three-dimensional. It is assumed that the total projection of the contact areas in positive and negative z-direction (fig. 3.5a) are equal, so that equilibrium in z-direction is automatically guaranteed and no torsional effects have to be taken into account.
For an arbitrary xy-plane the probability density function for the distribution of the diameters of the intersection circles will be derived. With this equation for every diameter the probability of being crossed by the crack plane is derived. Then it is possible to establish the mathematical expectation of the contribution of an arbitrary circle to the total contact area both in x- and y-direction, for a unit width of the xy-section (fig. 3.5c). By integration over the variation interval of the intersection circle diameters, the expected value of the sum of all contact areas in x- and y-direction for a crack of a length 1 and a width 1 can then be obtained, and as a result the restraint stresses for a unit crack area, by using eq. (3.2).

3.2 Determination of the distribution of intersected circles in a crack in an xy-plane

To be able to calculate the distribution of intersected circles, an assumption has to be made for the distribution of aggregate particles in the mixture. For this distribution a Fuller-curve is adopted. This curve results in a gradation of aggregate particles which leads to an optimum density and strength and is therefore often used in practice. Other reasons for this choice are that the Fuller-curve is described by a simple and handy mathematical expression, and that the concrete for the experimental part of this investigation was designed according to a Fuller-curve, so that a direct comparison between theory and experiments could be made.

The Fuller-curve is expressed with the relation:

\[ p = 100 \sqrt[3]{\frac{D}{D_{\text{max}}}} \]  

(3.3)

in which \( p \) denotes the percentage of weight passing a sieve with opening diameter \( D \). \( D_{\text{max}} \) is the diameter of the greatest aggregate particle.

This relation is used as the cumulative distribution function of spherical aggregate particles with a diameter \( D \) (fig. 3.6).
The probability than an arbitrary point in the concrete is situated within an aggregate particle is denoted with \( p = p_k \). Actually \( p_k \) is the ratio between the total volume of the aggregate and the concrete volume. The probability that a point, if situated within a particle, is also situated in a particle with a diameter smaller than an arbitrary choosen value \( D \), can be expressed using eq. (3.3) as:

\[
p(D < D) = p_k \left( \frac{D}{D_{max}} \right)^{\frac{1}{3}}
\]  

By differentiation of this function the probability density function \( p'(D) \) is obtained:

\[
p'(D) = p_k \frac{3}{D} p(D < D) = \frac{1}{3} p_k \left( \frac{D}{D_{max}} \right)^{-\frac{4}{3}} \cdot \frac{1}{D_{max}} = C D^{-\frac{1}{3}}
\]

with

\[
C = \frac{1}{3} p_k D_{max}^{-\frac{1}{3}}
\]

Subsequently an analysis is made of the probability that an arbitrary point is situated in an intersection circle with a diameter \( D_0 \), if the concrete volume is intersected by a randomly choosen xy-plane. The probability that a point, if situated within a particle with a
diameter $D_0$, also lies in an intersection circle with a diameter $D > D_0$ ($D_0 < D$), is equal to the ratio between the volume of the separated sphere-segment A (fig. 3.7) and the volume of half the sphere.

Fig. 3.7

The volume of the sphere segment $B$ is equal to:

$$ V_B = \frac{1}{6} \pi h (3a^2 + h^2) \quad \text{with} \quad h = R - \sqrt{R^2 - a^2} \quad (3.6) $$

The volume of half the sphere is:

$$ V = \frac{2}{3} \pi R^3 \quad (3.7) $$

The probability that a point within this sphere lies in an intersection circle with $D > D_0$ can be derived from (3.6) and (3.7).

$$ P_D (D > D_0) = \frac{\frac{2}{3} \pi R^3 - \frac{1}{6} \pi (R - \sqrt{R^2 - \frac{1}{4} D_0^2})(3(\frac{1}{2} D_0)^2 + (R - \sqrt{R^2 - \frac{1}{4} D_0^2})}{\frac{2}{3} \pi R^3}} \quad (3.8) $$

Substitution of $R = \frac{1}{2} D$ and elaboration results in:

$$ P_D (D > D_0) = 1 - \frac{D_0^2}{D^2} - \frac{D_0^2}{D^2} \sqrt{1 - \left(\frac{D_0}{D}\right)^2} \quad (3.8) $$
So the probability that an arbitrary point in an xy-plane lies in an intersection circle with a diameter \( D > D_o \) is obtained by integrating the product of (3.5) and (3.8) over the interval \( D_o \to D_{\max} \):

\[
P_c(D > D_o) = \int_{D_o}^{D_{\max}} p'(D^*) \cdot p(D > D_o) \, dD^*
\]

\[
= \int_{D_o}^{D_{\max}} C \cdot D^{-\frac{1}{2}} (1 - \frac{D_o^2}{D^2} - \frac{D_o^2}{D^2} \sqrt{1 - \frac{(D_o^2)^2}{D^2}}) \, dD^*
\]

\[
= \int_{D_o}^{D_{\max}} C \cdot D^{-\frac{1}{2}} \, dD^* - \int_{D_o}^{D_{\max}} C \cdot D_o^{2} \cdot D^{-2.5} \, dD^* - \int_{D_o}^{D_{\max}} \frac{1}{2} C \cdot D_o^{2} \cdot D^{-2.5} \sqrt{1 - \frac{(D_o^2)^2}{D^2}} \, dD^*
\]

**Integral I**

\[
C \int_{D_o}^{D_{\max}} D^{-\frac{1}{2}} \, dD^* = 2 C \cdot D^{\frac{1}{2}} \bigg|_{D_o}^{D_{\max}} = p_k \left( 1 - \frac{D^2}{D_{\max}^2} \right)
\]

**Integral II**

\[
C \int_{D_o}^{D_{\max}} D_o^{2} \cdot D^{-2.5} \, dD^* = C \cdot D_o^{2} \left[ - \frac{2}{3} D^{1.5} \right]_{D_o}^{D_{\max}} = p_k \left( \frac{1}{3} D_o^{\frac{3}{2}} D_{\max}^{-\frac{1}{2}} - \frac{1}{3} D_o^{2} D_{\max}^{-2} \right)
\]

**Integral III**

\[
\int_{D_o}^{D_{\max}} \frac{1}{2} C \cdot D_o^{2} \cdot D^{-2.5} \sqrt{1 - \frac{(D_o^2)^2}{D^2}} \, dD^*
\]
The function \( \sqrt{1 - \left(\frac{D_o}{D_c}\right)^2} \) is written as \( \sqrt{1 + \left(\frac{D_o}{D_c}\right)^2} \) and is then expanded in a Taylor series

\[
\sqrt{1 + \frac{D^2}{D_o^2}} = 1 + \frac{1}{2} \left(\frac{D^2}{D_o^2}\right) - \frac{1}{2 \times 4} \left(\frac{D^2}{D_o^2}\right)^2 + \frac{1}{2 \times 4 \times 6} \left(\frac{D^2}{D_o^2}\right)^3 - \frac{1.3}{2 \times 4 \times 6} \left(\frac{D^2}{D_o^2}\right)^4 + \ldots
\]

\[
= 1 - 0.5 D^2 D_o^{-2} - 0.125 D^4 D_o^{-4} - 0.063 D^6 D_o^{-6} - 0.039 D^8 D_o^{-8} + \ldots
\]

The integral can then be written as:

\[
I_{III} = \frac{1}{4} C D^2 \max \int_{D_o}^{D^\infty} (D^2 - 2.5 D^2 D_o^{-4.5} - 0.125 D^4 D_o^{-6.5} - 0.063 D^6 D_o^{-8.5} - 0.039 D^8 D_o^{-10.5} + \ldots) d D
\]

Integration results in:

\[
I_{III} = \frac{1}{4} p_k D^0 \max \int_{D_o}^{D^\infty} \left[ -0.667 D^0 D_o^{-1.5} + 0.143 D^2 D_o^{-3.5} + 0.023 D^4 D_o^{-5.5} + 0.008 D^6 D_o^{-7.5} + 0.004 D^8 D_o^{-9.5} \right] d D
\]

\[
= p_k \left[ -0.167 D^2 D_o^{-2} + 0.036 D^4 D_o^{-4} + 0.006 D^6 D_o^{-6} + 0.002 D^8 D_o^{-8} + 0.001 D^{10} D_o^{-10} \right]
\]

Combination of I, II and III gives:

\[
p_c (D > D_o) = p_k (1 - 1.455 D^0 D_o^{0.5} D_o^{-0.5} + 0.50 D^2 D_o^{-2} D_o^{0.5} - 0.036 D^4 D_o^{-4} + 0.006 D^6 D_o^{-6} - 0.002 D^8 D_o^{-8} - 0.001 D^{10} D_o^{-10})
\]
Then the cumulative distribution function, representing the probability that an arbitrary point in the concrete body, lying in an xy-intersection plane, is situated in an intersection circle with a diameter $D < D_o$, is

$$p_c(D < D_o) = p_k(1.455 \frac{D^0.5}{D_{\text{max}}^{0.5}} - 0.50 \frac{D^2}{D_{\text{max}}^{-2}} + 0.036 \frac{D^4}{D_{\text{max}}^{-4}}$$

$$+ 0.006 \frac{D^6}{D_{\text{max}}^{-6}} + 0.002 \frac{D^8}{D_{\text{max}}^{-8}} + 0.001 \frac{D^{10}}{D_{\text{max}}^{-10}})$$

(3.9)

This function is graphically represented in fig. 3.8.

---

Fig. 3.8 Cumulative distribution function for the diameter of intersection circles

The density function for the probability that an arbitrary point in the concrete body, lying in a plane of intersection xy, is situated in a circle of intersection with a diameter $D = D_o$ is obtained by differentiation of (3.9) to $D_o$:

$$p_c'(D_o) = p_k(0.727 \frac{D^{-0.5}}{D_{\text{max}}^{0.5}} - D_o \frac{D^{-2}}{D_{\text{max}}^{2}} + 0.144 \frac{D^3}{D_{\text{max}}^{-4}} + 0.036 \frac{D^5}{D_{\text{max}}^{-6}}$$

$$+ 0.016 \frac{D^7}{D_{\text{max}}^{-8}} + 0.010 \frac{D^9}{D_{\text{max}}^{-10}})$$

(3.10)
Since the distribution of circles of intersection in the xy-plane is now known, it is possible to establish the frequency of circles which both lie in the xy-plane and are crossed by the crack (fig. 3.9).

![Diagram of circles of intersection crossed by the crack](image)

Fig. 3.9 Circles of intersection crossed by the crack

The average length of the intersection line AB for a circle with diameter $D_o$, crossed by the crack is established using fig. 3.10.

![Diagram of determination of average length of intersection](image)

Fig. 3.10 Determination of the average length of intersection for a circle with diameter $D_o$

The surface area of the circle is $\frac{1}{4} \pi D_o^2$ and the surface area of the equivalent rectangle is $D_o s$. The average intersection length is then

$$s = \frac{\frac{1}{4} \pi D_o^2}{D_o} = \frac{\pi D_o}{4}$$

(3.11)
If the length of the crack line (intersection of the crack plane and the xy-plane, fig. 3.5c) is denoted 1, the probability density function for the expected length of that part of the line, the points of which form also part of an intersection circle with a diameter $D_o$, is obtained by multiplying 1 with $p'_c(D_o)$ (eq. 3.10), so

$$1(D_o) = p'_c(D_o) \cdot 1 \quad (3.12)$$

Subsequently the probability density function for the expected number of intersection circles with a diameter $D_o$ in the xy-plane, which intersect also the crack line 1 is obtained from (3.11) and (3.12)

$$n(D_o) = \frac{1(D_o)}{\frac{2\pi}{D_o}} = \frac{p'_c(D_o)}{\frac{\pi}{D_o}}$$

Per unit length of the crack line 1, this expression is simplified to:

$$n(D_o) = \frac{p'_c(D_o)}{\frac{\pi}{D_o}} \quad (3.13)$$

### 3.3 Analysis of the contribution of a single intersection circle in the xy-plane to the transfer of stresses in the crack

In the previous part the number of intersection circles with an arbitrary diameter $D_o$, crossed by a unit crack length, has been determined. However, the position of these circles with regard to the plane of cracking has not yet been taken into account. The distance from the crack line to the center of the circle is denoted $u$. This value is assumed to be a random variable within the interval $0 < u < R$. (For $-R < u < 0$ the circle is in the opposite crack face, for which the same considerations are valid).

If $w + u > R$, then for no value of $\Delta$ any contact area can be obtained. If $w + u < R$, then two characteristic values for $\Delta$ can be found. For $\Delta < \Delta_o$ no contact area is found (see also fig. 3.12a). If $\Delta_o < \Delta < \Delta_b$ a contact area is obtained, which increases for increasing value of $\Delta$. 
This phase is denoted as the "growing contact phase" (fig. 3.12b). If \( \Delta > \Delta_b \) no increase of contact area by further shear displacement can be obtained. This phase is denoted as the "maximum contact phase" (fig. 3.12c).

\[
\Delta_b < \Delta < \Delta_o = \sqrt{R^2 - u^2} - \sqrt{R^2 - (u+w)^2}
\]  

(3.14)

**Fig. 3.11** Position of intersection circle characterized by \( u \), randomly varying between 0 and \( R \)

**Fig. 3.12** Different phases for \( w + u < R \)

- a. Phase no contact \( 0 < \Delta < \Delta_o \)
- b. Phase growing contact \( \Delta_o < \Delta < \Delta_b \)
- c. Phase maximum contact \( \Delta_b < \Delta \)
\( \Delta_b \) is obtained by calculating the intersection point of circle II with the line \( y = R \)

\[
x^2 + (R - w)^2 = R^2
\]

so that \( x = -\sqrt{2 R w - w^2} \)

or \( \Delta_b = \sqrt{2 R w - w^2} \) (3.15)

For the interval \( \Delta_0 < \Delta < \Delta_b \) the values \( a_x \) and \( a_y \), characterizing the contact area, can be expressed as a function of \( u, w, \Delta \) and \( R \). This derivation is found in Appendix I;

\[
a_y = \sqrt{R^2 - \frac{1}{2} (w^2 + \Delta^2)} \frac{\Delta}{\sqrt{w^2 + \Delta^2}} - \frac{1}{2} w - u \quad (3.16)
\]

\[
a_x = \frac{1}{2} \Delta - \sqrt{R^2 - \frac{1}{2} (w^2 + \Delta^2)} \frac{w}{\sqrt{w^2 + \Delta^2}} + \sqrt{R^2 - (u + w)^2} \quad (3.17)
\]

For \( \Delta > \Delta_b \) it is easily deduced that

\[
a_y = R - (u + w) \quad (3.18)
\]

\[
a_x = \sqrt{R^2 - (u + w)^2} \quad (3.19)
\]

3.4 Determination of the expected average contribution of an intersection circle with radius \( R \) to the contact lengths \( a_x \) and \( a_y \) for an arbitrary displacement combination \((w, \Delta)\), taking into account a variable embedment depth

In the previous section the contact-parameters \( a_x \) and \( a_y \) have been calculated for a single intersection circle with a radius \( R \), which resulted in expressions containing the variables \( w, \Delta \) and \( u \). In the following considerations an analysis is made of the question: What
is the average contribution of an intersection circle with a radius R to the contact lengths \( a_x \) and \( a_y \) if \( \Delta \) and \( w \) have an arbitrary, constant value? If the answer on this question is known it is possible to find the total projections of the contact lengths \( \Sigma a_x \) and \( \Sigma a_y \) by integrating the contributions of all single circles over the full range of variation of \( R \).

