COMPUTATIONAL MODELLING OF FAILURE IN FIBRE REINFORCED PLASTIC
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FAILURE IN
FIBRE REINFORCED PLASTIC

Proefschrift

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SUMMARY

Fibre reinforced laminated plastics are becoming increasingly popular as structural materials. The material combines a light weight with a high performance and is therefore ideal for application in e.g. racing bicycles and aeroplanes. Especially in the last industry weight reduction at maintained or even increased strength compared to conventional materials is an important money saving fact. The lower the weight of the aeroplane itself, the higher the freight or passenger carrying capacity or the flight range. However, the high costs of the material and the maintenance are still a major drawback. Maintenance periods are small, since damage often cannot be detected with the naked eye, while it does decrease the load bearing capacity enormously. To increase the time span between maintenance sessions, damage prediction methods have to be developed.

In this thesis a number of material models is discussed capable of describing two of the three major failure mechanisms of fibre reinforced laminated graphite-epoxies with long unidirectional fibres in the individual plies, namely delamination and transverse matrix cracking. The third failure mechanism, fibre breaking, mainly occurs after the other two and is thus of less importance for the utilisation demands of a structure.

The delamination can be divided into two groups, namely pure mode-I delamination as it occurs in the mid-plane of a symmetric laminate and mixed-mode delamination. For both types material models have been developed, which have been implemented in a finite element code and then applied to the analyses of uniaxially loaded graphite-epoxy laminated composites. For mode-I delamination the material model is based on the formalism of orthotropic damage mechanics. Mixed-mode delamination models are based on either plasticity or damage mechanics to study the effect of both principles on the analysis time and robustness.

Driven by the comparison of plasticity and damage mechanics, the transverse matrix cracking model is again based on the formalism of orthotropic damage mechanics. With this model uniaxially loaded strips containing a centre hole are analysed.

The models are verified by comparison of the analyses results with experimental results found in literature and numerical results of other authors.
SAMENVATTING

Numerieke modellering van bezwijken in vezelversterkte kunststof

Vezelversterkte gelamineerde kunststoffen worden steeds populairder als constructiemateriaal. Het materiaal combineert een licht gewicht met goede prestaties en is daarom ideaal voor toepassing in bijvoorbeeld racefietsen en vliegtuigen. Vooral in de laatste industrie biedt gewichtsreductie bij gelijkblijvende of zelfs hogere sterkte dan conventionele materialen een belangrijke kostenbesparing. Bij een lager vliegtuiggewicht kan er meer vracht- of personenvervoer plaatsvinden of een standaard vracht kan over een grotere afstand vervoerd worden. De grootste bezwaren tegen het toepassen van het materiaal zijn de hoge fabricagekosten en de relatief korte perioden tussen onderhoud. Deze perioden zijn zo kort aangezien beschadigingen aan het materiaal vaak niet met het blote oog te detecteren zijn, terwijl de sterke aanzienlijk kan zijn aangetast. Om de onderhoudsintervallen te kunnen vergroten, zijn schade voorspellende methoden noodzakelijk.

In dit proefschrift worden enkele materiaalmodellen besproken die twee van de drie belangrijkste bezwijkmechanismes, namelijk delaminatie en transversale matrixschuurbuing, van gelamineerde grafit-epoxy composieten met unidirectionele vezels kunnen beschrijven. Het derde faalmechanisme, vezelbreuk, treedt voornamelijk op na de andere twee en is dus minder belangrijk vanuit het oogpunt van de gebruikseisen van de constructie.

Twee typen delaminatie kunnen worden onderscheiden, pure mode-I delaminatie zoals die optreedt in het middenvlak van een symmetrisch laminaat en een mengvorm van de drie breukmodes, zogenaamde mixed-mode delaminatie. Voor beide typen zijn materiaalmodellen ontwikkeld, die geïmplementeerd zijn in een eindige elementen pakket en toegepast zijn in de analyses van uniaxiale belaste, gelamineerde grafit-epoxy’s. Het materiaalmodel voor mode-I delaminatie is gebaseerd op het principe van orthotrope schademecanica. De mixed-mode delaminatie modellen zijn gebaseerd op de principes van plasticiteit of schademecanica, waarbij het verschil met betrekking tot analysetijd en robuustheid bestudeerd is.

Op basis van de vergelijking tussen plasticiteit en schademecanica voor delaminatie, is voor de beschrijving van transversale matrixschuurbuing een model ontwikkeld op basis van het principe van schademecanica. Met dit model zijn gelamineerde composieten met een gat in het midden van de plaat berekend.

De modellen zijn geverifieerd door vergelijking van de resultaten met experimentele gegevens zoals deze in de literatuur gevonden zijn en numerieke resultaten van andere auteurs.
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1. INTRODUCTION

More and more traditional materials in structures are replaced by lightweight, high performance materials such as fibre reinforced plastics. For example in surfboards and racing bicycles much more use is made nowadays of graphite fibre composites. Also the aeroplane industry tends towards the use of laminated fibre reinforced plastics (Figure 1.1), since weight reduction at maintained or even increased strength is an important money saving factor in this industry. Less weight leads to less fuel usage or to more passenger or freight carrying capacity. The main disadvantage of the material is the high cost, both of the material and of the maintenance. Up till now the periods between maintenance are still kept very small, since damage of the material often cannot be detected with the naked eye, while it does lead to an enormous reduction of the load bearing capacity due to the fairly brittle behaviour of the material.

![Figure 1.1 The new large aeroplane A3XX by Airbus](image)

To increase the time span between maintenance a better understanding of the failure mechanisms of the material is needed. Therefore, fibre reinforced laminated composites have been the subject of much research, both experimental and numerical, since the early 1970s. Although this research covers a large field ranging from static to dynamic behaviour for all types of fibre reinforced plastics, the historical overview given here will focus on laminates with long unidirectional fibres in the individual plies and the two major failure mechanisms that occur in these materials under static uniaxial loading, namely delamination and transverse matrix cracking, since these are the subjects concentrated on in this dissertation.

One of the first experiments performed on laminated fibre reinforced plastics (Pagano and Pipes 1973) showed the existence of a rather unusual failure mode, namely delamination under uniform uniaxial load. Subsequently the cause of this phenomenon, the stress peak occurring perpendicular to the plies at the free edge, was studied analytically (Pagano
1974, Pagano and Soni 1989) and numerically (Wang and Crossman 1977, Chang et al. 1988). Since the stress peak can only predict the first failure of the laminate and not the total load bearing capacity attention was soon focused on both the onset and the growth of the delamination. Experiments were performed (e.g. Wilkins et al. 1982, O'Brien 1982, Wang 1989) and failure criteria were developed to numerically investigate the delamination behaviour (e.g. Hashin 1980, Ladevèze 1992, Corigliano 1993, Schellekens 1992). The experiments also showed cracking in the matrix material in certain layers of the laminate (e.g. Crossman and Wang 1982, Laws et al. 1983, Renard et al. 1993, Guild et al. 1993, Nguyen 1997). Matrix cracking influences the delamination behaviour, since it occurs normally before the actual development of delamination and it speeds up the growth of delamination and simplifies the jumping of delamination between several layers.

1.1 Material description and manufacturing techniques

Fibre reinforced plastic composites is a collective term for a large family of materials ranging from short glass fibre reinforced polyesters to unidirectional graphite fibre epoxies. A first distinction is that between thermostet and thermoplastic materials, in which a thermostet material is defined as a material that cannot be reformed upon further application of heat and pressure. In this dissertation only thermostet materials will be studied. Of course, the different combinations of fibre and matrix materials lead to differences in material characteristics of the composite. Also the manifestation of the fibres and the manufacturing techniques of the composites have their influence. Although for the rest of this dissertation only long unidirectionally reinforced laminates are considered, a few words will be spent here on other types of fibre reinforced plastics to give an impression of the vast possibilities of the material family and the diversity of material characteristics. The majority of the material and manufacturing description will deal with the materials and techniques used for the examples in this dissertation.

1.1.1 fibre and matrix materials

The first fibres used in fibre reinforced plastics were made of glass. Already in 1931 the glass fibres were produced at an industrial scale. Although the virgin strength of glass is very high, the actual strength is limited by microscopic defects at the surface of the fibre. The same holds for the strain at failure. Since this is relatively low, the fibre cannot be bend over a short radius, which makes it less useful for some applications.

Although Edison made graphite fibres from cotton as early as 1880, it took till 1958 for the first strong graphite fibres to appear at the commercial market. The material properties of graphite fibres are very anisotropic since they have a laminar structure. Especially the
Young's modulus in the fibre direction is very high. In the transverse direction values of only 10% of that in longitudinal direction are mentioned. Graphite fibres maintain their strength even at high temperatures as long as they have a protective layer around them. The binding between the fibres and the matrix material can be improved by a surface treatment of the fibres.

More recent are the aramid fibres. These fibres are characterised by a relatively high strain at failure and high strength. A disadvantage of the material is the low Young's modulus in compression, which shows especially in bending. A combination with graphite fibres in the compressive zone can offer a solution.

Two types of plastics are normally used as matrix materials, namely unsaturated polyesters and epoxies. The use of unsaturated polyesters is restricted to temperatures up to approximately 100 °C and the resistance to chemicals is moderate. However, the main disadvantage of the material is the high amount of shrinkage at hardening, which causes high internal stresses and thus decreases the strength of the material.

The material characteristics of epoxy resins are superior to those of unsaturated polyester resins. The shrinkage at hardening is smaller than 2%, so the strength of the material is hardly influenced. The binding with all types of fibres is good and the resin can keep its strength up to a temperature of 250 °C. A disadvantage of epoxy resins compared to unsaturated polyesters is the high price, but when used in combination with expensive graphite or aramid fibres this aspect can be neglected.

1.1.2 manufacturing techniques

The manufacturing techniques depend on the form in which the fibres are used. Individual fibres can be either cut to small fibres or combined to fibre bundles, woven mats, mats with randomly distributed long curled fibres and binding between the fibres (uncut mats), etc. In Figure (1.2) the most common appearance forms of fibres are shown.

![Fig 1.2 Appearance form of fibre reinforcement in 2D](image)

For short fibres the method of spraying in which the cut fibres are sprayed in a mould together with the matrix resin is often used. The orientation of the fibres in this process is
random, leading to a nearly isotropic material. If on the other hand unidirectional fibre mats or woven mats are placed in the mould and resin is added the resulting material will be highly orthotropic. A special type of reinforcement common for graphite fibres combined with epoxy resin are the so called prepregs in which fibres are already combined with a matrix material and hardened till they are dried. They can then be stored in a dry form till the moment of usage.

The hardening of the composite can take place under high pressure, high temperature or simply at room temperature for all procedures in which the resin is added to the fibres in the mould, the so called wet process. For prepregs the fibres are combined under higher temperature or high pressure to let the matrix material completely harden as one, a process know as the dry process. The sort of hardening process used influences the strength of the material. For example the high temperature hardening leads to residual stresses in the material due to the temperature drop when the product is cooled to room temperature.

Besides the fibre appearance and hardening technique also the binding of the fibres and the resin influences the strength of the material and thus the failure behaviour. This binding can be improved by a surface treatment of the fibres, e.g. chemical treatment or oxidation, which increases the fibre specific surface and its roughness (Kaverov et al. 1995). A fourth factor influencing the material properties is the binding between the different layers in a laminated composite. This can also depend on the fabrication process. In case of layers that are already hardened somewhat before the next layer is applied, the binding between the layers will be less good. When all these factors are considered for unidirectionally reinforced laminated prepreg composites it is seen that the weak spots in the material will be the interface between the plies, since the individual layers are already hardened when the next layer is made, and the matrix material in the plies, because the bond between fibres and resin is fairly good.

1.2 Aims and scope of this research
As was argued, the two weak spots in prepreg laminates are the interfaces between the plies and the matrix material in the plies, leading to two major failure mechanisms, namely delamination and transverse matrix cracking. This dissertation will focus on the numerical simulation of these failure mechanisms, by means of:

- Development of material models for interface elements for the description of free edge delamination in strips subjected to a uniaxial loading. In the models the orthotropic character of the material is taken into account and use is made of the ideas of plasticity and damage mechanics. Complete mixed-mode delamination will be described as well as the special case of pure mode-I delamination occurring in the midplane of a symmet-
ric laminate.
- Checking the assumptions made in the two-dimensional analyses of the strips by complete three-dimensional analyses.
- Development of a material model to describe transverse matrix cracking in the plies. This model is based on damage mechanics and developed for continuum elements.
- Applying the developed models for delamination and matrix cracking to laminated strips to test their applicability with respect to numerical aspects, such as stability and computation time, and agreement with the real material behaviour.

1.3 Outline of the dissertation

In Chapter 2 the finite elements are discussed which have been used for the analyses in this research. They can be subdivided in the continuum elements for the plies, namely generalised plane strain elements for the two-dimensional analyses and solid elements for the three-dimensional analyses, and interface elements to describe the behaviour of the interfaces between the adjacent plies. For the description of matrix cracking without delamination this distinction between plies and interfaces is not taken into account and layered shell elements are utilised. Since the analyses of the laminated strip show major snap-back phenomena, the arc-length method is used. In case of strong localisations this method fails and a better solution will be indirect displacement control. Both methods are discussed in Chapter 2.

The basic principles behind the constitutive models used in Chapter 4 to 6 will be introduced in Chapter 3.

Chapter 4 discusses pure mode-I delamination of laminated composite strips. An orthotropic damage model is described and the numerical implementation of this model in a finite element code is discussed. The model is applied to a uniaxially loaded graphite-epoxy laminated composite. The mesh sensitivity of the model is analysed as well as the influence of temperature and ply thickness. An assessment of the assumption of uniform delamination in the strip is performed with complete three-dimensional calculations. The results are compared to experimental and numerical results found in the literature.

Delamination which is of a mixed-mode type is dealt with in the next chapter. Two material models are discussed and applied to two laminates of the same material but with a different stacking sequence, leading to different ratios between the individual failure modes. The first material model is an orthotropic plasticity model with orthotropic hardening. The model is also adjusted to a simpler model with isotropic hardening and the differences in the results and the computational aspects between these models are discussed. The second model is based on orthotropic damage mechanics. The plasticity and the damage model are compared with respect to computational stability, calculation speed and results.
In Chapter 6 the other main failure mode of laminated composites, namely matrix cracking of the plies parallel to the fibres, is discussed. A continuum model based on damage mechanics is described, and several analyses of a laminated strip with a central hole are discussed.

The conclusions, resulting from this project, are presented in the final chapter.

1.4 Notations and symbols

The tensor notation will be used in the main part of this dissertation. When it will clarify matters index notations or a full definition of a matrix will be given next to the tensor notation. Symbols will be explained when they first appear in the text.

For the notation of second order tensors and matrices bold-faced capitals are used, whereas vectors are indicated by lower-case bold-faced characters. An exception to this rule is the symbol for the Green-Lagrange strain tensor $\gamma$, used in Chapter 2, which is a second order tensor. However, since this second-order tensor is symmetric and has therefore only six independent components a vector representation of the tensor is often applied. Throughout this report the symbol $\delta$ is used to denote a variation of a quantity. An iterative change of a scalar or a vector is denoted by the symbol $\delta$ preceding this scalar or vector. The incremental value in the current load step is indicated by the symbol $\Delta$.

In this dissertation attention will focus on the behaviour of laminated strips subjected to an in-plane uniaxial tensile or compressive load. The lay-up of these strips is defined by $\alpha$, the angle of the fibres in the individual plies with respect to the loading direction. The notation starts with the angle of the fibres in the top layer of the laminate and works downwards. In case of a laminate which is symmetrical in thickness direction, this symmetry is shown by a subscript 's' added to half of the laminate lay-up. An example of the notation is given in Figure 1.3.
Figure 1.3 Notation of the laminate lay-up

\begin{equation}
[90, 0, 0, 90] = [90, 0]_s
\end{equation}
2. FINITE ELEMENT DESCRIPTIONS AND NUMERICAL SOLUTION TECHNIQUES

The analyses of all structures in this study have been performed by means of the finite element method. In this method the structure is divided into a number of finite elements, which are described by a number of nodes, the connectivities between these nodes and the material properties of the element. The elements are numerically integrated and therefore contain a number of integration points. The constitutive relations between the stresses and the strains are known in these integration points as well as the relation between the integration points and the nodes. For each element the relation between the nodal displacements and the nodal forces can then be obtained. After assembling all the elements in the structure and taking into account the boundary conditions and the external loading applied to the structure, a system of equations arises which can be solved to obtain the nodal displacements.

In this chapter the four types of finite elements used in this study will be discussed briefly. This description is restricted to physically linear elastic behaviour. The non-linearities occurring in the material will be discussed in the next chapter. To solve the physically non-linear behaviour of the structures under consideration advanced solution methods are needed, such as arc length methods and crack opening displacement (COD) control. These methods will also be discussed.

2.1 Finite elements

For the analysis of delamination in a laminated strip subjected to an in-plane tensile or compressive load, the laminate can be divided into plies in which no non-linear behaviour occurs and interfaces between the plies in which the delamination takes place. The finite elements used in the analyses are likewise divided in continuum elements to model the plies, namely generalised plane strain elements for two-dimensional analyses of a cross section of the strip and solid elements for the three-dimensional analyses, and interface elements utilised between the adjacent plies.

The second main failure mechanism mentioned in Chapter 1, matrix cracking, will be treated without delamination and thus no interfaces are needed. To limit the amount of elements while still being able to model a complete plate, layered shell elements are utilised in these analyses.

2.1.1 Generalised plane strain elements

For the finite element simulation of the plies of a uniaxially loaded laminated composite strip a generalised plane strain element has been formulated. In this paragraph the
geometrically non-linear formulation of this element will be given. The formulation includes hygro-thermal effects, which occur in the forming process of the laminate and which can cause significant initial stresses in the structure (as will be shown in Chapter 4).

