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A multigrid method
for an invariant formulation
of the incompressible Navier-Stokes
equations in general coordinates

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1 Introduction

An important topic in computational fluid dynamics is to compute the solution of the incompressible Navier-Stokes equations in complex geometries in non-orthogonal coordinate systems using a finite difference or a finite volume method. The solution of the discretized equations can be found with iterative methods. With the rapidly growing speed and storage possibilities of (mini-) supercomputers the solution of the 3-D equations comes within reach.

To develop computational methods for the incompressible Navier-Stokes equations in general non-orthogonal coordinates a number of choices have to be made, for example:

1) velocity unknowns: Cartesian, covariant or contravariant components;

2) staggered or non-staggered arrangement of the variables;

3) the iterative solution method to solve the discrete equations.

Recently a number of articles have appeared in which some of these alternatives are used. At the moment it is not clear, which approach is to be favored. A review is given in [8], where Cartesian velocity components on a non-staggered grid and a multigrid solution method are used. In [4] covariant velocity components on a staggered grid and an iterative solution method called SIMPLEC are used. On one point many authors seem to agree: one should try to avoid the occurrence of additional body force terms, which arise from the curvature of grid lines. In these terms the so-called Christoffel symbols occur. These depend on second derivatives of the coordinate mapping, and tend to be expensive and difficult to compute accurately.

However, Christoffel symbols occur in an invariant (i.e. coordinate system independent) formulation of the Navier-Stokes equations, and indeed in the invariant formulation of many physical conservation laws. To discretize these laws as they stand seems a general, robust and natural approach, which will be followed here. The only publications that the authors know of in which an invariant form of the incompressible Navier-Stokes equations is discretized directly are [10] and [5]. In [10] Gibbs' vector notation is used, Christoffel symbols do not appear as such; instead other geometric quantities appear. In [5] physical tensor components are used as unknowns. In our approach the Christoffel symbols appear explicitly. In fact, one of the aims of our research is to show how the Christoffel symbols can be discretized such that an accurate discretization in general coordinates results. We will present an invariant discretization in a non-orthogonal boundary fitted coordinate system, using contravariant velocity components, a staggered grid arrangement and a multigrid solution technique. The smoother used is the Symmetric Coupled Gauss-Seidel method (SCGS) introduced in [15]. In Cartesian coordinates properties of this smoother have been investigated in [9] and [12]. A comparison between SCGS and several uncoupled solution methods is presented in [1].

We present results for the two-dimensional case with Dirichlet boundary conditions. The outline of the paper is as follows:
In section 2 the equations are presented in a tensor formulation. Some basic tools of tensor
analysis are given together with some insights in choices for computing geometric quantities and the choice of variables.

In section 3 the Nonlinear Multigrid algorithm ([3], [6]) in the structured goto-free version presented in [16], the prolongation and restriction operator, the smoother and the choice for the coarse grid operator are given.

Some results are given in section 4. A comparison between reduction factors for a rectangular and a skewed driven cavity flow is given. The flow through an L-shaped pipe with a low Reynolds number is presented, and the flow through a nozzle is computed to show the accuracy of the discretization in the presence of appreciable grid irregularity.
2 General Curvilinear Coordinates

The physical domain $\Omega$ is mapped onto a rectangular block, called the computational domain, resulting in a boundary-fitted grid. It is assumed that the transformations $x = x(\xi)$ and $\xi = \xi(x)$ between the two domains are admissible, which implies that the Jacobian of the transformation does not vanish; $x$ are Cartesian coordinates, $\xi$ are boundary-conforming curvilinear coordinates.

Covariant base vectors $a_{(\alpha)}$ are defined as tangent vectors to the curvilinear coordinate lines $\xi^\alpha = \text{constant}$, i.e.

$$a_{(\alpha)} = \frac{\partial x}{\partial \xi^\alpha}$$

(2.1)

Contravariant base vectors $a^{(\alpha)}$ are defined as normal vectors to the surfaces on which $\xi^\alpha$ is constant, i.e.

$$a^{(\alpha)} = \frac{\partial \xi^\alpha}{\partial x}$$

(2.2)

We have $a_{(\alpha)} \cdot a^{(\beta)} = \delta^\beta_\alpha$ with $\delta$ the Kronecker delta. The covariant and contravariant metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are defined by

$$g_{\alpha\beta} = a_{(\alpha)} \cdot a_{(\beta)}; \quad g^{\alpha\beta} = a^{(\alpha)} \cdot a^{(\beta)}$$

(2.3)

