PIETER HUYBERS
POLYHEDROIDS
THAT FAMILY OF POLYHEDRA
POLYHEDROIDS: THAT FAMILY OF POLYHEDRA

PIETER HUYBERS
To my wife Bep
without whose enduring encouragement
this book may never have come to an end

Copyright Dr. Pieter Huybers
Westland, August 2014
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Extra, if on DVD: Stereoscopic pictures in MPO-format for presentation on 3D colour television.
Polyhedroids

In the context of this book, polyhedra are defined as portions of space that entirely are surrounded by regular polygons. The main emphasis is placed on the so-called uniform polyhedra. Polyhedra are called uniform if the faces meet in the same manner at every vertex: is face transitive [0.5]. Under this definition fall: the 5 Platonic solids, the 13 Archimedean solids (see Chapter 1), the 4 regular stellated polyhedra, called Kepler-Poinsot polyhedra, and 53 uniform star polyedra [1.5] (Chapter 7). Besides these there are two infinite rows of uniform prisms and antiprisms (Chapter 6). Only a limited group is convex: the Platonic and Archimedean solids and the prisms and antiprisms. Many other configurations are derived from these polyhedra or have very much in common. It is therefore important to have a proper knowledge of their basic geometry. In this book attention will be paid to the Platonic and Archimedean solids themselves, but also to a number of polyhedron based structures, in particular domes (Chapter 8) and space frames. In general we have called this group of forms 'Polyhedroids'. They together form a great family, hence the title of this book.

Fig. 0.1. Many members of the 'Family of Polyhedra'

In fact this study started with the wish to try to understand, what all these figures have in common and how they can be materialized in numbers and in visual form. For the polyhedra, composed of regular polygons, the author found a basic approach in the book Vielecke und Vielfläche by M. Brückner. From this starting point many other groups could be derived, such as
Polyhedroids

their reciprocals or duals, prisms and antiprisms, stellated polyhedra and finally sphere subdivisions. A proper knowledge of their geometry learns how these can be manipulated or be combined in spatial configurations: packings and spatial structures (Chapter 13).

Most of the pictures in this book have been made with the computer programme CORDIN, developed by the author in close co-operation with Gerrit van der Ende. The first versions of this programme date from as long ago as 1975 and it was originally only meant to have the availability of a means, with which polyhedral figures could be calculated, either resulting in numeric data or in pictures. But it gradually evolved into a quite versatile instrument and in many previous occasions, in conference papers and in articles, examples were shown proving its potential. The projection of vector files on the surface of polyhedra is a recent new extension, although it is still a rather laborious process in its present form. The author therefore decided to provide so many data that most of the processes described can be traced following the course that the author himself went, he not being a mathematician but - as he preferably calls it - a simple structural designer with an architectural background. Its understanding therefore does not require a thorough knowledge of mathematics but only the availability of a sufficient amount of common sense - and patience of course. He describes this from his personal background and experiences and it does not claim to cover the total area. Much additional information can be found on Internet or from other sources.

In this book thus not only the geometrical aspects are discussed but also great emphasis is given to illustrations. In Chapter 16, a number of stereo pictures is provided that can be seen as so-called anaglyphs (with additionally available blue/red-glasses). If this book is provided on DVD disk these pictures and a few more are provided as bonus material, in the form of MPO-files that are fit for presentation on a regular 3D-television set. When printed, the anaglyphs loose some in quality, as it is difficult to print the colours exactly as needed. Printing in colour may be considered, but as such this is costly and therefore we choose in the first instance to publish it in the form of an electronic book to be read on I-pads or tablets. This opened the possibility to eventually put it on one DVD together with the bonus material mentioned before. This has the extra advantage, that hyperlinks can be included in the text that allow direct access to Internet. As an example, the below link is given which connects to the own website of the author: http://www.pieterhuybers.nl

In printed form these links do not work this way of course, but they are still worth looking at.

Some of the exercizes described, may at first glance seem playful and not very serious, originating from the mind of the author, but he believes that such an attitude is required to come to new ideas and to new experiments. As examples of this thesis may serve the development of spatial structures as described in Chapter 13, that almost by definition are based on polyhedral geometry, as well as the new soccer ball design that is shown in Chapter 12, that deals with isodistant polyhedra.

References

Chapter 1. THE CONVEX UNIFORM POLYHEDRA

1. 1. The geometry of polyhedra

First of all we must agree upon a workable definition of what we consider in this context as a convex uniform polyhedron [Ref. 1.5]. We assume that:

1) They are covered with a closed pattern of plane, regular polygons.
   At this point we shall look only at the so-called Platonic solids, that are composed of identical polygons, and at the Archimedean solids, that consist of two or three different polygons. Both groups are called after the ancient scientists to which their discovery is usually ascribed [Refs. 1.1, 1.3, 1.6, 1.9]. The different polygons that occur in these solids have either 3, 4, 5, 6, 8 or 10 edges. The endless rows the prisms and antiprisms also fall in this category, but their two parallel sides can have any number of edges, and they will be treated separately in Chapter 6.

2) All vertices of a polyhedron lie on one circumscribed sphere.

3) These vertices all are identical. This is so because around each vertex of a particular polyhedron the polygons are grouped in the same number, kind and order of sequence.

4) The polygons meet in pairs at a common edge.

5) The dihedral angle at such an edge is always convex. This means that the dihedral angle between two adjacent polygons is less than 180°, if seen from the interior, or in other words: the sum of the polygon face angles that meet at a vertex is always smaller than 360° (see Table 1.6).

1.2. The different kinds

![Diagram of polyhedra](image)

Fig. 1.1. Review of the 5 regular polyhedra and the 13 semi-regular polyhedra, of which two have a left-handed as well as a right-handed version.
Chapter 01

These are:
The 5 regular polyhedra: 1) Tetrahedron, 2) Cube, 3) Octahedron, 4) Dodecahedron, 5) Icosahedron

It is easy to understand that under these conditions the minimum total number of polygons around a vertex is 3, the maximum number 5 and it is also simple to prove, that not more than 5 totally regular polyhedra can exist (Fig. 15.7). These are the regular or Platonic solids and they are each composed of one kind of face. Polyhedra are called semi-regular, or Archimedean, if more than one kind of polygon is used for their construction. According the first condition of the previous definition - namely, that the polygon has no more than 3, 4, 5, 6, 8 or 10 edges - a group of 15 principally different semi-regular polyhedra is found. For more information see the links: http://en.wikipedia.org/wiki/Platonic_solid and http://en.wikipedia.org/wiki/Archimedean_solid

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Table 1.1. Some characteristic aspects of the Platonic and Archimedean polyhedra.

In the table the few characteristics of polyhedra are given, where table P = polyhedron index, Vertex code = side-numbers of respective polygons that meet in a vertex; V, E and F = number of vertices, edges and faces. Radius = radius of circumscribed sphere at unit edge length. The formula of Euler is applicable, which means that: V - E + F = 1.

The different polyhedra are further referred to as P#, with # for the index number. These index numbers are useful to indicate the individual polyhedra, in order to avoid the need to use their mostly difficult and sometimes long scientific names. In computer programmes for the calculation of their geometry or for their visual presentation it is necessary to indicate them by a unique number.
That is why they are numbered here in a certain order of sequence, that is dictated by the
following consecutive criteria:

[1] Number of faces
[2] Number of edges
[3] Radius of the circumscribed sphere

If only criteria 1 and 2 were applied, the truncated dodecahedron and the truncated icosahedron would have obtained the same number. The left- and right-handed snubs have the same identification numbers, because they are topologically identical, although they have different coordinates. They are sometimes called ‘chiral’ (see Chapter 4).

1.3. The Platonic solids

There is a direct geometric relation between the regular polyhedra. P1, P3, P4 and P5 are for instance inscribable in Cube P2, following Figs. 1.2 and 1.3. This picture also gives the clues for the definition of their positions in the Euclidean space. The mutual relations are expressed in numerical form or in the form of simple expressions in Table 1.2.
Chapter 01

In this table the value of $\tau$, also known as the Golden Section, is defined as:

$\tau = (1 + \sqrt{5}) / 2 = 1.6180339887499...$

This has a few odd characteristics, a.o.:

$\tau^2 = \tau + 1 = 2.61803399$

$1/\tau = \tau - 1 = 0.61803399$

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<tr>
<th>cube</th>
<th>tetrahedron</th>
<th>octahedron</th>
<th>dodecahedron</th>
<th>icosahedron</th>
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Table 1.2. The relations of the 5 regular polyhedra.

1.4. The Archimedean solids

The names of the semi-regular solids indicate that they are generally considered as to be derived from the regular solids by truncation. If this truncation is done so that the original face edges are divided in three parts, the original faces convert into polygons with double the number of sides (i.e. triangle becomes hexagon, square becomes octagon and pentagon becomes decagon). Thus five new polyhedra are found: the truncated versions of the regular solids (P6, P8, P9, P13 and P14).
Fig. 1.5. The 15 Archimedean polyhedra. P15 and P18 have a left-handed and a right-handed version.

Fig. 1.6. The truncation of the octahedron P3 at one third of its side length, forming P8

The truncation procedure can be carried out a little bit further so that the original edges are exactly bisected. This gives two new solids (see Fig. 1.5): the Cuboctahedron (No. 7) and the Icosidodecahedron (No. 12). These two are peculiar ones and they are called quasi-regular, solids because they can be considered, as their names already suggest, to be compounds of two pairs of regular solids. P7, the Cuboctahedron, is composed of 6 squares (like the cube P2) and 8 triangles (like the octahedron P3). P12, the Icosidodecahedron, is composed of 20 triangles (like the Icosahedron P5) and 12 pentagons (like the Dodecahedron P4).

Truncation can also take place parallel to the edges. This generally produces square extra faces and it yields four new semi-regular solids (Nos. 10, 11, 16 and 17)
Fig. 1.7. The formation of the truncated icosahedron at one third of the sides

There are two other solids that are found by truncation of the corners and a double truncation of the edges. There are in fact four of them, as they occur in a right-handed as well in a left-handed (enantiomorphic) version. These are the Snub Cube (No. 15) and the Snub Dodecahedron (No. 18). These two are called after their circumscribed figures. The Snub Cube has 6 squares, each one completely surrounded by triangles, whereas the Snub Dodecahedron has 12 pentagons in a corresponding situation.

1.5. Regular polygons

Fig. 1.8. A) The six different polygons in polyhedra. B) Two adjacent sectors in a polygon.

The central angle of a regular polygon with n sides: \( \varphi_n = \frac{\pi}{n} \) \{1.1\}

The radius of the circumcircle: \( R_2 = \frac{1}{2 \sin \varphi_n} \) \{1.2\}

The distance of the center to the mid-point of a side: \( m_n = \frac{1}{2 \tan \varphi_n} = \sqrt{(R_2^2 - 0.25)} \) \{1.3\}

Two alternate corners of a polygon (P and S in Fig. 1.8B) have the distance: \( b_n = 2 \cos \varphi_n \) \{1.4\}

The area of an n-gon

\[
A_n = \frac{1}{2} nm_n = \frac{1}{2} \sqrt{(R_2^2 - 0.25)} = \frac{n}{4 \tan \varphi_n}
\] \{1.5\}
1.6. Vertex situations

If one connects the other ends of the edges, meeting in a vertex of a polyhedron a so-called 'vertex figure' is found. It forms the basis of a pyramid with the original vertex as its apex. This cap is called 'vertex pyramid' [Ref. 1.2].

![Diagram of a vertex figure](image)

**Fig. 1.9.** A) Polyhedron P13 (code 5-6-6), showing the situation in a vertex. B) The three polygons in P13 with their 'small diagonals' \( b \).

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<th>( n )</th>
<th>( \Phi )</th>
<th>( R_2 )</th>
<th>( m_n )</th>
<th>( b_n )</th>
<th>( \text{Area}_{\text{total}} )</th>
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Table 1.3. Relevant data of the first series of regular polygons with 3 to 20 sides.

A vertex figure has as many edges as the number of polygons that meet in the vertex of a polyhedron. It has therefore either 3, 4 or 5 edges and its form may be regular or not.
1.6.1. The triangular vertex figures

The triangular figure has in its most general form the three different sides u, v and w. The values of these are equal to $b_n$ in the respective adjacent polygon.

The vertex figure can thus be scalene (P11 and P17), isosceles (P6, P8, P9, P13, P14 and the prisms) or equilateral (P1, P2 and P4).

$$R_3 = \frac{uvw}{\sqrt{s(s-u)(s-v)(s-w)}} \Rightarrow s = \frac{1}{2}(u+v+w)$$ \{1.6\}

Fig. 1.10. Principal vertex figure with 3 sides

Fig. 1.11. All triangular vertex figures

1.6.2. The quadrangular vertex figures

The quadrangular vertex figure principally has the shape of a trapezoid (P7, P10 and P16 and the antiprisms). $R_3$ in this case is again determined by the fact that the circle has to pass through the three corners A, B and C. Subsequently, the vertex figure can be reduced to a basic triangle with the sides u, v and $d_m$ with

$$d_m = \sqrt{v^2 + uw}$$ \{1.7\}
and similarly: \( R_3 = \frac{u v d_m}{\sqrt{t(t-u)(t-v)(t-d_m)}} \Rightarrow \) with \( t = \frac{1}{2}(u + v + d_m) \) \{1.8\}

For antiprisms a general expression for \( d_m \) can be derived:
\[
d_m = \sqrt{(1 + 2 \cos \varphi_n)} \quad \{1.9\}
\]

where \( n \) is the number of the sides of the two parallel variable polygons [1.4].

Fig. 1.11. *Principal vertex figures with 4 sides.*

Two polyhedra can be found where \( u = w \) (the so-called 'quasi-regular' solids P7 and P12) and where the vertex figure is a rectangle with the diagonal:
\[
d_m = \sqrt{(u^2 + v^2)} = 2R_3
\]

For the octahedron (P3): \( u = v = w = 1 \), so that the vertex figure is a square with:
\[
d_m = 2R_3 = \sqrt{2}
\]

Fig. 1.13. *All quadrangular vertex figures*
1.6.3. The pentagonal vertex figures

The vertex figures of the Icosahedron P5 and of the two snub solids, P15 and P18, are pentagonal. In all three cases 4 equilateral triangles meet, the fifth meeting polygon being either a triangle, a square or a pentagon. Here again radius $R_3$ of the circumscribed circle is found from the basic triangle A-B-C, in which $AB = BC = 1$. The length of $AC = d_m$ has to be derived with the help of a third power equation.

\[ 4\varphi_1 + \varphi_2 = 180^\circ \implies \varphi_2 = 180^\circ - 4\varphi_1 \]

\[
\sin \varphi_1 = \frac{1}{2R_3} \quad \text{and} \quad \sin \varphi_2 = \frac{b}{2R_3} = \frac{\cos \varphi_n}{R_3}
\]

\[
R_3 = \frac{\cos \varphi_n}{\sin \varphi_2} = \frac{1}{2\sin \varphi_1}
\]

\[ 2\sin \varphi_1 \cos \varphi_2 = \sin \varphi_2 = \sin (4\varphi_1) = 2\sin (2\varphi_1) \cos (2\varphi_1), \quad \text{so that:} \]

\[ 4\cos^3 \varphi_1 - 2\cos \varphi_1 - \cos \varphi_n = 0 \]

This equation can be solved for various values of $n$. This is done in Table 1.4 [see Ref. 1.2].

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<th>$\varphi_1$</th>
<th>$\varphi_2 = 180^\circ - 4\varphi_1$</th>
<th>$R_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>0.8091699</td>
<td>36</td>
<td>36</td>
<td>0.85065080</td>
</tr>
<tr>
<td>P15</td>
<td>0.8425092</td>
<td>31.59396276</td>
<td>49.62414896</td>
<td>0.92819138</td>
</tr>
<tr>
<td>P18</td>
<td>0.8577807</td>
<td>30.93168860</td>
<td>56.27324558</td>
<td>0.97273285</td>
</tr>
</tbody>
</table>

Table 1.4. Values of $\varphi_n$ and $\varphi_1$ in the basic pentagonal vertex figure.

A pentagonal vertex figure can be reduced to a basic quadrangle u-u-v-d$_m$ by the introduction of a diagonal with the value:

\[ d_m = 2\cos \varphi_1 \]

There are ten polyhedra with a triangular vertex figure: P1, P2, P4, P6, P8, P9, P11, P13, P14 and P17. All other eight vertex figures can be reduced also to the form of a triangle. Two sides have the length of a short diagonal in either a triangle, a square or a pentagon and the third side has a length of the form $d_m$, as derived in the foregoing. They are summarized in the following table 1.5.
<table>
<thead>
<tr>
<th>P</th>
<th>Code</th>
<th>diagonal $d_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P3</td>
<td>3-3-3-3</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>P7</td>
<td>3-4-3-4</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>P12</td>
<td>3-5-3-5</td>
<td>$\sqrt{(1+4\cos^236^\circ)}$</td>
</tr>
<tr>
<td>P10</td>
<td>3-4-4-4</td>
<td>$\sqrt{2+\sqrt{2}}$</td>
</tr>
<tr>
<td>P16</td>
<td>3-4-5-4</td>
<td>$\sqrt{2+2\cos36^\circ}$</td>
</tr>
<tr>
<td>P5</td>
<td>3-3-3-3-3</td>
<td>$2\cos(36^\circ)$</td>
</tr>
<tr>
<td>P15</td>
<td>3-3-3-3-4</td>
<td>$2\cos(31.59396280^\circ)$</td>
</tr>
<tr>
<td>P18</td>
<td>3-3-3-3-5</td>
<td>$2\cos(30.93168860^\circ)$</td>
</tr>
</tbody>
</table>

Table 1.5. *Values of basic diagonals*

For a complete review of all uniform polyhedra by vertex figure see [http://en.wikipedia.org/wiki/List_of_uniform_polyhedra_by_vertex_figure](http://en.wikipedia.org/wiki/List_of_uniform_polyhedra_by_vertex_figure)

### 1.7. Characteristic radii of a polyhedron

Apart from the radius of the vertex figure’s circum-circle $R_3$, a polyhedron has a few other characteristic radii. The first is that of the circumscribed sphere of a polyhedron, which has to pass through the circle with the radius $R_3$ and its respective vertex T. This is shown in detail in Fig. 1.17A. $R_1$ is the circle around the triangle with the sides 1, 1 and 2 $R_1$.

Half the sum of the sides: $s = R_3 + 1$.

$$R_1 = \frac{2R_1}{4\sqrt{(R_3 + 1)(-R_3 + 1)R_3R_3}} = \frac{1}{2\sqrt{(1-R_3^2)}}$$  \hspace{1cm} \{1.13\}

![Fig.1.16. Derivation of $R_1$ as a circle around the triangle A-B-T.](image)
The radius of the 'inter-sphere': \( R_5 = \sqrt{R_1^2 - 0.25} \) \{1.14\}

The distance of the n-gon from the center M: \( z_n = \sqrt{R_1^2 - R_2^2} \) \{1.15\}

Fig. 1.17. A) represents 1/n-th part of a polyhedral pyramid, which has a polygon with n sides as its basis and M (the center of the polyhedron) as its summit. MPKN is also called orthoscheme or quadirectangular tetrahedron \[1.5\]. P and Q are the two ends and K the mid-point of a side. N is the center of the polygon and it has the distance \( z_n \) from the polyhedron center M. B) shows what is meant with 'deficient angle'.

The following table 1.6 gives a review of the most relevant values, derived in the foregoing, including the total angles of the polygons that meet in a corner, and also the deficient angle (Fig. 1.17B) which represents the missing part that must be cut from the flat plane in order to give the polyhedron its spatial form.

<table>
<thead>
<tr>
<th>P</th>
<th>( R_1 )</th>
<th>( R_3 )</th>
<th>( R_5 )</th>
<th>( R_6 )</th>
<th>Total angle</th>
<th>Deficient angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.61237244</td>
<td>0.57735027</td>
<td>0.35355339</td>
<td>0.20412415</td>
<td>180°</td>
<td>180°</td>
</tr>
<tr>
<td>2</td>
<td>0.86602540</td>
<td>0.81649658</td>
<td>0.70710678</td>
<td>0.57735027</td>
<td>270°</td>
<td>90°</td>
</tr>
<tr>
<td>3</td>
<td>0.70710678</td>
<td>0.70710678</td>
<td>0.50000000</td>
<td>0.35355339</td>
<td>240°</td>
<td>120°</td>
</tr>
<tr>
<td>4</td>
<td>1.40125854</td>
<td>0.93417236</td>
<td>1.30901699</td>
<td>1.22284749</td>
<td>324°</td>
<td>36°</td>
</tr>
<tr>
<td>5</td>
<td>0.95105652</td>
<td>0.85065081</td>
<td>0.80901699</td>
<td>0.68819096</td>
<td>300°</td>
<td>60°</td>
</tr>
<tr>
<td>6</td>
<td>1.17260394</td>
<td>0.90453403</td>
<td>1.06066017</td>
<td>0.95940322</td>
<td>300°</td>
<td>60°</td>
</tr>
<tr>
<td>7</td>
<td>1.00000000</td>
<td>0.86602540</td>
<td>0.86602540</td>
<td>0.75000000</td>
<td>300°</td>
<td>60°</td>
</tr>
<tr>
<td>8</td>
<td>1.58113883</td>
<td>0.94868330</td>
<td>1.50000000</td>
<td>1.42302495</td>
<td>330°</td>
<td>30°</td>
</tr>
<tr>
<td>9</td>
<td>1.77882365</td>
<td>0.95968298</td>
<td>1.70710678</td>
<td>1.63828133</td>
<td>330°</td>
<td>30°</td>
</tr>
<tr>
<td>10</td>
<td>1.3986633</td>
<td>0.93394883</td>
<td>1.30656296</td>
<td>1.22026295</td>
<td>330°</td>
<td>30°</td>
</tr>
<tr>
<td>11</td>
<td>1.31761091</td>
<td>0.97645098</td>
<td>1.26303344</td>
<td>1.20974121</td>
<td>345°</td>
<td>15°</td>
</tr>
<tr>
<td>12</td>
<td>1.61803399</td>
<td>0.95105652</td>
<td>1.53884177</td>
<td>1.46352495</td>
<td>336°</td>
<td>24°</td>
</tr>
<tr>
<td>13</td>
<td>1.47801866</td>
<td>0.97943209</td>
<td>1.42705098</td>
<td>1.37713161</td>
<td>348°</td>
<td>12°</td>
</tr>
<tr>
<td>14</td>
<td>1.96944902</td>
<td>0.98572192</td>
<td>1.92705098</td>
<td>1.88525831</td>
<td>348°</td>
<td>12°</td>
</tr>
<tr>
<td>15</td>
<td>1.3471337</td>
<td>0.92819138</td>
<td>1.24722317</td>
<td>1.15766179</td>
<td>330°</td>
<td>30°</td>
</tr>
<tr>
<td>16</td>
<td>1.2395051</td>
<td>0.97460776</td>
<td>1.17625090</td>
<td>1.12099102</td>
<td>348°</td>
<td>12°</td>
</tr>
<tr>
<td>17</td>
<td>3.80239450</td>
<td>0.99131669</td>
<td>3.76937713</td>
<td>3.73664646</td>
<td>354°</td>
<td>6°</td>
</tr>
<tr>
<td>18</td>
<td>1.15583738</td>
<td>0.97273285</td>
<td>1.09705384</td>
<td>1.03987315</td>
<td>348°</td>
<td>12°</td>
</tr>
</tbody>
</table>

Table 1.6. Characteristic radii and angles of the uniform solids
1.8. The dihedral angles

Table 1.7. Dihedral angles between face and plane through edge PQ and centre M

<table>
<thead>
<tr>
<th>P</th>
<th>n₁</th>
<th>ξ₁</th>
<th>n₂</th>
<th>ξ₂</th>
<th>n₃</th>
<th>ξ₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3)</td>
<td>35°15'51,8029'</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4)</td>
<td>45°00'00,0000'</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3)</td>
<td>54°44'08,1971'</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5)</td>
<td>58°16'57,0921'</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3)</td>
<td>69°05'41,4332'</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3)</td>
<td>74°12'24,5914'</td>
<td>6)</td>
<td>35°15'51,8029'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3)</td>
<td>70°31'43,6057'</td>
<td>4)</td>
<td>54°44'08,1971'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4)</td>
<td>70°31'43,6057'</td>
<td>6)</td>
<td>54°44'08,1971'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>3)</td>
<td>80°15'51,8029'</td>
<td>8)</td>
<td>44°59'60,0000'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>3)</td>
<td>77°14'08,1971'</td>
<td>4)</td>
<td>67°29'60,0000'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>4)</td>
<td>77°14'08,1971'</td>
<td>6)</td>
<td>67°29'60,0000'</td>
<td>8)</td>
<td>57°45'51,8029'</td>
</tr>
<tr>
<td>12</td>
<td>3)</td>
<td>79°11'15,6589'</td>
<td>5)</td>
<td>63°26'05,8158'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>5)</td>
<td>73°31'40,0415'</td>
<td>6)</td>
<td>69°05'41,4332'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>3)</td>
<td>84°20'24,3826'</td>
<td>10)</td>
<td>58°16'57,0921'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>3)</td>
<td>76°37''02,2579'</td>
<td>4)</td>
<td>66°21'58,0904'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>3)</td>
<td>82°22'38,5253'</td>
<td>4)</td>
<td>76°43'02,9079'</td>
<td>5)</td>
<td>71°33'54,1842'</td>
</tr>
<tr>
<td>17</td>
<td>4)</td>
<td>82°22'38,5253'</td>
<td>6)</td>
<td>76°43'02,9079'</td>
<td>10)</td>
<td>65°54'18,5668'</td>
</tr>
<tr>
<td>18</td>
<td>3)</td>
<td>82°05'15,6589'</td>
<td>5)</td>
<td>70°50'32,0541'</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1.8. The dihedral angles between all meeting pairs of polygons. The numbers of their sides are indicated in brackets.

The dihedral angle between the n-gon and the centri-plane PQM of Fig. 1.8B:
\[ \xi_n = \arctan \frac{z_n}{m_n} \]  
\{1.16\}

\( m_n \) follows from equation \{1.3\).

At any edge of a polyhedron always two polygons meet. The total dihedral angle is therefore composed of two parts, each of which is defined by its adjacent polygon.

### 1.9. Areas of the polyhedra

The total areas of a polyhedron are found by the summation of the areas of all polygons, occurring in a polyhedron.

\[ \text{Area } P_{\text{TOT}} = \sum (q_{n1}\text{Area } P_{n1} + q_{n2}\text{Area } P_{n2} + q_{n3}\text{Area } P_{n3}) \]  
\{1.17\}

### 1.10. Volumes of the polyhedra

The volume of a polyhedron can be found by adding the volumes of all centri-pyramids. The volume of such a pyramid:

\[ \text{Vol } P_n = \frac{1}{3} \times \text{area of polygon} \times \text{height} = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} \times m_n \times z_n \times 2 \times n \]

\[ = \frac{1}{6} \times m_n \times z_n \times n \]  
\{1.18\}

The total volume of the polyhedron:

\[ \text{Vol } P_{\text{TOT}} = \sum (q_{n1}\text{Vol } P_{n1} + q_{n2}\text{Vol } P_{n2} + q_{n3}\text{Vol } P_{n3}) \]  
\{1.19\}

A summary of the values for the areas and volumes is given in the following table together with those of the reciprocals, the derivation of which is explained in the next paragraph.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \text{Volume } P_{\text{TOT}} )</th>
<th>( \text{Area } P_{\text{TOT}} )</th>
<th>( \text{Volume } R_{\text{TOT}} )</th>
<th>( \text{Area } R_{\text{TOT}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.11785113</td>
<td>1.73205080</td>
<td>0.11785113</td>
<td>1.73205080</td>
</tr>
<tr>
<td>2</td>
<td>1.00000000</td>
<td>6.00000000</td>
<td>1.33333333</td>
<td>6.92820323</td>
</tr>
<tr>
<td>3</td>
<td>0.47140452</td>
<td>3.46410161</td>
<td>0.3535533</td>
<td>3.00000000</td>
</tr>
<tr>
<td>4</td>
<td>7.66311896</td>
<td>20.64572880</td>
<td>9.24180829</td>
<td>21.67283942</td>
</tr>
<tr>
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<td>1.18169499</td>
<td>8.66025403</td>
<td>1.80901699</td>
<td>7.88596668</td>
</tr>
<tr>
<td>6</td>
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<td>5.72756493</td>
<td>17.9077386</td>
</tr>
<tr>
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<td>1.38648539</td>
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<td>67.42484815</td>
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<td>19.29940656</td>
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<td>41.25536942</td>
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</tr>
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<td>37.61664996</td>
<td>55.28674495</td>
<td>37.58842367</td>
<td>55.28053092</td>
</tr>
</tbody>
</table>

Table 1.9. The areas and volumes of polyhedra and of their reciprocals
1.1. Prisms and antiprisms

There are numbers of other figures that also answer the definition that was given in paragraph 1.1, namely some of the prisms and antiprisms. In fact they form endless rows, the prisms having two parallel polygonal sides and a mantle of squares, where the antiprisms instead have a mantle of equilateral triangles. Following definition 1) only those are considered to be uniform polyhedra, that have parallel sides with 3, 4, 5, 6, 8, or 10 edges.

![Fig. 1.18. Row of the uniform prisms, following the definition in paragraph 1.1.](image)

![Fig. 1.19. The uniform antiprisms](image)

These solids have similar characteristics as the Archimedean polyhedra, in this case consisting of two kinds of polygons. They also have similar vertex figures as in Fig. 1.20. In Chapter 6 an overview is given of the whole group of prismatic or prism based figures and structural forms. They will be treated in this Chapter 6 separately and following a more general approach.

![Fig. 1.20. Vertex figures of the prisms and antiprisms.](image)
1.12. Isomeres

Four members of the semi-regular polyhedra allow different arrangements of parts of them that in fact answer most of the definitions of the uniform convex polyhedra on page 1.1, but with the exception of point 3, which says that all vertices are identical. The four solids in Fig. 1.18 have parts, which can be turned with respect to the rest of the solid. P7, P10 and P12 have two possible variants, whereas P16 has maybe five possible different arrangements. These are called Isomeres.

![Fig. 1.21. The four Archimedean solids, that allow different arrangements](image)

![Fig. 1.22. The pyramidal caps that can be turned around.](image)

![Fig. 1.23. Some isomeric forms](image)
Section 1.13. Pyramidization

On each polygonal plane of any polyhedron shallow pyramid can be erected, of which the apex, just like all corners of the polygon of the polyhedron, all lie on the circumscribed sphere. We have introduced here the term 'pyramidization'.

Fig. 1.24. The six pyramidized polygons, shown in plan.

Height of the pyramid: \( h_n = R_1 - z_n = R_1 - \sqrt{R_1^2 - R_2^2} \) \{1.20\}

Length of inclined edge: \( e_n = \sqrt{h_n^2 + R_2^2} \) \{1.21\}

Height of isosceles triangle: \( h_t = \sqrt{e_n^2 - 0.25} \) \{1.22\}

Basis angle of triangle: \( \lambda = \arctan \left( 2\sqrt{e_n^2 - 0.25} \right) \) \{1.23\}

Fig. 1.25. A) Pyramidization of polygon sector, B) Characteristic aspects of triangle on sphere

For the determination of the dihedral angles along the edges of the triangular sides of this pyramid a general approach can be used, where the corners of a triangle with the sides \( a, b \) and \( c \) lie on the sphere with the radius \( R_1 \). Around this triangle a circle can be circumscribed, of which the radius is called here \( R_4 \) with \( N \) as the centre of this circle.

\[
m_t = \sqrt{R_4^2 - \left( \frac{c}{2} \right)^2} = \sqrt{R_4^2 - 0.25c^2}
\]

\[
y = \sqrt{R_1^2 - R_4^2}
\] \{1.24\}
Chapter 01

\[ \psi_a = \arctan \left( \frac{y}{m_a} \right), \; \psi_b = \arctan \left( \frac{y}{m_b} \right) \text{ and } \psi_c = \arctan \left( \frac{y}{m_c} \right) \]

In the shallow pyramid the triangle is isosceles and the sides are \( e_n, e_n \) and 1.

Half the sum of the sides: \( S = \frac{2e_n + 1}{2} = e_n + 0.5 \)

Area of the triangle: \( O = \sqrt{(e_n + 0.5)(0.5)(e_n - 0.5)} = 0.5\sqrt{e_n^2 - 0.25} \)

Radius of circumscribed circle: \( R_4 = \frac{e_n}{4O} \psi = \arctan \left( \frac{y}{m_i} \right) = \frac{e_n^2}{\sqrt{4e_n^2 - 1}} \)

Two different dihedral angles occur:

1) On an inclined edge:

\[ m_c = \sqrt{R_4^2 - \left( \frac{e_n}{2} \right)^2} = \sqrt{R_4^2 - 0.25e_n^2} \quad \{1.25\} \]

\[ \psi_c = \arctan \left( \frac{y}{m_c} \right) \quad \{1.26\} \]

2) On the edge with length=1:

\[ m_i = \sqrt{R_4^2 - 0.25} \quad \{1.27\} \]

\[ \psi_i = \arctan \left( \frac{y}{m_i} \right) \quad \{1.28\} \]

Fig. 1.26. Models of pyramidized Platonic polyhedra; it is interesting to see that the tetrahedron converts into a cube.
1.14. Deltahedra

A class of figures, of which all faces are regular triangles, is called Deltahedra. Only eight of these are convex. The most obvious of these are of course the three triangular Platonic solids: the Tetrahedron, the Octahedron and the Icosahedron (P1, P3 and P5). The Octahedron can be seen as two square pyramids that are posed opposite to each other against their common square faces. There are two more of these: the triangular pyramid (enumerated in Fig. 1.28 as D2) and the one composed of two pentagonal pyramids, D4.
These different forms are called after their numbers of faces:
1. Tetradeltahedron (4 triangles), more familiar as the tetrahedron.
3. Octadeltahedron (8 triangles): the octahedron.
5. Dodekadeltahedron (12 triangles).
6. Tetrakaidekadeltahedron (14 triangles): a triangular prism with pyramids on its square faces.
7. Hexadekadeltahedron (16 triangles): a 4-sided antiprism with pyramids on its square faces.
8. Icosadeltahedron (20 triangles), synonymus to the regular Icosahedron but also similar to two pentagonal pyramids, placed on the parallel sides of a pentagonal antiprism.

1.15. References

[1.4] Albrecht Dürer, Unterweysung der Messung mit dem Zirkel und Richtscheyt (Nürnberg 1525)
Chapter 2. THE INDIVIDUAL POLYHEDRA

The total number of the regular polyhedra is 5 and not more than 5. This can easily be understood from table 1.5, where the deficient angles of all uniform polyhedra are given (See also Fig. 15.7). The total sum of the top angles of the polygons that meet in a vertex, must always be less than 360° in order to give the polyhedron its round form. If this angle is equal to 360° the plane becomes flat and if its larger, too much material is available so that the figure becomes wrinkled. Following the conditions in Chapter 1 we only use the polygons with 3, 4, 5, 6, 8, and 10 sides. In each vertex at least three polygons must meet in order to form a space angle.

In the previous Chapter 1 an algebraic approach was followed to obtain the geometric data of the polyhedra. In Chapter 14 all numeric data are given following this approach. But in special cases it is often desirable to have a formula available to calculate a particular property as exactly as possible. M. Brückner derived many of these and these are given in the following chapter, and in more concise form in the second part of Chapter 14. These formulae are given in coherence with the respective polyhedra and their net, if folded out. Some of the data derived by Brückner were corrected, if necessary and if indicated, with the help of Chapter 14 and computed earlier by Huybers in 1976 [2.2].

2.1. Tetrahedron P1

Fig. 2.1. The Tetrahedron and its net

\[ a = \text{edge length of the n-gons} \]

Dihedral angle between the 3-gon and the centriplane through the unit edge:
\[ \xi_1 = \arccos \left( \frac{1}{3} \right) = 35^\circ 15' 36.8" \]

Radius of the circumscribed sphere: \[ R_1 = \frac{a}{4} \sqrt{6} \]

Total surface area of the polyhedron: \[ A = a^2 \sqrt{3} \]

Total volume of the polyhedron: \[ V = \frac{a^2}{12} \sqrt{2} \]

Distance of a 3-gon from the centre of the circumscribed sphere: \[ z_3 = \frac{a}{12} \sqrt{6} \]
2.2. Cube P2

Fig. 2.2. The Cube and its net

Dihedral angle between the 4-gon and the centriplane through the unit edge: 
\[ \xi_4 = 45^\circ \]

Radius of the circumscribed sphere: 
\[ R_1 = \frac{a}{2} \sqrt{3} \]

Total surface area of the polyhedron: 
\[ A = 6a^2 \]

Total volume of the polyhedron: 
\[ V = a^3 \]

Distance of a 4-gon from the centre of the circumscribed sphere: 
\[ z_4 = \frac{a}{2} \]

2.3. Octahedron P3

Fig. 2.3. The Octahedron and its net

Dihedral angle between the 3-gon and the centriplane passing through the unit edge: 
\[ \xi_3 = \arctan(\sqrt{2}) = 54^\circ 29' 8.2'' \]

Radius of the circumscribed sphere: 
\[ R_2 = \frac{a}{2} \sqrt{2} \]

Total surface area of the polyhedron: 
\[ A = 2a^2 \sqrt{3} \]

Total volume of the polyhedron: 
\[ V = \frac{a^2}{3} \sqrt{2} \]

Distance of a 3-gon from the centre of the circumscribed sphere: 
\[ z_3 = \frac{a}{6} \sqrt{6} \]
2.4. Dodecahedron P4

Fig. 2.4. The Dodecahedron and its net

Dihedral angle between the 5-gon and the centriplane through the unit edge:
\[ \xi_{3,5} = \arccos\left( -\frac{1}{\sqrt{5}} \right) = 58^o16'42.1" \]

Radius of the circumscribed sphere: \( R_1 = \frac{a}{4} \sqrt{18 + 6\sqrt{5}} \)

Total surface area of the polyhedron: \( A = 3a^2 \sqrt{25 + 10\sqrt{5}} \)

Total volume of the polyhedron: \( V = \frac{a^3}{4} \left( 15 + 7\sqrt{5} \right) \)

Distance of a 5-gon from the centre of the circumscribed sphere: \( z_5 = \frac{a}{2} \sqrt{\frac{25 + 11\sqrt{5}}{10}} \)

2.5. Icosahedron P5

Fig. 2.5. The Dodecahedron and its net

Dihedral angle between the 3-gon and the centriplane through the unit edge:
\[ \xi_3 = \arcsin \left( \frac{\sqrt{15} + \sqrt{3}}{6} \right) = 69^o16'42.1" \]

Radius of the circumscribed sphere: \( R_1 = \frac{a}{4} \sqrt{10 + 2\sqrt{5}} \)
Chapter 02

Total surface area of the polyhedron: \( A = 5a^2 \sqrt{3} \)

Total volume of the polyhedron: \( V = \frac{a^3}{4} (15 + 7\sqrt{5}) \)

Distance of the 3-gon from the centre of the circumscribed sphere: \( z_3 = \frac{a}{12} (3 + \sqrt{5}) \sqrt{3} \)

\[ \text{2.6. Truncated Tetrahedron P6} \]

Fig.2.6. The Truncated Tetrahedron and its net

Dihedral angle between a 3-gon and the centriplane through the unit edge: \( \xi_3 = \arctan \left( \frac{5}{2} \sqrt{2} \right) \)

Dihedral angle between a 6-gon and the centriplane through the unit edge: \( \xi_6 = \arctan \left( \frac{1}{2} \sqrt{2} \right) \)

Radius of the circumscribed sphere: \( R_1 = \frac{a}{2} \sqrt{22} \)

Total surface area of the polyhedron: \( A = 7a^2 \sqrt{3} \)

Total volume of the polyhedron: \( V = \frac{23}{12} a^3 \sqrt{2} \)

Distance of a 3-gon from the centre of the circumscribed sphere: \( z_3 = \frac{5}{12} a \sqrt{6} \)

Distance of a 6-gon from the centre of the circumscribed sphere: \( z_6 = \frac{a}{4} \sqrt{6} \)
2.7 Cuboctahedron P7

Fig. 2.7. The Cuboctahedron and its net

Dihedral angle between the 3-gon and the centriplane through the unit edge:

\[ \xi_3 = \arctan(2\sqrt{2}) \]

Dihedral angle between the 4-gon and the centriplane through the unit edge:

\[ \xi_3 = \arctan(\sqrt{2}) \]

Radius of the circumscribed sphere: \( R_i = a \)

Total surface area of the polyhedron: \( A = 2a^2 \left(3 + \sqrt{3}\right) \)

Total volume of the polyhedron: \( V = \frac{5}{3}a^3 \sqrt{2} \)

Distance of the 3-gon from the centre of the circumscribed sphere: \( z_3 = \frac{a}{3} \sqrt{6} \)

Distance of the 4-gon from the centre of the circumscribed sphere: \( z_4 = \frac{a}{2} \sqrt{2} \)
2.8. Truncated Octahedron P8

Fig. 2.8. *The Truncated Octahedron and its net*

Dihedral angle between the 4-gon and the centriplane through the unit edge:
\[ \xi_4 = \arctan(2\sqrt{2}) \]

Dihedral angle between the 6-gon and the centriplane through the unit edge:
\[ \xi_6 = \arctan(\sqrt{2}) \]

Radius of the circumscribed sphere: \( R_i = \frac{a}{2} \sqrt{10} \)

Total surface area of the polyhedron: \( A = 6a^2 \left( 1 + 2\sqrt{3} \right) \)

Total volume of the polyhedron: \( V = 8a^3 \sqrt{2} \)

Distance of the 4-gon from the centre of the circumscribed sphere: \( z_4 = a\sqrt{2} \)

Distance of the 6-gon from the centre of the circumscribed sphere: \( z_6 = \frac{a}{2} \sqrt{6} \)
2.9. Truncated Cube P9

Fig. 2.9. The Truncated Cube and its net

Dihedral angle between the 3-gon and the centriplane through the unit edge: 
\[ \xi_3 = \arctan\left(1 + \sqrt{2}\right)^2 \]

Dihedral angle between the 8-gon and the centriplane through the unit edge: 
\[ \xi_8 = \arctan(1) = 45^\circ \]

Radius of the circumscribed sphere: 
\[ R_t = \frac{a}{2} \sqrt{7 + 4\sqrt{2}} \]

Total surface area of the polyhedron: 
\[ A = 2a^2 \left(\sqrt{3} + 6\left(1 + \sqrt{2}\right)\right) \]

Total volume of the polyhedron: 
\[ V = \frac{7}{3} a^3 \left(\sqrt{2} + 1\right)^2 \]

Distance of the 3-gon from the centre of the circumscribed sphere: 
\[ z_3 = \frac{a}{6} \sqrt{3} \left(1 + \sqrt{2}\right)^2 \]

Distance of the 8-gon from the centre of the circumscribed sphere: 
\[ z_8 = \frac{a}{2} \left(1 + \sqrt{2}\right) \]
2.10. Rhombicuboctahedron P10

Fig. 2.10. The Rhombicuboctahedron and its net

Dihedral angle between the 3-gon and the centriplane through the unit edge: \( \xi_3 = \arctan \left( 3 + \sqrt{2} \right) \)

Dihedral angle between the 4-gon and the centriplane through the unit edge: \( \xi_4 = \arctan \left( \frac{1}{\sqrt{2}} \right) \)

Radius of the circumscribed sphere: \( R_1 = \frac{a}{2} \sqrt{5 + 2 \sqrt{2}} \)

Total surface area of the polyhedron: \( A = 2a^2 \left( 9 + \sqrt{3} \right) \)

Total volume of the polyhedron: \( V = \frac{2}{3}a^3 \left( 6 + 5 \sqrt{2} \right) \)

Distance of the 3-gon from the centre of the circumscribed sphere: \( z_3 = \frac{a}{6} \sqrt{3} \left( 3 + \sqrt{2} \right) \)

Distance of the 4-gon from the centre of the circumscribed sphere: \( z_4 = \frac{a}{6} \sqrt{3} \left( 3 + \sqrt{2} \right) \)
2.11. Truncated Cuboctahedron P11

![Truncated Cuboctahedron and its net](image)

Fig.2.11. The Truncated Cuboctahedron and its net

Dihedral angle between the 4-gon and the centriplane through the unit edge: \( \xi_4 = \arctan(3 + \sqrt{2}) \)
Dihedral angle between the 6-gon and the centriplane through the unit edge: \( \xi_6 = \arctan(1 + \sqrt{2}) \)
Dihedral angle between the 8-gon and the centriplane through the unit edge: \( \xi_8 = \arctan(3 - \sqrt{2}) \)

Radius of the circumscribed sphere: \( R = \frac{a}{2} \sqrt{5 + 2\sqrt{2}} \)
Total surface area of the polyhedron: \( A = 12a^2 \left(2 + \sqrt{2} + \sqrt{3}\right) \)
Total volume of the polyhedron: \( V = 2a^3 \left(11 + 7\sqrt{2}\right) \)

Distance of the 4-gon from the centre of the circumscribed sphere: \( z_4 = \frac{a}{2} \left(3 + \sqrt{2}\right) \)
Distance of the 6-gon from the centre of the circumscribed sphere: \( z_6 = \frac{a}{2} \left(1 + \sqrt{2}\right) \sqrt{3} \)
Distance of the 8-gon from the centre of the circumscribed sphere: \( z_8 = \frac{a}{2} \left(1 + 2\sqrt{2}\right) \)
Fig.2.12. The Icosidodecahedron and its net

Dihedral angle between the 3-gon and the centriplane through the unit edge: $\xi_3 = \arctan(3 + \sqrt{5})$

Dihedral angle between the 5-gon and the centriplane through the unit edge: $\xi_5 = \arctan(2)$

Radius of the circumscribed sphere: $R_1 = \frac{a(1 + \sqrt{5})}{2} = a\tau$

where $\tau = \text{golden section}$

Total surface area of the polyhedron: $A = a^2 \left(5\sqrt{3} + 3\sqrt{5(5 + 2\sqrt{5})}\right)$

Total volume of the polyhedron: $V = \frac{a^3}{6} \left(45 + 17\sqrt{5}\right)$

Distance of the 3-gon from the centre of the circumscribed sphere: $z_3 = a\sqrt{3(3 + \sqrt{5})}$

Distance of the 5-gon from the centre of the circumscribed sphere: $z_5 = a\sqrt{\frac{5 + 2\sqrt{5}}{5}}$
2.13. Truncated Icosahedron P13

Fig. 2.13. The Truncated Icosahedron and its net

Dihedral angle between the 5-gon and the centriplane through the unit edge:
\[ \xi_5 = \arctan\left(\frac{1}{2}(9 - \sqrt{5})\right) \]

Dihedral angle between the 6-gon and the centriplane through the unit edge:
\[ \tan \xi_6 = \tau^2 \]

Radius of the circumscribed sphere:
\[ R_1 = \frac{a}{4} \sqrt{2(29 + 9\sqrt{5})} \]

Total surface area of the polyhedron:
\[ A = 15a^2 \left(2\sqrt{3} + \sqrt{1 + \frac{2\sqrt{5}}{5}}\right) \]

Total volume of the polyhedron:
\[ V = \frac{a^3}{4} (125 + 43\sqrt{5}) \]

Distance of the 5-gon from the centre of the circumscribed sphere:
\[ z_5 = \frac{a}{20} \sqrt{10(125 + 41\sqrt{5})} \]

Distance of the 6-gon from the centre of the circumscribed sphere:
\[ z_6 = \frac{a}{4} (3 + \sqrt{5}) \sqrt{3} \]
Chapter 02

2.14. Truncated Dodecahedron P14

Fig. 2.14. The Truncated Dodecahedron and its net

Dihedral angle between the 3-gon and the centriplane through the unit edge:
\[ \xi_3 = \arctan \left( \frac{1}{2} \left( 9 + 5\sqrt{5} \right) \right) \]

Dihedral angle between the 10-gon and the centriplane through the unit edge:
\[ \tan \xi_{10} = \arctan(\tau) = \arctan \left( \frac{1 + \sqrt{5}}{2} \right) \]

Radius of the circumscribed sphere: 
\[ R_1 = \frac{a}{4} \sqrt{2 \left( 37 + 15\sqrt{5} \right)} \]

Total surface area of the polyhedron: 
\[ A = 5a^2 \left( \sqrt{3} + 6\sqrt{5} + 2\sqrt{5} \right) \]

Total volume of the polyhedron: 
\[ V = \frac{5a^3}{12} \left( 99 + 47\sqrt{5} \right) \]

Distance of the 3-gon from the centre of the circumscribed sphere: 
\[ z_3 = \frac{a\sqrt{3}}{12} \left( 9 + 5\sqrt{5} \right) \]

Distance of the 10-gon from the centre of the circumscribed sphere: 
\[ z_{10} = \frac{a}{4} \sqrt{50 + 22\sqrt{5}} \]
2.15. Left-handed and right-handed Snub Cube P15L and P15R

Brückner gives only numerical values for the dihedral angles, which are however not quite accurate. The values that we found earlier in Chapter 1 are given here in brackets.