**Formulation of the virtual work equation**

Starting point for the formulation of the tangent stiffness matrix and the internal force vectors of a continuum element is the total equilibrium of a body

\[
\int_S t dS + \int_V \rho g dV = 0 \tag{2.1}
\]

where \( t \) denotes the tractions, \( \rho \) is the mass density and \( g \) the gravity. \( V \) and \( S \) are the actual volume and surface of the body, respectively. The tractions are defined as

\[
t = \tau n \tag{2.2}
\]

with \( \tau \) the Cauchy stresses and \( n \) the normal vector. Using Equation (2.2) and the divergence theorem, Equation (2.1) can be rewritten as

\[
\int_V \left[ \frac{\partial \tau}{\partial x} + \rho g \right] dV = 0 \tag{2.3}
\]

This equation can be used as a starting point for the formulation of the virtual work equation, in which it is required that for each kinematically admissible virtual displacement field the following equation holds

\[
\int_V \left[ \text{div} \tau + \rho g \right] \delta u dV = 0 \tag{2.4}
\]

Making use of the divergence theorem this leads to

\[
- \int_V \delta \varepsilon^T \tau dV + \int_V \rho \delta u^T g dV = \int_S \delta u^T t dS \tag{2.5}
\]

in which the vector representation of the symmetric strain tensor \( \varepsilon \) has been introduced

\[
\varepsilon = \frac{1}{2} \left( \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^T \right) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{2.6}
\]

The description of geometrically non-linear behaviour in this chapter is based on the Total Lagrange approach in which the variables are functions of the undeformed reference configuration. Therefore, it is necessary to transform Equation (2.5) to the reference configuration. A stress measure, related to the Cauchy stress and referring to the reference configuration is the Second Piola-Kirchhoff stress tensor \( \sigma \).
\[ \tau = \frac{\rho}{\rho_0} F \sigma F^T \]  

(2.7)

in which \( \rho_0 \) is the mass density in the original state. The deformation gradient \( F \) is obtained by differentiating the spatial coordinates of a point \( x \) with respect to the material coordinates of that point \( r \)

\[ F = \frac{\partial x}{\partial r} \]  

(2.8)

Between the components \( x \) and \( r \) the following relation exists

\[ x = r + u \]  

(2.9)

in which \( u \) is the displacement vector. Using this decomposition the deformation gradient can be written as

\[ F = I + \frac{\partial u}{\partial r} \]  

(2.10)

where \( I \) denotes the unit matrix. If conservation of mass of an elementary volume is considered

\[ \rho dV = \rho_0 dV_0 \]  

(2.11)

and using the vector representation of the Second Piola-Kirchhoff stress tensor the virtual work equation (Eq. 2.5) can be transformed to the reference configuration leading to

\[ \int_{V_0} \delta \gamma^T \sigma dV_0 = \int_{V_0} \rho_0 \delta u^T \sigma_0 dV_0 - \int_{S_0} \delta u^T t_0 dS_0 \]  

(2.12)

in which \( \gamma \) is written as the vector representation of the Green-Lagrange strain tensor defined by

\[ \gamma = \frac{1}{2} \left( F^T F - I \right) \]  

(2.13)

For later use the expression of the incremental strain tensor is also given:

\[ \Delta \gamma = \frac{1}{2} \left( F^T \Delta F + \Delta F^T F + \Delta F^T \Delta F \right) \]  

(2.14)

**Derivation of the tangent stiffness matrix and internal force vector**

If the length of a laminated strip, see e.g. Figure (2.1), is large compared to the width and the thickness the assumption is justified that at a certain distance from the edge the in-plane displacements in a cross section, the \( r_2 r_3 \)-plane, are independent of the coordinate defined in the length of the strip, \( r_1 \).
Figure 2.1 Uniaxially strained laminate

In a layered composite the individual layers will deform in a different manner, causing a warping of the cross-section of the specimen. To take this warping into account the generalised plane strain element has three translational degrees of freedom per node. The displacement field of a cross-section can then be described by (Pipes and Pagano 1970, Pagano 1974)

\[ u_1(r_1, r_2, r_3) = \mu \varepsilon_{11} r_1 + u_1(r_2, r_3) \]
\[ u_2(r_1, r_2, r_3) = u_2(r_2, r_3) \]
\[ u_3(r_1, r_2, r_3) = u_3(r_2, r_3) \]

(2.15)

with \( \varepsilon_{11} \) the prescribed normal strain, which acts as a load, in the \( r_1 \)-direction and \( \mu \) the load parameter. Using these equations and the engineering strains as a measure for the shear strains together with the symmetry of the strains, the following expression in vector representation can be obtained for the incremental strain tensor of Equation (2.14),

\[ \Delta y = \Delta g + \Delta \varepsilon + \Delta \eta + \Delta \mu \varepsilon_i \]

(2.16)

in which \( \Delta g \) is of the order zero in the displacement increments,
\[
\Delta g = \begin{bmatrix}
\Delta g_{11} \\
\Delta g_{22} \\
\Delta g_{33} \\
\Delta g_{12} \\
\Delta g_{23} \\
\Delta g_{31}
\end{bmatrix} = \begin{bmatrix}
(F_{11} - 1) \Delta \mu \varepsilon_{11} + \frac{1}{2} \Delta \mu^2 \varepsilon_{11} \\
0 \\
0 \\
F_{12} \Delta \mu \varepsilon_{11} \\
0 \\
F_{13} \Delta \mu \varepsilon_{11}
\end{bmatrix}
\]  
(2.17)

while \( \Delta \varepsilon \) and \( \Delta \eta \) are linear and quadratic in the displacement increments, respectively,

\[
\Delta \varepsilon = \begin{bmatrix}
F_{12} \Delta u_{1,r_2} + F_{22} \Delta u_{2,r_2} + F_{32} \Delta u_{3,r_2} \\
F_{13} \Delta u_{1,r_3} + F_{23} \Delta u_{2,r_3} + F_{33} \Delta u_{3,r_3} \\
(F_{11} + \Delta \mu \varepsilon_{11}) \Delta u_{1,r_1} \\
F_{12} \Delta u_{1,r_2} + F_{13} \Delta u_{1,r_3} + F_{22} \Delta u_{2,r_2} + F_{23} \Delta u_{2,r_3} + F_{32} \Delta u_{3,r_2} + F_{33} \Delta u_{3,r_3} \\
(F_{11} + \Delta \mu \varepsilon_{11}) \Delta u_{1,r_1}
\end{bmatrix}
\]  
(2.18)

and

\[
\Delta \eta = \begin{bmatrix}
0 \\
\frac{1}{2} \left( (\Delta u_{1,r_2})^2 + (\Delta u_{2,r_2})^2 + (\Delta u_{3,r_2})^2 \right) \\
\frac{1}{2} \left( (\Delta u_{1,r_3})^2 + (\Delta u_{2,r_3})^2 + (\Delta u_{3,r_3})^2 \right) \\
0 \\
\Delta u_{1,r_2} \Delta u_{1,r_3} + \Delta u_{2,r_2} \Delta u_{2,r_3} + \Delta u_{3,r_2} \Delta u_{3,r_3} \\
0
\end{bmatrix}
\]  
(2.19)

The notation \( \Delta u_{i,r_j} \) denotes differentiation of \( \Delta u_i \) with respect to \( r_j \). Finally, \( \varepsilon_l \) is given by

\[
\varepsilon_l = \begin{bmatrix}
\varepsilon_{11} \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]  
(2.20)

As mentioned before, hygro-thermal effects play an important role in the forming process.
of the laminate. To account for this an extra strain term occurs

$$\Delta \gamma_{ct} = \Delta T_i \alpha + \Delta C_i \beta$$  \hspace{1cm} (2.21)

Here $\Delta T_i$ and $\alpha$ are the incremental change in temperature in the current load step $i$ and the thermal expansion vector, respectively, while $\Delta C_i$ and $\beta$ are the incremental change in moisture content and the hygroscopic expansion vector. If there are no other non-linear effects in the plies, the stress increment at iteration $j$ is given by

$$\Delta \sigma_j = D_p(\Delta \gamma - \Delta \gamma_{ct}) = D_p(\Delta g_j + \Delta \varepsilon_j + \Delta \eta_j + \Delta \mu_j \varepsilon_i - \Delta T_i \alpha - \Delta C_i \beta)$$  \hspace{1cm} (2.22)

where $\Delta \mu_j$ is the incremental load parameter at iteration $j$ and $D_p$ is the elastic stress-strain relation for the ply. Subtraction of the stress at the end of iteration $j-1$ from the stress at the end of iteration $j$ leads to the expression for the stresses at iteration step $j$

$$\sigma_j = \sigma_{j-1} + D_p(dg_j + d\varepsilon_j + d\eta_j + d\mu_j \varepsilon_i)$$ \hspace{1cm} (2.23)

in which the hygro-thermal effects disappear, since they are constant during each load step. In the $r_2r_3$-plane no external loadings occur. Therefore, the equation of virtual work (Eq. 2.12) reduces to

$$\int_{V_0} \delta \gamma^T \sigma dV_0 = 0$$ \hspace{1cm} (2.24)

in which the variation of the Green-Lagrange strain tensor $\delta \gamma_j = \delta \gamma_{j-1} + \delta(d\gamma_j)$ is given by

$$\delta \gamma_j = \delta(dg_j) + \delta(d\varepsilon_j) + \delta(d\eta_j)$$ \hspace{1cm} (2.25)

After substituting Equations (2.23) and (2.25) into Equation (2.24) and linearising the expression needed for the application of Newton's method for the solution of the set of non-linear equations, the virtual work equation becomes

$$\int_{V_0} \delta(d\varepsilon_j)^T D_p d\varepsilon_j dV_0 + \int_{V_0} \delta(d\eta_j)^T (\sigma_{j-1} + D_p(dg_j + d\mu_j \varepsilon_i)) dV_0 =$$

$$- \int_{V_0} \delta(d\varepsilon_j)^T (\sigma_{j-1} + D_p(dg_j + d\mu_j \varepsilon_i)) dV_0$$ \hspace{1cm} (2.26)

For use in the finite element method Equation (2.26) has to be discretised. The linear part of the incremental strain tensor, $d\varepsilon_j$, is related to the incremental nodal displacement vector, $da_j$, through

$$d\varepsilon_j = B_L da_j$$ \hspace{1cm} (2.27)

where $B_L$ is given by
Here $n_{r,j}$ is the row vector of derivatives of the interpolation functions, $N_i$, to the $r_j$-direction.

For the non-linear strains a non-linear strain displacement matrix $B_{NL}$ is introduced.

\[
B_{NL} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
F_{12}n_{r2} & F_{22}n_{r2} & F_{32}n_{r2} \\
F_{13}n_{r3} & F_{23}n_{r3} & F_{33}n_{r3} \\
(F_{11} + d\mu_j\varepsilon_{11})n_{r2} & 0 & 0 \\
(F_{13}n_{r2} + F_{12}n_{r3}) & (F_{23}n_{r2} + F_{22}n_{r3}) & (F_{33}n_{r2} + F_{32}n_{r3}) \\
F_{11} + d\mu_j\varepsilon_{11}n_{r3} & 0 & 0
\end{bmatrix}
\]

(2.28)

Using the expressions for $B_L$ and $B_{NL}$ and requiring that the virtual work principle holds for any virtual displacement increment, the discretised virtual work equation can be written as

\[
\left\{ \int_{V_0} B_L^T D_P B_L dV_0 + \int_{V_0} B_{NL}(\Sigma_{j-1} + E_j) B_{NL} dV_0 \right\} \quad da_j =
\]

\[
- \int_{V_0} B_L^T \sigma_{j-1} dV_0 - d\mu_j \int_{V_0} B_L^T D_P \varepsilon_1 dV_0 - \int_{V_0} B_L^T D_P d\gamma dV_0
\]

(2.30)

The matrices $\Sigma_{j-1}$ and $E_j$ are given by

\[
\Sigma_{j-1} = \begin{bmatrix}
\sigma_{11,j-1}I & \sigma_{12,j-1}I & \sigma_{13,j-1}I \\
\sigma_{21,j-1}I & \sigma_{22,j-1}I & \sigma_{23,j-1}I \\
\sigma_{31,j-1}I & \sigma_{32,j-1}I & \sigma_{33,j-1}I
\end{bmatrix}_{9x9}
\]

(2.31)

and
\[ E_j = \begin{bmatrix} \sigma_{g,11} & \sigma_{g,21} & \sigma_{g,31} \\ \sigma_{g,12} & \sigma_{g,22} & \sigma_{g,32} \\ \sigma_{g,13} & \sigma_{g,23} & \sigma_{g,33} \end{bmatrix} \] (2.32)

with

\[ \sigma_g = D_p (dg + d\mu \varepsilon_i) \] (2.33)

From Equation (2.30) can be seen that the element tangent stiffness matrix is given by

\[ K_j = \int_{V_0} B_L^T \Delta D P B_L dV_0 + \int_{V_0} B_{NL} (\Sigma_{j-1} + E_j) B_{NL} dV_0 \] (2.34)

and the internal force vector reads

\[ -\int_{V_0} B_L^T \sigma_{j-1} dV_0 - d\mu \int_{V_0} B_L^T D_P \varepsilon_i dV_0 - \int_{V_0} B_L^T D_P \sigma_j dV_0 \] (2.35)

### 2.1.2 Solid elements

Solid elements have three translational degrees of freedom per node and six strain and stress components. Similar to the generalised plane strain element, the incremental Green-Lagrange strain tensor can be written as a combination of linear and quadratic terms in the displacement increments (zero order terms do not occur since there exists no explicit linear relation between a displacement and one of the three directions \( r_1, r_2, r_3 \)).

\[ \Delta \gamma = \Delta \varepsilon + \Delta \eta \] (2.36)

with \( \Delta \varepsilon \) and \( \Delta \eta \) given by

\[ \Delta \varepsilon = \begin{bmatrix} F_{11} \Delta u_{1,r_1} + F_{21} \Delta u_{2,r_1} + F_{31} \Delta u_{3,r_1} \\ F_{12} \Delta u_{1,r_2} + F_{22} \Delta u_{2,r_2} + F_{32} \Delta u_{3,r_2} \\ F_{13} \Delta u_{1,r_3} + F_{23} \Delta u_{2,r_3} + F_{33} \Delta u_{3,r_3} \end{bmatrix} \] (2.37)
\[
\Delta \eta = \begin{bmatrix}
\frac{1}{2} (\Delta u^2_{1,r_1} + \Delta u^2_{2,r_1} + \Delta u^2_{3,r_1}) \\
\frac{1}{2} (\Delta u^2_{1,r_2} + \Delta u^2_{2,r_2} + \Delta u^2_{3,r_2}) \\
\frac{1}{2} (\Delta u^2_{1,r_3} + \Delta u^2_{2,r_3} + \Delta u^2_{3,r_3}) \\
(\Delta u_{1,r_1} \Delta u_{1,r_2} + \Delta u_{2,r_1} \Delta u_{2,r_2} + \Delta u_{3,r_1} \Delta u_{3,r_2}) \\
(\Delta u_{1,r_2} \Delta u_{1,r_3} + \Delta u_{2,r_2} \Delta u_{2,r_3} + \Delta u_{3,r_2} \Delta u_{3,r_3}) \\
(\Delta u_{1,r_3} \Delta u_{1,r_3} + \Delta u_{2,r_3} \Delta u_{2,r_3} + \Delta u_{3,r_3} \Delta u_{3,r_3})
\end{bmatrix}
\] (2.38)

The effects of temperature and moisture are again taken into account by applying (2.21). Following the steps of Eq. (2.23) to (2.26) the linearized form of the virtual work equation for solid elements without external loading can be written as

\[
\int_{V_o} \delta (d\varepsilon_j)^T D_p d\varepsilon_j dV_o + \int_{V_o} \delta (d\eta_j)^T \sigma_{j-1} dV_o = -\int_{V_o} \delta (d\varepsilon_j)^T \sigma_{j-1} dV_o
\] (2.39)

Introducing the tensors \( B_L \) and \( B_{NL} \) for the coupling between the linear and non-linear part of the incremental Green-Lagrange strain tensor and the incremental nodal displacements, in which \( B_L \) and \( B_{NL} \) are given by

\[
B_L = \begin{bmatrix}
F_{11} n_{r_1} & F_{21} n_{r_1} & F_{31} n_{r_1} \\
F_{12} n_{r_2} & F_{22} n_{r_2} & F_{32} n_{r_2} \\
F_{13} n_{r_3} & F_{23} n_{r_3} & F_{33} n_{r_3} \\
(F_{11} n_{r_1} + F_{12} n_{r_1}) & (F_{21} n_{r_1} + F_{22} n_{r_2}) & (F_{31} n_{r_1} + F_{32} n_{r_2}) \\
(F_{12} n_{r_2} + F_{13} n_{r_2}) & (F_{22} n_{r_2} + F_{23} n_{r_3}) & (F_{32} n_{r_2} + F_{33} n_{r_3}) \\
(F_{13} n_{r_3} + F_{11} n_{r_3}) & (F_{23} n_{r_3} + F_{21} n_{r_1}) & (F_{33} n_{r_3} + F_{31} n_{r_1})
\end{bmatrix}
\] (2.40)
leads to the discretised virtual work equation.

\[
\begin{bmatrix}
B_{NL}^T D_P B_L \, dV_o + \int_{V_o} B_{NL}^T \Sigma_{j-1} B_{NL} \, dV_o \end{bmatrix} \, da_j = - \int_{V_o} B_{NL}^T \sigma_{j-1} \, dV_o
\]

or

\[
K_j da_j = f_i
\]

with \( \Sigma_{j-1} \) given by Equation (2.31), \( D_P \) the linear-elastic stiffness matrix, \( K_j \) the element stiffness matrix and \( f_i \) the internal force vector.

### 2.1.3 Interface elements

The interface elements discussed here are continuous interface elements, such as line, plane and shell interfaces. The formulation of the interfaces will be given along the same line as the description of the continuum elements. First the virtual work equation will be treated, followed by the determination of the stiffness matrix and the internal force vector of the element. As stated in the introduction of this chapter, numerical integration will be used.

**Virtual work equation**

For an interface element the virtual work in the current configuration is given by

\[
\delta W = \int_S \delta \mathbf{u}_c^T \mathbf{t}_c \, dS
\]

in which

\( \mathbf{t}_c \) = Cauchy traction vector
\( \delta u_c \) = virtual relative displacement vector, defined as the difference between the displacement of the upper and the lower part of the interface element.

\( S' \) = current element surface

As stated in Paragraph 2.1.1, it is convenient to reformulate Equation (2.46) in the undeformed reference state. Therefore, the transformations of the variables to the reference configuration are introduced as

\[
dS = \left( \frac{\det J}{\det J_0} \right) dS_0
\]

(2.47)

\[
t_c = \left( \frac{\det J_0}{\det J} \right) R t
\]

(2.48)

and

\[
u_c = R u
\]

(2.49)

with \( \det J \) and \( \det J_0 \) the Jacobian matrices in the current and the reference state, respectively, and \( R \) the rotation matrix which describes the transformation of the reference configuration to the current configuration (Figure 2.2)

![Figure 2.2 Rotation matrix from the reference to the current configuration](image)

With these transformation rules the following formulation of the virtual work in the reference configuration is obtained

\[
\delta W = \int_{S_0} \delta u^T t dS_0
\]

(2.50)

Since there will be no external forces defined for the interface elements used in this report, the virtual work equation reduces to \( \delta W = 0 \).