The determinant of the covariant metric tensor $g_{\alpha\beta}$ is denoted by $g$; $\sqrt{g}$ equals the Jacobian of the transformation, given by

$$J = \sqrt{g} = a_{(1)} \cdot (a_{(2)} \wedge a_{(3)})$$

(2.4)

Tensor notation proves indispensable for formulating physical conservation laws in general coordinates. An introduction to tensor analysis can be found in e.g. [2], [13], [11]. For completeness we summarize some basic tools used. Tensors are mathematical objects that are independent of the coordinate system. We will only consider tensors of rank zero (scalars), one (vectors) or two. An example is a mixed relative tensor $Q^\delta_\beta$ of weight $w$ and rank two with Cartesian components $q^\gamma_\beta$, satisfying the following transformation law:

$$Q^\delta_\beta = (\sqrt{g})^w a^\gamma_{(\alpha)} a^\delta_{(\beta)} q^\gamma_\beta$$

(2.5)

where $Q^\delta_\beta$ are the components in the curvilinear coordinate system. Relative tensors of weight zero are called absolute; relative tensors of weight one are called densities.

A covariant derivative of a tensor is a tensor which reduces to a partial derivative of a tensor in Cartesian coordinates. For an absolute scalar $\sigma$, the covariant derivative is identical to the partial derivative, and is denoted by

$$\sigma_{,\alpha} = \frac{\partial \sigma}{\partial \xi^\alpha}$$

(2.6)

The fluid density $\rho$ is a tensor of weight 1 (a density). This means that $\rho$ is not invariant under coordinate transformation, but $\sqrt{g} \rho$ is. Its covariant derivative is given by

$$\rho_{,\alpha} = \frac{\partial \rho}{\partial \xi^\alpha} - \mathbb{A}^{\beta}_{\alpha} \rho = \sqrt{g} \frac{\partial}{\partial \xi^\alpha} (\frac{1}{\sqrt{g}} \rho)$$

(2.7)
where \( \{ \gamma \beta \} \) represents the Christoffel symbol of the second kind, defined by

\[
\{ \gamma \beta \} = a^{(\gamma)} \cdot \frac{\partial a_{(\beta)}}{\partial \xi^\gamma} = \frac{\partial \xi^\alpha}{\partial \xi^\gamma} \frac{\partial^2 x^\delta}{\partial \xi^\alpha \partial \xi^\beta}
\]

(2.8)

The covariant derivative of a contravariant tensor of rank one is defined by

\[
Q^\alpha_{\beta} = \frac{\partial Q^\alpha}{\partial \xi^\beta} + \{ \gamma \beta \} Q^\gamma
\]

(2.9)

The covariant derivative of a contravariant tensor of rank two is defined by

\[
Q_{\gamma \beta}^{\alpha \beta} = \frac{\partial Q_{\gamma \beta}^{\alpha \beta}}{\partial \xi^\gamma} + \{ \alpha \beta \}_{\gamma} Q^{\beta} + \{ \beta \gamma \} Q^{\alpha \delta}
\]

(2.10)

It can be shown that

\[
Q_{\gamma \beta}^{\alpha \beta} = \frac{1}{\sqrt{g}} \frac{\partial Q^{\alpha \beta}}{\partial \xi^\beta} + \{ \gamma \beta \} Q^{\gamma \delta}
\]

(2.11)

In tensor notation the divergence theorem is given by

\[
\int_{\Omega} Q_{\gamma \alpha} d\Omega = \oint_{S} Q^\alpha dS_{\alpha}
\]

(2.12)

where \( d\Omega \) is the infinitesimal volume element given by

\[
d\Omega = \sqrt{g} d\xi^1 d\xi^2 ... d\xi^d
\]

(2.13)

d being the number of spatial dimensions, and \( dS_{\alpha} \) represents the (physical) surface element. A fundamental geometric identity can be derived by applying the divergence theorem to a constant vector field, which gives

\[
\int_{\Omega} Q_{\gamma \alpha} d\Omega = \oint_{S} Q^\alpha dS_{\alpha} = q^\beta \oint_{S} a_{(\beta)} dS_{\alpha} = 0
\]

(2.14)

Since \( q^\beta \) is arbitrary, this leads to the following geometric identity:

\[
\oint_{S} a_{(\beta)} dS_{\alpha} = 0
\]

(2.15)

It is important to satisfy this identity also numerically, in order to prevent conservation errors in the numerical solution.

