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THE VECTOR POTENTIAL IN A BRANCHED RIEMANN SPACE

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G. Schouten  The Vector Potential in a Branched Riemann Space
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SUMMARY

An attempt is made to extend the use of the vector potential to problems involving a branched space. It turns out that adaptation of known branched solutions of the Laplace equation to the equations of the vector potential leads to complications. In branched space the elementary ring vortex has to be used in stead of the simple vortex element in ordinary space. This implies that in branched space a line vortex has to be represented as a surface distribution of ring vortices. In ordinary space it can be represented as a line integral over vortex elements (Biot-Savart).

As an example of a problem that is suitable to be solved entirely in terms of the vector potential the problem of a ring vortex element above a half-plane is worked out. The use of the scalar potential however remains slightly simpler.

Another example that is treated is the simple looking problem of a straight vortex line perpendicularly crossing the edge of a half-plane. The problem is formulated in two ways, once using the scalar potential $\phi$ the other time using the vector potential $A$. It turns out that it is preferable to use the scalar potential and the Green's function formulation above using the method of images for the vector potential involving the more complicated integrals of distributions of ring vortices.
1. INTRODUCTION

The leading thought in this paper is the intuitive notion from Helmholtz's theorem that the scalar potential $\phi$ would be the most appropriate for the description of vector fields with divergence and the vector potential $A$ for the description of fields with rotation. In two-dimensional fields, with isolated divergence and rotation, $\phi$ and $A$ are fully equivalent as $A$ has only one non-zero component. In three-dimensional fields the scalar potential is widely used for the solution of boundary value problems using Green's theorem. The vector potential with its three components in general yields problems not simpler than those in terms of the vector fields which it describes. Boundary value problems turn out to be even more complicated due to the intricacies of the vector Green's theorem. The use of the vector potential in three-dimensional flow problems is therefore limited in practice to the law of Biot & Savart for the description of the velocity field of known vortex configurations. In simple boundary value problems where the method of images is applicable, avoiding the surface integrals of Green's theorem, the scalar potential and the vector potential seem to be equally efficient.

A large class of boundary value problems encountered in practice involves vortices passing over flat surfaces. We think of the vortices from upstream stabilizing surfaces passing over wings of airplanes, vortices generated by flaps passing under tailplanes, etc. In these problems the image vortex must be placed in a virtual space as the bounding surface has not enough 'inside room' for images (or none et all).

In the following first the way of solution of the eq. for the vector potential in ordinary space is critically analyzed. Then the extension of this procedure to a branched space is tried with the aim to find a simple way to solve the above mentioned kind of flow problems. The extended procedure is worked out but found to be impractical.
2. THE SCALAR POTENTIAL $\phi$ AND THE VECTOR POTENTIAL $A$

2.1. The elementary solutions

Every vector field $\mathbf{v}$ can be decomposed by Helmholtz's theorem, [1], [2], in an irrotational part $\mathbf{v}_1$ and a solenoidal part $\mathbf{v}_2$,

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2; \quad \nabla \times \mathbf{v}_1 = 0, \quad \nabla \cdot \mathbf{v}_2 = 0$$ (2.1)

The irrotational part can be represented by the gradient of a scalar potential function $\phi$, it takes care of the divergence $\nabla$ of $\mathbf{v}$

$$\mathbf{v}_1 = \nabla \phi; \quad \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{v}_1 = \nabla^2 \phi = q$$ (2.2)

The solenoidal part can be represented by the rotation of a vector potential $\mathbf{A}$, it takes care of the rotation $\gamma$ of the vector field $\mathbf{v}$

$$\mathbf{v}_2 = \nabla \times \mathbf{A}; \quad \nabla \times \mathbf{v} = \nabla \times \mathbf{v}_2 = \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \gamma$$ (2.3)

When the divergence $q$ and the rotation $\gamma$ are prescribed, the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are to be determined from the solutions $\phi$ and $\mathbf{A}$ to the equations (2.2) and (2.3). The latter eq. can be reduced to a simpler form by application of the Coulomb gauge. This is a vector field $\mathbf{A}' = \nabla F$, chosen such that $\nabla \cdot \mathbf{A}' = \nabla^2 F = -\nabla \cdot \mathbf{A}$, which when it is added to a solution $\mathbf{A}$ of (2.3) reduces eq. (2.3) to the form

$$\nabla^2 \mathbf{A} = -\gamma$$ (2.4)

Now $\phi$ as well as the components of $\mathbf{A}$ must satisfy inhomogeneous Laplace equations which in the elementary form read

$$\nabla^2 \mathbf{f} = \delta(x-x_o) \delta(y-y_o) \delta(z-z_o)$$ (2.5)

The solution to this eq. is the elementary solution

$$\mathbf{f} = \frac{-1}{4\pi R} \quad \text{where } R = \left((x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2\right)^{1/2}$$ (2.6)

The solutions to the eqs. (2.2) and (2.4) then follow from the integrals

$$\phi = \iiint_{V_o} dV_o \frac{-q_o}{4\pi R} \quad \text{and} \quad \mathbf{A} = \iiint_{V_o} dV_o \frac{\gamma_o}{4\pi R}$$ (2.7)