Numerical integration of the interface elements

For a general 3D configuration the nodes of a nno-noded interface element (nno=number of nodes) each have three translational degrees-of-freedom. Ordering of these degrees-of-
freedom in the individual direction and with both the upper nodes of the element and the lower nodes grouped together, this leads to the following element nodal displacement vector $\mathbf{a}$

$$\mathbf{a} = \begin{pmatrix} a_{n1}^1, a_{n2}^2, \ldots, a_{nno}^n, a_{s1}^1, a_{s2}^2, \ldots, a_{sno}^n, a_{t1}^1, a_{t2}^2, \ldots, a_{tno}^n \end{pmatrix}^T$$

(2.51)

in which the n-direction is normal to the interface surface and the s and t directions are tangential to the interface surface.

The continuous displacement field is defined as

$$\mathbf{v} = \begin{pmatrix} v_n^u, v_n^l, v_s^u, v_s^l, v_t^u, v_t^l \end{pmatrix}^T$$

(2.52)

The superscripts u and l indicate the upper and lower side of the interface, respectively.

The continuous displacement field is related to the nodal displacements by means of the interpolation polynomials $\mathbf{n} = (N_1, N_2, \ldots, N_{nno/2})$.

$$\mathbf{v} = \mathbf{H} \mathbf{a}$$

(2.53)

where

$$\mathbf{H} = \begin{bmatrix} n & 0 & 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 & 0 & 0 \\ 0 & 0 & n & 0 & 0 & 0 \\ 0 & 0 & 0 & n & 0 & 0 \\ 0 & 0 & 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}$$

(2.54)

To link the continuous displacement field to the relative displacements an operator matrix $\mathbf{L}$ is introduced

$$\mathbf{L} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

(2.55)

leading to

$$\mathbf{u} = \mathbf{L} \mathbf{v}$$

(2.56)

Combining Equations (2.53) and (2.56), the relation between the nodal displacements and the relative displacements can be established
\( u = LH\alpha = Ba \) \hspace{1cm} (2.57)

with \( B \) given by

\[
B = \begin{bmatrix}
-n & n & 0 & 0 & 0 & 0 \\
0 & 0 & -n & n & 0 & 0 \\
0 & 0 & 0 & 0 & -n & n
\end{bmatrix}
\] \hspace{1cm} (2.58)

Evidently, the matrix \( B \) has to be transformed to the global coordinate system if the local coordinate system in the nodes does not coincide with the global system.

With use of the elastic traction-relative displacement matrix, which is the equivalent of the elastic stiffness matrix for interface elements,

\[
C = \begin{bmatrix}
C_{nn} & 0 & 0 \\
0 & C_{ss} & 0 \\
0 & 0 & C_{tt}
\end{bmatrix}
\] \hspace{1cm} (2.59)

the traction-relative displacement relation can be written as

\[ t = Cu \] \hspace{1cm} (2.60)

The traction vector at the end of iteration \( j \), in a non-linear analysis, can be calculated from

\[ t_j = t_{j-1} + C \Delta u_j \] \hspace{1cm} (2.61)

in which \( \Delta u_j \) is the iterative change of the relative displacement vector in iteration \( j \). Substituting the Equations (2.57) and (2.61) in the virtual work equation (Eq. 2.50) leads to

\[
\delta(\Delta a_j)^T \int_{S_0} B^T C B dS_0 \delta a_j = -\delta(\Delta a_j)^T \int_{S_0} B^T t_{j-1} dS_0
\] \hspace{1cm} (2.62)

Requiring that this relation holds for every kinematically admissible virtual displacement field, the following definitions of the element stiffness matrix \( K \) and the internal force vector \( f_{j-1} \), respectively, are obtained.

\[
K = \int_{S_0} B^T C B dS_0 \quad \text{and} \quad f_{j-1} = -\int_{S_0} B^T t_{j-1} dS_0
\] \hspace{1cm} (2.63)

For numerically integrated interface elements the integrations in the relations for \( K \) and \( f_{j-1} \) are replaced by an integration over the iso-parametric coordinates \( \xi \) and \( \eta \). For a plane interface element this leads to
\[ K = \int_{\xi=-1}^{\xi=+1} \int_{\eta=-1}^{\eta=+1} B^T C B \det J d\eta d\xi \quad \text{and} \]
\[ f_{j-1} = -\int_{\xi=-1}^{\xi=+1} \int_{\eta=-1}^{\eta=+1} B^T t_{j-1} \det J d\eta d\xi \]

(2.64)

For line interfaces the interpolation polynomials are independent of \( \eta \) and the integrals reduce to

\[ K = b \int_{\xi=1}^{\xi=+1} B^T C B \left( \left( \frac{\partial x_1}{\partial \xi} \right)^2 + \left( \frac{\partial x_2}{\partial \xi} \right)^2 \right)^{1/2} d\xi \]

(2.65)

\[ f_{j-1} = -b \int_{\xi=1}^{\xi=+1} B^T t_{j-1} \left( \left( \frac{\partial x_1}{\partial \xi} \right)^2 + \left( \frac{\partial x_2}{\partial \xi} \right)^2 \right)^{1/2} d\xi \]

where \( b \) is the width of the interface.

2.1.4 Layered shell elements

In the layered shell element several material layers can be defined in one element. This makes the element very suitable for the analysis of the ply behaviour in a three-dimensional analysis of a laminated composite plate, since only one element over the height of the plate is needed instead of at least one element for each layer as with solid elements. This limits the total amount of elements considerably.

The geometry of the shell is determined by the coordinates of the nodes, \( r \), and a vector, \( v_3 \), perpendicular to the reference plane in the nodes (Figure 2.3).

![Figure 2.3 Geometry of a quadrilateral layered shell element](image)
With nodal coordinates \((\xi, \eta, \zeta)\) and interpolation polynomials \(N_i\) for node \(i\) the geometry of a \(n\)-noded isoparametric shell element can be written as

\[
r(\xi, \eta, \zeta) = \sum_{i=1}^{n} N_i r_i + \sum_{i=1}^{n} \frac{\zeta}{2} t_i N_i v_3
\] (2.66)

After defining orthogonal base vectors, \(v_{1,i}\) and \(v_{2,i}\) perpendicular to the normal of the element in node \(i\), \(v_{3,i}\), the displacements of the elements are defined by the nodal displacements, \(a_i\), and the two rotations in the reference plane, \(\phi_i^T = (\alpha_i, \beta_i)\)

\[
u(\xi, \eta, \zeta) = \sum_{i=1}^{n} N_i a_i + \sum_{i=1}^{n} \frac{\zeta}{2} N_i t_i \begin{bmatrix} v_{1,i} \, v_{2,i} \end{bmatrix} \phi_i
\] (2.67)

In incremental form the previous equation is given by

\[
\Delta \nu(\xi, \eta, \zeta) = \sum_{i=1}^{n} N_i \Delta a_i + \sum_{i=1}^{n} \frac{\zeta}{2} N_i t_i s_i
\] (2.68)

In which \(s_i\) for finite rotations is defined as

\[
s_i = \Delta \alpha_i \frac{\partial f_i}{\partial \alpha_i} + \Delta \beta_i \frac{\partial f_i}{\partial \beta_i}
\] (2.69)

where \(f_i\) is the incremental change of the normal vector of node \(i\).

In Hsiao and Chen (1989) various formulations for the description of finite rotations, with different functions for \(f_i\), were evaluated and the conclusion was reached that the type of formulation does not have a significant influence on the results, not even in analyses with displacements and rotations up to the order of magnitude of the major structural dimensions or deflections in the order of several times the thickness. Also for the development of the general theory of the shell elements the exact formulation of \(f\) is not needed and will therefore not be elaborated here. It is sufficient to take into account that \(f_i\) is a non-linear function of the rotational degrees of freedom. Interested readers can find several formulations in e.g. Ramm (1977), Parisch (1978), Surana (1983) and Oliver and Oñate (1994).

For the derivation of the tangent stiffness relation and the internal force vector we again start with the equation for virtual work, as given in Eq. (2.24). However, since a non-linear relation exists between the displacement field and the nodal degrees-of-freedom it is not sufficient simply to take the linearised form of the left hand side of the virtual work equation as is done for standard continuum elements. The left hand side of the virtual work equation has to be linearised with respect to the discrete nodal variables to take into account the stiffness contribution of the finite rotation description. To arrive at this linearised form a series expansion is developed of the left-hand side of the virtual work equation. After omitting the terms of the second and higher order this leads to
\[ \delta W = \int_{V_0} \left( \delta \gamma^T \sigma_{j-1} + \delta \gamma^T \frac{\partial \sigma}{\partial a} \, da_j + \left( \frac{\partial \delta \gamma}{\partial a} \right)^T \sigma_{j-1} \, da_j \right) dV_0 \]  

(2.70)

where \( da \) is the change of the nodal degrees-of-freedom from iteration \( j-1 \) to iteration \( j \). A linearised version of the variation of the Green-Lagrange strain, \( \delta \gamma \), and of its derivative to the nodal degrees-of-freedom is given by

\[ \delta \gamma = \frac{\partial \gamma}{\partial a} \, \delta da \]  

(2.71)

\[ \frac{\partial \delta \gamma}{\partial a} = \frac{\partial^2 \gamma}{\partial a^2} \, \delta da \]  

(2.72)

In a similar manner, the derivative of \( \sigma \) to \( a \) leads to

\[ \frac{\partial \sigma}{\partial a} = \frac{\partial \sigma}{\partial \gamma} \frac{\partial \gamma}{\partial a} = D_i \frac{\partial \gamma}{\partial a} \]  

(2.73)

in which \( D_i \) is the currently linearised stress-strain relation at integration point level, which depends on the material model that is used (consistent tangent tensor). Substitution of Equation (2.71), (2.72), (2.73) together with the expression for \( \gamma \) as given in Equation (2.13) into Equation (2.70) gives

\[ \delta W = \partial (da)^T \int_{V_0} \left( B_L^T \sigma_{j-1} + B_L^T D_i B_L \, da_j + \frac{\partial \bar{F}}{\partial a^T} \Sigma_{j-1} \frac{\partial \bar{F}}{\partial a} \right) \, da_j 

+ S_{j-1} P^T \frac{\partial^2 \bar{F}}{\partial a^2} \, da_j \right) dV_0 \]  

(2.74)

in which \( \Sigma_{j-1} \) is given by Equation (2.31) and \( B_L \) is given by

\[ B_L = P_1^T \frac{\partial \bar{F}}{\partial a} \]  

(2.75)

\( \bar{F} \) is a vector containing the components of the deformation gradient, \( F \), so the derivative with respect to \( a \) is the second order tensor.
\[
B_{NL} = \frac{\partial \hat{F}}{\partial a} = \begin{bmatrix}
  n_{r_1} & 0 & 0 & g_{1\alpha,r_1} & g_{1\beta,r_1} \\
  0 & n_{r_1} & 0 & g_{2\alpha,r_1} & g_{2\beta,r_1} \\
  0 & 0 & n_{r_1} & g_{3\alpha,r_1} & g_{3\beta,r_1} \\
  n_{r_2} & 0 & 0 & g_{1\alpha,r_2} & g_{1\beta,r_2} \\
  0 & n_{r_2} & 0 & g_{2\alpha,r_2} & g_{2\beta,r_2} \\
  0 & 0 & n_{r_2} & g_{3\alpha,r_2} & g_{3\beta,r_2} \\
  n_{r_3} & 0 & 0 & g_{1\alpha,r_3} & g_{1\beta,r_3} \\
  0 & n_{r_3} & 0 & g_{2\alpha,r_3} & g_{2\beta,r_3} \\
  0 & 0 & n_{r_3} & g_{3\alpha,r_3} & g_{3\beta,r_3}
\end{bmatrix}
\]  
(2.76)

and \( P_1 \) is given by

\[
P_1^T = \begin{bmatrix}
  F_{11} & F_{21} & F_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & F_{12} & F_{22} & F_{32} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & F_{13} & F_{23} & F_{33} & 0 \\
  F_{12} & F_{22} & F_{32} & F_{11} & F_{21} & F_{31} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & F_{13} & F_{23} & F_{33} & F_{12} & F_{22} & F_{32} & 0 \\
  F_{13} & F_{23} & F_{33} & 0 & 0 & 0 & F_{11} & F_{21} & F_{31} & 0
\end{bmatrix}
\]  
(2.77)

in which the following notations have been utilized

\[ n = (N_1, N_2, \ldots, N_{nno}) \]

\[ g_{k\alpha} = \left( \frac{\zeta_{1t_1}}{2} N_1 \frac{\partial f_{k,1}}{\partial \alpha}, \frac{\zeta_{2t_2}}{2} N_2 \frac{\partial f_{k,2}}{\partial \alpha}, \ldots, \frac{\zeta_{nno t_{nno}}}{2} N_{nno} \frac{\partial f_{k,nno}}{\partial \alpha} \right) \]

with \( f_{k,i} \) the component of the rotational function for node \( i \). In the virtual work expression (Eq. 2.74) the first term is the internal force vector. The second term gives the linear contribution to the element tangent stiffness matrix \( (K_L) \), while the third term is the geometrical contribution to the stiffness matrix \( (K_G) \). These three terms complete the formulation for a geometrically non-linear shell element restricted to small rotation increments, since in that situation the last term vanishes for two- and three-dimensional continuum elements. This is due to the linear relation that exists for the incremental relation of the element displacements with respect to the rotational degrees-of-freedom, leading to \( (\partial^2 \hat{F}/\partial a^2 = 0) \). For finite rotations however the last term has to be evaluated. In this case the second order derivatives of \( \hat{F} \) with respect to the rotational degrees-of-freedom are given by
\[
\frac{\partial^2 F_{kl}}{\partial \alpha^2} = F_{kl,\alpha \alpha} = \sum_{i=1}^{n_0} \frac{t_i}{2} \frac{\partial^2 f_{k,i}}{\partial \alpha^2} \left( \zeta \left( J_{i1}^{-1} \frac{\partial N_i}{\partial \xi} + J_{i2}^{-1} \frac{\partial N_i}{\partial \eta} \right) + J_{i3}^{-1} N_i \right) \tag{2.78}
\]

\[
\frac{\partial^2 F_{kl}}{\partial \alpha \partial \beta} = \frac{\partial^2 F_{kl}}{\partial \beta \partial \alpha} = F_{kl,\alpha \beta} = \sum_{i=1}^{n_0} \frac{t_i}{2} \frac{\partial^2 f_{k,i}}{\partial \alpha \partial \beta} \left( \zeta \left( J_{i1}^{-1} \frac{\partial N_i}{\partial \xi} + J_{i2}^{-1} \frac{\partial N_i}{\partial \eta} \right) + J_{i3}^{-1} N_i \right) \tag{2.79}
\]

\[
\frac{\partial^2 F_{kl}}{\partial \beta^2} = F_{kl,\beta \beta} = \sum_{i=1}^{n_0} \frac{t_i}{2} \frac{\partial^2 f_{k,i}}{\partial \beta^2} \left( \zeta \left( J_{i1}^{-1} \frac{\partial N_i}{\partial \xi} + J_{i2}^{-1} \frac{\partial N_i}{\partial \eta} \right) + J_{i3}^{-1} N_i \right) \tag{2.80}
\]

With these derivatives a vector can be assembled, which is defined as

\[
G_{\alpha \alpha}^T = (F_{11,\alpha \alpha}, F_{21,\alpha \alpha}, F_{31,\alpha \alpha}, F_{12,\alpha \alpha}, F_{22,\alpha \alpha}, F_{32,\alpha \alpha}, F_{13,\alpha \alpha}, F_{23,\alpha \alpha}, F_{33,\alpha \alpha}) \tag{2.81}
\]

Also the vectors \( G_{\alpha \beta} \) and \( G_{\beta \beta} \) can be defined in a similar fashion. Using these vectors a matrix \( F_{\alpha \alpha} \) is constructed, which for an nno-noded element is given by

\[
F_{\alpha \alpha} = \begin{bmatrix}
G_{\alpha \alpha,1} & 0 & \ldots & 0 \\
0 & G_{\alpha \alpha,2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & G_{\alpha \alpha,n_{no}}
\end{bmatrix} \tag{2.82}
\]

The second order derivative of \( F \) with respect to \( \alpha \) can now be written as

\[
\frac{\partial^2 F}{\partial \alpha^2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_{\alpha \alpha} & F_{\alpha \beta} \\
0 & 0 & 0 & F_{\alpha \beta} & F_{\beta \beta}
\end{bmatrix} \tag{2.83}
\]

The other variables used in the fourth term of the virtual work equation are defined as
\[ S_{j-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_{j-1} & 0 \\ 0 & 0 & 0 & 0 & \hat{S}_{j-1} \end{bmatrix} \]  \tag{2.84}

with the submatrices \( \hat{S}_{j-1} \) given by

\[ \hat{S}_{j-1} = \begin{bmatrix} \sigma_T & 0 & . & . & . & 0 \\ 0 & \sigma_T & . & . & . & 0 \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & \sigma_T \end{bmatrix}_{6 \times 6 \text{nno}} \]  \tag{2.85}

and

\[ P^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & 0 & P_2 \end{bmatrix} \]  \tag{2.86}

\[ P_2 = \begin{bmatrix} P_1^T & 0 & . & . & . & 0 \\ 0 & P_1^T & . & . & . & 0 \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & P_1^T \end{bmatrix}_{9 \times 9 \text{nno}} \]  \tag{2.87}

Summarising the linearised equilibrium equation becomes

\[ \left[ K_L + K_G + K_{\Theta} \right] d\hat{a}_j = \mu \hat{q} - f_{\text{in},j-1} \]  \tag{2.88}

with
\[ K_L = \int_{V_0} B_T^T D_i B_L dV_0 \]
\[ K_G = \int_{V_0} \frac{\partial \hat{F}}{\partial a^T} \Sigma_{j-1} \frac{\partial \hat{F}}{\partial a} dV_0 \]
\[ K_\Theta = \int_{V_0} S_{j-1} P^T \frac{\partial^2 \hat{F}}{\partial a^2} dV_0 \]
\[ f_{in,j-1} = \int_{V_0} B_L^T \sigma_{j-1} dV_0 \]
(2.89)

and \( \mu \hat{q} \) the load in the current load step.