The governing equations are the incompressible Navier-Stokes equations. Using contravariant tensor components for the fluid density, velocities, viscous stresses and pressure, the coordinate-invariant formulation of the governing equations becomes

\[
U_{i,\alpha} = 0
\]

(2.16)
and
\[ T_{\alpha\beta} = (\rho U^\alpha U^\beta)_{,\beta} + (g^\alpha{}_{\beta} p)_{,\beta} - \tau_{\alpha\beta} = \rho f^\alpha \]  
(2.17)
where \( \rho \) is the fluid density and \( \tau_{\alpha\beta} \) represents the deviatoric stress tensor given by
\[ \tau_{\alpha\beta} = \mu (g^\gamma{}_{\alpha} U^\beta_{,\gamma} + g^\gamma{}_{\beta} U^\alpha_{,\gamma}) \]  
(2.18)
with \( \mu \) the viscosity coefficient.

At the moment only Dirichlet boundary conditions are in the code. In forthcoming papers also Neumann boundary conditions will be implemented.

For accuracy reasons, the following requirements should be met:

(i) The geometric identity (2.15) should be satisfied exactly for all cells.

(ii) When representing a constant velocity field \( u \) on the staggered grid in terms of its contravariant components \( U^\alpha \), and recomputing \( u \) from \( U^\alpha \), the original vector field \( u \) should be recovered exactly.

(iii) Uniform flow fields should satisfy the discrete equations exactly.

These requirements can be met if one proceeds as follows. The base vectors \( a^{(\alpha)} \) are computed according to
\[ a^{(\alpha)}_{(1)} = \frac{\delta x^\alpha}{\delta \xi^1}, \quad a^{(\alpha)}_{(2)} = \frac{\delta x^\alpha}{\delta \xi^2} \]  
(2.19)
in the \( U^2 \)- and \( U^1 \)-points in the staggered grid respectively. Here \( \delta \) implies taking differences between the points where \( x(\xi) \) is given (i.e. the cell vertices) in the obvious way. Furthermore,
\[ \nabla a^{(\alpha)} = a^{(1)}_{(1)} a^{(2)}_{(2)} - a^{(1)}_{(2)} a^{(2)}_{(1)} \]  
(2.20)
taking averages where required. Finally,
\[ a^{(1)} = \frac{1}{\sqrt{g}} (a^{(2)}_{(2)}, -a^{(2)}_{(1)}), \quad a^{(2)} = \frac{1}{\sqrt{g}} (-a^{(1)}_{(2)}, a^{(1)}_{(1)}) \]  
(2.21)
taking averages where required.

For convenience we introduce the local cell coordinates given by Figure 2.1, which shows part of the computational grid in the \( \xi \)-plane. Integration of the incompressibility constraint over a pressure cell with center at \( (0,0) \) gives
\[ \int_{\Omega} U^\alpha d\Omega = \oint_{S} U^\alpha dS^\alpha \]  
(2.22)
Let \( \delta S^{(\alpha)} \) be the vector with direction along the outward normal to a face with a \( U^\alpha \) point as center, and with length equal to the length of that face. Hence
\[ \delta S^{(\alpha)} = \sqrt{g} a^{(\alpha)}(\delta \xi^\beta), \quad \alpha, \beta \text{ cyclic} \]  
(2.23)
To explain the discretization of the integral along \( S \) in (2.22) it suffices to consider the integral along the face with vertices \((1,1)\) and \((1,1)\). We have

\[
\int_{1,-1}^{1,1} U^\alpha \delta S_\alpha \cong (U^\alpha \delta S_\alpha^{(1)})|_{1,0}
\]

(2.24)

where \( \delta S_\alpha^{(1)} = a^{(\beta)}_{(\alpha)} \sqrt{g} a^{(1)}_{\beta} \delta \xi^2 = \sqrt{g} \delta \xi^2 \), since \( a^{(\beta)}_{(\alpha)} a^{(\gamma)}_{\beta} = \delta_{\gamma}^{\alpha} \). Hence

\[
\int_{1,-1}^{1,1} U^\alpha \delta S_\alpha \cong V_\alpha^{(1,0)} \delta \xi^2
\]

where \( V^\alpha = \sqrt{g} U^\alpha \), and finally

\[
\oint S U^\alpha \delta S_\alpha \cong V^1 |_{-1,0}^{1,0} \delta \xi^2 + V^2 |_{0,-1}^{0,1} \delta \xi^2
\]

(2.25)

Let \( u \) be a constant vector field. Substituting \( V^\alpha = \sqrt{g} a^{(\alpha)}_\beta u^\beta \) and using (2.19) and (2.20) one finds that

\[
V^1 |_{-1,0}^{1,0} \delta \xi^2 + V^2 |_{0,-1}^{0,1} \delta \xi^2 = 0
\]