It must be verified however that $\nabla \cdot \mathbf{A} = 0$, for if $\nabla \cdot \mathbf{A} \neq 0$, $\mathbf{A}$ does not satisfy eq. (2.3). The Coulomb gauge cannot help in trying to remove such a deficiency from a solution of (2.4), it only serves solutions of (2.3) to satisfy the simpler form (2.4). The verification can be performed in the integral expression (2.7) due to the symmetry of $R$ in source point $(x_o, y_o, z_o)$ and observation point $(x, y, z)$. Taking the divergence of the integral (2.7) we may write

$$\nabla \cdot \mathbf{A} = \frac{-1}{4\pi} \iiint_{V_o} dV_o \nabla \cdot \mathbf{v} \left(\frac{\gamma_o}{R}\right) = \frac{1}{4\pi} \iiint_{V_o} dV_o \gamma_o \cdot \mathbf{v} \left(\frac{1}{R}\right)$$

$$= \frac{1}{4\pi} \iiint_{V_o} dV_o \nabla \cdot \mathbf{v} \left(\frac{\gamma_o}{R}\right) - \frac{1}{4\pi} \iiint_{V_o} dV_o \frac{1}{R} (\nabla \cdot \gamma_o)$$

$$= \frac{1}{4\pi} \int d\mathbf{O}_o \frac{\gamma_o}{R} = 0$$ (2.8)
because \( \nabla \cdot \gamma = 0 \) and also \( \gamma \cdot n \) vanishes on the enclosing control surface \( 0 \) when a vorticity distribution of finite dimensions is considered.

It must be noted that (2.7) is an integral form of the law of Biot & Savart. This law states that the velocity field of an element \( ds \) of a vortex line with strength \( \Gamma \) (where \( \Gamma = \gamma \times \text{cross section} \)) is described by

\[
dv = \nabla \times \frac{\Gamma ds}{4\pi R} = -\frac{\Gamma}{4\pi} \frac{R \times ds}{R^3}
\]

(Biot & Savart) \hspace{1cm} (2.9)

This law implicitly states the vector potential \( dA \) of the element \( ds \) to be

\[
dA = \frac{\Gamma ds}{4\pi R}
\]

It must be noted also that this elementary vector potential does not satisfy the full equation (2.3) for the vector potential as \( \nabla \cdot dA \neq 0 \). (Of course the distribution \( \gamma(x,y,z) \) in the R.H.S. should be replaced by the element \( \Gamma ds \) and this is incorrect as \( \nabla \cdot ds \neq 0 \)) Only the integral of the contribution of the elements leads to a vector potential which is free of divergence as is shown in (2.8). The symmetry of \( R \) as to observation point and source point plays an essential rôle in the derivation of this result. In Sec. 3 elementary solutions different from (2.6) will be used, the condition \( \nabla \cdot A = 0 \) there requires special care.

The fact that a vector field is entirely determined by its divergence and its rotation leads to the suggestion that a velocity field generated by a source and sink distribution \( q \) be preferably described by the scalar potential \( \phi \), and that a velocity field generated by a vorticity distribution \( \gamma \) be preferably described by a vector potential \( A \).

In the following it will be demonstrated that this preference is not to be extended to the solution of boundary value problems. The scalar potential, apart from having only one component whereas the vector potential has three, is the more efficient as it has a much simpler Green's function technique at its disposal.

### 2.2. The method of images and the vector potential

So far the vector potential \( A \) of a prescribed solenoidal vorticity distribution has been determined using techniques analogous to those used with the scalar potential \( \phi \). When considering simple boundary value problems the analogy can be extended to the method of images. For potential flow problems requiring zero normal velocity on the boundaries the method of images is one of the first to be tried and it seems to be simply convertible to problems in terms of the vector potential. It does not seem practical to formulate general flow problems in terms of the vector potential. The three velocity components are tied up with the components of the vector potential in too complicated a way to allow for an acceptable way of solution (see also Morse & Feshbach [1] p.1767 ff.).

The method of images is directly applicable in those cases where the symmetry of the configuration allows all of the images to be situated in 'real' space, i.e. in problems inside rectangular cavities with plane walls. In these simple three-dimensional problems, whether they concern the scalar potential or the vector potential, the \( 1/R \)-function is to be used as an elementary solution.

When the symmetry would require images to be placed in 'virtual' space the \( 1/R- \)
function is no longer suitable as it has singularities in real and in virtual space. For potential flow problems the \(1/R\)-function is replaced in those cases by a function which is continuous, single-valued, and which has only one singular point in the whole of the union of the real and the virtual space. For the vector potential an analogous replacement is necessary but here the condition on the divergence of the solution requires special attention and care.

As an example of a problem requiring an image in virtual space the two-dimensional potential flow problem of a vortex in the neighbourhood of a half-plane is treated. For the solution of this problem it is transformed from the \(z\)-plane to the \(\zeta\)-plane by the conformal mapping \(z = \zeta^2\). In the \(\zeta\)-plane the method of images is applicable. The two sheets of the branched Riemann \(z\)-plane are mapped next to each other on the \(\zeta\)-plane. The complex potential \(\chi\) of the vortex \(\Gamma\) in \(z = z_0\), which reads

\[
\chi = \frac{4\Gamma}{2\pi} \ln(z - z_0) = \frac{4\Gamma}{2\pi} \ln(\zeta - \zeta_0)(\zeta + \zeta_0),
\]  

(2.11)

has singularities in the \(\zeta\)-plane in \(\zeta = \zeta_0\) and in \(\zeta = -\zeta_0\).