An intersection circle must be taken into account if it gives a contact in its most favourable position. It is evident that the most favourable position is obtained if the embedment depth \( u \) is zero. If a circle even in this extreme position is not in contact with the opposite crack face it may be excluded from the calculation.

The first demand, if contact is requested, is that \( R > w \); if \( R < w \) there is no contact for any value of \( \Delta \), even in the extreme case that \( u = 0 \) (fig. 3.13).

![Fig. 3.13 Intersection circle in the most favourable position (u = 0)](image)

If contact is not excluded in advance, so if \( R > w \), two fundamentally different possibilities must be distinguished.

**Case A: \( \Delta < w \)**

The value \( R_{\min} \), up to which the radius of the intersection circle has to "grow" to provide at least one point of contact (fig. 3.13), can be calculated from equation (3.14). For \( u = 0 \) this equation results in:

\[
R_{\min} = \frac{w^2 + \Delta^2}{2\Delta}
\]  

(3.20)
The value $R_{mc}$, providing a limit value for the range in which "maximum contact" (fig. 3.12c) is encountered is found from equation (3.15)

$$R_{mc} < \frac{w^2 + \Delta^2}{2w} \quad (3.21)$$

This value, however, is only theoretical, since the case A is characterized by $\Delta < w$, for which always $R_{mc} < R_{min}$.

Case B: $\Delta > w$

If $\Delta > w$, contact is guaranteed if $R > w$ (fig. 3.13). The value $R_{mc}$, below which "maximum contact" is found, is deduced in the same way as in the case A, so that $R_{mc}$ is expressed by equation (3.21).

In this case always $R_{mc} > R_{min}$, so that $R_{mc}$ is of practical importance.

The two cases A and B are schematically represented in fig. 3.14.

```
Δ< w  ————("growing contact") ———— Δ> w
(R_{mc} = \frac{w^2 + \Delta^2}{2w})

R_{min} = w

Δ< w  ————("maximum contact") ———— Δ> w
(R_{mc} = \frac{w^2 + \Delta^2}{2w})
```

fig. 3.14 Schematical representation of the fundamental contact modes for varying radius R

The fundamental cases A and B have to be distinguished if the average expected contributions of the intersection circles to the contact lengths are established. This is done in the following derivation.
Case A: $\Delta < w$

If there is any contact area, this is at least the case for the minimum embedment depth $u = 0$. Solutions are found up to an upper bound $u_{\text{max}}$. This value $u_{\text{max}}$ is deduced using fig. 3.15.

![Diagram](image)

Fig. 3.15 Calculation of maximum embedment depth $u_{\text{max}}$ for which still contact exists

For constant values of $\Delta$, $w$ and $R$, the variable $u$ is increased so far that only a single point of contact remains: in that stage $u_{\text{max}}$ is reached. In fig. 3.15 it is seen that

$$x_u = - \sqrt{R^2 - u^2}$$

To fulfil the condition that only one point of contact remains it is sufficient to demand that the point $(x_o, y_o) = (\Delta - \sqrt{R^2 - u^2}, u + w)$ lies on the circle, so

$$(u + w)^2 + (\Delta - \sqrt{R^2 - u^2})^2 = R^2$$

or

$$2uw + (w^2 + \Delta^2) = 2\Delta \sqrt{R^2 - u^2}$$

Quadratation of both members of this expression results in:
\[ u^2(4w^2 + 4\Delta^2) + 4uw(w^2 + \Delta^2) + (w^2 + \Delta^2)^2 - 4\Delta^2R^2 = 0 \]

which leads to:

\[ u_{\text{max}} = \frac{-\frac{1}{2} w(w^2 + \Delta^2) + \frac{1}{4}\sqrt{w^2(w^2 + \Delta^2)^2 - (w^2 + \Delta^2)((w^2 + \Delta^2)^2 - 4\Delta^2R^2)}}{(w^2 + \Delta^2)} \]  

(3.22)

So values for \(a_x\) and \(a_y\) are found for the range

\[ 0 < u < u_{\text{max}} \]

The probability density function for the occurrence of a value \(u\) is assumed to be equal to

\[ p(u) = \frac{1}{R} \]

(3.23)

The expected value for the average contribution of a circle with radius \(R\) to the contact lengths \(a_x\) and \(a_y\) can be formulated, using (3.23), as (fig. 3.16)

\[ a_{xR} = \frac{1}{R} \int_{u=0}^{u=u_{\text{max}}} a_{xR} \, du \]  

(3.24)

and

\[ a_{yR} = \frac{1}{R} \int_{u=0}^{u=u_{\text{max}}} a_{yR} \, du \]  

(3.25)

in which \(a_{yR}\) and \(a_{xR}\) are the contact lengths \(a_y\) and \(a_x\) for a circle with radius \(R\), according to (3.16 and 3.17).
Substitution of (3.16) in (3.24) results in:

\[ a_{yR} = \frac{1}{R} \int_0^{u_{\text{max}}} \left\{ \sqrt{R^2 - \frac{1}{4}(w^2 + \Delta^2)} \cdot \frac{\Delta}{\sqrt{w^2 + \Delta^2}} - \frac{1}{2} w - u \right\} \, du \]

\[ a_{yR} = \frac{1}{R} \int_0^{u_{\text{max}}} \left\{ \sqrt{R^2 - \frac{1}{4}(w^2 + \Delta^2)} \cdot \frac{\Delta}{\sqrt{w^2 + \Delta^2}} \cdot u_{\text{max}} - \frac{1}{2} \frac{w}{u_{\text{max}}} \cdot u_{\text{max}} - \frac{1}{4} \frac{u_{\text{max}}^2}{R} \right\} \, du \quad (3.26) \]

Substitution of (3.17) in (3.25) results in:

\[ a_{xR} = \frac{1}{R} \int_0^{u_{\text{max}}} \left\{ \frac{1}{2} \Delta - \sqrt{R^2 - \frac{1}{4}(w^2 + \Delta^2)} \cdot \frac{w}{\sqrt{w^2 + \Delta^2}} + \sqrt{R^2 - (u + w)^2} \right\} \, du \]

Integration of I results in:

\[ I_I = \left\{ \frac{1}{2} \Delta - \sqrt{R^2 - \frac{1}{4}(w^2 + \Delta^2)} \cdot \frac{w}{\sqrt{w^2 + \Delta^2}} \right\} \cdot u_{\text{max}} \quad (3.27) \]

Integration of II:

\[ I_{II} = \int_0^{u_{\text{max}}} \sqrt{R^2 - (u + w)^2} \, du \quad (3.28) \]

Since \( du = d(u + w) \) (3.28) can be formulated as:

\[ I_{II} = \int_w^{u_{\text{max}}} \sqrt{R^2 - (u + w)^2} \cdot d(u + w) = \int_w^{u_{\text{max}}} \sqrt{R^2 - t^2} \, dt \]

\[ = \left[ \frac{t}{2} \sqrt{R^2 - t^2} + \frac{R^2}{2} \arcsin \frac{t}{R} \right]_w^{u_{\text{max}}} \]

\[ = \frac{u_{\text{max}} + w}{2} \sqrt{R^2 - (u_{\text{max}} + w)^2} - \frac{w}{2} \sqrt{R^2 - w^2} + \frac{R^2}{2} \arcsin \left( \frac{w + u_{\text{max}}}{R} \right) - \frac{R^2}{2} \arcsin w \]
The final expression for $a_{xR}$ is then written as:

$$a_{xR} = \left( \frac{\Delta}{\sqrt{2}} - \frac{w}{\sqrt{w^2 + \Delta^2}} \right) \frac{w}{\sqrt{w^2 + \Delta^2}} \frac{u_{\text{max}} + u_{\text{max}} + w}{2R} \sqrt{R^2 - (w + u_{\text{max}})^2}$$

$$- \frac{w}{2R} \sqrt{R^2 - w^2} + \frac{R}{2} \arcsin \frac{w + u_{\text{max}}}{R} - \frac{R}{2} \arcsin \frac{w}{R} \quad (3.29)$$

**Case B: $\Delta > w$**

In fig. 3.14 it is seen that for $R > (w^2 + \Delta^2)/2w$ the calculation can be carried out in the same way as in case A. For the range

$$w < R < \frac{w^2 + \Delta^2}{2w} \quad (3.30)$$

the "maximum contact" phase is valid. Similarly as in the "growing contact" phase a circle is in contact with the opposite crack face if the embedment depth $u$ is greater than zero and smaller than a certain upper bound. In fig. 3.12c it can easily be seen that this upper bound is obtained for $u = u_{\text{max}} = R - w$.

For values of $R$ in the range indicated in (3.30) $a_{yR}$ is obtained by substituting (3.18) in (3.24):

$$a_{yR} = \frac{1}{R} \int_0^{(R-w)} (R - u - w) du = \frac{1}{2R} (R - w)^2 \quad (3.31)$$

$a_{xR}$ is obtained by substituting (3.19) in (3.25):

$$a_{xR} = \frac{1}{R} \int_0^{R-w} \sqrt{R^2 - (u + w)^2} \, du \quad (3.32)$$

Substituting $t$ for $(u + w)$, eq. (3.32) can be written as:
\[
\bar{a}_{xR} = \frac{1}{R} \int_{-R}^{R} \sqrt{R^2 - t^2} \, dt
\]

so that

\[
\bar{a}_{xR} = \frac{1}{R} \left[ \frac{t}{2} \sqrt{R^2 - t^2} + \frac{R^2}{2} \arcsin \frac{t}{R} \right]_{-R}^{R}
\]

\[
= \frac{\pi}{4} \cdot R - \frac{w}{2R} \sqrt{R^2 - w^2} - \frac{R}{2} \arcsin \frac{w}{R}
\]

(3.33)

For the range

\[ R > \frac{w^2 + \Delta^2}{2w} \]

the formulas (3.26) and (3.29) are valid.

3.5 Determination of the sum of all contact areas in x- and y-direction for a unit area of the crack plane

For a unit length of the crack line (line of intersection of the crack plane and an xy-plane (fig. 3.5c) it was shown that the probability density function for the expected number of circles with a diameter \(D_o\), crossing this length can be expressed by eq. (3.13). The total contact lengths in x- and y-direction, provided by all circles of intersection with a radius \(R\) crossing the unit crack length, can be expressed as:

\[
\Sigma_{a_{yR}} = \int_{R_{\text{min}}}^{R_{\text{max}}} n(R) \bar{a}_{yR} \, dR
\]

(3.34)

\[
\Sigma_{a_{xR}} = \int_{R_{\text{min}}}^{R_{\text{max}}} n(R) \bar{a}_{xR} \, dR
\]

(3.35)

in which \(n(R)\) can be calculated with eq. (3.13) and \(\bar{a}_{yR}\) and \(\bar{a}_{xR}\) can be taken from (3.26) and (3.29) in the case that \(\Delta < w\), or from (3.31) and (3.33) in the case that \(\Delta > w\).
The expected values for the sum of all contact lengths in x- and y-direction for a unit length of the crack line, are obtained by the summation of the contributions of all circles which have such a radius that contact is made, formulated otherwise: by integrating the expressions (3.34) and (3.35) over the full interval of circles making contact;

Case A: \( A < w \)

Contact is obtained if \( R > (w^2 + \Delta^2)/2\Delta \) or \( D > (w^2 + \Delta^2)/\Delta \), therefore:

\[
a_{yt} = \frac{D_{\text{max}}}{w^2 + \Delta^2} \int \frac{n(D) \cdot a_{yD}}{\Delta} dD \tag{3.36}
\]

\[
a_{xt} = \frac{D_{\text{max}}}{w^2 + \Delta^2} \int \frac{n(D) \cdot a_{xD}}{\Delta} dD \tag{3.37}
\]

in which \( n(D) \) is taken from (3.13), \( a_{yD} \) from (3.26) and \( a_{xD} \) from (3.29). \( D_{\text{max}} \) is the diameter of the greatest aggregate particle. No contact is possible if \( D_{\text{max}} < (w^2 + \Delta^2)/\Delta \) or, formulated otherwise, if

\[
\Delta < \frac{1}{2}(D_{\text{max}} - \sqrt{D_{\text{max}}^2 - 4w^2}).
\]

No contact is possible either if \( w > \frac{1}{2} D_{\text{max}} \). The physical background of these conditions is shown in fig. 3.17.

![Fig. 3.17 Minimum value of \( \Delta \) providing contact for the most favourable intersection circle (\( D = D_{\text{max}} \)) and the most favourable embedment depth (\( u = 0 \))](image-url)
Case B: $\Delta > w$

Contact is obtained if $D > 2w$. Two modes of contact are distinguished: "maximum contact" is found for $D < (w^2 + \Delta^2)/w$; growing contact is found for $D > (w^2 + \Delta^2)/w$.

This results in:

$$a_{yt} = \frac{w^2 + \Delta^2}{w} \int_{2w} n(D) \cdot \bar{a}_{yD_1} \cdot dD + \int_0^{\frac{w^2 + \Delta^2}{w}} n(D) \cdot \bar{a}_{yD_2} \cdot dD$$

$$a_{xt} = \frac{w^2 + \Delta^2}{w} \int_{2w} n(D) \cdot \bar{a}_{xD_1} \cdot dD + \int_0^{\frac{w^2 + \Delta^2}{w}} n(D) \cdot \bar{a}_{xD_2} \cdot dD$$

(3.38)

(3.39)

In these equations $n(D)$ is obtained from (3.13), $\bar{a}_{yD_1}$ from (3.31), $\bar{a}_{xD_1}$ from (3.33), $\bar{a}_{yD_2}$ from (3.26) and $\bar{a}_{xD_2}$ from (3.29).