(2.26)

so that requirement (i) is satisfied. Requirement (ii) is verified as follows. Let \( w \) be a constant vector field. Its representation in terms of \( V^\alpha \) on the staggered grid is \( V^\alpha = \sqrt{g} a^{(\alpha)}_\beta w^\beta \). Hence, using (2.21),

\[
V^1 = a^{(2)}_2 w^1 - a^{(1)}_2 w^2, \quad V^2 = -a^{(2)}_1 w^1 + a^{(1)}_2 w^2
\]

(2.27)

Now recompute the Cartesian components \( u^\alpha \) from (2.27) in the cell vertices:

\[
u^\alpha |_{1,1} = \frac{1}{2\sqrt{g}} |_{1,1} \left( \Sigma_1 (a^{(\alpha)}_1 V^1) + \Sigma_2 (a^{(\alpha)}_2 V^2) \right)
\]

(2.28)
where $\Sigma_1$ indicates summation over grid points (1,0) and (1,2), and $\Sigma_2$ indicates summation over (0,1) and (2,1). Substitution of (2.27) in (2.28), and evaluation of $\sqrt{g}$ according to (2.20) results in

$$u_i^\alpha|_{1,1} = u_i^\alpha|_{1,1}$$  \hspace{1cm} (2.29)$$

We also have (2.29) in cell centers (0,0). Hence, requirement (ii) is satisfied. If $U^\alpha$ is used as primary unknown instead of $V^\alpha$ (2.29) would not hold exactly, which is why the use of $V^\alpha$ is to be preferred. Imposing boundary conditions such that the exact solution is uniform and solving the discrete equations numerically results in the correct solution within machine precision, showing empirically that (iii) is satisfied.

As a preparation for the discretization of the momentum equations we discuss the discretization of a general conservation law of the form

$$T_{\beta}^{\alpha} = f^\alpha$$  \hspace{1cm} (2.30)$$

This equation is to be integrated over finite volumes. On the staggered grid used here, integration takes place over cells with vertices in $U^\alpha$-points and center in a $U^\rho$-point for $\alpha = 1$, and vice-versa for $\alpha = 2$. Taking a cell with center at (1,0) as an example, using equation (2.11) and partial integration gives

$$\int_0^1 T_{\beta}^{\alpha} \, d\Omega = \int_0^1 \frac{\partial g_{\beta}^{\alpha}}{\partial x^\alpha} \, d\xi^2 + \int_0^1 \{ \gamma^\rho \} \, T_{\gamma\rho}^{\alpha} \, \sqrt{g} \, d\xi^2$$

$$\approx (\sqrt{g} T^{11})_{\beta}^{\alpha} \delta \xi^2 + (\sqrt{g} T^{12})_{\beta}^{\alpha} \delta \xi^2$$

$$+ \sqrt{g} \{ \gamma_{\beta} \} T_{\gamma\rho}^{\alpha} |_{1,0} \delta \xi^2$$  \hspace{1cm} (2.31)$$

With $T_{\beta}^{\alpha}$ from (2.17) this is the discretization used for the momentum equations. It is found that the variable $V^\alpha = \sqrt{g} U^\alpha$ appears naturally in many places in (2.31).

In order to obtain equations suitable for multigrid solution some form of upwind discretization has to be used for the convection terms. In Cartesian coordinate systems the so-called hybrid scheme [14] is popular. For the convection terms a central difference scheme (CDS) is used, if the mesh Reynolds number is smaller than 2, and a first order upwind scheme (UDS), if it exceeds 2. In the latter case the diffusion term in the coordinate direction(s) in which the upwind discretization is applied is dropped. For further discussion of the hybrid scheme, see [7].

In general coordinates it is not at all trivial how a hybrid scheme should be formulated and how the mesh Reynolds number should be defined, because of the occurrence of source terms and mixed derivatives in the viscous stress term.

The tensor $\tau_{\alpha\beta}$ is reduced from a sum over four terms to a sum over two terms using the incompressibility constraint. From (2.18) one verifies that:

$$\tau^{11} = 2\mu (g^{11} U^1_1 + g^{12} U^1_2)$$  \hspace{1cm} (2.32)$$

$$\tau^{12} = \mu (g^{11} U^2_1 + g^{22} U^2_2)$$  \hspace{1cm} (2.33)$$
\[ r^{22} = 2\mu(g^{12}U_{i1}^2 + g^{22}U_{i3}^2) \]  

(2.34)

The criterion for switching between CDS and UDS will be generalized to general coordinates by requiring that the sum of coefficients arising from the viscous terms and the \textit{flux} part (i.e. the part not involving Christoffel symbols) of the convection terms multiplying \( V^\alpha \) in neighbouring points should be non-positive. This implies that a suitable definition of the mesh Reynolds number \( Re^{(i,j)} \) is that it is the ratio of the absolute magnitudes of the viscous term and the \textit{flux} part of the convective term, discretized with a central scheme in point \((i,j)\). If \( Re^{(i,j)} > 1 \) UDS is used, if \( Re^{(i,j)} < 1 \) CDS is used. Unlike the original hybrid scheme the contribution of the viscous stress tensor will be kept for all mesh-Reynolds numbers. The volume integral remaining in the convective terms in general coordinates is discretized with a central scheme.