One of these singularities is situated in the mapping of the 'physical' \(z\)-plane, the other is situated in the mapping of the 'virtual' \(z\)-plane. For this problem the elementary solution \(\ln(z - z_0)\) must be replaced by \(\ln(\sqrt{z - z'_0})\) as this latter function has only one singular point (apart from the branch point \(z = 0\)) in the two sheets of the \(z\)-plane. The solution to the flow problem in terms of the complex potential \(\chi\) then reads

\[
\chi = \frac{4\Gamma}{2\pi} \ln \left( \frac{\zeta - \zeta_0}{\zeta + \zeta_0} \right) = \frac{4\Gamma}{2\pi} \ln \left( \frac{\sqrt{z - z'_0}}{\sqrt{z - z'_0}} \right)
\]

\[
= \frac{4\Gamma}{2\pi} \ln(z - z_0) - \frac{4\Gamma}{2\pi} \ln(\sqrt{z - z'_0})(\sqrt{z + z'_0})
\]  

(2.12)

![Fig. 1: The mapping of a half-plane problem in \(z\) on a whole-plane problem in \(\zeta\) with the mapping \(z = \zeta^2\).](image)

A generalized three-dimensional problem requiring similar imaging techniques for the vector potential is that of an infinitely long straight vortex line parallel to the \(x\)-axis, in the plane \(z = 0\), and crossing the edge of the half-plane \(y = 0, x > 0\). This problem is treated in sec. 3.3, but first in sec. 3.2 the appropriate elementary solution is put forward.
3. THE VECTOR POTENTIAL IN A TWO-FOLD RIEMANN SPACE

3.1. The vector potential of a vortex element in a two-fold space

We consider a Riemann space which has the z-axis for branch line. Then, with cylindrical coordinates, we take the range \( 0 < \theta < 2\pi \) for the physical space, and \( 2\pi < \theta < 4\pi \) for the imaginary space, the two building up a two-fold Riemann space. A function which is single-valued in this Riemann space will be double-valued in ordinary space.

Various problems concerning the potential eq., the wave eq., the diffusion eq. have been treated by Sommerfeld [4], Carslaw [5], using a Riemann space. For the vector potential such a treatment seems new.

In the following an attempt is made to formulate and solve the difficulties that occur with the vector potential in branched space. In sec. 2.1 it was shown that in ordinary space the elementary solution of the Laplace eq., the 1/R-function, could be used for the vector potential eq. as well. We therefore first consider the elementary solution for the Laplace eq. in a two-fold Riemann space. The delta-function in the R.H.S. must be single-valued in this space, leading to

\[
\nabla^2 \phi = \frac{\delta(r-r_0)}{4\sqrt{rr_0}} \delta(\sin(\frac{\theta\theta_0}{4})) \delta(z-z_0) \tag{3.1}
\]

The solution to this equation, as to be found in [4] and [6], is

\[
\phi = -\frac{1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{rr_0}} \right) \tag{3.2}
\]

The equation for the vector potential in branched space must be used in its complete form (2.3) as the reduction to the form (2.4) is prohibited when the 1/R-function is not available. For simplicity we consider a singular vorticity in x-direction only, represented by the double-valued delta-function of (3.1) in the x-component of the R.H.S. The other components of the R.H.S. are zero in the resulting eq. for the vector potential

\[
\frac{\partial}{\partial x} (\nabla \cdot A) - \nabla^2 A_1 = \frac{\delta(r-r_0)}{4\sqrt{rr_0}} \delta(\sin(\frac{\theta\theta_0}{4})) \delta(z-z_0) \\
\frac{\partial}{\partial y} (\nabla \cdot A) - \nabla^2 A_2 = 0 \\
\frac{\partial}{\partial z} (\nabla \cdot A) - \nabla^2 A_3 = 0 \tag{3.3}
\]

Again the note following (2.10) applies, this R.H.S. is not allowable. Generalizing the procedure of sec. 2.1 to a branched space we tentatively put the elementary function (3.2) as a kernel in an integral representation of a solution to the eq. for the vector potential in two-fold Riemann space. The vector potential of a line vortex in x-direction in branched space would then become

\[
A_1 = \int dx_0 \left( \frac{1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{rr_0}} \right) \right) , A_2 = 0, A_3 = 0 \tag{3.4}
\]
In sec. 2.1 it could be proved, without performing the integration, that the resulting vector potential was free of divergence. The kernel (3.2) does not allow this proof to be performed here. Unfortunately one is left with the result that (3.4) does not satisfy the full equation for the vector potential with a line singularity in the branched space. There remains a distributed vorticity of the form

\[
\gamma_{\text{distr}} = \nabla (\nabla \cdot A) = \nabla \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \left( -\frac{1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{\theta - \theta'}\right) \right) \neq 0
\]

That this integral does not vanish is to be seen by changing \( \frac{\partial}{\partial x} \) into \( \frac{\partial}{\partial x_0} \) and correcting for that change so that one gets

\[
\gamma_{\text{distr}} = \nabla \int_{-\infty}^{\infty} dx_0 \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x_0} \right) \left( -\frac{1}{4\pi R} \right) \text{(etc.)}
\]

\[
= -\nabla \int_{-\infty}^{\infty} dx_0 \frac{r + r_0}{(r + r_0)^2 + (z - z_0)^2} \cos \left( \frac{\theta + \theta'}{2} \right) \neq 0
\]

The integrand in the latter integral is, for a certain value of \( \theta \), definitely positive or definitely negative but certainly not zero.