In the formulas (3.36-3.39) expressions are given for the most probable contact lengths in $x$- and $y$-direction for a crack line with a unit length, lying in an xy-plane (fig. 3.5). However, the crack plane can be considered as a collection of an infinite number of parallel lines, all providing similar expected $a_{yt}$ and $a_{xt}$ values. Hence the most probable contact areas for a unit crack area can be simply obtained by multiplying the expressions (3.36-3.39) with a unit crack width.

Substitution of (3.13, 3.26, 3.29, 3.31 and 3.33) in (3.36-3.39) results in the final set of equations, representing the contact areas in $x$- and $y$-direction for a unit crack surface area.

Case A: $\Delta < w$

$$A_y = \int_{\frac{w^2 + \Delta^2}{\Delta}}^{D_{\text{max}}} p_k \cdot \frac{4}{\pi} \cdot F\left(\frac{D}{D_{\text{max}}}\right) \cdot G_1(\Delta, w, D) \cdot dD$$

(3.40)
\[
A_x = \int_{\frac{w^2+\Delta^2}{\Delta}}^{D_{\text{max}}} p_k \cdot \frac{4}{\pi} \cdot F\left(\frac{D}{D_{\text{max}}}\right) \cdot G_2(\Delta, w, D) \cdot dD \\
(3.41)
\]

**Case B: \( \Delta > w \)**

\[
A_y = \int_{\frac{2w}{w^2+\Delta^2}}^{D_{\text{max}}} p_k \cdot \frac{4}{\pi} \cdot F\left(\frac{D}{D_{\text{max}}}\right) \cdot G_3(\Delta, w, D) \cdot dD
\]

\[
A_x = \int_{\frac{2w}{w^2+\Delta^2}}^{D_{\text{max}}} p_k \cdot \frac{4}{\pi} \cdot F\left(\frac{D}{D_{\text{max}}}\right) \cdot G_4(\Delta, w, D) \cdot dD
\]

\[
G_1(\Delta, w, D) = D^{-3} \left( \sqrt{D^2 - (w^2 + \Delta^2)} \frac{\Delta}{\sqrt{w^2 + \Delta^2}} \cdot u_{\text{max}} - w \cdot u_{\text{max}} - u_{\text{max}}^2 \right)
\]

\[
G_2(\Delta, w, D) = D^{-3} \left( \Delta - \sqrt{D^2 - (w^2 + \Delta^2)} \frac{w}{\sqrt{w^2 + \Delta^2}} \right) \cdot u_{\text{max}} + (u_{\text{max}} + w) \cdot \sqrt{D^2 - (w + u_{\text{max}})^2} - w \sqrt{\frac{1}{4} D^2 - u_{\text{max}}^2} + \frac{1}{4} D^2 \arcsin \frac{w + u_{\text{max}}}{D} \right)
\]

\[
- \frac{D^2}{4} \arcsin \frac{2w}{D} \right) dD
\]
\[ G_3(\Delta, w, D) = D^{-3} \left( \frac{1}{2} D - w \right)^2 \]

\[ G_u(\Delta, w, D) = D^{-3} \left( \frac{7}{8} D^2 - w \sqrt{\frac{1}{4} D^2 - w^2} - \frac{D^2}{4} \arcsin \frac{2w}{D} \right) \]

\[ F \left( \frac{D}{D_{\text{max}}} \right) = 0.727 \left( \frac{D}{D_{\text{max}}} \right)^{0.5} - \left( \frac{D}{D_{\text{max}}} \right)^2 + 0.144 \left( \frac{D}{D_{\text{max}}} \right)^4 + 0.036 \left( \frac{D}{D_{\text{max}}} \right)^6 + 0.016 \left( \frac{D}{D_{\text{max}}} \right)^8 + 0.010 \left( \frac{D}{D_{\text{max}}} \right)^{10} \]

\[ u_{\text{max}} = \frac{-\frac{1}{2} w (w^2 + \Delta^2) + \frac{1}{4} \sqrt{w^2 (w^2 + \Delta^2) - \Delta^2 (w^2 + \Delta^2)^2}}{w^2 + \Delta^2} \]

\[ p_k = \text{volume aggregate/volume concrete.} \]

Fig. 3.18 Contact areas \( A_y \) and \( A_x \) as a function of \( w \) and \( \Delta \), calculated with (3.40-3.43)
Integration of (3.40-3.43) was carried out in a numerical way. The text of the program (in Algol) is represented in Appendix IV. A stepwise integration in 10 steps appeared to be accurate enough (differences < 2%). In this way the contact areas in x- and y-direction for a unit crack area of 1 mm² are obtained for varying \((w, \Delta)\) combinations.

Fig. 3.18 shows a result of a calculation for a concrete mix with a maximum aggregate particle diameter of 32 mm and a \(p_k\) value equal to 0.75.

3.6 Final derivation of the relation between stresses and displacements in a crack

The relations between the stress conditions in a crack on the one side and the displacement components on the other side were expressed by the equations (3.1) and (3.2).

\[
\sigma = \sigma_{pu} (A_x - \mu A_y) \tag{3.44}
\]

\[
\tau = \sigma_{pu} (A_y + \mu A_x) \tag{3.45}
\]

in which \(A_x\) and \(A_y\) depend on \(\Delta\) and \(w\) (eq. 3.40-3.43). The parameters \(\sigma_{pu}\), the matrix yielding strength under the actual stress combination and \(\mu\), the coefficient of friction between matrix and aggregate are established by fitting the equations (3.44) and (3.45) to the experimental results, obtained in the tests on specimens with external restraint bars.

It appeared that the best results are obtained for a friction coefficient \(\mu = 0.50\) for all mixes. The matrix yielding strength which has to be inserted is somewhat higher than the strength of the hardened concrete itself. This is also in agreement with the normally observed material behaviour; the weakest link of a hardened concrete is the interface between aggregate-particles and matrix, where micro cracks initiate the deterioration of the concrete; as a result the concrete strength is lower than the strength of its constituting components.
Fig. 3.19 Comparison between experimental values for mix 3

\( f'_{cc} = 59.1 \text{ N/mm}^2 \), \( D_{\text{max}} = 16 \text{ mm} \) and the theoretical model, with \( p_k = 0.75 \), \( \mu = 0.50 \) and \( \sigma_{pu} = 80 \text{ N/mm}^2 \)
Fig. 3.20 Comparison between experimental values for mix 1
\( f'_{cc} = 37.6 \text{ N/mm}^2, D_{max} = 16 \text{ mm} \) and theoretical model, with \( p_k = 0.75, \mu = 0.50, \sigma_{pu} = 60 \text{ N/mm}^2 \)
Fig. 3.21 Comparison between experimental values for mix 5
($f'_{cc} = 33.4 \text{ N/mm}^2$, $D_{max} = 32 \text{ mm}$) and theoretical model, with $p_k = 0.75$, $\mu = 0.50$, $\sigma_{pu} = 48 \text{ N/mm}^2$

Fig. 3.22 Comparison between experimental values for mix 4
($f'_{cc} = 13.4 \text{ N/mm}^2$, $D_{max} = 16 \text{ mm}$) and theoretical model, with $p_k = 0.75$, $\mu = 0.50$, $\sigma_{pu} = 31 \text{ N/mm}^2$
The relation between the matrix strength \( \sigma_{pu} \) and the cube compression strength \( f'_{cc} \), which gives the best fitting results can approximately be represented by

\[
\sigma_{pu} = 5.83 f_{cc}^{0.63}
\]  

(3.46)

It must be emphasized that this is only a provisional, approximating relation. Actually the relation between \( \sigma_{pu} \) and \( f'_{cc} \) is not unique.

---

**Fig. 3.23** The influence of the maximum particle size on the prism compression strength for various water-cement ratios, according to CORDON and GILLESPIE [2].

An example is represented in fig. 3.23, which shows that for the same matrix the prism compression strength varies as a function of the scale of the aggregate. Such effects are not taken into account in order to avoid too great complexity. In expression (3.46) it is seen that the value \( \sigma_{pu} \), and as a result \( \tau_{pu} (= \mu \sigma_{pu}) \) does not increase proportionally with the cube crushing strength of the concrete, but less. Apparently the resistance of crack faces against shear displacements is relatively less favourable for higher concrete strengths. This tendency is also observed for concrete specimens subjected to quite different types of loading. REINHARDT [27] pointed out that this general feature has to be attributed to the more brittle character of the matrix in high strength concretes, giving rise to higher stress concentrations. Fig. 3.24 confirms this point of view, representing the results of tests on multi-axially loaded concrete specimens, carried out by
BREMER [26]. In this type of tests lateral compressive stresses reduce the stress concentrations in the concrete always in the same way. Hence this effect is greater for concretes with a lower strength, and accordingly a less brittle character.

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</table>

Fig. 3.24 Results of multiaxial compression tests by BREMER [26]

3.7 Analysis of some aspects of shear transfer in cracks on basis of the model developed

The model that has been developed enables a further analysis of the phenomenon. Some aspects will be dealt with.

a. The role of friction between the aggregate particles and the matrix material.

It was shown that equilibrium in the contact area was obtained by combinations of normal (yielding-) stresses and shear (friction-) stresses. It was shown that a friction coefficient equal to 0.5 led to the best fitting of the curves to the experimental results. By conducting a calculation with a friction coefficient $\mu = 0$ the influence of friction can be visualized. A calculation is carried out for a mixture with maximum aggregate size $D_{max} = 16 \text{mm}$, $\sigma_{pu} = 48 \text{N/mm}^2$, $p_k = 0.75$ and $\mu = 0.0$ resp. 0.5. The results of this calculation are shown in fig. 3.25 for some crack widths ($w = 0.2, 0.6$ and $1.0 \text{mm}$).

It is seen that the friction increases the shear stress with values up to among 50%, whereas the normal restraint stresses to provide equilibrium are reduced.
Fig. 3.25 The role of friction between aggregate and matrix in the transfer of stresses in a crack

b. The contribution of the various aggregate fractions to the transfer of stresses in a crack.

By a slight modification in the derivation of the equations, representing the relations between stresses and displacements in the crack, it is possible to find the contribution of only a part of the aggregate particles.

The equation (3.9) was found by integrating the product of eq. (3.5) and (3.8) over the interval $D_o \rightarrow D_{\text{max}}$. By integrating only over the range $D_o \rightarrow nD_{\text{max}}$ ($n < 1$) the cumulative distribution function is obtained, representing the probability than an arbitrary point in the concrete body, lying in an $xy$-intersection plane, is situated
in an intersection circle with a diameter \( D < D_o \), with as an additional condition that it must also be situated in a sphere with a diameter \( D < n.D_{\text{max}} \). An elaboration of this integration results in a modified expression (3.9):

\[
p_c(D < D_o) = p_k (1 - n + 1.455 \frac{D^0.5}{D_{\text{max}}} - 0.5 n^{-1.5} \frac{D}{D_{\text{max}}} - 0.5 n^{-3.5} \frac{D^6}{D_{\text{max}}^6} + 0.036 n^{-7.5} \frac{D^8}{D_{\text{max}}^8} + 0.006 n^{-5.5} \frac{D^4}{D_{\text{max}}^4} + 0.002 n^{-9.5} \frac{D^{10}}{D_{\text{max}}^{10}})
\]

The probability density function is obtained by differentiation to \( D_o \):

\[
p_c'(D_o) = p_k (0.727 \frac{n^{-0.5}}{D_{\text{max}}} - n^{-1.5} \frac{D}{D_{\text{max}}} - 0.144 n^{-3.5} \frac{D^4}{D_{\text{max}}^4} + 0.036 n^{-5.5} \frac{D^6}{D_{\text{max}}^6} + 0.016 n^{-7.5} \frac{D^8}{D_{\text{max}}^8} + 0.001 n^{-9.5} \frac{D^{10}}{D_{\text{max}}^{10}})
\]

Inserting this expression in (3.13), and (3.13) in (3.36-3.39) the same set of final equations (3.40-3.43) is obtained, with only a modified expression for \( F(D_{\text{max}}) \):

\[
F(D_{\text{max}}) = 0.727(D_{\text{max}})^{0.5} - n^{-1.5}(D_{\text{max}})^2 + 0.144 n^{-3.5}(D_{\text{max}})^4 + 0.036 n^{-5.5}(D_{\text{max}})^6 + 0.016 n^{-7.5}(D_{\text{max}})^8 + 0.001 n^{-9.5}(D_{\text{max}})^{10}
\]
With these equations the contributions of a number of fractions have been established and represented for a small (0.1 mm), an average (0.6 mm) and a great (1.0 mm) crack width. Two mixtures have been analyzed, both with $\sigma_{pu} = 48 \text{ N/mm}^2$, $\mu = 0.50$, $p_k = 0.75$, but different maximum aggregate size ($D_{\text{max}} = 16 \text{ mm}$ and $D_{\text{max}} = 32 \text{ mm}$). (Fig. 3.26 and 3.27).

The lines, represented in these figures, describe the relations between $\sigma$, $t$, $W$ and $\Delta$ if only the particles with a diameter between 0 and a varying fraction of $D_{\text{max}}$ are considered. It is seen that the small aggregate fractions lose importance if the crack width becomes greater.

Fig. 3.26 Contributions of the various aggregate fractions to the transfer of stresses in cracks for crack widths $w = 0.1$, 0.6 and 1.0 mm (Mix: $\sigma_{pu} = 48 \text{ N/mm}^2$, $p_k = 0.75$, $\mu = 0.50$, $D_{\text{max}} = 16 \text{ mm}$)
c. The influence of the scale of the aggregate.

To obtain an impression about the influence of the scale of the aggregate, two mixtures have been compared. Both have the same properties, except for the maximum particle diameter, which is once 16 mm and once 32 mm. The results of this comparison are shown in fig. 3.28. It is seen that the normal stress $\sigma$ is not very susceptible for this variation, but that the shear stress $\tau$ is influenced, the more if the crack width is greater. This tendency is confirmed by the results of the experimental part of this investigation (fig. 3.20 and 3.21).