Convergence problems for the multigrid method are avoided by using a "smooth" switch from central to upwind discretization:

\[ CONV.TERM = (1 - \alpha^{(i,j)}) CENTRAL + \alpha^{(i,j)} UPWIND \]  

(2.35)

where \( \alpha^{(i,j)} = \alpha(Re^{(i,j)}) \) may be defined by

\[
\begin{align*}
Re^{(i,j)} &< 0.9 ; & \alpha^{(i,j)} &= 0 \\
0.9 \leq Re^{(i,j)} \leq 1.1 ; & \alpha^{(i,j)} &= 5Re^{(i,j)} - 4.5 \\
Re^{(i,j)} > 1.1 ; & \alpha^{(i,j)} &= 1
\end{align*}
\]  

(2.36)

Figure 2.2 shows the structure of the resulting stencils; (a), (b), (c) and (d) refer to the continuity equation and the contributions of the inertia, pressure and viscous terms, respectively. The stencils for the \( U^2 \)-momentum equation are obtained from those for the \( U^1 \).

![Stencils](image)

Figure 2.2: Stencil structure.

equation by rotation over 90°. The total number of variables linked together in a momentum equation is 19.
Christoffel symbols occur. These are computed in the obvious way by computing $a^{(o)}$ and $a_{(o)}$ in the way described before, and taking differences. The Christoffel symbols involve second derivatives of the mapping $x = x(\xi)$, which may not be approximated accurately by the method just described. On an irregular grid discretizing according to (2.8) was found to cause a non-physical pressure distribution. This is avoided by eliminating the Christoffel symbols from the pressure term in (2.17) and (2.31) with the relation:

$$
\sqrt{g} \left( \frac{\alpha_{\gamma}}{\beta_{\gamma}} \right) g^{\alpha_{\gamma}} = -\frac{\partial(\sqrt{g} a^{\alpha_{\gamma}})}{\partial \xi^\beta}
$$

(2.37)

Figure 2.3 gives an example of a Poiseuille flow on an irregular grid. (a) shows the grid, (b) gives the pressure distribution, when the Christoffel symbols in the pressure term are kept and discretized according to (2.8). The pressure distribution is not correct. Figure (c) shows correct distribution obtained by using (2.37).

Figure 2.3. A Poiseuille flow, Re=100, (a) the $13 \times 11$-grid, (b) the non-physical pressure using (2.8), (c) the physical pressure using (2.37).
3 The multigrid algorithm

The standard nonlinear multigrid algorithm is used in the form presented in [6]. A non-recursive well structured version (requiring only one GOTO statement in FORTRAN) will be presented, which is an improved version [17] of the algorithm described in [16]. Iteration should start with nested iteration for best efficiency, but this has not yet been implemented. Figure (3.1) gives the structure diagram.

- THE SMOOTHING METHOD

The smoothing method considered here is the Symmetric Coupled Gauss-Seidel method (SCGS) introduced in [15]. The variables are updated collectively cell by cell. The five discretized equations to be solved simultaneously for an interior cell are:

\[(A^1_c)_{i-1/2,j} V^1_{i-1/2,j} = F^1_{i-1/2,j} \]  \hspace{1cm} (3.1)

\[(A^1_c)_{i+1/2,j} V^1_{i+1/2,j} = F^1_{i+1/2,j} \]  \hspace{1cm} (3.2)

\[(A^2_c)_{i-1/2,j} V^2_{i-1/2,j} = F^2_{i-1/2,j} \]  \hspace{1cm} (3.3)

\[(A^2_c)_{i+1/2,j} V^2_{i+1/2,j} = F^2_{i+1/2,j} \]  \hspace{1cm} (3.4)

and:

\[(V^1_{i+1/2,j} - V^1_{i-1/2,j})/d\xi^1 + (V^2_{i,j+1/2} - V^2_{i,j-1/2})/d\xi^2 = 0 \]  \hspace{1cm} (3.5)

where, for example:

\[F^1_{i+1/2,j} = A^1_{sw} V^1_{i-1/2,j-1} + A^1_{sv} V^1_{i+1/2,j-1} + A^1_{se} V^1_{i+1/2,j+1} + A^1_{si} V^1_{i+3/2,j+1} + A^1_{st} V^1_{i+3/2,j} + A^1_{sw} V^1_{i-1/2,j+1} + A^1_{sv} V^1_{i+1/2,j+1} + A^1_{se} V^1_{i+1/2,j-1} + A^1_{si} V^1_{i+3/2,j-1} + A^1_{st} V^1_{i+3/2,j+1} \]  \hspace{1cm} (3.6)