The branched solution (3.2) of the Laplace equation (3.1) thus probably is not suitable as an elementary solution of the equation for the vector potential in branched space, even not if it is only used in an integral representation. The problem remains to find a function that is suitable for that purpose. As the divergence of the vector potential is causing the trouble above, one is led to look for an elementary solution that is free of divergence itself, thereby abandoning the simple Biot & Savart like approach. A correct element to start with would be a ring vortex element. In the following the branched vector potential of such an element will be constructed. This will be done in a special, not straightforward, way in order to completely avoid the appearance of distributed vorticity.

### 3.2. The vector potential of a ring vortex element in branched space

We consider the same Riemann space as in sec. 3.1 and we put the ring vortex element in the x-y-plane, fig. 2.

![Fig. 2: A ring vortex element in the x-y-plane.](image)

The z-axis is the branch line.

The vector potential of such a ring vortex element in a two-fold space must satisfy the eqns.
\[ \frac{\partial}{\partial x} (\nabla \cdot \mathbf{A}) - \nabla^2 A_1 = - \frac{\partial}{\partial y_o} \frac{\delta}{4\pi r_o} \frac{\delta(r-r_o)}{4\pi r_o} \delta(\sin(\frac{\theta-\theta_o}{4})) \delta(z-z_o) \]

\[ \frac{\partial}{\partial y} (\nabla \cdot \mathbf{A}) - \nabla^2 A_2 = \frac{\partial}{\partial x_o} \frac{\delta}{4\pi r_o} \frac{\delta(r-r_o)}{4\pi r_o} \delta(\sin(\frac{\theta-\theta_o}{4})) \delta(z-z_o) \] 

(3.7)

\[ \frac{\partial}{\partial z} (\nabla \cdot \mathbf{A}) - \nabla^2 A_3 = 0 \]

In finding the solution to these eqns. we have the advantage that the velocity field of a dipole in ordinary space is identical to that of a ring vortex. The double-valued potential of a dipole follows from (3.2) by differentiation

\[ \phi = \frac{\partial}{\partial z_o} \left\{ \frac{-1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{\frac{\theta-\theta_o}{2}}} \right) \right\} \] 

(3.8)

The equivalence of this scalar potential and the vector potential satisfying (3.7) requires that \( \nabla \times \mathbf{A} = \nabla \phi \), or

\[ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial}{\partial x} \frac{\delta \phi}{\delta x}, \quad \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = \frac{\partial}{\partial y} \frac{\delta \phi}{\delta y}, \quad \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = \frac{\partial}{\partial z} \frac{\delta \phi}{\delta z} \] 

(3.9)

For the solution of these eqns., see Bricard [3] p. 136, we have an infinity of possibilities due to the Coulomb gauge. Any solution of (3.9), \( \phi \) satisfying (3.8), will satisfy (3.7) as the R.H.S. of (3.9) is free of divergence (except at the singularity of course). The formal procedure of solution is to find an arbitrary solution to (3.9) and then add a solution of the form \( \mathbf{A}' = \mathbf{V} \mathbf{F} \) chosen such that it annihilates the divergence of \( \mathbf{A} \).

Intuitively conjecturing that the vector potential of a ring vortex in the x-y-plane has no z-component (as the R.H.S. of (3.7) has none) directly leads to the solenoidal solution

\[ A_1 = \int \frac{\partial \phi}{\partial y} \, dz, \quad A_2 = \int -\frac{\partial \phi}{\partial x} \, dz, \quad A_3 = 0 \] 

(3.10)

The still existing symmetry in \( z \) and \( z_o \) of (3.8) allows the \( \partial / \partial z_o \) to be replaced by \( -\partial / \partial z \), so that the solution to (3.6)' becomes

\[ (A_1, A_2, A_3) = \left( -\frac{\partial}{\partial y_o} \frac{\delta}{\delta x_o} 0 \right) \left( \frac{-1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{\frac{\theta-\theta_o}{2}}} \right) \right) \] 

(3.11)

[It should be noted that the more direct generalization as pursued in sec. 3.1 would have led to an incorrect expression. When we bluntly had taken for \( A \) the form suggested by the R.H.S. of (3.7), we would have got

\[ (A_1, A_2, A_3) = \left( \frac{\delta}{\delta y_o}, -\frac{\delta}{\delta x_o}, 0 \right) \text{(etc. ...)} \]

This expression does not satisfy (3.7), it has distributed vorticity.]