Fig. 3.27 Contributions of the various aggregate fractions to the transfer of stresses in cracks for crack widths $w = 0.1$, 0.6 and 1.0 mm (Mix: $\sigma_{pu} = 48$ N/mm$^2$, $p_k = 0.75$, $\mu = 0.50$, $D_{max} = 32$ mm)
Fig. 3.28 The influence of the maximum aggregate particle size on the transfer of stresses in a crack. Maximum diameter $D_{\text{max}} = 16$ and $32$ mm. Mix properties: $p_k = 0.75$, $\mu = 0.50$, $\sigma_{pu} = 48$ N/mm$^2$, Fuller aggregate distribution.

d. **The influence of the sieve line.**

In the previous analyses and in the experiments always a Fuller-curve was adopted. However, in practice the most Codes allow an admissible sieve line area. The ideal Fuller-curve is near to the lower limit, bounding this area. To study the influence of the sieve line a line is chosen which approximates the upper limit given in the Dutch Code of Practice, the VB'74, for $D_{\text{max}} = 32$ mm (line B, fig. 3.29).
Fig. 3.29 Admissible sieve line area in the VB'74 for $D_{\text{max}} = 32$ mm

$\tau (\text{N/mm}^2)$ Theoretical Model $D_{\text{max}} = 32$ mm, $\mu = 0.5$, $p_k = 0.75$, $\alpha_{pu} = 48 \text{ N/mm}^2$

Between dotted lines: admissible sieve line area in Dutch Code VB'74
A Ideal Fuller curve
B Upper limit curve

Fig. 3.30 The influence of the sieve line on the transfer of stresses in a crack
The relations between stresses and displacements in a crack in a concrete, composed according to the sieve line B (and with $D_{\text{max}} = 32 \text{ mm}$, $p_k = 0.75$, $\mu = 0.50$ and $\sigma_{\text{pu}} = 48 \text{ N/mm}^2$) are constructed with the aid of the results obtained under b, where the contributions of the individual fractions to the transfer of stresses in a crack were established. The results of this calculation are compared with those obtained in an earlier calculation for a mixture with the same properties ($D_{\text{max}} = 32 \text{ mm}$, $p_k = 0.75$, $\mu = 0.50$ and $\sigma_{\text{pu}} = 48 \text{ N/mm}^2$) but composed according to an ideal Fuller-curve (fig. 3.30).

It is seen that the influence of the sieve line on the normal stresses $\sigma$ is not great, but is significant for the shear stresses. The most pronounced differences are obtained for greater crack widths. This can be explained by the fact that the mixture according to sieve line B contains much more small particles.

e. **Cyclic loading.**

From tests [6, 24] it is known that in the case of cyclic loading a considerable difference exists between the behaviour of the crack plane during the first loading cycle and the subsequent cycles. The shear stress - shear displacement relationship of the initial cycle is nearly linear and after unloading a considerable amount of hysteresis can be observed. The shear stress - shear displacement relationship for the later loading cycles is highly non-linear and a hardening type of behaviour is observed. This overall-behaviour can be explained with the theory developed. As an example a fictitious specimen is considered according to fig. 3.31.

![Fig. 3.31 Fictitious specimen considered](image)
The specimen consists of two parts, separated by a crack, the initial width of which is supposed to be \( w_0 = 0.50 \text{ mm} \). The two halves are loaded by shear forces, while an enlargement of the crack width is counteracted by external restraint rods. The stiffness of these rods is assumed to be such that an increase of the crack width of \( \Delta w = 0.1 \text{ mm} \) results in an increase of the normal stress on the crack plane of \( \Delta \sigma = 0.5 \text{ N/mm}^2 \). The top shear stress applied is assumed to be \( \tau = 3 \text{ N/mm}^2 \).

The assumed concrete quality is \( f'_{cc} = 33.4 \text{ N/mm}^2 \) with \( D_{\text{max}} = 32 \text{ mm} \), so that the relations represented in fig. 3.21 can be used to construct the ascending branch \( OA \) of the \( \tau-\Delta \) relation (fig. 3.32).

![Fig. 3.32](image)

**Fig. 3.32** Response to cyclic loading of fictitious specimen, calculated with theoretical model

A description of this procedure is given in Appendix II. The unloading branch is calculated in another way: just before unloading, the relations (3.44) and (3.45) are valid:

\[
\sigma_1 = \sigma_{pu} \left( A_x - \mu A_y \right) \quad (3.48)
\]

\[
\tau_1 = \sigma_{pu} \left( A_y + \mu A_x \right) \quad (3.49)
\]
At that moment the friction is still a maximum ($\mu = 0.50$). This situation can be compared with the loading case of a body on an inclination, with frictional forces acting between body and supporting surface, and the resultant of horizontal force $V$ and vertical force $N$ lying in the direction of the line $b-b'$.

Unloading is done by diminishing the horizontal (shear) force $V$. Movement of the two halves of the specimens related to each other can only occur if $V$ is so small that the maximum frictional resistance in the other direction is reached (comparable with the situation in fig. 3.33 that the resultant is in the direction of $a-a'$).

![Fig. 3.33 Body on a rough inclination as an analogon for the behaviour of a crack under cyclic loading](image)

Movement of the crack faces occurs if $\mu$ has turned to $-\mu$. At that moment $\sigma$ is still the same as at the onset of unloading, since the external restraint rods have not been subjected to any change in length during this stage. Hence:

$$\sigma_2 = \sigma_{p_1} (A_x + \mu A_y) \quad (3.50)$$

In which $\sigma_{p_1}$ is the unknown normal stress on the surface of the aggregate particles.
However, since $\sigma_1 = \sigma_2$ it turns out

$$\sigma_{p1} = \frac{\sigma_{pu} (A_x - \mu A_y)}{(A_x + \mu A_y)}$$

(3.51)

If the crack planes are about to move, the shear stress $\tau_2$ can be formulated by replacing $\mu$ in (3.49) by $-\mu$:

$$\tau_2 = \sigma_{p1} (A_y - \mu A_x)$$

(3.52)

Substitution of (3.51) in (3.52) results in:

$$\tau_2 = \sigma_{pu} \frac{A_x - \mu A_y}{A_x + \mu A_y} (A_y - \mu A_x)$$

(3.53)

$\tau_2$ can be expressed as a function of $\tau_1$, comparing (3.49) with (3.53):

$$\tau_2 = \xi \tau_1$$

(3.54)

with

$$\xi = \frac{A_x - \mu A_y}{A_x + \mu A_y} \cdot \frac{A_y - \mu A_x}{A_y + \mu A_x}$$

In this way it is possible to determine the position of point B in fig. 3.32. For the case considered it was calculated that $A_x = 3.13 \times 10^{-2}$ mm$^2$ and $A_y = 4.49 \times 10^{-2}$ mm$^2$. With $\mu = 0.50$ a value $\xi = 0.080$ is obtained.

If the frictional resistance of the crack faces is exceeded a displacement will occur. This displacement will continue until no areas of contact and no external forces remain. To reach this stage the crack width has to attain its original value, but the shear displacement needs not to come back to zero. This is explained with the aid of fig. 3.34.

The position of the crack faces before loading is represented in fig. 3.34a. At peak stress level the crack width has been increased with $\Delta w$ and the shear displacement with $\Delta$. As a result of the
rigid-plastic character of the matrix material a cavitation has been formed (shaded area in fig. 3.34b). Due to this cavitation the "no contact phase" after unloading is reached before the shear displacement is back to zero (fig. 3.34c).

Fig. 3.34 Three characteristic stages during the first loading cycle
a. Before loading  
b. Peak stress  
c. After unloading

The "no contact phase" is considered to be reached, if in the most unfavourable case \( R = \frac{1}{2} D_{\text{max}} \) and \( u = 0 \) no contact exists any more. For the case considered \( D_{\text{max}} = 32 \text{ mm}, \ w_0 = 0.5 \text{ mm} \) it can easily be calculated (Appendix II, using the formulas of Appendix I) that the remaining shear displacement \( \Delta_r \) is equal to 0.41 mm (point C in fig. 3.32). To bring the two halves of the specimen back in their neutral position a small shear force may be necessary, since the rubble between the crack faces due to the deterioration of matrix material during loading, may cause some frictional resistance (point D in fig. 3.32). If the shear force is imposed in the other direction the same type of behaviour can be expected since those parts of the crack surfaces which get in touch in this reversed cycle are not yet damaged (fig. 3.34c). Hence a similar loading and unloading curve can be expected (fig. 3.32, points A', B', C', D').

In the subsequent loading cycles the presence of the cavitations, worn out in the first cycle of loading, influence the behaviour
of the specimen considerably. At first a shear displacement will occur under a low shear force, until contact between the opposing areas is obtained ($\Delta > 0.41 \text{ mm (point E)}$). Then in a short interval of $\Delta$ full contact between the cavitations will be obtained. In this short interval a proceeded wearing off will take place at places of high contact stresses (point X in fig. 3.34c). Hence a steeply ascending branche (EFG) may be expected, slightly shifted from the fore-going loading line. At unloading a similar behaviour as in the first cycle may be expected (GHI - fig. 3.32).

A comparison of the $\tau$-$\Delta$ relation, constructed with the aid of the theoretical model (fig. 3.32) with experiments, carried out by LAIBLE, WHITE and GERGEHY [9], shows a rather good agreement in behaviour (fig. 3.35). This proofs that the assumption of rigid-plastic matrix behaviour (fig. 3.3) is a fairly good approximation of reality.

![Graph showing experimental results for cyclic loading](image)

**Fig. 3.35** Experimental result for a specimen subjected to cyclic loading; $f'_c = 21 \text{ N/mm}^2$, $D_{\text{max}} = 38 \text{ mm}$, $w_o = 0.75 \text{ mm}$, restraint stiffness $\Delta \sigma = 0.3 \text{ N/mm}^2$ for $\Delta w = 0.1 \text{ mm}$ [24]
f. **Lightweight concrete.**

One of the test series carried out consisted of lightweight specimens (mix 6). This mix was composed of Korlin lightweight particles and a sand fraction with particles from 0.1-4 mm, distributed according to a Fuller-curve. Since the lightweight particles are intersected by the crack, whilst the sand particles which possess a much greater strength, are not intersected but exceed from the crack plane, the experimental results were compared with the theoretical results, calculated with the developed model, in which only the sand fraction was taken into account and the contribution of the lightweight particles was assumed to be zero. However, in this way the theoretical model resulted in shear and normal stresses which were about 50% too low. It is probable that the lightweight aggregate particles, intersected by the crack plane contribute also to the transfer of stresses, due to their rough surface of intersection.

g. **Comparison with other static tests.**

Tests on cracks in plain concrete were a.o. carried out by LOEBER and PAULAY \[17\]. These tests were carried out with constant crack widths \(w = 0.13, 0.25\) and \(0.51\) mm. As a result of the way of testing rather high shear stresses could be obtained, in contradiction to other tests \[4, 5\] in which the formation of secondary cracks disturbed the measurements in an early stage of loading. The concrete type, used in \[17\] had a \(D_{max}\) of 19 mm and an average cube compression strength of \(f'_{cc} = 37\) N/mm\(^2\). The experimental results have been compared with the results of the theoretical model, with the assumptions that \(D_{max} = 19\) mm, \(u = 0.50\), \(p_k = 0.75\) and \(\sigma_{pu} = 57\) (according to eq. 3.46). This comparison is represented in fig. 3.36. The agreement between experimental and theoretical values, taking into consideration the differences in sieve line and experimental set-up between both investigations, is satisfying.
h. **Simplified equations for the relations between** $\tau$, $\Delta$, $w$ and $\sigma$, $\Delta$, $w$

On basis of a regression analysis simplified equations have been derived fitting the experimental results. In these equations only the cube crushing strength has been considered as a variable. The influence of the scale of the aggregate has not been taken into account, since it has only scarce influence for the range tested. The formulas derived are apted in principle for the type of mixes used (Fuller-curves), the interval of $D_{\text{max}}$ ($16 < D_{\text{max}} < 32$ mm), and for the range of $\Delta$ and $w$-values tested. The curves which fitted the results with the greatest accuracy are:

$$
\tau = -\frac{f'_{cc}}{30} + (1.8 w^{-0.80} + (0.234 w^{-0.707} - 0.20) f'_{cc}) \Delta \ (\tau > 0) \ (3.55)
$$

and

$$
\sigma = -\frac{f'_{cc}}{20} + (1.35 w^{-0.63} + (0.191 w^{-0.552} - 0.15) f'_{cc}) \Delta \ (\sigma > 0) \ (3.56)
$$

A comparison of these bilinear approximations with the experimental results of the various series are represented in the figures 3.37-3.40.
Fig. 3.37 Comparison of eq. (3.55) and (3.56) with experimental results (Mix 1)

Fig. 3.38 Comparison of eq. (3.55) and (3.56) with experimental results (Mix 3)
Fig. 3.39 Comparison of eq. (3.55) and (3.56) with experimental results (Mix 4)

Fig. 3.40 Comparison of eq. (3.55) and (3.56) with experimental results (Mix 5)
A regression analysis of the results of the experiments with lightweight concrete (mix 6) led to the equations:

\[ \tau = -\frac{f'_{cc}}{80} + (1.495 w^{-1.233} - 1).A \quad (\tau > 0) \]  
\[ (3.57) \]

and

\[ \sigma = -\frac{f'_{cc}}{40} + (1.928 w^{-0.87} - 1).A \quad (\sigma > 0) \]  
\[ (3.58) \]

A comparison of these equations with the experimental results is presented in fig. 3.41.
4.1 General

In chapter 3 it was shown, that the behaviour of a crack, subjected to shear loading, can be predicted if the restraint stiffness is known. In reinforced cracks the restraint stiffness is provided internally, by the axial stiffness of the reinforcing bars against pull out forces. Furtheron the bars contribute to the shear stiffness by dowel action. The equilibrium of forces, if one part of a specimen is considered, is represented in fig. 4.1.

![Equilibrium of forces in a reinforced crack](image)

**Fig. 4.1 Equilibrium of forces in a reinforced crack**

- $V_e$ = External shear force
- $H_i$ = Axial tensile force in a reinforcing bar
- $V_{di}$ = Dowel force of reinforcing bar
- $I_{cv}$ = Vertical aggregate interlock component
- $I_{ch}$ = Horizontal aggregate interlock component

In the experiments the external load $V_e$ was measured by a load cell. The axial force in the reinforcing steel was not measured: the force after yielding of the steel ($w > 0.3$-$0.5$ mm) is known from control tests (Part I, fig. 2.12), the force before yielding can be approximated by a method proposed by REHM and MARTIN \[11, 18\]. Also the
dowel force has to be estimated by means of data found in literature. These estimations are carried out prior to further analysis of the behaviour of reinforced cracks.

4.2 Calculation of the contribution of the steel reinforcement to the equilibrium of forces in a "reinforced crack"

As has been stated before, the contribution of the steel bars to the equilibrium of forces in a "reinforced crack" must be subdivided into an axial restraint force and a dowel force. Both contributions are approximately established in the following.