Since for shell elements it is necessary that the stress-strain relation, \( D_i \), is evaluated in the local coordinate system of the integration point, the stresses and strains have to be transformed from the global to the local coordinate system by a rotation matrix \( R \). The local stress-strain relation can then be determined by
\[ \overline{D}_3 = R^T D_i R \]
(2.90)
so that
\[ \bar{\sigma} = \overline{D} \bar{\varepsilon} \]
(2.91)
in which a barred quantity refers to the quantity in the local coordinate system. The discussed layered shell element is based on the degenerated solid element, containing the shell assumption that the stress normal to the shell surface is zero. By static condensation the new stress-strain relation, \( \overline{D} \), can be determined as
\[
\begin{bmatrix}
\overline{D}_{11} & \overline{D}_{12} & 0 & 0 & 0 & \overline{D}_{16} \\
\overline{D}_{21} & \overline{D}_{22} & 0 & 0 & 0 & \overline{D}_{26} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{D}_{44} & 0 & 0 \\
0 & 0 & 0 & \overline{D}_{55} & 0 & 0 \\
\overline{D}_{61} & \overline{D}_{62} & 0 & 0 & 0 & \overline{D}_{66}
\end{bmatrix}
\begin{bmatrix}
\bar{\varepsilon}
\end{bmatrix}
\]
(2.92)
in which
\[ \overline{D}_{ij} = \bar{D}_{ij} - \frac{\bar{D}_{i3} \bar{D}_{3j}}{\bar{D}_{33}} \quad \text{for } i = 1, 2, 6 \]
(2.93)
and
\( \bar{D}_{ii} = \bar{D}_{ii} \) \hspace{1cm} \text{for} \ i = 4, 5 \hspace{1cm} (2.94) 

### 2.2 Solution methods

All stress-strain relations discussed in this dissertation have a non-linear character, meaning that the stresses depend in some complex, non-linear way on the strains and the history of the strain. To solve the equilibrium equations of a structure with these material laws, a system of non-linear equations has to be solved. The most common way to do this is by means of an iterative application of linear principles until the solution is achieved within the required accuracy. The final load is applied in small steps to ensure that the relation between the stresses and the strains, which might be path dependent, is accurately captured and that the physically meaningful solution is obtained.

The set of equations that has to be solved in a load step reads

\[
\Psi_{n+1} = \Psi(a_{n+1}) = p(a_{n+1}) - f_{n+1} = 0
\]

in which \( p \) are the internal forces in the structure, \( f \) the external forces and \( \Psi \) are the residual forces, which have to be equal to zero at the end of the load step. The nodal displacements, \( a \), are the fundamental unknowns of the problem. The external forces, \( f \), are obtained by adding a small step to the external forces from the previous step.

\[
f_{n+1} = f_n + \Delta f_{n+1}
\]

In the same manner the nodal displacements in this step can be written as the sum of the nodal displacement corresponding to the previous step and a new part \( \Delta a_{n+1} \). It is this new part of the nodal displacements that is calculated in an iterative manner by means of a Newton-Raphson method, resolving into the determination of the root of an approximation of the relation for the residual forces at the current state

\[
\Psi(a_{n+1}^{i+1}) = \Psi(a_{n+1}^i) + \left( \frac{\partial \Psi}{\partial a} \right)^i_{n+1} \delta a_{n+1}^{i+1} = \\
\Psi(a_{n+1}^i) + (K^{ig})_{n+1} \delta a_{n+1}^{i+1} = 0
\]

where \( K^{ig} \) is the tangent stiffness matrix. If after successive iterations the residual forces are close enough to zero, the \( \Delta a \) is obtained by adding all the \( \delta a \) of the iteration process.

Since in the calculations of the laminated strips a snap-back phenomenon can occur, usual methods of force or displacement control techniques cannot be used. To overcome the problems of snap-back that occur in the analyses, the arc-length method, introduced by Riks (1970) and Wempner (1971) and later adjusted by Crisfield (1981) and Ramm (1981) is used. The adjustments of Crisfield and Ramm where aimed at maintaining the possible
banded nature and symmetry of the system. In short the system of the arc length control can be written as

\[
\delta a_{n+1}^{i+1} = -(K^{i+1})_{n+1}^{-1} \Psi_{n+1}^i = -(K^{i+1})_{n+1}^{-1}(p_{n+1}^i - f_{n+1} + 1)
\]

\[
= -(K^{i+1})_{n+1}^{-1}(p_{n+1}^i - f_{n+1} - \Delta \lambda_{n+1}^{i+1} f_0)
\]

\[
= \left( (K^{i+1})_{n+1}^{-1}(f_n - p_{n+1}^i) \right) + \Delta \lambda_{n+1}^{i+1} \left( (K^{i+1})_{n+1}^{-1} f_0 \right)
\]

\[
= \delta^I a_{n+1}^{i+1} + \Delta \lambda_{n+1}^{i+1} \delta^II a_{n+1}^{i+1}
\]

in which \(\delta^I a_{n+1}^{i+1}\) is the result of the external loading at the end of the previous load step and the internal forces after iteration \(i\), while the second part, \(\delta^II a_{n+1}^{i+1}\), is the result of the part of the external load that is applied in this load step. The load factor, \(\Delta \lambda_{n+1}^{i+1}\), can be determined by a method based on the orthogonality between the tangent vector and the update vector, leading to

\[
(\Delta a_{n+1}^i)^T \delta a_{n+1}^{i+1} = 0
\]

from which it follows that \(\Delta \lambda_{n+1}^{i+1}\) is equal to

\[
\Delta \lambda_{n+1}^{i+1} = -\frac{(\Delta a_{n+1}^i)^T \delta^I a_{n+1}^{i+1}}{(\Delta a_{n+1}^i)^T \delta^II a_{n+1}^{i+1}}
\]

In case of strong localisations this method will fail since the deformations in the localisation zone are "compensated" with the deformations in the areas in which elastic unloading takes place. In such a situation a better approach is to limit the number of degrees-of-freedom in the constraint equation. This method, called indirect displacement control is often applied in this dissertation in which the localisation phenomena are concentrated in a small area. In the case where only one deformation is taken as a control parameter Equation (2.100) becomes

\[
\Delta \lambda_{n+1}^{i+1} = -\frac{\delta^I a_{n+1}^{i+1}}{\delta^II a_{n+1}^{i+1}}
\]

in which \('a'\) is the chosen nodal variable.
3. GENERAL CONSTITUTIVE FORMULATIONS

For the analysis of the response of a structure to a certain load the relation between the stresses and strains in the integration points of the finite elements has to be known. For graphite-epoxy laminates these constitutive relations start off linearly, but become non-linear at increased loading. The non-linearity occurs through the appearance of microcracks in the material, which grow to macro-cracks and eventually lead to complete rupture of the material. To describe this degradation process of the material two families of constitutive relations will be applied, namely plasticity and damage mechanics. Of these, damage mechanics is actually based on a degradation of a material by the occurrence of micro-cracks, while plasticity is normally associated with the slip of dislocations at a crystal level in metals. However, also in polymers dislocations in the in the crystallites can be present, or a reorientation of the molecular bonds in an amorphous region can take place, leading to permanent inelastic deformations. These deformations can be seen as the plastic deformations in the theory of plasticity.

Due to the orthotropic character of graphite-epoxy laminates the degradation process will in general be non-isotropic, causing the need for anisotropic constitutive models. The growth of the cracks in the material will cause a gradual decrease in stresses at increasing strains or softening. Therefore, in this chapter the basic concepts of single surface plasticity with anisotropic softening and anisotropic damage mechanics are described.

3.1 Plasticity

In the theory of plasticity the yield function plays an important role. Stresses and strains are assumed to remain elastic if they remain inside the volume spanned by the yield function, \( \Phi = \Phi(\sigma) = 0 \). Stresses on the yield surface can cause plastic deformations. Stress situations outside the yield surface cannot exist. The yield function can be extended by a parameter, called the softening/hardening parameter, which causes the yield surface to shrink/expand respectively during loading. The function then becomes \( \Phi = \Phi(\sigma, \kappa) = 0 \), in which \( \kappa \) determines the amount of change of the yield surface.

After occurrence of plasticity a part of the strains will no longer disappear on load removal. This irreversible part of the strains is called the plastic strain. In a strain rate formulation the complete strain rate is therefore composed of a reversible, elastic, part and the irreversible, plastic, part

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \]  \hspace{1cm} (3.1)

The stress rate in the material is related to the elastic strain rate, by means of the elastic stiffness matrix, \( C \), of the material which is assumed to be independent of the strains in this dissertation.
\[ \dot{\sigma} = C \dot{\varepsilon} = C(\dot{\varepsilon} - \dot{\varepsilon}^p) \]  \hspace{1cm} (3.2)

For the plastic strain rate an associated flow rule is assumed here, yielding

\[ \dot{\varepsilon}^p = \dot{\lambda} \frac{\partial \Phi}{\partial \sigma} \]  \hspace{1cm} (3.3)

The solution of the stress-strain situation in an integration point of a finite element is an iterative procedure. The procedure starts by determining a so called trial stress, which is computed by multiplying the new strains in the integration point by the linear elastic stiffness

\[ \sigma_{n+1}^{\text{trial}} = \sigma_n + C \Delta \varepsilon_{n+1} \]  \hspace{1cm} (3.4)

If the yield function for this trial stress is larger than zero, implying the stress is outside the yield surface, a return mapping routine will be executed to return the stress to the yield surface. In this return mapping algorithm the yield function is rewritten solely as a function of the plastic multiplier, \( \dot{\lambda} \). This nonlinear function is then solved by means of a Newton-Raphson method for which the derivative of the yield function is needed with respect to the plastic multiplier. This procedure is summarised in Figure (3.1). The procedure will be treated in detail in Chapter 5, in which a single surface plasticity model with anisotropic softening is discussed.

For the determination of the tangent stiffness matrix needed in the Newton-Raphson method, the derivatives of the update equations for the stresses and the hardening parameter are needed. If these update equations are given by

\[ \sigma_{n+1} = \sigma_n + C(\Delta \varepsilon_{n+1} - \Delta \lambda_{n+1} \frac{\partial \Phi}{\partial \sigma}) \]  \hspace{1cm} (3.5)

\[ \kappa = \kappa(\sigma_{n+1}, \Delta \lambda_{n+1}) \]  \hspace{1cm} (3.6)
Figure 3.1 Flow diagram for plasticity analysis
the derivatives are given by

$$\dot{\sigma}_{n+1} = H^{-1}(\dot{\epsilon}_{n+1} - \dot{\lambda} \frac{\partial \Phi}{\partial \sigma})$$

(3.7)

$$\dot{\kappa}_{n+1} = \left(\frac{\partial \kappa}{\partial \sigma}\right)^T \dot{\sigma}_{n+1} + \frac{\partial \kappa}{\partial \Delta \lambda} \dot{\lambda}_{n+1}$$

(3.8)

where the matrix $H$ is given by

$$H = C^{-1} + \Delta \lambda_{n+1} \frac{\partial^2 \Phi}{\partial \sigma^2}$$

(3.9)

Substituting the derivatives of the update equations (Eq. 3.7 and 3.8) in the consistency condition, $\Phi = 0$, leads to a relation for $\dot{\lambda}$

$$\dot{\lambda}_{n+1} = \frac{\gamma^T H^{-1}}{h + \gamma^T H^{-1} \frac{\partial \Phi}{\partial \sigma}} \dot{\epsilon}_{n+1}$$

(3.10)

in which

$$\gamma = \frac{\partial \Phi}{\partial \sigma} + \frac{\partial \Phi}{\partial \kappa} \frac{\partial \kappa}{\partial \sigma}$$

(3.11)

$$h = -\frac{\partial \Phi}{\partial \kappa} \frac{\partial \kappa}{\partial \Delta \lambda}$$

(3.12)

Substitution of this relation for $\dot{\lambda}$ in Equation 3.7 leads to

$$\dot{\sigma}_{n+1} = \left[H^{-1} - \frac{H^{-1} \frac{\partial \Phi}{\partial \sigma} \gamma^T H^{-1}}{h + \gamma^T H^{-1} \frac{\partial \Phi}{\partial \sigma}} \hat{\epsilon}_{n+1}\right] C^{\text{tg}} \hat{\epsilon}_{n+1}$$

(3.13)

This equation demonstrates that even with an associated flow rule, the tangent stiffness matrix can become non-symmetrical if the hardening parameter, $\Delta \kappa$, is not a linear function of the plastic multiplier, $\Delta \lambda$. 
3.2 Damage mechanics

The principle of damage mechanics was first described by Kachanov (1958) in the limited context of creep failure in metals. In the 1970s the principle was extended to other types of loading and failure by others, e.g. Lemaitre and Chaboche (1978). More recently, the formalism is increasingly used in the failure analyses of laminated composites (e.g. Allix and Ladevèze 1989 and Nguyen and Fleury 1997). It is based on the degradation of a material subjected to loading. This degradation is caused by the development of microcracks in the material. Damage is described by a damage parameter, which is defined by the ratio of the cracked volume over the total volume of a piece of material (Figure 3.2).

\[ \text{damage} = \frac{\text{cracked}}{\text{cracked} + \text{uncracked}} \]

Figure 3.2 Definition of the damage parameter

In principle there are two ways to take the degradation of the material into account in the constitutive relations. The first is through the principle of equivalent stress, in which the strain in a damaged material will be higher at equal stress than in a virgin material. The second is by means of the principle of equivalent strain (Figure 3.3). As can be seen from the figure the stress in the damaged material is now lower than in the virgin material at equal strain. Since the classical finite element method starts from the assumption of known strains in the integration points and stresses which are calculated from these strains, the principle of strain equivalence will be adopted in this study.
The damage is taken into account as a reduction of the stiffness of the material by multiplying the stiffness with a term equal to one minus the damage parameter. In the case of an isotropic material this damage parameter can be represented by a scalar variable. The constitutive relation then becomes

\[ \sigma = (1 - d)C \varepsilon \]  \hspace{1cm} (3.14)

in which \( \sigma \) and \( \varepsilon \) are the stresses and the strains in the integration point respectively. The damage parameter of the material is given by \( d \).

In general for anisotropic materials a scalar damage variable is no longer sufficient, since the material behaviour, and thus the occurrence of damage, is different in different directions. Cordebois and Sidoroff (1982) proposed to take a damage matrix in these situations, so the constitutive relation changes to

\[ \sigma = (I - D)C \varepsilon \]  \hspace{1cm} (3.15)

where \( I \) is the unit matrix and \( D \) the damage matrix. The main problem with such a damage matrix is that for continuum elements with a Poisson's ratio not equal to zero an indirect coupling occurs between the degradation of the stiffnesses in the individual directions. This makes it hard to determine the individual damage parameters which occur in the matrix. For the interface elements described in Chapter 2 this problem does not occur since the stiffness matrix for these elements is diagonal, so each component of the stiffness matrix can individually be tuned to the arising damage.

Whether or not damage occurs is determined by a function, \( f \), which usually depends in
some way on the deformations in the integration point. If the value of this function exceeds a threshold value, \( f_{\text{tres}} \), damage will occur in the integration point. The threshold value is then adjusted to the new situation and growth of damage is determined by exceeding this new value in next steps. This procedure is shown in Figure (3.4). If damage does not grow during a load step, secant unloading will take place. The growth law of the damage variable can depend on many different things, for example the fracture energy or on the actual deformations and some characteristic deformations of the integration point.

Figure 3.4 Flow diagram for damage analysis

As was stated in the previous chapter the Newton-Rhapson method is used in the calculations so that the derivative of the constitutive relation is needed. For the constitutive relation with anisotropic material behaviour this derivative is given by

\[
\dot{\sigma} = (I - D)C\dot{\varepsilon} - \dot{D}C\varepsilon
\]  

(3.16)

in which \( \dot{D} \) is the derivative of the damage matrix with respect to a virtual time. For the derivation of a consistent tangent stiffness it is necessary that this derivative is written as a function of the derivatives of the deformations \( \dot{D}\varepsilon = D^*\dot{\varepsilon} \), leading to

\[
\dot{\sigma} = (I - D)C\dot{\varepsilon} - \dot{D}C\varepsilon = (I - D)C\dot{\varepsilon} - D^*C\dot{\varepsilon} = C^{18}\dot{\varepsilon}
\]  

(3.17)

Whether or not this tangent stiffness matrix is symmetrical depends on the formulation of the damage growth law.
4. MODE-I DELAMINATION IN FIBRE REINFORCED LAMINATED STRIPS

Delamination is one of the main failure mechanisms of fibre reinforced laminated strips. Depending on the stacking sequence of the laminate and the position of the delamination zone in the laminate, this delamination occurs purely as mode-I delamination or as delamination due to a combination of several cracking modes, called mixed-mode delamination.

In this chapter attention is focused on mode-I delamination, which is the dominant mode if delamination occurs in the midplane of a symmetric laminate. The described material model is based on the principle of orthotropic damage mechanics discussed in Chapter 3. A complete description of the model is given in Paragraph 4.1.

In Paragraph 4.2 a delamination calculation of a section of a unidirectionally fibre reinforced laminated strip subjected to a uniaxial strain is discussed and compared with failure strains found in experiments. The influences of ply thickness and temperature are discussed as well as the mesh sensitivity of the model. Each of these analyses has been performed on a two-dimensional model of a cross-section of the laminate. Such a two-dimensional approach is normally justified by the assumption of uniform delamination at a certain distance from the loaded edge. In the last part of this chapter the validity of this assumption is checked by performing three-dimensional analyses of the total strip and comparing these results with the two-dimensional calculations.

4.1 Mode-I damage model

To describe the mode-I delamination process an orthotropic damage model for the interface elements is introduced. Recalling the principle of damage mechanics described in Chapter 3 and rewriting the stress-strain relation and its derivative (Eq. 3.15 and 3.17) in terms of tractions and relative displacements (Eq. 4.1 and 4.2), it is seen that the complete description of the material model depends on the formulation of the damage matrix $D$.

\[ t = (I - D)Cu \]  \hspace{1cm} (4.1)

so that

\[ t = (I - D)C\dot{u} - \dot{D}Cu = (I - D)C\dot{u} - D^*C\dot{u} = C^{\text{rel}\dot{u}} \]  \hspace{1cm} (4.2)

with $D^*$ the derivative of the damage matrix rewritten in such a way that it is independent of $\dot{u}$ (compare Eq. 3.17).
Mode-I damage of an interface takes place if the normal deformation in an integration point of the interface exceeds the damage threshold deformation in this direction, \( u_i \), or in terms of the diagram given in Figure (3.4) if \( f(\varepsilon) = u_n > f_{\text{res}} = u_i \). After damage has been initiated the stiffness in mode-I is gradually reduced to zero, such that a linear softening traction-relative displacement relation occurs. In the other directions an immediate drop of the strength to zero occurs after damage initiation. The damage growth in the proposed mode-I damage model is completely determined by the deformation in the normal direction of the delaminating interface. To overcome the numerical problems that occur at an abrupt transition between the linear softening branch and the stress-free geometry that remains after complete damage, the end of the softening branch is smoothed by an exponential curve (Figure 4.1).