One must conclude that the vector potential of a solenoidal vorticity distribution in two-fold space in principle can be formed as integrals with elementary ring vortex potentials of the form (3.11) as kernels. In sec. 4 such integrals
will be used in examples involving vortex lines.
To help the imagination computed streamline patterns of a ring vortex element in two-fold space are shown in fig. 3. The ring vortex is situated on the y-axis \((\theta = \pi/2)\) and has its components in the x-y-plane. The location of the branch plane of course may be chosen at will, as long as the z-axis is the edge. In the figure the branch plane is put through the singularity so that half of the flow takes place in the branch \(-3\pi/2 < \theta < \pi/2\) and the other half in the branch \(\pi/2 < \theta < 5\pi/2\). In this way symmetrical pictures result.

![Diagram](image)

**Fig. 3a:** The branches used below.

![Diagram](image)

**Fig. 3b:** Streamlines of a ring vortex in two-fold space.

3.3. The velocity field of a ring vortex in the x-y-plane above a half-plane \(y=0\), \(0 < x\) with the z-axis as an edge.

The flow problem is solved by the method of images in the two-fold space of which the half-plane is a branch-membrane. We consider the scalar potential \(\phi\) since from the way in which the components \((3.10)\) of the vector potential are to be obtained it is clear that using the vector potential implies a useless detour. When the ring vortex (or the dipole) is situated at \((r_o, \theta, z)\) an image ring vortex (or dipole) must be put at \((r_o, 4\pi-\theta, z)\). The potential field of this flow problem is described by
\[ \phi = \frac{\partial}{\partial z_o} \left\{ -\frac{\gamma ds}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{r_o} \cos \left( \frac{\theta - \theta_o}{2} \right)} \right) \right\} \]
\[ -\frac{\gamma ds}{4\pi R^*} \left( 1 - \frac{1}{\pi} \arctan \frac{R^*}{2\sqrt{r_o} \cos \left( \frac{\theta - \theta_o}{2} \right)} \right) \]

where \( R = \left( (x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2 \right)^{\frac{1}{2}} \) and \( R^* = \left( (x-x_o)^2 + (y+y_o)^2 + (z-z_o)^2 \right)^{\frac{1}{2}} \).

The components of the velocity then are, multiplied by \( 4\pi/\gamma \),

\[ \frac{4\pi}{\gamma} v_x = \frac{3(z-z_o)}{R^5} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{F_1} (x-x_o) + \frac{3}{\pi R^4} (z-z_o) \frac{F_1}{F_2} (x-x_o) \right) \]
\[ - \frac{2}{\pi R^2} \frac{(z-z_o) r_o}{F_1 F_2} \left( \frac{x}{r} + \frac{x_o}{r_o} \right) + \frac{z-z_o}{\pi R^2} \frac{F_1}{(F_2)^2} 2(r+r_o) \frac{x}{r} + 4\pi v^* \]  

\[ \frac{4\pi}{\gamma} v_y = \frac{3(z-z_o)}{R^5} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{F_1} (y-y_o) + \frac{3}{\pi R^4} (z-z_o) \frac{F_1}{F_2} (y-y_o) \right) \]
\[ - \frac{2}{\pi R^2} \frac{(z-z_o) r_o}{F_1 F_2} \left( \frac{y}{r} + \frac{y_o}{r_o} \right) + \frac{z-z_o}{\pi R^2} \frac{F_1}{(F_2)^2} 2(r+r_o) \frac{y}{r} + 4\pi v^* \]  

\[ \frac{4\pi}{\gamma} v_z = \left( \frac{1}{R^3} - \frac{3(z-z_o)^2}{R^5} \right) \left( 1 - \frac{1}{\pi} \arctan \frac{R}{F_1} \right) + \frac{3}{\pi R^4} (z-z_o)^2 \frac{F_1}{F_2} \]
\[ - \frac{1}{\pi R^2} \frac{F_1}{F_2} + \frac{2(z-z_o)^2}{\pi R^2} \frac{F_1}{(F_2)^2} + 4\pi v^* \]  

where \( F_1 = 2 \sqrt{r_o} \cos \left( \frac{\theta - \theta_o}{2} \right) \) and \( F_2 = (r+r_o)^2 + (z-z_o)^2 \).

The velocities of the image field are indicated by asterisks. The differences in the expressions for the fields of the original and the image occur in \( y_o, R \) and \( F_1 \). The double-valuedness of the expressions is taken care of by \( F_1 \). A natural situation of the branch-cut is at \( \theta = \theta_o \pm \pi (\pm 2\pi) \). With this location of the cut we have \( \phi_1 \) in the first branch of \( \phi \) space where \( \theta - \theta_o < \pi \), and \( \phi_2 \) in the second branch where \( \pi < \theta - \theta_o < 2\pi \). The argument of the \arctan should be taken in the region \( 0 < \arctan < \pi \).

Since in this flow problem the branch cut must coincide with the half-plane this has to be taken care of in the selection of the branches of \( \phi \) and \( \phi^* \). The choice is shown schematically in fig. 4.
Fig. 4: The selected branches of $\phi$ and $\phi^*$ in the flow problem.

