A direct-numerical-simulation-based second-moment closure for turbulent magneto-hydrodynamic flows

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A magnetic field, imposed on turbulent flow of an electrically conductive fluid, is known to cause preferential damping of the velocity and its fluctuations in the direction of Lorentz force, thus leading to an increase in stress anisotropy. Based on direct numerical simulations (DNS), we have developed a model of magneto-hydrodynamic (MHD) interactions within the framework of the second-moment turbulence closure. The MHD effects are accounted for in the transport equations for the turbulent stress tensor and energy dissipation rate—both incorporating also viscous and wall-vicinity nonviscous modifications. The validation of the model in plane channel flows with different orientation of the imposed magnetic field against the available DNS (Re=4600, Ha=6), large eddy simulation (Re=2.9×10^4, Ha=52.5,125) and experimental data (Re=5.05×10^4 and Re=9×10^4, 0≤Ha≤400), show good agreement for all considered situations. © 2004 American Institute of Physics. [DOI: 10.1063/1.1649335]

I. INTRODUCTION

Predicting flow field, heat and mass transfer in electrically conductive fluid when subjected to the magnetic (Lorentz) force, is an important prerequisite for design and optimization of many processes in various industrial applications: metal processing (continuous casting), crystal growth, heat exchangers in nuclear fusion reactors, etc. In most of full-scale applications, the fluid occupies a complex three-dimensional domain and the magnetic field may vary both in time and space. In general, a magnetic field imposed on turbulent flows of conductive fluids suppresses the fluid velocity and its fluctuations in the direction of Lorentz force. This leads to a reorganization of the vortical structure towards the alignment of the vorticity vectors with the direction of the imposed magnetic field, which in turn results in an increase in the anisotropy of the turbulent stress field. The eddy-structure reorganization cannot be directly reproduced by using the Reynolds-averaged approach but their indirect effects on turbulence can be modelled. Among the first attempts to model magneto-hydrodynamic (MHD) wall-bounded turbulent flows was reported by Lykoudis and Brouillette where the mixing-length model was modified in order to introduce additional damping of turbulence due to the magnetic field. In more recent studies, the two-equations k–ε and the second-moment closures have been used as standard framework for modeling of additional electromagnetic effects on turbulence. For example, Naot et al. derived an eddy viscosity suitable for the k–ε model from the stress transport model for the theoretical case of shear flow in local equilibrium subjected to three differently oriented magnetic fields. In addition, an algebraic stress model was developed for channel flows with longitudinal (Naot et al.) and transversal (Naot) magnetic fields. Similar to this approach, Ji and Gardner introduced modifications to the standard low-Re number k–ε model to account for effects of a transversal magnetic field on turbulent flow in a pipe. The new terms were expressed as a product of the inverse magnetic time scale σB^2/ρ and an exponential damping function f^M=exp(-C^M/N), where N is the bulk interaction parameter (N=σB^2L/ρU). In addition, they introduced identical damping function for the turbulent viscosity, ν_t=CDf_μk^2/ε. The model was applied to the simulations of turbulent pipe flows in a transversal magnetic field over a range of Reynolds and Hartmann number, 1.5×10^4≤Re≤1.5×10^5 and 0≤Ha≤375, resulting in good agreement with the available experimental data. In order to avoid the use of the bulk flow parameters and of the additional explicit damping of the turbulent viscosity as proposed in model of Ji and Gardner, Kenjeres and Hanjalic expressed this bulk flow parameter (N) via the local time scales N=σB^2/ρε and omitted the damping function in ν_t. As a result, a more general formulation is obtained which can be used for complex geometries and for nonhomogenous magnetic fields. The model was tested first in a priori mode using the direct numerical simulation (DNS) data of Noguchi et al. and then for a duct flow of liquid metal in an inhomogeneous magnetic field (experiments by Tananaev, in Branover). The new model produced results in good agreement with both the DNS and the experimental data. The same model was later applied as a subscale model in the transient RANS approach to very-large eddy simulations (VLES) for

