On actuator disc force fields generating wake vorticity

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1 Introduction:

Actuator disc calculations can be divided in two categories: force models where, for a prescribed force field, the flow is calculated using a CFD method, and kinematic models, where the wake is calculated based on wake boundary conditions and the force field is known when the velocities are known. In both categories, but specifically for the kinematic models, results are reported that differ some 10% from momentum models. Furthermore, most calculations which give details about the flow through the disc do not satisfy the condition derived by Xyros & Xyros (2007) that the axial velocity through the disc is uniform for discs with a uniform surface load. Apart from this, the inconsistency in the momentum models discussed by van Kuik (2003) is still unresolved. These observations raise the questions: what is the relation between force- and flow field, what are the requirements for a steady axisymmetric force field to generate vorticity in an Euler flow?

2 The equations of motion:

A new formulation for the equation of motion for actuator disc flow fields is obtained by merging of three equations: the Euler equation of motion,

\[ \mathbf{f} = \nabla H - \rho \mathbf{v} \times \mathbf{w} \]  

the power conversion equation

\[ \mathbf{f} \cdot \mathbf{v} = \Omega \rho f_\phi \]  

which equates torque times rotational speed to the work done by the force field, and the expression for axisymmetric vorticity \( \mathbf{w} \) in a cylindrical system \((x,r,\phi)\). This results in:

\[ \mathbf{f} = -\rho \left( \mathbf{v}_{rot} \times \mathbf{w}_{rot} + \mathbf{v}_{rot} \times \mathbf{e}_r 2\Omega \right) \]  

\[ = -\rho \mathbf{v}_{rot} \times \mathbf{w}_{rot} \]  

where the transformation from the co-rotating reference frame to the inertial frame is \( \mathbf{v}_{rot} = \mathbf{v} - \mathbf{e}_r \Omega \) and \( \mathbf{w}_{rot} = \mathbf{w} - \mathbf{e}_r \Omega \), \( \mathbf{e}_r \) being the unit vector. The first term at the right hand side of (3) is the Kutta-Joukowsky force on the bound vorticity, where the second term is the Coriolis force. Both are perpendicular to the local velocity, so do not perform work in the rotating system. In eq. (4) the subscript rot distinguishes this expression from the expression of a Kutta-Joukowsky force. Compared to Euler equation (1) the Bernoulli constant \( H \) is absent since the conversion of power is now expressed in kinematical terms. This enables a much easier interpretation of the flow and force field, and makes a comparison possible of the disc force and flow field with the force and flow field of real rotor blade, see section 4. Eq. (4) is consistent with Wu (1962), who derived the flow equation expressed in the streamfunction, showing the occurrence of a force component perpendicular to the streamtube. According to (4) this is the load on disc-bound vorticity.
3 The generation of a Rankine vortex

The new formulation of the actuator disc force field is confirmed by an analytical, exact, solution: the generation of a Rankine vortex by a force field in a flow that is irrotational upstream of the disc. The disc has thickness ε. The vorticity that is generated in the vortex kernel (radius R) is axial, see figure 1. Eq. (4) gives the force field, showing axial and azimuthal components as expected. For \( r < R \), so where vorticity is produced, also a radial component is present. Interpretation of this component in the inertial frame shows that it is needed to satisfy the centripetal momentum balance in the disc volume. In the rotating system it is the Kutta-Joukowsky and Coriolis force acting on the bound vorticity.

The impact of this load is verified by numerical calculations using Fluent. For a thick disc, \( \varepsilon = R \), figure 2 shows the radial velocity. Indeed application of only the appropriate thrust and torque is not sufficient. For a thin disc the effect of the absence of a radial force vanishes, as confirmed by the analytical solution.

4 The force field of a rotor blade

Before discussing the classical disc with uniform axial surface load, the correspondence between a disc- and a rotor force field is treated. Although eq. (4) is derived for the disc, is also the equation used to determine blade loads. This is usually done in the rotating reference frame, applying the Kutta-Joukowsky law \( \mathbf{L} = -\rho \mathbf{v}_{\text{rot}} \times \Gamma_{\text{rot}} \), where \( \Gamma = \oint \omega dC \) is the circulation bound in the blade cross section \( C \). Since \( \omega_{\text{rot}} = \omega - \mathbf{e}_z \Omega \), only the axial component of \( \Gamma \) is affected by the transformation rotating-to-inertial. For blades with a zero pitch angle, \( \Gamma_x = 0 \) so \( \Gamma_{\text{rot}} \equiv \Gamma \). Then the blade load is \( \mathbf{L} = -\rho \mathbf{v}_{\text{rot}} \times \Gamma \) which is the integrated version of (4). Figure 3 shows a rotor blade vortex system, including the
loads given by (4). At the tip, bound azimuthal vorticity exists, carrying a normal and spanwise load when \( \nu_{rot} \neq 0 \). The normal tip load is known as the lift on the bound part of the tip vortex. A discussion on the occurrence of in-plane loads is found in Milne-Thomson (1966) §10.61. Now for two wing flows the spanwise load is determined using Kutta-Joukovsky’s expression. For the elliptic wing it is proportional, like the induced drag, to \( \Gamma_{max}^2/A \) (\( A \) is aspect ratio, \( \Gamma_{max} \) at mid-span) and for Prandtl’s horseshoe vortex to \( \Gamma^2/b \) (\( b \) is the span). In general it is shown that the spanwise load is equivalent to the pressure integrated on the spanwise-projected surface. The conclusion is that finite lifting surfaces, like rotor blades, carry a span-wise load. For wings this load is always neglected. The question is whether this is also allowable for rotors or actuator discs.

5 The classical actuator disc

For the flow case with only azimuthal vorticity \( \omega_\phi \) (no torque, no swirl) there is no analytical solution available to check the possible impact of the load on the bound vorticity. All results published in literature do not account for \( f_{edge} \), except Greenberg & Powers (1970). For e.g. the ‘hover’ disc they have published analytical results including a singularity in \( \omega_\phi \) at the disc edge. The results deviate 10% from momentum theory results, which they could not explain. When the force at the singularity is calculated and included in the momentum balance, the results agree. CFD calculations show that without applying an edge force similar results as reported in Sörensen e.a. (1998) are obtained. Calculations including the edge load derived from Greenberg & Powers are ongoing.

6 Conclusions

- The disc force field cannot be chosen arbitrarily, as shown by the force field generating a Rankine vortex. The load on disc-bound vorticity has to be included.
- For the classical actuator disc the radial component hereof corresponds to the radial load on a real blade, which is non-zero.
- One result reported in literature implicitly accounts for the load on the disc-bound vorticity. Accounting for this load in momentum theory gives agreement with the results, whereas the authors observed disagreement.

Keywords: actuator disc, force field, vorticity, wake

References

Wu, T.Y., 1962, Flow through a heavily loaded actuator disc, Schiffstechnik, 9, 47
Xyros, M.I., Xyros, N.I., 2007, Remarks on wind turbine power absorption increase by including the axial force due to the radial pressure gradient in the general momentum theory, Wind Energy, 10, 99.