Dihedral angle between the 3-gon and the centripedal plane through the unit edge:
\[ \xi_3 = 76^0 27' 2'' (76^0 37' 02.2579'') \]

Dihedral angle between the 4-gon and the centripedal plane through the unit edge:
\[ \xi_4 = 66^0 21' 58.2'' (66^0 21' 58.0904'') \]

Radius of the circumscribed sphere: \[ R_i = a * 1.29461...(a * 1.34371337) \]

This differs quite much from the value of Table 1.6 in the foregoing Chapter.

Total surface area of the polyhedron: \[ A = 2a^2 \left( 4\sqrt{3} + 3 \right) \]

Total volume of the polyhedron: \[ V = a^3 * 7.889472(a^3 * 7.44739519) \]

No distances in the form of formulae of the 3-gon and the 4-gon from the centre of the circumscribed sphere are given by Brückner.
2.16. Rhombicosidodecahedron P16

Fig. 2.17. The Rhombicosidodecahedron and its net

Dihedral angle between the 3-gon and the centriplane through the unit edge:
\[ \xi_3 = \arctan(3 + 2\sqrt{5}) \]

Dihedral angle between the 4-gon and the centriplane through the unit edge:
\[ \xi_4 = \arctan(2 + \sqrt{5}) \]

Dihedral angle between the 5-gon and the centriplane through the unit edge:
\[ \xi_5 = \arctan(3) \]

Radius of the circumscribed sphere:
\[ R = \frac{a}{2} \sqrt{11 + 4\sqrt{5}} \]

Total surface area of the polyhedron:
\[ A = 5a^2 \left( 6 + \sqrt{3} + 3 \sqrt{1 + \frac{2\sqrt{5}}{5}} \right) \]

Total volume of the polyhedron:
\[ V = \frac{a^3}{3} \left( 60 + 29\sqrt{5} \right) \]

Distance of the 3-gon from the centre of the circumscribed sphere:
\[ z_3 = \frac{\sqrt{3}}{6} \left( 3 + 2\sqrt{5} \right) \]

Distance of the 4-gon from the centre of the circumscribed sphere:
\[ z_4 = \frac{a}{2} (2 + \sqrt{5}) \]

Distance of the 5-gon from the centre of the circumscribed sphere:
\[ z_5 = \frac{3a}{2} \sqrt{\frac{5 + 2\sqrt{5}}{5}} \]
2.17. Truncated Icosidodecahedron

Fig. 2.18. *The Truncated Icosidodecahedron and its net*

- Dihedral angle between the 4-gon and the centriplane through the unit edge: \( \xi_4 = \arctan(3 + 2\sqrt{5}) \)
- Dihedral angle between the 6-gon and the centriplane through the unit edge: \( \xi_6 = \arctan(2 + \sqrt{5}) \)
- Dihedral angle between the 10-gon and the centriplane through the unit edge: \( \xi_{10} = \arctan(\sqrt{5}) \)

Radius of the circumscribed sphere: \( R_t = \frac{a}{2}\sqrt{31+12\sqrt{5}} \)

Total surface area of the polyhedron: \( A + 30a^2\left(1 + \sqrt{3} + \sqrt{5} + 2\sqrt{5}\right) \)

Total volume of the polyhedron: \( A + 30a^2\left(1 + \sqrt{3} + \sqrt{5} + 2\sqrt{5}\right) \)

Distance of the 4-gon from the centre of the circumscribed sphere: \( z_4 = \frac{a}{2}(3 + 2\sqrt{5}) \)

Distance of the 6-gon from the centre of the circumscribed sphere: \( z_6 = \frac{a}{2}(2 + \sqrt{5})\sqrt{3} \)

Distance of the 10-gon from the centre of the circumscribed sphere: \( z_{10} = \frac{a}{2}\sqrt{5(5 + 2\sqrt{5})} \)
2.18. Left-handed and right-handed Snub Dodecahedron P18L and P18R

Fig. 2.19. The left-handed and the right-handed Snub Dodecahedron P18L and P18R

Fig. 2.20. Net of the left-handed Snub Dodecahedron P18L

Fig. 2.21. Net of the right-handed Snub Dodecahedron P18R
Dihedral angle between the 3-gon and the centriplane through the unit edge:
\[ \xi_3 = 82^\circ 05' 14.3'' (82^\circ 05' 15.6589'') \]

Dihedral angle between the 5-gon and the centriplane through the unit edge:
\[ \xi_5 = 70^\circ 50' 29.2'' (70^\circ 50' 32.0541'') \]

Radius of the circumscribed sphere: \( R = a \ast 2.7654 (a \ast 2.15583738) \)

The difference here is quite significant from the one found in Table 2.6.

Total surface area of the polyhedron:
\[ A = 5a^2 \left( 4\sqrt{3} + 3\sqrt{1 + \frac{2\sqrt{5}}{5}} \right) \]

Total volume of the polyhedron:
\[ V = a^3 \ast 37.61549 (a^3 \ast 37.61664996) \]

Fig.2.22. P18 composed of 12 pentagons and 20 large triangles, each consisting of 4 triangles of the unit edge length
2.19. References


Chapter 3. RECIPROCAL POLYHEDRA

3.1. The reciprocal figure

The reciprocal or dual figure of a polyhedron is found by interconnecting the midpoints $K$ of all edges that meet in a vertex. The thus found figure has the same shape as the vertex figure but has half its size. Its circumscribed circle is therefore $R_3/2$. The plane of this reduced vertex figure can be expanded until it meets planes of adjacent semi-vertex figures, equally expanded. The section lines between these planes bisect the original edges of the polyhedron perpendicularly and are also perpendicular to the line that connects the mid-edge $K$ with center $M$. This is shown in Fig. 3.1.

Fig.3.1. Derivation of a reciprocal polyhedron.

Fig. 3.2. Construction of a reciprocal face.
3.2. The reciprocal faces.

The face of a reciprocal figure can have either 3, 4 or 5 sides, depending on the number of n-gons, that in the original polyhedron meet in each vertex. The 5 regular polyhedra are self-reciprocal, i.e. tetrahedron-tetrahedron, octahedron-cube and dodecahedron-icosahedron. The reciprocals in this case have regular polygon faces. The reciprocal faces of the semi-regular polyhedra however are more or less irregular. They can be constructed by drawing tangent lines around the circle with the radius $R_3/2$ in the original midpoints of the unit edges $K_a$, $K_b$ and $K_c$ (see Fig. 3.2). This construction is known as the Dorman-Luke construction [Ref. 3.1]. The faces of such a reciprocal figure are composed of a number of quadrangular sectors $K_a-L_n-K_b-U$. These sectors are directly related to the n-gons in the corresponding polyhedron, with $K_a - K_b = 1/2 b_n = \cos \varphi_n$

Half the central face angle: $\zeta_n = \arctan \frac{\cos \varphi_n}{\sqrt{(R_3^2 - \cos^2 \varphi_n)}}$ \hspace{1cm} \{3.1\}

The face angle: $\chi_n = 180^\circ - 2\zeta_n$ \hspace{1cm} \{3.2\}

Each polygon at any of the vertices gives a specific contribution to the edge length of the reciprocal face. For the polygon with n edges:

$$g_n = \frac{\cos \varphi_n}{2 \cos \zeta_n} = \frac{R_3 \tan \zeta_n}{2}$$ \hspace{1cm} \{3.3\}

The distance of a face corner to the center of the inscribed circle:

$$j_n = \frac{g_n}{\sin \zeta_n}$$ \hspace{1cm} \{3.4\}

The distance of the face corner from the system center $M.$:

$$R_7 = \sqrt{(R_6^2 + j_n^2)}$$ \hspace{1cm} \{3.5\}

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Table 3.1. Distances $R_7$ from centre
Fig 3.3. Review of the reciprocal figures

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<td>6</td>
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<td>8</td>
<td>109°28'16.39&quot;</td>
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<tr>
<td>3</td>
<td>Cube (edge = 1/2 √2)</td>
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<td>12</td>
<td>6</td>
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</tr>
<tr>
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<td>Icosahedron (edge = τ)</td>
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<td>60</td>
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</tr>
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<td>60</td>
<td>154°07'16.91&quot;</td>
</tr>
<tr>
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<td>180</td>
<td>120</td>
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<td>18</td>
<td>Pentagonal Hexecontahedron</td>
<td>90</td>
<td>150</td>
<td>60</td>
<td>153°10'43.44&quot;</td>
</tr>
</tbody>
</table>

Table 3.2. Names and numerical data of the reciprocal figures (V = vertices, E = edges, F = faces).
Fig. 3.4. Models of the reciprocal solids

The names in this table give an indication of the number of faces. The suffix ‘-kis’ means: number of subdivision. And further: \( \tau = (1 + \sqrt{5})/2 \), or the Golden Section. The numbers in the first column of this table refer directly to that of their related polyhedra.

Some characteristics of these duals are given in Table 3.2. The index numbers for the polyhedra and consequently for their reciprocals in this table were introduced by the author [Ref. 1.10]. They are useful to give reference to the individual polyhedra, without having to make use of their uncomfortably difficult scientific names, but they can also be used in a more or less “administrative sense”. If one utilises computer programmes for the calculation of their geometry or for their visual presentation, it is often necessary to indicate them by a unique number (see also Chapter 1, p.8 and 9).

Here in particular their scientific names are difficult. In the latest column the dihedral angles are given. This angle occurs at each edge of the reciprocal. Two of the reciprocals are special. These are the ones that are derived from the so-called quasi-regular polyhedra: the Rhombic Dodecahedron and the Rhombic Triacontahedron, numbers 7 and 12. Their sides are rhombic. There is a certain analogy with the Cube, being here the reciprocal figure of the Octahedron, as its sides are also rhombic.

Fig. 3.5. The reciprocity of the regular polyhedra
Fig. 3.6. The triangular reciprocal faces with their vertex figures and the in-scribed circles with the radius $R_3$.

Fig. 3.7. The quadrangular and the pentagonal reciprocal faces, also with vertex circles and in-scribed circles.
3.3 The dihedral angles

Fig. 3.8A shows the plane MQP through the center M of the circum-sphere and through the unit edge PQ of the polyhedron. This plane is perpendicular to the edge of the reciprocal which passes through K. If one drops a perpendicular from K to the point U₁ on the line MQ, than this has the length \( \frac{1}{2} R_1 \). U₁ is the center of the inscribed circle in the reciprocal face. The point U₂ is the corresponding center of an adjacent face. Thus, two adjacent faces of a reciprocal figure cut the drawing plane perpendicularly according the lines U₁-K and U₂-K and they intersect each other according a line in point K perpendicular to the plane of drawing. The angle U₁-K-U₂, or \( \theta \), in Fig. 3.8B is the dihedral angle between the faces in a reciprocal figure (see also Table 1.8 on page 19).

\[
R_6 = \sqrt{(R_5^2 - 0.25R_3^2)} \quad \text{and as} \quad R_1 : R_5 = R_5 : R_6 \quad \text{and hence:}
\]

\[
R_6 = \frac{R_1^2 - 0.25}{R_1}
\]

\[
\theta = 2\arctan \left( \frac{2R_6}{R_3} \right)
\]

3.4. Volumina and surface areas of reciprocals

The volume of a reciprocal figure can be understood as to be composed of \( n \) pyramids with the basis \( K_n-L_n-K_n-U \) and the height \( R_6 \). Each pyramid has the volume:

\[
\text{Vol}_R = \frac{1}{3} \ast \frac{1}{2} \ast R_1 \text{g}_n \ast R_6 = \frac{R_1 \text{g}_n R_6}{6}
\]

Total volume: \( \text{Vol}_R = \sum (q_{n1} n1 \text{Vol}_R n1 + q_{n2} n2 \text{Vol}_R n2 + q_{n3} n3 \text{Vol}_R n3) \)

Total surface area: \( \text{Area}_R = \sum \frac{1}{2} R_3 (q_{n1} n1 g_{n1} + q_{n2} n2 g_{n2} + q_{n3} n3 g_{n3}) \)
planes, equally expanded. The section lines between these planes bisect the original edges PQ of the polyhedron perpendicularly and are also perpendicular to the line that connects the mid-edge K with the center of the circumscribed sphere. This is shown in Fig. 3.8B. A reciprocal figure is composed of a number of quadrangular sectors $K_n^\perp K_0^\perp U$. These sectors are directly related to the n-gons in the corresponding polyhedron, with $K_n^\perp K_0^\perp = \frac{1}{2} \ b_n = \cos \phi_n$.

The central angle of a regular polygon with $n$ sides: $\phi_n = \frac{\pi}{n}$ \hfill (3.11)

The radius of the circumscribed circle: $R_2 = \frac{1}{2 \sin \phi_n}$ \hfill (3.12)

The distance of the mid-point of a side from the center N:
$$m_n = \frac{1}{2 \tan \phi_n} = \sqrt{\left(\frac{R_2}{2} \cdot 0.25\right)}$$ \hfill (3.13)

Half the central face angle: $\zeta_n = \arctan \left(\frac{\cos \phi_n}{\sqrt{\left(\frac{R_2}{2} - \cos^2 \phi_n\right)}}\right)$ \hfill (3.14)

The face angle: $\chi = 180^\circ - 2\zeta_n$ \hfill (3.15)

Each polygon at any of the vertices gives a specific contribution to the edge length of the reciprocal face. For the polygon with $n$ edges:
$$g_n = \frac{\cos \phi_n}{2 \cos \zeta_n} = \frac{R_1 \tan \zeta_n}{2}$$ \hfill (3.16)

<table>
<thead>
<tr>
<th>R</th>
<th>$\chi^1$</th>
<th>$\chi^2$</th>
<th>$\chi^3$</th>
</tr>
</thead>
<tbody>
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<td>5} 33°33'26.3151&quot;</td>
<td>8} 37°46'24.0243&quot;</td>
</tr>
<tr>
<td>2</td>
<td>4} 60°00'00.0000&quot;</td>
<td>4} 70°31'43.6057&quot;</td>
<td>10} 37°43'06.2941&quot;</td>
</tr>
<tr>
<td>3</td>
<td>3} 90°00'00.0000&quot;</td>
<td>6} 48°11'22.8664&quot;</td>
<td>12} 36°14'37.0465&quot;</td>
</tr>
<tr>
<td>4</td>
<td>5} 60°00'00.0000&quot;</td>
<td>8} 31°23'58.9733&quot;</td>
<td>14} 35°22'26.3731&quot;</td>
</tr>
<tr>
<td>5</td>
<td>3} 108°00'00.0000&quot;</td>
<td>6} 81°34'44.1908&quot;</td>
<td>16} 34°14'37.0465&quot;</td>
</tr>
<tr>
<td>6</td>
<td>3} 112°53'07.3697&quot;</td>
<td>4} 55°01'28.9061&quot;</td>
<td>18} 33°33'26.3151&quot;</td>
</tr>
<tr>
<td>7</td>
<td>3} 109°28'16.3943&quot;</td>
<td>2} 70°31'43.6057&quot;</td>
<td>20} 32°14'37.0465&quot;</td>
</tr>
<tr>
<td>8</td>
<td>4} 83°37'14.2672&quot;</td>
<td>6} 48°11'22.8664&quot;</td>
<td>22} 31°23'58.9733&quot;</td>
</tr>
<tr>
<td>9</td>
<td>3} 117°12'02.0534&quot;</td>
<td>8} 31°23'58.9733&quot;</td>
<td>24} 30°28'16.3471&quot;</td>
</tr>
<tr>
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<td>26} 29°23'58.9733&quot;</td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
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<td>4} 86°58'26.9598&quot;</td>
<td>38} 23°16.3471&quot;</td>
</tr>
<tr>
<td>17</td>
<td>4} 88°59'30.4869&quot;</td>
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<td>5} 67°27'12.6323&quot;</td>
<td>42} 21°16.3471&quot;</td>
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</table>

Table 3.3. Face angles: the value of $\chi$ is related to the polygon in the polyhedron
Table 3.4. *Contribution of n-gon to edge length. The value of $g_n$ is related to the polygon in the polyhedron*.

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<tr>
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<th>$g_3$</th>
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Table 3.5 *Distance of face centre from corner; the value of $j_n$ is related to the polygon in the polyhedron*.

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</table>

The distance of a face corner from the center of the inscribed circle:

$$j_n = \frac{g_n}{\sin \zeta_n} \quad \{3.17\}$$

The distance of the face corner from the system center $M$: $z_r = \sqrt{(R_\phi^2 + j_n^2)} \quad \{3.18\}$
### Table 3.6

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Distance $Z_7$ of face corner from system center $M$; the value of $Z_7$ is related to the polygon in the polyhedron.

### 3.4. The reciprocal faces

In the following chapter the co-ordinates of the face corners and the nets of the reciprocal solids are given.

#### R1, Tetrahedron (edge=1)

Fig. 3.9. The reciprocal Tetrahedron $R1$

<table>
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</tr>
</thead>
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</tr>
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</tr>
<tr>
<td>3</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Chapter 03

\[ Z = R_6 = 0.20412415 \]

Edge 1-2 = edge 2-3 = edge 1-3 = 2 \( g_3 = 1 \)

**R2, Octahedron (edge=\( \sqrt{2} \))**

\[ Z = R_6 = 0.57735027 \]

Edge 1-2 = edge 2-3 = edge 1-3 = 2 \( g_3 = 1.41421357 \)

**R3, Cube (edge =\( \frac{1}{2} \sqrt{2} \))**

\[ Z = R_6 = 0.35355339 \]

Edge 1-2 = edge 2-3 = edge 3-4 = edge 1-4 = 2 \( g_3 = 0.70710678 \)
**R4, Icosahedron (edge=τ)**

Fig. 3.12. *The Reciprocal Icosahedron R4*

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>j₅ = 0.93417236</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-g₅ = -0.80901699</td>
<td>-R₃/2 = -0.46708618</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>g₅ = 0.80901699</td>
<td>-R₃/2 = -0.46708618</td>
<td></td>
</tr>
</tbody>
</table>

Z = R₆ = 1.22284749  
Edge 1-2 = edge 2-3 = edge 1-3 = 2 g₅ = 1.61803398

Fig. 3.13. *Net of the reciprocal R4*

**R5, Dodecahedron (edge=1/τ)**

Fig. 3.14. *The Reciprocal Dodecahedron R5*
Chapter 03

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$g_3 = 0.30901699$</td>
<td>$R_3/2 = 0.42532540$</td>
</tr>
<tr>
<td>2</td>
<td>$-g_3 = -0.30901699$</td>
<td>$R_3/2 = 0.42532540$</td>
</tr>
<tr>
<td>3</td>
<td>$-c = -0.5$</td>
<td>$-h = -0.16245985$</td>
</tr>
<tr>
<td>4</td>
<td>$0$</td>
<td>$-j_3 = -0.52573111$</td>
</tr>
<tr>
<td>5</td>
<td>$c = 0.5$</td>
<td>$-h = -0.16245985$</td>
</tr>
</tbody>
</table>

$Z = R_6 = 0.68819096$

$\kappa = 3 \zeta_3 - 90^\circ = 18^\circ$

$c = j_3 \cos \kappa = 0.5$

$h = j_3 \sin \kappa = 0.16245985$

Edge 1-2 = edge 2-3 = edge 3-4 = edge 4-5 = edge 1-5 = 0.61803398

---

Fig. 3.15. *Net of reciprocal $R5$*

---

**R6, Triakis Tetrahedron**

<table>
<thead>
<tr>
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<th>X</th>
<th>Y</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$0$</td>
<td>$j_3 = 0.54272042$</td>
</tr>
<tr>
<td>2</td>
<td>$-g_6 = -1.5$</td>
<td>$-R_3/2 = -0.45226702$</td>
</tr>
<tr>
<td>3</td>
<td>$g_6 = 1.5$</td>
<td>$-R_3/2 = -0.45226702$</td>
</tr>
</tbody>
</table>

$Z = R_6 = 0.95940322$

Edge 1-2 = edge 1-3 = $g_3 + g_6 = 1.8$

Edge 2-3 = $2g_6 = 3$

---

56
**R7, Rhombic Dodecahedron**

Fig. 3.18. *The Rhombic Dodecahedron R7*

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>( j_3 = 0.53033009 )</td>
</tr>
<tr>
<td>2</td>
<td>-0.75</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>( -j_3 = -0.53033009 )</td>
</tr>
<tr>
<td>4</td>
<td>0.75</td>
<td>0</td>
</tr>
</tbody>
</table>

\( Z = R_5 = 0.75 \)

Edge 1-2 = edge 2-3 = edge 3-4 = edge 1-4 = \( g_3 + g_4 = 0.91855866 \)

Fig. 3.17. *Net of the reciprocal R6*

Fig. 3.18. *Net of the reciprocal R7*
**R8, Tetrakis Hexahedron**

Fig. 3.19. *Tetrakis Hexahedron*

<table>
<thead>
<tr>
<th></th>
<th>X (2)</th>
<th>Y (3)</th>
<th>1</th>
<th>0</th>
<th>j₄ = 0.71151247</th>
<th>-R₃/2 = -0.47434165</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>g₆ = -1.06066017</td>
<td>R₆/2 = 1.06066017</td>
<td>2</td>
<td>-1.06066017</td>
<td>-0.47434165</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>g₆ =  1.06066017</td>
<td>R₆/2 =  1.06066017</td>
<td>3</td>
<td>1.06066017</td>
<td>0.47434165</td>
<td></td>
</tr>
</tbody>
</table>

Z = R₆ = 1.42302495

Edge 1-2 = edge 1-3 = g₄ + g₆ = 1.77217264

Edge 2-3 = 2 g₆ = 2.12132034
R9, Triakis Octahedron

Fig. 3.20. The Triakis Octahedron

\[
\begin{align*}
X & = \begin{pmatrix} 1 & 0 \\ 2 & -g_8 = -1.70710678 \\ 3 & g_8 = 1.70710678 \end{pmatrix} & Y & = \begin{pmatrix} j_3 = 0.56216928 \\ -R_3/2 = -1.70710678 \\ -R_3/2 = -0.47984149 \end{pmatrix} \\
Z & = R_6 = 1.63828133 \\
\text{Edge 1-2} & = \text{edge 1-3} = g_3 + g_8 = 2 \\
\text{Edge 2-3} & = 2 \cdot g_8 = 3.41421356
\end{align*}
\]

R10, Trapezoidal Icositetrahedron

Fig. 3.21. *The Trapezoidal Icositetrahedron*
<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>j₄ = 0.71481349</td>
</tr>
<tr>
<td>2</td>
<td>-c = -0.70710679</td>
<td>-h = -0.10468199</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-j₃ = -0.55287898</td>
</tr>
<tr>
<td>4</td>
<td>c = 0.70710679</td>
<td>-h = -0.10468199</td>
</tr>
</tbody>
</table>

\[ Z = R₆ = 1.22026295 \]
\[ \kappa = 2 \zeta₄ - 90° = 8.42105612° \]
\[ c = j₄ \cos \kappa = 0.70710679 \]
\[ h = j₃ \sin \kappa = 0.16245985 \]

Edge 1-2 = edge 1-4 = 2 g₄ = 0.59197996
Edge 2-3 = edge 3-4 = g₃ + g₄ = 0.83718608

Fig. 3.22. *Net of the reciprocal R10*

**R11, Hexakis Octahedron**

Fig. 3.23. *The Hexakis Octahedron*
$X$

$Y$

1  \(-f = -0.10617737\)  \(d = 0.69994335\)
2  \(-g_6 = -0.93737914\)  \(-R_3/2 = -0.48822549\)
3  \(g_8 = 1.42707327\)  \(-R_3/2 = 0.48822549\)
1' \(-f = -0.10617737\)  \(-d = -0.69994335\)
2' \(-g_6 = -0.93737914\)  \(R_3/2 = 0.48822549\)
3' \(g_8 = 1.42707327\)  \(R_3/2 = 0.48822549\)

$Z = R_6 = 2.20974121$

$\kappa = 180^\circ - 2 \zeta_6 - \zeta_4 = 8.62567802^\circ$

d = $j_4 \cos \kappa = 0.69994335$

f = $j_4 \sin \kappa = 0.10617737$

Edge 1-2 = $g_4 + g_6 = 1.45004882$

Edge 2-3 = $g_6 + g_8 = 2.36445241$

Edge 1-3 = $g_4 + g_8 = 1.93974295$

Fig. 3.24. *Net of the reciprocal R11*

**R12, Rhombic Triacontahedron**

Fig. 3.25. *The Rhombic Triacontahedron*
Chapter 03

Fig. 3.2. Net of the Rhombic Triacontahedron

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>j₃ = 0.55901699</td>
</tr>
<tr>
<td>2</td>
<td>-j₅ = -0.90450850</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-j₃ = -0.55901699</td>
</tr>
<tr>
<td>4</td>
<td>j₅ = 0.90450850</td>
<td>0</td>
</tr>
</tbody>
</table>

Z = R₆ = 1.46352549
Edge 1-2 = edge 2-3 = edge 3-4 = edge 1-4 = g₃ + g₅ = 1.06331351

R13, Pentakis Dodecahedron

Fig. 3.27. The Pentakis Dodecahedron

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>j₅ = 0.86881307</td>
</tr>
<tr>
<td>2</td>
<td>-g₆ = -0.92705098</td>
<td>-R₃/2 = -0.48971604</td>
</tr>
<tr>
<td>3</td>
<td>g₆ = 0.92705098</td>
<td>-R₃/2 = -0.48971604</td>
</tr>
</tbody>
</table>

Z = R₆ = 2.37713161
Edge 1-2 = edge 1-3 = g₅ + g₆ = 1.64469598
Edge 2-3 = 2 g₆ = 1.85410195
Fig. 3.28. *The Pentakis Dodecahedron*

**R14, Triakis Icosahedron**

Fig. 3.29. *The Triakis Icosahedron*

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$j_3 = 0.57189489$</td>
</tr>
<tr>
<td>2</td>
<td>$-g_{10} = -1.80901699$</td>
<td>$-R_3/2 = -0.49286096$</td>
</tr>
<tr>
<td>3</td>
<td>$g_{10} = 1.80901699$</td>
<td>$-R_3/2 = -0.49286096$</td>
</tr>
</tbody>
</table>

$Z = R_6 = 2.88525831$

Edge 1-2 = edge 1-3 = $g_3 + g_{10} = 2.09910635$

Edge 2-3 = 2 $g_{10} = 3.61803398$
Chapter 03

Fig. 3.30. *Net of the Triakis Icosahedron*

**R15, Pentagonal Icositetrahedron**

Fig. 3.31. *The left- and right-handed Pentagonal Icositetrahedron*

\[
\begin{array}{ll}
X & Y \\
1 & g_3 = 0.29673268 \\
2 & -g_3 = -0.29673268 \\
3 & -c = -0.54577649 \\
4 & 0 \\
5 & c = 0.54577649 \\
\end{array}
\]

\[
\begin{array}{ll}
R_3/2 & = 0.46409569 \\
R_3/2 & = 0.46409569 \\
h & = -0.07458632 \\
-j_4 & = -0.71641942 \\
h & = -0.07458632 \\
\end{array}
\]

\[
\begin{array}{ll}
Z & = R_6 = 1.15766179 \\
\kappa & = 3 \zeta_3 - 90^\circ = 7.78188828^\circ \\
c & = j_3 \cos \kappa = 0.54577649 \\
h & = j_3 \sin \kappa = 0.07458632 \\
\text{Edge 1-2} & = \text{edge 2-3} = \text{edge 1-5} = 0.59346536 \\
\text{Edge 3-4} & = \text{edge 4-5} = 0.84250916 \\
\end{array}
\]
Fig. 3.32. *Net of the left-handed Pentagonal Icositetrahedron*

Fig. 3.33. *Net of the right-handed Pentagonal Icositetrahedron*
Chapter 03

**R16, Trapezoidal Hexecontahedron**

![Diagram of the Trapezoidal Hexecontahedron](image)

**Fig. 3.34. The Trapezoidal Hexecontahedron**

<table>
<thead>
<tr>
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<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>(j_5 = 0.87389636)</td>
</tr>
<tr>
<td>2</td>
<td>(-c = -0.69098301)</td>
<td>(-h = -0.15472421)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>(-j_3 = -0.56770675)</td>
</tr>
<tr>
<td>4</td>
<td>(c = 0.69098301)</td>
<td>(-h = -0.15472421)</td>
</tr>
</tbody>
</table>

\[Z = R_6 = 2.12099102\]
\[\kappa = 3 \zeta_3 + \zeta_4 - 90^\circ = 12.62141648^\circ\]
\[c = j_4 \cos \kappa = 0.69098301\]
\[h = j_4 \sin \kappa = 0.15472421\]

Edge 1-2 - edge 1-4 = \(g_4 + g_5 = 1.23916012\)

Edge 2-3 = edge 3-4 = \(g_3 + g_4 = 0.80499199\)

**Fig. 3.35 Net of the Trapezoidal Hexecontahedron**
R17, Hexakis Icosahedron

Fig. 3.36. The Hexakis Icosahedron

<table>
<thead>
<tr>
<th></th>
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<th>Y</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>-f = -0.15588603</td>
<td>d = 0.68982201</td>
</tr>
<tr>
<td>2</td>
<td>-g₆ = -0.88982924</td>
<td>-R₃/2 = -0.49565834</td>
</tr>
<tr>
<td>3</td>
<td>g₁₀ = 1.68571670</td>
<td>-R₃/2 = -0.49565834</td>
</tr>
<tr>
<td>1'</td>
<td>-f = -0.15588603</td>
<td>-d = -0.68982201</td>
</tr>
<tr>
<td>2'</td>
<td>-g₆ = -0.88982924</td>
<td>R₃/2 = 0.49565834</td>
</tr>
<tr>
<td>3'</td>
<td>g₁₀ = 1.68571670</td>
<td>R₃/2 = 0.49565834</td>
</tr>
</tbody>
</table>

Z = R₆ = 3.73664646
κ = 180° - 2 ζ₆ - ζ₄ = 12.73382057°
d = j₄ cos κ = 0.68982201
f = j₄ sin κ = 0.15588603
Edge 1-2 = g₄ + g₆ = 1.39428702
Edge 2-3 = g₆ + g₁₀ = 2.57554594
Edge 1-3 = g₄ + g₁₀ = 2.19017448

Fig. 3.37. Net of the Hexakis Icosahedron
R18, Pentagonal Hexecontahedron

Fig. 3.38. The left- and right-handed Pentagonal Hexecontahedron

<table>
<thead>
<tr>
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<th>X</th>
<th>Y</th>
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<tbody>
<tr>
<td>1</td>
<td>g_3 = 0.29144977</td>
<td>R_3/2 = 0.48636643</td>
</tr>
<tr>
<td>2</td>
<td>-g_3 = -0.29144977</td>
<td>R_3/2 = 0.48636643</td>
</tr>
<tr>
<td>3</td>
<td>-c = -0.56633099</td>
<td>-h = -0.02764932</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-j_5 = -0.87596839</td>
</tr>
<tr>
<td>5</td>
<td>c = 0.56633099</td>
<td>-h = -0.02764932</td>
</tr>
</tbody>
</table>

Z = R_6 = 2.03987315
κ = 3 \zeta_3 - 90° = 2.79506586°
c = j_3 \cos \kappa = 0.56633099
h = j_3 \sin \kappa = 0.02764932
Edge 1-2 = edge 2-3 = edge 1-5 = 2g_3 = 0.58289954
Edge 3-4 = edge 4-5 = g_3 = g_5 = 1.01998825

Fig. 3.39. Net of the left-handed Pentagonal Hexecontahedron
### 3.5. Volumina and surface areas of reciprocals

The volume of a reciprocal figure can be understood as to be composed of \( n \) pyramids with the basis \( K_a-L_n-K_b-U \) and the height \( R_6 \) (see Fig. 3.7B). Each pyramid has the volume:

\[
\text{VolR}_n = \frac{1}{3} \cdot \frac{1}{2} \cdot R_3 \cdot g_n \cdot R_6 = \frac{R_3 \cdot g_n \cdot R_6}{6}
\]

\[\{3.21\}

Total volume:

\[
\text{VolR}_{\text{tot}} = \sum (q_{n1} \cdot n1 \cdot \text{VolR}_{n1} + q_{n2} \cdot n2 \cdot \text{VolR}_{n2} + q_{n3} \cdot n3 \cdot \text{VolR}_{n3})
\]

\[\{3.22\}

Total surface area:

\[
\text{AreaR}_{\text{tot}} = \sum \frac{1}{2} \cdot R_3 \cdot (q_{n1} \cdot n1 \cdot g_{n1} + q_{n2} \cdot n2 \cdot g_{n2} + q_{n3} \cdot n3 \cdot g_{n3})
\]

\[\{3.23\}

<table>
<thead>
<tr>
<th>( P )</th>
<th>\text{VolumeR}_{\text{TOT}}</th>
<th>\text{AreaR}_{\text{tot}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.11785113</td>
<td>1.73205080</td>
</tr>
<tr>
<td>2</td>
<td>1.33333333</td>
<td>6.92820323</td>
</tr>
<tr>
<td>3</td>
<td>0.35355339</td>
<td>3.00000000</td>
</tr>
<tr>
<td>4</td>
<td>9.24180829</td>
<td>22.67283942</td>
</tr>
<tr>
<td>5</td>
<td>1.80901699</td>
<td>7.88596668</td>
</tr>
<tr>
<td>6</td>
<td>5.72756493</td>
<td>17.90977386</td>
</tr>
<tr>
<td>7</td>
<td>2.38648539</td>
<td>9.54594154</td>
</tr>
<tr>
<td>8</td>
<td>14.31891232</td>
<td>30.18691769</td>
</tr>
<tr>
<td>9</td>
<td>23.3170850</td>
<td>42.69176749</td>
</tr>
<tr>
<td>10</td>
<td>8.75069057</td>
<td>21.51345464</td>
</tr>
<tr>
<td>11</td>
<td>49.66382185</td>
<td>67.42484815</td>
</tr>
<tr>
<td>12</td>
<td>14.80021243</td>
<td>30.33813728</td>
</tr>
<tr>
<td>13</td>
<td>59.87641488</td>
<td>75.5654470</td>
</tr>
<tr>
<td>14</td>
<td>111.14946533</td>
<td>115.56968557</td>
</tr>
<tr>
<td>15</td>
<td>7.44739519</td>
<td>19.29940656</td>
</tr>
<tr>
<td>16</td>
<td>42.25536942</td>
<td>59.76739510</td>
</tr>
<tr>
<td>17</td>
<td>228.17899489</td>
<td>183.19554518</td>
</tr>
<tr>
<td>18</td>
<td>37.58842367</td>
<td>55.28053092</td>
</tr>
</tbody>
</table>

Table 3.7. Areas and volumes of polyhedra and their reciprocals
Chapter 03

<table>
<thead>
<tr>
<th>P</th>
<th>Volume $P_{\text{tot}}$</th>
<th>Area $P_{\text{tot}}$</th>
<th>Volume $R_{\text{tot}}$</th>
<th>Area $R_{\text{tot}}$</th>
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</thead>
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<td>1.73205081</td>
</tr>
<tr>
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<td>6.92820323</td>
</tr>
<tr>
<td>3</td>
<td>0.47140452</td>
<td>3.46410162</td>
<td>0.35355339</td>
<td>3.00000000</td>
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<tr>
<td>4</td>
<td>7.66311896</td>
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<td>9.24180829</td>
<td>22.67283942</td>
</tr>
<tr>
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<td>2.18169499</td>
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</tr>
<tr>
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<td>7</td>
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<td>2.38648539</td>
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<tr>
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<td>11.31370850</td>
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<td>30.18691770</td>
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<tr>
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<td>13.59966329</td>
<td>32.43466436</td>
<td>23.31370850</td>
<td>42.69176750</td>
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<tr>
<td>10</td>
<td>8.71404521</td>
<td>21.46410161</td>
<td>8.75069057</td>
<td>21.51345465</td>
</tr>
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<td>55.28674496</td>
<td>37.58842367</td>
<td>55.28053092</td>
</tr>
</tbody>
</table>

Table 3.8. Volumes and surface areas of polyhedra ($P$) and reciprocals ($R$).

![Fig. 3.41. All uniform reciprocals with their reproducible parts.](image)
Fig. 3.42. *Compounds of the polyhedra and their reciprocals*

Fig. 3.43. *Models of P6 and R6 and of their compound.*
Models of some compounds of polyhedra and their duals. Below at right: those of the pentagonal prism and antiprism.

3.6. Duals of prisms and antiprisms

Prisms and antiprisms also can be considered following the definition in Chapter 1.1 and they also have reciprocal or dual counterparts, that can be found in a similar way. Looking at the vertices of these solids, vertex figures can be constructed as is done in Chapter 1.12. From these the faces of their reciprocals can be derived, similarly to which is done in Figs 3.5 and 3.6. This is described in more detail in Chapter 6.
Fig. 3.46. A number of prisms and their dual figures. A choice is made of those prisms, that are composed of polygons that also occur in the Platonic and Archimedean solids, these with 3, 4, 5, 6, 8 or 10 sides.