4.2.1 Derivation of the axial restraint action of the reinforcing bars

Bond between steel bars and surrounding concrete depends predominantly on the profiling of the reinforcing bar, characterized by the ratio between the area of the ribs \( F_R \) and the shear area \( F_S \): the "related rib area \( f_R \)" (REHM [18, 19], MARTIN [11], NOAKOWSKI [15] - fig. 4.2).

\[
f_R = \frac{F_R}{F_S}
\]

Fig. 4.2 Definition of the related rib area \( f_R \)

Average values of \( f_R \) for conventional steel bars are (KOCH [7])

\[
\begin{align*}
f_R & \approx 0.045 \to 0.060 \quad \text{for } \phi = 4 \to 11 \, \text{mm} \\
& \approx 0.065 \quad \quad \quad \quad \quad \quad \quad \text{for } \phi > 12 \, \text{mm}
\end{align*}
\]

(4.1)

Tests on reinforcing bars, embedded over a short length, demonstrated that for a great range of slip values a proportionality between the bond stress \( \tau \) and the concrete strength \( f'_{cc} \) exists. In order to find
functions describing the relation between the pull out force and the slip of a bar, it seemed therefore appropriate to base oneself on the value \( \tau / f'_{cc} \). Experiments showed that the basic relation between \( \tau / f'_{cc} \) and the slip \( \Delta_s \) can be represented by

\[
\frac{\tau}{f'_{cc}} = a_0 + b_0 \frac{\Delta_s}{\Delta_s} \quad (a_0, b_0 \text{ and } \beta \text{ are constants})
\]

(4.2)

However, this equation results in a complicated differential equation. MARTIN [11] showed that an approximating solution can be obtained by a difference calculation. Therefore the reinforcing bar is divided into elements with a length \( \Delta x \). Besides equation (4.2) for all elements two other conditions have to be fulfilled:

- **Equilibrium of forces** (fig. 4.3)

  \[
  \frac{d\sigma_s}{dx} = \frac{\Sigma u}{A_s} \cdot \tau_x
  \]

  (4.3)

  In this equation is:

  \( \Sigma u \) = circumference of the reinforcing bar

  \( A_s \) = cross sectional area of the reinforcing bar

- **Compatibility of deformations**

  \[
  \frac{d\Delta x}{dx} = \frac{\sigma_{sx}}{E_s} \cdot \left( 1 + \frac{\sigma_{cx}}{\sigma_{sx}} \cdot \frac{E_s}{E_c} \right)
  \]

  (4.4)
In words: the difference in strain between steel and concrete over the length of the element has to result into a slip $\Delta x$. SCHIESSL [20] demonstrated that the concrete strain is of minor influence and can be neglected without making a significant error. With the equations (4.2 - 4.4) a difference calculation was carried out according to the scheme, represented in fig. 4.4.

Fig. 4.4 Flow scheme of difference calculation to obtain the relation between the pull-out force in a bar and the slip
This procedure enables a sufficient accurate estimation of $\sigma_{s,i}$, $\tau_i$ and $\Delta_{s,i}$ over the length of a bar, starting from given initial values for $\Delta_{s,o}$, $\sigma_{s,o}$ and $\tau_o$ on the condition that the length of the elements is small enough. Calculations with a variable value of $\Delta x$ demonstrated that a length, equal to the distance between the ribs, results in a fair degree of accuracy [11].

MARTIN [11] carried out tests in which the bar was embedded in the concrete over a length of 7-10 $\phi$. He determined in an iterative way, with the procedure described, the basic equation (4.2). Inaccuracies $< 0.5\%$ were accepted. The values of $a_o$, $b_o$ and $\beta$ resulting from this equation are given in Appendix III, Table A. Limit values for the slip $\Delta_{s,max}$ as obtained in the tests are represented in Appendix III, Table B.

In the way described in fig. 4.4, the pull-out characteristics of the reinforcing bars in the own shear specimens have been calculated (a computer program in Algol is represented in Appendix III). The $f_R$-values, necessary for the calculation were established by measurements on the bars in a way described in [1]. The pull-out characteristics of the single bars, obtained by the calculation, are represented in Appendix III, fig. III.1. The maximum bond length of the stirrup legs in the shear specimens was 170 mm (Part I, fig. 2.2).
Fig. 4.5 Development of restraint stresses (total axial steel force per unit crack area) as a function of crack opening

For some of the specimens this length was exceeded by the calculated values. However, a slightly modified calculation (taking different values for $\sigma_o$ (fig. 4.4) and calculating the pull-out force at the crack) showed that this had scarcely any influence on the pull-out characteristics. The calculated pull-out characteristics have been used to establish the restraint stiffness of the crack against crack opening in normal direction. The restraint stresses (total force in the steel per unit crack area) have been represented as a function of the crack width in fig. 4.5a-d.

Actually, it must be realized that the loading conditions in the tested shear specimens differ from those obtained in the tests of REHM and MARTIN \[11, 18, 19\] in this respect, that additional stresses act perpendicular to the bars as a result of the external load. These stresses might influence the relations before yielding. Up to now no data are available enabling an estimation of this effect. However, it appeared from tests, carried out by MEHLHORN et.al. \[13\] (see also \[23\], pp. 45-46), that high concentrated concrete stresses, due to dowel action, did not affect the bond-slip relations significantly.
4.2.2 Derivation of an empirical relation to establish the contribution of dowel action

In the State of the Art Report [23, p. 44] it was shown that the dowel action of a single bar can be expressed by:

\[ F_d = \beta^3 EI \Delta \text{ with } \beta = \sqrt{\frac{\phi G_f}{4EI}} \]  \hspace{1cm} (4.5)

in which

- \( EI \) = bending stiffness of the bar
- \( \phi \) = bar diameter
- \( G_f \) = foundation modulus of the concrete

Substitution of \( I = \frac{\pi \phi^4}{64} \) in (4.5) results in the equation

\[ F_d = 3.56 \phi^{1.75} G_f^{0.75} \Delta \]  \hspace{1cm} (4.6)

\( G_f \) is assumed to be a function of \( \Delta \) ([23], fig. 3.13) and of the concrete strength. To obtain an expression for \( G_f \), eq. (4.6) has been compared with the results of a series of tests, carried out by PAULAY [16], in which the dowel force was measured as a function of \( \Delta \), for bars with various diameters, embedded in a concrete with a strength of \( f_{cc} \approx 30 \text{ N/mm}^2 \). The values of \( G_f \), resulting from this comparison, can be described by

\[ G_f = 188 \Delta^{-0.85} \]  \hspace{1cm} (4.7)

(See also [23], p. 39, fig. 39). However, \( G_f \) has also to be related with the concrete strength. Since the modulus of elasticity \( E_c \) is generally related to the concrete strength as

\[ E_c = C_1 f_{cc}^{1/2} \]

a similar relation has been adopted for the foundation modulus:

\[ G_f = C_2 f_{cc}^{1/2} \]
Taking this relation into account (4.7) is modified to

\[
G_f = 34 \sqrt{ f'_{cc} \Delta^{-0.85}}
\]

This relation, however, is only based on experiments without axial extension forces, so for \( w = 0 \). Tests, carried out by ELEIOTT [6, 24], showed that an axial tensile force in the reinforcing bar reduces the dowel force considerably; a tensile stress of 175 N/mm\(^2\) in a bar with \( \phi \) 12.8 mm reduced the dowel stiffness with about 50\%, whilst an increase to 350 N/mm\(^2\) resulted again in a decrease of 40\%. For the concretes used in the experiments on the shear specimens, a stress level of 175 mm is approximately obtained for a crack width of \( w = 0.2 \) mm and a stress of 350 N/mm\(^2\) for \( w = 0.4 \) mm.

In this way a reduction factor for the dowel force is obtained, approximately equal to

\[
a = 0.20 \ (w + 0.2)^{-1}
\]

so that the dowel force can, approximately be described by

\[
F_d = 10 \ (w + 0.2)^{-1} \ \Delta^{0.36} \ \phi^{1.75} \ f'_{cc}^{0.38}
\]  

(4.8)

Fig. 4.6 Contribution of dowel action, calculated with eq. (4.8) to the total shear stress in a crack, for the specimens 250208, 250408, 250608, 250808
Comparing the values obtained by (4.8) for the actual crack opening path, found in the experiments, with the total shear force, it is seen that dowel action is anyhow of minor importance (fig. 4.6).

4.3 Analysis of the contribution of the aggregate interlock forces in a crack

4.3.1 Qualitative analysis

For cracks in plain concrete the behaviour under shear loading can be predicted if the restraint stiffness is known. The way in which this is done is indicated in fig. 4.7.

If the restraint stress normal to the crack is known as a function of the crack width, \((\sigma, w)\) combinations can be subscribed in the \(\sigma, \tau, w, \Delta\) intersection diagram (fig. 4.7). Subsequently these \((\sigma, w)\) combinations are related to values of the shear displacement \((\Delta)\) and the shear stress \((\tau)\). Carrying out this procedure for several values of \(w\), the related \(\tau-\Delta\) and \(\Delta-w\) functions can be constructed. In this way it can be studied if the behaviour of a reinforced crack is similar to that of a crack in plain concrete: the \(\sigma-w\) relations.
can be taken from fig. 4.5 so that the according \( \Delta \) and \( \tau \) values be read. Subsequently the \( \tau \) value is increased with a dowel term, obtained from equation (4.8). The results of this comparison for a number of tests with concrete strengths \( f'_{cc} = 20, 30, 38 \) and 56 N/mm\(^2\) are represented in fig. 4.8a-d.

\begin{align*}
\text{w(mm)} & \quad \text{\tau(N/mm}^2) \\
0 & \quad 0 \\
0.5 & \quad 0.5 \\
1 & \quad 1 \\
1.5 & \quad 1.5 \\
2 & \quad 2 \\
\Delta (mm) & \quad \Delta (mm) \\
\end{align*}
Fig. 4.8a-d Comparison between experimental results and results, calculated if assumed that the behaviour of reinforced and unreinforced cracks under shear loading is similar

--- experiment

--- calculation
It is seen that generally bad agreement exists between the experimental curves and the ones, calculated with the described method. Good agreement is found only for low values of \( \rho_o \) (2 stirrups \( \phi 8 \text{ mm} = 0.56\% \) and 2 stirrups \( \phi 4 \text{ mm} = 0.14\% \)): for these reinforcement ratios the calculated relations are rather similar to the experimental ones. However, for increased values of \( \rho_o \) growing deviations are observed, most of all for the crack opening path.

Apparently there is a fundamental difference in behaviour between cracks which are crossed by reinforcing bars and others, which are not. It is believed that this has to be attributed to local disturbance of the crack structure around the reinforcing bars. This disturbance is caused by local splitting forces, originating from the ribs of the reinforcing bars, when these are pulled out of the concrete by axial tensile forces (fig. 4.9).

In this way concentrations of loose asperities are formed, which contribute in a different way to the transmission of forces over the crack (fig. 4.10). The actual mechanism of transmission may be rather complicated. Not only crushing of matrix material occurs, but also sliding friction at the contact points between the aggregate particles.
and rolling friction, due to which the particle shape may have an influence. Furthermore the volume of loose particles increases with increasing extension of the reinforcing bars. The difference in crack structure can also be observed after opening of the specimens.

Fig. 4.11 shows a crack plane of a specimen, tested with external restraint bars; only a small amount of fine material was found to be teared off from the crack faces. So the interlocking forces had been transmitted over particles, embedded in the crack faces of the specimen.
Fig. 4.12 Crack faces of a specimen, reinforced with 2 stirrups Ø 8 mm, with crater-shaped holes around the reinforcing bars (after removing of loose material)

Fig. 4.12 shows a crack face of a specimen, reinforced with two stirrups Ø 8 mm, after opening of the crack. A considerable amount of loose particles was released. After removing of remaining particles crater-shaped holes around the bars became visible. It is likely that these holes have mainly been formed before yielding of the bars, during the actual test, and only for a minor part during crack opening after the tests, since the greatest increase in steel stress, and attended slip of the reinforcing bars out of the concrete combined with splitting action, takes place before yielding.

The fact that this splitting mechanism is responsible for the difference in behaviour between the cracks in plain concrete and the cracks, crossed by reinforcement, was confirmed by the experiments on reinforced specimens, in which the bars were draped with soft sleeves over 20 mm on both sides of the crack. These sleeves eliminate splitting forces as represented in fig. 4.9, and as such also the activation of the additional mechanism of aggregate interlock over
loose particles, at least for low crack widths. Indeed it appeared that the crack opening path was dependent on the restraint stiffness of the reinforcing bars just like for the specimens with external restraint bars. The restraint stiffness of this type of reinforcement was calculated with the method, presented in 4.2.1, taking into account the free, unbonded length. The effect of this length on the restraint stiffness is comparatively small, as shown in fig. 4.13.

![Diagram](image)

Fig. 4.13 Effect of sleeves on the restraint stress for specimens with 2, 4, 6 and 8 stirrups \( \phi 8 \) mm

However, a calculation of the \( w-\Delta \) and \( \tau-\Delta \) relation for the series with sleeves, in the same way as done before (fig. 4.8), showed that for both the crack opening path and the \( \tau-\Delta \) relation, much better agreement was obtained (fig. 4.14).

In spite of the apparently complex character of the mechanism of transmission of forces around the reinforcing bars, the experiments on reinforced specimens reveal two characteristic modes of behaviour.

1. The mechanism is not active for low values of the reinforcement ratio (fig. 4.8, specimens with 2 stirrups \( \phi 4 \) or \( \phi 8 \) mm). It seems that, if the "natural" crack opening direction does not exceed a certain, critical value, the loose particles around the bars do not lock-up and as such do not influence the behaviour.
Fig. 4.14 Comparison between experimental values of the tests on specimens with soft sleeves around the bars and the results, calculated under the assumption that the behaviour of reinforced and unreinforced cracks under shear loading is similar.

2. If locking-up of the loose particles occurs, apparently struts are formed with relatively great stiffness, since for all reinforcing percentages greater than about 0.6% the parts of the specimen are forced to follow the same crack opening path.

4.3.2 Quantitative analysis of the aggregate interlock forces in a reinforced crack

Considering the equilibrium of forces in a crack it is concluded that, apart from the contribution of aggregate interlock over loose particles, all contributions are known, or approximately known (fig. 4.15): the external shear force $V_e$ was measured by a load cell, the dowel force $\Sigma V_d$ can be estimated with eq. (4.8), the
restraint force in the reinforcing bars $\Sigma H$ can be taken from the figures 4.5, the aggregate interlock forces $I_{1v}$ and $I_{1h}$ can be estimated using the equation (3.55) and (3.56). The forces needed to close the polygon must be provided by the unknown mechanism of aggregate interlock, acting in the direct vicinity of the reinforcing bars. Since the crack opening pathes appeared to be approximately constant (fig. 2.15-2.19), the unknown force is supposed to be resolved into directions normal and parallel to the direction of crack opening for all $(w, \Delta)$ combinations. The equilibrium of forces in the cracks has been analyzed in this way for crack widths between 0 and 1 mm (step 0.1 mm), for four series with different concrete qualities.