A computed picture of the streamline pattern of a dipole in $z$-direction above the edge of a half-plane is shown in fig. 5.

Fig. 5: Streamlines of a ring vortex in the $x$-$y$-plane above the edge of a half-plane $y = 0, 0 < x$. 
Interesting phenomena in this velocity field are the stagnation points and the singular flow around the edge. The location of the stagnation points is not to be found in a simple analytical way from (3.13a) and (3.13b) as the expressions are of a mixed algebraic-transcendental type. A numerical iteration is the only means to localize them. The singular flow around the edge is investigated more simply. Both in (3.13a) and in (3.13b) a term occurs with $F_1$ in the denominator, leading to a $1/\sqrt{r}$-singularity in the velocities $v_x$ and $v_y$. The magnitude of the singular flow is found from an expansion around $x = 0, y$ of the relevant terms, yielding

$$v_{x,\text{sing.}} = -\frac{\gamma}{\pi^2} \frac{z/\rho}{(r^2 + z^2)^2} \cos(\theta/2) \cos(\theta/2) \frac{1}{\sqrt{r}}$$

$$v_{y,\text{sing.}} = -\frac{\gamma}{\pi^2} \frac{z/\rho}{(r^2 + z^2)^2} \cos(\theta/2) \sin(\theta/2) \frac{1}{\sqrt{r}}$$

(3.14)

The magnitude varies with the distance from the plane of symmetry, it is anti-symmetric around $z = 0$ and has extremes at

$$z = \pm r_0/\sqrt{3}$$

(3.15)
4. THE DOUBLE VALUED VECTOR POTENTIAL OF VORTEX LINES

4.1. The double valued vector potential of a vortex parallel to the branch line

This essentially two-dimensional problem should yield as a result a vector potential identical to the stream function of the complex potential \( \chi \), (2.12). Omitting the image vortex from (2.12) it reads

\[
\phi = \frac{1}{4\pi} \ln\left( r + r_0 \right) + 2 \left( \frac{\theta - \theta_0}{2} \right) \cos\left( \frac{\theta - \theta_0}{2} \right) \tag{4.1}
\]

We construct this vector potential as an integral of ring vortex elements in the plane \( y = y_0 \) as sketched in fig. 6.

![Fig. 6: A ring vortex in a plane \( y = y_0 \).](image)

The vector potential of such a ring vortex element is constructed in analogy to the construction of (3.11) from (3.8). Conjecturing \( A_2 = 0 \) we then get

\[
d\mathbf{A} = \left( -\frac{\partial}{\partial z}, 0, \frac{\partial}{\partial x} \right) \cdot \frac{1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{rr_0} \cos\left( \frac{\theta - \theta_0}{2} \right)} \right) \tag{4.2}
\]

The vector potential of the line vortex than becomes

\[
A = \int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{\infty} dx_0 \frac{dA}{r_0 \cos\theta_0} \tag{4.3}
\]

yielding \( A_1 = 0, A_2 = 0 \) and

\[
A_3 = \int_{-\infty}^{\infty} dz_0' \int_{-\infty}^{\infty} dx_0' \int dy \frac{\partial}{\partial x} \frac{\partial}{\partial y} \left[ -\frac{1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{rr_0} \cos\left( \frac{\theta - \theta_0}{2} \right)} \right) \right] \tag{4.4}
\]

This integration is easiest performed when \( \theta_0 = 0 \), that means putting the ring vortex distribution in the plane through the branch-line and the vortex line. The result, restoring \( \theta_0 \neq 0 \), is

\[
A_3 = \frac{1}{2\pi} \ln\left( r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) \right) \tag{4.4}
\]

\[
= \frac{1}{4\pi} \ln\left( r + r_0 + 2\sqrt{rr_0} \cos\left( \frac{\theta - \theta_0}{2} \right) \right) - \frac{1}{4\pi} \ln\left( r + r_0 - 2\sqrt{rr_0} \cos\left( \frac{\theta - \theta_0}{2} \right) \right) \tag{4.4}
\]
4.2. The double-valued vector potential of a vortex line crossing the branch line of a two-fold space at right angles

The vortex line is represented as the edge of a half infinite distribution of ring vortex elements in the plane $z = 0$ of the Riemann space described in sec. 3.1.

Fig. 7: The configuration when the vortex crosses the edge at right angles.

The vector potential of the vortex line is represented by the integral

$$ A = \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dy' \frac{dA}{a} $$

where $dA$ is the branched vector potential (3.11) of an element. The components of (4.5) then are:

$$ A_1 = -\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\partial}{\partial y} \left[ \frac{Y}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2} \cos \left( \frac{\theta - \theta'}{2} \right) \right) \right] $$

$$ A_2 = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\partial}{\partial x} \left[ \frac{Y}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2} \cos \left( \frac{\theta - \theta'}{2} \right) \right) \right] $$

$$ A_3 = 0 $$

It should be noted that the scalar potential of the vortex line, in the two-fold space, formed as an integral over a distribution of dipoles has a form analogous to (4.6a, b).

$$ \phi = -\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\partial}{\partial z} \left[ \frac{Y}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2} \cos \left( \frac{\theta - \theta'}{2} \right) \right) \right] $$