Fig. 3.47. Models of antiprisms and their duals.
Chapter 03

3.7. References

For many beautiful pictures of catalan solids see http://flickrhivemind.net/Tags/math.solid/Interesting

Chapter 4. THE CHIRAL POLYHEDRA

4.1. Chirality

'Chirality' of a certain form means that it is not super-imposable on its mirror image. Most of the regular and semi-regular polyhedra have identical original and mirror images but there are two that have a mirror image that is different. This means that they have a left-handed and a right-handed version. These two polyhedra are the so-called 'snub solids', the snub cube and the snub dodecahedron, that consist of a number of triangles and a number of either squares or pentagons respectively. They have interesting characteristics, that - on closer examination - appear to be applicable to a few other forms as well. The semi-regular polyhedra are generally considered as to be found by truncation of the regular solids, either according their vertices or according their edges or both. In the snub solids this truncation is done twice per edge and under a certain angle with respect to this edge and to each other, so that two triangular faces per original edge occur. The snub solids both have a left-handed and a right-handed version, that are 'enantiomorphic' (see Fig. 4.1). This group of chiral polyhedra is so interesting, that it deserves special attention. For that purpose it was unavoidable to repeat for clarity sake some of the basic notions of Chapter 1.

Fig. 4.1. The right- and left-handed versions of the snub solids for n having the value of 3, 4 or 5. The icosahedron is a snub solid with n=3. It also can have two distinct positions in space.

They are much more complicated than the other polyhedra. It is therefore interesting to consider their geometry in more detail, particularly as some representatives are often used as a basis for a further subdivision of spherical surfaces. The present chapter is an elaboration of a contribution by the author to the IASS conference of 1992 in Toronto [4.2] and of a paper that he wrote in collaboration with H.S.M. Coxeter [4.1] and which was further developed in 1998 in Warsaw [4.3].

The two snub solids are usually called after their circumscribed figures, the cube and the dodecahedron. The snub cube has 6 squares that each are completely surrounded by triangles, whereas the snub dodecahedron has 12 pentagons in a similar situation. They have the common characteristic, that they all are based on a polygon or n-gon with a variable number (n) of the sides. In the two known snub figures always 4 triangles meet at the corner of the n-gon with 4 or 5 sides. However, the row of possible snub polyhedra can theoretically be extended in both directions with n being an integer number ranging from 1 to 6. The row of snub figures can theoretically be extended by letting the sides number of this variable n-polygon range from 6 (the planar tesselation of triangles) to 3 (icosahedron), to 2 (octahedron), or even to 1 (tetrahedron).

It is possible to derive the snub cube and the snub dodecahedron by truncation from the octahedron and from the icosahedron respectively. They are the Platonic solids that are composed of triangles only. These regular polyhedra differ in the fact that in each vertex a variable number of triangles meet. The latter derivation - from the triangular regular solids - is even more logical than the first. This would mean, that the two snubs might as well be called 'snub octahedron' and 'snub icosahedron'. The two different approaches are in the following discerned as Approach A (see Fig. 4.2) and B (Fig. 4.3).
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Fig. 4.2. The three different snub solids, including the Icosahedron, with their two possible circumscribed figures, according approach A.

Fig. 4.3. The three snub solids with their circumscribed figures according approach B. The faces of the trigonal or deltahedric circumscribed solids now always touch a triangle of the snub solid. This particular triangle maybe called 'specific triangle', which will be referred to later, in paragraph 4.6.

4.2. General characteristics

Since the vertices of a so-called 'uniform polyhedron' are identical, any polyhedron is characterised by its vertex figure that is found by cutting this solid - at a certain distance from this vertex - by a plane that is perpendicular to the connection line between this vertex and the polyhedron centre (See Fig. 4.4)

Fig. 4.4. Vertex figure in a polyhedron
The basic form of a pentagonal vertex figure

This vertex figure is different - and thus characteristic - for any polyhedron and can have either three, four or five sides, depending on the number of the polygons that meet in each vertex of a particular polyhedron. The vertex figures of the two snub solids - and also that of the icosahedron - all are pentagonal.

If the plane of the vertex figure is chosen midway between the original vertex and the neighbouring vertices, this vertex figure can be said to have a circumscribed circle with the radius $R_3$. This position is taken for convenience, since all relevant data can be derived from the pyramidal cap with this vertex figure as its basis and the remaining halves of the polyhedron edges as the inclined edges. Considering the planar vertex figure, one can imagine that it is subdivided according the contributions of the respective polygons in the polyhedron itself (Fig. 4.6).

In the case of the snub solids, the vertex figure consists of 5 sectors with the ratios:

$$4\varphi_1 + \varphi_2 = 180^\circ$$

If $n = \text{the number of sides of the n-gon}$:

$$bn = 2 \cos(\pi/n)$$

$$\sin(\phi_1) = \frac{1}{2R_3} \quad \text{and} \quad \sin(\phi_2) = \frac{\cos(\pi/n)}{R_3}$$

$$R_3 = \frac{\cos(\pi/n)}{\sin(\phi_2)} = \frac{1}{2\sin(\phi_2)}$$

Thus:

$$2 \sin(\phi_1) \cos(\pi/n) = \sin(\phi_2) = \sin(4\varphi_1) = 2 \sin(2\varphi_1) \cos(2\varphi_1)$$

It can therefore be found that

$$4 \cos^3(\varphi_1) - 2 \cos(\varphi_1) - \cos(\pi/n) = 0$$

{4.3}
Brückner [Ref. 1.2] has a general equation that gives similar results:

$$\cos \varphi_1 = \frac{1}{2} \left( \cos \left( \frac{\pi}{n} \right) + \sqrt{\cos^2 \left( \frac{\pi}{n} \right) - \left( \frac{2}{3} \right)^3} + \sqrt{\cos \left( \frac{\pi}{n} \right) - \sqrt{\cos^2 \left( \frac{\pi}{n} \right) - \left( \frac{2}{3} \right)^3}} \right) \quad \{4.4\}$$

This equation can be solved for various values of n. It is also clear that for n all integers from 2 to 6 are imaginable. This is done in Table 4.1.

<table>
<thead>
<tr>
<th>polyhedron</th>
<th>n</th>
<th>(\cos(\varphi_1))</th>
<th>(\varphi_1)</th>
<th>(\varphi_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>octahedron</td>
<td>2</td>
<td>0.707106781187°</td>
<td>45.0000000000°</td>
<td>0.0000000000°</td>
</tr>
<tr>
<td>icosahedron</td>
<td>3</td>
<td>0.809016994375°</td>
<td>36.0000000000°</td>
<td>36.0000000000°</td>
</tr>
<tr>
<td>snub cube</td>
<td>4</td>
<td>0.842509162445°</td>
<td>32.5936276104°</td>
<td>49.62414895582°</td>
</tr>
<tr>
<td>snub dodecahedron</td>
<td>5</td>
<td>0.857780749849°</td>
<td>30.9316862069°</td>
<td>56.27324551725°</td>
</tr>
<tr>
<td>plane tesselation</td>
<td>6</td>
<td>0.866025403784°</td>
<td>30.0000000000°</td>
<td>60.0000000000°</td>
</tr>
</tbody>
</table>

Table 4.1. Central angles in vertex figures.

Brückner finds the value of \(\cos(\varphi_1) = 0.842508\ldots\) for P15, which becomes inaccurate from the 6th digital and = 0.857779\ldots\ for P18.

The circumscribed sphere of a polyhedron has the radius:

$$R_1 = \frac{1}{2 \sqrt{1 - R_3^2}} \quad \{4.5\}$$

4.3. The circumscribed figures

In approach A of Fig. 4.2 the circumscribed figure is composed of a number of n-gons of the same kind as those that occur in the snub figure which has been derived from it. The larger n-gons have the edge length

$$F_A = \left( z_s / z_c \right)_n \quad \{4.6\}$$

z follows from Fig. 4.7. This shows a sector of the polyhedron, and it consists of 2 so-called 'orthoschemes', that each are composed of 4 rectangular triangles [Ref. 2].

Fig. 4.7. A section of a uniform polyhedron with unit edge length.

P and Q are two neighbouring vertices, K = the middle of an edge, M = the centre of the polyhedron and N = the middle point of one of the polygons. This point N - and thus the polygon - has the distance \(z_n\) from the centre M. \(z_c\) in equation \{4.6\} refers to the n-gon of the circumscribed figure and \(z_s\) to the snub solid, both of unit edge length.
Further:
\[ \varphi = \frac{\pi}{n}, \text{ where } n \text{ is the number of sides of the polygon} \]
\[ R_1 = \text{radius of circumsphere.} \]
\[ R_2 = \text{radius of circumsphere of the } n\text{-gon} = \frac{1}{2} \csc(\pi/n) \]
\[ R_5 = \rho \text{ or radius of inter-sphere} = \sqrt{R_1^2 - 0.25} \]
\[ \zeta = \text{dihedral angle between } n\text{-gon and centriplane VNC} \]
\[ z_n = \text{distance of } n\text{-gon to centre } M = \sqrt{R_1^2 - r_n^2} \]
\[ m_n = \text{distance of } n\text{-gon centre } N \text{ to mid-edge } K = \frac{1}{2} \cotan(\pi/n) \]

4.4. Rotation angles

In both approaches the polygon of the snub solid that touches the larger polygon of the circumscribed figure is rotated over a certain angle with respect to this latter polygon. This angle is called \( \Upsilon_A \) in approach A and is clock-wise for the dextro or right-handed variant. In approach B (Fig. 4.3) this angle is called \( \Upsilon_B \) and it is anti clock-wise for the levo or left-handed variants. If one considers the two derivation methods of the 3 relevant snub solids, it is interesting to notice that they are basically each others reciprocal or dual cases. The two circumscribed figures in each snub solid can be made to intersect (Fig. 4.8). Fig. 4.8B shows the parts that the circumscribed figures have in common. They consist of a number of regular \( n \)-gons with the side length \( g \) and a number of irregular hexagons with the side lengths \( g \) and \( d \), where

\[ d = 2 \tan \zeta_A (F_A \rho_A - F_B \rho_B) \]
\[ g = \frac{1}{2} (F_B - d) \]

Upon these faces the relevant polygons of the snub figures can be drawn, the right- as well as the left-handed versions, and their corners lie on the edges with length \( g \) of these faces. This Fig. 8B forms therefore the basis for the derivation of the rotation angles, because the positions of these corners can be found by determining the section points of the circumscribed circles of the respective polygons with these edges. Lines [Ref. 3] describes the principle of this method. (Fig. 4.8).

Fig. 4.8. The intersections of the circumscribed figures of the snub solids.
Fig. 4.9. Determination of the section points on the edges of the p-gons and the hexagons in Fig. 4.8.B, corresponding to the approaches A and B.

\[ \gamma_A = \arccos \left( \frac{g m/n}{r_n} \right) \]
\[ = \arccos \left( \frac{g}{\frac{1}{2} \cot \left( \frac{\pi}{n} \right)} \right) \]
\[ = \arccos \left( g \cos \left( \frac{\pi}{n} \right) \right) \] \{4.7\}

\[ \gamma_B = \arccos \left( (g + 2d) \cos \left( \frac{\pi}{n} \right) \right) \]
\[ = \arccos \left( \frac{1}{2} g + d \right) \] \{4.8\}

In the case that \( d = 0 \Rightarrow g = \frac{1}{2} F_A = \frac{1}{2} F_B. \)

This is the situation in the case of the snub tetrahedron, which produces the icosahedron.

4.5. Simplified construction method according approach B

In the case of approach B a simple construction method can be found, starting from the circumscribed figure. If one edge of the triangle STU, which belongs to the snub solid and which lies in the larger - so-to-say 'circumscribed' - triangle VWX, is extended it intersects the opposite edge \( F_B \) in the point \( V' \) at a distance

\[ \ell = (F_B \cot \varepsilon + \cosec \varepsilon)^{1/3}, \]

with \( \varepsilon = 60^\circ \angle \gamma_B \) (see Fig. 4.10.A)

Fig. 4.10. Determination of the section points on line \( F_B \)

From numerical elaboration follows:
\[ \ell = \frac{1}{2} F_B \]

This means that the extended edge meets the other triangle exactly in one of the vertex points, so that the small triangle can be constructed starting from these vertices. The opposite edge of the triangle VWX is bisected in the point \( Y \) according the ratio:
\[ V = \frac{1}{2} F_B - \tan(30^\circ - \gamma_B) \frac{1}{2} \sqrt{3}/F_B \]

Or:
\[ V = \frac{1}{2} (1 - \sqrt{3} \tan(30^\circ - \gamma_B)) \]

The magnitude of the various values, that were derived in the foregoing, are given in Table 4.2. They can relatively easily be determined with the help of a spreadsheet program.

<table>
<thead>
<tr>
<th>aspect</th>
<th>icosahedron</th>
<th>snub cube</th>
<th>snub dodecahedron</th>
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<td>( R_3 )</td>
<td>0.85065081</td>
<td>0.92819138</td>
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<td>( R_1 )</td>
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<td>1.34371337</td>
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<td>( R_{2A} = r_n )</td>
<td>0.57735027</td>
<td>0.70710678</td>
<td>0.85065081</td>
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<tr>
<td>( R_{1A} )</td>
<td>0.61237244</td>
<td>0.86602540</td>
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<tr>
<td>( \rho_A )</td>
<td>0.35355339</td>
<td>0.70710678</td>
<td>1.30901699</td>
</tr>
<tr>
<td>( m_n )</td>
<td>0.28867513</td>
<td>0.50000000</td>
<td>0.68819096</td>
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<tr>
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<td>1.11351636</td>
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<tr>
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<td>1.61589952</td>
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<td>0.95105652</td>
</tr>
<tr>
<td>( \rho_B )</td>
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<td>0.57735027</td>
<td>0.57735027</td>
</tr>
<tr>
<td>( m_3 )</td>
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<td>0.28867513</td>
<td>0.28867513</td>
</tr>
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<td>69.09484255°</td>
</tr>
<tr>
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<td>0.34060243</td>
</tr>
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</tr>
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</tr>
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<td>20.31501434°</td>
<td>19.51792257°</td>
</tr>
<tr>
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<td>39.68498566°</td>
<td>40.48207743°</td>
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<td>1.37417040</td>
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<tr>
<td>( V )</td>
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<td>0.35220113</td>
<td>0.33977187</td>
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</table>

Table 4.2. Relevant values of the snub cubes and their circum-scribed figures.

For the two extremes, the octahedron and the tesselation pattern of hexagons surrounded by triangles a few values can be found, that have a similar background (see Fig. 4.11 and table 4.3):

<table>
<thead>
<tr>
<th>octahedron</th>
<th>tesselation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_B )</td>
<td>30°</td>
</tr>
<tr>
<td>( \ell )</td>
<td>( \approx )</td>
</tr>
<tr>
<td>( V )</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 4.3. Some of the basic values, applicable to the extremes.
Fig. 4.11. The two extreme cases: at left the octahedron and at right the plane tessellation of hexagons and surrounding triangles.

Fig. 4.12. The parts that correspond to the circumscribed faces and that must be rotated in space, as many times as they occur.

Note that the group of four triangles in Fig. 4.12, that connect the significant faces - if one thinks of the construction of a material model - can eventually be made in one piece, that is folded along the central triangle. This part is repetitive and occurs as many times as those that occur in the B-stadium. The yellow triangle P-Q-R, called 'specific', in the middle of the three others always connects three of the larger polygons around it with its corners.
Fig. 4.13. The left- and right-handed versions of the ‘three’ snub solids, 1) the Icosahedron, 2) the Snub Cube and 3) the Snub Dodecahedron. They are shown together with their reciprocal or dual figures and with the compounds of both.
4.6. Workshops

From time to time the author organizes workshops at various occasions, where the participants are invited to build material models with the help of thick paper following the model plates in Fig 4.14. The parts must be cut from these plates, the folding lines scored, the holes at the side ends punched with a special tool and finally the parts must be connected with elastic bands. The author’s website [4.5] shows some more pictures.

Fig. 4.14. Model plates for the workshop exercises (see Chapter 15). The parts have a side length of 10 cm. Using these parts all chiral polyhedra and many of the other polyhedra can be built.

Part B in fact combines 4 triangles, of which the triangle in the centre is accentuated by a circumscribed circle with partial black infill. This is done to indicate this as the so-called 'specific' triangle, which is one of the triangles that touch the circumscribed deltahedric polyhedra: the tetrahedron, the octahedron and the icosahedron.

Originally three polyhedra were said to be chiral: the snub dodecadron, in which four triangles meet the pentagon in one point, the snub cube with the four triangles meeting a square and the icosahedron, where four triangles meet another triangle. In the paragraph 4.5 was shown, that the row of chiral solids can theoretically be extended in one direction toward the flat situation of four triangles that meet the hexagon, and also in the other direction with the four triangles that meet a line (digon) thus forming the octahedron, and finally the four triangles meeting even in one point (monogon). This is shown in the following Fig. 4.15, where the model cards were used.

A complete description of the instructions and full size sketches of the panels are is given in Chapter 15.
Fig. 4.1. Layouts of the various chiral solids. Note that there is also a flat version: the hexagonal form.
Fig. 4.15. Models of the chiral solids: 1) the 1-gonal one or the tetrahedron, 2) the 2-gonal one or the octahedron, 3) the 3-gonal one or the Icosahedron, 4) the 4-gonal one or the Snub Cube, 5) the pentagonal one or the Snub Dodecahedron, 6) the 6-gonal solution or the flat plane.

4.7. References

Chapter 5. FORM GENERATION OF POLYHEDRIC SHAPES

The Platonic - or regular - and the Archimedean - or semi-regular - polyhedra can be considered as portions of space that are completely surrounded by one or more kinds of regular polygons. The numbers and positions in space of these polygons are strictly ruled by universal criteria. It is therefore possible to form these polyhedra by placing polygons around the centre of the coordinate system in distinct numbers, at certain distances and under certain angles in accordance with these rules. This is called here 'rotation' and in this Chapter a method is described where this is done for the regular and semi-regular polyhedra and for related figures that are found by derivation from these polyhedra. The figures that are rotated have not necessarily to be regular polygons, nor do they have to be strictly planar. This method thus allows the rotation of arbitrary figures - also spatial ones - and the rotation procedure can even be used repeatedly, so that very complex configurations can be described.

5.1. An automatic generation routine for polyhedric shapes

The surface of the so-called 'uniform' polyhedra is formed by a closed pattern of regular polygons. All regular polyhedra consist of a set of polygons of one kind, whereas the semi-regular figures are formed either by two or by three different kinds of polygons. These polygons are organized in specific arrangements and numbers. [5.2]

Such a polyhedron can be generated by firstly placing a polygon in the centre of the system and in the XY-plane, then translating it over the distance $z_n$ along the Z-axis and in the end rotating it around the $Z$, $X$ and $Y$-axis respectively (angles $\alpha$, $\beta$ and $\gamma$ in Fig. 5.2). This has to be done for each of the polygons occurring in the polyhedron and also for each kind of polygon, that compose the polyhedron. (Fig. 5.1).

![Fig. 5.1 Example of a polyhedron, consisting of 3 kinds of polygons and each kind having its own specific number and arrangement.](image-url)
Chapter 05

The coordinates after rotation are \([5.7]\):

\[
\begin{align*}
P_x &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \sin \gamma - \sin \alpha \cos \beta + \cos \alpha \sin \beta \sin \gamma + \sin \beta \cos \gamma \\
P_y &= \sin \alpha \cos \gamma + \cos \alpha \cos \gamma - \sin \gamma \\
P_z &= -\cos \alpha \sin \beta + \sin \alpha \cos \beta \sin \gamma + \sin \alpha \sin \beta + \cos \beta \cos \gamma
\end{align*}
\]

All vertices in a particular polyhedron lie on a circumscribed sphere, which is unique for this polyhedron. Each polygon has a distance to the centre of the coordinate system of

\[
z_n = \sqrt{R_1^2 - R_2^2}
\]

with

\[
R_1 = \text{radius of the circumscribed sphere} \quad \text{and} \quad R_2 = \text{radius of the circumscribed circle of polygon} = 1/2 \sin(\pi/n),
\]

where \(n\) is the number of edges of the considered polygon \([5.1]\).

The value of \(z_n\) can thus be calculated for each kind of polygon occurring in a polyhedron, if \(R_1\) and \(n\) are known. It is necessary to know also the numbers in which each of the different polygon kinds occurs and the angles according which they have to be rotated. The answer to this question is quite obvious for the cube: it counts 6 squares that first are translated along the Z-axis over a distance of \(z_n = 0.5\) (times the unit edge length) and then are successively rotated around the Z-, X- and Y-axis over the angles: \(0^\circ-0^\circ-0^\circ\), \(0^\circ-0^\circ-180^\circ\), \(90^\circ-0^\circ-90^\circ\), \(90^\circ-0^\circ-270^\circ\), \(90^\circ-270^\circ-0^\circ\) and \(90^\circ-90^\circ-0^\circ\). With 7 different rotation cases, that each represent a specific combination of rotation angles and number of faces, all known polyhedra, can be formed.

---

**Fig. 5.2. Principle of the translation and rotation procedure of polygons for the formation of polyhedra in 4 stages:**

1. The \(n\)-sided polygon is formed in the centre of the coordinate system and in the XY-plane
2. The polygon is translated along the Z-axis over distance \(z_n\) and rotated over angle \(\alpha\) around the Z-axis
3. Rotation over the angle \(\gamma\) around the X-axis
4. Rotation over the angle \(\beta\) around the Y-axis
5.2. Rotation cases

The required rotation angles for all uniform polyhedra, that are necessary to enable the formation of the primary polyhedra are summarized in the next captions.

The rotations take place according:
1. The 4 triangles in the tetrahedron
2. The 6 squares in the cube
3. The 8 triangles in the octahedron
4. The 12 pentagons in the dodecahedron
5. The 20 triangles in the icosahedron
6. The 12 squares in the truncated cuboctahedron
7. The 30 squares in the truncated icosidodecahedron

**Case 1** (tetrahedron):

\[ \zeta = \arctan(1/2 \sqrt{2}) \approx 35.26438968^\circ \]

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<th>( \beta )</th>
<th>( \alpha )</th>
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Table 5.1. Rotation angles of case 1

Fig. 5.4. Case 1.
Chapter 05

Case 2 (cube):

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Table 5.2. Rotation angles of case 2

Case 3 (octahedron):

$\xi = \arctan(\sqrt{2}) = 54.73561032^\circ$

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Table 5.3. Rotation angles of case 3

Case 4 (dodecahedron):

$\lambda = \arctan(\tau) = 58.28252559^\circ$

with $\tau = (1+\sqrt{5})/2$

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Table 5.4. Rotation angles of case 4

Fig. 5.5. Case 2

Fig. 5.6. Case 3

Fig. 5.7. Case 4
Case 5 (icosahedron):

\[ \xi = \arctan(\sqrt{2}) = 54.73561032^\circ \]
\[ \psi = \arctan(\tau^2) = 69.09484255^\circ \]
\[ \epsilon = \arctan(\sqrt{3}/5) = 37.76124390^\circ \]

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<td>-(\epsilon)</td>
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Table 5.5. Rotation angles of case 5

Case 6 (truncated cuboctahedron)

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<td>0</td>
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<td>90</td>
<td>270</td>
</tr>
</tbody>
</table>

Table 5.6. Rotation angles of case 6
Chapter 05

**Case 7** (truncated icosidodecahedron):

\[ \mu = \arctan\left(\frac{1}{\tau}\right) = 31.71747441^\circ \]
\[ \delta = \arctan\left(\frac{1}{\tau^2}\right) = 20.90515745^\circ \]

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<td>270+( \delta )</td>
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Table 5.7. Rotation angles of case 10

![Image](image_url)  
**Fig. 5.10. case 7**
5.3 formation of polygons

Instead of the mentioned polygons, that form the original polyhedra, any other applicable polygon can be rotated in the same way, using such a case. It is suitable to arrange these data in sets, that are called in this context 'rotation cases', and that can be treated as entities. With one rotation case and one kind of polygon various solids can be described, just by varying distance $z_n$ and the initial rotation $\alpha$ of the polygon. For the regular polyhedra the rotation of one kind of polygon suffices, whereas the semiregular solids usually require more than one run. The polyhedron shown in Fig. 1 is in this way composed of 30 squares rotated according case 7, 20 hexagons according case 5 and 12 decagons according case 4. The coordinates of the corners in a polygon can be computed in the way as is shown in Fig. 5.11. The corner with the sequence number $v$ has as its coordinates in the XY-plane:

$$
\begin{align*}
    y_v &= R_2 \cos (v.2\phi) = R_2 \cos (v.2\pi/n) \\
    x_v &= R_2 \sin (v.2\pi/n) \\
    z_v &= 0
\end{align*}
$$

Fig. 5.11. The formation of a polygon in the XY-plane

Fig. 5.12. The different polygons that constitute the uniform polyhedra

5.4. Rotation data of the uniform polyhedra

The various solids can be generated, using the following data as input:

- $n_1$ = number of edges of polygon
- $n_3$ = number of rotation case
- $\alpha, \gamma, \beta$ = initial rotation of polygon around Z-, X- or Y-axis
- $d_{(x,y,z)}$ = initial translation of polygon along X-, Y- or Z-axis

In the next caption the relevant data for the rotation of the familiar solids are summarized. The polyhedron is indicated with a sequence number P.
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Table 5.8. Rotation variables of the uniform polyhedra
Fig. 5.13. Review of the 20 uniform polyhedra, 4 of which are left- and right-handed in pairs (P15 and 18). The colours and the numbers refer to the respective rotation cases for the polygons in question (see also Fig. 2.1).

The snub solids with the numbers 15 and 18 have a left- and a right-handed version. The signs of the value $\alpha$ refer to these. They both have two kinds of triangles: 8 (resp. 20) which touch only the corners of the p-gons and 24 (resp. 60) which have one common edge with one of the p-gons. These solids can not be formed in one operation. Therefore triangular caps, that are composed of four equilateral triangles, have to be made in advance. Such a cap is shown in Fig. 5.14.

Fig. 5.14. A typical cap of a snub polyhedron.

At first a triangle has to be generated and to this can be given the translation $z_n$ and the successive rotation $\gamma = 180^\circ - 2 \xi_3$ (around the X-axis). Both are different for the two snubs 15 and 18 [Ref. 5].

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Table 5.9. Tilting angles of the snub solids

A cap is finally formed by the composition of three of thus shifted triangles (ABC, BDE and EFC) with respective additional rotations around the Z-axis of $\alpha = 60^\circ$, 120$^\circ$ or 180$^\circ$. To these three surrounding triangles is then added a fourth triangle BCE in the middle at a distance $z_n$ and placed in an upside down position by the rotation $\alpha = 180^\circ$. The caps and the n-gons are organized in the way that is shown in Fig. 15.
Fig. 5.15. Position of the triangular caps in the snub solids P15 and P18. P5 (icosahedron) is added as it can be considered as a snub tetrahedron (Ref. 5).

Fig. 5.16. The Snub Dodecahedron, formed by the rotation of 12 pentagons (case 4) and 20 caps (case 5)

5.6. The formation of reciprocal figures by rotation

The 5 regular duals and also R7 and R12 (the duals of the quasi-regular polyhedra) can be found directly by the rotation of their faces, using an appropriate rotation case. For all other duals, first a cap has to be formed. This cap can be composed of 3, 4 or 5 faces. These caps are indicated in Figs. 4 to 21 and are formed by giving the face an initial 'tilt' or rotation $\gamma$ around the X-axis and by then reproducing it in a circular sense a number of times around the Z-axis. For Nos. 11 and 17 an additional tilt $\beta$ around the Y-axis is required before the reproduction phase can take place. The whole cap is then rotated in space according to one of the rotation cases, sometimes after an initial rotation $\alpha$ around the Z-axis.
Fig. 5.17. Examples of initial tilts $\gamma$ (rotation around X-axis) or $\gamma + \beta$ (successively around X- and Y axis).

Fig. 5.18. Review of the reciprocal polyhedra. The green parts are compositions of the basic faces, described in Chapter 4. These are then rotated following the cases as indicated by the numbers.
Table 5.10. Rotation data of reciprocal solids.

### 5.7. Augmentation

Augmentation means that a spatial figure is added to each polygon, which has the same basis as this polygon. Simon Stevin [5.4] introduced in his work *Problemata Geometrica* (1582 - 1586) the term 'auctare', or augmenting and added 3-, 4- or 5-sided regular pyramids to the polyhedra.

Emmerich [5.5] spoke of 'composite polyhedra' and he added to the polyhedra, parts of other polyhedra having a polygonal basis (Fig. 5.19.1 to 5.19.7). The polyhedron can be covered partially or totally by polyhedral caps. Emmerich thus discovered 102 variations.

![Polyhedral caps](image-url)

Fig.5.19. Polyhedral caps, used for the augmentation of regular polyhedra to form composite polyhedra. Number 1 is in fact the Tetrahedron P1; 2 is half of an octahedron P3; number 4 is a pentagonal cap of the Icosahedron P5; number 5 is octagonal and is a part of the Rhombicuboctahedron P10. Numbers 6 and 7 are both decagonal, where 6 is half of an Icosidodecahedron P12, and 7 is part of a Rhombicosidodecahedron P16. Caps 4 to 7 can be placed in a different position, which makes the number of possible variations very great.
Fig. 5.20. Three figures in which triangular pyramids are added to the truncated cube P9, the Icosidodecahedron P12 and the Truncated Dodecahedron P14.

Fig. 5.21. Square pyramids on Truncated Cuboctahedron P11, Rhombicosidodecahedron P16 and Truncated Icosidodecahedron P17.

Fig. 5.22. Decagonal caps on a Truncated Dodecahedron P14 and two kinds of caps, octagonal and triangular ones, on a Truncated Cube P9. Both the decagonal and the octagonal caps can be placed in two different positions; in the last case two neighbouring square faces are in a direct line with each other.
Fig. 5.23. *Square, hexagonal and decagonal caps on the Truncated Icosidodecahedron P17.*

### 5.8. References


Chapter 6. PRISMS AND ANTIPRISMS

Prismoids form a group of mathematical figures, that have found wide-spread application in many disciplines, but especially in architecture and in building structures. Many of these applications are trivial, but modifications and combinations can lead to a specific form language for this family of forms. Their geometry is based on that of prisms and antiprisms, which have two identical parallel polygonal faces, that are kept apart by a closed ring of squares or of triangles. The two polygons and the square or triangular faces of the mantle enclose a portion of space, that is completely surrounded by regular polygons. They have therefore very much in common with the Platonic and Archimedean - often called 'uniform' - polyhedra. Both groups form endless rows as the parallel polygons can have any number of sides. They are also counted to the convex uniform solids and they were first mentioned and shown in sketch by Kepler in the 16th century [Ref. 6.1]. The present paper deals in detail with these figures and their duals, as well as with similar solids, having polygrams (or star-shaped) parallel faces. Attention will be paid to practical applications in architecture or in engineering of some representants, and particularly to antiprismatic structures. These are concertina-like folded planes, formed by a parallel arrangement of antiprisms. They lend themselves to be adapted to practical and aesthetic demands.

6.1. Geometrical properties

![Rows of prisms (E) and antiprisms (F) with 2 to 8 sides. Note that E3 is identical to a Cube, F2 to a Tetrahedron and F3 to an Octahedron.](image)

The geometrical properties of prisms and antiprisms can be determined in a similar way as is done for the uniform polyhedra. These properties are all laid down in the pyramidal caps that can be cut off from the corners of these solids at a distance of the unit edge length. The basis of such a 'vertex pyramid' is called: 'vertex figure'. This is identical for all vertices of any polyhedron and it is therefore characteristic, as it contains all geometric data that are specific for such a polyhedron (see Fig. 6.3).
6.2. Prisms

A polygonal prism has \( n \) sides and has a triangular vertex figure with the sides: \( \sqrt{2} - \sqrt{2} - b_n \), where \( b_n \) is the so-called 'lesser diagonal' which connects two alternate corners in one of the two parallel regular polygons.

\[
b_n = 2 \cos \varphi_n
\]  \( \{6.1\} \)

Fig. 6.2. Models of a row of prisms with 3, 4, 5, 6, 8 10 and 12 sides.

Fig.6.3. Principal data of prismatic polyhedra.

with \( \varphi_n = \frac{180^\circ}{n} \)  \( \{6.2\} \)

so that the circumscribed circle of a polygon with \( n \) sides is:

\[
(R_2)_n = \frac{1}{2 \sin \varphi_n}
\]  \( \{6.3\} \)

The radius of the inscribed circle:

\[
M_n = (R_2)_n \cos \varphi_n
\]  \( \{6.4\} \)
Fig. 6.4. Models of a row of antiprisms with 3, 4, 5, 6, 8, 10 and 12 sides.

Fig.6.5. Vertex figures of n-sided prisms and antiprisms.

The radius of the circumscribed circle of the vertex figure, that connects the other ends of the edges that meet in the vertex of a prism:
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\[ R_3 = \frac{abc}{4 \times \text{Area}(ABC)} \]
\[ = \frac{b_n \sqrt{2}}{4 \sqrt{(\sqrt{2} + \cos \phi_n) \cos \phi_n \cos \phi_n (\sqrt{2} + \cos \phi_n)}} \]
\[ = \frac{1}{\sqrt{2 - \cos^2 \phi_n}} \]  

\{6.5\}

The radius of the circumscribed sphere of the prism:

\[ R_E = \sqrt{R_3^2 + 0.25} = \sqrt{\frac{1}{4 \sin^2 \phi_n} + 0.25} = \sqrt{\frac{1 + \sin^2 \phi_n}{2 \sin \phi_n}} \]
\[ \{6.6\} \]

The radius of the 'inter-sphere' [Ref. 3]:

\[ R_s = \sqrt{(R_E^2 - 0.25)} \]  
\[ \{6.7\} \]

The inter-sphere connects the mid-points of the edges. The radius of the 'in-sphere', which is the sphere that touches the reciprocal faces:

\[ R_s = \sqrt{(R_s^2 - 0.25R_3^2)} \]  
\[ \{6.8\} \]

\[ \zeta_n = \arcsin \frac{\cos \phi_n}{R_3} \]  
\[ \{6.9\} \]

where \( \chi_n = 180^\circ - 2 \zeta_n \)

---

**Figure 6.6. Faces of the reciprocals or duals.**

\[ g_n = 0.5R_3 \tan \zeta_n \]  
\[ \{6.10\} \]

\[ j_n = \frac{g_n}{\sin \zeta_n} \]  
\[ \{6.11\} \]

The distance of a vertex from the system centre:

\[ (R_j)_n = \sqrt{(R_s^2 + j_n^2)} \]  
\[ \{6.12\} \]
6.3. Antiprisms

The antiprism has a trapezoidal or quadrangular vertex figure with the sides: 1 - 1 - 1 - bₙ. This has a diagonal with the length:

\[ dₙ = \sqrt{1 + 2\cos \varphiₙ} \quad \text{\{6.13\}} \]

The circumscribed circle of this vertex figure:

\[ R₃ = \frac{dₙ}{4 \cdot \text{Area}(BCD)} = \frac{dₙ}{4 \sqrt{(1 + 0.5dₙ)0.5dₙ^2(1 - 0.5dₙ)}} \]
\[ = \frac{1}{2\sqrt{1 - (0.5dₙ)^2}} = \frac{1}{\sqrt{3 - 2\cos \varphiₙ}} \quad \text{\{6.14\}} \]

The radius of the circumscribed sphere:

\[ R_F = \frac{1}{2\sqrt{1 - R₃^2}} = \frac{1}{2\sqrt{1 - \frac{1}{3 - 2\cos \varphiₙ}}} = \frac{\sqrt{3 - 2\cos \varphiₙ}}{2\sqrt{2 - 2\cos \varphiₙ}} \quad \text{\{6.15\}} \]

As \( 1 - \cos 2\varphiₙ = 2\sin^2 \varphiₙ \):

\[ R_F = \frac{\sqrt{3 - 2\cos \varphiₙ}}{4\sin \frac{\varphiₙ}{2}} \quad \text{\{(Cundy e.a. [3])\} \{6.16\}} \]

By substitution of the cosine, the equation is obtained that can be found at Brückner [1.2]:

\[ R_F = \frac{\sqrt{3 - 2 + 4\sin^2 \frac{\varphiₙ}{2}}}{4\sin \frac{\varphiₙ}{2}} = 0.5 \sqrt{1 + \left( \frac{1}{2\sin \frac{\varphiₙ}{2}} \right)^2} \quad \text{\{6.17\}} \]

The values of \( Rₙ, Rₙ^₆, \zeta_n, gₙ, jₙ \) and \( (Rₙ)^7 \) as well as those for \( n=3 \), can correspondingly be calculated with the formulas \( \{6.7\} \) to \( \{6.12\} \). The dihedral angles in an antiprism are a summation of one part which is contributed by the triangle and another part by the \( n \)-gon, thus

\[ \zetaₙ = \arctg \frac{Zₙ}{Mₙ} \quad \text{\{6.18\}} \]

Brückner:

\[ \zetaₙ = \arctg \frac{\sin \frac{\varphiₙ}{2}}{\cos \varphiₙ \sqrt{1 + 2\cos \varphiₙ}} \]
\[ \zeta₃ = \arctg \frac{\sqrt{1 + 2\cos \varphiₙ}}{2\sin \frac{\varphiₙ}{2}} \quad \text{\{6.19\}} \]
\[ \zeta_{\text{total}} = \zetaₙ + \zeta₃ \quad \text{\{6.20\}} \]

The dihedral angles in the reciprocal figure of an antiprism are all alike and equal to:

\[ \Theta = 2\arctg \frac{2R₆}{R₃} \quad \text{\{6.21\}} \]
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Some of the previously derived data are given for a few number of sides in the following tables.

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### Table 6.2. Angles and distances in prisms and antiprisms

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### 6.4 Star-shaped or polygrammatic versions

The two parallel polygonal faces can be substituted by regular stars or polygrams. This produces two new families: star-prisms and star-antiprisms. In the first group a mutual distance, equal to the unit edge length can be chosen, as in the normal prism. The resulting figure has a mantle, consisting of rectangles.

![Fig. 6.7. Row of star polygons or polygrams](image)

Polygrams are stellations of the regular polygons, that are formed by the extension of their sides, until they intersect again. The triangle and the square therefore do not have stellated forms.
The star-antiprisms have a somewhat unexpected appearance [Ref. 3]. At closer examination the forms with even numbers of sides seem to be composed out two antiprisms of half the number of sides, but with an edge length $b_n$ (see Fig. 6.1).

### 6.5. Form generation

Forms like these can be generated in a fully automatic way, requiring the input of only very few parameters. If the number of sides is known, all other geometric data can be computed, using the previous formulas, for any of the four families that were indicated before. The two parallel faces are first generated and placed at a distance $y = +$ or $-z_n$ from the Z-X-plane, with

$$z_n = \sqrt{(R_1^2 - R_2^2)} \quad \text{(see Fig. 6.2)} \quad \{6.22\}$$

With $R_1 = R_e$ or $R_f$

The value of $z_n$ is equal to 0.5 for prisms.

The mantle is formed by rotational reproduction of a square or a triangle. For the formation of a prism, a square is generated in the XY-plane, then translated over the distance $M_n = z_n$ along the Z-axis and finally placed $n$ times under mutual angles of $2\phi$ around the vertical Y-axis. The mantle of an antiprism is formed in two stages. One triangle is placed point-down at a distance $z_3$, under an angle of $-\alpha = 90^\circ - \vartheta/2. = \arccos j_n/(R_2)_n$. Compounds of the original figures and their reciprocals are made with the same routine and using the faces of the reciprocals, upon which small vertex pyramids are placed.
Fig. 6.9. The reciprocal prisms and antiprisms (ER and FR and their compounds (EC and FC) with the original figures. Note that ER4 is an octahedron, FR2 a tetrahedron and FR3 a cube, FC2 is the Stella Octangula as FS4.

Fig. 6.10. Models of prism reciprocals
Fig. 6.11 Models of antiprism reciprocals

Fig. 6.12 Compounds of the prisms and the antiprisms with their reciprocal versions.
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6.6. Prismatic forms

The most simple structural forms are the prismatical shapes. They fit usually well together and they allow the formation of close-packings in many varieties. The accompanying figure shows examples of such applications: matrices can be formed with regular or deformed prisms, parts can be linked up in rows to make cylinders or elongated prisms can serve as the elements of space frames.

Fig. 6.13. Prismatic shapes.

Fig. 6.14. Rectangular frame constructed of tubular prisms.
Fig. 6.15. Dome structure formed from prismatically shaped struts, and the nodes also have the form of shallow prisms. The way in which these elements were formed will be dealt with in greater detail in Chapter 13.

6.7. Antiprismatic folding structures

If a number of antiprisms is put together according to their polygonal faces, a geometry is obtained which is often used as the basis for structural applications—usually in a more or less adapted form. The outer mantle has the appearance of a cylindrical, concertina-like folded plane. It can be described by a combination of three angles: $\alpha$, $\phi_h$, and $\gamma$. The element in Fig. 6.16 represents two adjacent triangles, that in this case are not taken equilateral, so that a more general character is obtained.

Fig. 6.16. Geometric data of antiprismatic structures.

The variables that define the shape of an antiprismatic structural shape are [6.2]:

$\alpha = \frac{1}{2}$ the top angle of the isosceles triangle ABC with height $a$ and base length $2b$. 

\[ a \cdot \cos \gamma \] 

\[ a \cdot \tan \alpha \] 

\[ 2b \] 

\[ 2b \] 

\[ a \] 

\[ a \]
\[ \gamma = \text{half the dihedral angle between the 2 triangles along the basis.} \]
\[ \varphi_n = \text{half the angle under which this basis with the length } 2b \text{ is seen from the cylinder axis.} \]

Fig. 6.17. *Continuous vaults composed of antiprismatic cylinders*

Fig. 6.18. *The relation of } \alpha, \gamma \text{ and } \varphi \text{ for antiprismatic forms.}*

The relation of these angles \( \alpha, \gamma \text{ and } \varphi_n \) [6.2]:

112
b = a \cdot \tan \alpha = a \cdot \cos \gamma \cdot \cot g \varphi_n + \frac{a \cdot \cos \gamma}{\sin \varphi_n}

\tan \alpha = \frac{\cos \gamma (\cos \varphi_n + 1)}{\sin \varphi_n}

\tan \alpha = \cos \gamma \cdot \cot g \frac{\varphi_n}{2}

\{6.23\}

These three parameters define together with the base length (or scale factor \(2b\)) the shape and the dimensions of a section in such a structure. This provides an interesting tool to describe any antiprismatic configuration. Two additional data must be given: the number of elements in transverse direction (\(p\)) and that in length direction (\(q\)).

The fat line in the graph gathers all possible regular antiprisms (see numbers at left side of picture). 'I' is the cylinder of Fig. 6.12 and 'II' that of Fig. 6.13 and following figures.

Fig. 6.19. Antiprismatical form with: \(\alpha = 65^\circ\), \(\varphi = 15^\circ\), and \(\gamma = 73.6^\circ\), \(p = 6\), \(q = 10\). This particular form has been applied by Renzo Piano for the roof of a GRP sulphur factory in Milan [6.4].
Another configuration in the form of a prototype, composed of sandwich panels with a polyurethane core and GRP outer layers, has been investigated by B.S. Benjamin in the Laboratory of Space Structures in Guildford [6.3]. This was based on the data: \( \alpha = 65^\circ, \varphi = 0^\circ, \gamma = 90^\circ, p = 4 \) and \( q = 8 \).