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modelling deterministic eddy structure in flows simultaneously subjected to thermal buoyancy and Lorentz force, Hanjalić and Kenjereš.8,9 The simulations reproduced well the reorganization of the large-scale vortical structures and consequent modifications of the mean and turbulence parameters and wall heat transfer. More recently, Smolentsev et al.10 reported on computation of flow in an open channel subjected to a magnetic field using the \( k-e \) model modified in a very similar manner as Ji and Gardner,4 but introduced additional terms with different coefficients for the \( k \) and \( e \) contributions. These coefficients were tuned on the basis of the available experimental data for the friction factor in a pipe subjected to a magnetic field of different orientations. As a result, an inconsistent model was obtained with different sets of coefficients for different orientations of a magnetic field. This model was subsequently applied to compute turbulent MHD flow down an inclined flat surface with a heat flux applied to the free surface. The authors stated that this configuration—despite simplifications—served as a model for a film flow over the divertor plate in a fusion reactor. Widlund et al.11 introduced an additional scalar variable (\( \alpha \)) representing the length scale anisotropy to be used for modeling Joule dissipation (\( \mu = 2 \sigma B^2 \alpha k/\rho \)). The new second-moment closure model was tested by performing simulations of decaying homogeneous turbulence in presence of a magnetic field, resulting in a good agreement with DNS data. Widlund12 applied this model to the computation of channel flow subjected to a transversal magnetic field of a final length.

In this study, we developed an extended variant of the second-moment closure that accounts for effects of the electromagnetic force—together with the near-wall viscous and wall-reflection modifications. The attention is first focused on careful examination of the two key prerequisites for achieving the satisfactory level of model generality: the model response to different orientations of the imposed magnetic field (without any specific modification for different orientation) and the requirement that the model should cover a wide range of \( \text{Re}(=U D/\nu) \), \( \text{Ha}(=B_0 D/\sigma/\rho \nu) \), and \( N(=\text{Ha}^2/\text{Re}) \) numbers.

II. THE PRESENT MODEL

The motion of electrically conducting fluid subjected to a magnetic field is described by the equation set consisting of Navier–Stokes and Maxwell’s equations, complemented with Ohm’s law for moving media, Branover,7 Moreau,13 The coupling between hydrodynamic and electromagnetic fields is through the Lorentz force. In the momentum equations the Lorentz force per unit mass is defined as \( \mathbf{F} = \mathbf{J} \times \mathbf{B} \), where \( \mathbf{J} \) is the total electric current density and \( \mathbf{B} \) is the imposed magnetic flux density. By applying Ohm’s law for moving media, \( \mathbf{J} = \sigma (-\nabla \Phi + \mathbf{U} \times \mathbf{B}) \), the Lorentz force can be expressed in the following form (using index notations):

\[
F_i^L = \sigma \left( -\epsilon_{ijk} B_k \frac{\partial \Phi}{\partial x_j} + U_k B_j \frac{U_j}{B_k} - U_k B^2 \right),
\]

where \( \Phi \) is the electric potential and \( \epsilon_{ijk} \) is the permutation symbol. By applying the inductorless approximation (effects of the fluid flow on the magnetic field are neglected, i.e., the magnetic Reynolds number is small, \( \text{Re}_m = \mu \sigma U L \ll 1 \)) and by using Kirchhoff continuity condition \( \nabla \cdot \mathbf{J} = 0 \), lead to the Poisson equation for the electric potential in following form:

\[
\nabla^2 \Phi = \nabla \cdot \mathbf{U} \times \mathbf{B}.
\]

Hence, for Reynolds averaged motion the momentum equation can be written as

\[
\frac{D\mathbf{U}_i}{Dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] u_{ij} - u_{ij} F_i^L,
\]

where the second velocity moment \( u_{ij} \) is the turbulent
stress tensor. In addition to the direct interaction with the mean velocity field through the electromagnetic force \( F_i^{el} \), which will itself influence the turbulence through the modification of the mean rate of strain, the magnetic field affects also the velocity fluctuations by the fluctuating Lorentz force \( f_i^L \), even if the magnetic flux density \( B \) is assumed invariable in time and space.