![Figure 4.1 Traction-relative displacement relation for interface elements](image)

To obtain this softening behaviour the following form of the damage matrix is needed

\[
D = \begin{cases} 
0 & \text{if } u_n < u_i \\
\begin{bmatrix}
u_i (u_n - u_i) & 0 & 0 \\
\frac{u_i}{u_n} (u_n - u_i) & 0 & 1 \\
0 & 1 & 0 \\
1 - \alpha \frac{u_i}{u_n} & \exp \left( \frac{(1-\alpha) u_e + \alpha u_i - u_n}{\alpha (u_e - u_i)} \right) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} & \text{if } u_i \leq u_n \leq (1-\alpha) u_e + \alpha u_i \\
\end{cases} 
\]

(4.3)
with its derivative given by

\[
\dot{D} = \begin{bmatrix}
    0 & & & \\
    \frac{u_e u_i \dot{u}_n}{u^2(u_e - u_i)} & 0 & 0 & \\
    0 & 0 & 0 & \\
    0 & 0 & 0 & \\
\end{bmatrix}
\]

\[
\begin{aligned}
    & u_n < u_i \\
    & u_i \leq u_n \leq (1 - \alpha)u_e + \alpha u_i \\
    & u_n > (1 - \alpha)u_e + \alpha u_i
\end{aligned}
\] (4.4)

Substitution of this derivative into Eq. (4.2) leads to

\[
i = C^{i*} \dot{u}
\] (4.5)

in which the tangent stiffness matrix, \(C^{i*}\), is given by

\[
C^{i*} = \begin{bmatrix}
    C & & & \\
    \left[ \begin{array}{cccc}
        \frac{u_e u_i}{u(u_e - u_i)} & 0 & 0 \\
        0 & 0 & 0 \\
        0 & 0 & 0
    \end{array} \right] & & & \\
    u < u_i \\
    u_i \leq u \leq (1 - \alpha)u_e + \alpha u_i \\
    u > (1 - \alpha)u_e + \alpha u_i
\end{bmatrix}
\] (4.6)

\(C_{nn}\) is the elastic stiffness of the material in the normal direction of the interface. The factor \(\alpha\) determines the transition point between the linear and the exponential softening branch. The choice for \(\alpha\) influences the fracture energy that is used in the analyses, but between certain limits the effect is negligible and \(\alpha\) can be chosen arbitrarily. In the
analyses in this chapter \( \alpha \) was chosen as 0.1.

### 4.2 Two-dimensional analyses of free-edge delamination of a \([25, -25, 90]_s\) laminate

The described damage model is used for the free edge delamination analyses of several laminated strips, each with a laminate lay-up of \([25, -25, 90]_s\), and constructed of an AS-3501-06 graphite-epoxy of which the material characteristics are given in Table (4.1). The initiation of delamination occurs at a threshold deformation of \( u_i = 51.6 \cdot 10^{-7} \text{mm} \) in the normal direction of the interface. This threshold deformation is determined from the tensile strength of the material and the applied dummy stiffness for the interface elements of \( C_{ii} = 10^{+8} \, \text{N/mm}^3 \) (\( i = n, s, t \)). This high stiffness was used to limit the influence of the interface elements on the elastic behaviour of the strip. The use of the dummy stiffness in the determination of \( u_i \) makes the initiation of damage rather arbitrary, but since the opening of the interface after damage initiation is controlled by the fracture energy of the material, \( G_f = 0.175 \, \text{N/mm} \), propagation of the delamination is properly predicted. The analysed strips vary in ply thickness to study the effect of this parameter on the response. Further study involves the mesh sensitivity of the model and the influence of the temperature drop that occurs during the manufacturing process. In this process the laminates are autoclaved at a temperature of \( T = 150^\circ \text{C} \). Afterwards the strips are cooled to 25°C, causing initial stresses in the laminate.

<table>
<thead>
<tr>
<th>Young's Mod. [MPa]</th>
<th>Shear Mod. [MPa]</th>
<th>Pois. Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{11} )</td>
<td>140 \cdot 10^3</td>
<td>( G_{12} ) 5.5 \cdot 10^3 ( \nu_{12} ) 0.29</td>
</tr>
<tr>
<td>( E_{22} )</td>
<td>11 \cdot 10^3</td>
<td>( G_{13} ) 5.5 \cdot 10^3 ( \nu_{13} ) 0.29</td>
</tr>
<tr>
<td>( E_{33} )</td>
<td>11 \cdot 10^3</td>
<td>( G_{23} ) 5.5 \cdot 10^3 ( \nu_{23} ) 0.3</td>
</tr>
</tbody>
</table>

**Table 4.1** Material parameters AS-3501-06 graphite-epoxy

The analyses were performed on two-dimensional meshes of a cross-section of a laminate subjected to a uniaxial strain under the assumption of uniform delamination at a certain distance away from the loaded edges (Figure 4.2). For the individual plies generalised plane strain elements were utilised, while the non-linear behaviour of the interfaces between the plies was modelled by means of line interface elements as described in Chapter 2.
Figure 4.2 Laminated strip subjected to uniaxial loading

The first analysis was performed on a strip with an individual ply height of 0.132 mm and a specimen width of 25 mm. For this specimen three different finite element discretisations were taken, with element lengths of 0.05, 0.1 and 0.2 mm near the free edge. This fine discretisation was only used at the 5 mm closest to the free edge of the specimen. The remainder of the laminate was modelled with a coarser mesh with an element length of 2.5 mm. For the two-dimensional analyses symmetry is assumed, so only one quarter of the cross section of the laminate is modelled. This assumption was made to reduce the problem size and thus the computation time. Of course, the assumed symmetry axis in the midplane of the laminate is valid due to the symmetrical ply lay-up. The other assumed symmetry axis is not an actual symmetry axis of the laminate due to the fibre orientation in the plies. At the end of this chapter this assumption will be checked for a complete three-dimensional analyses of a strip. At the symmetry edges of the model supports were added according to the symmetry conditions, namely \( u_1(0, 0, r_3) = 0, u_2(0, 0, r_3) = 0 \) and \( u_3(0, r_2, 0) = 0 \).

The average axial stress (= total axial load divided by the surface of the cross section) versus the applied axial strain is shown in Figure (4.3) for the three meshes. As can be seen from the figure no mesh sensitivity occurs even when looking to the enlarged graph of the snap-back behaviour.

Comparison of the obtained ultimate strain, \( \varepsilon_u = 0.518\% \), with the results of numerical analyses performed by Schellekens (1992) and experimental results by Wang et al. (1985), \( \varepsilon_u = 0.516\% \) and \( \varepsilon_u = 0.53\% \) respectively, shows a good agreement.

If the temperature drop occurring in the manufacturing process is not taken into account the ultimate strain of the laminate is highly overestimated as can be seen in Figure (4.4) in which the delamination length from the free edge versus the applied axial strain is shown for the mesh with an element length of 0.1 mm near the free edge.
Figure 4.3a Average axial load versus applied axial strain \([25, -25, 90]_s\) laminate

Figure 4.3b Enlarged graph of the snap-back behaviour
Figure 4.4 Delamination versus applied strain with and without temperature drop

If the thickness of the plies is enlarged the ultimate strain should decrease according to the formula previously reported by Crossman and Wang (1982) and O’Brien (1982) which is based on experiments performed by Rodini and Eisenmann (1980)

$$\%\varepsilon_I(n) = \%\varepsilon_I(1)/\sqrt{n}$$ (4.7)

in which $n$ is the factor by which the ply-thickness is increased. In Figure (4.5) it can be seen that there is a reasonable agreement between the numerical results and Equation (4.7).
**Figure 4.5** Effect of the ply thickness on the ultimate strain

### 4.3 Three-dimensional analyses rectangular plate

To check the assumption of uniform delamination at a distance of the loaded edges three-dimensional calculations of the laminated strip were performed with the package ScaFiEP, a scalable finite element package for parallel computing. For more information on this package and the specific numerical tools used for these calculations the reader is referred to Lingen (2000) and Lingen and Schipperen (2000). To limit the amount of elements in the three-dimensional analyses a smaller laminate, total width 10 mm, than the one described in the previous section was taken for all analyses in this paragraph. Since temperature loading could not be taken into account in the ScaFiEP package, the analyses were performed without the initial temperature drop. For the loading of the laminate a uniform stress was taken, which is the loading as it is applied in experiments.

For the first three-dimensional calculations only the part of the T-bone specimen in which uniform delamination would occur (Figure 4.2) is modelled as a rectangular plate (5 mm wide and 10 mm long) subjected to a uniform stress. To limit the mesh size and thus the computation time the same assumption of symmetry as in the two-dimensional calculations was used and even extended to a symmetry in the length of the plate, leading to a modelling of one eighth of the plate. The effect of all these assumed symmetry axes and the modelling of only part of the T-bone specimen will be discussed in the remainder of this chapter.

The delamination zone at a load level of 193 $N/mm^2$ is shown in Figure (4.6). This delamination zone is all but uniform, which gives the impression that the two-dimensional
modelling is incorrect. To explain the delamination behaviour of the plate, the stress profile in the linear-elastic stage of the interface layer was examined (Figure 4.7).

![Figure 4.6 Delamination zone rectangular plate subjected to a uniaxial stress](image)

**Figure 4.6 Delamination zone rectangular plate subjected to a uniaxial stress**

![Figure 4.7 Stress profile linear-elastic calculation (stresses in MPa)](image)

**Figure 4.7 Stress profile linear-elastic calculation (stresses in MPa)**

As can be seen a large stress peak occurs at the free corner of the laminate, leading to the delamination profile as shown in Figure (4.6). This stress peak is caused by the load introduction in the plate and the fact that in this free corner the difference in the deformation behaviour of the individual plies of the laminate will be the largest.

In an experimental set-up the load will be gradually introduced into the specimen by either thickening or widening the specimen towards the loaded edge. Therefore, the previous approach does not simulate the behaviour of the laminate as it occurs in experiments. To improve the simulation a load introduction zone was introduced at the beginning of the rectangular plate in which the interfaces remain linearly elastic. A second elastic zone was added at the end of the plate near the supports to limit the influence of the boundary conditions, since these boundary conditions force the edge of the plate to remain straight, while in a T-bone specimen warping of this cross-section will occur since it is no real symmetry axis (Figure 4.8).
Three different meshes were used in these calculations to study the mesh sensitivity. The coarse mesh consisted of 20 elements over the width and 25 elements over the length of the plate. For the two finer meshes the element distribution over the width was not equidistant. For the 2.5 mm of the width of the plate closest to the free edge a finer mesh was used. This leads to 10+25 elements over the width and 25 over the length for the second mesh and to 20+50 elements over the width and 50 over the length for the finest of the three meshes. The crack opening displacement of a node approximately in the middle of the free edge has been plotted versus the applied uniform axial stress for all three meshes in Figure (4.9). Since no mesh sensitivity is noticeable all further calculations were carried out with element sizes either equal to or comparable with the medium mesh, since this mesh combines a fast analysis with a smooth curve.

![Figure 4.8 Linear elastic zones in plate](image)

![Figure 4.9 Mesh sensitivity study: Crack opening displacement (COD) at the free edge](image)
versus axial stress for the three-dimensional analyses

A comparison of the results of a two-dimensional analyses of this laminate and the three-dimensional analysis is given in Figure (4.10). From this figure it seems that the two-dimensional approximation is correct. However, this is still under the assumption of uniform delamination. In Figure (4.11) the delamination zone of the plate is shown at several stages during the computation. Up to the peak load the delamination is uniform (the waving of the delamination zone is purely due to visualisation aspects). In the softening regime the delamination zone starts to become more and more non-uniform.

Figure 4.10 Comparison two- and three-dimensional analysis

![Diagram showing delamination stages](image)

Figure 4.11 Delamination zone in the 3-dimensional analysis at several stages, using elastic zones

This deviation from uniform delamination can be explained by the use of the elastic zones, which limits the crack opening near these zones. This limitation of the crack opening near
the elastic zones will prevent the decrease of the load carrying capacity of the plate in this area. Since the load carrying capacity will remain intact, the delamination near the elastic zones will not propagate, since no stress redistribution takes place.

To check the symmetry assumptions used in the calculations, two additional analyses were performed. In the first analysis the complete width of the plate was modelled. The deformation profile in a section of the plate approximately in the middle of the free edge shows no large difference between the two sides of the plate (Figure 4.12). This deformation profile was obtained in the pre-peak regime of the calculation. In the post-peak regime asymmetrical behaviour was observed where the delamination zone grew on one side of the plate, while at the other side unloading occurred. The delaminating and unloading side depended on the fibre orientation. So, the conclusion can be drawn that up to the peak load the assumption of symmetry is reasonable. In the second analysis the complete length of the plate was modelled. Also here the results of the analyses with the symmetry assumption agreed with those from this analysis.

![Deformation Profile](image)

**Figure 4.12** Pre-peak deformation profile of a central plate section

The length of the elastic zones in the plate has been chosen rather arbitrarily. If this length is chosen correctly the load introduction zone and the influence of the boundary conditions on the stresses, strains and warping should be restrained to these elastic zones. As can be seen from the Figures (4.13) and (4.14) this is indeed the case. The elastic zone at the boundary could even have been chosen smaller.
Figure 4.13 Strains over length of the plate at 9 different heights

Figure 4.14 Warping of the plate along the free edge (5 elements over elastic zone at each side)

Of course, linear elastic ends of the plate are a simplification of the real situation in an experimental set-up. At the edge of the linear part an unrealistic stress situation will occur. As was already stated the elastic zones also influence the crack opening near the zones. To assess the influence of this simplification on the behaviour of the plate, a quarter of a strip was modelled which widens towards the loaded edge (Figure 4.15). To limit the computation time, the radius of the transition zone was taken fairly small in comparison to the numbers normally suggested in norms.
Figure 4.15 Quarter of a T-bone specimen, geometry and element mesh

In this strip the load is gradually introduced into the actual plate. The COD versus axial stress shows the same behaviour as for the rectangular plate with linear elastic zones (Figure 4.16). The axial stress for the strip in this figure is the average stress as it occurs in the narrow part of the strip. The delamination zone in Figure (4.17) shows a gradual development of the zone along the length of the strip.
Figure 4.16 COD versus axial stress for both the rectangular plate and the T-bone specimen

load = 348 MPa  load = 437 MPa (peak)  load = 421 MPa

Figure 4.17 Development of the delamination zone in a T-bone specimen

4.4 Conclusions

If free edge delamination occurs in the midplane of a symmetric laminate the main delamination mode is completely mode-I. Comparison with experimental results show that a two-dimensional analysis with a mode-I delamination model suffices to describe this failure mode of the plate. Complete three-dimensional calculations are not necessary to obtain the ultimate load of the plate. However, to obtain reliable information on the shape of the delamination zone three-dimensional analyses are required. By three-dimensional analyses it is shown that the free edge delamination zone in a plate is uniform up to the peak load.
Only after the peak a deviation from this uniform delamination occurs.

To limit the computational time three-dimensional analyses can be performed on an eighth of the rectangular mid-section of a T-bone specimen. Although the symmetry axes are no real symmetry axes due to the fibre orientation in the plies, the results are not significantly influenced by the assumption of these symmetry axes. The limitation to a rectangular plate only shows correct results if linear elastic zones are embedded in the finite element mesh. These elastic zones are needed in the areas of load introduction in the plate and near the boundary conditions, which are a result of the assumed symmetry in the plate.
5. MIXED-MODE DELAMINATION IN FIBRE REINFORCED LAMINATED STRIPS

While in the previous chapter attention was focused on mode-I delamination, which occurs in the midplane of a symmetric laminate, this chapter will deal with delamination that consists of a combination of cracking modes, so-called mixed-mode delamination. The three cracking modes that can occur are shown in Figure (5.1).

Two different formalisms, plasticity and damage mechanics, are used for the material models described in this chapter. The different material models will be discussed and the results will be compared on the basis of computed laminate behaviour and numerical aspects, such as computation time and the simplicity with which the equilibrium path can be followed.

In Paragraphs 5.1 and 5.2 two plasticity models are described. Both are based on the same initial orthotropic yield surface but the softening behaviour of the models is different. While the first model shows completely orthotropic softening, the second model assumes a more simplified isotropic softening behaviour. The influences of the simplification will be discussed by the calculation of two different laminate lay-ups. Paragraph 5.3 deals with an orthotropic material model based on damage mechanics, comparable to the plasticity model discussed in the previous paragraph. The same laminates are analysed again. A comparison between the plasticity models and the damage model will be given in Paragraph 5.4. The best performing model is then used to analyse the influence of the laminate thickness and some parameter variations on the ultimate strain. In the last paragraph some conclusions are drawn.

![Figure 5.1 Cracking modes](image-url)
5.1 Orthotropic plasticity model

The first part of this paragraph deals with the model description and the numerical implementation of the plasticity model with orthotropic softening. Special attention will be given to the derivation of the tangent stiffness matrix, which is not as straightforward as in the general description of plasticity in Chapter 3. In the second part a description of the two laminates that will be used for the verification of all the material models in this chapter will be given as well as the analyses of these laminates with the orthotropic plasticity model.