6.8. Manipulation of prismatic and antiprismatic forms

O.L. Tonon describes methods to modify the general shape of such antiprismatically folded planes [6.6]. In the present Chapter this is being worked for radial and triangular transformations, so that toroidal and spherical overall forms can be composed on the basis of polygonal or star-formed prisms and antiprisms. Parts of these can in turn be combined for larger compounds. This procedure can of course be applied to any eligible form.
Fig. 6.23. A) Radial and B) triangular compression of antiprismatic forms

\[ \varphi_1 = \arctan\left(\frac{s}{y_0}\right) \]

\[ \varphi_2 = \frac{x_1}{s} \varphi_1 \]

\[ x_2 = y_1 \sin \varphi_2 = y_1 \sin \left(\frac{x_1}{s} \varphi_1\right) \]

\[ y_2 = y_1 \cos \left(\frac{x_1}{s} \varphi_1\right) \]

\{6.24\}

Fig.6.24. Triangular and radial compression of Piano's cylindrical form.
Fig. 6.25. Triangularly and circularly compressed cylinders put together to form larger structural forms.

Fig. 6.26. Isometric views of the structures in Fig. 6.25.

Fig. 6.27. The structures of Fig. 6.15, but with smaller central opening.
Fig. 6.28. The cylinders can be triangularly compressed in length direction also, either in one direction or even in two directions, and successively combined in the form of circular vaults.
Chapter 06

6.9. Spherical extensions and further modifications

As a next step the whole pattern can be projected upon a sphere. This is done by converting the cartesian co-ordinates into polar co-ordinates and by substituting the direction vector for the radius of the circumsphere. This is demonstrated in Fig. 6.28, where a sector with an antiprismatic subdivision is circularly deformed in two perpendicular directions. This results in a hemisphere and parts of these can be combined with a circular vault in order to form a closed shape, maybe useful as a building envelope.

Fig. 6.29. Antiprismatic sector, of which four specimens are converted to a hemisphere.

Fig. 6.30. Two different ways to split the 8-frequency antiprismatic dome of the previous figure in two equal halves.
Fig. 6.31. Half 6-frequency antiprismatic dome, used for the following pictures.

Fig. 6.32. The hemisphere of Fig. 6.29 can be split and added to both sides of the standard cylinder of Fig. 6.24.

Such combinations of spherical and cylindrical parts can in turn be twisted or bent in several ways. Figure 6.33 shows a circular set-up of such worm-like creatures. The cylindrical part is made to taper in length direction. The cylindrical part in Figs. 6.32 and 6.33 is in fact identical to the one in Fig. 6.24.

Similar things can be done with star shaped prisms or antiprisms. Fig. 6.34 shows such an experiment with triangularly compressed octagonal starprisms, arranged in the form of a torus.
Fig. 6.33. Arrangement of radially deformed cylinders according Figure 19
Fig. 6.34. Torus made of 24 triangularly compressed octagonal starprisms.
Chapter 06

6.10. References


Chapter 7. STELLATED POLYHEDRA

7.1. History

Star polyhedra are found by extending the faces of polyhedra so that they intersect again. There are three regular polyhedra, that lend themselves to this procedure: Octahedron, Dodecahedron and Icosahedron. Sometimes this process can be repeated; with the Icosahedron even as many as 58 times. Many most interesting forms are thus found with amazing characteristics and with a rather complex geometry. Reference must be made here to the work of Magnus Wenninger [http://en.wikipedia.org/wiki/Magnus_Wenninger](http://en.wikipedia.org/wiki/Magnus_Wenninger)

Fig. 7.1. The 5 regular stellations: the Stella Octangula, the Great Dodecahedron, the Great Icosahedron, the Small Stellated Dodecahedron and the Great Stellated Dodecahedron.

Some of the regular, or Platonic, polyhedra also exist in a stellated or star-shaped form. Kepler (1571-1630) was the first who mentioned two regular star-figures, the Small and the Great Stellated Dodecahedron [7.1]. L. Poinset (1777-1859) discovered two others: the Great Dodecahedron and the Great Icosahedron [7.3]. These four star-polyhedra are therefore sometimes called “The Kepler-Poinset Polyhedra”. Some of these star-polyhedra are extremely complicated. At the time of the publication of [7.6] in 1938 the authors Coxeter e.a. did not have the aid of the computer and all their drawings had to be made by hand. The same with [7.8], written in 1979 by the present author. Brückner (1900) shows numerous figures based on model studies [1.1]. Wenninger produced many magnificent but very elaborate material models and shows them in photograph [7.4]. Since then however much has changed with the introduction of the computer. Maeder's home pages [7.7] give a complete overview of all uniform polyhedra and also of the stellated regular polyhedra, illustrated with small coloured pictures.

Fig. 7.2. Eligible forms for stellation, three of the five regular polyhedra: the Octahedron P3, the Dodecahedron P4, the Icosahedron P5, and the two quasi-regular ones: the Cuboctahedron P7 and the Icosidodecahedron P12.

Two semi-regular or Archimedean polyhedra, P7 and P12, can also be stellated. The two that are the most relevant in this respect are the so-called quasi-regular polyhedra: , the cuboctahedron (6 squares + 5 triangles) and the icosidodecahedron (20 triangles + 12 pentagons), which - as their names suggest - can be considered as compilations of either the cube and the octahedron or of the icosahedron and the dodecahedron.
The stellation process becomes much more complicated for these solids, as in one figure two geometries have to be combined. Wenninger explains a procedure for the construction of models of the two quasi-regular polyhedra. This procedure will be worked out in the next paragraphs.

### 7.2. Polygrams or stellated polygons

A polygram or star polygon is a stellated polygon. This means that it is formed from an ordinary regular polygon by 'stellation': the sides are extended until they meet again in pairs.

![Polygrams of polygons](image)

**Fig. 7.3. Polygrams of polygons**

The triangle and the square have no stellated forms, but with the pentagon we find a figure, that is well-known as the ‘pentagram’, to which special metaphysical properties were ascribed by the ancients. This pentagram occurs in many of the stellated solids. Polygrams are considered as genuine polygons as they share one specific property, which is that the sides can be traced in one continuous movement around the centre. The interior points of intersection are disregarded.

The entire geometry of pentagrams is defined by the Golden Ratio: \( \tau = \frac{1 + \sqrt{5}}{2} \) (see Fig.7.4).

**Fig. 7.4. Geometrical properties of the pentagram**

The triangle and the square have no stellated forms, but with the pentagon we find a figure, that is well-known as the ‘pentagram’, to which special metaphysical properties were ascribed by the ancients. This pentagram occurs in many of the stellated solids. Polygrams are considered as genuine polygons as they share one specific property, which is that the sides can be traced in one continuous movement around the centre. The interior points of intersection are disregarded.

The entire geometry of pentagrams is defined by the Golden Ratio: \( \tau = \frac{1 + \sqrt{5}}{2} \) (see Fig.7.4).

### 7.3. The stellation of the uniform polyhedra

If a polyhedron is placed on a flat plane, some of its faces will cut this plane when produced in space. The intersection lines form a pattern, that includes all possible star-forms. Stellation of the tetrahedron and the cube do not lead to new figures, as the extended sides do not intersect. The process becomes interesting for polyhedra with higher numbers of sides and it can in some cases
be carried out repeatedly, so that figures are found of great complexity. In fact all other convex regular polyhedra can be stellated. Of these the octahedron, the dodecahedron and the icosahedron will be discussed in detail.

### 7.4. The regular star solids

The star-solids are technically speaking considered as regular polyhedra. They bring therefore the total number of regular polyhedra, together with the Platonic solids, up to nine. As their names indicate, they are stellations of the dodecahedron and the icosahedron. Of the dodecahedron a total of three stellations are possible, which all are regular. Coxeter a.o. discovered that there are as many as 58 stellations of the icosahedron. Only one of these is completely regular: the Great Icosahedron. But many of the others are also interesting. There is a fifth star-figure, which has many aspects in common with the regular star-polyhedra: the first and only stellation of the octahedron. This was also discovered by Kepler and he called it the "Stella Octangula": the eight-pointed star.

#### 7.4.1. The stellated octahedron

There exists only one stellation of the octahedron and it consists of 24 triangular faces. These triangles are arranged in eight planes: three small triangles form one larger triangle with an open, triangular centre.

![The stellated octahedron or 'Stella Octangula'](image)

The large triangles coincide with the faces of the octahedron and the stellated figure can therefore be generated by the rotation of these planes according the faces of an octahedron at a distance from the centre of:

\[ z = \frac{a}{6} \sqrt{6} \]  \hspace{1cm} \text{(see Brückner²)} \hspace{1cm} \text{(7.1)}

In this figure the large triangles meet in pairs and it has identical vertex situations. This makes it comparable to the other uniform polyhedra, but in fact it is not just one polyhedron but a compound of two intersecting tetrahedra. Therefore it is not considered as a new solid.

### 7.4.2. Stellations of the dodecahedron

If the dodecahedron is placed on a flat plane, its faces if extended form a net of 10 section lines that include two larger pentagrams around the basic pentagon and it thus has three subsequent stellated forms:

1. The Small Stellated Dodecahedron, which can be thought to be constructed by placing pentagonal pyramids on the faces of the basic dodecahedron. It results in 12 interpenetrating pentagrams.
2. The Great Dodecahedron is identical to an icosahedron with the side length \( \phi \) and having triangular dimples.
3. The Great Stellated Dodecahedron can be imagined by placing sharp triangular pyramids on the icosahedron, which results in a compound of 12 larger pentagrams.

They all are considered to be regular. The faces have a distance from the centre of:

\[ z = \frac{a}{2} \sqrt{\frac{25 + 11\sqrt{5}}{10}} \]

\{7.2\}

![Fig. 7.5. The rotation model of the stellated dodecahedra](image)

![Fig. 7.6. The three stellated forms of the Dodecahedron (0): 1) Small Stellated Dodecahedron, 2) Great Dodecahedron and 3) Great Stellated Dodecahedron. The colours indicate the way they are formed.](image)

In Fig. 7.7, the relevant geometric data are given of the three stellations of the pentagon that form the above figures, where \( \tau \) is the golden section, which has a few odd characteristics:

\[ \tau = \frac{1 + \sqrt{5}}{2} = 1.61803399 \]

\[ \tau^2 = \tau + 1 = 2.61803399 \]

\[ \tau^3 = \tau^2 \times \tau = (\tau + 1)\tau = \tau^2 + \tau = 4.23606798 \]

Side length of the great pentagon:

\[ L_{28} - L_{29} = \tau^4 = 6.85410197 \]

The radius of the circle, circumscribed around the points 26 to 30:

\[ R = \frac{\tau^4}{2\sin 36^\circ} = 5.83044738 \]

\{7.3\}
Fig. 7.7. Geometric data of the three stellations of the dodecahedron

The stellation plane of the small stellated dodecahedron ('1' in Fig. 7.6) is formed by 5-fold rotation of part 1-5-16; that of the great dodecahedron (2) is part 1-16-17 and that of the great stellated dodecahedron (3) is part 16-26-17.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>R</th>
</tr>
</thead>
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<td>0</td>
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<td>5.83044738</td>
</tr>
<tr>
<td>27</td>
<td>(5(\tau^2 + 3))/2</td>
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<td>R\sin18^\circ</td>
</tr>
<tr>
<td>28</td>
<td>-(\tau^4 / 2)</td>
<td>-3,42705098</td>
<td>-R\sin54^\circ</td>
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<tr>
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<td>(\tau^4 / 2)</td>
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<td>-R\sin54^\circ</td>
</tr>
<tr>
<td>30</td>
<td>(5(\tau^2 + 3))/2</td>
<td>5,54508497</td>
<td>R\sin18^\circ</td>
</tr>
</tbody>
</table>

Table 7.1. Coordinates of the relevant points in Fig. 7.7.
Chapter 07

7.4.3. Stellations of the icosahedron

Of all 58 possible stellated forms of the icosahedron, only one is regular: the Great Icosahedron. It is a composition of 20 greater triangles with a side length of \( \tau + 2\tau^2 = 3\tau + 2 \) at a distance from the centre of:

\[
z = \frac{a}{12} (3 + \sqrt{5}) \sqrt{3}
\]

The interior part of these triangles is not visible and this part can be constructed by connecting the subdivision points of the sides. The total stellation pattern of the icosahedron consists of 18 section lines: the number of non-parallel faces. Coxeter [7.2] gave indices to the different parts of the triangular subdivision. In Fig. 7.8 these parts are shown with their index number, the left-handed versions are indicated by an accent.

He also formulated five conditions for what he considered as a valid stellation:
1. The faces must lie in the 20 bounding planes of the regular icosahedron.
2. All parts of the faces must be in the same plane, although they may be disconnected.
3. All parts in one plane must have trigonal symmetry.
4. All parts in any plane must be accessible from the outside.
5. Excluded are cases where parts form two sets, each giving a solid that is symmetric to the whole figure.

The various stellations can be indicated by combinations of the faces shown in Fig.7.8. In the further text the numbering, suggested by Maeder [7.7], is followed. He counts as the first stellation the icosahedron itself, consisting of 20 faces with No. 0. Some of the stellations form compounds: of five or ten tetrahedra and of five octahedra (see Figs.7.18, 7.17 and 7.12 resp.).

Fig. 7.8. The great triangular stellation pattern around the basic triangle.
<table>
<thead>
<tr>
<th></th>
<th>( \tau / 2 )</th>
<th>( -\tau / 2 )</th>
<th>(-\tau^2 / 2 )</th>
<th>( \tau^2 / 2 )</th>
<th>((2\tau+1) / 2 )</th>
<th>( \tau^2 + \tau / 2 )</th>
<th>(-\tau^2 + \tau / 2 )</th>
<th>0</th>
</tr>
</thead>
<tbody>
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<td>31</td>
<td>0.80901699</td>
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<td>1.30901699</td>
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<td>3.42705098</td>
<td>-3.42705098</td>
<td>0.00000000</td>
</tr>
<tr>
<td>32</td>
<td>((3\tau + 2) / 2\sqrt{3})</td>
<td>((3\tau + 2) / 2\sqrt{3})</td>
<td>(-3\tau + 1) / 2\sqrt{3})</td>
<td>((3\tau + 1) / 2\sqrt{3})</td>
<td>(-\sqrt{3} / 6)</td>
<td>((3\tau + 2) / 2\sqrt{3})</td>
<td>((3\tau + 2) / 2\sqrt{3})</td>
<td>(\sqrt{3} (3\tau + 2) / 3)</td>
</tr>
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<td>1.97860881</td>
<td>1.97860881</td>
<td>-1.68993367</td>
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<td>-0.28867513</td>
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<td>1.97860881</td>
<td>1.97860881</td>
<td>-3.95721762</td>
</tr>
</tbody>
</table>

Table 7.2. The coordinates of the exterior points

Fig. 7.9. The inner points are found by intersecting lines between the nine exterior points
Chapter 07

7.8. Enumeration of the 59 Icosahedra

Coxeter e.a. developed a basic system for the enumeration of the stellated icosahedron. Maeder however did this later in a more systematic way, and it is therefore that his suggestions are followed in the next caption, where a selection of these is shown. For a full overview of the stellation process is refered to the websites below, where many details and beautiful pictures are published.

http://mathworld.wolfram.com/IcosahedronStellations.html

![Fig. 7.10. Maeder's enumeration system of all 59 Icosahedra (including P5 itself)](image)

In the next part an eligible choice of examples is given from the above table, giving an idea of how the individual stellated polyhedra are formed. All these stars are based on the points in the triangular pattern of Fig. 7.9. Parts of this triangle must be rotated following the faces of the icosahedron (case 5 of Fig. 5.8). As the total face to be rotated has trigonal symmetry, only one third has to be indicated by node numbers. It will automatically be rotated three time in its plane and put at a distance z from the systems centre. The face numbers are following Coxeter and refer to Fig. 7.8.

![Fig. 7.11. Second stellation of the icosahedron, face 1.](image)
Fig. 7.12. 3rd stellation of the icosahedron: compound of 5 octahedra, faces 2 and 2’.

Fig. 7.13. 4th stellation of the icosahedron, faces 3, 4 and 4’.

Fig. 7.14. 5th stellation of the icosahedron, faces 3, 6, 6’ and 7.

Fig. 7.15. 8th stellation of the icosahedron, faces 4, 4’, 5 and 5’.
Fig. 7.16. 9th stellation of the icosahedron, faces 5, 5’, 6, 6’ and 7.

Fig. 7.17. 12th stellation of the icosahedron: compound of 10 tetrahedra, faces 7, 9, 9’, 10 and 10’.

Fig. 7.18. 17th stellation of the icosahedron: compound of 5 tetrahedra, faces 5, 6, 7, 9, and 10.

Fig. 7.19. 22nd stellation of the icosahedron, faces 7, 9, 9’ and 12.
Fig. 7.20. 31st stellation of the icosahedron, faces 3, 6, 6’ and 8.

Fig. 7.21. 34th stellation of the icosahedron, faces 5, 5’, 6, 6’ and 8.

Fig. 7.22. 35th stellation of the icosahedron, faces 8, 9, 9’, 10 and 10’.

Fig. 7.23. 36th stellation of the icosahedron, faces 5, 6, 8, 9, 10.
Fig. 7.24. 39th stellation of the icosahedron, faces 8, 9, 9’ and 12.

Fig. 7.25. 45th stellation of the icosahedron, faces 11, 9, 11’, 9’ and 7. This figure has open spaces in its interior.

It is beyond the scope of this book to show all 59 stellated icosahedra in detail. Therefore we restricted ourselves to show a selection. Most of the icosahedral stars shown have a closed exterior surface, but most others have openings right through the solid. No. 45 of the above figure is the only shown here of these with interior spaces, that are accessible from the outside.

Fig. 7.26. 47th stellation of the icosahedron, faces 10, 10’, 11 and 11’.
Fig. 7.27. 54th stellation of the icosahedron: the Great Icosahedron, faces 11, 11' and 12.

This Great Icosahedron (1) is the most particular one of all 59, as it has 20 intersecting great triangles and is therefore considered as one of the five regular star polyhedra, together with the Stella Octangula, Small Stellated Dodecahedron (3), the Great Dodecahedron and the Great Stellated Dodecahedron.

Fig. 7.28. The five regular stars.
Fig. 7.29. 59th and final stellation of the icosahedron, faces 13, 13' and 14.

The 54th icosahedron is the only regular stellation of the icosahedron: this is an intersection of 20 large triangles. If the stellation pattern is extended beyond the large triangle, the final stellation (No. 59) is found (Fig. 7.29).

Some of the stellated icosahedra have hollow parts, as f.i. the 45th of Fig. 7.25. But in this chapter we have mostly confined ourselves to the solid ones.

Fig. 7.30. A few models of stellated icosahedra, made by Fletcher in collaboration with Coxeter, Du Val and Petrie and that are kept in the library of the Dept. of Pure Mathematics and Mathematical Statistics in Cambridge, where they were photographed by the author.
7.9. The stellation of quasi-regular solids

The same stellation process can be followed for the semi-regular or Archimedean solids. This will be done here according the methodology suggested by Wenninger [7.4], but only for the two so-called quasi-regular polyhedra, the Cuboctahedron (6 squares + 5 triangles) and the Icosidodecahedron (20 triangles + 12 pentagons), which - as their names indicate - can be considered as compilations of either the Cube and the Octahedron or the Icosahedron and the Dodecahedron. They combine in one figure both geometries. The other semi-regular solids can of course also be stellated, but this will not be treated here because of their great complexity.

![Image of stellations of Cuboctahedron and Icosidodecahedron]

**Fig. 7.30. The quasi-regular polyhedra: the Cuboctahedron and the Icosidodecahedron.**

7.10. The cuboctahedron

The stellation of the Cuboctahedron takes place on the hand of two different patterns: a triangular and a square based set of lines. The patterns are again found by placing the solid either on one of its constituting triangles or of its squares. In both cases 12 stellation lines are found. The Cuboctahedron has four stellated forms, where the first stellation expectedly is a compound of the Cube and the Octahedron.

The triangular pattern has to be rotated according the faces of the Octahedron and has a distance from the centre: $z_3 = \frac{\sqrt{6}}{3}$

The orthogonal pattern must be rotated according the faces of the Cube at the distance: $z_4 = \frac{\sqrt{2}}{2}$

![Image of stellations of Cuboctahedron]

**Fig. 7.31. 1st stellation of the Cuboctahedron, leading to a compound of the Cube and the Octahedron.**
Fig. 7.32  Stellation patterns of the Cuboctahedron: A) Square and B) Triangular.
The coordinates of the relevant nodal points of the stellation planes are:

<table>
<thead>
<tr>
<th>Orthogonal pattern</th>
<th>Triangular pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nodes</strong></td>
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</tr>
<tr>
<td>1</td>
<td>b</td>
</tr>
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<td>a</td>
</tr>
<tr>
<td>3</td>
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<td>9</td>
<td>-a</td>
</tr>
<tr>
<td>10</td>
<td>a</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 7.4. Coordinates of stellation planes.**

In Fig. 7.32.A: Orthogonal pattern

\[
\begin{align*}
a &= 0.5\sqrt{2} \\
b &= \sqrt{2} \\
c &= 1.5\sqrt{2} \\
d &= 2\sqrt{2}
\end{align*}
\]

In Fig. 7.32B: Triangular pattern

\[
\begin{align*}
e &= \frac{\sqrt{3}}{3} \\
f &= \frac{5\sqrt{3}}{6} \\
g &= \frac{4\sqrt{3}}{3}
\end{align*}
\]

Fig. 7.33 2\textsuperscript{nd} stellation of the Cuboctahedron

The stellation planes are formed either by 4-fold rotation or by 3-fold rotation of the indicated parts.
7.11. The stellation of the icosidodecahedron

In the case of the icosidodecahedron one stellation plane is formed by 3-fold rotation of the indicated part of the triangle and the other by 5-fold rotation.

The triangular pattern must be rotated according the faces of the Icosahedron (case 5 in chapter 5) at the distance:

\[ z_3 = \frac{(3+\sqrt{5})\sqrt{3}}{6} \]  \{7.4\}

The pentagonal pattern has to be rotated according case 4, the Dodecahedron at the distance:

\[ z_5 = \sqrt{\frac{5+2\sqrt{5}}{5}} \]  \{7.5\}
Fig. 7.36. *The relevant nodal points in the triangular stellation pattern.*

The part indicated in yellow is basically identical to that in Fig. 7.9, but it extends to the larger surrounding triangle. All new points are found by interpolation or by extrapolation from the old points. The numbering in the yellow part is also identical to that in Fig. 7.9.
Fig. 7.37. The pentagonal pattern. The green part is basically the same as that in Fig. 7.7. Here again all other points are found by extending connection lines between old points. Plane 26-54-55-53 is part of the final stellation that will be treated later.

Fig. 7.39. *2nd Stellation of the Icosidodecahedron*

Fig. 7.40. *3rd Stellation of the Icosidodecahedron*
Fig. 7.41. $4^{th}$ Stellation of the Icosidodecahedron.

Fig. 7.41. $5^{th}$ Stellation of the Icosidodecahedron: compound of the Small Stellated Dodecahedron and 5 Octahedra.

Fig. 7.43. $12^{th}$ Stellation of the Icosidodecahedron
Fig. 7.44. 14th Stellation of the Icosidodecahedron

Fig. 7.45. 15th Stellation of the Icosidodecahedron

Fig. 7.46. Stellation pattern of the 19th and final stellation of the Icosidodecahedron
7.12. Final remarks

This programme CORDIN, developed for the computation of polyhedral forms, appeared also well suitable for the production of alphanumeric data and figures of star-solids, as it is based on the rotation of planar objects in Euclidean space. To this purpose the section points of the stellation patterns were calculated and kept in the background of the programme. For each figure the repetitive part of the plane pattern has to be defined by the input of the respective corner numbers. Each part has either 3-, 4- or 5-fold symmetry and is therefore calculated internally by rotation in its plane. As a next step, this pattern is rotated in space at the required distances and according the appropriate rotation cases. Apart from the 45th stellation of the icoshedron in Fig.7.25, of all others only the convex ones were shown. This work gives an impression of the study that the author did in this respect, but [7.4, 7.6, 7.7 and 7.10] show a far more detailed and complete review of all possible stellated forms. This present chapter is not meant to compete here with these, but maybe that it contributes to a better understanding.

7.13. References

Chapter 8. SPHERES

8.1. Introduction

For the further subdivision of spherical surfaces in most cases the icosahedron is used as a starting point, because it consists of twenty equilateral triangles that can be easily covered with a suitable pattern and then projected upon a sphere. This leads to economic kinds of subdivision up to high frequencies and with relatively small numbers of different member lengths. There are two other triangular regular solids that can be used similarly, the tetrahedron and the octahedron, although this is not often done. The geometry of the pattern is generally obtained in the form of Cartesian co-ordinates of the nodes. These must be converted into polar co-ordinates and then made spherical, which means that the radii of all points are equal. In this context special mention must be made of Richard Buckminster "Bucky" Fuller (1895 – 1983), who was very famous for his work in this field. He developed the geodesic dome principle, which in fact had already been invented some 30 years earlier by Dr. Walther Bauersfeld. Carbon molecules known as fullerenes were later named after him. We will come back to that in Chapter 11. He is credited for popularizing this type of structure and upon his directives many huge ultra-light domes have been constructed.

Fig. 8.1. Richard Buckminster Fuller, the spiritual father of spherical subdivisions, as published on the front cover of Times magazine in 1964, and as a post stamp in 2004.
Richard Buckminster Fuller's famous 9 meter diameter Fly's Eye Dome. Circular cupolas are placed on the vertices of a 'pyramidized' dodecahedron. These cupolas are 32 in number: 20 on the corners of the original dodecahedron and 12 in the centres of the pentagons.

Fig. 8.5. The 15 meter diameter Fly's Eye Dome. based on the truncated icosahedron P13. Cupolas on the corners and in the centres of the 12 pentagons and of the 20 hexagons, 92 in total.

Fig. 8.6. Biosphere built in 1967 at the occasion of the world Expo in Montreal, 67 m in diameter and 72 m high. [8.9]


8.2. First frequency subdivision or pyramidization

If a figure is added in the form of a shallow pyramid with a vertex lying on the circumscribed sphere, we prefer to speak of 'pyramidization'. This pyramid can have any of the occurring polygons as its basis. This can in fact also be called: first order subdivision of polyhedral spherical surface. In the next part the various characteristic values are deduced.

Fig. 8.2. The six pyramidized polygons seen from above.
Fig. 8.3. Central subdivision of polygons in the polyhedra P1 to P10, coloured in pink of 1 to 5 and in green for 6 to 18, and their conversion to spherical coordinates, all coloured in purple.

This central subdivision forms in fact a relatively simple method to construct pictures of the pyramidized version of polyhedra, that was mentioned already in Chapter 1.13 and shown in Figs. 1.26 and 1.27.
Fig. 8.4. The first frequency subdivision of the polyhedra P11 to P18.

Height of the pyramid: \( h_n = R_1 - z_n = R_1 - \sqrt{R_1^2 - R_2^2} \) \{8.1\}

Length of inclined edge: \( e_n = \sqrt{h_n^2 + R_2^2} \) \{8.2\}

Height of isosceles triangle: \( h_n = \sqrt{e_n^2 - 0.25} \) \{8.3\}

Basis angle of triangle: \( \lambda = \arctan \left( \frac{e_n}{2\sqrt{e_n^2 - 0.25}} \right) \) \{8.4\}
8.3. Dihedral angles of pyramidal additions

The corners of the shallow pyramid that is erected at each polygon of a polyhedron are all on the circumscribed sphere. For the determination of the dihedral angles along the edges of the triangular sides of this pyramid a general approach can be used, where a the corners of a triangle with the sides $a$, $b$ and $c$ lie on the sphere with the radius $R_1$. Around this triangle a circle can be circumscribed, of which the radius is called here $R_4$ with $N$ as the centre of this circle.

$$m_c = \sqrt{R_4^2 - \left(\frac{c}{2}\right)^2} = \sqrt{R_4^2 - 0.25c^2} \quad \{8.5\}$$

$$y = \sqrt{R_1^2 - R_4^2} \quad \{8.6\}$$

In the shallow pyramid, the triangle is isosceles and the sides are $e_n$, $e_n$ and 1.

Half the sum of the sides: $S = \frac{2e_n + 1}{2} = e_n + 0.5 \quad \{8.7\}$

Area of the triangle: $O = \sqrt{(e_n + 0.5)^2 (e_n - 0.5)} = 0.5\sqrt{e_n^2 - 0.25} \quad \{8.8\}$

Radius of circumscribed circle: $R_4 = \frac{e_n}{4O} \quad \{8.9\}$

$$\psi_i = \arctan \left(\frac{y}{m_i}\right) = \frac{e_n^2}{\sqrt{4e_n^2 - 1}} \quad \{8.10\}$$

Two different dihedral angles occur:

On inclined edge:

$$m_e = \sqrt{R_4^2 - \left(\frac{e_n}{2}\right)^2} = \sqrt{R_4^2 - 0.25e_n^2} \quad \{8.11\}$$

$$\psi_e = \arctan \left(\frac{y}{m_e}\right) \quad \{8.12\}$$

On the unit edge with length=1:

$$m_i = \sqrt{R_4^2 - 0.25} \quad \{8.13\}$$

$$\psi_i = \arctan \left(\frac{y}{m_i}\right) \quad \{8.14\}$$
8.4. Methods of subdivision for the triangular regular polyhedra

Spherical grids are usually generated on the basis of one of the regular polyhedra with triangular faces: the tetrahedron, the octahedron or the icosahedron.

Fig. 8.6. The three triangular, regular polyhedra: tetrahedron (4 faces), octahedron (8 faces) and icosahedron (20 faces).

The originally flat, triangular polyhedron face has to be subdivided in the desired frequency, so that elements are produced of the required maximum or minimum length. This can be done according a number of different methods, each with its own specific advantages [8.6].

Generally two methods are used:

1. Subdivision of the polyhedron edge in equal parts and successive interconnection of corresponding points on opposite edges of the triangular polyhedron face, so that a pattern of regular small triangles is found. This pattern is then projected from the centre onto the sphere.
2. Subdivision of the polyhedron edge in equal parts of the spherical angle under which this edge is seen from the centre, so that in the case of the sphere, equal chords are found. The parts into which the edge is subdivided are no longer equal and, if in this case opposite points are interconnected, the connection lines do therefore not intersect in points but they form small triangular 'windows' [8.1]. The centres of these windows can successively be projected onto the envelope.

Fig. 8.7. Two principally different methods for the subdivision of the polyhedron triangle.

In Chapter 5 is described how the regular as well as the semi-regular polyhedra can be generated by rotating polygons around the centre of the XYZ coordinate system in certain numbers, at certain distances and under certain angles. These polygons are defined by a given set of coordinates for their corners and by a listing of the successive corner numbers, that follow the circumference of the enclosed plane.
Fig. 8.8. Examples of sphere subdivisions according 3 classes, 2 methods and in various frequencies.

Since the formation of polygons can be done analytically and as the rotation data of the various polyhedra can be laid down in integer numbers in arrays, it is possible to carry out such a formation procedure completely automatically and very accurately. 7 data sets are sufficient to describe all different polyhedra. These were previously called 'rotation cases'. The various solids can be generated, requiring only the following data as the input for each of the occurring polygon types:
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n1 = number of edges of polygon  
n3 = number of rotation case  
k(x,y,z) = initial rotation of polygon  
d(x,y,z) = initial translation of polygon

In most cases an initial translation along the Z-axis and/or a rotation around this Z-axis is sufficient for the formation of the various polyhedra. This method can in fact be carried out for objects of any form, if they are described in the proper format, and it is therefore also usable for polygons that are covered by a subdivision pattern or tesselation on their surface.

8.5. Economy of subdivision methods for the triangular polyhedra

In the following paragraph the economy of the 3 main classes and subdivision patterns according the 2 methods of Fig. 8.6 are compared. The different solutions lead to a diversity in the results. In this paragraph six cases are compared on a statistical basis. In each case a specific frequency is chosen in order to make this comparison valuable. These frequencies were based on the angles under which the original polyhedron edges are seen from the centre:

In each case a specific frequency is chosen in order to make this comparison valuable. These frequencies were based on the angles under which the original polyhedron edges are seen from the centre:

<table>
<thead>
<tr>
<th></th>
<th>Angle (°)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Tetrahedron</td>
<td>109.47121950</td>
<td>10</td>
</tr>
<tr>
<td>2. Octahedron</td>
<td>90.00000000</td>
<td>8</td>
</tr>
<tr>
<td>3. Icosahedron</td>
<td>63.43494854</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 8.1. Angles and frequencies per polyhedron edge.

The different solutions lead to a diversity in the results. In this paragraph six cases are compared on a statistical basis: the 3 main classes and two different subdivision patterns according the 2 methods.

Fig. 8.9. Spherical subdivisions on the basis of the 3 triangular solids, according the 2 methods of subdivision
The data that are found can be compared for the various cases on a statistical basis. This is done in table 8.2 for the specimens in Fig. 8.9 in relation to the areas of their triangular faces. In this case a radius=70 was chosen [8.3].

<table>
<thead>
<tr>
<th>Method</th>
<th>Tetrahedron</th>
<th>Octahedron</th>
<th>Icosahedron</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Minimum area</td>
<td>24.96</td>
<td>77.90</td>
<td>49.24</td>
</tr>
<tr>
<td>Maximum area</td>
<td>471.50</td>
<td>274.69</td>
<td>192.88</td>
</tr>
<tr>
<td>Average area</td>
<td>150.43</td>
<td>151.34</td>
<td>118.70</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>106.50</td>
<td>51.80</td>
<td>37.70</td>
</tr>
<tr>
<td>Variation coefficient</td>
<td>70.79</td>
<td>34.22</td>
<td>31.76</td>
</tr>
</tbody>
</table>

Table 8.2. Variation in area for the different methods of subdivision.

8.6. Some more classes of subdivision

The different classes that were discerned by the author are:

1. Tetrahedron
2. Octahedron
3. Icosahedron
4. Similar to 1, 2 and 3 but all vertices on horizontal rings.
5. Intersecting meridians and parallel circles.
6. Schwedler-type domes: similar to group 5 but with either left- or right-handed diagonals.
7. Isozonahedra, sometimes also called lamella domes.
8.4. The truncation of spheres

For practical purposes, parts of the sphere have often to be cut off in order to make it fit on horizontal or against vertical planes. This can be done as demonstrated in Fig. 8.7. A certain value for the angle $\theta$ of the desired truncation plane has to be chosen and an area around it ($+\Delta\theta$ or $-\Delta\theta$). All nodes occurring in this zone have to be transferred to this truncation plane in order to obtain a properly closed lower boundary. The pattern itself can be rotated or translated before the projection upon the sphere takes place. The particular example in question shows the cap of an icosahedron, hence the angle of $\theta = 63.43494892^\circ$.

Fig. 8.12. Truncation of the sphere and adaptation of the lower boundaries.
8.7. Variable connection methods

The 'triangular' polyhedra (tetra-, octa- and icosahedron) have 3-, 4- or 5-axial symmetry (Fig. 8.2). This means that they have also 3, 4 or 5 sectors, if seen on one of their vertices. This number can be changed, which offers another tool for modification of the surface (Ref. 11). A next possibility is, that the interconnections between the nodes of a triangular subdivision pattern are made in a different way. In Fig. 8.13 is shown that thus a new form of tessellation is found reminding of the antiprismatic forms. This aspect can be used in many ways, eventually in combination with variations in the sector number or with parts of spheres.

Fig. 8.13. Different ways of connecting the nodal points in a triangular network: horizontal or vertical.

Fig. 8.14. Hemispheres following the specifications:
   A). Class 4, Freq. 8, 4 sectors, vertical connections and with footline
   B). Class 7, Freq. 12, 48 sectors, horizontal connections and footline

The specification 'footline' is to be given if the lower boundary of the hemisphere must be completed by a surrounding base line.
8.8. Modification by combination or addition

Fig. 8.15. Combination of two parts following different classes in order to reduce the number of struts meeting at the top (compare with Fig. 8.14B).

In the hemisphere of Fig. 8.15B it may be difficult to find a proper solution for the meeting of so many struts in the top. A subdivision according Class 4 has all its nodes on horizontal rings. This has it in common with Class 7. This offers the opportunity to combine parts of both under certain conditions in order to reduce the number of struts that meet in the top.

The hemisphere of Fig. 8.15A is made following these two classes so that the number of meeting struts can be reduced from 16 to 8. The bottom part is done as Class 7 (4 top rows are discarded), frequency \(8, n_1 = n_2 = 2\), number of sectors \(n_3 = n_4 = 32\), horizontal connections and footline. The top part: class 4, Freq. = 8, \(n_3 = n_4 = 8\), 4 bottom rows skipped. This procedure can in fact be done at any height.

Fig. 8.17. Model, made by students of the TU-Delft from electricity pipe. reminding of the structure shown in Fig. 8.15
The structure in Figs. 8.17 and 8.18 is composed of two parts following the specifications:

Bottom part: Class 7, Frequency 8, numbers of sectors $n_3=n_4=30$, 3 toplayers skipped.
Top Part: Class 4, Frequency 8, $n_3=n_4=10$, 5 bottom layers skipped.

The frequency counts for the vertical subdivision in all cases and for the horizontal subdivision in case 4. This subdivision is done following the arc (Method 2 in Fig. 8.7)

Fig. 8.18. Sketch of structure in Fig. 8.17.
8.9. Special sphere forms

The author discovered a surprising phenomenon of a Class 7 subdivision that he thinks may be worth mentioning in this context. If the frequency of subdivision - which is the number of rows in vertical direction - is said to have a certain magnitude and is based on a mutual distance given by equal arcs (as in Fig. 8.3B), and if the number of sectors chosen to be double this number, the projection of this pattern in groundfloor plan has a very special characteristic. In fact an endless row of such forms can be found, but in Fig. 8.19 a few out of the lower part of this series is shown: with 4, 6, 8, 12 and 16 sectors.

Fig 8.19. Five examples of special hemispheres in zonhedral subdivision (class 7): Frequency 4, 6, 8, 12 and 16.
The vertical projections of the hemispheres of Fig. 8.20, show that all nodes can be found by the intersection of circles that have halve the diameter of the sphere itself. The circles indicated in this picture therefore all have the same diameter. The numbers of these circles depend of course on the number of sectors. These hemispheres are characterized by the data: Frequency = 2, 3, 4, 6 or 8, $E_1 = E_2 = 1$, Class = 7, sector numbers: $n_3 = n_4 = 4, 6, 8, 12$, or 16. The row distance along the sphere surface is given by an equal arc subdivision. Meurant [8.5] calls these figures Polygonal or Circular Polar Zonagon Mandalas, leaving away the concentric connections.

Fig. 8.21. Polygonal and circular mandala following Meurant. The centres of the circles lie on a concentric circle of the same diameter and these are called primary points.
8.10. References

[8.8] http://www.youtube.com/watch?v=sVGXpc5EIUc
[8.9] http://www.youtube.com/watch?v=8Ibk9ZI0Z 4
Chapter 9. ELLIPSOIDS

9.1. Ellipsoidal transformation

According to Kenner [9.7] the equation of the sphere can be transformed into a set of two expressions, describing it in a more general way:

\[ R_1 = E_1 / (E_1^{n_1} \sin^{n_1} \phi + \cos^{n_1} \phi)^{1/n_1} \]  \{9.1\}

\[ R_2 = R_1 \cdot E_2 / (E_2^{n_2} \sin^{n_2} \theta + R_1^{n_2} \cos^{n_2} \theta)^{1/n_2} \]  \{9.2\}

From \{9.1\} and \{9.2\} follow the main form variables. The horizontal axes are equal to 1 and \( E_1 \), the vertical axis is \( E_2 \), the horizontal exponent to \( n_1 \) and finally the vertical exponent to \( n_2 \). The input variables are:

1). With respect to the form,
   - \( E_1 \) = ratio of the axes of the ground ellipse
   - \( E_2 \) = ratio of the axes of the vertical ellipse
   - \( n_1 \) = exponent of the ground ellipse
   - \( n_2 \) = exponent of the vertical ellipse
   - Factor = magnification ratio

2). With respect to the subdivision pattern,
   - Class = 1 to 7 (0 = input of arbitrary figure) \( \rightarrow \) see Fig. 8.10.
   - Frequency = 1 to 50
   - Method = 1 (= edge) or 2 (= chord) \( \rightarrow \) see Fig. 8.3
   - \( F_{corr} \) = optional redistribution factor for extreme values of \( E \)
   - \( n_3 \) = number of sectors in ground plan \( \rightarrow \) see Fig. 8.14
   - \( n_4 \) = number of sectors to be actually calculated
   - Figtype = horizontal (1=geodesic) or vertical (2=antiprismatic) connection lines in two adjacent triangles \( \rightarrow \) see Fig. 8.13.

Fig. 9.1. General form of ellipse and ellipsoid.
These variables offer many alternatives to influence the shape of the sphere, leading to a number of transformations. The curvature is a pure ellipse for $n = 2$ but if $n$ is raised a form is found, which approximates the circumscribed rectangle. If $n$ is decreased, the curvature is flattened until $n = 1$ and the ellipse then has the form of a pure rhombus with straight sides, connecting the maximal values on the co-ordinate axes. For $n < 1$ the curvature becomes concave and obtains a shape, reminiscing a hyperbola. For $n = 0$ the figure coincides completely with the X- and Y-axes. By changing the value of both the horizontal and the vertical exponent the visual appearance of a hemispherical shape can be altered considerably.

**Fig. 9.2. The influence of the variation in the factor $n$ on the shape of the ellipse**

**Fig. 9.3. Combinations of pairs of ellipses with different values of $n$. The diagonal line indicates the variants that have the same $n$-values in both directions**
The pure sphere forms in fact only one specific representative out of a great number of possible shapes that are formed by a combination of different horizontal and vertical ellipses. Some of these do not even remind of the original convex ellipsoidal shape, yet are very familiar such as the pyramid, the cone, the cylinder, the cube, etc. The subdivision of the surface of such an ellipsoidal shape may be based on the same methods as previously described. The thus formed pattern is projected onto the surface of the ellipsoid from the inside, using the centre of the system as the projection centre.

Fig. 9.4. Schematic review of possible ellipsoidal forms with different values for the exponents of the horizontal ellipse ($n_1$) and of the vertical ellipse ($n_2$).

Fig. 9.5. Conversions of spheres using different variables, and subdivision patterns explained in Chapter 10 (see also Fig. 10.27 on page 197)
Chapter 09

9.2. Playing with sectors

The 'triangular' polyhedra (tetra-, octa- and icosahedron) have 3-, 4- or 5-axial symmetry (Fig. 8.8). This means that they have also 3, 4 or 5 sectors, if seen on one of their vertices. This number can be changed, which offers another tool for modification of the surface [9.8].

Fig. 9.5. A complex spherical object, making use of the various modification methods described in the foregoing.

It consists of the following parts, which are in fact all spherical elements either three-dimensional or flat:

- 2 hemi-spheres, factor 1 and 5
- 2 quarter spheres, factor 1 and 5
- 2 hemi-circles, outer diameter 5
- 4 eights of spheres of 1 unit thickness and outer diameters

Geometrical data of the spheres: \( n_1 = n_2 = 1 \), frequency = 5, \( E_1 = E_2 = 1 \), factor is 1 or 5, \( n_3 = 4 \), method = 2, class = 5.