A. Source/sink (direct) electromagnetic contributions

The implementation of the fluctuating Lorentz force \( f_i^L \) into the transport equations for the Reynolds stress \( u_i u_j \) and the energy dissipation rate \( \varepsilon \) leads to the following extra source terms, Kenjereš and Hanjalić [5] [a summary of all model equations (Tables I and II) is given in the Appendix]:

\[
S_{ij}^M = \frac{\sigma}{\rho} \left( -\epsilon_{ijk} B_k \frac{\partial \phi}{\partial x_k} - \epsilon_{ijk} \frac{B_k \varepsilon_{ij}}{\partial x_k} + B_i B_j \frac{\partial u_k}{\partial x_k} + B_i B_k \frac{\partial u_j}{\partial x_k} - 2 B_i^2 \frac{\varepsilon_{ij}}{\partial x_k} \right)
\]

\[
S_{e}^M = \frac{2 \nu \sigma}{\rho} \left( -\epsilon_{ijk} B_k \frac{\partial u_i}{\partial x_k \partial \varepsilon_{ij}} - \epsilon_{ijk} \frac{B_k \varepsilon_{ij}}{\partial x_k} + B_i B_k \frac{\partial u_j}{\partial x_k} B_k^2 \frac{\partial \varepsilon_{ij}}{\partial x_k} + B_i \frac{\partial B_k}{\partial x_k} \frac{\partial u_j}{\partial x_k} + 2 B_k \frac{\partial u_k}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right)
\]

For channel flows subjected to uniform magnetic fields, the \( S_e^M \) term can be written in the following form:

\[
S_{e}^M = \frac{1}{2} S_{e}^M \frac{\varepsilon}{k} = \frac{\sigma}{\rho} \left( -\epsilon_{ijk} B_k \frac{\partial \phi}{\partial x_k} - 2 k B_i \frac{\varepsilon_{ij}}{\partial x_k} + B_i B_k \frac{\partial u_k}{\partial x_k} \right)
\]
where $B_i$ is the strength or the magnetic field and $\phi$ is the fluctuating electric potential.

In order to simplify the closure of these new terms, only the contributions by fluctuating velocity-electric field correlations ($u_i e_j = -u_i \partial \phi / \partial x_j$) are considered since the remaining terms [$S_{ij}^{M2}$ in Eq. (4) and $S_{e}^{M2}$ in Eq. (6)] can be treated in an exact manner in the second-moments closure approach (exact in the sense that no modelling is applied to these terms since all stress components are provided from the solution of the modelled equations for $u_i u_j$). The consistent approach would require the derivation and modelling of the full transport equation for the $u_i e_j$ correlation. Multiplying the equation for the fluctuating velocity $u_i$ by $e_j$ and the equation for the fluctuating electric field $e_i$ by $u_j$, then summing these equations and, finally, performing the Reynolds averaging, yields the exact equation for $u_i e_j$. Unfortunately, none of the available DNS studies of MHD flows (Noguchi et al., $^{6}$ Lee and Chou, $^{14}$ Satake et al.$^{15,16}$) provide any information about the $u_i e_j$ budget. Hence, a direct term-by-term analysis of the $u_i e_j$ equation is at present not possible. Besides, even if one could model the differential transport equation for $u_i e_j$, this would lead to a cumbersome and complex model that would not be appealing to solving complex MHD flows. Hence we have opted for a simpler approach that should provide reasonable approximation of the terms $S_{ij}^{M1}$ and $S_{e}^{M1}$. Using specific simplifications for the fully developed channel flow, the first order approximation leads to

$$\frac{\partial u_i}{\partial x_j} = C_\lambda e_{jk} B_i u_k.$$  

(7)

It remains to evaluate the proportionality coefficient $C_\lambda$. Different possibilities have been explored using an a priori approach and the DNS database of Noguchi et al.$^6$ for Re$_T$ = $U_D/2\nu$ = 150 and transversal (Ha = 6) and longitudinal (Ha = 20) orientations of the imposed magnetic field. The a priori evaluation of the electromagnetic contributions to the budgets of the normal Reynolds stress components ($S_{ij}^{M}$) for different orientations of the imposed magnetic field, shown in Figs. 1 and 2, indicate that good agreement can be obtained for both orientations of the magnetic field with Eq. (7) and $C_\lambda$ = 0.6. The same conclusion can be drawn on the electromagnetic contributions to the dissipation rate of turbulent kinetic energy ($S_{e}^{M}$), as shown in Fig. 3. It is interesting to note the difference in $S_{e}^{M}$ contributions for the different