5.1.1 model description

For the description of the plasticity model the outline as given in Paragraph 3.1 will be followed. However, for interface elements, the general relations have to be rewritten in terms of tractions and relative displacements. The yield function represents an ellipsoid in the three-dimensional traction space (Figure 5.2), which is given by

$$
\Phi(t, \kappa) = \frac{1}{2} t^T P t + t^T p - \bar{t}^2(\kappa) = 0
$$

(5.1)

in which

$t^T = \text{traction vector of the interface } (t_n, t_s, t_t)$

$$
P = \begin{bmatrix}
2C_{nn} & 0 & 0 \\
0 & 2C_{ss} & 0 \\
0 & 0 & 2C_{tt}
\end{bmatrix}
$$

$p^T = (C_n, 0, 0)$

$$
\bar{t}(\kappa) = \bar{t}_0 \left(1 - \frac{\kappa}{G_f}\right)
$$

$\bar{t} = \text{current equivalent yield traction}$

$\kappa = \text{hardening parameter}$

$\bar{t}_0 = \text{initial equivalent yield traction}$

$G_f = \text{fracture toughness}$

The variables $C_{nn}, C_{ss}, C_{tt}, C_n$ are functions of the initial and current equivalent yield traction and the yield tractions in the various directions according to

$$
C_{nn} = \frac{\bar{t}}{\bar{t}_n}, \quad C_{ss} = \frac{\bar{t}_0}{\bar{t}_s}, \quad C_{tt} = \frac{\bar{t}_0}{\bar{t}_t}, \quad C_n = \bar{t} - \frac{\bar{t}^2}{\bar{t}_n}
$$

(5.2)
where $T_n^C$ is the compressive yield traction in the direction normal to the interface plane, while $T_s$ and $T_t$ are the shear yield tractions in the $s$ and $t$ direction of the interface, respectively.

![Figure 5.2 Yield surface in the three-dimensional stress space](image)

An indication of the softening behaviour is given in Figure (5.3), which provides a two-dimensional representation of the yield surface for different values of the equivalent yield traction. As can be seen from this figure the softening behaviour does not influence the compressive strength of the model in the direction normal to the interface element.

If the yield condition is satisfied the total relative displacements are decomposed into an elastic and an inelastic part.

$$\dot{u} = \dot{u}^e + \dot{u}^{ie}$$  \hspace{1cm} (5.3)

Different from the general description in Chapter 3 the inelastic displacement rates are not given by $\dot{u}^p$ since in the model only the inelastic relative displacements in the normal direction of the interface under compression are regarded as plasticity. All other inelastic deformations are considered to represent cracking behaviour and will be referred to with the superscript $cr$ instead of $p$. 
Figure 5.3 Two-dimensional representation of the yield surface and the softening behaviour

The work hardening hypothesis used in the mixed-mode delamination model is only influenced by the cracking behaviour of the model. For the plastic deformation a perfect plasticity model is assumed. Therefore, the work hardening hypothesis can be written as

$$
\dot{k} = t^T \dot{\lambda} = \lambda t^T \frac{\partial \Phi}{\partial \lambda} \bigg|_{cr} 
$$

(5.4)

In variance with Chapter 3, no explicit relation for the determination of the tractions, $\tilde{t}$ and $\Delta \lambda$ can be found, due to the interaction of $\tilde{t}$, the tractions, the yield function and $\Delta \lambda$. Therefore, a system of four equations with four independent variables will be solved simultaneously. For the derivation of this system the incremental work hardening relation is rewritten as

$$
\Delta \lambda = \Delta \lambda \left( t^T \frac{\partial \Phi}{\partial \lambda} \bigg|_{cr} \right)^{-1} 
$$

(5.5)
The four equations consist of the residuals of the update of the tractions and the yield function. If an implicit Euler backward integration scheme is used, the update of the stress is given by

\[
t_{n+1} = t_n + \Delta t_{n+1} = t_n + C \Delta u^e_{n+1} = t_n + C \left( \Delta u_{n+1} - \Delta u^e_{n+1} \right) = t_{n+1}^{trial} - \Delta \lambda_{n+1} C \left. \frac{\partial \Phi}{\partial t} \right|_{n+1}
\]

\[
= C u^e_n + C \Delta u_{n+1} - \Delta \lambda_{n+1} C (p_{t_{n+1}} + p)
\]

(5.6)

Substitution of Equation (5.5) in Equation (5.6) leads to the following system of residual equations

\[
g = C^{-1} t_{n+1} - \left( u^e_n + \Delta u_{n+1} \right) + \Delta \kappa \left( t^T \frac{\partial \Phi}{\partial t} \right)_{cr}^{-1} (p_{t_{n+1}} + p) = 0
\]

(5.7)

The fourth equation in the system is the yield function in which \( \vec{t} \) is replaced by

\[
\vec{t} = t_0 \left( 1 - \frac{\kappa_{cr} + \Delta \kappa}{G_f} \right)
\]

(5.8)

In this system the tractions and \( \Delta \kappa \) are taken as independent variables. For the solving of the system a Newton iteration scheme is used

\[
\delta_{n+1} = \delta_n - J^{-1} r_n
\]

(5.9)

with \( \delta^T = (t, \Delta \kappa) \), \( r^T = (g, \Phi) \) and \( J \) the matrix containing the derivatives of the residual functions with respect to the independent variables. For the system described above these derivatives are given by

\[
\frac{\partial r(1)}{\partial t_n} = \frac{1}{a_n} + \frac{2 \Delta \kappa \vec{t}}{t_n^T \frac{\partial \Phi}{\partial t}} - \frac{\Delta \kappa (P(1, 1) t_n + p(1))(2P(1, 1) t_n + p(1))}{\left( t^T \frac{\partial \Phi}{\partial t} \right)^2}
\]

(5.10a)

\[
\frac{\partial r(1)}{\partial t_s} = -2 \frac{\Delta \kappa (P(1, 1) t_n + p(1)) P(2, 2) t_s}{\left( t^T \frac{\partial \Phi}{\partial t} \right)^2}
\]

(5.10b)

\[
\frac{\partial r(1)}{\partial t_i} = -2 \frac{\Delta \kappa (P(1, 1) t_n + p(1)) P(3, 3) t_s}{\left( t^T \frac{\partial \Phi}{\partial t} \right)^2}
\]

(5.10c)
\[
\frac{\partial r(1)}{\partial \Delta \kappa} = \frac{P(1,1) t_n + p(1)}{t^T \frac{\partial \Phi}{\partial t}} + \frac{\Delta \kappa}{G_f} \frac{t_n}{t_n} \left(-2 \frac{t_n}{t_n} - 1 + P(1,1)\right) \frac{t_0}{G_f} + \frac{\Delta \kappa}{G_f} \left(-2 \frac{t_n}{t_n} - 1 + P(1,1)\right) \frac{t_n}{G_f} \frac{t_0}{G_f} \tag{5.10d}
\]
\[
\frac{\partial r(2)}{\partial t_n} = -\frac{\Delta \kappa (2 P(1,1) t_n + p(1)) P(2,2) t_s}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} \tag{5.10e}
\]
\[
\frac{\partial r(2)}{\partial t_s} = \frac{1}{d_s} + \frac{\Delta \kappa P(2,2) t_s}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} - 2 \frac{\Delta \kappa (P(2,2) t_s)^2}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} \tag{5.10f}
\]
\[
\frac{\partial r(2)}{\partial t_i} = -2 \frac{\Delta \kappa P(2,2) t_s P(3,3) t_i}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} \tag{5.10g}
\]
\[
\frac{\partial r(2)}{\partial \Delta \kappa} = \frac{P(2,2) t_s}{t^T \frac{\partial \Phi}{\partial t}} - \frac{\Delta \kappa P(2,2) t_s}{G_f} \frac{t_0}{G_f} \left(-2 \frac{t_n}{t_n} - 1 + P(1,1)\right) t_n \frac{t_0}{G_f} \frac{t_0}{G_f} \tag{5.10h}
\]
\[
\frac{\partial r(3)}{\partial t_n} = -\frac{\Delta \kappa (2 P(1,1) t_n + p(1)) P(3,3) t_i}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} \tag{5.10i}
\]
\[
\frac{\partial r(3)}{\partial t_s} = -2 \frac{\Delta \kappa P(2,2) t_s P(3,3) t_i}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} \tag{5.10j}
\]
\[
\frac{\partial r(3)}{\partial t_i} = \frac{1}{d_i} + \frac{\Delta \kappa P(3,3) t_i}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} - 2 \frac{\Delta \kappa (P(3,3) t_i)^2}{\left(t^T \frac{\partial \Phi}{\partial t}\right)^2} \tag{5.10k}
\]
\[
\frac{\partial r(3)}{\partial \Delta \kappa} = \frac{P(3, 3)t_i}{t^T \frac{\partial \Phi}{\partial t}} \frac{\Delta \kappa P(3, 3)t_i}{G_f} \left( -2 \frac{t_n}{t_n} - 1 + P(1, 1) \right) t_n
\]  
\[\text{(5.10i)}\]

\[
\frac{\partial r(4)}{\partial t_n} = P(1, 1)t_n + p(1)
\]  
\[\text{(5.10m)}\]

\[
\frac{\partial r(4)}{\partial t_s} = P(2, 2)t_s
\]  
\[\text{(5.10n)}\]

\[
\frac{\partial r(4)}{\partial t_t} = P(3, 3)t_i
\]  
\[\text{(5.10o)}\]

\[
\frac{\partial r(4)}{\partial \Delta \kappa} = \frac{\tau_0}{G_f} \left( -\frac{t_n^2}{t_n^2} - t_n + P(1, 1)t_n + 2\tau \right)
\]  
\[\text{(5.10p)}\]

The tangent stiffness matrix can be found by differentiation of the residual functions with respect to \( u \). This leads to

\[
\frac{dr}{du} = \frac{\partial r}{\partial \delta} \frac{d\delta}{du} + \frac{\partial r}{\partial u} = J \begin{bmatrix} \frac{dt}{du} \\ \frac{d\Delta \kappa}{du} \end{bmatrix} - \begin{bmatrix} [I]_{3 \times 3} & 0 \\ 0^T & 0 \end{bmatrix} = 0
\]  
\[\text{(5.11)}\]

After rewriting this equation it can be seen that the tangent stiffness matrix is equal to the upper \([3, 3]\) part of the inverse of \( J \).

To test the model a one element test has been performed for three different loading conditions, pure mode-I, pure mode-II and a mixed-mode loading condition. The element is subjected to a uniform displacement in each of the situations. The load-displacement curves of the tests can be seen in Figure (5.4a to c). The small peak in Figure (5.4b) for a small displacement can be explained by the direction of the normal to the yield surface under pure shear which is not exactly parallel to the shear axis, but also contains a small component in the normal direction. In the return mapping, this leads to a small compressive traction in the normal direction and therefore, to a larger shear traction.

From Figure (5.4a) it can be seen that the unloading or reloading of mode-I in the compression regime is governed by the initial elastic stiffness. In the other situations of unloading and reloading, the secant stiffness matrix is used.
Figure 5.4a Mode-I loading

Figure 5.4b Mode-II loading

Figure 5.4c Mixed mode loading
5.1.2 description and analyses of two T300-5208 graphite-epoxy laminates

Similar to the mode-I analyses of the previous chapter the laminates analysed in this chapter are strips subjected to a uniform strain in the axial direction. The two laminates used for the evaluation of the material models are both constructed of a T300-5208 epoxy prepreg. The difference in the laminates is the lay-up of the layers. As was mentioned in Chapter 1 the lay-up of a laminate is described by the angle of the fibres in each layer with respect to the loading direction, starting from the top most layer and continuing downwards. The first laminate analysed is a $[30, -30, 30, -30, 90, 30]_2$ laminate, in which the bar denotes a layer of half the thickness of the other layers. The second laminate lay-up analysed is a $[45, -45, 0, 90]_s$. The material properties for the T300-5208 prepreg are listed in Table (5.1).

<table>
<thead>
<tr>
<th>Young's Mod. [MPa]</th>
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<th>Strengths</th>
</tr>
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<tbody>
<tr>
<td>$E_{11}$</td>
<td>$G_{12}$</td>
<td>$\nu_{12}$</td>
<td>$t^*_n$</td>
</tr>
<tr>
<td>$138 \cdot 10^3$</td>
<td>$5.9 \cdot 10^3$</td>
<td>0.21</td>
<td>40.0</td>
</tr>
<tr>
<td>$E_{22}$</td>
<td>$G_{13}$</td>
<td>$\nu_{13}$</td>
<td>$t^*_n$</td>
</tr>
<tr>
<td>$15 \cdot 10^3$</td>
<td>$5.9 \cdot 10^3$</td>
<td>0.21</td>
<td>246.0</td>
</tr>
<tr>
<td>$E_{33}$</td>
<td>$G_{23}$</td>
<td>$\nu_{23}$</td>
<td>$\tilde{t}_c, \tilde{t}_f$</td>
</tr>
<tr>
<td>$15 \cdot 10^3$</td>
<td>$5.9 \cdot 10^3$</td>
<td>0.21</td>
<td>68.0</td>
</tr>
</tbody>
</table>

Table 5.1 Material parameters T300-5208 prepreg

Neither of the analyses will start with a temperature analyses since the fracture energy used in the calculations was determined with the effect of the initial thermal stresses already included (O'Brien, 1982). Another aspect with regard to the fracture energy is that the value given by O'Brien, $G_c = 0.137 \text{N/mm}$, corresponds to a delamination which "jumps" between two similar interface layers as shown in Figure (5.5). Since in the analyses only the upper half of the cross-section of a laminated strip is analysed and thus only half of the delamination, the value for the fracture energy taken in the analyses is half of the value found by O'Brien.

Figure 5.5 Jumping of delamination between two similar interfaces
The total width of the $[30, -30, 30, -30, 90, 90]_s$ laminated strip was 38.0 mm while the individual ply thickness was 0.14 mm. The delamination takes places at the interface between the -30 and 90 degree plies. In Figure (5.6) the delamination versus the applied axial strain can be seen. At an applied strain of 0.34% the first delaminated integration points had almost completely softened. In this situation the curvature of the yield surface in these integration points is very large, making the return-mapping problematic. The convergence-radius of the step size is now extremely small and the analyses could no longer be continued with reasonable step-sizes. Although no horizontal level of the strain has yet been reached a tendency towards such a line can be observed from the diagram. The maximum strain found in the analyses, $\varepsilon_{ult} = 0.34\%$, is in good agreement with the experimentally obtained value of 0.35% by O'Brien (1982).

![Graph](image)

**Figure 5.6** Delamination versus applied uniaxial strain for the $[30, -30, 30, -30, 90, 90]_s$ laminate

The second laminate analysed with the plasticity model is a $[45, -45, 0, 90]_s$ laminated strip also with a width of 38.0 mm. The ply thickness was equal to 0.15 mm. The delamination in this laminate takes places at the interface between the 0 and 90 degree plies. In Figure (5.7) the delamination is given versus the applied axial strain, while Figure (5.8) shows the COD in mode-I direction versus the axial stress. The found ultimate strain of 0.57% is in good agreement with both the experimental result, $\varepsilon_u = 0.57\%$, of O'Brien (1982, 1984) and the numerical result, $\varepsilon_u = 0.57\%$, found by Schellekens (1992) using a comparable plasticity model.
Figure 5.7 Delamination versus applied uniaxial strain $[45, -45, 0, 90]_s$ laminate

Figure 5.8 COD versus average axial stress (axial force/undeformed area of the strip)
5.2 Plasticity model with isotropic softening behaviour

Although the results obtained with the material model described in the previous paragraph were good, the model has two major drawbacks. First, the analyses took much time and secondly, the step size needed to follow the equilibrium path was very small due to the small convergence radius. To avoid these difficulties a simplification of the softening behaviour was implemented. The initial yield surface remains the same, but the variables in the matrix \( P \) and in vector \( p \) are now constants depending only on the initial equivalent yield traction and the tensile and compressive strengths of the material:

\[
\Phi(t, \kappa) = \frac{1}{2} t^T P t + t^T p - \tilde{F}^2(\kappa) = 0
\]  \hspace{1cm} (5.12)

with

\[ P = \text{diag} \left[ 2C_{nn}, 2C_{ss}, 2C_{n} \right], \quad p^T = (C_n, 0, 0) \]  \hspace{1cm} (5.13)

\[ C_{nn} = \frac{\bar{T}_n}{\bar{T}_{n}^2}, \quad C_{ss} = \frac{\bar{T}_s}{\bar{T}_{s}^2}, \quad C_n = \frac{\bar{T}_0}{\bar{T}_n}, \quad C_s = \frac{\bar{T}_0}{\bar{T}_s} \]  \hspace{1cm} (5.14)

A two-dimensional representation of the softening behaviour with this description of the plasticity model is given in Figure (5.9).

![Figure 5.9 Two-dimensional representation of the yield surface and the softening behaviour](image)

Since the interaction of \( \bar{T} \), the matrix \( P \) and the vector \( p \) in the yield function has disappeared in the isotropic softening formulation, the outline given in Paragraph 3.1 can be followed for the determination of the tangent stiffness matrix. The general formulation for the tangent stiffness matrix, rewritten in traction-relative displacement formulation, is given by

\[
C^T_s = H^{-1} - \frac{H^{-1} \frac{\partial \Phi}{\partial t} \gamma^T H^{-1}}{h + \gamma^T H^{-1} \frac{\partial \Phi}{\partial t}} \]  \hspace{1cm} (5.15)
in which, for this specific yield and hardening criterion,

\[
\frac{\partial \Phi}{\partial t} = Pt + p \tag{5.16}
\]

\[
H = C^{-1} + \Delta \lambda_{n+1} P \tag{5.17}
\]

\[
\gamma^T = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial t} \frac{\partial \varepsilon^T}{\partial \varepsilon} \cdot \frac{\partial \varepsilon}{\partial t} = (Pt + p) + 2\Delta \lambda T \frac{\tilde{\tau}_0}{G_f} \left( Pt + p + \varepsilon^T P \right) \tag{5.18}
\]

and

\[
h = -\frac{\partial \Phi}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \Delta \lambda} = -2\Delta \lambda T \frac{\tilde{\tau}_0}{G_f} \varepsilon^T (Pt + p) \tag{5.19}
\]

With this plasticity model the two laminates discussed in the previous paragraph have been analysed again. In Figure (5.10) and Figure (5.11) the delamination versus the applied uniaxial strain for the \([30, -30, 30, -30, 90, 90]_s\) and the \([45, -45, 0, 90]_s\) laminates are given, respectively.

![Image](figure.png)

**Figure 5.10** Delamination versus applied uniaxial strain \([30, -30, 30, -30, 90, 90]_s\) laminate, analysed with the plasticity model with isotropic softening
Figure 5.11 Delamination versus applied uniaxial strain [45, -45, 0, 90], laminate, analysed with the plasticity model with isotropic softening

The ultimate loads of 0.36% for the [30, -30, 30, -30, 90, 90], laminate and 0.57% for the [45, -45, 0, 90], laminate are still in good agreement with the experimentally found values of 0.35% and 0.57% (O’Brien, 1982), respectively.