In trying to evaluate the above integrals by contour integration in the complex plane expressions with multiple square roots arise. It does not therefore seem useful to pursue this further. For an investigation of the velocity field the above double-valued potentials do not seem to be very practical either. In the following paragraph the double-valued scalar Green's function will be used, which is easier to handle.
4.3. The scalar potential of a line vortex crossing the edge of a half-plane expressed by means of a scalar Green's function integral

Green's integral representation reads

\[
\phi(x', y', z') = \iiint q(x, y, z) G(x', y', z'; x, y, z) \, dx \, dy \, dz + \int\int\int \frac{\partial}{\partial n} G(x', y', z'; x, y, z) \, \frac{\partial \phi}{\partial n} (x, y, z) \, d\sigma
\]

(4.8)

The volume integral here formally represents the potential \( \phi_0 \) of the vortex line parallel to the \( x \)-axis through \( y_0, z_0 \)

\[
\phi_0(x', y', z') = \frac{\Gamma}{2\pi} \arctan \frac{y'-y_0}{z'-z_0}
\]

(4.9)

It must be understood that the observation point is primed in (4.9) and (4.8). In (4.8) this means of course that the source point of the Green's function becomes the observation point of \( \phi \). The normal is pointing outward so that

\[
\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial y} \quad \text{on the upper surface}
\]

\[
\frac{\partial \phi}{\partial n} = -\frac{\partial \phi}{\partial y} \quad \text{on the lower surface}
\]

(4.10)

We split the potential \( \phi \) in a \( \phi_0 \) of the original vortex and a \( \phi^* \) of the image vortex which is to be represented by the surface integrals. In the surface integrals we thus should use \( \phi^* \). The value of \( \frac{\partial \phi^*}{\partial n} \), to be inserted in (4.8) should annihilate the normal velocity of the original vortex so that we have on the half plane \( y = 0 \),

\[
\frac{\partial \phi^*}{\partial y} = -\frac{\partial \phi_0}{\partial y} = -\frac{\Gamma}{2\pi} \frac{(y-y_0)}{(y-y_0)^2+(z-z_0)^2}
\]

(4.11)

For simplicity we put \( z = 0 \). The Green's function for Neuman conditions consists of the two-valued potential of a source in \((r', \theta', 0)\) and an image source in \((r', 4\pi-\theta', 0)\),

\[
G = -\frac{1}{4\pi R} \left( 1 - \frac{1}{\pi} \arctan \frac{R}{2\sqrt{rr'}} \cos\left(\theta-\theta'\right) \right)
\]

\[
-\frac{1}{4\pi R^*} \left( 1 - \frac{1}{\pi} \arctan \frac{R^*}{2\sqrt{rr'}} \cos\left(\theta+\theta'\right) \right)
\]

(4.12)

where \( R = (r^2 + r'^2 - 2rr' \cos(\theta-\theta') + z^2)^{\frac{1}{2}} \)

and \( R^* = (r^2 + r'^2 - 2rr' \cos(\theta+\theta') + z^2)^{\frac{1}{2}} \)

The integral representation for the scalar potential distribution in space thus can be written out. The expressions are complicated however and we limit ourselves to the potential distribution in and on the plane \( y = 0 \). The kernel (4.12) then reduces to a simpler form. For convenience we first write down the forms to which (4.12) reduces when observation point and integration point are on the same or on different sides of the half-plane. First, when they are on the same side, we have
\[ \theta = 0 \quad \theta' = 0 \quad G = -\frac{1}{2\pi R_s} \left( 1 - \frac{1}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \right) \] (4.13)

or \[ \theta = 2\pi, \quad \theta' = 2\pi \]

and when they are on different sides

\[ \theta = 0, \quad \theta' = 2\pi \quad G = -\frac{1}{2\pi R_s} \frac{1}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \] \[ \theta = 2\pi, \quad \theta' = 0 \quad G = -\frac{1}{2\pi R_s} \frac{1}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \] (4.14)

where \( R_s = R(y = 0) = R^*(y = 0) = ((x-x')^2 + (z-z')^2)^{1/2} \), and \( r' = x' \) subs. \( r = x \).

The potential distribution on the upper surface then becomes

\[ \phi_u(x', o, z') = -\frac{\Gamma}{2\pi} \arctan \frac{z'}{y_o} \]

\[ - \int \int \text{dxdz} \frac{1}{2\pi R_s} \left( 1 - \frac{1}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \right) \frac{-\Gamma}{2\pi} \frac{z'}{y_o^2 + z'^2} \]

\[ - \int \int \text{dxdz} \frac{1}{2\pi R_s} \left( \frac{1}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \right) \frac{\Gamma}{2\pi} \frac{z}{y_o^2 + z^2} \] (4.15)

And on the lower surface \( \phi \) follows from

\[ \phi_l(x', o, z') = -\frac{\Gamma}{2\pi} \arctan \frac{z'}{y_o} \]

\[ - \int \int \text{dxdz} \frac{1}{2\pi R_s} \left( \frac{1}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \right) \frac{-\Gamma}{2\pi} \frac{z}{y_o^2 + z^2} \]

\[ - \int \int \text{dxdz} \frac{1}{2\pi R_s} \left( 1 - \frac{1}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \right) \frac{\Gamma}{2\pi} \frac{z}{y_o^2 + z^2} \] (4.16)