FIG. 4. Distributions of turbulent stresses for different orientations of imposed magnetic field: a priori testing of algebraically truncated SMC: Re$_T$ = 150, Ha = 6 (B$\parallel$ y) and Ha = 20, (B$\parallel$x). The dotted lines indicate neutral state (Ha = 0).
orientations of the imposed magnetic field. For the transversal orientation, the both $S_{eM1}^M$ and $S_{eM2}^M$ have the same sign resulting in significant value of the total $S_{eM}^M$. In contrast, for the longitudinal orientation of the magnetic field, $S_{eM1}^M$ and $S_{eM2}^M$ contributions have opposite signs, resulting in a negligible total $S_{eM}^M$. The proposed model reproduces closely both of these opposite trends, Fig. 3.

B. Redistributive (indirect) electromagnetic contributions

The indirect contribution of the imposed magnetic field comes from the pressure-strain correlation due to the electromagnetic force $f_i^L$ in the Poisson equation for the fluctuating pressure:

![Graphs showing mean velocity, Reynolds stress components, and their invariants obtained with full simulations with the proposed model, $Re=150$, $Ha=6$. DNS of Noguchi et al. (Ref. 6): longitudinal magnetic field, $B(1,0,0)\parallel uu$.](image1)

![Graphs showing mean velocity, Reynolds stress components, and their invariants obtained with full simulations with the proposed model, $Re=150$, $Ha=6$. DNS of Noguchi et al. (Ref. 6): transversal magnetic field, $B(0,1,0)\parallel uv$.](image2)
After integration of Eq. (8), multiplication by the fluctuating rate of strain and averaging, the contribution to the pressure strain \( \Phi_{ij} \) due to fluctuating magnetic force can be written as

\[
\Phi_{ij} = \frac{1}{4 \pi} \int_V \left[ \frac{\partial f_{ij}}{\partial x_i} \right] \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dV(x^*). \tag{9}
\]

The modelling of \( \Phi_{ij} \) can proceed in analogy with the modelling of effects of other body forces, e.g., thermal buoyancy. We adopt here a simple IP (isotropization of production) type model:

\[
\Phi_{ij}^M = -C_4 (S_{ij} - \frac{1}{2} S_{kk} \delta_{ij}) \tag{10}
\]

and its wall blocking modification in analogy with the “wall reflection” model of Gibson and Launder:

\[
\Phi_{ij}^{M,w} = C_4^{w} f_w (\Phi_{ik}^M n_k n_i \delta_{ij} - \frac{1}{2} \Phi_{ik}^M n_k n_j - \frac{1}{2} \Phi_{kj}^M n_k n_i), \tag{11}
\]

where the \( n_i \) stands for the normal unit vector. Now the two new coefficients, \( C_4 \) and \( C_4^{w} \) need to be estimated. The best approach will be to split the pressure strain into velocity and electromagnetic contributions and to test each of these parts separately against the DNS data. Unfortunately, in the database of Noguchi et al., only the total pressure strain is given so that the direct verification of each of the pressure-strain terms is not possible. Since we already know that the model performs reasonably well in nonmagnetic flows, we decided to perform the calibration and validation of all above mentioned electromagnetic contributions by performing the simulations of the turbulent channel flows subjected to a magnetic field of different orientations and strengths, over a range of Re, Ha, and \( N \) numbers.