This is a straightforward generalization of the Cartesian case discussed in [15]. Instead of updating the variables by themselves, corrections to a current solution are calculated. In terms of corrections (indicated by primes) and residuals (3.1) to (3.5) are rewritten as follows:

\[(A^3_c)_{i-1/2,j} (V^1)'_{i-1/2,j} - (A^3_c)_{i-1/2,j} P^1_{i,j} = R^1_{i-1/2,j} = F^1_{i-1/2,j} - (A^1_c)_{i-1/2,j} V^1_{i-1/2,j} \]  \hspace{1cm} (3.7)
\[(A_c^1)_{i+1/2,j} (V^1')_{i+1/2,j} - (A^3_w)_{i+1/2,j} p_{i,j} = R^1_{i+1/2,j} = F^1_{i+1/2,j} - (A^2_c)_{i+1/2,j} V^1_{i+1/2,j} \quad (3.8)\]

\[(A^2_c)_{i,j-1/2} (V^2')_{i,j-1/2} - (A^2_w)_{i,j-1/2} p_{i,j} = R^2_{i,j-1/2} = F^2_{i,j-1/2} - (A^2_c)_{i,j-1/2} V^2_{i,j-1/2} \quad (3.9)\]

\[(A^2_c)_{i,j+1/2} (V^2')_{i,j+1/2} - (A^3_w)_{i,j+1/2} p_{i,j} = R^2_{i,j+1/2} = F^2_{i,j+1/2} - (A^2_c)_{i,j+1/2} V^2_{i,j+1/2} \quad (3.10)\]

\[
\frac{(V^1_{i+1/2,j} - V^1_{i-1/2,j})}{dx^1} + \frac{(V^2_{i,j+1/2} - V^2_{i,j-1/2})}{dx^2} = R^3_{i,j} = \quad (3.11)\]

\[
\frac{(V^1_{i-1/2,j} - V^1_{i+1/2,j})}{dx^1} + \frac{(V^2_{i,j-1/2} - V^2_{i,j+1/2})}{dx^2} \quad (3.12)\]

In matrix notation (3.7) to (3.12) are given by:

\[
\begin{bmatrix}
(A^2_c)_{i-1/2,j} & 0 & 0 & 0 & -(A^3_w) \\
0 & (A^2_c)_{i+1/2,j} & 0 & 0 & -(A^3_w) \\
0 & 0 & (A^2_c)_{i,j-1/2} & 0 & -(A^3_w) \\
0 & 0 & 0 & (A^2_c)_{i,j+1/2} & -(A^3_w) \\
-1/dx^1 & 1/dx^1 & -1/dx^2 & 1/dx^2 & 0
\end{bmatrix} \cdot
\begin{bmatrix}
(V^1')_{i-1/2,j} \\
(V^1')_{i+1/2,j} \\
(V^2')_{i,j-1/2} \\
(V^2')_{i,j+1/2} \\
p_{i,j}
\end{bmatrix} =
\begin{bmatrix}
R^1_{i-1/2,j} \\
R^1_{i+1/2,j} \\
R^2_{i,j-1/2} \\
R^2_{i,j+1/2} \\
R^3_{i,j}
\end{bmatrix} \quad (3.13)\]

At the boundaries (3.13) is modified, because velocities at a Dirichlet boundary are not being updated.

Underrelaxation is introduced in (3.13) by the following replacements:

\[(A^2_c)_{i\pm1/2,j} = (A^1_c)_{i\pm1/2,j}/\alpha_1 \quad (3.14)\]

\[(A^3_w)_{i\pm1/2,j} = (A^2_c)_{i\pm1/2,j}/\alpha_2 \quad (3.15)\]

\[(A^3_w)_{i\pm1/2,j} = (A^2_c)_{i\pm1/2,j}/\alpha_2 \quad (3.16)\]

After the system is solved by an explicit formula the calculated correction is added to the current solution.
PROLONGATION AND RESTRICTION.