Fig. 4.15 Equilibrium of forces in a crack in a reinforced specimen, considering two types of aggregate interlock ($I_1$ and $I_2$)

Although inaccuracies may occur due to the fact that $\Sigma V_d$, $\Sigma H$ (before yielding) and $I_{1v}$ and $I_{1h}$ are calculated on basis of other experiments and are not the result of direct measurements on the actual specimens that were tested (which can hardly be done in an undisputable way), it is at least possible to trace tendencies with this method. The values $I_{2n}$ and $I_{2t}$ have been calculated for a number of specimens with different concrete strengths and reinforcing ratios. They are
represented as $\sigma_N = I_{2n}/A_c$ and $\tau_N = I_{2t}/A_c$ ($A_c =$ crack area) in the figures 4.16a-d.

---

**a)**

- $f'_{cc} = 19.9$ N/mm$^2$ (mix 4)
- $\sigma_N = I_{2n}/A_c$

---

**b)**

- $f'_{cc} = 30$ N/mm$^2$ (mix 1)
- $\sigma_N = I_{2n}/A_c$

---

**c)**

- $f'_{cc} = 38.2$ N/mm$^2$ (mix 5)
- $\sigma_N = I_{2n}/A_c$

---

**d)**

- $f'_{cc} = 56.1$ N/mm$^2$ (mix 3)
- $\sigma_N = I_{2n}/A_c$

---

**Fig. 4.16a-d** Calculated $\sigma_N$, $\tau_N$ values for the additional mechanism of aggregate interlock.

It is seen that, apart from the results for one specimen (230208), positive values for $\sigma_N$ are obtained, in combination with relatively
small values for $\tau_N$. For the lower concrete qualities and high reinforcement ratios apparently negative values for $\tau_N$ are obtained, which is from a physical point of view highly improbable. This can be explained by the fact that a part of the concrete adjacent to the crack failed locally under biaxial stress combinations: spalling of the concrete was observed in the top and bottom parts of the specimens for which negative $\tau_N$-values were found. Fig. 4.16c and d show that the negative $\tau_N$-values disappear for higher concrete qualities, for which local failure was not observed. The upper limit to the strength of the specimens due to the biaxial strength of the concrete should be discussed shortly. MATTOCK [12] pointed out already that the strength of reinforced cracks subjected to shear-loading would be limited by the biaxial concrete strength if high reinforcing ratios were applied. The shear resistance as a function of the total restraint stress normal to the crack plane may hence be represented by a combination of two curves (fig. 4.17); one curve governed by the behaviour of a reinforced crack and one curve, governed by failure of the concrete besides the crack under biaxial stress conditions.

![Graph showing shearing behaviour](image)

**Fig. 4.17** Supposed limitation of shear strength of a reinforced crack by the monolithic concrete strength [12]

It would not be correct to construct the concrete failure envelope for the specimens, comparing the average stresses in the crack with the results of tests on biaxially loaded concrete [8, 10, 14].
This was demonstrated by SCHWING [21], who carried out numerical calculations on the stresses in joints, assuming certain joint properties and varying the specimen geometry. The stresses appeared to be far from uniformly distributed, with the most unfavourable combinations of stresses occurring at the top and bottom of the joint (fig. 4.18). (The deviations in the own tests cannot have been great, according to the measurements).

\[ l/b = 1 \quad l/b = 3 \quad b = 0.6 \text{m (const)} \]

\[ l/b = 2 \quad l/b = 4 \]

Fig. 4.18 Influence of the specimen geometry on the distribution of stresses in a joint, according to SCHWING [21]

On basis of these figures it must be concluded that there must be a more gradual transition from curve I (fig. 4.17) to the point of ultimate resistance, due to a proceeding reduction of that part of the shear plane, which is actively contributing to the total shear resistance.

Furtheron it was observed that the specimens type 1 (Report I, fig. 2.2a) were slightly more susceptible to the occurrence of stress concentrations at the ends of the crack than specimens type 2 (Report I, fig. 2.2b), which could be an explanation for the comparatively low values for \( \tau_N \) for \( f'_{cc} = 30 \text{ N/mm}^2 \) (fig. 4.16).
Although the average values of $\tau_N$ (fig. 4.16) seem to increase with increasing concrete qualities, it seems not to be appropriate to formulate this by a mathematical expression; small misestimations of the dowel force, the bond slip relation, the yield strength of the steel or aggregate interlock would influence this relation. As will be demonstrated in 4.4, neglection of $\tau_N$ does not have a very great influence on the total shear value.

4.4 General formulation of the behaviour of the specimens with reinforcement crossing the crack plane

4.4.1 Specimens with reinforcing bars normal to the crack plane

In 4.3 it was shown that for low reinforcing ratios the relations between stresses and displacements during crack opening are well described by the functions, derived for cracks in plane concrete (fig. 4.8). For higher reinforcing ratios the additional mechanism of aggregate interlock around the reinforcing bars is activated: "locking up" of the crack occurs and the crack faces are forced to follow a certain crack opening path, apparently characteristic for the type of concrete, which is used.

![Diagram of forces in a reinforced crack](image-url)
This characteristic crack opening path being known, the equilibrium in a crack can reasonably well be described by distinguishing a major and a minor roughness level. The major roughness level is defined by the characteristic crack opening path, the minor roughness level is defined by what happens on "particle level". Schematically the interaction of macro- and micro-level is represented in fig. 4.19.

The macro-roughness is represented by the infinitely stiff compression struts $S$ (interlock over loose particles), the micro-roughness by the circular element in the middle (interlock by contact forces over embedded particles). For low reinforcing ratios the tangent of the crack opening direction $\frac{d\Delta}{dw}$, as a result of the interaction of $F_{iv}$, $F_{ih}$, $F_d$ and $F_s$, is small and the compression strut $S$ is not activated. If the reinforcement ratio is great a crack opening could be obtained, greater than a critical value $(\frac{d\Delta}{dw})_{cr}$. This is however prevented by the struts, which force the crack faces to follow the critical crack opening path (the crack is locked-up). In this case the behaviour of the crack is a result of the interactions between $S$, $F_d$, $F_s$, $F_{ih}$, $F_{iv}$.

Studying the experiments it is recognized that the critical crack opening path is characteristic for every concrete mixture. Varying of reinforcement ratios for specimens made of the same mixture exhibited that the differences in crack opening path were negligible (fig. 2.10-2.19). Comparing the average crack opening paths for the different mixes (fig. 2.20), it is observed that the deviations between the mixes 1, 2, 4 and 5 with $19.9 < f'_{cc} < 38.2$ N/mm$^2$ are small. Mix 3, with $f'_{cc} = 56.1$, for which the crack will intersect a number of aggregate particles, deviates from the others.

The critical crack opening paths (or major roughness) are described by mathematical expressions obtained from curve fitting. For the low and medium strength concrete the critical crack opening path is described by

$$\Delta = 1.4 w^{1.18}$$  \hspace{1cm} (4.9)

and for the high strength concrete by

$$\Delta = 1.87 w^{1.4}$$  \hspace{1cm} (4.10)

(fig. 4.20)
In the following it is shown that the experimental results can be described on basis of the system, represented in fig. 4.19. At first it has to be found out whether the compression strut S is activated or not. This is done in a way as described in fig. 4.7; the relation between the restraint stress and the crack width is taken from fig. 4.7, subsequently the value $\Delta$ is determined. The combinations of $(w, \Delta)$ reveal whether the critical crack opening path is exceeded or not. If not the value $\tau(w, \Delta)$ can be assessed from the diagram and the total external shear stress is obtained by adding the dowel term (eq. 4.8). If the critical crack opening path is indeed exceeded, the strut S is activated and the crack faces are forced to follow this path. The way in which the external shear stress is constructed is shown in fig. 4.21.
The polygon of forces can be constructed for every value of \( w \), starting from point A. For a value \( w \) the accompanying value of \( \Delta \) can be calculated with eq. (4.9) or (4.10). The dowel force is obtained from eq. (4.8). The horizontal restraint stress, caused by the reinforcement, can be taken from the figures 4.5. The vertical and horizontal components of aggregate interlock (Type I) are found from expressions (3.55) and (3.56). The direction of the normal on the crack opening path is obtained from (4.9) or (4.10). Consequently the external shear force is found as AB. This calculation has been carried out for many specimens (fig. 4.22-24) and shows a satisfying agreement with the experimental results. In some of the more heavily reinforced specimens the calculated lines reach a higher top than the experimental ones, which can be explained by the occurrence of spalling regions at top and bottom of the crack, observed during testing which weaken the ultimate resistance (fig. 4.22c, h, j, k, p). Only in one of the specimens the struts S were not found to be stressed (specimen 230208, fig. 4.24r/v). If the influence of the stresses normal to the bars on the bond characteristics could be taken into account, this would probably even increase the accuracy of the approximation, since slightly higher values for \( \tau \) would be obtained for values \( w < 0.4-0.5 \) mm.
Fig. 4.22 Comparison between calculated relations (dotted lines) and experiments (solid lines)
Fig. 4.24 Comparison between calculated relations (dotted lines) and experimental ones (solid lines).
4.4.2 Specimens with reinforcing bars inclined to the crack plane

It can be shown that the behaviour of specimens with reinforcing bars inclined to the crack plane is not essentially different from that of specimens with reinforcement perpendicular to the crack plane. To be able to construct the $T-W$ and $A-W$ relations it is necessary to calculate the restraint stress normal to the crack plane, and the dowel action of the bars. The restraint stress normal to the crack plane is, in the case of inclined bars, not only a function of the crack width $w$, but also of the shear displacement $\Delta$ (fig. 4.25).

Fig. 4.25 Pull out slip for an inclined reinforcing bar

For a displacement $(w, \Delta)$ the total pull out slip of the reinforcing bar is equal to $w = w \sin \theta + \Delta \cos \theta$. The total steel force $F_{s,\theta}$ in the direction $\theta$ can be calculated using the fig. 4.5, replacing $w$ by $w_\perp$ and multiplying $\sigma$ with $A_c$, and is subsequently resolved in a restraint force normal to the crack plane equal to $F_{s\theta} \cdot \sin \theta$, and a shear force parallel to the crack plane, equal to $F_{s\theta} \cdot \cos \theta$. For the dowel action of the inclined bars an expression suggested by MATTOCK [12] has been used, which related this force to the dowel action of a bar perpendicular to the crack, according to the formula

$$F_{d,\theta} = F_{d,90} \cdot \sin^2 \theta$$

(4.11)
The $\tau-w$ and $w-\Delta$ curves can be constructed, requiring equilibrium in the direction normal to the crack. The stress normal to the crack, due to aggregate interlock, is formulated as a function of $w$ and $\Delta$ in eq. (3.55) and (3.56). The restraint stress normal to the crack, due to the tensile force in the reinforcement is calculated as described previously, also as a function of $w$ and $\Delta$ (dashed lines in fig. 4.26). Combinations of $(w, \Delta)$ for which equilibrium is obtained can be graphically assessed with interaction diagrams (fig. 4.26).

![Interaction Diagram](image)

**Fig. 4.26 Interaction diagram for $f' = 34 \text{ N/mm}^2$ and 2 stirrups, $\phi$ 8 mm inclined to the crack plane with $\theta = 112^\circ$**

The according value of $\tau$, only as a result of aggregate interlock can be read in the upper part of the diagram. To obtain the total shear force this value has to be increased with a term, resulting from the axial steel force and a term resulting from dowel action. Furthermore, it has to be checked whether the critical crack opening path is not exceeded. The results of this calculation are represented in fig. 4.28. The agreement between calculated and experimental results is satisfying. In the calculation it was found that even for $\theta = 135^\circ$ the reinforcement was subjected to a tensile stress. In the cases of $\theta > 90^\circ$ this axial tensile force itself has a
negative influence on the shear resistance, but acts positively by providing a restraint stiffness against crack opening and, as such, activating aggregate interlock (fig. 4.27).

Fig. 4.27 Equilibrium in a crack with reinforcement inclined with \( \theta = 135^\circ \)
Fig. 4.28 Comparison between calculated (dotted) and experimental (solid) $\tau$-w and w-$\Delta$ relations for specimens with inclined bars.
4.5 Hypothesis for the behaviour of reinforced cracks subjected to general combinations of external loads and displacements

In the previous section it was shown that the behaviour of the test specimens could be described by assuming two levels of roughness, a minor one related with the contact stresses provided by the embedded particles and a major one, related with the mechanism of transfer of stresses over loose particles around the reinforcing bars. For the tests carried out, the level of major roughness seemed to be a characteristic property, since great repeatability of crack opening pathes was observed. However, nearly all specimens had an initial crack width < 0.04 mm, and all specimens were only subjected to (external) shear forces. Due to these restrictions the information, which was obtained, was limited. It may for instance be wondered what crack opening path would be obtained if the initial crack width would be greater, or, if the external shear force would be combined with an axial tensile force.

It seems not unlikely that also in other points of the $w, \Delta$ plane critical crack opening directions exist due to locking up of loose particles. At the moment only scant evidence is available to support this supposition. Besides a few tests carried out with slightly greater initial crack widths, represented in fig. 2.3b and 2.4b, only a series, carried out by MATTOCK [12] was found in literature, yielding complementary results. In this investigation comparable, precracked specimens, with reinforcing ratios between 0.4-2.3%, were subjected to an external shear load. During precracking the crack width reached an average maximum value of 0.28 mm. When the line loads were removed a residual crack width of about 0.23 mm remained, this being the average width of the crack in the shear plane before the shear transfer test.

Fig. 4.29 shows the crack opening path (supposing that for all specimens $w_o = 0.23$ mm) for this series, with $f'_{ccyl} \approx 28$ N/mm$^2$ (which agrees with a cube crushing strength of about $f'_c = 35$ N/mm$^2$), and $D_{max} = 19$ mm. It is seen that after a short vertical branch an approximately constant slope is followed (other tests, on other types of concrete didn't exhibit the vertical branche, see also [23, p. 85]), so that also here a characteristic crack opening path seems to exist.
The average crack opening path obtained in the own tests for intermediate concrete strengths, is indicated with a dashed line in fig. 4.29. It may be assumed that in every point of the \( w, \Delta \) plane a critical crack opening direction exists, which cannot be exceeded, and that these critical directions can be represented by the definition of a continuous vector field. An example is given in fig. 4.30, in which the expression

\[
\frac{d\Delta}{dw} = w^{0.18} (1.65 - 2.10 w) - 1.5 \Delta \tag{4.12}
\]

is used as a definition-formula for the critical crack opening direction.