III. RESULTS

As a basis we use the second-moment closure with low-Re modifications (the near-wall viscous and blockage effects) proposed by Hanjalić and Jakirlić\textsuperscript{18} for nonmagnetic turbulent flows. This model was earlier tested in a range of pressure-driven flows: boundary layers (with constant, favorable, and adverse pressure gradients), in pipes (stationary and axially rotating, without and with imposed swirl), channels (plane and with symmetric expansion, with a fence, imposed oscillations), and others—all resulting in good agreement with the available experimental or DNS data, Hanjalić et al.\textsuperscript{19} It is important to note, however, that this choice of

![FIG. 7. Comparison of the present low-Re second-moment closure (—) with LES of Shimomura (Ref. 20) with two different subgrid models [classical Smagorinsky (△), extended Smagorinsky (○)] and experiments of Brouillette and Lykoudis (Ref. 21) (■); Re = 2.9 × 10^4, B(0,1,0)|uv, Ha = 52.5 and 125 (top). Blow-up of fully developed region for different values of Ha (middle); Lumley’s “flatness” parameter (bottom).](image)
the basic model does not limit the generality of proposed modelling of electromagnetic effects—these can be simply used in conjunction with any any second-moment closure model.

A. *A priori* testing in fully developed channel flow using algebraic truncation

As the first step in the model validation, we used the DNS results of Noguchi et al. for fully developed flow of a conductive fluid in plane channel subjected to magnetic fields of different orientation for \( \text{Re}_t = U_t D/2 \nu = 150 \) (Re = 4600). It is noted that different orientations of the magnetic field have different physical mechanisms of coupling the electromagnetic and velocity fields. The longitudinal magnetic field influences directly only the velocity fluctuations (through direct and indirect contributions to the budgets of \( u_i u_j \)) because the horizontal component of the mean Lorentz force is zero. In contrast, the transversal magnetic field affects also directly the mean velocity through the horizontal Lorentz force in addition to the effects on velocity fluctuations.

In order to test the performance of the new model when all terms are lumped together, we performed *a priori* testing using the algebraically truncated version of the proposed low-Re SMC in which all unknowns are directly provided from the DNS database. By assuming proportionality between turbulent stresses and turbulent kinetic energy (weak equilibrium hypothesis)

\[
\frac{D u_i u_j}{D t} - \bar{D}_{ij} = \frac{u_i u_j}{k} \left( \frac{D k}{D t} - D_k \right) \rightarrow P_{ij} + \Phi_{ij} - \varepsilon_{ij} + S_{ij}^M
\]

the partial differential equations for \( u_i u_j \) are reduced to algebraic expressions from which the turbulent stresses can be directly evaluated from the DNS database. The results are shown in Fig. 4. It can be concluded that *a priori* tested algebraic stress model produced good agreement with the DNS data for both orientations of the imposed magnetic field. The agreement with DNS data is particularly good in the near-wall region which is the focus of this study. The small deviation from the DNS profile in this region is observed only for the spanwise stress component \( \bar{w} \bar{w}_1 \) but the general trend is captured satisfactorily (Fig. 4).

B. Simulations

Next we present full simulations with the electromagnetic extensions of the low-Re SMC. The finite-volume numerical code for orthogonal 2D geometries with collocated variable arrangement (Hanjalić and Jakirlić) is used for simulations of flow of electrically conductive fluid in a channel with electrically insulated walls. Since the fully developed turbulent flows are in focus of this study, the periodic boundary conditions are applied in the streamwise direction thus reducing the number of control-volumes in this

![Fig. 8. Effect of strength of the magnetic field (Ha) on the normal and shear stress components: Re=2.9×10^4.](image)
second-moment closure with the LES of Shimomura\(^{20}\) and experimental data of Brouillette and Lykoudis\(^{21}\) (Re\(=2.9 \times 10^4\), Ha\(=52.5,125\)) is shown in Fig. 7. The results of Shimomura’s LES with Smagorinsky subgrid scale model show significant deviations from the experimental values for both intensities of the magnetic field considered. This subgrid model gives too large \(U^+\) at Ha\(=52.5\) and too small at Ha\(=125\) (denoted by ∆ in Fig. 7). The author concluded that this unsatisfactory behavior originates from excessive eddy viscosity. In order to overcome this problem, Shimomura suggested an extension of the Smagorinsky model to account for magnetic contributions to turbulent viscosity. This modification is introduced in the form of a locally determined damping factor which depends on the strength of the imposed magnetic field, \(v_{SGS} = \nu_s \exp[-(\sigma/\rho)(C_m\Delta)^2|B_0|^2/\nu_s]\), where \(\nu_s\) is the standard Smagorinsky eddy viscosity, \(\nu_s = (C_s\Delta)(S_{ij}S_{ij})^{1/2}\). The additional coefficient \(C_m\) is calibrated against the experimental data of Brouillette and Lykoudis\(^{21}\) With this new modification, significant improvements of the velocity \(U^+\) profiles are reported (symbols ○ in Fig. 7).