Fig. 9.6. Combinations of sphere sectors and flat parts of spheres with \( E = 0 \) [9.8].
Fig. 9.7. *The constituent parts of the spherical form in Fig. 9.5.*

The flat circles that are actually also spheres have: $n_1 = 2$, $n_2 = 1$, $E_1 = 1$, $E_2 = 0.00000001$ (not quite zero to prevent the calculation from being derailed), frequency = 5, $n_3 = 32$. method = 1, class = 5, correction factor = 0.00000001, scale factor = 5. The choice of this correction factor leads to concentric circles in the ground floor plan. For the hemi-circle 16 circles were executed, with 1 top layer eliminated. For the quarter circles 8 sectors were made and either one or more top or bottom layers were discarded.

Fig. 9.8. *The end result*
9.3. Special effects

With the use of the various tools described in the previous pages many interesting sphere based shapes can be described and actually formed. The below figure is such an example, which is the result of a graduation project from a Civil Engineering student.

Fig. 9.9. Cigar-like structure composed of a cylindrical part (consisting of prisms) following the procedure described in Chapter 6 and crowned with 2 hemispheres following Class 5. In the end the whole configuration is radially compressed as is explained in Chapter 6, Fig. 6.23B on page 115. This is the subject of a project by a Civil Engineering student.

As shown in Chapter 8 the interconnections between the nodes of a triangular subdivision pattern (Class 2) are made such that a form of tesselation is found, very much reminding of the antiprismatic forms. This aspect can be used in many ways, eventually in combination with variations in the sector number (Fig. 9.10).

Fig. 9.10. Ellipsoids with extreme forms with variations in 'E' or exponent 'n' of ground ellipse or vertical ellipse.
Fig. 9.11. Low frequency ellipsoidal 'antiprismatic' spheres, forming folded circular shells, just by varying the exponents of the horizontal and vertical ellipses as well as the number of subdivisions and of the sectors.

Fig. 9.12. A few antiprismatically subdivided spheroids with different values for the exponents $e_1$ and $e_2$. 
9.5. Hemispherical additions to basic polyhedra

Fig. 9.13. P13 covered with 20 hexagonal and 12 pentagonal hemispheres pointing outward or inward.

In the previous figure the pentagonal and hexagonal faces of P13 are replaced by ditto spherical caps. For a greater uniformity they are initially computed as hemispheres but reduced later in height to keep them at the same height. This formation is done in a number of successive steps, as these additional forms can not be computed by merely using one of the previous classes. The process can be started from a class 1 or 4 subdivision, the octahedron. The spherical cap of Fig 9.14 is generated with the data: frequency of subdivision $F = 6$, $n_1 = 1$, $n_2 = 2$, $E_1 = E_2 = 1$, method $= 2$, $n_3 = 4$, $n_4 = 4$. These data produce a square cap with straight sides, as $n_1 = 1$ and with a radius $= 1$ (unit edge length, so that the diagonal in ground floor plan $= 2$ and the side length $= \sqrt{2}$.

Fig. 9.14 Square spherical cap.

For the formation of a pentagonal cap or a hexagonal cap, only one sector of the square cap is calculated: $n_4 = 1$. In order to build these caps with 5 or with 6 sides, the unit side length has to be reduced in X direction with the factors:

$F_{x5} = \tan 36^\circ = 0.72654253$

$F_{x6} = \tan 30^\circ = 0.57735027$
Fig. 9.15. One quarter sector of the square cap and 1/5th or 1/6th of the two other hemispheres.

In order to make all side lengths = 1, these sectors have to be reduced in X direction by the factors above but after this all of them must be multiplied in all directions with the factors \( F_{XYZ} = 1/\sqrt{2} = 0.70710678 \), \( \tan 36^\circ/\sqrt{2} = 0.51374315 \) and \( \tan 30^\circ/\sqrt{2} = 0.40824829 \).

Fig. 9.16. The same sectors after multiplication which makes

Fig. 9.17. Schematic representation of the three sectors.

The complete caps are found by rotating the thus found triangular sections around the centre: 4, 5 or 6 times respectively.
Fig. 9.18. *The pentagonal cap and the hexagonal cap in plan and in front view.*

Fig. 9.19. *The pentagonal and the hexagonal cap in isometric projection.*

To all caps can be given the same height, as was the aim in the composed forms of Fig. 9.13, the Y value must in the end be multiplied in Z-direction by the factors $F_{x5} = \tan 36^\circ = 0.72654253$ and $F_{x6} = \tan 30^\circ = 0.57735027$. 
Fig. 9.20. Spikey caps on faces of a Truncated Icosahedron, directed outwards or inwards. In the 3-dimensional Fig. 16.35 (p.338) the left figure is crowned with small spheres on the tops of the spikes.

Fig. 9.21. Hemispherical caps of frequency 6 and with $E_2 = 0.5$, placed in the previous figure on the faces of a Truncated Icosahedron.

Fig. 9.22. Basic low exponent sphere with $E_1 = E_2 = 0.3$, frequency = 2, placed here in a row
Fig. 9.23. A number of ‘spikeys’ in ocahedral arrangement.

9.24. The same items in a cubical arrangement
9.6. A practical example

The apparent elasticity in the combinatory sense of the horizontal and vertical cross-sections of such 'pseudo-hemispherical' structures offers the opportunity to follow the outlines of the required space more closely than pure spheres do [9.3]. Apart from the exponent there is still the ratio of the axes that may be varied. In order to demonstrate the potentials of the optimization procedure, that this method offers, a practical example has been worked out. It was assumed that a rectangular prismatic space form, having dimensions of 40 x 60 x 60 units length, has to be enclosed. Point P of this prism has the polar coordinates \( R \), with the angle \( \theta \) in the horizontal plane and the angle \( \phi \) in the vertical plane. \( R \), \( \theta \) and \( \phi \) can be derived from Fig. 9.23.

\[
R = \sqrt{(20^2 + 30^2 + 60^2)} = 70 \tag{9.3}
\]

\[
\theta = \arctan \left( \frac{\sqrt{20^2 + 30^2}}{60} \right) = 31.00271913^\circ \tag{9.4}
\]

\[
\phi = \arctan \left( \frac{20}{30} \right) = 33.69006753^\circ \tag{9.5}
\]

with \( d = \sqrt{20^2 + 30^2} = 36.05551275 \) and \( E_3 = \frac{E_2}{E_1} \) of which the latter represents the ratio of the axes of the vertical ellipse passing through point P \((X,Y,Z) = P(30,20,60)\). For a start this ratio was chosen here as \( E_3 = 60/d = 1.66410059 \).

In further approximations this value may be varied as well. A number of ellipses passing through the point P with different values of the exponent is shown in Fig. 9.26. The most characteristic values are gathered in table 9.1, where

\[
F_v * R_1 = R_p /k = 70/k, \quad \tag{9.6}
\]

\[
k = E_3 / \left( E_3^{n_2} \sin^{n_2} \theta + \cos^{n_2} \theta \right)^{1/n_2}, \quad \tag{9.7}
\]

\[
F_v * E_3 * R_1 = F_v * E_2 \quad \tag{9.8}
\]
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Table 9.1. Characteristic values of ellipses through point $P$

<table>
<thead>
<tr>
<th>$N_2$</th>
<th>$E_{n}^{n^2}$</th>
<th>$k$</th>
<th>$F_v * R_1$</th>
<th>$F_v * E_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>0.0000</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2900</td>
<td>0.4854</td>
<td>144.2221</td>
<td>240.0000</td>
</tr>
<tr>
<td>1</td>
<td>1.6641</td>
<td>0.9707</td>
<td>72.1110</td>
<td>120.0000</td>
</tr>
<tr>
<td>2</td>
<td>2.7692</td>
<td>1.3728</td>
<td>50.9902</td>
<td>84.8528</td>
</tr>
<tr>
<td>2.5</td>
<td>3.5723</td>
<td>1.4713</td>
<td>47.5755</td>
<td>79.1705</td>
</tr>
<tr>
<td>10</td>
<td>162.8530</td>
<td>1.8114</td>
<td>38.6433</td>
<td>64.3064</td>
</tr>
<tr>
<td>100</td>
<td>1.31E+22</td>
<td>1.9280</td>
<td>36.3063</td>
<td>60.4173</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\infty$</td>
<td>1.9415</td>
<td>36.0555</td>
<td>60.0000</td>
</tr>
</tbody>
</table>

Fig.9.26. Ellipses of varying exponent through point $P$

Similar exercises as with the ellipse in one plane can be made in the third dimension. A number of possible shapes, all of them touching the space prism to be enclosed in its outer most corner has been worked out in Table 9.2 and shown in Figs. 9.27 and 9.28. The table gathers some physical magnitudes of the thus found shapes also: the covered floor area, the surface area of the dome and its volume. These data give information, that might be used in further evaluation phases.
Table 9.2. Data of ellipsoidal shapes passing through point P in Fig. 9.26 according Class 2, Method 1. It is remarkable that the volume of No. 11 appears to be smaller than that of the inscribed prism, which is 14400. This also counts for the area, which is 16800 (see Fig. 9.28-11). This is due to the triangular subdivision of the ellipsoidal surface, which at the edges often gives connections behind this surface.

<table>
<thead>
<tr>
<th>No</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>Factor $F_v$</th>
<th>Floor Area</th>
<th>Dome Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1.0</td>
<td></td>
<td>1.6641</td>
<td>50.9902</td>
<td>8110</td>
<td>23548</td>
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<tr>
<td>2</td>
<td>2</td>
<td>2.5</td>
<td>1.0</td>
<td></td>
<td>1.6641</td>
<td>47.5755</td>
<td>7097</td>
<td>22267</td>
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<tr>
<td>3</td>
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<td></td>
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<td>38.6433</td>
<td>4827</td>
<td>18255</td>
</tr>
<tr>
<td>4</td>
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<td>100</td>
<td>1.0</td>
<td></td>
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<td>36.3063</td>
<td>4320</td>
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<td>1.5</td>
<td></td>
<td>2.1213</td>
<td>37.3213</td>
<td>6547</td>
<td>21764</td>
</tr>
<tr>
<td>8</td>
<td>2.5</td>
<td>2.5</td>
<td>1.5</td>
<td></td>
<td>2.2736</td>
<td>34.8220</td>
<td>6144</td>
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</tr>
<tr>
<td>9</td>
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<td>10</td>
<td>1.5</td>
<td></td>
<td>2.1213</td>
<td>30.3143</td>
<td>4459</td>
<td>17806</td>
</tr>
<tr>
<td>10</td>
<td>2.5</td>
<td>10</td>
<td>1.5</td>
<td></td>
<td>2.2736</td>
<td>28.2843</td>
<td>4243</td>
<td>1137</td>
</tr>
<tr>
<td>11</td>
<td>100</td>
<td>100</td>
<td>1.5</td>
<td></td>
<td>2.9793</td>
<td>20.2792</td>
<td>2724</td>
<td>12725</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>2</td>
<td>1.0</td>
<td></td>
<td>1.0000</td>
<td>70.0000</td>
<td>15284</td>
<td>30389</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>1</td>
<td>1.0</td>
<td></td>
<td>1.0000</td>
<td>103.3542</td>
<td>21364</td>
<td>37004</td>
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<td>14</td>
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<td>30.2087</td>
<td>3863</td>
<td>10174</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>2</td>
<td>1.5</td>
<td></td>
<td>2.1213</td>
<td>40.0000</td>
<td>7482</td>
<td>23024</td>
</tr>
</tbody>
</table>

Fig.9.27. A number of shapes generated around the prismatic shape of 20 x 30 x 60 units.
The figures are reproduced in the same scale to make them comparable.

Fig. 9.28. Another group of ellipsoidal approximations.
9.7. Distribution of the nodal points

The subdivision of the surface of such an ellipsoidal shape may be based on the same Methods and Classes as previously described. The thus formed pattern is projected onto the surface of the ellipsoid from the inside, using the origin as the projection centre. However, when the ellipsoid has a very much distorted shape, for instance if large values of $E$ are used, in that case a correspondingly uneven distribution of the nodal points is found. This is demonstrated in Table 9.3 and Fig. 9.29 for three ellipsoids of revolution with $E_2 = 1.6641006$ and subdivided according Class 2 and Method 2. In the second and third case a Correction Factor is introduced. This changes the value of $\vartheta$ to $\vartheta'$:

$$\vartheta' = \arctan \left( \tan(\vartheta) \frac{1}{F_{\text{corr}}} \right)$$  \{9.9\}

A value for $F_{\text{corr}}$ of $\sqrt{E_2}$ gives good results. Their effect is compared in Table 9.4 and Fig.9.29 for dome No. 1 in Table 9.3.

<table>
<thead>
<tr>
<th>Correction Factor</th>
<th>(1) $E_2$</th>
<th>(2) $1.6641006$</th>
<th>(3) $1.2900002$</th>
</tr>
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<tbody>
<tr>
<td>Minimum area</td>
<td>50.36</td>
<td>50.79</td>
<td>64.70</td>
</tr>
<tr>
<td>Maximum area</td>
<td>149.76</td>
<td>117.13</td>
<td>115.88</td>
</tr>
<tr>
<td>Average area</td>
<td>91.90</td>
<td>92.31</td>
<td>92.20</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>30.67</td>
<td>15.53</td>
<td>13.59</td>
</tr>
<tr>
<td>Largest difference</td>
<td>99.30</td>
<td>66.34</td>
<td>51.19</td>
</tr>
<tr>
<td>Variation coefficient</td>
<td>33.37</td>
<td>16.83</td>
<td>14.74</td>
</tr>
</tbody>
</table>

Table 9.3. Area compensation of ellipsoid (with $E_1 = 1$, $E_2 = 1.6641006$, $n_1 = n_2 = 2$, Method = 2, Frequency = 8) by the use of the correction factors $E_2$ and $\sqrt{E_2}$.

Fig.9.29. The effect on the subdivision pattern of the correction filters 1, 1.6641006 and 1.2900002.

9.8. References


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Fig.9.30. The Thinker of Rodin puzzles what in this context is the real shape of Mother Earth. Is this just a matter of exponents? (Picture by unknown artist)
Chapter 10. POLYHEDRAL PATTERNS

10.1. The radial projection of patterns on the sphere surface

In Chapter 5 a rotation procedure is described enabling the automatic generation of polyhedra, given a few general characteristics. With the same technique any pattern can be projected upon the faces of such a polyhedron and eventually be converted into spherical co-ordinates. The original Cartesian coordinates, that are found as a result of the previously described rotation procedure have to be converted into polar coordinates and to be centrally projected upon the sphere.

In Fig. 10.1:

\[
\begin{align*}
    z &= R \cos \theta \\
    x &= R \sin \theta \sin \varphi \\
    y &= R \sin \theta \cos \varphi
\end{align*}
\]

so that

\[
R = 1/ \sqrt{\sin^2 \theta \sin^2 \varphi + \sin^2 \theta \cos^2 \varphi + \cos^2 \theta}
\]

Fig. 10.1. Conversion of Cartesian coordinates into polar coordinates

This leads to interesting new configurations. Schattschneider and Walker constructed 3-D models of M.C. Esscher’s tessellations [10.5], but this can be done by computer as well, and maybe with less effort. To this purpose, bitmaps – or rather vector files – must be rotated in space. The pattern must not necessarily be flat, but can even have a three-dimensional structure. Some eligible examples of this technique will be shown.

10.2. Sphere subdivisions on the basis of polyhedra

Previously it was mentioned that the polyhedra, being considered here, are composed of polygons with 3, 4, 5, 6, 8 or 10 sides. Kitrick, [10.7] discerns mainly three so-called Classes of subdivision of the triangular faces:

- Class I is the basic subdivision type for triangles and is generally used in combination with co-sahedra (only rarely with octahedra)
- Class II is in fact reciprocal (the rhombic triacontahedron)
- Class III is the snub dodecahedron type. In this chapter a number of specific subdivision types has been worked out for the different polygons, more or less on the analogy of the general concept suggested by Kitrick.
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Fig. 10.2. The three subdivision classes of the spherical triangle.

Class I, radial type:
The triangle is taken as the starting point. This can be subdivided into smaller triangles. 2 basically different methods are in use: edge based and arch based [10.8, 10.9]. In Chapter 8 these are called Method 1 and 2 and it has been worked out for the tetrahedral, the octahedral and icosahedral geometry. In this chapter for simplicity reasons only the edge based method has been used. The polygon is thus first subdivided in its plane and then each intersection is projected radially onto the sphere. In the next part, by way of example, triangular patterns with frequencies of 3 or 6 are used and also a particular hexagonal pattern. This, basically equilateral, triangle can be shifted, transformed and subsequently reproduced by rotation around the axis perpendicular to its plane in order to fill the radial patterns of the polygons with more than 3 sides.

Fig. 10.3. Class I type subdivisions of polygons

Class II, parallel type:
The square can be subdivided rectangularly, f.i. in smaller squares. They can eventually be provided with diagonals. A further subdivision of the polygon is found by placing in the middle another polygon but with half the number of sides and connecting this to alternate edges by rectangles. The remaining parts are triangular. The triangular and the rectangular parts can be filled in with transformations of the regular versions. The basic polygon must have an even number of sides. The square is trivial and this leaves only the polygons with 6, 8, and 10 sides having a triangle, a square or a pentagon in the centre.

Fig. 10.4. Class II type subdivisions of polygons

Class III, chiral type:
This is comparable to Class II but with the central - smaller - polygon slightly rotated over the angle $\pi/n$ ($n =$ number of the sides). The remaining part can be made up out of smaller triangles, which can again be subdivided subsequently. If seen from above this type reminisces very much of the snub polyhedra. This is the reason why it also here has been called 'chiral'. Polygons with 4, 6, 8 or 10 sides are suitable for this type, but the square leads to a somewhat trivial solution.
Spherical grids are usually generated as in Chapter 8 on the basis of a regular polyhedral pattern with triangular faces: the tetrahedron, the octahedron or the icosahedron. The original polyhedron triangle is subdivided by a three-way grid into smaller triangles to the required frequency, so that elements are produced of a given maximum or minimum length.
Fig. 10.7. Spherical transformation of Class I subdivision patterns, frequency 1 of the polyhedra P1 to P10. This also demonstrates a method to visualize the pyramidization process as described in Chapter 1.5.

The polygons can at first be centrally subdivided as in Fig. 10.3 and then subsequently rotated to form polyhedra as is shown in Figs. 10.7 and 10.8: the pink and the green forms.
Fig. 10.8. Spherical transformation, Class I and frequency 1, of the ten other polyhedra: P11 to P18R
10.3. The formation of patterns

These patterns can be formed with the help of orthogonal basic grids, of which the nodes are numbered systematically. The nodes of square panels are numbered as in Fig. 10.9. The faces can thus be subdivided in rectangles, eventually with additional diagonals.

![Diagram](image)

Fig. 10.9. Generating arbitrary figures on a square pattern. This arrow for instance was used to indicate in Figs. 5.4 to 5.10 on page 89 to 92 the direction of the planes in the 7 different rotation cases.

In other cases the squares are firstly subdivided in four parts according their great diagonals. For the further subdivision, a triangular pattern can be generated upon which a triangular partitioning is defined by consecutive numbering.

This also is of use for the formation of hexagonal patterns.
Fig. 10.10. *Numbering scheme in a triangular grid. In this case it is used for the formation of a hexagonal pattern.*

Such a hexagonal pattern can also be composed of singular elements. This is demonstrated in Fig. 10.10 for half a regular triangle. This can in its turn be used to also form parts of squares or the sides of the dodecahedron in Fig. 10.11 or of rhombic dodecahedra as in Fig. 10.21.

Fig. 10.11. *Dodecahedron with hexagonal subdivision.*
Fig. 10.12. Generating hexagonal patterns for the subdivision of regular triangles

Fig. 10.13. Vertical compression in the ratio \( s = \frac{1}{\sqrt{3}} \) to fit in a square sector
Fig. 10.14. Hexagonal subdivision of the cuboctahedron $P7$ and its successive conversion to spherical coordinates.

Fig. 10.15. Compression of the regular hexagonal pattern in vertical direction in the ratio \( \frac{\cotan(36^\circ)}{2} = 0.68819096 \) to fit in a regular pentagon as the starting point for the further subdivision of spheres so as for instance the dodecahedron of Fig. 10.11 or the cuboctahedron of 10.14.

All other regular and semi-regular solids, their reciprocals and even the regular prisms and antiprisms can be treated this way, their faces being subdivided in an appropriate way. The Cartesian co-ordinates, that are found as a result of the rotation process have to be converted into polar co-ordinates and to be projected from the centre upon the sphere.

The following Figs. 10.16, 10.17 en 10.18 are black and white pictures from the paper that the author wrote with Gerrit van der Ende for a conference in Milan 1995 [10.4]. They were meant to picture some possible sphere subdivisions based on the described procedure. But here is the place to reproduce some of these in colour and in a larger scale.
The possibilities are in fact numerous and it is difficult to make a representative choice. The above figure only gives one option for each of the regular polyhedra. The same is done in the next two figures, but from Fig. 10.19 on some more pattern types are shown for the more complex polyhedra.

Fig. 10.16. *The tesselated regular polyhedra and their radial projections on the sphere*
Fig. 10.17. Review of a number of tesselated semi-regular polyhedra
Fig. 10.18. A number of polygonal subdivisions of the larger polyhedra
Fig. 10.19. Two different patterns on the surface of the Truncated Tetrahedron P8.

Fig. 10.20. As shown before (see fig. 10.6), any polyhedron can be subdivided in many ways and successively converted into a sphere. This figure shows some of the ways in which the truncated cube P9 can be subdivided. The possibilities in variation thus are numerous.

Fig. 10.21. Hexagonal subdivision of R12, the dual of the Icosidodecahedron P12 (See also Fig.
Fig. 10.22. Hexagonal subdivision of P13, the Truncated Tetrahedron
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Fig. 10.23. Triangular subdivision of the snub Dodecahedron P18: a very regular pattern.

Fig. 10.24. Triangular subdivision in the frequency 6 of the Truncated Icosidodecahedron P17, producing a very complex pattern.

10.4. Reciprocal subdivisions

The dual or reciprocal versions of the polyhedra are also applicable for further subdivision. This is easily understandable for those, that have a triangular composition.

Two of these are most interesting in this respect: the Rhombic Dodecahedron R7 and the Triaccontahedron R12 (Fig. 10.21). R3 (or the Cube) which can be considered as the dual of the Octahedron also belongs to this category of rhombic polyhedra by analogy. Thus there are three reciprocals that act as the counterparts of the 3 triangular regular polyhedra.
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Fig. 10.25. *Polyhedral and spherical transformation of reciprocal patterns.*

10.5. Prismatic patterns

The prisms and antiprisms may in their turn also serve as the starting point for a further subdivision of spherical surfaces. To that purpose all faces are covered with a pattern, that is in the examples of Fig. 10.9 composed of triangles. This leads for the n-gonal parallel faces to a somewhat uneven distribution of the newly found points on the circumscribed sphere. This can be improved by adding n-gonal pyramids to these before the projection takes place.

Fig. 10.26. *Sphere subdivisions on the basis of prisms or antiprisms. At right: the pentagonal antiprism with pyramidal caps is in fact identical to the icosahedron.*
Fig. 10.27. A few ellipsoids with subdivision patterns, that are based on some of the semi-regular solids.

10.6. Star-shaped figures

Fig. 10.28 shows the projection of the so-called Dodecadodecahedron on the sphere. This figure is based on twelve pentagrams that meet at their corners and thus form a pattern of 10 continuous circles around the sphere. This pattern is nowadays used as the logo of the European Champions League of soccer ball. But this logo itself does not properly fit on a spherical surface. It is therefore that this dodecadodecahedron pattern for the soccer balls that are used in this competition.

Fig. 10.28. The Dodecadodecahedron in its various components: 12 pentagons and 60 parallelograms. This is not a regular stellation, but it is produced similarly [10.9, p.112].
Fig. 10.29. The spherical transformation of the dodecadodecahedron (at left) and at right: the logo which is in use for the European Champions League. Note that alternatively 3 or 4 pentagons are interconnected here. Although suggested otherwise, this logo is insofar wrong that this configuration does not lead to a closed pattern on the sphere.

Fig. 10.30 The Dodecadodecahedron in various projections.

Fig. 10.31. Projection of the dodecadodecahedron on a football and the same with the small dritrigonal icosidodecahedron [1.9, p.106]
10.7. Projections of variable patterns

As already suggested in Fig. 10.8 a routine has been developed to enable a quick generation - of more or less - arbitrary surface patterns, based on a two-way, or square grid of orthogonal system lines with numbers for each crossing. Patterns can be defined by a sequence of numbers following the boundaries of the planes to be described. This is applicable in many situations, as it represents a rectangular planar co-ordinate system. If this grid is fine enough, the chosen surface pattern can be approximated closely. It was first used for the definition of planes in the form of characters and ciphers. Fig. 10.32 shows the formation of such a character on a standard grid. The input is done in the form of a list of ciphers indicating the vertex points. Fig. 10.34 gives the logo of the Structural Morphology Working Group, composed this way. The text was firstly projected upon a cube and then made spherical. To make it follow the sphere curvature more closely, all straight lines had at first been subdivided in a number of equal parts.

Fig. 10.32. The formation of characters on a square grid.

Fig. 10.33. The text 'POLYHEDROIDS' shows, that to the characters also a certain thickness can be given and that it can eventually be pictured in perspective.
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Fig. 10.34. *The SMG-logo, projected on the faces of a cube and later converted into a sphere.*

10.8. *Escher patterns*

Even more complex patterns can be generated this way and projected on a polyhedron surface. Schattschneider and Wallace show some of these in the form of cardboard models [10.6]. Suitable patterns derived from Escher’s paintings were scanned and printed on triangular and square or rhombic faces, forming a polyhedron. In the present paper one of these has been translated by projecting a three-way or triangular grid upon the original picture. Only one third part has to be described in detail; the other parts are found by rotating this in its plane around the centre of the triangle. Once the triangular pattern is defined, it can be rotated in space according the tetrahedron (Case 1), the octahedron (Case 3) or the icosahedron (Case 5). It is interesting to see, that the pattern always fits and that the lizards in this picture appear undisturbed all around any of these three different polyhedra. This provides a comfortable way to produce Escher patterns on a polyhedron, without the necessity to build material models. They can also be converted into spherical variants (Fig. 10.39).
Fig. 10.35 The projection of a triangular grid upon the Escher picture and its translation.

Fig. 10.36. The tetrahedron in colour (at left) and the corresponding Schattschneider model (at right).
Fig. 10.37. The layouts of the three regular triangular polyhedra with Escher pattern.

Fig. 10.38. The three regular triangular solids transformed into spatial Escher figures.
Fig. 10.39. *Spherical transformations of three spatial Escher figures.*

Fig. 10.40. Two spherical objects, made by M.C. Escher [10.11].

The following pictures demonstrate the projection of 3-dimensional forms on a dodecahedron. A tangram figure with a certain thickness was generated and used as an example.
Fig. 10.41. A tangram figure with a given thickness on the faces of a dodecahedron.
Fig. 42. The set-up for the pattern of the tangram figure

Fig. 10.43 Great circles around the sphere following R7, R8 and R11

Fig. 10.44. Models of great circles around the sphere following [10.10]
10.9. References

Chapter 11. POLYHEDRON PACKINGS

Some polyhedra lend themselves to be put together in tightly packed formations. In this way quite complex configurations can be formed. It is obvious that cubes and rectangular prisms can be stacked most densely. But some of the other polyhedra can, in certain combinations, also form close packings. The Truncated Octahedron and the Rhombic Dodecahedron are self-filling on their own and both packings are basically identical to the honeycomb. A honeycomb is more closely imitated by a packing of the dual of the Cuboctahedron, the Rhombic Dodecahedron. The Truncated Cuboctahedron, the Truncated Octahedron and the Cube together form other example of tight packings. If arbitrary figures or polyhedron related figures, such as prisms, spheres or ellipsoids, are translated or rotated in space, interesting configurations can be obtained. This can be used for the formation and the realistic presentation of spatial structures, where polyhedron related elements are used on different levels: macroform or overall shape, microform or internal structure, space frames, nodes and struts. [11.8]

11.1. Means of Manipulation

The most important programming tools for the modification or manipulation of polyhedra are:

11.1.1. Translation

A) Translation, shifting along straight lines without copying.
B) Linear progression, moving along straight lines, leaving a copy at the place of departure.

![Translation Diagram](image)

Figure 11.1. A) Principle of translation with and B) without copying.

11.1.2. Rotation

C) Rotation, moving along circular lines without copying.
D) Circular progression, moving along circular lines leaving a copy at the place of departure.

![Rotation Diagram](image)

Figure 11.2. C) Principle of rotation with and D) without copying.
11.1.3. Matrix Building
E) Matrix production, building a cubical set with given numbers at given distances in three directions.

Figure 11.3. A) Cubic matrix of separated cubes. B) Matrix of cubes with initial rotation.

11.1.4. Helix Formation
Helicoidal progression, moving along circular lines while simultaneously being lifted at a given distance from the circular plane and leaving a copy at the place of departure.

Figure 11.4. A cylindrical and a helicoidal translation.

In Figure 11.4:
A: Cylinder of 96 cubes, either a circle is formed and put in a matrix of only two dimensions. here at a vertical distance of 1.41, or a vertical column of 6 cubes (matrix of 1,6,1) is rotated over an angle of 22.5°. (all distances mentioned are expressed in a unit edge length = 1 of the used polyhedra)
B: Helix of 96 cubes with a rotation angle of \((360° + 360°/32):16 = 23.203°\), translation per step 2.41: 16 = 0.088.
11.1.5. Polyhedral Construction

The construction of a polyhedron is a complex product of reproduction: one polygonal face of a certain kind is placed in the centre of the XY-plane, shifted along the Z-axis, rotated around the X- and Y-axis respectively and left there. This is done as many times as this specific polygon occurs in the polyhedron and each time a copy is left in this point under certain angles. The formation of a particular polyhedron is done with a given set of rotation angles for each of the polygon types that occur in this polyhedron. This described in more detail in Chapter 5.

Seven sets of rotation angles are sufficient for the construction of all known polyhedra: the 5 regular or Platonic polyhedra and the 13 (or actually 15, as two of them occur in a left-handed and a right-handed version) semi-regular or Archimedean polyhedra. We talk explicitly of polyhedral construction, as other figures can be constructed similarly using the same technique. For instance the duals of the polyhedra and the star polyhedra can be formed with the same sets of rotation angles, but this time the faces are not regular polygons. All uniform polyhedra can be formed using any of the following sets of rotation angles belonging to the faces of the Tetrahedron, the Cube, the Octahedron, the Dodecahedron, the Icosahedron, the Truncated Cuboctahedron or the Truncated...
Icosidodecahedron. The input can either be: number of polygon or polygram sides, geometric data of dual faces and of star-faces. The distance of each of these faces from the rotation centre must be known and also the value of an eventual initial extra rotation of this face around the Z-axis.

11.2. Close Packings of Polyhedra

Rotation and translation is also used for the construction of packings. Polyhedra can be packed in the form of cubic matrices in various combinations to fill space entirely. Matrices of Cubes or Truncated Octahedra are self-filling on their own. In combination with Truncated Cuboctahedra and cubes in three different positions, they form another tight space filling.

Figure. 11.7. Close packing of Truncated Octahedra, placed in two matrices of 3 x 3 x 3 at distances of 2.83, the second matrix shifted along 1.415 units in three directions.
Figure 11.8. 3x3x3 matrices of Truncated Cuboctahedra (A) and Truncated Octahedra (B), both at distances of 3.83 and the second shifted along the axes: $X = Y = -Z = 1.915$

Figure 11.9. 3x3x3 matrices of cubes at 3.83 distance with initial $X$, $Y$- or $Z$- rotation $= 45^\circ$ and shifted: C) $X = 3.83, Y = -Z = 1.915$, D) $X = -Z = 1.915, Y = 0$, E) $X = Y = 1.915, Z = 0$. 
Figure 11.10. Combination of the 5 matrices in Figs.11.8 and 11.9 to a fully closed packing of 135 solids.

11.3. Sphere packings

11.3.1. Regular sphere packings
Spheres can be arranged in various ways. In the first place, spheres can be rotated following the procedure described as in Chapter 5, using the so-called 'rotation cases'. Fig. 11.11 and 11.1A give examples of this idea, where this is done in such way that the spheres touch each other and form a close conglomerate.

Fig.11.11. Spheres arranged in the corners of the five regular solids and in the corners of the Truncated Tetrahedron P6, the Cuboctahedron P7, the Icosidodecahedron P12.
Packings of spheres can also be done so that a close packing is reached. The five Platonic solids have the property that all their sides are at the same distance from the center so that an inscribed sphere touches all its faces. The cube is the only one of these that allows a space filling packing. Fig. 11.12A shows a simple packing (SCP) of 64 equal spheres in a so-called cubic lattice, which means, that these are the inscribed spheres in a matrix of 4 x 4 cubes.

This leads to the ratio of filled space versus available space, which is for sphere radius $R = .1$:

$$\rho = \frac{\text{Sphere Volume}}{\text{Cube Volume}} = \frac{4}{3} \frac{\pi R^3}{8} = \frac{\pi}{6} = 0.52359878 \text{ (see Fig. 11.13A)}$$

This is by far not the closest possibility; as it can be done much more economically.

The problem of the closest stacking of spheres is known as the Kepler’s Conjecture, dating from 1611. It says that no arrangement of equally sized spheres filling space has a greater average density than that of the cubic close packing (CCP) and the hexagonal close packing (HCP)[11.11]. In this context the Cannonball Stacking Problem must be mentioned. This has to do with the question to quickly calculate the total number of cannonballs in a square stack without having to count them (See Fig. 11.12B). Thomas Herriot solved this problem for his superior Sir Walter Raleigh and found the equation for a square pyramid: $n = 1/6 k(1+k)(1+2k)$, with $k$ = number of balls along the side of the bottom layer. For the stack below right in the figure $k = 7$ and the resulting total number is $n = 140$. For triangular pyramids this number $n = (k(k+1)(k+2))/6$. In the example of Fig.11.12B: $n = 84$.

Gauss proved mathematically that by any regular lattice arrangement no greatest density can be achieved than $rac{\pi}{3\sqrt{2}}$

Instead of the cube, one of the reciprocal figures can also be stacked so that a complete space filling system is achieved. One main characteristic of any reciprocal figure is that all its faces have the same distance from the centre, so that each has an inscribed sphere that touches all its faces. One of these reciprocols has the special property, that it can fill space fully in itself, as the cube does. This is the Rhombic Dodecahedron, the reciprocal figure of the Cuboctahedron, which consists of 12 rhombic faces, that are arranged in such a way and under such angles, that they allow fully closed packings. In the foregoing we have called the radius of this sphere $R_6$, which in this case $R_6 = 0.75$. The CCP as well as the HCP-packing can be seen as a collection of spheres ar-

Figure. 11.12. A) Simple cubical arrangement of 64 spheres (SCP) and B) Demonstration of the cannonball stacking problem [11.10] with examples of a square and a triangular pile of spheres.
ranged following a close packing of rhombic dodecahedra. Such an arrangement has great resemblance of a bee hive, where the cells are arranged similarly. The density of this packing can be expressed as:

\[ \rho = \frac{\text{Vol}_{\text{inscribed sphere}}}{\text{Vol}_{R7}} = \frac{\frac{4}{3} \pi R_6^3}{\frac{2.38648539}{3}} = \frac{1.76714586}{0.70480480} \approx 2.50 \] (see Table 1.6 and 3.1 and [114])

![Figure 11.13](image1.png)

**Fig. 11.13.** The Cube P2 and the Rhombic Dodecahedron R7 with their inscribed spheres. A packing of spheres as A) leads to a simple rectangular matrix of cubes, the one in B) to the bee hive.

![Figure 11.14](image2.png)

**Fig. 11.14.** Top view of a square pyramidal packing as in Fig. 1.12B, consisting of 140 spheres. This triangular pyramid B) consists of 84 spheres.
Fig. 11.15. *Four interwoven $4\times4\times4$ matrices forming a honeycomb of 256 Rhombic Dodecahedra.*

### 13.3.2. Random sphere pack

Random close sphere packings do not have a precise geometric definition [11.11]. Numerical data of the density are defined statistically or empirically. If spheres are loosely poured into a container, the density or volume fraction will lie between 0.609 - 0.625. If the container is vibrated, this density will presumably not exceed 0.634. This counts for equal spheres (monodisperse), but for polydispersed objects with different diameters the so-called volume fraction depends on the size-distribution and can thus theoretically become close to 1, depending on the amplitude and frequency of vibration.

Fig.11.16  *A) Spheres rotated following the Truncated Icosahedron P13. The latter consists of 60 spheres and is also called 'Buckyball'.*[11.9] and B) a spiral packing of spheres
Spheres with a very low exponent of the two main elliptical generators – in this case 0.3 in stead of 2 – and a frequency $F = 2$ give the 'spikey' figure A. This is rotated here, once (B) and twice (C) according the Octahedron. It has also been put in a 3x3x3 cubic matrix (D).

11.6 References
Chapter 12. ISOHEDRA

12.1. Introduction

If any of the polygons of a polyhedron is extended until it cuts the surface of the circumscribed sphere of the polyhedron, a circle is found in which this polygon fits. As polygons with greater side numbers have circumscribed circles of greater diameter, they lie closer to the centre of the system than those with smaller numbers of sides, given that their corners all lie on the same sphere. It is interesting to see, what happens if one brings all faces in one particular polyhedron at the same distance from the centre. Such a polyhedron will be called 'isodistant' and in this case all faces of one polyhedron have the same circumscribed circle.

![Fig. 12.1. Review of the 5 regular polyhedra and the 12 semi-regular polyhedra, of which two have a left-handed as well as a right-handed version.](image)

Table 12.1. Some characteristic aspects of the Platonic and Archimedean polyhedra.

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Table 12.1. Some characteristic aspects of the Platonic and Archimedean polyhedra.

In the table a few characteristics of polyhedra are given, that are relevant for the understanding of the present chapter. P = polyhedron index; R1 = radius of circumscribed sphere. n1, n2 and n3
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give the side numbers of the polygon kinds that occur in the specific polyhedron; z1, z2 and z3 are the distances of these polygons from the centre; the columns %diff give the ratio of \( \frac{R}{z} \) in percents. The definition of the term 'deficient angle' is shown in the next Fig. 12.2.

Fig. 12.2. The different kinds of polygons with their circumscribed circles.

### 12.2. Isodistant groups

If one studies the numbers of faces that constitute the different polyhedra, it becomes apparent that a certain kind of clustering appears. Apart from the five regular ones, that are isodistant in themselves as they each consist of regular polyhedra, the semi-regulars form groups with the same numbers of faces. Of the semi-regular solids P6 is the only one that has the same number of faces as one of the regular solids, the Octahedron P3, and thus it has the Octahedron P3 as its isodistant counterpart.

Fig. 12.3. The 5 regular polyhedra

#### 12.2.1. The octahedron or 8-hedron

In this light it has sense to further indicate the individual isodistant polyhedra after their numbers of faces. Hence the regular isodistant polyhedra maybe called: tetra- or 4-hedron, cube or hexa-6-hedron, octa- or 8-hedron and icosa- or 20-hedron. And further we will find: the 14-hedron, the 26-hedron, the 32-hedron, the 38-hedron, the 62-hedron and the 92-hedron.

Fig.12.4. The Truncated Tetrahedron P6 and its isodistant version, the Octahedron P3
12.2.2. The 14-hedron

Fig. 12.5. The group of polyhedra with 14 faces

P7, -8 and -9 have 14 faces and thus have the same isodistant form. This can be seen as a further truncation of the octahedron P3. The process starts from P3 with the edge length = 1.

Distance of the triangles from the centre: \( z_3 = \frac{\sqrt{6}}{6} = 0.40824829 \)

Fig. 12.6. The formation of P8 by truncation through 1/3 of the sides.

Radius of the circumsphere: \( R_1 = \frac{\sqrt{2}}{2} = 0.70710681 \)

Fig. 12.7. The truncation process of an octahedron and the variables in a pyramid sector
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The height of a regular pyramid: \( h = 0.5\sqrt{\tan^2\alpha - \cot^2\varphi} \)

If the truncation takes place at the same distance \( z_3 \) as that of the triangles, the triangles are transformed to irregular hexagons with the side lengths \( A \) and \( B \).

We find a form consisting of 8 of such hexagons and 6 squares with the side length \( A \) and we call this the 14-hedron.

In such a triangle with the edge length = 1

\[
A + B = \frac{z_3}{R_1} = \frac{\sqrt{6}}{6} : \frac{\sqrt{2}}{2} = \frac{\sqrt{3}}{3} = 0.57735027
\]

Fig. 12.8. The truncation of P8 at equal distances from the centre

Fig.12.9. The two kinds of faces fit on one circumscribed circle

As \( 2A + B = 1 \)

\[
A = 1 - \frac{\sqrt{3}}{3} = 0.42264973
\]

\[
B = 0.15470054
\]

The two faces fit in a circle with the radius: \( R_c = \frac{A\sqrt{2}}{2} = 0.29885849 \)

The angles under which the sides are seen: \( A \rightarrow 2\cdot\arcsin\left(\frac{A}{2} : R_c\right) = 90^\circ \)

\( B \rightarrow 2\cdot\arcsin\left(\frac{B}{2} : R_c\right) = 30^\circ \)

\( A : B = \sin 45^\circ : \sin 30^\circ \)

It is remarkable, that these angles can be expressed in integers.

The circumscribed sphere: \( R_1 = \sqrt{R_2^2 + z_3^2} = 0.50594769 \)
To this hexagon must be given an initial rotation around the Z-axis (which is perpendicular to this plane) \( dkz = 180^\circ \) and then rotated following the polar coordinates of the faces of the octahedron P3 (called case3 in Chapter 5) at a distance of \( z_3 \). The square face will have side length \( A = 0.42264973 \) and must successively be rotated over the angle \( dkz = 45^\circ \) and following the polar coordinates of the cube P2 (case 2).

\[ \begin{array}{c|cc}
\text{X} & \text{Y} \\
1 & \frac{A}{2} & \frac{A}{2} \\
2 & -\frac{A}{2} & \frac{A}{2} \\
3 & \frac{A}{2} + \frac{B}{2} & -\frac{A}{2} + \frac{B\sqrt{3}}{2} \\
4 & -\frac{B}{2} & \frac{B}{2}/\tan(15^\circ) \\
5 & \frac{B}{2} & \frac{B}{2}/\tan(15^\circ) \\
6 & \frac{A}{2} + \frac{B}{2} & -\frac{A}{2} + \frac{B\sqrt{3}}{2} \\
\end{array} \]

Table 12.2 and Fig. 12.10. The coordinates of the hexagon

12.2.3. The 26-hedron

Fig. 12.11. The rhombicuboctahedron P10 and the truncated cuboctahedron P11 both have 26 faces

In order to get this 26-hedron, the 14-hedron has to be truncated parallel to the edge B at the distance \( z_3 \). By this truncation shallow pyramids are cut off, that have a rectangular bottom plane with the side lengths D and C. The squares will convert into octagons with the sides D and E; the hexagons will become smaller hexagons with the sides E and C.