This formula is constructed in such a way that both for the own tests, and for MATTOCKS tests, a fitting crack opening path is obtained (see also fig. 4.31).
Fig. 4.30 Vector field according to eq. (4.12), defined in such a way that both for MATTOCKS [12] and the own tests a good approximation of the experimental crack opening path is obtained.

Fig. 4.31 Comparison between experimental values of tests by MATTOCK [12] (solid lines) with calculated relations (dashed lines)
It can be demonstrated that also MATTOCKS test results [12] can be reasonably well described using the procedure described in section 4.4, with the formulas (3.55), (3.56), (4.8) and the figures 4.5, if eq. (4.12) is used to define the critical crack opening path. A comparison between calculated and experimental relations for these tests is represented in fig. 4.31.

Since, however, only a few test results are available, further experiments on reinforced specimens are necessary, focusing on the existence of critical crack opening paths for a wider range of \( w, \Delta \) values. Also aspects of load history should be taken into account.
CONCLUSIONS

1. Aggregate interlock has not only to be associated with the transfer of shear stresses, as has been done in the utmost part of literature on this subject, but also with stresses normal to the crack faces.

2. Both shear and normal stresses resulting from aggregate interlock in cracks in plain concrete are mainly a function of crack width, shear displacement and concrete quality.

3. Stresses due to aggregate interlock in cracks in plain concrete are not proportional to the scale of the aggregate particles. Variation of the scale with a factor 2 \( (D_{\text{max}} = 16 \text{ mm to } 32 \text{ mm}) \) showed that the normal stress was scarcely influenced, and that the influence on the shear stress was slight for small crack widths, but increased for greater crack widths.

4. According to the experiments the influence of the global ondulation of the crack face is not significant. Aggregate interlock is mainly a function of the minor roughness of the crack plane, provided by the aggregate particles protruding from the crack face.

5. Variation of the aggregate particle distribution (sieve line) within practical limits has only a slight influence on the development of the normal stress in the crack. The influence on the shear stress is slight for small values of the crack width, but increases for increasing crack widths.

6. The aggregate interlock action in a crack in plain concrete is characterized by crushing of matrix material and overriding of the crack faces, during which the friction between particles and matrix contributes significantly to the resistance against crack opening. The brittle character of the matrix is confirmed by tests on specimens subjected to cyclic loading, reported in literature, exhibiting a considerable difference in behaviour between the first and the subsequent loading cycles.
7. The behaviour of reinforced cracks under shear forces cannot merely be described with the relations, derived for aggregate interlock in cracks in plain concrete, even if dowel action and axial steel stress would be accurately known. A fundamental difference in behaviour is caused by modifications of the crack structure around the reinforcing bars, due to splitting forces caused by the pull out forces in the bars, resulting in local concentrations of loose particles. The crack opening path for reinforced cracks is not any more dependent on the restraint stiffness against crack opening, as for cracks in plain concrete, but remains constant for a great range of reinforcement ratios. Only for low reinforcement ratio ($\phi_c < 0.6\%$) a relation with the restraint stiffness seems to be obtained. The crack opening paths for different types of concrete of intermediate strengths ($20 \text{ N/mm}^2 < f'_c < 38 \text{ N/mm}^2$) exhibit only slight differences. For concretes in which the crack may intersect the aggregate particles (high strength - or lightweight concrete), deviating crack opening paths were obtained, exhibiting greater relative shear displacements.

8. Variations of the number of bars (and their diameters) at constant reinforcement ratio has a negligible influence on the behaviour of cracks subjected to shear loading.

9. Unloading and reloading of reinforced cracks gives rise to considerable hysteresis effects. This can be explained by the existence of frictional forces between aggregate particles and matrix material.

10. The response of reinforced cracked specimens subjected to shear loading can be described considering the crack opening path as a characteristic property and assuming that the loose particles around the reinforcing bars provide only a reaction perpendicular to the actual crack opening direction.

11. Further experiments are necessary to study the effect of critical crack opening directions in reinforced concrete for a greater range of displacements.
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APPENDICES

Appendix I

Determination of the contact area for an intersection circle with radius R as a function of the crack width w and the shear displacement Δ.

A contact area can only exist if \( R < w \). If \( R > w \) three possibilities are distinguished:

1. \( \Delta < \Delta_o \); Phase "No contact"; \( a_x = 0; a_y = 0 \)

2. \( \Delta_o < \Delta < \Delta_b \); Phase "Growing contact";

![Diagram of contact components](image1)

**Fig. I.1** Contact components \( a_x \) and \( a_y \) for \( w < R \) and \( \Delta_o < \Delta < \Delta_b \)

The coordinates of S and T can be calculated. To simplify this calculation the xy-axes are rotated over an angel \( \alpha \), so that in the new situation the displacement is not characterized by \( (\Delta, w) \) but by \( (v, 0) \) with \( v = \sqrt{\Delta^2 + w^2} \) (fig. I.2).

![Diagram of rotated axes](image2)

**Fig. I.2**
The coordinates of S can be calculated using fig. I.2. S must fulfil the conditions

\[ x_1^2 + y_1^2 = R^2 \]

\[ x_1 = \frac{1}{2} v \]

Which results in

\[ y_1 = \sqrt{R^2 - \frac{1}{4} v^2} \]

So that:

\[ (x_1, y_1) = \left( \frac{1}{2} v, \sqrt{R^2 - \frac{1}{4} v^2} \right) \]

The relations between the new and the old coordinated are expressed by:

\[ x_o = x_1 \cos \alpha - y_1 \sin \alpha \]

\[ y_o = x_1 \sin \alpha + y_1 \cos \alpha \]

So the coordinates of S in the main xy-system are:

\[ x_S = \frac{1}{2} v \cos \alpha - \sqrt{R^2 - \frac{1}{4} v^2} \sin \alpha \quad (I.1) \]

\[ y_S = \frac{1}{2} v \sin \alpha + \sqrt{R^2 - \frac{1}{4} v^2} \cos \alpha \]

The coordinates of point T can be established immediately (fig. I.1)

\[ y_T = u + w \quad (I.2) \]

\[ x_T = - \sqrt{R^2 - (u + w)^2} \]

By subtracting T from S it is found that:

\[ a_y = y_S - y_T = \frac{1}{2} v \sin \alpha + \sqrt{R^2 - \frac{1}{4} v^2} \cos \alpha - u - w \quad (I.3) \]

\[ a_x = x_S - x_T = \frac{1}{2} v \cos \alpha - \sqrt{R^2 - \frac{1}{4} v^2} \sin \alpha + \sqrt{R^2 - (u + w)^2} \]
Furtheron \( v \) and \( \alpha \) are related to \( w \) and \( \Delta \) by:

\[
\begin{align*}
    v \sin \alpha &= w \\
    v \cos \alpha &= \Delta \\
    v &= \sqrt{w^2 + \Delta^2}
\end{align*}
\]  

So \( \sin \alpha = \frac{w}{\sqrt{w^2 + \Delta^2}} \) and \( \cos \alpha = \frac{\Delta}{\sqrt{w^2 + \Delta^2}} \)  

Substitution of (1.4) and (1.5) in (1.3) results in:

\[
a_y = \sqrt{R^2 - \frac{1}{4}(w^2 + \Delta^2)} - \frac{\Delta}{\sqrt{w^2 + \Delta^2}} - \frac{1}{2} w - u \tag{1.6}
\]

\[
a_x = \frac{1}{2} \Delta - \sqrt{R^2 - \frac{1}{4}(w^2 + \Delta^2)} - \frac{w}{\sqrt{w^2 + \Delta^2}} + \sqrt{R^2 - (u + w)^2}
\]

3. \( \Delta_b < \Delta \); Phase "Maximum contact";

It is easily deduced (fig. I.1) that:

\[
\begin{align*}
    a_y &= R - (u + w) \tag{1.7} \\
    a_x &= \sqrt{R^2 - (u + w)^2}
\end{align*}
\]
Appendix II

Construction of the ascending branch of the τ-Δ relationship represented in fig. 3.32.

The relations between \( w, \Delta, \tau \) and \( \sigma \), according to fig. 3.21, are represented in fig. II.1 for the crack widths \( w = 0.5, 0.6 \) and 0.7 mm.

The initial crack width is \( w_o = 0.50 \) mm. It is seen that for \( w = 0.50 \) mm and \( \Delta = 0.1, 0.2 \) and 0.3 mm no increase of crack width can be expected, since no normal stress \( \sigma \) is developed. Furtheron it is known that an increase of crack width of \( \Delta w = 0.1 \) mm results in an increase of the normal stress with \( \Delta \sigma = 0.1 \) N/mm\(^2\). So the following points form part of the ascending branch.
Calculation of point C in fig. 3.32

<table>
<thead>
<tr>
<th>$w$ (mm)</th>
<th>$\Delta$ (mm)</th>
<th>$\sigma$ (N/mm$^2$)</th>
<th>$\tau$ (N/mm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.1</td>
<td>0</td>
<td>0.25</td>
</tr>
<tr>
<td>0.50</td>
<td>0.2</td>
<td>0</td>
<td>0.85</td>
</tr>
<tr>
<td>0.50</td>
<td>0.3</td>
<td>0</td>
<td>1.75</td>
</tr>
<tr>
<td>0.60</td>
<td>0.52</td>
<td>0.5</td>
<td>3.10</td>
</tr>
</tbody>
</table>

Fig. II.2 Calculation of the residual shear displacement $\Delta_r$ after unloading in the first cycle

At peak stress level $w_0 = 0.6$ mm and $\Delta = 0.5$ mm. Substituting these values in combination with $R = 16$ mm in eq. (I.6), it is found that $a_y = 9.95$ mm.

In fig. II.2 it is seen that

$$\Delta_r = \Delta_2 - \Delta_1$$

Circle I: $x_1^2 + y_1^2 = R^2$

$$\Rightarrow x_1 = 12.53 \text{ mm} \quad \Delta_1 = R - x = 3.47 \text{ mm}$$

$y_1 = a_y$

Circle II: $x_2^2 + y_2^2 = R^2$

$$\Rightarrow x_2 = 12.12 \text{ mm} \quad \Delta_2 = R - x_2 = 3.88 \text{ mm}$$

$y_2 = w_0 + a_y$

So:

$$\Delta_r = \Delta_2 - \Delta_1 = 3.88 - 3.47 = 0.41 \text{ mm}.$$
Appendix III

<table>
<thead>
<tr>
<th>$f'_R$</th>
<th>$a_o$</th>
<th>$b_o$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.0320</td>
<td>0.129</td>
<td>2.34</td>
</tr>
<tr>
<td>0.010</td>
<td>0.0317</td>
<td>0.300</td>
<td>2.00</td>
</tr>
<tr>
<td>0.025</td>
<td>0.0317</td>
<td>0.680</td>
<td>1.85</td>
</tr>
<tr>
<td>0.050</td>
<td>0.0314</td>
<td>0.872</td>
<td>2.10</td>
</tr>
<tr>
<td>0.100</td>
<td>0.0315</td>
<td>1.135</td>
<td>2.31</td>
</tr>
<tr>
<td>0.200</td>
<td>0.0322</td>
<td>1.353</td>
<td>2.53</td>
</tr>
<tr>
<td>0.400</td>
<td>0.0316</td>
<td>1.308</td>
<td>2.85</td>
</tr>
</tbody>
</table>

Table A Constants to be inserted into basic bond stress-slip relation according to \[11\]

<table>
<thead>
<tr>
<th>$f'_R$</th>
<th>limit slip value $\Delta_{s,\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f'_{cc} = 20 , \text{N/mm}^2$</td>
</tr>
<tr>
<td>0.005</td>
<td>5.00</td>
</tr>
<tr>
<td>0.010</td>
<td>3.06</td>
</tr>
<tr>
<td>0.025</td>
<td>1.50</td>
</tr>
<tr>
<td>0.050</td>
<td>0.88</td>
</tr>
<tr>
<td>0.100</td>
<td>0.51</td>
</tr>
<tr>
<td>0.200</td>
<td>0.30</td>
</tr>
<tr>
<td>0.400</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Table B Limit values for the slip according to \[11\]
Fig. III.1 Pull-out characteristic, calculated on basis of \([11]\).