The present low-Re SMC model (full lines in Fig. 7) shows very good agreement both with the experimental and the LES (magnetically extended subgrid model) results. It is interesting to note the strongly nonmonotonic behavior of the velocity profiles and the stress flatness parameter \(A = 1 - 9/8(A_2 - A_3)\) (where \(A_2\) and \(A_3\) are the stress anisotropy second and third invariant, respectively, see Appendix) for different intensities of the magnetic field, Fig. 7. For a relatively weak magnetic field \((0<\text{Ha} \leq 30)\), an increase in the velocity profiles over the standard logarithmic profiles is observed. Above a certain value of the magnetic field \((30 \leq \text{Ha} < 40)\) this trend towards the laminarization stops and in the range of the intermediate magnetic field \((40 \leq \text{Ha} \leq 80)\) the velocity profiles are more damped in the central region of the channel \((y^+ \geq 10^3)\). For strong magnetic fields \((\text{Ha} > 80)\) the velocity profiles take the characteristic shape of the Hartmann solution. The flatness parameter \(A\) shows the expected response in different ranges of Ha. Here the trend towards the two-components turbulence in the central part of channel \((y^+ > 400)\) for two highest values of Ha \((\text{Ha} = 60,80)\) can be observed [Fig. 7 (bottom)].

The effect of a transversal magnetic field of different intensity on the distributions of turbulent stresses is shown in Fig. 8. Generally, all turbulent stresses are significantly reduced compared to the neutral (nonmagnetic) state. It is interesting to see that \(u^+\) component of the turbulent stress is less suppressed in the near-wall region compared to the remaining two components. The characteristic peaks of this stress components at approximately \(y^+ = 20\) show smaller differences for different values of Ha compared to \(v^+\) and \(w^+\), indicating the strong anisotropy prior and during the laminarization process. In the central region of the channel, all components are strongly suppressed.

In order to demonstrate the potentials of the new model to qualitatively capture effects of different magnetic field orientations, we performed additional simulations starting from the neutral case for the highest simulated value of Re\(_T\) = 590 (which correspond to Re\(=2.2 \times 10^4\)) of DNS pre-
The effects of the differently oriented magnetic fields on the mean velocity, friction factor and Reynolds stresses are illustrated in Figs. 9 and 10. The modifications of the mean velocity profiles and significant deviation from the standard log-law behavior is observed for both orientations of the magnetic field, Fig. 9, top. It is noted that for a longitudinal magnetic field with high Ha, a kind of "premature" laminarization takes place. Unfortunately, beside the already mentioned low-Re DNS data, there are no additional studies that can be used for more detailed examination of this behavior. In contrast to the longitudinal magnetic field orientation, the transversal orientation produces...
more complicated behavior of the mean velocity (as already mentioned for the previous test case, $Re=2.9\times10^4$). The effects of different magnetic field orientation can be further seen in Fig. 8 (bottom), where the friction factor ($c_f = 2\tau_w/\rho U^2$) dependence on $Ha$ is shown. The longitudinal magnetic orientation reduces slightly the $c_f$ until $Ha=40$ when a very significant drop towards the laminar value occurs. Further intensification in the magnetic field will not result in any additional reduction in $c_f$ since all velocity fluctuations are damped and there is no direct interaction between the Lorentz force and the mean velocity.