5.3 Damage mechanics model

The third material model implemented for the determination of the material response in case of mixed-mode delamination is based on orthotropic damage mechanics. Following the outline of Chapter 3, the traction-relative displacement relation for this situation is given by

\[ t = (I - D)Cu \]  \hspace{1cm} (5.20)

Similar to the situation described in Chapter 4, damage occurs if the equivalent relative displacement, \( \bar{r} \), exceeds a threshold value, \( \bar{r}_f \), and will grow afterwards if the equivalent relative displacement resulting from the current relative displacements is higher than all previously obtained values.

\[ \bar{r}(u, t) = \max[\bar{r}(u, \tau) | \tau \leq t, \bar{r}_f] \]  \hspace{1cm} (5.21)
The equivalent relative displacement used in the damage formulation of the material model can be obtained by rewriting the yield function of the plasticity model with isotropic softening in a strain-based format. The requirement that the yield value, \( \Phi \), equals zero for yielding leads to the following expression for the equivalent relative displacement in the damage formulation.

\[
\vec{\Delta} = \sqrt{\frac{1}{2} \vec{u}^T \vec{P}^* \vec{u} + \vec{p}^* \vec{u}}
\]  

(5.22)

where

\[
\vec{p}^* = \text{diag} \left[ 2 \frac{\vec{u}_n'}{\vec{u}_n^2}, 2 \left( \frac{\vec{u}_n'}{\vec{u}_n^2} \right)^2, 2 \left( \frac{\vec{u}_n'}{\vec{u}_n^2} \right)^2 \right]
\]

and

\[
\vec{p}^* = \begin{bmatrix} \left( \vec{u}_n' - \frac{\vec{u}_n}{\vec{u}_n^2} \right)^2, 0, 0 \end{bmatrix}^T
\]

In the plasticity model the softening branch resembles an exponential branch. To obtain such a branch for the damage model the damage growth law has to have an exponential form. For the material model described here, the following damage growth law was utilised:

\[
d_{ji} = 1 - \frac{\vec{\Delta}}{\vec{u}} \exp \left( - \frac{2C_{ii} \vec{\Delta}}{G_f} (\vec{\Delta} - \vec{\Delta}_i) \right)
\]  

(5.23)

in which \( d_{ji} \) and \( C_{ji} \) are the damage in the j-direction and the j,j component of the stiffness tensor, respectively, while \( |u| \) is the norm of the relative displacements. The tangent stiffness can be obtained by differentiation of the traction-relative displacement relation to a virtual time, which leads to

\[
C_{ji} = (1 - d_{jk})C_{jk} - \frac{dd_{ji}}{du_k} C_{ji}u_j
\]

(5.24)

where it should be noted that \( d_{jk} \) equals zero for all \( j \) not equal to \( k \).

Again the [30, -30, 30, -30, 90, 90], and the [45, -45, 0, 90], laminates were analysed. A damage threshold value, \( \vec{\Delta}_i \), of 4.0 \( \cdot \) 10\(^{-7}\) was taken. The results of the analyses can be seen in Figure (5.12) and (5.13), respectively.
Figure 5.12 Delamination versus applied uniaxial strain \([30, -30, 30, -30, 90, 90]\), laminate, analysed with the damage model

Figure 5.13 Delamination versus applied uniaxial strain \([45, -45, 0, 90]\), laminate, analysed with the damage model

Again, the numerically found ultimate strains are in good agreement with the experimental results by O'Brien (1982). For the \([30, -30, 30, -30, 90, 90]\), laminate an ultimate strain of
0.34\% was found compared to 0.35\% in the experiments. For the \([45, -45, 0, 90]_s\), laminate the ultimate strain was 0.61\%, while the experiments gave an ultimate load of 0.57\%.

From the last three paragraphs it can be concluded that each of the three material models gives results that are in good agreement with the experiments. However, from a computational perspective there are considerable differences as will be discussed in the next paragraph.

### 5.4 Comparison of the material models

In order to compare the results of the different material models not only with respect to results but also with respect to computational efficiency, the \([30, -30, 30, -30, 90, 90]_s\), laminate was analysed with large fixed steps in the applied strain. For each model the total strain level that could be reached with these steps as well as the average calculation time for the first few steps was computed. The step-sizes used in all the analyses were: first a step of \(\varepsilon = 0.1\%\), followed by four steps of 0.03\%, 10 steps of 0.005\% and finally as many steps of 0.0005\% as could be taken. In Table (5.2) a comparison between the material models is made on the basis of the strain level reached, the average computation time over the first ten steps, the delamination length at the end of the analysis and the COD in mode-I at the free edge.

<table>
<thead>
<tr>
<th>model</th>
<th>(% \varepsilon_u)</th>
<th>average t [s]</th>
<th>delamination [mm]</th>
<th>COD [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>plasticity anisotropic</td>
<td>0.26</td>
<td>53</td>
<td>0.092</td>
<td>2.20e-4</td>
</tr>
<tr>
<td>plasticity isotropic</td>
<td>0.33</td>
<td>53</td>
<td>0.492</td>
<td>3.08e-3</td>
</tr>
<tr>
<td>damage</td>
<td>0.34</td>
<td>55</td>
<td>1.008</td>
<td>4.15e-3</td>
</tr>
</tbody>
</table>

**Table 5.2** Comparison of the computational aspect of the three material models

From this table it becomes clear that from the plasticity models that with isotropic softening performs best. This is mainly due to the fact that the curvature of the yield function is much larger for the plasticity model with anisotropic softening (compare Figure 5.3 and Figure 5.9) which makes the return mapping more cumbersome and the convergence radius of the analysis smaller. Overall it can be said that the damage mechanics model performs best. The equilibrium path is easily followed almost up to complete failure even with large steps. Therefore, this model will be used in the next paragraph to determine the influence of parameter changes on the laminate behaviour.
5.5 Parameter influence on the material behaviour

In this paragraph the effect of a change in ply thickness, a variation of 10% on the fracture energy and a variation in the initial damage threshold also of 10% will be discussed. First, the ply thickness of the \([45,-45,0,90]_t\) laminate was taken twice, respectively three times the original value of 0.15 mm. The effect on the load versus delamination curve of this change in thickness is shown in Figure (5.14). Crossman and Wang (1982) and O’Brien (1982) reported a decrease of the ultimate strain with increasing ply thickness following the n-law. This was based on experiments performed by Rodini and Eisenmann (1980).

\[
\varepsilon_u(n) = \frac{\varepsilon_u(1)}{\sqrt{n}}
\]  
(5.25)

This effect is also seen in the numerical results (Figure 5.15).

![Figure 5.14 Delamination versus applied load for \([45_n,-45_n,0_n,90_n]_t\) laminates](image)

The parameters needed in the damage material model have been determined from experiments. These values are therefore sensitive to measurement errors. To assess the effect of a variation of the parameters, analyses were performed in which either the fracture energy or the initial threshold value for damage, and thus the tensile strength of the material, are varied by 10% (Figure 5.16 and 5.17). A variation of the fracture energy of 10% leads to a variation in the ultimate strain of 5%. It is therefore necessary to determine this value accurately from the experiments, since considerable scatter can exist in the experimental values.
Figure 5.15 Comparison of the ultimate loads with the n-law

Figure 5.16 Influence of a variation of 10% of $G_f$ on the material behaviour
Figure 5.17 Influence of a variation of 10% of the $R_i$ on the material behaviour

At first sight, a variation in the initial damage threshold shows a strange behaviour. One would expect the ultimate strain to be lower with a lower value of the $R_i$. However, since the fracture energy remains unchanged the softening curve for the material model with lower $R_i$ will be less steep since the area under the complete stress-strain curve has to remain equal (Figure 5.18). This less steep softening curve leads to a higher load bearing capacity in the softening regime of an integration point and thus to a higher overall load bearing capacity of the structure. This explains the earlier start of damage, but higher end value of the strain.
Figure 5.18 Stress-strain curves for different values of $\kappa_i$

5.6 Conclusions

Although plasticity models can be used to determine the mixed-mode delamination behaviour of laminated composites, their convergence behaviour appears to be very sensitive to stepsizes. Mixed mode delamination models based on damage mechanics are better suited for the analyses since the equilibrium path can easily be followed with relatively large loading steps.

The $n$-law curve dependence of the ultimate loading strain on the thickness of the plies, as described in Crossman and Wang (1982) and O'Brien (1982), is also observed in the numerical analyses.

It is necessary to have an accurate determination of the fracture energy of the material since a variation of this value by 10% alters the computed ultimate strain by 5%. The influence of a variation of the initial damage threshold is smaller since this effect is diminished by the flatter or steeper softening branch that occurs due to the lower or higher initial damage threshold, respectively.
6. TRANSVERSE MATRIX CRACKING

Delamination is not the only important failure mechanism in fibre reinforced laminated plastics. Although normally not a failure cause on its own, matrix cracking also plays an important role in the failure behaviour of the material. Matrix cracks can trigger delamination and therefore lower the load bearing capacity of the plate. Normally, a plate without matrix cracks will delaminate at a higher load level than one with matrix cracks.

Matrix cracks occur mainly in between and parallel to the fibres in a ply, which is the weakest part of the ply. In Figure (6.1) an example of matrix cracking in a laminated strip can be seen.

![Figure 6.1 X-radiographs of matrix cracking in the 90 degree ply of a [0₃, (35, -35)₃, 90₃]s laminate (Globevnik, 1992)](image)

In this chapter a material model is discussed to describe the transverse matrix cracking. Based on the experience obtained in the development of the material models for delamination, the formalism of damage mechanics is used as a basis of the material model.

Two laminates, a [(30, -30)₆], T300/976 graphite epoxy and a [0, 90, 45, -45], AS4/3502, both with a central hole are considered in the verification of the model. The first laminate is loaded in uniaxial compression. A comparison is made with the load-deformation behaviour found in experiments by Lessard and Chang (1991). Parameter studies have been performed. The laminate is modelled by the layered shell elements described in Chapter 2. For the second laminate the damage patterns in the individual layers are analysed and the results are compared with Tan (1991) and Nguyen (1996).

6.1 Orthotropic damage model for transverse matrix cracking

For the formulation of the damage model for transverse matrix cracking the strategy utilised in the previous chapter is again applied. A plasticity model is rewritten in a strain-based formulation to serve as the strain measure, \( \varepsilon \), which governs the occurrence and growth of damage (compare with Paragraph 5.3). Since the behaviour of the individual plies is
orthotropic, the Hoffman plasticity model is used as a basis for the developed damage model. For a continuum material the yield function with isotropic softening of the Hoffman model is given by

\[ \Phi(\sigma, \sigma) = \frac{1}{2} \sigma^T P \sigma + \sigma^T p - \sigma^2 = 0 \]  
\[ p^T = (\alpha_1, \alpha_2, \alpha_3, 0, 0, 0) \]  
\[ P = \begin{bmatrix} 
2(\alpha_4 + \alpha_6) & -2\alpha_4 & -2\alpha_6 & 0 & 0 & 0 \\
-2\alpha_4 & 2(\alpha_4 + \alpha_5) & -2\alpha_5 & 0 & 0 & 0 \\
-2\alpha_6 & -2\alpha_5 & 2(\alpha_6 + \alpha_5) & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_7 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_8 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_9 
\end{bmatrix} \]  

and

\[ \sigma^T = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}) \]  

The softening of the model is represented by the softening parameter \( \sigma \), while \( \alpha_i \) depends on the yield values of the material according to

\[ \alpha_1 = \frac{\sigma_0^2}{\bar{\sigma}_i^{\text{t}} - \bar{\sigma}_i^{\text{c}}} \left( \frac{\bar{\sigma}_i^{\text{c}}}{\bar{\sigma}_i^{\text{t}}} \right) \]
\[ \alpha_4 = \frac{\sigma_0^2}{2} \left( \frac{1}{\bar{\sigma}_{11}^{\text{t}} \bar{\sigma}_{11}^{\text{c}}} + \frac{1}{\bar{\sigma}_{22}^{\text{t}} \bar{\sigma}_{22}^{\text{c}}} - \frac{1}{\bar{\sigma}_{33}^{\text{t}} \bar{\sigma}_{33}^{\text{c}}} \right) \]
\[ \alpha_2 = \frac{\sigma_0^2}{\bar{\sigma}_{22}^{\text{t}} - \bar{\sigma}_{22}^{\text{c}}} \]
\[ \alpha_5 = \frac{\sigma_0^2}{2} \left( \frac{1}{\bar{\sigma}_{22}^{\text{t}} \bar{\sigma}_{22}^{\text{c}}} + \frac{1}{\bar{\sigma}_{33}^{\text{t}} \bar{\sigma}_{33}^{\text{c}}} - \frac{1}{\bar{\sigma}_{11}^{\text{t}} \bar{\sigma}_{11}^{\text{c}}} \right) \]
\[ \alpha_3 = \frac{\sigma_0^2}{\bar{\sigma}_{33}^{\text{t}} - \bar{\sigma}_{33}^{\text{c}}} \]
\[ \alpha_6 = \frac{\sigma_0^2}{2} \left( \frac{1}{\bar{\sigma}_{33}^{\text{t}} \bar{\sigma}_{33}^{\text{c}}} + \frac{1}{\bar{\sigma}_{11}^{\text{t}} \bar{\sigma}_{11}^{\text{c}}} - \frac{1}{\bar{\sigma}_{22}^{\text{t}} \bar{\sigma}_{22}^{\text{c}}} \right) \]
\[ \alpha_7 = \frac{\sigma_0^2}{3\bar{\sigma}_{12}^2} \quad \alpha_8 = \frac{\sigma_0^2}{3\bar{\sigma}_{23}^2} \quad \alpha_9 = \frac{\sigma_0^2}{3\bar{\sigma}_{13}^2} \]

The parameters \( \bar{\sigma}_i^{\text{t}}, \bar{\sigma}_i^{\text{c}} \) and \( \bar{\sigma}_{ij} \) are the yield values in tension, compression and shear, respectively.

To obtain the equivalent strain measurement of the damage formulation the yield function has to be rewritten into a strain-based formulation by means of Hooke's law.

\[ \bar{k} = \frac{1}{C_{11}} \left( \frac{1}{2} e^T C^T P C e + e^T C^T p \right)^{1/2} \]  

The tensor \( P \) and the vector \( p \) are the same as in the plasticity formulation, only the softening parameter \( \sigma_0 \) is substituted by the equivalent strain measure threshold \( \bar{k}_0 \), for consistency with the damage formalism given in Chapter 3.

Transverse matrix cracking influences the stiffness parameters in the transverse
direction, the off-diagonal terms in the stiffness matrix related to the transverse direction \((C_{12} \text{ and } C_{23})\) as well as, to a lower extent, the other off-diagonal term, \(C_{13}\), and the shear stiffnesses in the 12 and 23-direction, \(C_{44}\) and \(C_{55}\) (Lessard and Chang 1991). The stiffnesses in the fibre direction, the off-ply direction and the shear stiffness in the 13-direction, \(C_{66}\), are not influenced by the matrix cracking (Allix and Ladevèze 1989, Chang and Lessard 1991, Nguyen 1997 and Nguyen and Fleury 1997). Therefore, the damage formulation is not taken as for isotropic damage. However, the formulation of orthotropic damage, \(\sigma = (I - D)C\varepsilon\) can cause problems, since for non-diagonal stiffness matrices this causes an undesirable influence of the damage parameter in one direction on the other directions by means of the off-diagonal terms in the elastic stiffness matrix. To describe the damage in all directions correctly the damage parameters are directly implemented into the stiffness matrix, leading to

\[
C = \begin{bmatrix}
C_{11} & (1-d)C_{12} & (1-\alpha d)C_{13} & 0 & 0 & 0 \\
(1-d)C_{12} & C_{22} & (1-d)C_{23} & 0 & 0 & 0 \\
(1-\alpha d)C_{13} & (1-d)C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & (1-\alpha d)C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & (1-\alpha d)C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix} \tag{6.7}
\]

For the damage growth law an exponential softening law is chosen

\[
d = 1 - \exp\left(-\frac{\bar{R}}{\bar{R}_0}\right) \tag{6.8}
\]

The factor \(\alpha\) in Eq. (6.7) determines the degree of damage of the less influenced directions with respect to the damage parameter. The derivative of this damage growth law is given by

\[
\dot{d} = \frac{1}{\bar{R}_0} \exp\left(-\frac{\bar{R}}{\bar{R}_0}\right) \tilde{\kappa} \tag{6.9}
\]

The derivative of \(\bar{R}\) has components in all the strain directions, and is given by

\[
\kappa = \frac{\partial \kappa}{\partial \varepsilon} \tilde{\varepsilon} = \left(\frac{1}{2\bar{R}C_{11}} \left(C_{ji}P_{jk}C_{kl}\varepsilon_l + C_{ji}P_j \right)\right) \tilde{\varepsilon}_i \tag{6.10}
\]

For clarity the total derivation of the first row of the tangent stiffness matrix will be given, starting from the individual stress-strain relation. For the other components only the resulting terms are given, which have been derived in a similar manner. To simplify the formulation the following abbreviation has been used

\[
b_i^* = \frac{1}{\bar{R}_0} \exp\left(-\frac{\bar{R}}{\bar{R}_0}\right) \left(\frac{1}{2\bar{R}C_{11}} \left(C_{ji}P_{jk}C_{kl}\varepsilon_l + C_{ji}P_j \right)\right) \tilde{\varepsilon}_i \tag{6.11}
\]

Then,
\( \sigma_1 = C_{11} \dot{e}_1 + (1 - d)C_{12} \dot{e}_2 + (1 - ad)C_{13} \dot{e}_3 \)  \( \text{(6.12)} \)

\( \dot{\sigma}_1 = C_{11} \ddot{e}_1 + (1 - d)C_{12} \ddot{e}_2 + \dot{a}C_{12} \dot{e}_2 + (1 - ad)C_{13} \ddot{e}_3 - a \dot{a}C_{13} \dot{e}_3 \)  \( \text{(6.13)} \)

so that,

\( C_{11}^{tg} = \frac{\partial \dot{\sigma}_1}{\partial \dot{e}_1} = C_{11} - b_1^* C_{12} e_2 - ab_1^* C_{13} e_3 \)  \( \text{(6.13a)} \)

\( C_{12}^{tg} = \frac{\partial \dot{\sigma}_1}{\partial \dot{e}_2} = (1 - d)C_{12} - b_2^* C_{12} e_2 - ab_2^* C_{13} e_3 \)  \( \text{(6.13b)} \)