The forms (4.15) and (4.16) can be unified in

\[ \phi(x', o, z') = -\frac{\Gamma}{2\pi} \arctan \frac{z'}{y_o} \pm \phi^* \quad ; \quad y' = \pm 0, \quad x' > 0 \] (4.17)

where

\[ \phi^* = -\frac{\Gamma}{(2\pi)^2} \int \text{dxdz} \frac{1}{R_s} \left( 1 - \frac{2}{\pi} \arctan \frac{R_s}{2\sqrt{xx'}} \right) \cdot \frac{z}{y_o^2 + z^2} \] (4.18)

It must be noted that in the half plane \( x' < 0, \quad y' = 0 \), the potential \( \phi \) is the undisturbed potential \( \phi_u \). The Green's function there has the value \( G = -1/(4\pi R_s) \) for \( \theta' = \pi, \theta = 0 \) and \( \theta'_2 = 2\pi \).

The integrals over the upper and over the lower surface of the half-plane have different signs in \( \partial\phi/\partial n \) and thus annihilate each other.

The expression (4.18) does not lend itself to a complete evaluation. We therefore have to restrict ourselves to some special cases where approximate expressions can be found. In these cases the observation point is near the edge, or far away from the edge and near the plane of symmetry. It is not difficult to show that far from the edge on the half-plane, for small values of \( z' \), the expression (4.18) reduces to
\( \phi^* = \frac{\Gamma}{2\pi} \arctan \frac{z'}{y_o}; \quad x' \gg y_o \text{ and } z' < y_o \) \hspace{1cm} (4.19)

This means that far from the edge the vortex 'does not see' the edge. The half-plane 'reflects' the vortex line as if it were a whole plane. On the lower side of the half-plane the image potential (4.19) to first order neutralizes the potential of the vortex and far from the edge in first approximation no velocity is felt.

Near the edge of the half-plane the integral (4.18) can be approximated to yield a square root singularity in the velocity field

\[
\frac{\partial \phi^*}{\partial x'} = -\frac{\Gamma}{2\pi \sqrt{x'}} \frac{i z'}{y_o^2 + z'^2} \left( \frac{y_o^2 + z'^2}{y_o + (y_o^2 + z'^2)^{\frac{1}{2}}} \right)^{\frac{1}{2}}
\]

The minus sign must be chosen on the upper surface, the plus sign on the lower. There is a maximum in the singular flow around the edge at \( z' = \pm y_o \sqrt{3} \), where the strength of \( \frac{\partial \phi^*}{\partial x} \) is

\[
\left( \frac{\partial \phi^*}{\partial x} \right)_{\text{max}} = \frac{i}{4\pi \sqrt{2} y_o} \frac{1}{\sqrt{x'}}; \quad z' = \pm y_o \sqrt{3}
\]

(4.4)

In fig. 8 a sketch of the streamline pattern on the surface is presented.

![Streamline pattern](image-url)

**Fig. 8:** Streamline pattern on the surface of the half-plane.
5. REFERENCES

1. Morse, P.M. and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill, 1953.


6. LIST OF SYMBOLS

A - vector potential with Cartesian components $A_1$, $A_2$, $A_3$
A' - vector potential from Coulomb gauge
ds - line element
f - function
F - scalar field in Coulomb gauge, $F = F(x, y, z)$
$F_1, F_2$ - shorthand for expressions defined at p. 13
G - Green's functions, specified at p. 18
L.H.S. - Left Hand Side
q - source strength or divergence of vector field, $q = q(x, y, z)$
r - radial coordinate
R - metric distance $R = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$\textsuperscript{1/2}
R* - metric distance to image
R.H.S. - Right Hand Side
v - vector field, built up from $v_1$ and $v_2$
$v_1$ - irrotational part of $v$
$v_2$ - solenoidal part of $v$
$\nabla$ - velocity field with Cartesian components $u$, $v$, $w$
x, y, z - Cartesian coordinates
z - complex number $z = x + \imath y$ or its mapping on the $z$-plane
$\Gamma$ - strength of vortex line
$\chi$ - strength of vorticity or rotation of vector field, $\chi = \chi(x, y, z)$
$\delta$ - deltafunction
$\zeta$ - complex number $\zeta = \xi + \imath \eta$, or its mapping on the $\zeta$-plane
$\theta$ - angular coordinate
$\phi$ - scalar potential function, $\phi = \phi(x, y, z)$
$\phi^*$ - scalar potential of image
$\chi$ - complex potential function $\chi = \phi + \imath \psi$
$\psi$ - stream function
$\nabla$ - nabla operator with components $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$

Subscripts, superscripts and indices

x, y, z - no subscripts, refers to observation point
$x_0, y_0, z_0$ - refers to source point, incidentally it refers to the integration parameter
$x', y', z'$ - refers to integration parameter, at p. 18 it refers to the source point which is after the integration playing the role of observation point
* - asterisk refers to image
$\phi_u, \phi_x$ - u and x refer to upper and lower side