The transversal magnetic field causes first a slight reduction in the $c_f$ ($10\leq Ha\leq 40$). Then, $c_f$ starts to increase ($40\leq Ha\leq 65$) until the transitional point is reached with a characteristic “dip” region ($Ha=65$). When the intensity of the transversal magnetic field is increased further, $c_f$ continues to grow following the Hartmann line. It can be concluded that significant and quite opposite effects in terms of $c_f$ can be achieved by changing the orientations of the imposed magnetic field. This finding can be especially useful for flow control of electrically conductive fluids in many industrial applications.

It is interesting to note that very similar distribution of the friction coefficient $c_f$ is shown in Gardner and Lykoudis\(^{23}\) for a pipe flow subjected to transversal magnetic field. This “transitional” region was explained in terms of the “final damping of the fluctuations near the wall.” The same conclusion can be made from the presented distributions of turbulent stresses as shown in Fig. 10. The transversal magnetic field suppresses totally the velocity fluctuations for $Ha>60$ and this is exactly the same value for which the characteristic local minimum of $c_f$ appears (the first point on Hartmann line), Fig. 9 (bottom). It can be concluded that the transversal magnetic field [Fig. 10 (left)] suppresses significantly all stress components. In contrast, the longitudinal magnetic field dampens the wall-normal stress ($\tau_{ww}$) and enhances the streamwise one ($\tau_{uu}$), leaving the spanwise component ($\tau_{ww}$) almost unaffected, Fig. 10 (right). The corresponding profiles of the flatness parameter ($A$) for both magnet orientations are shown in Fig. 11. Here the typical nonmonotonic behavior in damping of turbulence and its strong anisotropic nature are nicely reflected in the $A$ profiles for the transversal orientation, Fig. 11 (left). In contrast to the $B||y$ orientation, the $B||x$ magnetic field introduces just
small changes in the central region of the channel, Fig. 11 (right). Different states of turbulence can be identified in terms of the second (II = \(-1/2b_{ij}b_{ij}\)) and third (III = \(-1/3b_{ij}b_{jk}b_{ki}\)) invariants of the Reynolds stress anisotropy tensor \((b_{ij} = u_i u_j/2k - 1/3\delta_{ij})\), Lumley.\(^{24}\) It can be seen once more that the \(B_{uu}y\) orientation promotes more significantly two-components turbulence state compared to \(B_{uu}x\) orientation.

Typical values of the characteristic nondimensional parameters for turbulent flows subjected to magnetic fields in real industrial situations are \(Re = \mathcal{O}(10^5)\) and \(Ha = \mathcal{O}(10^2)\). In order to demonstrate the applicability of the proposed model to “real-life” industrial situations, we compared the model performance against the experimental data for a channel flow of Brouillette and Lykoudis\(^{21}\) for a range of high \(Ha\) and \(Re\) numbers, Figs. 12 and 13. Despite high values of \(Re\), the integration up to the wall did not produce any additional difficulties thanks to the fine mesh (162 CV’s) clustered towards the wall. The mean velocity profiles show very good agreement with the available experimental data over the entire \(10^4 \leq Re \leq 10^5\) and \(0 \leq Ha \leq 500\) range, giving additional proofs of the potentials of the proposed model.

IV. CONCLUSIONS

The low-\(Re\)-number second-moment closure (Hanjalić et al.\(^{19}\)) has been extended to account for magnetic effects in flows of electrically conducting fluids. In addition to including exact terms that appear in the transport equations, several terms need to be modelled. We followed here the analogy with the practice in modelling effects of a general body force, while accounting for specific nature of the magnetic force. The model was first validated against the available DNS data at relatively low \(Re\) and \(Ha\) numbers, both for transverse and longitudinal magnetic fields, showing good agreement with \textit{a priori} term-by-term comparison and full simulation. Subsequently, the model was applied to flows with intermediate \(Re\) and stronger magnetic fields yielding the results in good agreement with the available LES data and showing also trends towards the asymptotic and limiting solutions (laminarization and Hartmann line). Finally, the model was applied to high \(Re\) and high \(Ha\) channel flows and results compared with experiments, showing good agreement.