Fig. 12.12. The formation of the 26-hedron by the truncation of the 14- hedron at equal distances

The new figure again has all its faces at the distance \( z_3 \) of the octahedron P3, and it is in fact a further truncation of the 14-hedron.
Fig. 12.13. The faces of the 26-hedron and the same in their circumscribed circle

Fig. 12.14. The further truncation of the 14-hedron for the formation of the 26-hedron

In Fig. 12.14 one of the square panels of the 26-hedron is shown with four of its adjacent hexagons. In order to get this situation, the 14-hedron has to be truncated parallel to the edge B at the distance $z$. By this truncation shallow pyramids are cut of that have a rectangular bottom plane with the side lengths D and C. The squares will convert into octagons with the sides D and E; the hexagons will become smaller hexagons with the sides E and C.

Fig. 12.15. Layout of 26-hedron

Considering the pyramid, the dihedral angle in side B: $\xi_{\text{dih}} = \arctan(\sqrt{2}) = 54.73561032^\circ$
Distance of side B from origin: \((R_b)_B = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = 0.5\)

Height of pyramid: \(h = 0.5 - \frac{\sqrt{6}}{6} = 0.09175171\)

Shortest side of pyramid: \(D = 2 * x = 2 * h * \tan(\xi) = (1 - \frac{\sqrt{6}}{3}) = 0.25951302\)

Cut off length of square sides: \(\frac{D}{\sqrt{2}} = 0.18350342\)

Shortest side of new octagon: \(E = A - \frac{2D}{\sqrt{2}} = 0.05564289\)

\(C = B + 2 \frac{D}{2\sqrt{2}} = B + \frac{D}{\sqrt{2}} = 0.338203959\)

All three thus found faces fit in one circumscribed circle, which has the radius:
\[R_C = \sqrt{\left(\frac{C}{2}\right)^2 + \left(\frac{D}{2}\right)^2} = 0.21314838\]

The central angles under with the sides are seen:
\[C \rightarrow 2 * \arcsin\left(\frac{C}{2} : R_C\right) = 105°\]
\[D \rightarrow 2 * \arcsin\left(\frac{D}{2} : R_C\right) = 75°\]
\[E \rightarrow 2 * \arcsin\left(\frac{E}{2} : R_C\right) = 15°\]

Again these angles are expressed in integer numbers.

\[C : D : E = \sin 52.5° : \sin 37.5° : \sin 7.5°\]

12.2.4. The 32-hedron

![32-hedron diagram](image)

Fig. 12.16. The group with 32 faces of which \(P_{12}\) is the most well-known

The polyhedron \(P_{12}\) can be found by the truncation of the icosahedron through the mid-points of its sides.
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Fig. 12.17. Truncation process leading from the 20-hedron or icosahedron $P_5$ to $P_{13}$

The isodistant version of the polyhedra in this group can be derived from the icosahedron by truncation of the vertices at the same distance $z_3$ from the centre as the original triangles. In this case the hexagons will have a slightly irregular form and have two different side lengths instead that they are equilateral [12.2].

Fig. 12.18. Truncation of Icosahedron having all its faces at distance $z_3$ and the faces with their circumscribed circle.

For an Icosahedron with the side length = 1:

Both polygon distances: $z_5 = z_6 = 0.75576131 = z_3$ of the icosahedron

Height of pyramidal cap $= h = 0.95105651 - 0.75576131 = 0.19529521$

and also: $h = 0.5A \sqrt{\left(\tan^2 60^\circ - \cot^2 36^\circ\right)} = 0.52573111A$

Hence $A = \frac{0.19529521}{0.52573111} = 0.37147355$

And $B = 1 - 2A = 0.2570529 = 0.69198171A$

The ratio of the sides: $\frac{B}{A} = \frac{\sin 24^\circ}{\sin 36^\circ} = 0.69198171$

And thus: $A : B = \sin 36^\circ : \sin 24^\circ$

Fig. 12.19. The 32-hedron formed by the truncation of the 20-hedron at the distance $z_3$
The two polygons have the same circumcircle, which both are equal to:

Circumscribed circle of the pentagon: $R_2 = \frac{A}{2\sin36^\circ} = 0.85065081A = 0.31599427$

The radius of the circumscribed sphere: $R_1 = \sqrt{R_2^2 + z_3^2} = 0.81916271$

12.2.5. The 62-hedron

The 62-hedron is the isodistant truncation of the 32-hedron parallel to its sides.

The isodistant version of the rhombicosidodecahedron is obtained by truncating the isodistant truncated icosahedron (the 32-hedron) once more by planes, parallel to the edges that connect
two adjacent hexagons in a direction perpendicular to the radius. This can be done through the midpoints of these sides or at one-third.

Fig. 12.23. *The isodistant truncation of the 32-hedron*

The truncation through the midpoints of the edges does not lead to a third set of planes at the same distance from the centre.

As before:

\[ A = 0.37147355 \]
\[ B = 0.25705290 \]

\[ L_1 = \frac{A}{2} + B = 0.44278968 \]
\[ L_2 = \frac{A}{\tau} = 0.30052841 \]
\[ L_3 = \sqrt{(L_1^2 + L_2^2)} = 0.53514486 \]
\[ L_4 = \tau L_2 = 0.48625189 \]

In order to bring the rectangle at the same distance \( z \) of the hexagon and the decagon from the centre, the section lines have to be shifted outwards.

The dihedral angle: \( \xi = \arcsin\left(\frac{2}{3} \sqrt{\frac{3}{2}} \times \frac{\tau}{2}\right) = \arcsin \frac{\tau}{\sqrt{3}} = \arcsin \frac{1 + \sqrt{5}}{2\sqrt{3}} = 69.09484255^\circ \)

Distance of midpoint of triangle edge to the centre: \( R_3 = \frac{\tau}{2} = 0.80901699 \)

Height of rectangular pyramid: \( h = \frac{\tau}{2} - z_3 = 0.05325568 \)
Fig. 12.25. *Cross-section through rectangular pyramid and schematic top view*

Half cord: \( x = h \tan \xi = 0.13942518 \)
Width of rectangle: \( w = 2x = 0.27885036 = D \)
Cut-off length from A: \( q = \frac{w}{2 \cos 36^\circ} = 0.17233900 \)
E = 2t = A - 2q = 0.02679555
Resulting lengths:
C = \( L_1 - 2t \sin 60^\circ = 0.42939190 \)
D = \( L_2 - 2t \cos 36^\circ = 0.27885036 \)
\( \delta = \arctan \frac{t}{r} = 3.00000000^\circ \) (!)
\( \epsilon = \arctan \frac{C}{D} = 57.00000000^\circ \)

Fig. 12.26. *The three kinds of faces, fitting in one circumcircle*
Fig. 12.27. *The lay-out of the new form.*

Ratio of the sides \( C : D : E = \sin57^\circ : \sin33^\circ : \sin3^\circ = 16.025 : 10.407 : 1 \)

Radius of the circumcircle: \( R_2 = \sqrt{\frac{(C^2 + D^2)}{2}} = 0.25599557 \)

Radius of the circumsphere: \( R_1 = \sqrt{(R_2^2 + z_3^2)} = 0.79782792 \)

### 12.2.6. The 38-hedron

Fig. 12.28. *The 38-hedron has a left-handed as well as a right-handed version*

\[
A = 1 - \frac{\sqrt{3}}{3} = 0.42264973
\]

\[
B = 0.15470054
\]

The two faces fit in a circle with the radius: \( R_2 = \frac{A\sqrt{2}}{2} = 0.29885849 \)

Radius of the circumscribed sphere: \( R_1 = \sqrt{R_2^2 + z_3^2} = 0.50594769 \)

Distance of mid edge A to centre of system: \( R_3 = \sqrt{R_1^2 - \left(\frac{A}{2}\right)^2} = 0.56759914 \)
The isodistant snub cube, or the 38-hedron is directly related to the isodistant cuboctahedron, or the 14-hedron.

The square face of the 38-hedron and also one of its triangular faces (the blue ones in Fig. 12.28), are placed on the respective faces of the 14-hedron and enlarged until their corners (numbered 1 in the figure) lie on one of its edges. These two figures must be rotated and simultaneously enlarged, keeping the corners still on this edge. The edge length 2-3 is the one to be found. The rotation must be continued until the sides 1-3, 1-2 and 2-3 have a certain relation with respect to each other.

This relation follows from the fact, that all three different faces of the isodistant 38-hedron lie within the same circumscribed circle. This is shown in Fig. 12.30.

From Fig. 12.30, it can be reasoned quite easily that the ratio of the sides must be:

\[ A : B : C = \sin 45^\circ : \sin 75^\circ : \sin 60^\circ \]
<table>
<thead>
<tr>
<th>ϕ</th>
<th>by iteration</th>
<th>1st trial</th>
<th>Second trial</th>
<th>final</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$1 - \sqrt{3}/3$</td>
<td>0.42264973</td>
<td>0.42264973</td>
<td>0.42264973</td>
</tr>
<tr>
<td>B$_{isoP8}$</td>
<td>$1 - 2A$</td>
<td>0.15470054</td>
<td>0.15470054</td>
<td>0.15470054</td>
</tr>
<tr>
<td>z$_{3p5}$</td>
<td>$\sqrt{6}/6$</td>
<td>0.40824829</td>
<td>0.40824829</td>
<td>0.40824829</td>
</tr>
<tr>
<td>R$_{isoP8}$</td>
<td>$\sqrt{z_{3p5}^2 + R^2}$</td>
<td>0.46105567</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R$<em>5$ (on L$</em>{13}$)</td>
<td>$\sqrt{R_{isoP15}^2 - B_{isoP8}^2}$</td>
<td>0.42207000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi_4$</td>
<td>$\arctan(2\sqrt{2})$</td>
<td>70.52877937</td>
<td>70.52877937</td>
<td>70.52877937</td>
</tr>
<tr>
<td>$\xi_6$</td>
<td>$\arctan(\sqrt{2})$</td>
<td>54.73561032</td>
<td>54.73561032</td>
<td>54.73561032</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>$A\sqrt{2}/2$</td>
<td>0.29885849</td>
<td>0.29885849</td>
<td>0.29885849</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$A/2$</td>
<td>0.21122487</td>
<td>0.21122487</td>
<td>0.21122487</td>
</tr>
<tr>
<td>R$_2 = m_2$</td>
<td>$ml/\cos(\phi)$</td>
<td>0.21122487</td>
<td>0.21295906</td>
<td>0.21425606</td>
</tr>
<tr>
<td>t = $X_1$</td>
<td>$ml\tan(\phi)$</td>
<td>0.00000000</td>
<td>0.03347057</td>
<td>0.03531944</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>0</td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>0.00000000</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$A/2$</td>
<td>0.21122487</td>
<td>0.21122487</td>
<td>0.21122487</td>
</tr>
<tr>
<td>$Y_2'$</td>
<td>$A/2 - t$</td>
<td>0.21122487</td>
<td>0.17785429</td>
<td>0.17600543</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$B/2 + A/4 + t/2$</td>
<td>0.18301270</td>
<td>0.19974799</td>
<td>0.20067242</td>
</tr>
<tr>
<td>$Y_3'$</td>
<td>$(B + A/2 - t)\sqrt{3}/2$</td>
<td>0.31698730</td>
<td>0.28800093</td>
<td>0.28639977</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>$Y_2'\sin(\xi_4)$</td>
<td>0.19923899</td>
<td>0.16768264</td>
<td>0.16593951</td>
</tr>
<tr>
<td>$Y_3$</td>
<td>$Y_3'\sin(\xi_6)$</td>
<td>0.25881905</td>
<td>0.23515178</td>
<td>0.23384443</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$Y_2'\cos(\xi_4)$</td>
<td>0.07044162</td>
<td>0.05928476</td>
<td>0.05866848</td>
</tr>
<tr>
<td>$Z_3$</td>
<td>$Y_3'\cos(\xi_6)$</td>
<td>0.18301270</td>
<td>0.16627742</td>
<td>0.16535298</td>
</tr>
<tr>
<td>L$_{23}$</td>
<td>$\sqrt{(X_2 - X_3)^2 + (Y_2 + Y_3)^2 + (Z_3 - Z_2)^2}$</td>
<td>0.47253676</td>
<td>0.41696165</td>
<td>0.41291093</td>
</tr>
<tr>
<td>L$_{12}$</td>
<td>$A_1^* m_2 / m_1$</td>
<td>0.29885849</td>
<td>0.30258380</td>
<td>0.30300383</td>
</tr>
<tr>
<td>L$_{13}$</td>
<td>$(A + 2B)^* m_2 / 2m_1$</td>
<td>0.36602540</td>
<td>0.37058796</td>
<td>0.37110239</td>
</tr>
<tr>
<td>Also : L$_{23}$</td>
<td>$\sqrt{L_{12}^2 + L_{13}^2 - 2*L_{12}*L_{13}\cos(75^o)}$</td>
<td>0.40824829</td>
<td>0.41233716</td>
<td>0.41291093</td>
</tr>
<tr>
<td>s</td>
<td>$(L_{12} + L_{13} + L_{23})/2$</td>
<td>0.56871033</td>
<td>0.54506670</td>
<td>0.54400858</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$2\arcsin(\sqrt{(s-L_{12})^2 * (s-L_{13})} / L_{12} L_{13})$</td>
<td>90.00000000</td>
<td>75.79451273</td>
<td>74.99999997</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$2\arcsin(\sqrt{(s-L_{13})^2 * (s-L_{23})} / L_{13} L_{23})$</td>
<td>39.23152048</td>
<td>44.70826172</td>
<td>45.00000000</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$2\arcsin(\sqrt{(s-L_{12})^2 * (s-L_{23})} / L_{12} L_{23})$</td>
<td>50.76847952</td>
<td>59.49722455</td>
<td>60.00000002</td>
</tr>
<tr>
<td>R$_{isoP15}$</td>
<td>$\sqrt{z_{3p5}^2 + R^2}$</td>
<td>0.45970084</td>
<td>0.46091772</td>
<td>0.46105567</td>
</tr>
</tbody>
</table>

Table 12.3. The derivation of angle $\phi$ by iteration.

$\phi$ is the requested angle of rotation and it can be found by iteration. Three trials are shown above, where $\phi$ consecutively is chosen as 0.00000000, 3.00000000 and the final value. In reality of course many more steps had to be done to find it as accurate as is found in the end. Most of the values in the foregoing table depend from this angle $\phi$, and the best approximation is reached as soon the angles $\alpha, \beta$ and $\gamma$ have their required value as close as possible, in this case the 8th decimal.

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Fig. 12.31. One of the two basic triangular faces, in which all side lengths occur and which is one of 80 in the whole solid.

The coordinates of the triangle 1-2-3:

\[
\begin{align*}
x_1 &= L_{13} / 2 = & 0,18555119 \\
y_1 &= L_{13} \sin(30^\circ) / 2 = & 0,10712803 \\
x_2 &= L_{13} - L_{12} \sin(15^\circ) = & 0,10712803 \\
y_2 &= L_{12} \cos(15^\circ) - y_1 = & 0,18555119 \\
x_3 &= L_{13} / 2 = & 0,18555119 \\
y_3 &= y_1 = & 0,10712803 \\
L &= L_{23} + L_{12} = & 0,71691477
\end{align*}
\]

The rotation angles of triangle 1-2-3:

\[
\begin{align*}
\xi_{13} &= \arccos(y_1 / R5) = & 75,29657312^\circ \\
\text{rotation angle } \psi &= 180^\circ - 2\xi_{13} = & 29,40685373^\circ \\
\text{also } \xi_{13} &= \arctan(z_{3_{p5}} / y_1) = & 75,29657312^\circ \\
\text{also } \xi_{13} &= \arcsin(z_{3_{p5}} / R5_{old,13}) = & 75,29657312^\circ
\end{align*}
\]

If all these conditions are answered at the same time, the angle for the initial rotation of the square resp. the regular triangle that lie on the faces of the isodistant 14-hedron, is found:

\[\varphi = 9.48832590^\circ\]

This solution could be found using the spreadsheet programme EXCEL.

There is a remarkable aspect in this configuration, that three of the triangles 1-2-3, shown in Fig. 12.30, gathered around the equal-sided triangle with the side length 1-3, form a regular triangle with the side length \(L = L_{12} + L_{23}\). Some of the items, such as the value of \(L_{23}\) and \(\xi_{12}\), were calculated in two and sometimes three different ways just for control.
Fig. 12.32. Net of the 38-hedron and two combinations, in the semi-regular polyhedron P15 and in the 38-hedron; four triangles are combined: three white ones around the blue 'specific'one.

The construction of the 38-hedron can be realized following the next ten phases:
1. The triangle 1-2-3 is placed in Z-direction at a distance from the systems center:
   \[ z_{3 \text{P3}} = \sqrt{6} / 6 = 0.40824829 \]
2. Triangle 1-2-3 is rotated over angle \( \varphi = 29.40685373^\circ \) around the X-axis.
3. Then this triangle is rotated and copied 3 times around the Z-axis over the angle 120°
4. An equilateral triangle is formed with the side length \( L_{23} = 0.37110239 \)
5. This triangle is put at the distance \( z \) from the systems centre.
6. This triangle and three scalene triangles of Phase 3 are put together.
7. A square is formed with the side length \( L_{12} = 0.30300383 \).
8. The square is placed at the distance \( z \) from the centre.
9. The assembly of Phase 6 gets an initial rotation around the Z-axis of \( -\varphi = -9.48832590^\circ \) and is thereupon rotated following Case 3 of Chapter 05, and put in the position KKX = 290°, KKY = 0° and KKZ = 40°.
10. The square of Phase 8 is treated similarly but now following Case 2.

These phases are shown in Fig. 12.33, where the variant with the open seems is made by adding the distance \( dz = 0.05 \) to those in Phase 6 and Phase 8, before Phase 9 and 10 have taken place. From this picture it becomes clear that the part of Phase 6 can be folded flat and that it forms an equilateral triangle with the side length \( L = A+B = 0.71691477 \). The values that are used here can be found back in the foregoing Table 12.2.
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Fig. 12.33. The construction of the 38-hedron in various phases.

12.2.7. The 92-hedron

Fig. 12.34. The left- and right-handed version of P18. Four triangles at a time can be combined to form larger equilateral triangles. This is also the case in the isodistant version.

Fig. 12.35. The composition of a snub dodecahedron from pentagons and larger triangles, each consisting of four basic triangles at left the situation in the regular version and at left in its isodistant version.
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Fig. 12.36. The layout of the 92-hedron, composed of pentagons and large triangles.

This 92-hedron is derived from the isodistant 32-hedron.

In the 32-hedron:

\[ A = 0.37147355 \]
\[ B = 1-2A = 0.25705290 \]

Fig. 12.37. Projection of the 32-hedron upon the 32-hedron, both isodistant.
Fig. 12.38. The two basic triangles and the pentagon in their circumscribed circle

From this figure it is apparent that the sides relate as follows:

\[ \frac{A}{B} : \frac{C} = \sin 36^\circ : \sin 84^\circ : \sin 60^\circ \]

Dihedral angles:

\[ \xi_\alpha = \arctan \left( \frac{9 - \sqrt{5}}{2} \right) = 73.52773931^\circ \]

\[ \xi_\beta = \arctan \left( \tau^2 \right) = 69.09484255^\circ \]

If all these conditions are answered at the same time, the angle for the initial rotation of the regular triangle that lies on the faces of the isodistant 32-hedron, is found:

\[ \phi = 3.34264307 \]
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<table>
<thead>
<tr>
<th>item</th>
<th>1st trial</th>
<th>further trial</th>
<th>final values</th>
</tr>
</thead>
<tbody>
<tr>
<td>τ</td>
<td>(1 + (5^{1/2}))/2</td>
<td>1,61803399</td>
<td>1,61803399</td>
</tr>
<tr>
<td>Ψ</td>
<td>By iteration</td>
<td>0,00000000</td>
<td>3,00000000</td>
</tr>
<tr>
<td>A (IsoP12)</td>
<td>0,37147355</td>
<td>0,37147355</td>
<td>0,37147355</td>
</tr>
<tr>
<td>B (IsoP12)</td>
<td>1 - 2B</td>
<td>0,25705290</td>
<td>0,25705290</td>
</tr>
<tr>
<td>z3(P5)</td>
<td>(3 + (5^{1/2})/(12))</td>
<td>0,75576121</td>
<td>0,75576121</td>
</tr>
<tr>
<td>R2(IsoP12)</td>
<td>A / 2 / \sin36^0</td>
<td>0,31599428</td>
<td>0,31599428</td>
</tr>
<tr>
<td>R1(IsoP12)</td>
<td>\sqrt{R2^2 + z^2}</td>
<td>0,81916271</td>
<td>0,81916271</td>
</tr>
<tr>
<td>R5(on edgeA)</td>
<td>\sqrt{R1^2 - \alpha^2 / 4}</td>
<td>0,76649444</td>
<td>0,76652371</td>
</tr>
<tr>
<td>ζ5</td>
<td>\arctan((9 - (5^{1/2}))/2)</td>
<td>73,52778931</td>
<td>73,52778931</td>
</tr>
<tr>
<td>ζ6</td>
<td>\arctan(1 + (1/2))</td>
<td>69,09484255</td>
<td>69,09484255</td>
</tr>
<tr>
<td>A1</td>
<td>Aτ / 2</td>
<td>0,30052841</td>
<td>0,30052841</td>
</tr>
<tr>
<td>m1</td>
<td>A / 2 / \tan36^0</td>
<td>0,25564474</td>
<td>0,25564474</td>
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<tr>
<td>m2</td>
<td>ml / \cosφ</td>
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<td>t</td>
<td>ml / tanφ</td>
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<td>0,01239777</td>
</tr>
<tr>
<td>X1</td>
<td>t</td>
<td>0,00000000</td>
<td>0,01239777</td>
</tr>
<tr>
<td>Y1</td>
<td>0</td>
<td>0,00000000</td>
<td>0,00000000</td>
</tr>
<tr>
<td>X2</td>
<td>(A / 2 - t)sin18^0 + A / 2</td>
<td>0,24312259</td>
<td>0,23899246</td>
</tr>
<tr>
<td>Y2</td>
<td>(A / 2 - t)cos18^0</td>
<td>0,17664617</td>
<td>0,16390412</td>
</tr>
<tr>
<td>X3</td>
<td>(A + 2B - 2t) / 4</td>
<td>0,22129482</td>
<td>0,22809372</td>
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<tr>
<td>Y3</td>
<td>X3\sqrt{3}</td>
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<td>0,37186430</td>
</tr>
<tr>
<td>Y2</td>
<td>Y2 \sinξ5</td>
<td>0,16939615</td>
<td>0,15717707</td>
</tr>
<tr>
<td>Y3</td>
<td>Y3 \sinξ6</td>
<td>0,35822437</td>
<td>0,34738535</td>
</tr>
<tr>
<td>Z2</td>
<td>R5 - Y2 \cosξ5</td>
<td>0,05008807</td>
<td>0,04647506</td>
</tr>
<tr>
<td>Z3</td>
<td>R5 - Y3 \cosξ6</td>
<td>0,12682953</td>
<td>0,12268939</td>
</tr>
</tbody>
</table>

Length L23, also

\[ \sqrt{(X2 - X3)^2 + (Y2 - Y3)^2 + (X2 - Z3)^2} \]

\[ \sqrt{(L12^2 + L13^2 - 2L12\cos84^0)} \]

\[ \beta \]

2 arcsin \( \sqrt{(s - L12) \ast (s - L13)} / L12 / L13 \)

\[ 90,00000000 \]

\[ 84,61796009 \]

\[ 84,00000000 \]

\[ \alpha \]

2 arcsin \( \sqrt{(s - L13) \ast (s - L23)} / L13 / L23 \)

\[ 34,16536393 \]

\[ 35,81669307 \]

\[ 36,00000012 \]

\[ \gamma \]

2 arcsin \( \sqrt{(s - L12) \ast (s - L23)} / L12 / L23 \)

\[ 55,83463607 \]

\[ 59,56534684 \]

\[ 59,99999989 \]

R1(IsoP18) \[ \sqrt{m2^2 + z3^2} \]

\[ 0,79782792 \]

\[ 0,79794041 \]

\[ 0,79796763 \]

Length A+2B \[ L12 + L23 = \]

\[ 0,83567328 \]

\[ 0,81293199 \]

\[ 0,81039572 \]

Table 12.4. Data of the 92-hedron.

The angle φ is found by iteration. By gradually increasing its value from zero through three and further the optimum value is reached, where α, β and γ arrive as closely as possible at the theoretically required 36°, 84° and 60°.
Co-ordinates of the corners in triangle 1-2-3:

\[ x_1 = \frac{L_{13}}{2} = 0.22177214 \]

\[ x_2 = x_1 - L_{12} \sin 6^\circ = 0.19030483 \]

\[ x_3 = x_1 = 0.22177214 \]

\[ y_1 = x_3 \tan 30^\circ = 0.12804020 \]

\[ y_2 = h_1 - y_1 = 0.17125124 \]

\[ y_3 = y_1 = 0.12804020 \]

The rotation angles are:

\[ \xi_{1,3} = \arccos\left(\frac{y_1}{R5}\right) = 80.38432060 \]

\[ \psi = 180^\circ - 2\xi_{13} = 19.23125880 \]

Also:

\[ \xi_{1,3} = \arccos\left(\frac{x_2}{y_1}\right) = 80.38432059 \]

Also:

\[ \xi_{1,3} = \arccos\left(\frac{z_3}{R5}\right) = 80.38432044 \]

The two latest values were calculated for control.

The isodistant polyhedron is formed in the following phases:

1. The triangle 1-2-3 is placed in Z-direction at a distance from the systems centre:
   \[ z_{3,P3} = \sqrt{6}/6 = 0.40824829 \]

2. Triangle 1-2-3 is rotated over angle \( \varphi = 29.40685373^\circ \) around the X-axis.

3. Then this triangle is rotated and copied three times around the Z-axis over the angle 120°

4. An equilateral triangle is formed with the side length \( L_{23} = 0.37110239 \)

5. This triangle is put at the distance \( z \) from the systems centre.

6. This triangle and three scalene triangles of Phase 3 are put together.

7. A square is formed with the side length \( L_{12} = 0.30300383 \).

8. The square is placed at the distance \( z \) from the centre.

9. The assembly of Phase 6 gets an initial rotation around the Z-axis of \( -\varphi = -9.48832590^\circ \) and is thereupon rotated following Case 3 of Chapter 05, and put in the position KXX = 290°, KKY = 0° and KKZ = 40°.

10. The square of Phase 8 is treated similarly but now following Case 2.

These phases are shown in the next pictures, where the variant with the open seems is made by adding the distance \( dz = 0.05 \) to those in Phase 6 and Phase 8, before Phase 9 and 10 have taken place. From this picture it becomes clear that the part with four coupled triangles of Phase 6 can be folded flat and that it forms an equilateral triangle with the side length \( L = 0.71691477 \). The values that are used here can be found back in the foregoing Table 12.4.
Fig. 12.40  *The first two phases in the rotation process, leading to the formation of the 92-hedron.*

Fig. 12.41. *The final phases of this formation process. At right the ‘exploded’ version.*
Chapter 12

Fig. 12.42. Review of the isodistant polyhedra and their relatives

5 regulars

8-hedron

14-hedron

26-hedron

32-hedron

62-hedron

38-hedron

92-hedron
Chapter 12

12.3. The roundness of polyhedra

The roundness of a polyhedron can be defined by the following conventions:
1. All faces touch the same inscribed sphere: have the same distance from the systems center.
2. All corners lie on one circumscribed sphere.
3. The radius of the inscribed sphere is closest to that of the circumscribed sphere.
4. It has the smallest deficient angle: the missing part of the full 360°, if all faces that meet at a corner are put together.

12.3.1. Fair dice

A characteristic of fair dice is that if these are rolled over a plane, horizontal surface in the end they will not have a preference for which plane will point upwards. The use of dice can serve many different purposes; this can range from simple games to poker or backgammon, where in fact great economic issues can be at stake. Since the old times dice were also used for fortune spelling or to foresee the ordeal of gods. This is called Cleromancy and it is a form of divination using sortition, casting of lots, or casting bones or stones, in which an outcome is determined by means that what normally would be considered random, such as the rolling of dice, but are sometimes believed to reveal the will of God, or other supernatural entities.

12.3.2. Polyhedral dice

An example of a traditional dice is a rounded cube, with each of its six faces showing a different number of (pips) from 1 to 6. When thrown or rolled, the die comes to rest showing on its upper surface a random integer from one to six, each value being equally likely.

The corners are generally chamfered. The sides are numbered from 1 to 6 by the addition of circular marks called pips, in a contrastive colour, sometimes in the form of dimples. The distribution of the six numbers of pips is generally done as in B and this can be either right- or left-handed. For a proper equilibrium, dimples must in the end be filled with the same basic material.

A variety of similar devices are also described as dice; such specialized dice may have polyhedral or irregular shapes and may have faces marked with symbols instead of numbers. They may be used to produce results other than one through six. Loaded and crooked dice are designed to favor some results over others for purposes of cheating or amusement.
The full geometric set of uniform fair dice is:

1. Platonic dice

![Image of Platonic dice]

Fig 12.44. Platonic solids, the five regular polyhedra: 4, 6, 8, 12, 20 sides. The cubical die of Fig. 12.43 is one of these. The one in the right top is the dual of a pentagonal antiprism.

<table>
<thead>
<tr>
<th>Hedron</th>
<th>R2</th>
<th>z</th>
<th>R1</th>
<th>100*R1/z</th>
<th>Deficient angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0,57735027</td>
<td>0,20412415</td>
<td>0,6123724</td>
<td>300,00000000</td>
<td>180°</td>
</tr>
<tr>
<td>6</td>
<td>0,50000000</td>
<td>0,50000000</td>
<td>0,8660254</td>
<td>173,20508076</td>
<td>90°</td>
</tr>
<tr>
<td>8</td>
<td>0,57735027</td>
<td>0,40824829</td>
<td>0,7071068</td>
<td>173,20508076</td>
<td>120°</td>
</tr>
<tr>
<td>12</td>
<td>0,85065081</td>
<td>1,11351636</td>
<td>1,4012585</td>
<td>125,84085787</td>
<td>36°</td>
</tr>
<tr>
<td>20</td>
<td>0,57735027</td>
<td>0,75576131</td>
<td>0,9510565</td>
<td>125,84085841</td>
<td>60°</td>
</tr>
</tbody>
</table>

Table.12.5. Relevant data of the Platonic polyhedra.

2. Duals of Archimedean polyhedra.

![Image of Catalan solids]

Fig. 12.45. Catalan solids, the duals of the 13 Archimedean solids: 12, 24, 30, 48, 60, 120 sides
Table 12.6. Data of the Archimedean duals (See for the meaning R6 and R7 Fig. 3.8 in Chapter 3).

3. Duals of prisms, any even number above 3

![Diagram of bipyramids]

Fig. 12.46. Row of bipyramids, a selection from the duals of the infinite set of prisms (E) with triangular faces and their compounds with the original prisms. Only the coloured ones are useful as dice as these always have one face pointing upwards when resting on a flat surface.

4. Duals of antiprisms any odd number above 4

![Diagram of antiprisms]

Fig. 12.47. Duals of the antiprisms (F) with kite faces together with their compounds.

In Fig. 12.46 and 12.47 a series of duals are shown of the prisms (E in table 12.3) and of the antiprisms (F in table 12.3). It is apparent from this picture that the 4-sided prism dual is equal to...
the octahedron (P3) and that the 2- and 3-sided antiprism duals are identical to the tetrahedron and the cube respectively. Bipyramids and trapezohedra are of course onlyrollable around their length axis.

<table>
<thead>
<tr>
<th>Numbers of sides</th>
<th>R71</th>
<th>R72</th>
<th>R6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>E1</td>
<td>60,85569534</td>
<td>0,66666667</td>
</tr>
<tr>
<td>5</td>
<td>E3</td>
<td>68,76234859</td>
<td>1,44721360</td>
</tr>
<tr>
<td>6</td>
<td>E4</td>
<td>71,47053667</td>
<td>2,00000000</td>
</tr>
<tr>
<td>8</td>
<td>E5</td>
<td>75,40321688</td>
<td>3,41421356</td>
</tr>
<tr>
<td>10</td>
<td>E6</td>
<td>78,11278108</td>
<td>5,23606798</td>
</tr>
<tr>
<td>3</td>
<td>F2</td>
<td>56,71032871</td>
<td>1,01505177</td>
</tr>
<tr>
<td>4</td>
<td>F3</td>
<td>60,98940663</td>
<td>1,53884177</td>
</tr>
<tr>
<td>6</td>
<td>F4</td>
<td>64,47514507</td>
<td>2,18095618</td>
</tr>
<tr>
<td>8</td>
<td>F5</td>
<td>69,69010322</td>
<td>3,81760399</td>
</tr>
<tr>
<td>10</td>
<td>F6</td>
<td>73,36790367</td>
<td>5,92294761</td>
</tr>
</tbody>
</table>

Table 12.7. Data of a number of prismatic duals (E) and antiprismatic duals (F).

5. Stretched tetrahedra

![Disphenoids](image12.48)

Fig. 12.48. Disphenoids, or stretched tetrahedra made from four congruent non-regular triangles.

6. Stretched and rounded even sided prisms

![Elongated even-sided prisms](image12.49)

Fig. 12.49. Elongated even-sided prisms with rounded or capped ends.

They are based on an infinite set of - even sided - prisms. All the (rectangular) faces they may actually land on are congruent, so they are equally fair. (The other 2 sides of the prism are rounded or capped with a pyramid, designed so that the die never actually rests on one of those faces.)
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7. Isohedra

Fig. 12.50. The seven Archimedean isohedra.

<table>
<thead>
<tr>
<th>Hedron</th>
<th>R2</th>
<th>z</th>
<th>R1</th>
<th>100*R1/z</th>
<th>Deficient angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0,57735027</td>
<td>0,40824829</td>
<td>0,7071068</td>
<td>173,20508076</td>
<td>120</td>
</tr>
<tr>
<td>14</td>
<td>0,29885849</td>
<td>0,40824829</td>
<td>0,50594769</td>
<td>123,93136746</td>
<td>30</td>
</tr>
<tr>
<td>26</td>
<td>0,21314838</td>
<td>0,40824829</td>
<td>0,46054196</td>
<td>112,80928104</td>
<td>15</td>
</tr>
<tr>
<td>32</td>
<td>0,31599428</td>
<td>0,75576131</td>
<td>0,81916271</td>
<td>108,38907679</td>
<td>12</td>
</tr>
<tr>
<td>38</td>
<td>0,29885849</td>
<td>0,75576131</td>
<td>0,81270632</td>
<td>107,53478704</td>
<td>30</td>
</tr>
<tr>
<td>62</td>
<td>0,25599557</td>
<td>0,75576131</td>
<td>0,79794040</td>
<td>105,58100741</td>
<td>6</td>
</tr>
<tr>
<td>92</td>
<td>0,25564473</td>
<td>0,75576131</td>
<td>0,79782792</td>
<td>105,56612363</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 12.8. Data of the Archimedean isohedra.

Considering the previous data, the isohedra are in many respects the roundest polyhedra of all. And of these the 62-hedron has the smallest deficient angle, which means that this one most closely approaches the theoretical circumscribed sphere with the radius R1. The ratio of R1 over the face distance z also is an important aspect, where both the 62- and the 92-hedron score best.

12.4. Soccer Ball Construction

12.4.1. A new soccer ball design

It is interesting to note, that the polyhedra from Table 12.8 can be ordered following their number of faces and that all members of the thus formed groups have their own isodistant representative. A striking phenomenon is that the ratio of the sides in almost all cases is based on a very simple expression with the sinus of angles in integer numbers. The only exceptions are the 26-hedron and the 38-hedron, that are accurate in no more detail than 0.5 degrees.

Although the before considerations seem quite theoretical, some of the outcome can be applied for very practical purposes. The most obvious characteristic of the found figures is, that they have all their faces as well as their vertices at the same distance from the centre. Generally spoken, this means that they more closely approximate the sphere, that can be circumscribed through the vertices, than their counterparts. This is even better, when the figure is composed of a greater number of faces, such as in the 62-hedron and the 92-hedron.

The roundness of any spherical object, that is composed of flat faces, can thus be improved and therefore the author studied the use of this idea in the construction of soccer balls. These are made from a flexible fabric reinforced membrane, which is basically produced in flat sheets. Most balls are made in the form of a truncated icosahedron, which is - as in the foregoing is mentioned - composed of 20 flat regular hexagons and 12 ditto pentagons, that are sewn together - usually by hand. The ball becomes spherical by the inflation of a rubber bladder inside. The flat sheets must be stretched in order to reach this sphere surface. The isodistant version of the 32-hedron might thus form a better alternative.
As said before: the greater the number of faces the closer the approximation of the sphere. The one that is the uppermost eligible is the 94-hedron, or the isodistant snub dodecahedron. In order to reduce the number of sides, four triangles at a time can be combined to form larger equilateral triangles. This is also the case in the isodistant version, although the ratio of the side lengths is different from that of the mother version.

This could be materialized in the form of a soccer ball, but in actual practice this proved to be difficult to make. Always one seem meets the other one under 60°, which easily causes secondary deviations from the sphere and this must be avoided. This idea was therefore abandoned. A more promising result was found in the 62-hedron. Here again the total length of the seams was considered to be unacceptably great and a transformation was introduced where the rectangular panels were cut in four parts, that were added to adjacent 'triangular' and 'pentagonal' panels. In this way a solution was found of twelve slightly folded 'pentagons' twenty hexagons (see the yellow picture C and the paper model D in Fig.12.54).
When inflated a ball is formed with as many panels as the standard ball, but closer to the ideal sphere surface and with panels that have almost the same area (5.2% difference vs 50%). Upon each panel a circle can be inscribed, that is identical. This means that the span and hence the stiffness of all panels is the same. This gives an optimal material and stress distribution, so that the best accuracy in ball handling is guaranteed.[12.4]

A similar trick can be done with the 26-hedron consisting of 26 parts: 12 squares, 8 hexagons and 6 octagons. Its faces can also be brought at equal distances from the centre (Fig. 12.56.A and -B); the rectangular faces can be subdivided and added to the adjacent faces. The new figure consists of only 14 parts; 8 ‘hexagons’ and 6 ‘squares’ with slightly curved sides (the blue lines in Fig. 12.56C). The two types of faces again have almost the same magnitude: 8.2% difference and they have the same inscribed and circumscribed circles. This principle has been applied in a few prototypes. The Truncated Cuboctahedron, consisting of 6 octagons, 8 hexagons and 12 squares. This number of 26 panels can thus be reduced to 14 by the redistribution of the rectangles.
15.4.2. Evaluation of design criteria

In Table 12.9 a number of data are given, that were considered the most determining for four eligible ball designs, two commercially available types: A (standard ball) and D (Hyperball) and two others that mainly are of theoretical value: B (isodistant 32-hedron) and C (32-hedron with 6- and 5-sided faces of equal area), all in the un-inflated state.

<table>
<thead>
<tr>
<th>Radius of sphere: R = 110 mm</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Main values</strong></td>
<td>32-hedron</td>
<td>Iso-32-hedron</td>
<td>Equal faces</td>
<td>Iso-62-hedron</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
<td>44.39030317</td>
<td>49.88275193</td>
<td>50.91998522</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>44.39030317</td>
<td>34.51795190</td>
<td>32.62420190</td>
</tr>
<tr>
<td>3</td>
<td>b/a</td>
<td>1.00000000</td>
<td>0.69198171</td>
<td>0.64069543</td>
</tr>
<tr>
<td>4</td>
<td>Distance Z_i in mm</td>
<td>5</td>
<td>103.31569831</td>
<td>All</td>
</tr>
<tr>
<td>Distance Z_i in mm</td>
<td>6</td>
<td>100.64542102</td>
<td>All</td>
<td>101.48624147</td>
</tr>
<tr>
<td>5</td>
<td>Ave. distance Z_{ave}</td>
<td>101.64677501</td>
<td>101.48624147</td>
<td>101.43156465</td>
</tr>
<tr>
<td>6</td>
<td>R - Z_{ave}</td>
<td>8.35322499</td>
<td>8.51375853</td>
<td>8.56843535</td>
</tr>
<tr>
<td>7</td>
<td>Area A_{5} in mm²</td>
<td>3390.19902407</td>
<td>4281.04488683</td>
<td>4460.93104420</td>
</tr>
<tr>
<td>Area A_{6} in mm²</td>
<td>5119.50661670</td>
<td>4575.72301994</td>
<td>4460.93104420</td>
<td>4595.3350063</td>
</tr>
<tr>
<td>8</td>
<td>Ratio A_{5}/A_{6} in %</td>
<td>51.00902868</td>
<td>6.88332267</td>
<td>1.00000000</td>
</tr>
<tr>
<td>9</td>
<td>Diam. incircle Di5</td>
<td>5</td>
<td>61.09801073</td>
<td>All</td>
</tr>
<tr>
<td>Diam. incircle Di6</td>
<td>6</td>
<td>76.88626045</td>
<td>All</td>
<td>67.06985067</td>
</tr>
<tr>
<td>10</td>
<td>Di6/Di5</td>
<td>1.25840857</td>
<td>1.00000000</td>
<td>1.0446054</td>
</tr>
<tr>
<td>11</td>
<td>Area_{tot} in mm²</td>
<td>143072.52062285</td>
<td>142886.99904067</td>
<td>142749.79341430</td>
</tr>
<tr>
<td>12</td>
<td>Area_{tot}/Area_{theor}</td>
<td>0.94093797</td>
<td>0.93971786</td>
<td>0.93881551</td>
</tr>
<tr>
<td>13</td>
<td>Volume_{tot}, mm³</td>
<td>4836075.77751844</td>
<td>4833688.13083265</td>
<td>4826444.69630832</td>
</tr>
<tr>
<td>14</td>
<td>Vol_{tot}/Vol_{theor}</td>
<td>0.86741401</td>
<td>0.86698575</td>
<td>0.86568660</td>
</tr>
<tr>
<td>15</td>
<td>C</td>
<td>0.96662189</td>
<td>0.96755834</td>
<td>0.96752057</td>
</tr>
<tr>
<td>16</td>
<td>IQ</td>
<td>0.90317097</td>
<td>0.90597826</td>
<td>0.90569220</td>
</tr>
<tr>
<td>17</td>
<td>Seam length, mm</td>
<td>3995.1278529</td>
<td>4028.50367250</td>
<td>4033.9261989</td>
</tr>
<tr>
<td>18</td>
<td>Panels number</td>
<td>32</td>
<td>32</td>
<td>32</td>
</tr>
<tr>
<td>19</td>
<td>Deficient angle</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>20</td>
<td>Stretch in%</td>
<td>6.27693129</td>
<td>6.41491910</td>
<td>6.51720104</td>
</tr>
</tbody>
</table>

Table 12.9. Review of design criteria for various ball types.

For R = 110mm:

\[
\text{Area}_{\text{theoretical}} = 4\pi R^2 = 152053.08443375 \text{ mm}^2
\]

\[
\text{Volume}_{\text{theoretical}} = \frac{4\pi R^3}{3} = 5575279.76257068 \text{ mm}^3
\]

Comments:
Row 1 and 2) Gives lengths of sides. Note that A in case D is the developed length of this side, as it is actually slightly curved.
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Row 4) \(Z_5\) and \(Z_6\) are the distances of the pentagon and of the hexagon from the centre. B and D have all their faces at the same distance from the centre.