\( C_{13}^{tg} = \frac{\partial \dot{\sigma}_1}{\partial \dot{e}_3} = (1 - ad)C_{13} - b_3^* C_{12} e_2 - ab_3^* C_{13} e_3 \)  \( \text{(6.13c)} \)

\( C_{14}^{tg} = \frac{\partial \dot{\sigma}_1}{\partial \dot{e}_4} = -b_4^* C_{12} e_2 - ab_4^* C_{13} e_3 \)  \( \text{(6.13d)} \)

\( C_{15}^{tg} = \frac{\partial \dot{\sigma}_1}{\partial \dot{e}_5} = -b_5^* C_{12} e_2 - ab_5^* C_{13} e_3 \)  \( \text{(6.13e)} \)

\( C_{16}^{tg} = \frac{\partial \dot{\sigma}_1}{\partial \dot{e}_6} = -b_6^* C_{12} e_2 - ab_6^* C_{13} e_3 \)  \( \text{(6.13f)} \)

The other terms of the tangent stiffness matrix are given by

\( C_{21}^{tg} = (1 - d)C_{12} - b_1^* C_{13} e_1 - b_1^* C_{22} e_2 - b_1^* C_{23} e_3 \)  \( \text{(6.14a)} \)

\( C_{22}^{tg} = (1 - d)C_{22} - b_2^* C_{13} e_1 - b_2^* C_{22} e_2 - b_2^* C_{23} e_3 \)  \( \text{(6.14b)} \)

\( C_{23}^{tg} = (1 - d)C_{23} - b_3^* C_{13} e_1 - b_3^* C_{22} e_2 - b_3^* C_{23} e_3 \)  \( \text{(6.14c)} \)

\( C_{24}^{tg} = -b_4^* C_{12} e_1 - b_4^* C_{22} e_2 - b_4^* C_{23} e_3 \)  \( \text{(6.14d)} \)

\( C_{25}^{tg} = -b_5^* C_{12} e_1 - b_5^* C_{22} e_2 - b_5^* C_{23} e_3 \)  \( \text{(6.14e)} \)

\( C_{26}^{tg} = -b_6^* C_{12} e_1 - b_6^* C_{22} e_2 - b_6^* C_{23} e_3 \)  \( \text{(6.14f)} \)

\( C_{31}^{tg} = (1 - ad)C_{13} - ab_1^* C_{13} e_1 - b_1^* C_{23} e^2 \)  \( \text{(6.15a)} \)

\( C_{32}^{tg} = (1 - d)C_{23} - ab_2^* C_{13} e_1 - b_2^* C_{23} e^2 \)  \( \text{(6.15b)} \)

\( C_{33}^{tg} = C_{33} - ab_3^* C_{13} e_1 - b_3^* C_{23} e^2 \)  \( \text{(6.15c)} \)

\( C_{34}^{tg} = -ab_4^* C_{13} e_1 - b_4^* C_{23} e^2 \)  \( \text{(6.15d)} \)

\( C_{35}^{tg} = -ab_5^* C_{13} e_1 - b_5^* C_{23} e^2 \)  \( \text{(6.15e)} \)

\( C_{36}^{tg} = -ab_6^* C_{13} e_1 - b_6^* C_{23} e^2 \)  \( \text{(6.15f)} \)

\( C_{4i}^{tg} = -ab_i^* C_{44} e_4 \) for \( i = 1, 2, 3, 5, 6 \)  \( \text{(6.16a)} \)

\( C_{44}^{tg} = C_{44} - ab_4^* C_{44} e_4 \)  \( \text{(6.16b)} \)
\[ C_{5i}^{f} = -\alpha b_{i}^{*}C_{55}\varepsilon_{5} \]  
for \( i = 1, 2, 3, 4, 6 \) \hfill (6.17a)

\[ C_{55}^{f} = C_{55} - \alpha b_{3}^{*}C_{55}\varepsilon_{5} \]  
\hfill (6.17b)

\[ C_{6i}^{f} = 0 \]  
for \( i = 1, 2, 3, 4, 5 \) \hfill (6.18a)

\[ C_{66}^{f} = C_{66} \]  
\hfill (6.18b)

Evidently, the tangent stiffness matrix is non-symmetric.

### 6.2 T300/976 graphite-epoxy with central hole subjected to in-plane compressive loading

A T300/976 graphite-epoxy laminate with a central hole and a laminate lay-up of [(30, -30)\(_{6}\)]\(_{1}\) was tested in in-plane compression by Lessard and Chang (1991). The X-rays of their experiments showed pure in-plane failure parallel to the fibres in all layers. Delamination did not occur. The load extensometer curve from the experiments was fairly linear up to failure, which occurred suddenly and without warning. In the analysis described by Chang and Lessard (1991) the dominant failure mode appeared to be tensile matrix cracking. The laminate geometry is given in Figure (6.2), while the material parameters, taken from Nguyen (1996), can be found in Table (6.1). All values, except the Poisson’s ratios which are dimensionless, are in MPa.

![Figure 6.2 Geometry of the T300/976 graphite-epoxy laminate containing a central hole](image-url)

**Figure 6.2** Geometry of the T300/976 graphite-epoxy laminate containing a central hole
For a first analysis $\alpha = 0$ was taken. The equivalent strain measure threshold value is taken equal to the tensile strength of a ply in the fibre direction divided by the elastic stiffness in this direction, $\bar{\sigma}_{11}^t/\bar{C}_{11}$. In Figure (6.3) the load $P$ is plotted versus the extensometer reading. As can be seen the analysis differs considerably from the experimental results found by Lessard and Chang (1991).

![Figure 6.3 Comparison of the damage analysis with the experimental results of Lessard and Chang (1991)](image)

A possible explanation for this large difference can be found in the highly non-linear behaviour of especially the shear stresses and strains previous to failure. This effect was experimentally found by Hahn and Tsai (1973), who developed an third power relation between the shear strain, $\gamma$, and the shear stress, $\tau$ (Eq. 6.19). In publications by Chang and Lessard (1991) and Nguyen (1996) this relation was successfully used.

$$\gamma_{ii} = \frac{\tau_{ii}}{G_{ii}} + \eta \tau_{ii}^3 \quad i = 4, 5, 6 \quad (6.19)$$
The value of $\eta$ determines the amount of non-linearity and was taken equal to $2.4408E-8 \, MPa^{-1}$ for this material (Nguyen 1996). A disadvantage of such a non-linear relation is the iterative procedure which is needed in the loading steps to get the correct shear stiffness for each stress situation. Since one of the aims of the matrix cracking model described in this chapter is that it should be easy to use, the non-linearity of the shear stiffness is taken into account by using a linear relation between the shear stresses and strain, but with a lower shear stiffness. As the real linear-elastic shear stiffness gave a load extensometer curve with a too stiff material behaviour, the secant stiffness of the stress-strain relation up to failure, $G_{ij} = 2380 \, MPa$, should give a too flexible behaviour. With this secant relation the stiffness is underestimated largely (Figure 6.4). A better solution would be to take the secant stiffness with respect to half of the ultimate strain, $G_{50\%}$. This stiffness, $G_{ij} = 3400 \, MPa$, appears to be a good trade-off between the underestimation at the beginning of the curve and the overestimation at the end.

![Figure 6.4 Non-linear shear stress-strain curve approximation by straight line.](image)

The laminate was again analysed with $\alpha = 0$ for the secant stiffness and for $G_{50\%}$. The comparison with the experimental results are shown in Figure (6.5). As is seen the secant stiffness underestimates the load extensometer curve, as was expected, while the $G_{50\%}$ shows a small overestimation of the experiment. Since both the secant and the $G_{50\%}$ curve agree reasonably well with the experiments the influence of the degradation of the shear stiffnesses is studied for both these stiffnesses.
Figure 6.5 Load extensometer curve of the [(30, -30)_6], T300/976 laminate for different shear stiffnesses.

The value of $\alpha$ is now varied to study the effect on the interaction of damage in the shear directions on the load extensometer curve. From research conducted by Laws et al. (1983) and Nguyen (1997b) it can be seen that also the shear stiffnesses in the 44- and 55-direction degrade with increasing matrix cracking. From Nguyen (1996) it appears that the maximum reduction of the shear stiffnesses was approximately 30%. To obtain this reduction a factor $\alpha = 0.3$ is needed. Since this is the maximum reduction, the analysis is also carried out with $\alpha = 0.15$. From Figure (6.6) can be concluded that an increase in the contribution of damage in the shear direction, that is an increase of $\alpha$, leads to a decrease slope of the load extensometer curve.
Figure 6.6 Effect of the degradation of the secant and the $G_{50\%}$ shear stiffnesses

The best agreement with the experimental results of Chang and Lessard (1991) is given by an analysis with $G_{50\%}$ and $\alpha = 0.3$.

6.3 AS4/3502 [0,90,45,−45], laminate containing a central hole

A graphite-epoxy AS4/3502 laminate containing a central hole is studied next with the described matrix cracking model. The lay-up of the laminate is [0,90,45,−45], and it is subjected to an in-plane tensile loading. This laminate was tested experimentally by Tan (1991) and analysed numerically by e.g. Tan (1991) and Nguyen (1997b). From the results found by these authors matrix cracking is assumed to occur in all the layers except for the 0 degree layers, in which the loading is parallel to the fibres and stresses perpendicular to the fibres are negligible. The material parameters, taken from Nguyen (1997b) and Tan (1991), are given in Table (6.2). The stiffnesses and the yield strengths are given in MPa.
| $E_{11}$ | 143900. | $G_{12}$ | 6700. | $\nu_{12}$ | 0.326 |
| $E_{22}$ | 11900. | $G_{13}$ | 6700. | $\nu_{13}$ | 0.326 |
| $E_{33}$ | 11900. | $G_{23}$ | 3800. | $\nu_{23}$ | 0.54 |
| $\sigma_{11}^f$ | 1862. | $\sigma_{11}^f$ | 1482. | $\bar{\sigma}_{12}$ | 64.8 |
| $\sigma_{22}^f$ | 51.7 | $\sigma_{22}^f$ | 206.8 | $\bar{\sigma}_{23}$ | 64.8 |
| $\sigma_{33}^f$ | 51.7 | $\sigma_{33}^f$ | 206.8 | $\bar{\sigma}_{13}$ | 64.8 |

Table 6.2 Material parameters AS4/3502 graphite-epoxy

For the nonlinear behaviour of the shear stiffnesses no parameters were found, so the $\eta = 2.4408 \cdot 10^{-8}$ MPa$^{-3}$ used in the T300/976 analysis was adopted. Also the influence of the damage on the shear directions, $\alpha = 0.3$, was taken from this laminate.

The thickness of the individual plies is 0.132 mm, which makes the total laminate thickness equal to 1.056 mm, while the length of the specimen was 50.8 mm, the width equalled 25.4 mm and the diameter of the hole was taken to be 7.62 mm. Due to symmetry considerations only one quarter of the laminate was modelled by layered shell elements. The laminate was first subjected to a temperature drop of 125 °C, to account for the initial stresses occurring from the autoclaving process. Afterwards a uniformly distributed load was applied to the short edge of the laminate. Figure (6.7) to (6.9) show the damaged area in each of the damaged plies at several stages during the analysis. At a load of 323 MPa considerable damage has occurred and complete rupture will be assumed (Note: no complete rupture will be found since the 0 degree layers remain linearly elastic). This load is in good agreement with the ultimate load found by Tan (1991) and Nguyen (1997b) of 326 MPa and 265 MPa, respectively.
Figure 6.7 Damage pattern at a distributed load of 170 MPa
Figure 6.8 Damage pattern at a distributed load of 296 MPa
Figure 6.9 Damage pattern at a distributed load of 323 MPa
In Figure (6.10) the axial displacement of a point on the edge of the central hole is plotted versus the load.

![Graph showing load versus displacement for point A of the [0, 90, 45, −45]_t laminate.](image)

**Figure 6.10** Load versus displacement point A of the [0, 90, 45, −45]_t laminate

### 6.4 Conclusions

Since in the individual directions the laminate responds differently, transverse matrix cracking is directly implemented into the stiffness matrix via the softening relations. This prevents undesirable influences of the damage behaviour in one direction onto the other directions by means of off-diagonal terms in the stiffness matrix. A limiting effect is introduced via a reduction factor.

The described materials show a highly non-linear stress-strain relation in the shear directions, with a steep first part which gradually flattens at increasing strain. If this behaviour is not taken into account in the analyses of laminates and the linear elastic stiffness of the first part of the curve is taken in the complete analysis, the load deformation curves will be overstiff. However, good results are obtained when a secant stiffness is taken, which passes through the non-linear stress-strain curve at 50% of the ultimate strain level, \( G_{50\%} \).
7. CONCLUSIONS AND FINAL REMARKS

To obtain a better understanding of the behaviour of fibre reinforced laminated composites, material models were developed that describe the two major failure mechanisms, namely delamination and transverse matrix cracking. The models were implemented into a finite element code and tested in free edge delamination analyses of composite strips subjected to a uniform uniaxial strain and in matrix cracking analyses of laminated strips containing a central hole.

The delamination analyses were split into two groups. The analyses in which delamination occurs in the midplane of a symmetric laminate, where the delamination mode can be assumed to be completely in mode-I, and mixed-mode delamination analyses as it occurs in other interfaces or non-symmetric laminate lay-ups. For the mode-I delamination analysis a material model based on the formalism of orthotropic damage mechanics was developed. The free edge delamination growth in the midplane of a [25, −25, 90], laminate was studied. The ultimate strain applied to the strip corresponded well with the results from experiments performed by Wang et al. (1985). To obtain this result it is necessary to take into account the temperature drop which occurs in the manufacturing process of the laminate. If this is not taken into account the ultimate strain is highly overestimated. The results were found in a two-dimensional analysis of a cross-section of the laminate. The nonlinear behaviour in this analysis took place in interface elements, while the plies were modelled with generalised plane strain elements. Since it is normally assumed that the delamination at a certain distance from the loaded edge is uniform, a two-dimensional, less time consuming, analysis should suffice. To assess the validity of this assumption, the results were compared with three-dimensional analyses of the same laminate. If elastic zones at the ends of the strip are taken into account to avoid load introduction effects, it was seen that the free edge delamination is indeed fairly uniform and that a two-dimensional analysis suffices to obtain the load bearing capacity of the strip. Only for an overall view of the damage pattern a complete three-dimensional analysis is needed.

For the description of mixed-mode delamination three different material models were developed. The first two based on the formalism of plasticity while the last was again based on orthotropic damage mechanics. Each model was based on the Hoffman yield criterion. The difference between the two plasticity models comes down to a difference in the softening behaviour. The first model shows a complete orthotropic softening behaviour, while for the latter model a simplified isotropic softening behaviour was assumed. For the damage model the Hoffman criterion was rewritten in a strain-based form. The three models were applied to two different laminate lay-ups. All three models perform well with respect to the ultimate load which was found. However, when compared on the basis of computational robustness and computation time it was observed that the damage model shows a far better behaviour. Parameter studies showed that the n-law curve for the thickness effect, described in e.g. O’Brien (1982) is also seen in the numerical analyses. The determination of the
fracture energy of the material must be done accurately, since a variation of this parameter has a large effect on the computed ultimate strain.

Based on the experience obtained with the delamination analyses, a material model for transverse matrix cracking was constructed on the formalism of damage mechanics. Again the Hoffman yield criterion served as a basis, but now in the form used in continuum elements. The transverse matrix cracking in laminated strips containing a central hole was analysed. For the finite element analyses layered shell elements were utilised. A comparison with experiments shows a reasonable agreement.

The models described in this dissertation give a robust tool for the finite element analyses of laminated composites. Although the examples discussed here are all restricted to laminated strips subjected to uniaxial loading, the models have no limitation to such a geometry or loading type. Therefore also structural parts could be analysed with the models. With the developed tools more insight can be gained in the damage development and the load bearing capacity of graphite-epoxy materials, which leads to a better damage prediction in composite structures. Also, the analyses of free edge delamination show that this is a serious failure mode, which could simply be avoided in structures by avoiding free edges. From analyses with different laminate lay-ups, the effects of fibre orientation and stacking sequence on the delamination and matrix cracking behaviour can be seen, which can lead to better lay-ups in structural parts.
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**Stellingen**

theses

1. Teneinde voorspellende berekeningen te doen aan vezelversterkte kunststoffen moet er een experimentele database worden opgezet, met materiaalgegevens van verschillende typen vezelversterkte composieten met inbegrip van vezelorientatie.
   *In order to carry out predictive analyses of fibre reinforced plastics, an experimental database should be established with material parameters of several types of fibre reinforced composites including the effect of fibre orientation.*

2. De stelling die men daad’lijk begrijpt, is het begrijpen meestal niet waard.
   (vrij naar Multatuli idee 68)
   *The thesis that is immediately understood, is mostly not worth understanding.*
   (free after Multatuli idea 68)

3. Het is een geruststelling voor promovendi om te weten dat de soep die verplicht onderwijsprogramma heet niet zo heet wordt gegeten als hij wordt opgediend.
   *It is a relieve for PhD students to know that the obligatory educational programme is sure to simmer down.*

4. De overheidslogan: "De maatschappij dat ben jij" onderschrijft het idee dat kritiek op de maatschappij feitelijk zelfkritiek is.
   *The slogan of the Dutch government: “You are the society” endorses the idea that criticism on the society is actually selfcriticism.*

5. Vanwege de eindige toegang tot artikelen moeten verwijzingen naar andere artikelen voor parameters noodzakelijk voor de reproductie van analyses worden beperkt tot een minimum.
   *Due to the finite access to papers, references to other papers for parameters needed for reproduction of analyses should be reduced to a minimum.*
6. Er bestaat een sterke overeenkomst tussen een promotieproject en origami. De individuele stappen (= vouwen) hebben vaak geen duidelijke link met het te verwachten eindresultaat, maar zijn voor het bereiken hiervan wel noodzakelijk. A strong comparison exists between a PhD project and origami. The individual steps (= folds) often do not have a clear link with the expected grand result, but are necessary for achieving it.

7. Trucs, toegepast om een numeriek stabiel proces te verkrijgen zijn alleen toegestaan indien er een fysische verklaring voor de truc gegeven kan worden. Tricks, applied to obtain a numerically stable process are allowed only if a physical explanation can be given.

8. Rekenpakketten die veelvuldig gebruik maken van opslag op harde schijf verliezen de concurrentieslag door de opkomst van steeds snellere processoren en parallele mogelijkheden. Computational software that frequently uses the hard disc for input/output operations will lose the competitive edge because of the advent of increasingly faster processors and parallel possibilities.