Prolongation and restriction operators are more or less dictated by the staggered grid arrangement. Prolongation operators are derived for all variables using bilinear interpolation. The restricted coarse grid fluxes \( V^\alpha \) are defined to be the mean of their two neighbouring fine grid fluxes. Coarse grid pressures are defined to be the mean of the four neighbouring fine grid pressures. In evaluating the coarse grid right-hand side area weighting is used for the fine grid residuals, as follows

\[
ger_{1,j}^{(1)k-1} = \frac{1}{8}(r_{2i-1,2j}^{(1)k} + r_{2i-1,2j-1}^{(1)k} + r_{2i+1,2j}^{(1)k} + r_{2i+1,2j-1}^{(1)k} + \frac{1}{4}(r_{2i,2j}^{(1)k} + r_{2i,2j-1}^{(1)k})
\]

(3.17)

\[
ger_{1,j}^{(2)k-1} = \frac{1}{8}(r_{2i,2j-1}^{(2)k} + r_{2i-1,2j}^{(2)k} + r_{2i-1,2j+1}^{(2)k} + r_{2i,2j+1}^{(2)k} + \frac{1}{4}(r_{2i,2j}^{(2)k} + r_{2i-1,2j}^{(2)k})
\]

(3.19)

\[
ger_{1,j}^{(3)k-1} = \frac{1}{4}(r_{2i-1,2j}^{(3)k} + r_{2i-1,2j-1}^{(3)k} + r_{2i,2j}^{(3)k} + r_{2i,2j-1}^{(3)k} + \frac{1}{2}(r_{2i,2j}^{(1)k} + r_{2i,2j-1}^{(1)k})
\]

(3.20)

COARSE GRID CORRECTION.

On a coarse grid the even cell vertices from the fine grid are removed and the geometric quantities are calculated as was done on the finest grid. Therefore requirement (i) derived in Section 2 is satisfied also on a coarse grid. The relation \((d\zeta^\alpha)^{k-1} = 2(d\zeta^\alpha)^k\) holds not only for \(d\zeta^\alpha\) occurring in (2.31), but also for the derivation of the coarse grid geometric quantities, of course.
Choose $(\tilde{V})^L$ and cycle $((V)^L = (V^1, V^2, p)^T)$

**Comment:** $\gamma = 1 : V$-cycle; $\gamma = 2 : W$-cycle

$f^L = b^L$; $k = L$; $n_L =$init

if (cycle = F) $\gamma = 2$

while ($n_L \geq 0$) do

- $n_k = 0$ or $k = 1$

A

- $k = k - 1$
- $n_k = \gamma$

F

T

k = 1

F

T

$S(\tilde{V}, V, f, ncrs, k)$

- $n_k = n_k - 1$
- if (cycle = F) $\gamma = 1$

k < L

F

T

k = k + 1

- $n_k = n_k - 1$

B

$A$: 

- $S(\tilde{V}, V, f, npre, k)$
- $r_k = f^k - L^k(V^k)$
- Choose $V^{k-1}, s_{k-1}$
- $f^{k-1} = L^{k-1}(\tilde{V}^{k-1}) + s_{k-1}R^{k-1}r^k$

$B$: 

- $V^k = V^k + \frac{1}{s_{k-1}}p_k(V_v^{k-1} - \tilde{V}^{k-1})$
- $S(\tilde{V}, V, f, npost, k)$

Figure 3.1: The structure diagram of the non-recursive multigrid algorithm including the V, F and W cycle.
4 Test problems, some results

The first test problem investigated is the driven cavity flow. Average reduction factors \( r \) are compared, defined as

\[
r = \left( \frac{||\text{res}||_2}{||\text{res}||_0} \right)^{1/2}
\]

i.e. the 2-norm of the residual after 20 iterations divided by the 2-norm of the starting residual.

For the driven cavity flow, \( r \) is computed for a skewed cavity (i.e. a parallelogram) and a unit rectangular cavity, for several Reynolds numbers (Re).

The starting vector is the zero-solution on the finest grid. The number of pre-smoothing iterations \( npre \) is 1, the number of post-smoothing iterations \( npost \) is 1, the number of coarse grid relaxation iterations \( ncrs \) is 10. The coarse grid correction parameters \( s_k \) are chosen 1.

The results given are reduction factors for the W-cycle, which showed the best reduction factors, followed by the F and V cycle. As in [9] different relaxation parameters \( \alpha_k \) are needed for different Reynolds numbers. For high \( Re \) we need to choose \( \alpha_k \) different from [9], probably because the stress term is not neglected in our hybrid scheme.

For low \( Re \) (\( Re < 400 \)) the SCGS-smoother is lexicographical (i.e. sweeping along horizontal lines), for high \( Re \) the lexicographical sweep is followed by a sweep along vertical lines as in [9].