Row 5) In B and D the faces have the same distance from the centre. In A and C the distances are averaged as: 
\[
(12 \times Z_5 + 20 \times Z_6)
\]

Row 4): D has its faces by far at the smallest distance from the sphere surface.

Row 8) Note the great differences in panel areas. C has by definition equal faces, but D is second.

Row 10): The more mutually equal the diameters of the inscribed circles the more even the stress distribution.

Row 15 and 16) For the expression of the roundness sometimes one of the two following values are used:

Compactness: 
\[
C = \frac{3\sqrt[3]{36 \pi V^2}}{A}
\]

Isoperimetric quotient: 
\[
IQ = \frac{36 \pi V^2}{A^3}
\]

If Volume and Area are equal to the theoretical values, then both values are equal to 1. The closer to 1 the value for any polyhedron, the rounder it is considered to be.

Row 19) Deficient angle of D is two times smaller than all others: the smaller this is, the smoother its curvature.

Row 20) The lower the necessary amount of stretch to reach the ideal sphere after inflation, the lower the overall stress in the material.

To these results scores are given: yellow for the highest ranking and blue for the second place. They are related to the theoretical values:

\[
\text{Yellow} \Rightarrow \text{First score} \quad \text{Blue} \Rightarrow \text{Second score}
\]

As a whole rows 6, 8, 10, 12, 14, 15, 18, 19 and 20 are considered as the most decisive for the initial roundness. In these 9 categories D scores 8 times as highest and 1 time as second.

12.5. References


Chapter 13. POLYHEDRAL BUILDING STRUCTURES

13.1. Full scale cardboard buildings

The form of a polyhedron can lead to interesting building shapes, especially as these lend themselves to be composed of mutually equal and planar polygonal panels. A solution in multilayer cardboard has been worked out together with groups of students at the Civil Engineering Dept. of the Delft University of Technology.

Fig. 13.1. Fig. 13.2.

Fig. 13.3. The above figures give an impression of a few kinds of full scale modular houses of multilayer cardboard, built by students.
Fig. 13.4. The construction of the model in Fig. 13.1. The connections are realized with plywood stripes at both sides of two matching flanges, kept together by bolts.

Fig. 13. 5. One of the models is being erected.
Fig. 13.6. Impression of potential building forms, based on polyhedral geometry, with the help of a limited number of different panel forms.
Chapter 13

13.2 Prefab panel building system

The cardboard models of Chapter 13.1 were in fact meant as a study object for a prefabricated house building system in emergency situations [13.2].

Fig. 13.7. Review of the different panel forms, used for the composition of the building shapes, suggested in Fig. 13.6, but translated in actual panel size with \( a = 125 \) cm.

Fig. 13.8. Idea of a simple house.

Fig. 13.9. Scale model of a part of this house.
Fig. 13.10. *Just another idea of a possible house, based on the same panels as in the previous model.*

Fig. 13.11. *Elevated view of the ground floor of the house in Fig. 13.10.*

Two other prototype buildings have been built from sandwich panels with impregnated paper honeycomb cores and linings of water resistant hardboard (Fig. 13.14) and a metal hinges.
Fig. 13.12 and Fig. 13.13. Full scale prototype of a part of the previous modular house.

Fig. 13.14. Panel construction.  

Fig. 13.15. Interconnection with metal hinges.

Figs. 13.16. A second prototype (see below)  

Fig. 13.17. Erection of the prototype.
13.3. Office building in Bamako, Mali

In Bamako Mali in the year 1988 an office building had been constructed by the author and his colleagues from the Delft University of Technology from palmwood. This is a very rough and hard, hence termite resistant, material that is gained by splitting the stems of palm trees. Such trees have a quite irregular form with a thicker ‘waist’ in the middle. The thus formed beams or trusses have a roughly triangular cross-section and show great differences in diameter in length direction. The local techniques allow no further machining of the wood. This therefore is a rather difficult material to work with and it must more or less be used as hinged trusses, allowing only longitudinal loads. The connections were done in a, for western countries, traditional way with threaded ends, washers and nuts. The building has the form of three adjoined rhombicuboctahedra.
13.4. Plastics space structures

In the sixties great enthusiasm arose for building structures that are completely made of plastics panels or spatial elements, mainly of glass fibre reinforced polyester (GRP) that was considered to be the material of the future. The author himself experimented in this area with more or less success. The durability of this material, its sustainability and its load to the environment in many aspects remains a matter of concern. But nevertheless a few buildings had been realized during this period that maybe worth to be mentioned here [13.1].

13.4.1. Sandwich panel demonstration system
Fig. 3.24. Half of P13, the truncated icosahedron, consisting of 15 triangles, 6 pentagons and 10 hexagons.

Fig. 3.25. Half of icosidodecahedron P12, composed of 10 triangles and 6 pentagons.

Fig. 3.26. Hexagonal prism: 6 squares and 1 hexagon.

Fig. 3.27. Pentagonal antiprism: 10 triangles and 1 pentagon.

Fig. 13.28. Configuration built of squares and triangles.

Fig. 13.29. Meeting of a number of panels with aluminium hinges.
The panels have the form of sandwich panels and are composed of GRP linings against a polyurethane foam core. The open slots in the panel sides contain aluminium hinges that fit around an aluminium tube. As this system was only designed for demonstration purposes no special precautions had to be taken to ensure a water tight connection.

13.4.2. GRP canopy roof at Arnhem, The Netherlands

The struts that interconnected the apexes of the pyramids were made of glassfibre polyester by filament winding in length direction around a polyurethane foam core.

The construction was dismantled around 1985 and does no longer exist.
13.4.3. GRP load-bearing roof structure at Delft, The Netherlands

Fig. 13.30. Birds eye view of an experimental building completely built of tetrahedral roof elements and of GRP wall panels

Fig. 13.31. Structure seen from ground floor level   Fig. 13.32. Interior view of the building.

The GRP material of the roof elements was to a great extent translucent. The structure was erected as a practical experiment at an outdoor site of the Stevin Laboratory of the Civil Engineering Department of Delft University. The roof of the building was composed of shallow all-sided closed tetrahedral elements [13.1] It was built in 1974 and taken apart again in 2010.
13.4.4. Entrance building in front of the Gist & Spiritus factory at Delft

Fig. 13.33. **Entrance building at Delft, 1980.**

Fig. 13.34. **The building is composed of pyramids in the form of half-octahedra.**

Fig. 13.35. **The apexes of the pyramids are interconnected with MERO struts against special flat nodes.**
13.5. The materialization of space frames

Structural forms can be described as solid, as planar or as linear. Space frames are composed of linear elements with hinge-like connections, usually with spherical or polyhedral node elements. Leonardo da Vinci and, more recently, M.C. Esscher are two famous artists who paid much attention in their works of art to polyhedra in their spatial appearance.

If the wish arises to visualise these space frames, it is necessary to ascribe to the theoretically derived linear elements realistic geometric properties in order to give them a material appearance. The author has been working since quite some time on the visualisation of structural forms, which are generated as a set of planes with the help of a computer program called Cordin, that has been specifically developed for the generation of polyhedron based forms. For the material presentation of spatial structures a conversion routine had been developed, translating these planar forms - as generated with this program - into wire frames. The struts can consecutively be shaped so that they obtain the appearance of real structural elements, eventually with additional nodes. In this chapter this method is described and some of the results are shown. This is demonstrated in the design of a.o. two tower structures of roundwood that have been constructed in The Netherlands in the years 1995 and 2000. Suggestions for various versions of towers varying in height are shown here, making use of a limited number of different kinds of elements.

13.5.1. Structure of data sets

The structural forms, found by the generation procedure of this programme, consist of planes [1.2]. The definition of these planes is given by the number of their vertices, that can in fact be an integer number in the case that the face is a flat, regular polygon, its shape is defined by the input of the number of vertices and of a certain scale factor.
In other cases such a face can be formed by the input of a text file containing the co-ordinates of the vertices, followed by a listing of the vertex numbers in the order in which they occur, according the circumference of the face. This results in a shape, consisting of a number of plane faces. The faces are generally stored in a more complex form than that of their input, that is to say: the number of vertices and the co-ordinates of these vertices. Thus formed - virtually 3-dimensional - shapes are stored in a condensed state as objects in a memory, ready for eventual further use. Such data sets are not compatible with most of the existing computing programmes and they must be converted according the desired format. A conversion routine is provided, where the faces are skipped and only one single set of boundary lines or system lines is maintained. These data are stored as a text file, consisting of a set of node numbers preceded by a serial number and by a set of connecting lines, that are defined by a serial number and the numbers of the two nodes that it connects. As this data set has a general character, such a set can directly be used as the input in computing programmes for the calculation of strut forces and deformations in the space frame, represented by these connection lines.

The description of these connection lines is basically theoretical, which means that the material properties are given to the struts in a later stage. Pictures of space frames, generated by computing programmes are normally of the wire frame type and often do not give a true image of the reality. Therefore a routine has been developed in order to give to these wire frames an appearance that is as close as possible to the final space structure. This process has been called "materialization".
13.5.2. Materialization method

The lines of the wire frame can at wish be materialised in such a way that they obtain the appearance of real struts as parts of spatial configurations. 18 directions meet in the central node.

Fig. 13.42. At left: pile of Cuboctahedra in 3 different projections with triangular pyramid on top and underneath and at right: pile of square antiprisms.
13.4. Shaping the struts

The struts themselves are defined as elongated prisms with the length $L$ and the diameter $d$ and they can be shaped further with the help of a number of variables such as:

- Number of sides ($n = 1$ to 50)
- Radius $r$ of the diameter $d$. If the faces themselves have a distance $m$ from the axis, then $r = m/\cos \varphi$, with $\varphi = 360^\circ/n$
- Cutting length $a$
- Choice of ending, straight with or without line extension, or conical
  - Eventual truncation $t$ of conical ends. The length of this extra truncation has generally to do with the diameter of an eventual, additional node. If this node is polyhedral, each of its faces has a distance from the centre: $z = \sqrt{R_1^2 - R_2^2}$, with $r_1$ being the radius of the circumsphere and $r_2$ the radius of the circumcircle of the face.

Fig. 13.43 Different types of strut endings, eventually in combination with a node.

Fig. 13.44. Details of the space frame with the polyhedral node.
Fig. 13.45. Space frame configuration.

Fig. 13.46. Perspective view of extended set-up. These configurations are shown in full 3D in the form of so-called anaglyphs (Chapter 16), that can be seen with blue-end-red-glasses.
In the case of the space frame of Figs. 13.44 to 13.46 the node has the form of a so-called Rhombicuboctahedron, which has 18 square faces and 8 triangular faces. This solution reminisces strongly of the well-known MERO-system. The struts are connected to this node by screwing it against one of the square faces, that all are provided with a threaded hole. Therefore 18 strut directions are present, offering a great variety of connection possibilities. If the modular length of the space frame is taken as 1, the other data are:

<table>
<thead>
<tr>
<th>Strut length = 1</th>
<th>Node: P10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of sides = 16</td>
<td>Side length = 0.04</td>
</tr>
<tr>
<td>D = 0.08</td>
<td>R₁ = 0.04x1.39896633 = 0.055958653</td>
</tr>
<tr>
<td>A = 0.15</td>
<td>R₂ = 0.04x0.70710678 = 0.028284271</td>
</tr>
<tr>
<td>T = 0.048</td>
<td>Z = 0.048284271</td>
</tr>
</tbody>
</table>

Fig. 13.47. Materialization of Tetrahedra in a Cube. The meeting of struts in the centre  counts 18 directions coinciding with the square faces of the Rhombicuboctahedron P10. This gives shape to the node that is shown here.
Chapter 13

13.6. Roundwood timber for space frames

In the recent past many types and forms of space structures have been executed and studied by many persons, institutes and firms. This here is not the place to give a complete overview of all different developments. But the author had the opportunity to apply some of the principles of space structures in the form of roundwood timber frames [13.13 and 13.14]. This material is greatly underestimated and it therefore was worth while to try to apply thin roundwood stems in space structures, as the struts are relatively thin and are mostly loaded in axial direction.

13.6.1. Square offset space frames

A number of structures of thin roundwood poles were built since 1986 on the principle of a space frame. They had overall dimensions of 10.8 x 16.2 m, 8.1 x 16.2 m, 8.1 x 18.9 m in plan, at a height of 3.8 to 6.0 m. The roof frames were all composed of poles with a nominal diameter of 10 cm. The columns are generally thicker and are of 15 cm diameter. The poles meet each other against circular or octagonal steel nodes and are interconnected with bolts. This connection method facilitated the construction of relatively large span load-bearing frames for buildings, made of roundwood members with diameters of generally not more than about 10 cm. A few of these structures have been realized during the recent years in the Netherlands and in England. Their structural behaviour is promising; they tend to behave strong and stiff. In most of these cases the wood was debarked only and required no other machining apart from the provisions at their ends, that were necessary for the connections.
The problem with thin roundwood is that sooner or later it may show radial shrinkage cracks. This of course is dangerous if larger loads have to be transferred, when the normal connection methods as with bolts or threaded ends have to be used. To overcome this a special solution had been developed: the bolt shaft is replaced by a metal tube, steel wires are led through it and these wires were then tightly wrapped at both sides of the strut with the help of a lacing tool, that was specifically designed for this by Jaap Lanser. This tool was called: ‘Delft Wire Lacing Tool’. Working drawings of this tool and manuals in various languages are available as a free download from the author’s own website www.pieterhuybers.nl.
Fig. 13.51. *Cross-section of a thin wooden pole with cracks caused by shrinkage*

Fig. 13.52. *Special wire lacing tool, developed at the Delft University*

Fig. 13.53. *The designer of the tool, Jaap Lanser, demonstrating its working with an earlier prototype.*

Working drawings of the tool and a small manual in the languages Dutch, English, French, German and Finnish can be downloaded at the link [http://www.pieterhuybers.nl/pagina37.html](http://www.pieterhuybers.nl/pagina37.html)

Fig. 13.54. *Detail view of structure in Lelystad*

Fig. 13.55. *Similar structures at Dronten, The Netherlands, and in Winchester, England*
13.6.2. Diagonal, nodeless space frame

A particular roundwood space frame has been built in 1990 in Rotterdam. This consists of two individual parts in a diagonal arrangement with respect to the groundfloor plan. The larger part forms the roof frame of a closed building and had been clad later on. The other part is free standing [13.8].

Fig. 13.56. Roundwood structure on a children's play ground in Rotterdam.

Fig. 13.57. Model of the nodeless roundwood structure.

Fig. 13.58 The connection with steel plates that are inserted in the pole ends

Fig. 13.56. The connection taken apart
Fig. 13.59. *Roundwood structure in Rotterdam*

Fig. 13.60. *Ending of the poles, which all have the same length and identical steel connectors.*
Figs. 13.61. *Standard detail of the building. The poles can be connected without additional node elements, apart from eventual filling plates or angled parts as in Fig. 13.58.*

The interconnection differs from the previous cases. The plates in the ends of the poles are provided with welded-on pieces of angle steel. They fit together in a special way, so that in fact no extra node elements are needed. Diagonal arrangements are easy to be realized with this solution.

Fig. 13.62. *Free standing part of structure in Rotterdam*
Fig. 13.63. *Later to be closed part, containing stables for small cattle.*

**13.6.3. Alternative application of the space frame system**

Fig. 13.64. *Model of a parallel space frame in the form of a saddle roof.*
13.6.4. Roundwood watch-tower at Kootwijk

In June 1999 a tower structure of thin round wood poles has been completed in Kootwijk, The Netherlands [Ref.13.10]. It is used as a watchtower and it is situated in a dune landscape. This tower structure was one of the main outcomes of the Dutch contribution to the international co-operation project Fair-2 [13.7]. This tower has the shape of three stacked octagonal antiprisms. The height of each antiprism is 3 m. The edge length of the top and the bottom octagon is 2.5 m. The two middle octagons have the smaller edge length of 1.91 m. So that a narrower ‘waist’ is formed at half height, giving the impression of a sand-glass. This shape was considered as to form a characteristic landmark in the surroundings. The octagon edges and the inclined struts were made of 140 mm thick poles. The central mast fulfils a supporting and binding function and has a diameter of 200 mm. The stairs were made of steel plate with timber clad steps. The building has been constructed of debarked, non-calibrated larch wood.

![Diagram of the tower structure](image)

Fig. 13.65. Tower structure of roundwood, built in 1999 in Kootwijk, The Netherlands.

Fig. 13.66B indicates the various strut types. The table below shows the used variables.

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>D in m</th>
<th>B=2m</th>
<th>A</th>
<th>Line = ●</th>
<th>Cone = ●</th>
<th>T = ●</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>0.424</td>
<td>0.3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>0.566</td>
<td>0.4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>C</td>
<td>8</td>
<td>0.141</td>
<td>0.139</td>
<td>0.14</td>
<td>X</td>
<td>X</td>
<td>0.13</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>± 0.028</td>
<td>± 0.02</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
<td>0.184</td>
<td>0.13</td>
<td>0.1</td>
<td>X</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>F</td>
<td>4</td>
<td>0.213</td>
<td>0.15</td>
<td>0.1</td>
<td>X</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>G</td>
<td>4</td>
<td>0.184</td>
<td>0.13</td>
<td>-</td>
<td>X</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>H</td>
<td>8</td>
<td>0.1</td>
<td>0.092</td>
<td>0.05</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>J</td>
<td>16</td>
<td>0.2</td>
<td>0.196</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 13.1. Review of the different struts that constitute the tower.
Fig. 13.66. The final tower structure and the generated struts, indicated in Table 13.1.

The occurring tapering was accepted if not greater than 10 mm/m. In all cases the interconnection of the struts takes place via steel plates in the ends of the wooden poles. The poles have a slit, in which the plate is embedded. In corresponding holes a steel dowel is put through pole and plate, two in each end of the thinner poles and four in those of the central mast. The dowel is fixed in the hole by nails and around the two opposite holes a 25 mm wide stainless steel band is wrapped and subsequently sealed. This band is prevented from shifting by two wood screws. The final appearance of such a connection is very neat, because nothing but the band straps are visible.

During the evaluation process after the completion of the tower, it was concluded, that the tower as a whole had a very good structural and visual impact. The connection detail appeared however to be quite complex and it was suggested to use in following cases the hollow sphere of Fig. 13.91. This might have formed a more practical and economic solution [13.11].
Fig.13.67 Sketch of elevations and floor plans of the tower structure.
Fig. 13.68. *Sketch of typical connection*

Fig. 13.69. Detail photograph mantle showing a mantle detail.

Fig. 13.70. *The completed tower structure.*
Fig.13.71. Sketch of the final situation
Fig. 13.72. *Interior views of the tower*

Fig. 13.73. *Sketch of the same 10 meter tower, but now with a wider foot storey.*
Fig. 13.74. Plan and elevation of wider foot storey: further referred to as Part D
Fig. 13.75. 12 meter high tower in two versions: with the original narrow footing and at right with the wider ground storey.

The process of materialisation deals at present only with the visual characteristics of the frame, although it is imaginable that it in a later stage this could be extended towards a real and actual form of materialisation. A computer programme like CORDIN may offer ways for the visual presentation of spatial structures without the need to build them in reality. It is based on the generation of facial objects, but these can be transformed into skeletal frames. Once these frames and their nodes having been formatted in a proper way, they can be materialised in the form of a real and realistic configuration of struts and nodes.
Fig. 13.76. *Suggestion for a 15 meter high tower based on the wider ground storey.*
13.6.5. Observation tower

In April 1995 the construction of an observation tower with a total height of 27 meter took place in Apeldoorn. It is composed of 13, 15 and 20 cm thick poles. It has connecting nodes, that basically consist of 4 identical steel elements, made out of standard angle steel. They fit together and allow a maximum number of 18 meeting poles.

Fig. 13.77. Sketch of the 27 meter high observation tower in Apeldoorn.
Fig. 13.78. Different combinations of the basic node element sometimes with an additional plate.

Fig. 13.79. Construction of the node from four basic elements, allowing the connection of 18 struts in directions under the polar angles of the rhombicuboctahedron P10.

Fig. 13.80. One particular meeting of struts in the framework.

Fig. 13.81. A complete node, made of steel.
Fig. 13.82. *View of completed tower.*

Fig. 13.83. *Tower seen from a helicopter during the official opening ceremony.*

Fig. 13.84. *View of tower from foot to top.*
In this case the plates in the poles are 10 mm thick and they are fixed to the poles with bolts and then secured with wire lacings. The tower has a stair case and is accessible to the public. Consulting Office De Bondt acted for this project as the supervisor for the construction [13.11].

FIG. 13.85. *View of tower from below.*
13.7. Spherical space structures

As a next step the material presentation of spherical structures is shown. The form of this structure has also been generated by the program corelli itself. The way in which this is done, has been previously described [13.6]. It is an Icosahedron, subdivided in the frequency 2. Nodes can be added. They have to be made in advance, so that they can be produced at wish. They are generated with the program and kept in a library.

Fig. 13.86. The Icosahedron and its 2-frequency subdivided version.

Fig.13.87. Sphere with a configuration, based on an icosahedron with a 2-frequency subdivision. The nodes have the same geometry, but are 30 times smaller (see also Fig. 16.16 in Chapter 16).
A little temporary dome structure with a diameter of 6 m and a total height of also 6 m, composed of 10 cm thick cylindrical poles had been built in the early beginning of the structural round wood research (1984). This dome was used several times as a pavilion for exhibition purposes [13.13]. Later it had been provided with a tent structure, suspended from the nodal points, and from 1999 to 2000 it was exposed at a building site in Kootwijk.

The design of a larger dome had been worked out with a span of 25 m. From the statical analysis it appeared, that poles of not more than 13 cm diameter would be sufficient. For the standard detail a somewhat different solution had been suggested: a hollow steel sphere against which the poles were connected with a central bolt. For this purpose, the plate in the end of the pole would be provided with a cylindrical part with an internally threaded hole.
Fig. 13.90. Plan of 25 meter diameter dome structure, designed to be made of thin roundwood poles: hemisphere on octagonal basis Class 2 with the frequency 8.

Fig. 13.91. Spherical steel nodes, suggested for the connection of the struts in the construction to be built of wooden poles.

Fig. 13.92. Cross-section of the spherical node.
Fig.13.93. Hemisphere, also of Class 2 subdivision with a frequency of 12.

13.8. References


Chapter 14. THE UNIFORM POLYHEDRA BY THE NUMBER

14.1 Notes on the polyhedron data

In the next part data are given per polyhedron, its pyramidized form and its reciprocal. They were calculated with the sub-programme HEDRON, which makes out an integral part of the main programme CORDIN. In this page is explained what is meant with their short names.

The geometry of a polyhedron is defined by the polygons of which it is composed. Each polygon gives a contribution to the magnitude of the values, that are relevant for this geometry. The ciphers in brackets refer to the formula, according to which these are calculated. All calculated data are related to the unit side length = 1. In brackets the formula, derived in the foregoing.

P = Index number of polyhedron
n = number of polygon edges

1. Polyhedron

R1 = radius of circumscribed sphere through the vertices \{1.13\}
R2 = radius of circumscribed circle around polygon with edge number n \{1.2\}
R3 = radius of circle through adjacent vertices around a given vertex \{1.6\}
R5 = distance of edge mid-point to polyhedron center \{1.14\}
z = distance of polygon from polyhedron center \{1.15\}
ksi = partial dihedral angle at mid-edge \{1.16\}
volp = partial volume of polyhedron \{1.18\}

2. Pyramidization

h = height of polygonal pyramid \{1.20\}
e = length of inclined edge of pyramid \{1.21\}
y = distance of triangular side plane to polyhedron center \{1.24\}
psi1 = dihedral angle between side plane and plane through polyhedron edge and polyhedron center \{1.26\}
psi2 = dihedral angle between side plane and plane through inclined edge and polyhedron center \{1.28\}
labda = base angle of isosceles triangular side plane \{1.23\}

3. Reciprocal figure

R6 = radius of internal sphere touching the faces of the reciprocal figure \{3.6\}
R7 = distance from face corner to polyhedron center \{3.5\}
g = contribution to side length of reciprocal \{3.2\}
j = distance of face center to face corner \{3.4\}
theta = dihedral angle on edge \{3.7\}
zeta = half the central angle in face portion defined by the specific n-gon \{3.14\}
chi = half the face angle in face portion defined by the specific n-gon \{3.15\}
volr = partial volume of reciprocal \{3.21\}
Fig. 14. 1. Review of the regular and the semi-regular polyhedra.

**P1, Tetrahedron**

\[
\begin{align*}
p & = 1 \\
n & = 3 \\
R1 & = 0.61237244 \\
R2 & = 0.57735027 \\
R3 & = 0.57735027 \\
R5 & = 0.35355339 \\
z & = 0.20412415 \\
ksi & = 35.26438968 \\
volp & = 0.02946278
\end{align*}
\]

\[
\begin{align*}
h & = 0.40824829 \\
e & = 0.70710678 \\
y & = 0.35355339 \\
psil & = 90.00000000 \\
psil2 & = 45.00000000 \\
labda & = 44.99999999 \\
R6 & = 0.20412415 \\
R7 & = 0.61237244 \\
g & = 0.50000000 \\
j & = 0.57735027 \\
theta & = 70.52877937 \\
zeta & = 59.99999999 \\
chi & = 60.00000000 \\
volr & = 0.02946278
\end{align*}
\]
P2, Cube or Hexahedron

\[ p = 2 \]
\[ n = 4 \]
\[ R_1 = 0.86602540 \]
\[ R_2 = 0.70710678 \]
\[ R_3 = 0.81649658 \]
\[ R_5 = 0.70710678 \]
\[ z = 0.50000000 \]
\[ ksi = 45.00000000 \]
\[ volp = 0.16666667 \]
\[ h = 0.36602540 \]
\[ e = 0.79622522 \]
\[ y = 0.69879436 \]
\[ psi_1 = 81.20602311 \]
\[ psi_2 = 65.31171830 \]
\[ labda = 51.10000202 \]
\[ R_6 = 0.57735027 \]
\[ R_7 = 1.00000000 \]
\[ g = 0.70710678 \]
\[ j = 0.81649658 \]
\[ theta = 109.47122063 \]
\[ zeta = 59.99999999 \]
\[ chi = 60.00000000 \]
\[ volr = 0.22222222 \]

Fig. 14. 3

P3, Octahedron

\[ p = 3 \]
\[ n = 3 \]
\[ R_1 = 0.70710678 \]
\[ R_2 = 0.57735027 \]
\[ R_3 = 0.70710678 \]
\[ R_5 = 0.50000000 \]
\[ z = 0.40824829 \]
\[ ksi = 54.73561032 \]
\[ volp = 0.05892557 \]
\[ h = 0.29885849 \]
\[ e = 0.65011517 \]
\[ y = 0.49126003 \]
\[ psi_1 = 79.27141688 \]
\[ psi_2 = 51.47231231 \]
\[ labda = 39.72735471 \]
\[ R_6 = 0.35355339 \]
\[ R_7 = 0.61237244 \]
\[ g = 0.35355339 \]
\[ j = 0.50000000 \]
\[ theta = 90.00000000 \]
\[ zeta = 44.99999999 \]
\[ chi = 90.00000000 \]
\[ volr = 0.04419417 \]

Fig. 14. 4
**Chapter 14**

**P4, Dodecahedron**

Fig. 14. 5

\[
p = 4 \\
n = 5 \\
R1 = 1.40125854 \\
R2 = 0.85065081 \\
R3 = 0.93417236 \\
R5 = 1.30901699 \\
z = 1.11351636 \\
ksi = 58.28252559 \\
volp = 0.63859325 \\
h = 0.28774217 \\
e = 0.89799908 \\
y = 1.29280420 \\
psi1 = 80.97300596 \\
psi2 = 76.89479525 \\
labda = 56.16566844 \\
R6 = 1.22284749 \\
R7 = 1.53884177 \\
g = 0.80901699 \\
j = 0.93417236 \\
theta = 138.18968510 \\
zeta = 60.00000000 \\
chi = 59.99999999 \\
volr = 0.77015069
\]

**P5, Icosahedron**

Fig. 14. 6

\[
p = 5 \\
n = 3 \\
R1 = 0.95105652 \\
R2 = 0.57735027 \\
R3 = 0.85065081 \\
R5 = 0.80901699 \\
z = 0.75576131 \\
ksi = 69.09484255 \\
volp = 0.10908475 \\
h = 0.19529520 \\
e = 0.60948630 \\
y = 0.78772539 \\
psi1 = 76.82592460 \\
psi2 = 60.96999005 \\
labda = 34.87885354 \\
R6 = 0.68819096 \\
R7 = 0.86602540 \\
g = 0.30901699 \\
j = 0.52573111 \\
theta = 116.56505118 \\
zeta = 36.00000000 \\
chi = 107.99999999 \\
volr = 0.09045085
\]
P6, Truncated Tetrahedron

\[ p = 6 \]
\[ n = 3 \]
\[ R_1 = 1.17260394 \]
\[ R_2 = 0.57735027 \]
\[ R_3 = 0.90453403 \]
\[ R_5 = 1.06066017 \]
\[ z = 1.02062073 \]
\[ ksi = 74.20683095 \]
\[ volp = 0.14731391 \]
\[ h = 0.15198321 \]
\[ e = 0.59701946 \]
\[ y = 1.03758610 \]
\[ psi_1 = 78.02702368 \]
\[ psi_2 = 66.20597013 \]
\[ lambda = 33.12360658 \]
\[ R_6 = 0.95940322 \]
\[ R_7 = 1.10227038 \]
\[ g = 0.30000000 \]
\[ j = 0.54272042 \]
\[ theta = 129.52119636 \]
\[ zeta = 33.55730976 \]
\[ chi = 112.88538048 \]
\[ volr = 0.13017193 \]

\[ p = 6 \]
\[ n = 6 \]
\[ R_1 = 1.17260394 \]
\[ R_2 = 1.00000000 \]
\[ R_3 = 0.90453403 \]
\[ R_5 = 1.06066017 \]
\[ z = 0.61237244 \]
\[ ksi = 35.26438968 \]
\[ volp = 0.53033009 \]
\[ h = 0.56023150 \]
\[ e = 1.14623703 \]
\[ y = 0.98455476 \]
\[ psi_1 = 68.16320244 \]
\[ psi_2 = 74.24178005 \]
\[ lambda = 64.13769007 \]
\[ R_6 = 0.95940322 \]
\[ R_7 = 1.83711731 \]
\[ g = 1.50000000 \]
\[ j = 1.56669890 \]
\[ theta = 129.52119636 \]
\[ zeta = 73.22134512 \]
\[ chi = 33.55730976 \]
\[ volr = 1.30171930 \]
Chapter 14

**P7, Cuboctahedron**

Fig. 14. 8

\[
\begin{align*}
p &= 7 \\
n &= 3 \\
R1 &= 1.00000000 \\
R2 &= 0.57735027 \\
R3 &= 0.86602540 \\
R5 &= 0.86602540 \\
z &= 0.81649658 \\
\text{ksi} &= 70.52877937 \\
\text{volp} &= 0.11785113 \\
\text{h} &= 0.18350342 \\
\text{e} &= 0.60581089 \\
\text{y} &= 0.84392481 \\
\text{psi1} &= 77.02814126 \\
\text{psi2} &= 62.31635679 \\
\text{labda} &= 34.37701645 \\
R6 &= 0.75000000 \\
R7 &= 0.91855865 \\
g &= 0.30618622 \\
j &= 0.53033009 \\
\text{theta} &= 120.00000000 \\
\text{zeta} &= 35.26438968 \\
\text{chi} &= 109.47122063 \\
\text{volr} &= 0.09943689 \\
\text{p} &= 7 \\
n &= 3 \\
\end{align*}
\]

\[
\begin{align*}
R1 &= 1.00000000 \\
R2 &= 0.70710678 \\
R3 &= 0.86602540 \\
R5 &= 0.86602540 \\
z &= 0.70710678 \\
\text{ksi} &= 54.73561032 \\
\text{volp} &= 0.23570226 \\
\text{h} &= 0.29289322 \\
\text{e} &= 0.76536686 \\
\text{y} &= 0.86285621 \\
\text{psi1} &= 85.09680372 \\
\text{psi2} &= 69.05897953 \\
\text{labda} &= 49.21052906 \\
R6 &= 0.75000000 \\
R7 &= 1.06066017 \\
g &= 0.61237244 \\
j &= 0.75000000 \\
\text{theta} &= 120.00000000 \\
\text{zeta} &= 54.73561032 \\
\text{chi} &= 70.52877937 \\
\text{volr} &= 0.26516504 \\
\end{align*}
\]
**P8, Truncated Octahedron**

\[
\begin{align*}
p &= 8 \\
n &= 4 \\
R1 &= 1.58113883 \\
R2 &= 0.70710678 \\
R3 &= 0.94868330 \\
R5 &= 1.50000000 \\
z &= 1.41421356 \\
ksi &= 70.52877937 \\
volp &= 0.47140452 \\
h &= 0.16692527 \\
e &= 0.72654253 \\
y &= 1.49976713 \\
psi1 &= 88.99039420 \\
psi2 &= 77.06068156 \\
labda &= 46.51292225 \\
R6 &= 1.42302495 \\
R7 &= 1.59099026 \\
g &= 0.53033009 \\
j &= 0.71151247 \\
theta &= 143.13010235 \\
zeta &= 48.18968510 \\
chi &= 83.62062979 \\
volr &= 0.47729708 \\
\end{align*}
\]

\[
\begin{align*}
p &= 8 \\
n &= 6 \\
R1 &= 1.58113883 \\
R2 &= 1.00000000 \\
R3 &= 0.94868330 \\
R5 &= 1.50000000 \\
z &= 1.22474487 \\
ksi &= 54.73561032 \\
volp &= 1.06066017 \\
h &= 0.35639396 \\
e &= 1.06161041 \\
y &= 1.46216612 \\
psi1 &= 77.10416278 \\
psi2 &= 79.03079113 \\
labda &= 61.90190364 \\
R6 &= 1.42302495 \\
R7 &= 1.83711731 \\
g &= 1.06066017 \\
j &= 1.16189500 \\
theta &= 143.13010235 \\
zeta &= 65.90515745 \\
chi &= 48.18968510 \\
volr &= 1.43189123 \\
\end{align*}
\]
Chapter 14

**P9, Truncated Cube**

![Diagram of P9, Truncated Cube]

\[
\begin{align*}
\text{p} & = 9 \\
\text{n} & = 3 \\
R1 & = 1.77882365 \\
R2 & = 0.57735027 \\
R3 & = 0.95968298 \\
R5 & = 1.70710678 \\
z & = 1.68252198 \\
\text{ksi} & = 80.26438968 \\
\text{volp} & = 0.24285113 \\
h & = 0.09630166 \\
e & = 0.58532670 \\
y & = 1.68740588 \\
\text{psi1} & = 81.28697661 \\
\text{psi2} & = 74.09414461 \\
\lambda & = 31.32591021 \\
R6 & = 1.63828133 \\
R7 & = 1.73205081 \\
g & = 0.29289322 \\
j & = 0.56216928 \\
\theta & = 147.35010013 \\
\zeta & = 31.39971481 \\
\chi & = 117.20057038 \\
\text{volr} & = 0.23024786 \\
\end{align*}
\]

\[
\begin{align*}
\text{p} & = 9 \\
\text{n} & = 8 \\
R1 & = 1.77882365 \\
R2 & = 1.30656296 \\
R3 & = 0.95968298 \\
R5 & = 1.70710678 \\
z & = 1.20710678 \\
\text{ksi} & = 44.99999999 \\
\text{volp} & = 1.94280904 \\
h & = 0.57171686 \\
e & = 1.42617213 \\
y & = 1.60762659 \\
\text{psi1} & = 70.34344055 \\
\text{psi2} & = 80.57217613 \\
\lambda & = 69.47666603 \\
R6 & = 1.63828133 \\
R7 & = 2.41421356 \\
g & = 1.70710678 \\
j & = 1.77326293 \\
\theta & = 147.35010013 \\
\zeta & = 74.30014260 \\
\chi & = 31.39971481 \\
\text{volr} & = 3.57862094 \\
\end{align*}
\]

Fig. 14. 10
P10, Rhombicuboctahedron

p = 10
n = 3

R1 = 1.39896633
R2 = 0.57735027
R3 = 0.93394883
R5 = 1.30656296
z = 1.27427369
ksi = 77.23561032
volp = 0.18392557

h = 0.12469263
e = 0.59066199
y = 1.28427766
psi1 = 79.40255862
psi2 = 69.91519622
labda = 32.16615072

R6 = 1.22026295
R7 = 1.33967042
g = 0.29598998
j = 0.55287898
theta = 138.11795906
zeta = 32.36841282
chi = 115.26317435
volr = 0.16866444

p = 10
n = 4

R1 = 1.39896633
R2 = 0.70710678
R3 = 0.93394883
R5 = 1.30656296
z = 1.20710678
ksi = 67.50000000
volp = 0.40236893

h = 0.19185954
e = 0.73267325
y = 1.30611091
psi1 = 88.49276145
psi2 = 75.32588042
labda = 46.96598286

R6 = 1.22026295
R7 = 1.41421356
g = 0.54119610
j = 0.71481349
theta = 138.11795906
zeta = 49.21052906
chi = 81.57894188
volr = 0.41118750
Chapter 14

P11, Truncated Cuboctahedron

\[ p = 11 \]
\[ n = 6 \]

\[ R_1 = 2.31761091 \]
\[ R_2 = 1.00000000 \]
\[ R_3 = 0.97645098 \]
\[ R_5 = 2.26303344 \]
\[ z = 2.09077028 \]
\[ \text{ksi} = 67.50000000 \]
\[ \text{volp} = 1.81066017 \]

\[ h = 0.22684064 \]
\[ e = 1.02540561 \]
\[ y = 2.24197690 \]
\[ \text{psil1} = 82.17790754 \]
\[ \text{psil2} = 82.72149510 \]
\[ \text{labda} = 60.81625955 \]

\[ R_6 = 2.20974121 \]
\[ R_7 = 2.44948974 \]
\[ g = 0.93737914 \]
\[ j = 1.05690292 \]
\[ \text{theta} = 155.08217962 \]
\[ \text{zeta} = 62.48765193 \]
\[ \text{chi} = 55.02469615 \]
\[ \text{volr} = 2.02258669 \]

Fig. 14. 12

\[ p = 11 \]
\[ n = 6 \]

\[ R_1 = 2.31761091 \]
\[ R_2 = 0.70710678 \]
\[ R_3 = 0.97645098 \]
\[ R_5 = 2.26303344 \]
\[ z = 2.0710678 \]
\[ \text{ksi} = 77.23561032 \]
\[ \text{volp} = 0.73570226 \]

\[ h = 0.11050413 \]
\[ e = 0.71568929 \]
\[ y = 2.26300203 \]
\[ \text{psil1} = 89.69811946 \]
\[ \text{psil2} = 81.23561032 \]
\[ \text{labda} = 45.68303314 \]

\[ R_6 = 2.20974121 \]
\[ R_7 = 2.44948974 \]
\[ g = 0.93737914 \]
\[ j = 1.05690292 \]
\[ \text{theta} = 155.08217962 \]
\[ \text{zeta} = 62.48765193 \]
\[ \text{chi} = 55.02469615 \]
\[ \text{volr} = 2.02258669 \]

\[ \text{R1} = 2.31761091 \]
\[ \text{R2} = 1.30656296 \]
\[ \text{R3} = 0.97645098 \]
\[ \text{R5} = 2.26303344 \]
\[ z = 1.91421356 \]
\[ \text{ksi} = 57.76438968 \]
\[ \text{volp} = 3.08088023 \]

\[ h = 0.40339735 \]
\[ e = 1.36741954 \]
\[ y = 2.19811630 \]
\[ \text{psil1} = 76.24326033 \]
\[ \text{psil2} = 83.03324606 \]
\[ \text{labda} = 68.55227266 \]

\[ R_6 = 2.20974121 \]
\[ R_7 = 2.67541744 \]
\[ g = 1.42707327 \]
\[ j = 1.50827791 \]
\[ \text{theta} = 155.08217962 \]
\[ \text{zeta} = 71.11332996 \]
\[ \text{chi} = 37.77334008 \]
\[ \text{volr} = 4.10560220 \]
Chapter 14

P12, Icosidodecahedron

\[ p = 12 \]
\[ n = 3 \]

\[
\begin{align*}
R1 &= 1.61803399 \\
R2 &= 0.57735027 \\
R3 &= 0.95105652 \\
R5 &= 1.53884177 \\
z &= 1.51152263 \\
\text{ksi} &= 79.18768304 \\
\text{volp} &= 0.21816950 \\
\theta &= 143.99999999 \\
\zeta &= 58.28252559 \\
\chi &= 63.43494882 \\
\text{volr} &= 0.89246119
\end{align*}
\]

\[ R6 = 1.46352549 \]
\[ R7 = 1.56665467 \]
\[ g = 0.29389263 \]
\[ j = 0.55901699 \]
\[ \text{labda} = 31.60796781 \]

\[ h = 0.10651136 \]
\[ e = 0.58709284 \]
\[ y = 1.51800245 \]
\[ \text{psi1} = 80.55996187 \]
\[ \text{psi2} = 72.55563648 \]

\[
\begin{align*}
R1 &= 1.61803399 \\
R2 &= 0.85065081 \\
R3 &= 0.95105652 \\
R5 &= 1.53884177 \\
z &= 1.37638192 \\
\text{ksi} &= 63.43494882 \\
\text{volp} &= 0.78934466 \\
\theta &= 143.99999999 \\
\zeta &= 58.28252559 \\
\chi &= 63.43494882 \\
\text{volr} &= 0.89246119
\end{align*}
\]

\[ \text{Fig. 14. 13} \]
Chapter 14

P13, Truncated Icosahedron

Fig. 14. 14

\[ p = 13 \]
\[ n = 5 \]

\[
\begin{align*}
R1 &= 2.47801866 \\
R2 &= 0.85065081 \\
R3 &= 0.97943209 \\
R5 &= 2.42705098 \\
z &= 2.32743844 \\
\text{ksi} &= 73.52778931 \\
\text{volp} &= 1.33476841 \\
\end{align*}
\]

\[
\begin{align*}
h &= 0.15058022 \\
e &= 0.86387569 \\
y &= 2.42074823 \\
\text{psi1} &= 85.86993152 \\
\text{psi2} &= 82.78237076 \\
\lambda &= 54.63472953 \\
\text{volr} &= 1.39237420 \\
\end{align*}
\]

\[
\begin{align*}
R6 &= 2.37713161 \\
R7 &= 2.53092687 \\
g &= 0.71764500 \\
j &= 0.86881307 \\
\theta &= 156.71855373 \\
\zeta &= 55.69063953 \\
\chi &= 68.61872093 \\
\text{volr} &= 1.39237420 \\
\end{align*}
\]

\[
\begin{align*}
p &= 13 \\
n &= 6 \\
R1 &= 2.47801866 \\
R2 &= 1.00000000 \\
R3 &= 0.97943209 \\
R5 &= 2.42705098 \\
z &= 2.26728394 \\
\text{ksi} &= 69.09484255 \\
\text{volp} &= 1.96352549 \\
\end{align*}
\]

\[
\begin{align*}
h &= 0.21073472 \\
e &= 1.02196337 \\
y &= 2.40775943 \\
\text{psi1} &= 82.77113973 \\
\text{psi2} &= 83.21072469 \\
\lambda &= 60.70841749 \\
\end{align*}
\]

\[
\begin{align*}
R6 &= 2.37713161 \\
R7 &= 2.59807621 \\
g &= 0.92705098 \\
j &= 1.04844901 \\
\theta &= 156.71855373 \\
\zeta &= 62.15468023 \\
\chi &= 55.69063953 \\
\text{volr} &= 2.15839622 \\
\end{align*}
\]
P14, Truncated Dodecahedron