Algol program calculating the slip of a bar as a result of an axial pull-out force, based on a difference method as described in \([11]\).

```
0 'BEGIN'
1 'REAL' FR, D, S, BW, BN, COEF, KR, SP, SL, T, DSP, DSL, X, SVL, EN, DEN, INTA, INTB, DEGR,
2 BET;
2 'INTEGER' I, J, HI, HJ;
3 'REAL' 'ARRAY' F(/1:7/), BWU(/1:3/), DEG(/1:3,1:7/), B(/1:7/), BETA(/1:7/);
4 INARRAY(0,F);
5 INARRAY(0,BWU);
6 INARRAY(0,DEG);
7 INARRAY(0,B);
8 INARRAY(0,BETA);
9 INREAL(0,D);
10 INREAL(0,FR);
11 INREAL(0,S);
12 INREAL(0,BW);
13 INREAL(0,SVL);
14 OUTSTRING(1,'('BARDIAMETER(MM)=')');FIX(1,2,0,D);LINE(1,2);
```
17 OUTSTRING(1,'("FR="') ; FIX(1,1,3,FR) ; LINE(1,2) ;
20 OUTSTRING(1,'("CUBE STRENGTH="') ; FIX(1,2,0,BW) ; LINE(1,5) ;
23 OUTSTRING(1,'("YIELD STRESS STEEL="') ; FIX(1,4,0,SVL) ; LINE(1,5) ;
26 OUTSTRING(1,'("FORCE (N) SLIP (MM) BONDLENGTH (MM) ENERGY (KN.MM'
27 ')")') ;
27 LINE(1,2) ;
28 I:=0; J:=0;
30 DOX: J:= J+1;
32 'IF' FR-F(/J/) > 0 'THEN' 'GOTO' DOX;
34 MOX: I:= I+1;
36 'IF' BW-BWU(/J/) > 0 'THEN' 'GOTO' MOX;
38 HI:= I-1;
39 HJ:= J-1;
40 INTA:= DEG(/HI,HJ/) - (FR-F(/HJ/)) / (F(/J/)-F(/HJ/)) * (DEG(/HI,HJ/) - DEG(/HI,J /
41 /));
41 INTB:= DEG(/I,HJ/) - (FR-F(/HJ/)) / (F(/J/)-F(/HJ/)) * (DEG(/I,HJ/) - DEG(/I,J/)
42 );
42 DEGR:= INTB + (BWU(/I/) - BW) / 20 * (INTA-INTB);
43 BN:= B(/HJ/) + (B(/J/) - B(/HJ/)) * (FR-F(/HJ/)) / (F(/J/)-F(/HJ/));
44 BET:= (BETA(/J/) - BETA(/HJ/)) * (FR-F(/HJ/)) / (F(/J/)-F(/HJ/)) + BETA(/HJ/);
45 COEF:= 1 / BET;
46 KR:= 0; X:= 0; SP:= 0; SL:= 0.001; EN:= 0;
51 NEXT: T:= (0.032 + BN*K(SL/10) + K*COEF*BW;
53 DSP:= 4*K*S/T/D;
54 KR:= 0.785*K*D*M*(SP+DSP);
55 DSL:= S*K*(2*SP+DSP)/420000;
56 DEN:= (DSL*K*0.785*K*D*M*(SP+0.5*K*DSP))/1000;
57 EN:= EN+DEN;
58 SL:= SL+DSL;
59 X:= X+S;
60 SP:= SP+DSP;
61 FIX(1,6,4,KR); BLANK(1,3);
63 FIX(1,1,4,SL); BLANK(1,4);
65 FIX(1,4,2,X); BLANK(1,5);
67 FIX(1,3,3,EN);
68 LINE(1,2);
69 'IF' SP<SVL 'THEN' 'GOTO' NEXT;
71 OUTSTRING(1,'("SLIP="') ;
72 FIX(1,1,4,DEGR); LINE(1,2);
74 'END'
0.005, 0.010, 0.025, 0.050, 0.100, 0.200, 0.400
20, 40, 60
5.00, 3.06, 1.50, 0.88, 0.51, 0.30, 0.217
3.60, 2.16, 0.62, 0.375, 0.21, 0.15
3.00, 1.80, 0.88, 0.52, 0.30, 0.18, 0.127
0.129, 0.300, 0.680, 0.872, 1.135, 1.353, 1.308
2.34, 2.00, 1.85, 2.10, 2.31, 2.53, 2.85
8, 0.043, 5, 38, 430;

The input of data is done in the last line. The sequence is:
bar diameter (mm) (8 in example)
related rib area f_R (0.043 in example)
step (rib distance, mm) (5 in example)
cube strength (150 mm - N/mm²) (38 in example)
yield strength steel, N/mm² (430 in example)
The program can be used for concretes with cube crushing strengths
20 ≤ f'_cc ≤ 60 N/mm².
Appendix IV

Algol program for stepwise numerical integration of eq. (3.40-3.43).

0 'BEGIN'
1 'REAL'DMAX,PK,W,A,DELTA,SIGMAY,DOND,DBOV,HU,TN,DG,UMAX,S,T,
2 HDX,HDY,FX,FY,DIX,DIY,AX,AY,NMAX,NC,MU,SU,SIGMAX;
2 'INTEGER'N,P;
3 'REAL'ARRAY'D(/0:100/),TERM(/1:100/);
4 DMAX:=16.03;PK:=0.75;
6 NMAX:=10;
7 SU:=48;
8 MU:=0.50;
11 OUTSTRING(1,'('PARAMETERS')');LINE (1,1);
13 OUTSTRINGd,'('SU=')');FIX(1,2,1,SU) ;LINE(1,1) ;
16 OUTSTRINGd,'('MU=') ') ;FIXd,l,2,MU) ;LINE(1 ,1) ;
19 OUTSTRINGd,'('PK=')');FIXd,l,2,PK);LINE(1,1);
22 OUTSTRINGd,'('NMAX=')');FIX(1,3,0,MMAX);LINE(1,3);
25 W:=0;
27 N:=0;
27 CROCK:N:=N+1;
29 D(/N/):=(N/NMAX)KDMAX-DMAX/(2nNMAX);
30 A:=D(/N/)/DMAX; .
31 TERM(/N/):=0.727KAM+0.5-AK+0.144AK+4+0.036AK+6+0.016AK+8+0.01
32 AK+10;
32 'IF'N<NMAX'THEN'GOTO'CROCK;
34 BROM:W:=W+0.1;
36 DELTA:=0;
37 SHOT:DELTA:=DELTA+0.1;
39 'IF'DELTA<W+0.01'THEN'
40 'BEGIN'
41 SIGMAY:=0;SIGMAX:=0;
43 DOND:=(W+2+DELTA+2)/DELTA;
44 N:=0;
45 CRJ:N:=N+1;P:=N-1;
48 D(/N/):=(N/NMAX)KDMAX;
49 'IF'D(/N/)<DOND'THEN'GOTO'CRU;
51 HU:=W+2+DELTA+2;
52 TN:=TERM(/N/); 
53 'IF' D(/N/)−DOND>DMAX/NMAX'THEN' 
54 DG:=(D(/N/) + D(/P/))/2 
55 'ELSE' 
56 'BEGIN' 
57 D(/P/):=DOND; 
58 A:=DG/DMAX; 
59 TN:=0.727KAMX0.5−AMX2+0.144KAMX4+0.036KAMX6+0.016KAMX8+0.01KAMX10; 
60 'END'; 
61 'END'; 
62 UMAX: = (-0.5KDWH+0.5KWSQRT(W2+WH2−HUx(HUx2−(DELTAxDG)x2))/HU; 
63 S:=(W+UMAX)x2/DG; 
64 T:=2xW/DG; 
65 HDY:=(SQRT(DGxDG−HU)xDELTAXUMAX/SQRT(HU)−WxUMAX−UXMAXx2)/DGx3; 
66 HDX:=((DELTAX−SQRT(DGx2+WH2)xHU/SQRT(HU))xUMAX+UMAX+W)x 
67 SQRT(0.25xDGxWTx−WxSQRT(0.25xDGx3x2−W2xW))+(UMAX+W)x 
67 +0.25xDGxWTxARCTAN(S/SQRT(1−SxS))−DGx2/4xARCTAN(T/SQRT(1−T2xT)))/DGx3 
67 3; 
67 FX:=PKx1.273xTNxHDX; 
68 FY:=PKx1.273xTNxHDY; 
69 DIX:=FXx(D(/N/)−D(/P/)); 
70 DIY:=FYx(D(/N/)−D(/P/)); 
71 SIGMAX:=SIGMAX+DIX; 
72 SIGMAY:=SIGMAY+DIY; 
73 'IF' N<NMAX'THEN' 'GOTO' CRU; 
75 'END' 
76 'ELSE' 
77 'BEGIN' 
78 SIGMAY:=0;SIGMAX:=0; 
80 DOND:=2xW; 
81 DBOV:=(WH2+DELTAxW2)/DELTA; 
82 N:=0; 
83 TOL:=N+1;P:=N−1; 
86 D(/N/) := (N/NMAX)xDMAX; 
87 'IF'D(/N/)−DOND'THEN' 'GOTO' TOL; 
89 HU:=WH2+DELTAxW2; 
90 TN:=TERM(/N/);
'IF' D(/N/) > DBOV' THEN'
'BEGIN'
'IF' D(/N/) - DBOV < DMAX/NMAX' THEN'
'BEGIN'
'IF' D(/N/) - DOND < DMAX/NMAX' THEN'
'D(/P/) := DOND;
DG := (D(/P/) + DBOV) / 2;
A := DG / DMAX;

TN := 0.727 * AN * 0.5 - AN * 2 + 0.144 * AN * 4 + 0.036 * AN * 6 + 0.016 * AN * 8 + 0.01 * AN * 10;
HDY := (0.5 * W * DG - W) / DG * 2;
T := 2 * W / DG;
HDX := (0.393 * DGM / W * SQRT(0.25 * DGM * W - W / 2) - 0.25 * DGM * W / ARCTAN(T / SQRT(1 - T^2))) / DGM * 3;
FX := PK * 1.273 * TN * HDX;
FY := PK * 1.273 * TN * HDY;
DIX := FX * (DBOV - D(/P));
DIY := FY * (DBOV - D(/P));
DG := (DBOV + D(/N)) / 2;
A := DG / DMAX;

TN := 0.727 * AN * 0.5 - AN * 2 + 0.144 * AN * 4 + 0.036 * AN * 6 + 0.016 * AN * 8 + 0.01 * AN * 10;
UMAX := (-0.5 * W * HU + 0.5 * SQRT(W * HU * 2 - HU * (HU - (DELTA / DG) * W / 2))) / HU;
S := (W + UMAX) / 2 / DG;
T := 2 * W / DG;
HDY := (SQRT(DGM / DG - HU) * DELTA / UMAX / SQRT(HU)) - W * UMAX - UMAX * S / SQRT(HU) + (UMAX + W) * S;
HDX := ((DELTA / SQRT(DGM / DG - HU) * SQRT(HU)) * UMAX + (UMAX + W) * S / SQRT(HU) - W * SQRT(0.25 * DGM / DG - W)) / DGM * 2 / 4 / ARCTAN(T / SQRT(1 - T^2))) / DGM;

FX := PK * 1.273 * TN * HDY;
FX := PK * 1.273 * TN * HDX;
DIY := DIY + FY * (D(/N) - DBOV);
DIX := DIX + FX * (D(/N) - DBOV);
'END'
'ELSE'
'BEGIN'
DG := (D(/N) + D(/P)) / 2;
UMAX := (-0.5 * W * HU + 0.5 * SQRT(W * HU * 2 - HU * (HU - (DELTA / DG) * W / 2))) / HU;
\[ S := (W + U_{\text{MAX}}) \frac{\kappa_2}{D_G}; \]
\[ T := 2 \frac{m}{W/D_G}; \]
\[ H_{DY} := \frac{(\text{SQRT} (D_G \Delta \kappa - H_u) \kappa_{\Delta \kappa} U_{\text{MAX}} / \text{SQRT} (H_u) - W \kappa_{\text{MAX}} - U_{\text{MAX}} \kappa_2)}{D_G \kappa_{\Delta \kappa} \kappa_{\Delta \kappa}}; \]
\[ H_{DX} := \frac{(\text{SQRT} (D_G \Delta \kappa^2 - H_u \kappa_{\Delta \kappa} / \text{SQRT} (H_u)) \kappa_{\Delta \kappa} (U_{\text{MAX}} + W) + 0.25 \kappa_D \kappa_{\Delta \kappa} - W H_{\text{MAX}} \text{ARCTAN} S / \sqrt{0.25 \kappa_D H_{\text{MAX}}^2 - W H_{\text{MAX}}^2})}{D_G \kappa_{\Delta \kappa} \kappa_{\Delta \kappa}^3}; \]
\[ F_Y := P K_{1.273 T_N} H_{DY}; \]
\[ F_X := P K_{1.273 T_N} H_{DX}; \]
\[ D_{\text{MAX}} := \frac{F_Y (D(N) - D(P))}{D(G_N - D(\text{DOND}) / 2; \}
\[ A := D_{\text{MAX}} / D_{\text{DOND}}; \]
\[ T_N := 0.727 \kappa_{\text{MAX}}^0.5 - A \kappa_{\text{MAX}} + 0.144 \kappa_{\text{MAX}}^4 + 0.036 \kappa_{\text{MAX}}^6 + 0.016 \kappa_{\text{MAX}}^8 + 0.01 \kappa_{\text{MAX}}^{10}; \]
\[ F_X := P K_{1.273 T_N} H_{DX}; \]
\[ F_Y := P K_{1.273 T_N} H_{DY}; \]
\[ D_{\text{MAX}} := \frac{F_X (D(\text{BOV}) - D(P))}{D(G_N - D(\text{DOND}) / 2; \}
\[ T := 2 \frac{m}{W/D_G}; \]
\[ H_{DX} := \frac{(0.393 \kappa_D \kappa_{\Delta \kappa}^2 - W H_{\text{MAX}} \text{SQRT} (0.25 \kappa_D \kappa_{\Delta \kappa}^2 - W H_{\text{MAX}}^2) - 0.25 \kappa_D \kappa_{\Delta \kappa} \text{ARCTAN} (T / \sqrt{1 - T W T}))}{D_G \kappa_{\Delta \kappa} \kappa_{\Delta \kappa}^3}; \]
\[ F_Y := P K_{1.273 T_N} H_{DY}; \]
\[ F_X := P K_{1.273 T_N} H_{DX}; \]
\[ D_{\text{MAX}} := \frac{F_X (D(\text{BOV}) - D(P))}{D(G_N - D(\text{DOND}) / 2; \}
\[ T := 2 \frac{m}{W/D_G}; \]
\[ H_{DX} := \frac{(0.393 \kappa_D \kappa_{\Delta \kappa}^2 - W H_{\text{MAX}} \text{SQRT} (0.25 \kappa_D \kappa_{\Delta \kappa}^2 - W H_{\text{MAX}}^2) - 0.25 \kappa_D \kappa_{\Delta \kappa} \text{ARCTAN} (T / \sqrt{1 - T W T}))}{D_G \kappa_{\Delta \kappa} \kappa_{\Delta \kappa}^3}; \]
158 \( AY := \Sigma u (\Sigma M - \Sigma u \Sigma m) \);
159 \( AX := \Sigma u (\Sigma M - \Sigma u \Sigma m) \);
160 \text{OUTSTRING}(1, 'W = ') \text{FIX}(1, 1, 1, W) \text{BLANK}(1, 3);
163 \text{OUTSTRING}(1, 'DELTA = ') \text{FIX}(1, 1, 2, DELTA) \text{BLANK}(1, 3);
166 \text{OUTSTRING}(1, 'SIGMA = ') \text{BLANK}(1, 1) \text{OUTREAL}(1, AY) \text{BLANK}(1, 3);
170 \text{OUTSTRING}(1, 'SIGMA = ') \text{BLANK}(1, 1) \text{OUTREAL}(1, AX) \text{LINE}(1, 2);
174 'IF' \text{DELTA} < W + 0.01 'THEN' 'GOTO' SHOT;
176 'IF' W < 0.99 'THEN' 'GOTO' BROM;
178 'END'

Comment: the parameters are inserted in the beginning of the program:

- line 4: DMAX is the maximum particle diameter (16 mm in the example)
PK is the ratio aggregate volume/concrete volume (0.75 in the example)

- line 6: NMAX is the number of integration steps (10 in the example)

- line 7: SU is the matrix normal crushing strength in N/mm² in this report denoted \( \sigma_{pu} \) (48 in the example)

- line 8: MU is the coefficient of friction between matrix and aggregate, in this report denoted \( \mu \) (0.50 in the example).
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