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APPENDIX: THE FINAL VERSION OF THE MODEL

1. Reynolds stress equations

\[
\frac{D u_i u_j}{Dt} = \frac{\partial}{\partial x_k} \left[ \nu \delta_{ij} + C_3 \frac{k}{u_k u_j} \frac{\partial u_i u_j}{\partial x_k} \right] - \left( \frac{u_i u_k}{\partial x_j} + u_j u_k \frac{\partial U_i}{\partial x_k} \right) + \Phi_{ij} + S_{ij}^m - \delta_{ij}
\]

\[
\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi_{ij,3} + \Phi_{ij,4} + \Phi_{ij,5}
\]

\[
\Phi_{ij,1} = -C_1 \varepsilon a_{ij}, \quad \Phi_{ij,2} = -C_2 \left( \frac{P_{ij}}{3} - \frac{1}{2} \delta_{ij} \right),
\]

\[
\Phi_{ij,3} = -C_4 \left( S_{ij}^m - \frac{2}{3} S_k^m \delta_{ij} \right)
\]

\[
\Phi_{ij,4} = C_5 \left( \frac{\varepsilon_{ij} a_{ij}}{k} - \frac{1}{2} \delta_{ij} \right)
\]

\[
\Phi_{ij,5} = C_6 \left( \frac{\varepsilon_{ij} a_{ij}}{k} + \frac{1}{2} \delta_{ij} \right)
\]

\[
S_{ij}^m = S_{ij}^m + S_{ij}^m, \quad S_{ij}^m = -\frac{\sigma}{\rho} \left( \varepsilon_{ij} B_{ik} B_{j} + \varepsilon_{ij} B_{ik} B_{j} - 2B_{ij}^2 \right)
\]

\[
S_{ij}^m = -\frac{\sigma}{\rho} \left( \varepsilon_{ij} B_{ik} B_{j} + \varepsilon_{ij} B_{ik} B_{j} - 2B_{ij}^2 \right)
\]

where

\[
C = 2.5AF^{1/4}, \quad F = \min\{0.6, A_2\}, \quad f = \min\left( \frac{Re}{150} \right)^{3/2}, \quad A = 1 - \frac{2}{3} (A_2 - A_3), \quad E = 1 - \frac{2}{3} (E_2 - E_3),
\]

\[
A_2 = a_{ij} a_{ij}, \quad A_3 = a_{ij} a_{jk} a_{ki}, \quad E_2 = e_{ij} e_{ij}, \quad E_3 = e_{ij} e_{jk} e_{ki}, \quad a_{ij} = \frac{u_i u_j}{k} - \frac{2}{3} \delta_{ij},
\]

\[
e_{ij} = \frac{e_{ij}}{e} - \frac{2}{3} \delta_{ij}.
\]

<table>
<thead>
<tr>
<th>Table I. Specification of coefficients in the $u_i u_j$ equation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_5$</td>
</tr>
<tr>
<td>0.22</td>
</tr>
<tr>
<td>Table II. Specification of coefficients in the $e$ equation.</td>
</tr>
<tr>
<td>---------------------------------</td>
</tr>
<tr>
<td>$C_5$</td>
</tr>
<tr>
<td>0.18</td>
</tr>
</tbody>
</table>
2. Dissipation rate of turbulent kinetic energy

\[
\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_i} \left[ \nu \frac{\partial \varepsilon}{\partial x_i} + C_{\varepsilon} \frac{k}{\varepsilon} \frac{\partial \varepsilon}{\partial x_j} \right] - C_{\varepsilon 1} \frac{k}{\varepsilon} \frac{\partial U_i}{\partial x_j} \frac{\partial \varepsilon}{\partial x_j} - C_{\varepsilon 2} \frac{k}{\varepsilon} \frac{\partial^2 U_i}{\partial x_j^2} + \frac{S_M^M}{\varepsilon} + \frac{S_M}{\varepsilon} \] 

where

\[
f_{_\varepsilon} = 1 - \frac{C_{\varepsilon 2} - 1.4}{C_{\varepsilon 2}} \exp \left[ -\frac{(Re_f)^2}{6} \right], \quad S_M^M = \frac{\varepsilon}{\kappa} S_k^M.
\]