Fig. 14. 15

\begin{align*}
\text{p} &= 14 \\
\text{n} &= 3 \\
R_1 &= 2.96944902 \\
R_2 &= 0.57735027 \\
R_3 &= 0.98572192 \\
R_5 &= 2.92705098 \\
z &= 2.91278117 \\
\text{ksi} &= 84.34010627 \\
\text{volp} &= 0.42042375 \\
\text{h} &= 0.05666785 \\
\text{e} &= 0.58012462 \\
\text{y} &= 2.91383742 \\
\text{psi1} &= 84.55377040 \\
\text{psi2} &= 80.39703705 \\
\text{labda} &= 30.47124447 \\
R_6 &= 2.88525831 \\
R_7 &= 2.94139071 \\
g &= 0.29008936 \\
j &= 0.57189489 \\
\text{theta} &= 160.61255221 \\
\text{zeta} &= 30.48032457 \\
\text{chi} &= 119.03935087 \\
\text{volr} &= 0.41251612 \\
\text{p} &= 14 \\
\text{n} &= 10 \\
R_1 &= 2.96944902 \\
R_2 &= 1.61803399 \\
R_3 &= 0.98572192 \\
R_5 &= 2.92705098 \\
z &= 2.48989828 \\
\text{ksi} &= 58.28252559 \\
\text{volp} &= 6.38593247 \\
\text{h} &= 0.47955073 \\
\text{e} &= 1.68760271 \\
\text{y} &= 2.83498016 \\
\text{psi1} &= 75.59117638 \\
\text{psi2} &= 84.72486633 \\
\text{labda} &= 72.76579649 \\
R_6 &= 2.88525831 \\
R_7 &= 3.44095480 \\
g &= 1.80901699 \\
j &= 1.87495451 \\
\text{theta} &= 160.61255221 \\
\text{zeta} &= 74.75983772 \\
\text{chi} &= 30.48032457 \\
\text{volr} &= 8.57492858
\end{align*}
Chapter 14

**P15L and -P15R, Left-handed and right-handed Snub Cube**

![Snub Cube Diagram]

\[ p = 15 \]
\[ n = 3 \]
\[\begin{align*}
p &= 15 \\
n &= 3 \\
R1 &= 1.34371337 \\
R2 &= 0.57735027 \\
R3 &= 0.92819138 \\
R5 &= 1.24722317 \\
z &= 1.21335580 \\
ksi &= 76.61729386 \\
volp &= 0.17513282 \\
h &= 0.13035757 \\
e &= 0.59188380 \\
y &= 1.22463971 \\
psi1 &= 79.08010911 \\
psi2 &= 69.11964519 \\
labda &= 32.35372511 \\
R6 &= 1.15766179 \\
R7 &= 1.28203585 \\
g &= 0.29673268 \\
j &= 0.55084943 \\
theta &= 136.30923289 \\
zeta &= 32.59396276 \\
chi &= 114.81207448 \\
volr &= 0.15942433 \\
r_1 &= 1.34371337 \\
r_2 &= 0.70710678 \\
r_3 &= 0.92819138 \\
r_5 &= 1.24722317 \\
z &= 1.14261351 \\
ksi &= 66.36613622 \\
volp &= 0.38087117 \\
h &= 0.20109986 \\
e &= 0.73514703 \\
y &= 1.24665868 \\
psi1 &= 88.27611428 \\
psi2 &= 74.70098650 \\
labda &= 47.14572431 \\
R6 &= 1.15766179 \\
R7 &= 1.36141015 \\
g &= 0.54577648 \\
j &= 0.71641942 \\
theta &= 136.30923289 \\
zeta &= 49.62414896 \\
chi &= 80.75170209 \\
volr &= 0.39096942
\end{align*}\]

Fig. 14. 16
P16, Rhombicosidodecahedron

\[ p = 16 \]
\[ n = 3 \]
\[ R1 = 1.61803399 \]
\[ R2 = 0.57735027 \]
\[ R3 = 0.95105652 \]
\[ R5 = 1.53884177 \]
\[ z = 1.51152263 \]
\[ ksi = 79.18768304 \]
\[ volp = 0.48511556 \]
\[ h = 0.16268730 \]
\[ e = 0.72558057 \]
\[ y = 1.53863594 \]
\[ psil = 89.06286198 \]
\[ psi2 = 77.36233950 \]
\[ labda = 46.44082673 \]
\[ R6 = 1.46352549 \]
\[ R7 = 1.62712707 \]
\[ g = 0.52868563 \]
\[ j = 0.71108060 \]
\[ theta = 144.00000000 \]
\[ zeta = 48.03008477 \]
\[ chi = 83.93983047 \]
\[ volr = 0.49058342 \]

\[ p = 16 \]
\[ n = 4 \]
\[ R1 = 1.61803399 \]
\[ R2 = 0.70710678 \]
\[ R3 = 0.95105652 \]
\[ R5 = 1.53884177 \]
\[ z = 1.45534669 \]
\[ ksi = 71.03929012 \]
\[ volp = 0.48511556 \]
\[ h = 0.16268730 \]
\[ e = 0.72558057 \]
\[ y = 1.53863594 \]
\[ psi1 = 89.06286198 \]
\[ psi2 = 77.36233950 \]
\[ labda = 46.44082673 \]
\[ R6 = 1.46352549 \]
\[ R7 = 1.62712707 \]
\[ g = 0.52868563 \]
\[ j = 0.71108060 \]
\[ theta = 144.00000000 \]
\[ zeta = 48.03008477 \]
\[ chi = 83.93983047 \]
\[ volr = 0.49058342 \]

\[ p = 16 \]
\[ n = 5 \]
\[ R1 = 1.61803399 \]
\[ R2 = 0.85065081 \]
\[ R3 = 0.95105652 \]
\[ R5 = 1.53884177 \]
\[ z = 1.37638192 \]
\[ ksi = 89.06286198 \]
\[ volp = 0.78934466 \]
\[ h = 0.16268730 \]
\[ e = 0.72558057 \]
\[ y = 1.53863594 \]
\[ psi1 = 89.06286198 \]
\[ psi2 = 77.36233950 \]
\[ labda = 55.56901179 \]
\[ R6 = 1.46352549 \]
\[ R7 = 1.72047740 \]
\[ g = 0.76942088 \]
\[ j = 0.90450850 \]
\[ theta = 144.00000000 \]
\[ zeta = 58.28252559 \]
\[ chi = 63.4394882 \]
\[ volr = 0.89246119 \]
Chapter 14

P17, Truncated Icosidodecahedron

Fig. 14.18

\begin{align*}
\text{p} & = 17 \\
\text{n} & = 6 \\
\text{R1} & = 3.80239450 \\
\text{R2} & = 1.00000000 \\
\text{R3} & = 0.99131669 \\
\text{R5} & = 3.76937713 \\
\text{z} & = 3.66854248 \\
\text{ksi} & = 76.71747441 \\
\text{volp} & = 3.17705098 \\
\text{h} & = 0.13385202 \\
\text{e} & = 1.00891841 \\
\text{y} & = 3.75777562 \\
\text{psi1} & = 85.50353104 \\
\text{psi2} & = 85.61990329 \\
\text{labda} & = 60.29198260 \\
\text{R6} & = 3.73646466 \\
\text{R7} & = 3.87298335 \\
\text{g} & = 0.88982924 \\
\text{j} & = 1.01856431 \\
\text{theta} & = 164.88789191 \\
\text{zeta} & = 60.88104019 \\
\text{chi} & = 58.23791962 \\
\text{volr} & = 3.29610545 \\
\text{p} & = 17 \\
\text{n} & = 4 \\
\text{R1} & = 3.80239450 \\
\text{R2} & = 0.70710678 \\
\text{R3} & = 0.99131669 \\
\text{R5} & = 3.76937713 \\
\text{z} & = 3.73606798 \\
\text{ksi} & = 82.37736814 \\
\text{volp} & = 1.24535599 \\
\text{h} & = 0.06632652 \\
\text{e} & = 0.71021068 \\
\text{y} & = 3.76937461 \\
\text{psi1} & = 89.93371127 \\
\text{psi2} & = 84.66462664 \\
\text{labda} & = 45.24986084 \\
\text{R6} & = 3.73664646 \\
\text{R7} & = 3.80298335 \\
\text{g} & = 0.50445778 \\
\text{j} & = 0.70721627 \\
\text{theta} & = 164.88789191 \\
\text{zeta} & = 45.50409905 \\
\text{chi} & = 88.99180191 \\
\text{volr} & = 1.24574167 \\
\text{R1} & = 3.80239450 \\
\text{R2} & = 1.00000000 \\
\text{R3} & = 0.99131669 \\
\text{R5} & = 3.76937713 \\
\text{z} & = 3.66854248 \\
\text{ksi} & = 76.71747441 \\
\text{volp} & = 3.17705098 \\
\text{h} & = 0.13385202 \\
\text{e} & = 1.00891841 \\
\text{y} & = 3.75777562 \\
\text{psi1} & = 85.50353104 \\
\text{psi2} & = 85.61990329 \\
\text{labda} & = 60.29198260 \\
\text{R6} & = 3.73646466 \\
\text{R7} & = 3.87298335 \\
\text{g} & = 0.88982924 \\
\text{j} & = 1.01856431 \\
\text{theta} & = 164.88789191 \\
\text{zeta} & = 60.88104019 \\
\text{chi} & = 58.23791962 \\
\text{volr} & = 3.29610545 \\
\text{R1} & = 3.80239450 \\
\text{R2} & = 1.00000000 \\
\text{R3} & = 0.99131669 \\
\text{R5} & = 3.76937713 \\
\text{z} & = 3.44095480 \\
\text{ksi} & = 65.90515745 \\
\text{volp} & = 8.82514162 \\
\text{h} & = 0.36143970 \\
\text{e} & = 1.65791213 \\
\text{y} & = 3.70165942 \\
\text{psi1} & = 79.12307085 \\
\text{psi2} & = 85.94819807 \\
\text{labda} & = 72.44722507 \\
\text{R6} & = 3.73664646 \\
\text{R7} & = 3.80298335 \\
\text{g} & = 0.50445778 \\
\text{j} & = 0.70721627 \\
\text{theta} & = 164.88789191 \\
\text{zeta} & = 45.50409905 \\
\text{chi} & = 88.99180191 \\
\text{volr} & = 1.24574167
\end{align*}
**P18L and -P18R, Left-handed and right-handed Snub Dodecahedron**

\[
\begin{align*}
p & = 18 \\
n & = 3 \\
R1 & = 2.15583738 \\
R2 & = 0.7735027 \\
R3 & = 0.97273285 \\
R5 & = 2.09705384 \\
z & = 2.07708966 \\
ksi & = 82.08768303 \\
volp & = 0.29980207 \\
h & = 0.07874772 \\
e & = 0.58269592 \\
y & = 2.07984074 \\
psi1 & = 82.65384844 \\
psi2 & = 76.82563780 \\
labda & = 30.89826144 \\
R6 & = 2.03987315 \\
R7 & = 2.11720990 \\
g & = 0.29144977 \\
j & = 0.56700553 \\
theta & = 153.17873256 \\
zeta & = 30.93168862 \\
chi & = 118.13662276 \\
volr & = 0.28915484 \\
p & = 18 \\
n & = 5 \\
R1 & = 2.15583738 \\
R2 & = 0.85065081 \\
R3 & = 0.97273285 \\
R5 & = 2.09705384 \\
z & = 1.98091595 \\
ksi & = 70.84223725 \\
volp & = 1.13604037 \\
h & = 0.17492143 \\
e & = 0.86844937 \\
y & = 2.08940046 \\
psi1 & = 85.10343568 \\
psi2 & = 81.67449790 \\
labda & = 54.84861248 \\
R6 & = 2.03987315 \\
R7 & = 2.22000070 \\
g & = 0.72853848 \\
j & = 0.87596839 \\
theta & = 153.17873256 \\
zeta & = 56.27324552 \\
chi & = 67.45350897 \\
volr & = 1.204669000
\end{align*}
\]

Fig. 14. 19
14.2. Formulas according to M. Brückner

This paragraph is meant to give a review of the formulas, that can be found Brückner's book 'Vielecke und Vielfläche - Theorie und Geschichte', that appeared in 1900 in Bautzen, Germany [1.2], but which is at present still available in the form of a reproduction.

These formulas were already shown in Chapter 2, but it seemed practical to give them here again as review in the form of a table, which may serve as an easily accessible reference.

The formulas given have the meaning:

- $\xi_n$ = dihedral angle between n-gon and centre plane passing through edge
- $\xi_{n1n2}$ = total dihedral angle between two adjacent n-gons
- $R_1$ = radius of circumscribed sphere
- $Z_n$ = perpendicular on n-gon through centre of circumscribed sphere
- $V$ = total volume of polyhedron
- $A$ = total surface area of polyhedron
- $a$ = edge length of n-gons

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\cos \xi_{3,3} = \frac{1}{\sqrt{3}}$</td>
<td>$\xi_{3,3} = 70^\circ 31' 43.6''$</td>
<td>$z_3 = \frac{a}{\sqrt{6}}$</td>
<td>$R_1 = \frac{a}{\sqrt{6}}$</td>
</tr>
<tr>
<td></td>
<td>$A = a^2 \sqrt{3}$</td>
<td>$V = \frac{a^2 \sqrt{2}}{12}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\tan \xi_{4,4} = 1$</td>
<td>$\xi_{5,4,4} = 90^\circ$</td>
<td>$z_4 = \frac{a}{\sqrt{3}}$</td>
<td>$R_1 = \frac{a}{\sqrt{3}}$</td>
</tr>
<tr>
<td></td>
<td>$A = 6a^2$</td>
<td>$V = a^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\tan \xi_3 = \sqrt{2}$</td>
<td>$\xi_{3,3} = 109^\circ 28' 16.4''$</td>
<td>$z_3 = \frac{a}{\sqrt{6}}$</td>
<td>$R_1 = \frac{a}{\sqrt{2}}$</td>
</tr>
<tr>
<td></td>
<td>$A = 2a^2 \sqrt{3}$</td>
<td>$V = \frac{a^2 \sqrt{2}}{3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\cos \xi_{3,5} = \frac{-1}{\sqrt{5}}$</td>
<td>$\xi_{5,5} = 116^\circ 33' 54.2''$</td>
<td>$z_5 = \frac{a}{2} \sqrt{\frac{25 + 11\sqrt{5}}{10}}$</td>
<td>$R_1 = \frac{a}{2} \sqrt{18 + 6\sqrt{5}}$</td>
</tr>
<tr>
<td></td>
<td>$A = 3a^2 \sqrt{25 + 10\sqrt{5}}$</td>
<td>$V = \frac{a^3}{4} \left(15 + 7\sqrt{5}\right)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\sin \xi_3 = \frac{\sqrt{15 + \sqrt{3}}}{6}$</td>
<td>$\xi_{3,3} = 138^\circ 11' 22.6''$</td>
<td>$z_3 = \frac{a}{12} \left(3 + \sqrt{5}\right)$</td>
<td>$R_1 = \frac{a}{4} \sqrt{10 + 2\sqrt{5}}$</td>
</tr>
<tr>
<td></td>
<td>$A = 5a^2 \sqrt{3}$</td>
<td>$V = \frac{5a^3}{12} \left(3 + \sqrt{5}\right)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\tan \xi_3 = \frac{5}{2} \sqrt{2}$</td>
<td>$\tan \xi_3 = \frac{1}{2} \sqrt{2}$</td>
<td>$z_6 = \frac{a}{4} \sqrt{6}$</td>
<td>$R_1 = \frac{a}{2 \sqrt{22}}$</td>
</tr>
<tr>
<td></td>
<td>$\xi_{3,6} = 109^\circ 28' 16.3''$</td>
<td>$\xi_{5,6} = 70^\circ 31' 43.6''$</td>
<td>$z_6 = \frac{a}{4} \sqrt{6}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$z_3 = \frac{5}{12} a \sqrt{6}$</td>
<td>$V = \frac{23}{12} a^3 \sqrt{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 7a^2 \sqrt{3}$</td>
<td>$R_1 = \frac{a}{2 \sqrt{22}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\tan \xi_3 = 2 \sqrt{2}$</td>
<td>$\tan \xi_3 = \sqrt{2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Chapter 14

<table>
<thead>
<tr>
<th>Section</th>
<th>Equation</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3,4</td>
<td>$\zeta = 125°15'15.5''$</td>
<td>$z_3 = \frac{a}{3} \sqrt{6}$</td>
<td>$A = 2a^2 (3 + \sqrt{3})$</td>
</tr>
<tr>
<td>4.4</td>
<td>$\tan\zeta = 2\sqrt{2}$</td>
<td>$z_4 = a\sqrt{2}$</td>
<td>$V = \frac{5}{3} a^3 \sqrt{2}$</td>
</tr>
<tr>
<td>4.4</td>
<td>$\zeta_{3,4} = 125°15'51.8''$</td>
<td>$z_4 = a\sqrt{2}$</td>
<td>$R_1 = a\sqrt{10}$</td>
</tr>
<tr>
<td>5.5</td>
<td>$\tan\zeta = \sqrt{2}$</td>
<td>$z_5 = a\sqrt{2}$</td>
<td>$V = 8a^3 \sqrt{2}$</td>
</tr>
<tr>
<td>5.5</td>
<td>$\zeta_{3,5} = 144°44'8.2''$</td>
<td>$z_5 = a\sqrt{2}$</td>
<td>$R_1 = a\sqrt{7 + 4\sqrt{2}}$</td>
</tr>
<tr>
<td>6.6</td>
<td>$\tan\zeta = 2$</td>
<td>$z_6 = a\sqrt{2}$</td>
<td>$V = 2a^3 (11 + 7\sqrt{2})$</td>
</tr>
<tr>
<td>6.6</td>
<td>$\zeta_{5,6} = 142°37'21.5''$</td>
<td>$z_6 = a\sqrt{2}$</td>
<td>$R_1 = a\sqrt{5 + 2\sqrt{5}}$</td>
</tr>
<tr>
<td>7.7</td>
<td>$\tan\zeta = 1$</td>
<td>$z_7 = a\sqrt{2}$</td>
<td>$V = a^3 (45 + 17\sqrt{5})$</td>
</tr>
<tr>
<td>8.8</td>
<td>$\zeta_{3,8} = 125°15'51.8''$</td>
<td>$z_8 = a\sqrt{2}$</td>
<td>$R_1 = a\sqrt{5 + 2\sqrt{5}}$</td>
</tr>
<tr>
<td>9.9</td>
<td>$\tan\zeta = 1$</td>
<td>$z_9 = a\sqrt{2}$</td>
<td>$V = 2a^3 (11 + 7\sqrt{2})$</td>
</tr>
<tr>
<td>10.10</td>
<td>$\zeta_{4,4} = 135°$</td>
<td>$z_9 = a\sqrt{2}$</td>
<td>$R_1 = a\sqrt{5 + 2\sqrt{5}}$</td>
</tr>
<tr>
<td>11.11</td>
<td>$\tan\zeta = 2$</td>
<td>$z_{10} = a\sqrt{2}$</td>
<td>$V = a^3 (45 + 17\sqrt{5})$</td>
</tr>
<tr>
<td>12.12</td>
<td>$\zeta_{3,5} = 142°37'21.5''$</td>
<td>$z_{10} = a\sqrt{2}$</td>
<td>$R_1 = a\sqrt{5 + 2\sqrt{5}}$</td>
</tr>
<tr>
<td>13.13</td>
<td>$\zeta_{5,6} = 142°37'21.5''$</td>
<td>$z_{10} = a\sqrt{2}$</td>
<td>$V = a^3 (45 + 17\sqrt{5})$</td>
</tr>
<tr>
<td>Chapter 14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 14 | \[
\begin{align*}
\tan \xi_3 &= \frac{1}{2}(9 + 5\sqrt{5}) \\
\xi_{3,10} &= 142^\circ 38^\prime 21.4^\prime (!) \\
z_3 &= \frac{a\sqrt{3}}{12}(9 + 5\sqrt{5}) \\
A &= 5a^2\left(\sqrt{3} + 6\sqrt{5} + 2\sqrt{5}\right) \\
\tan \xi_{10} &= \tau \\
z_{10} &= \frac{a}{4}\sqrt{50 + 22\sqrt{5}} \\
V &= \frac{5a^3}{12}(99 + 47\sqrt{5}) \\
R_i &= \frac{a}{4}\sqrt{2(37 + 15\sqrt{5})}
\end{align*}
\] |

| 15 | \[
\begin{align*}
\cos \varphi_1 &= \frac{1}{2}\left(3\cos(45^\circ) + \frac{11}{27}\right) + 3\cos(45^\circ) - \left(\frac{11}{27}\right) = 0.842508...
\end{align*}
\] |

| 16 | \[
\begin{align*}
\varphi_1 &= 32^\circ 35^\prime 38.3^\prime \\
\xi_3 &= 76^\circ 27^\prime 2^\prime \\
A &= 2a^2\left(4\sqrt{3} + 3\right) \\
\xi_4 &= 66^\circ 21^\prime 58.2^\prime \\
V &= a^3 7.889472 \\
R_i &= a \ast 1.29461....
\end{align*}
\] |

| 17 | \[
\begin{align*}
\tan \xi_3 &= 3 + 2\sqrt{5} \\
\xi_3 &= 82^\circ 22^\prime 38.5^\prime \\
z_3 &= \frac{a\sqrt{3}}{6}(3 + 2\sqrt{5}) \\
A &= 5a^2\left(6 + \sqrt{3} + 3\sqrt{1 + \frac{2\sqrt{5}}{5}}\right) \\
\tan \xi_4 &= 2 + \sqrt{5} \\
\xi_4 &= 76^\circ 43^\prime 2.9^\prime \\
z_4 &= \frac{a}{2}(2 + \sqrt{5}) \\
V &= a^3\left(60 + 29\sqrt{5}\right) \\
R_i &= \frac{a}{2}\sqrt{11 + 4\sqrt{5}}
\end{align*}
\] |

| 18 | \[
\begin{align*}
\cos \varphi_1 &= \frac{1}{2}\left(3\cos(36^\circ) + \sqrt{(\cos^2(36^\circ) - \frac{2}{3})^3} + 3\cos(36^\circ) - \left(\frac{2}{3}\right)^3\right) = 0.857779...
\end{align*}
\] |
Chapter 15. EXERCISES ON FOLDABLES

15.1. Examples of antiprismatic folding structures

In Chapter 6 a method is described to calculate the form of antiprismatic structures.

Such a structure is defined by:
\[ \alpha, \text{ half the top angle of the isosceles triangular sides with height } a \text{ and base length } 2b \]
\[ \phi, \text{ half the angle under which length } 2b \text{ is seen from the centre } \phi = \pi/n \]
\[ \gamma, \text{ half the dihedral angle between two triangles along their basis } 2b = \arccos(\tan \alpha \cdot \tan \phi/2) \]

Graph 6.18 shows all possible values of \( \phi \) and \( \gamma \) for \( \alpha \), increasing by 5°. Also two specific items are indicated, both on the line for \( \alpha = 65° \). The first is an experimental structure, investigated by B.S. Benjamin in England [6.3], and the other is the GRP roof of a sulphur factory by R. Piano in Genoa [6.4]. Both will be worked out here.

Benjamin’s structure had an octagonal cross-section, so that:
\[ \phi = 180\degree/8 = 22.5\degree. \]
For \( \alpha = 65\degree \rightarrow \gamma = \arccos(\tan \alpha \cdot \tan \phi/2) = 64.75\degree \]

Fig. 15.1. Cross-section and side elevation of Benjamin’s test structure.

\[ R = 2b/2\sin \phi = 2.6131 \ b \]

Fig. 15.2. Geometric data of test structure investigated by Benjamin.

\[ a = b/\tan \alpha = 0.4663 \ b \]
\[ h = a \cdot \cos \gamma = 0.1989 \ b \]
and \[ c = a \cdot \sin \gamma = 0.4218 \ b \]
Chapter 15

A similar element as in the foregoing, also with a top angle $\alpha = 65^\circ$, had been used by Renzo Piano for his factory roof construction in Genoa. This has a span $D = 10$ m but in this case with a cross-section in the form of a twelve-sided polygon. From these data the element length $2b$ can easily be calculated with the use of the preceding formulas.

As $R = \frac{2b}{2\sin 15^\circ} = 500$ cm $\Rightarrow 2b = 1000 \sin 15^\circ = 258.8$ cm and $\gamma = 73.6^\circ$

The specific data of a flat folding plan can be calculated with the program CORDIN, using $n$, $\alpha$, $q$ and $p$ (number of the elements in length and in transverse direction), but with $\gamma = 89.9999^\circ$ (nearing $90^\circ$). This provides a perfect network.

Fig. 15.3. Model and layout of structure in Genoa, built by Piano

Fig. 15.4. Model of a hemi-cylinder with sections of the same diameter but of different angles $\alpha$ ranging from $30^\circ$ (the regular antiprism) to $65^\circ$ (the structures of Benjamin and Piano)
The exercise implies that the magnitude of the various values must be calculated, and checked with the following table. The edge length $2b = 50\text{mm}$, $h = 4.9728092\text{mm}$, diameter $2R = 130.6656297\text{mm}$. Plates in the format A4 of the models in Figs. 15.3, 15.4 and 15.5 can be downloaded with the link [http://www.pieterhuybers.nl/pagina75.html](http://www.pieterhuybers.nl/pagina75.html).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Hoogte driehoek = $a$</th>
<th>$\gamma$</th>
<th>Length of cylindrical section in $\text{mm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>11.65</td>
<td>64.74998679</td>
<td>21.09</td>
</tr>
<tr>
<td>60</td>
<td>14.43</td>
<td>69.84711633</td>
<td>27.10</td>
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<tr>
<td>55</td>
<td>17.50</td>
<td>73.49635743</td>
<td>33.57</td>
</tr>
<tr>
<td>50</td>
<td>21.00</td>
<td>76.28723880</td>
<td>40.76</td>
</tr>
<tr>
<td>45</td>
<td>25.00</td>
<td>78.62663555</td>
<td>49.00</td>
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<tr>
<td>40</td>
<td>29.79</td>
<td>80.39194898</td>
<td>58.75</td>
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<td>35</td>
<td>35.70</td>
<td>81.99381842</td>
<td>70.71</td>
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<tr>
<td>30</td>
<td>43.30</td>
<td>83.40548178</td>
<td>86.03</td>
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</table>

<table>
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<tr>
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<th>Length of part 1</th>
<th>Length of part 2</th>
<th>Total length</th>
</tr>
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<tr>
<td></td>
<td>179.15</td>
<td>171.52</td>
<td>387.01</td>
</tr>
<tr>
<td></td>
<td>217.60</td>
<td>215.50</td>
<td></td>
</tr>
<tr>
<td></td>
<td>396.75</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 15.1. *Data of the cylinder model in Figs. 15.3, and 15.4*
15.2. Models of Polyhedra

Fig. 15.6. Row of all uniform polyhedra, composed of regular polygons.

For the specific properties of the polyhedra one may take notice of table 1.1. In this table a few characteristics of polyhedra are given, where table P = polyhedron index, Vertex code = side-numbers of respective polygons that meet in a vertex; V, E and F = number of vertices, edges and faces. Radius = radius of circumscribed sphere at unit edge length. These data enable the calculation of various geometric data starting from a given side length of any polyhedron. Many of these are given in Chapter 14. In Chapter 2 their nets are given.

The author once developed a construction method of models, called POLYHEDRON-SET based on a side length of 40 mm. This was described in Stevin Report 10-74-02, written in Dutch, and using 28 model cards in the A4 format, later succeeded by Report 01-83-19. These model cards were printed on paper of different colours, that were used to build a great number of the models shown in Fig. 0.1 in the Preface to this publication and in a number of photographs appearing in the other chapters. At that time the models were photographed in stereo in many combinations, and converted to MPO-files that can be shown on any appropriate television set. These stereo pictures accompany this publication in a separate set. The model cards and the reports are out of print, but they maybe reproduced if appropriate, as the basic plates are still available.

From material models it is quite easy to understand, that there are no more than 5 fully regular polyhedra, composed of just one kind of polygon. In the below Fig.15.8 this is shown.

Fig.15.7. The (only) five fully regular polyhedra
315

Chapter 15

15.4. Workshops on Snub Solids

In Chapter 4 was already announced, that a model set had been designed, allowing the complete row of chiral polyhedra. The snub cube P18 and the snub dodecahedron P15 are well known, both in a left- and right-handed version. To these were given the same identification numbers P15L and P15R, as well as P18L and P18R, because they are topologically identical, although they have different co-ordinates. In their vertices always a p-gon (with 4 or 5 sides) meets 4 triangles. In fact the icosahedron P5 has similar characteristics, but with the difference that the p-gon has 3 sides. In theory, we may also speak of a left-handed and a right-handed version of P5. The most extreme members have a 2-gon (P3) or even a 1-gon (P1). At the other end of the row the p-gon can have 6 sides, but this does not produce a polyhedron as the result in this case is flat plane.

In Fig. 15.10 the four plates are reproduced, that were used in a number of workshops enabling the construction of the various models. They are given here in a smaller, but when printed in full scale with 100 mm side length of the parts on thick paper, preferably 350 gr/m², these models can
be made, the biggest one having a diameter of 430 mm. In Plate 1 four triangles are combined to make all specific parts in the various models. A special print is applied to the triangle in the middle of part B to emphasize the specific character of this triangle. They all have surrounding tabs, that must be scored with a not too sharp knife (Fig. 15.10). The holes in the corners can be made with a normal punching tool (Fig. 15.11). They must be connected with elastic bands. At wish the plates on full scale of the cards can be downloaded with the link: 
http://www.pieterhuybers.nl/pagina35.html

Fig. 15.10. Plates of the snub parts and their discriminating polygons, with 1 to 6 sides.

Fig. 15.11. Scoring the tabs of the model parts with a not too sharp knife.
Fig. 15.12. The holes in the corners of the parts must be punched with a simple hand tool or with a normal office puncher.

Fig.15.13. Connecting the parts with a rubber band.
Fig. 15.14. The nets of the 6 variations of snubs

Fig. 15.15. Models of the complete series of snub figures: Tetrahedron, Octahedron, Icosahedron, Snub Cube and Snub Dodecahedron. In the corners of the parts a circular hole is punched, the tabs are folded outwards and connected with rubber bands.
15.5. Workshop at Interschool Hilversum 2011

Fig. 15.16. Scholars of Interschool busy working on models.

Fig. 15.17. One of the models, the biggest, being put together.
Fig. 15.18. Not just the snub solids were made with the same model cards, but also a few others, like the cube, the dodecahedron and the cuboctahedron.

Fig. 15.19. The four pictures above were taken during the workshop at Interschool.
Chapter 16. STEREOSCOPIC PICTURES - ANAGLYPHS

Fig. 16.1. *The condensed title of this book against a background of coupled cubes*

In this chapter a number of stereoscopic pictures are shown, called 'anaglyphs'. These are based on the principle, that two different pictures are put together, of which one shows an object or situation as seen with the left eye and the other with the right eye. The colours of the left picture are shifted towards blue and the right picture towards red. The stereoscopic view comes forward if this mixture is observed with spectacles with red glass at left and blue glass at right. Thus the two intermixed pictures are entangled again and seen in the proper way by each eye. Such a pair of glasses (in fact: transparant plastic foil in a cardboard frame) accompanies this DVD. On further pages a selection is shown of some anaglyphs that the author prepared in the many years that he worked on this study.

As can be noticed, these anaglyphs however have limited colouring possibilities. Therefore, together with this E-book a file is put on the DVD, with the same pictures and with many others in the so-called MPO format, which makes it possible to see these on a larger scale and in true colours on a standard 3D television screen.
Chapter 16

Fig. 16.2. *The Platonic polyhedra*

Fig. 16.3. *Relations of the five Platonic polyhedra with respect to the cube*
Fig. 16.4. *The formation of an Archimedean solid (P13) by truncation of an icosahedron (P5)*

Fig. 16.5. *The six different polygonal planes that constitute the regular and the semi-regular polyhedra*
Fig. 16.6. The spherical Cube, Octahedron and Dodecahedron

Fig. 16.7. Scale model of little roundwood dome structure with tent, later to be erected temporarily in Kootwijk (see Fig.14.89)
Fig. 16.8. 12 Dodecahedra (P4) in a dodecahedral arrangement

Fig. 16.9. Close packing of Truncated Octahedra (P8), forming together a honeycomb
Fig. 16.10. Close packing of Cubes (P2), Truncated Octahedra (P8) and Truncated Cuboctahedra (P11)

Fig. 16.11. Truncated icosidodecahedron (P17), following a suggestion by D.S. Emmerich augmented by the addition of parts of other polyhedra, in this case with halves of Octahedra (P3), Cuboctahedra (P7) and Icosidodecahedra. (P12)
Fig. 16.12. *54th stellation of the icosahedron: the Great Icosahedron*

Fig. 16.13. *17th stellation of the Icosahedron, forming a compound of five Tetrahedra.*
Fig. 16.14. 18 prismatic bars, attached to the square faces of the Rhombicuboctahedron (P10)

Fig. 16.15. Cube with inscribed Tetrahedron, made of long prisms and with P10 as node element.
Fig. 16.16. *Two cubic elements combined.*

Fig. 16.17. *Similar configuration as Fig. 16.13 from another standpoint.*
Chapter 16

Fig. 16.18. Tetrahedra and half-octahedra form a spatial arrangement, known as 'octet-truss' as called so by Richard Buckminster Fuller, which is mostly used in horizontal space frames. This picture shows the principle of the MERO System.

Fig. 16.19. Scale model of the roundwood space frame, erected in 1986 in Lelystad
Fig. 16.20. Model of saddle shaped large span space structure, envisaged to be executed in roundwood.

Fig. 16.21. Scale model of 27 m. high roundwood watch tower, built in 1995 in Apeldoorn.
Fig. 16.22. *Icosidodecahedron, with the twelve pentagons in pyramidized form.*

Fig. 16.23. *Cubes in helicoidal arrangement*
Fig. 16.24. *Antiprismatic folding structure*

Fig. 16.25. *Hexagonal vault of triangularly compressed antiprismatic cylinders*
Fig. 16.26. *Sphere subdivision following Class 5 and in a frequency of 12*

Fig. 16.26. *Hexagonal sphere subdivision based on the rhombic triacontahedron*
Fig. 16.28. *6 spheres, subdivided following class 2 and in frequency 3, in an octahedral arrangement.*

Fig. 16.29. *30 spheres, class 3, frequency 3, placed in the corners of the icosidodecahedron*
Fig. 16.30. *Four ellipsoids with different exponents*

Fig. 16.31. *Cubic arrangement of 64 ellipsoids with exponents 0.3 and frequency 2*
Fig. 16.32. Truncated Icosahedron (P13) augmented with 20 hexagonal and 12 pentagonal spherical caps

Fig. 16.33. P13 augmented with low exponent ellipsoidal caps
Fig. 16.34. *P13 with the low exponent caps of Fig. 16.28, but pointing inwards.*

Fig. 16.35. *Sphere of Fig. 16.28 with added on small spheres.*
Fig. 16.36. *Dodecadron with Tangram figures on its sides*

Fig. 16.37. *Review of the Polyhedral Family*
Fig. 16.38. Beads model of the C60 molecule

Fig. 16.39. The logo in 3D of the Structural Morphology working Group (SMG) of the International Association of Shell and Spatial Structures (IASS).
Chapter 17. EPILOGUE

17.1. Acknowledgements

Westland, June 2014

This document describes the outlines of the work done by the author as a member of the Building Technology Section at the Civil Engineering Department of the Delft University of Technology, mainly from 1962 to 2000, the year of his retirement. It did not actually end at that point, as he stayed active up to some amount in this field till now. Particularly the theoretical and development work on a new type of soccer ball described in Chapter 12 took place after his retirement and this in fact is still running.

As an architectural engineer employed at the Civil Engineering Dept., the author found a special place, that was not altogether easy, as his work was situated in a region between the two disciplines: Architecture and Civil Engineering. Nevertheless he was lucky to get and to find the opportunity to do the work described. It was at the Interdisciplinary Symposium on Symmetry of Structure at Budapest in 1989, that he met others who were in a comparable situation and it was then that it was decided to form a companionship. These others were Ture Wester from Denmark, Jean-Francois Gabriel from the USA and René Motro from France. The four together worked out a proposal for a working group under the aegis of IASS, the International Association of Shell and Spatial Structures. This was accepted as Working Group No. 15, under the name of 'Structural Morphology'. It encourages research on form, material and fabrication and it is focussed on the relationship between geometry and structures.

This group attracted many members of great international reputation. The group is still existing and is open to participation. Contact can be taken up via the blog http://structuralmorphology.org. The stereoscopic picture Fig. 16.39 in the previous chapter shows the original logo of the working group. This was made following the technique described in Chapter 10. The final logo that is presently in use, was formed by conversion of the first one into a spherical form.

Fig. 17.1. Two versions of the logo of the Structural Morphology Working Group of IASS

In the year 2000 a seminar, in the organization of which the author was narrowly engaged, was held by SMG at the Delft University of Technology with the title 'Structural Morphology - Bridge between Architecture and Civil Engineering', emphasizing the interdisciplinary character of its scope. (SMG Seminar 2000)
Chapter 17

Special mention must be made of those, who had an essential contribution to the realization of the structures mentioned in Chapter 13: Gerrit van der Ende, Caspar Groot, Rogier Houtman, Jaap Lanser, Sier van der Reijken, Erik Sluis, Peter de Vries (all from TU-Delft), Rico Golstein (Bureau De Bondt, structural advisor of the two tower structures), Peter Mulder (construction of the tower in Apeldoorn), Erik Klein-Lebbink (Staatsbosbeheer, construction of the watch tower in Kootwijk). Thanks to their joint scientific approach many valuable results were gained, that may help exploit a source of material that so far has been very much underestimated.

The computer drawings for the anaglyphs were made with the GFA programme Cordin, that was developed by Gerrit van der Ende in collaboration with the author as part of their work in the Building Technology Group at the Civil Engineering Department of the Delft Technical University. His work in this respect was of great importance for the outcome of this whole book. The author owes much appreciation to him for his great enthusiasm and never ending dedication.

The World of 3-D, A practical Guide to Stereo Photography by Jac. G. Ferwerda served as the leading handbook for the practical realization of these computer generated pictures and also of the photographs of models that are shown in Chapter 16. For the composition of the stereoscopic pictures, anaglyphs as well MPO-files, the latters made for 3D presentation on a suitable television screen, use was made of the programme StereoPhotomaker, that can be downloaded as Freeware from internet. Many valuable suggestions in this respect were done by Jaap van Loon and Johan Steketee, who are two outstanding members of the Nederlandse Vereniging voor Stereofotografie (Dutch Association for Stereo Photography).

Above all, the author is indebted much gratitude towards his wife Elisabeth (Bep for friends), who encouraged him to fulfill what he felt as a serious assignment, to gather all his publications on 'polyhedroids' from several sources - that were sometimes difficult of access, as they were at times written only in Dutch - and to put these together to form a coherent and continuous story.

17.2. Publications since retirement

At the occasion of his retirement in October 2000 the author made the TU-Report '38 Years of Morphology - an Anthology' with copies of a selection from his most relevant articles of the recent foregoing years and with a complete list of his publications till then. The next list reviews those, published after that date.


Soccer Ball Geometry, a Matter of Morphology, Chapter in Structural Morphology and Configuration Processing of Space Structures, Ed. René Motro, p.139-150.


CV OF THE AUTHOR

Pieter Huybers was born on 2-8-1935 in Alkmaar, The Netherlands. He studied Architecture, Faculty of Building Engineering, at the Technological University of Delft from 1954 to 1962. He worked one year at an architects office.

He was head of a research group on Building Technology at the Civil Engineering Faculty of Delft University and he did investigations on various materials and building methods. He obtained his doctor's degree in 1972 on the thesis: “See-through Structuring, a method of construction for large span plastics roofs ”. He designed a few structures in GRP, that were built in The Netherlands from 1968 to 1974. He directed a few courses on the design of plastics structures and he organized the IASS conference on Pneumatic Structures in 1972 at Delft.

Later he did research on houses of cardboard, concrete interlocking building blocks and an emergeney housing system using board and paper-honeycomb sandwich panels. He participated in construction projects in Mali (1977) and in Burkina Faso (1982). Since the beginning of the eighties much attention was paid to the use of thin roundwood poles for building structures. He developed new construction methods for this material and he designed a number of timber space structures, that were actually realized in The Netherlands and in England, among which a 27 m high space frame tower. A co-operation project on the use of thin roundwood, gained as a by- product of wood maintenance, as the primary material for building structures was started in January 1996 with partners from Finland, England, France and Austria in the context of the European FAIR-2 Programm. This had as its main goal to develop guidelines for engineers and it finally also resulted in a demonstration building in the form of a 10 m high watch-tower according a novel structural concept and with a striking appearance. Both tower structures were built following a design by the author. For this structure a set of components has been developed which facilitate the construction of a range of such towers, varying in height.

From the start he had a great interest in the morphology of structures and investigated in that respect the relevancy and the potentials of polyhedra and similar forms for building design. Spheroidal structures - as he called these - or domes, and space frames, being the most important representatives of this category.

He is one of the four founder members (jokingly called 'gang of four')of the IASS-Working Group No. 15 on Structural Morphology. From 1992 till 2000 he was the editor of the yearly SMG Newsletter and he was chairman of the Scientific Committee of an SMG-conference on
17-19 August 2000, at the Delft University of Technology on the subject “Structural Morphology, Bridge between Civil Engineering and Architecture”.
In the year 2000 he received from the International Association for Shell and Spatial Structures the Tsuboi Award for his publication 'The Chiral Polyhedra' in the Journal of IASS, Nr. 2, August 1999, p. 133-143. (http://www.pieterhuybers.nl/pagina718.html)

On 1-9-2000 he reached the date of his official retirement from the Delft University of Technology. From then on he developed the design of a new type of soccer ball, of which he claims that this is rounder than any other ball, due to its geometry. As his main sport he practiced speed skating and completed all three latest 200 km long 'Eleven Cities Tours' in the northern part of the country in the years 1985, 1986 and 1997. This is a great event in which some 20,000 skaters take part. The tour takes place only occasionally, as it requires a long period of severe frost, which is seldom in the Netherlands. Therefore alternative 200 km-tours are organized in other countries and he took part in a number of these. From the age of 60 in his spare time he became a professional skating teacher and introduced many young or other beginning skaters in the technique of this sport. He has photography as his hobby and made many pictures, with a special interest in stereophotography and in macrophotos of flowers. (http://www.pieterhuybers.nl/pagina78.html)