Table 4.1 to 4.4 give the reduction factors. Figures 4.1 to 4.4 show the streamlines and the pressure contours for the skewed driven cavity (skew angle 63°), obtained on the 64 x 64 grid. It looks as if \( r \) is well-bounded away from 1 independent of the number of levels, except for \( Re = 1000 \); this effect is less pronounced when the cavity is closer to a square. With a skew angle of 79° \( r = 0.526 \) on a 128 x 128 grid. Here the streamlines, given in figure 4.5, look more like the ones obtained for the square domain.

The following test problems show the potentialities and limitations of the discretization method at the present stage of development. A flow at low Reynolds number through a straight pipe (equidistant grid, square cells), an L-shaped pipe and a nozzle flow are considered. Parabolic velocity profiles are prescribed at in- and outflow. This is unphysical of course, but of no concern here. Average reduction factors \( r \) are compared for these geometries for several grids.

Table 4.5 shows the reduction factors for the W-cycle. The reduction factors for the other cycles did not differ much. The details for the MG-algorithm are: \( npre = 1, npost = 1, ncrs = 10, s_k = 1, Re = 1 \), lexicographical SCGS, \( \alpha_k = 0.7 \). Figures 4.6 and 4.7 show grids, flow patterns, streamlines and pressure contours for the L-shape and nozzle geometry.

It seems that for the nozzle reduction factors are not level-independent, but figure 4.7 shows that the 16 x 40 grid is much more non-uniform than the 8 x 20 grid, which may be of greater consequence for \( r \) than the number of levels. Finally, figure 4.8 shows a limitation of the discretization used. Refinement in y-direction results in a non-physical pressure behaviour, although relation (2.37) is used. Certain ideas to overcome this problem are being investigated.
<table>
<thead>
<tr>
<th>$Re = 1$</th>
<th>levels</th>
<th>grid</th>
<th>$r$, skewed</th>
<th>$r$, square</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k = 0.7$</td>
<td>4</td>
<td>16 x 16</td>
<td>0.331</td>
<td>0.288</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>32 x 32</td>
<td>0.349</td>
<td>0.268</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>64 x 64</td>
<td>0.360</td>
<td>0.316</td>
</tr>
</tbody>
</table>

Table 4.1: Driven cavity ($Re = 1$), lexicographical SCGS, npre = npost = 1, W-cycle.

<table>
<thead>
<tr>
<th>$Re = 100$</th>
<th>levels</th>
<th>grid</th>
<th>$r$, skewed</th>
<th>$r$, square</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k = 0.7$</td>
<td>4</td>
<td>16 x 16</td>
<td>0.390</td>
<td>0.328</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>32 x 32</td>
<td>0.355</td>
<td>0.315</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>64 x 64</td>
<td>0.345</td>
<td>0.310</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>128 x 128</td>
<td>0.338</td>
<td>0.313</td>
</tr>
</tbody>
</table>

Table 4.2: Driven cavity ($Re = 100$), lexicographical SCGS, npre = npost = 1, W-cycle.

<table>
<thead>
<tr>
<th>$Re = 400$</th>
<th>levels</th>
<th>grid</th>
<th>$r$, skewed</th>
<th>$r$, square</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k = 0.4$</td>
<td>4</td>
<td>16 x 16</td>
<td>0.502</td>
<td>0.454</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>32 x 32</td>
<td>0.445</td>
<td>0.446</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>64 x 64</td>
<td>0.463</td>
<td>0.398</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>128 x 128</td>
<td>0.310</td>
<td>0.310</td>
</tr>
</tbody>
</table>

Table 4.3: Driven cavity ($Re = 400$), alternating SCGS, npre = npost = 1, W-cycle.

<table>
<thead>
<tr>
<th>$Re = 1000$</th>
<th>levels</th>
<th>grid</th>
<th>$r$, skewed</th>
<th>$r$, square</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k = 0.3$</td>
<td>4</td>
<td>16 x 16</td>
<td>0.508</td>
<td>0.516</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>32 x 32</td>
<td>0.619</td>
<td>0.565</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>64 x 64</td>
<td>0.646</td>
<td>0.564</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>128 x 128</td>
<td>0.710</td>
<td>0.517</td>
</tr>
</tbody>
</table>

Table 4.4: Driven cavity ($Re = 1000$), alternating SCGS, npre = npost = 1, W-cycle.
<table>
<thead>
<tr>
<th>levels</th>
<th>grid</th>
<th>$r_{\text{pipe}}$</th>
<th>$r_{\text{L-shape}}$</th>
<th>$r_{\text{nozzle}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 =sg</td>
<td>4×10</td>
<td>0.835</td>
<td>0.848</td>
<td>0.843</td>
</tr>
<tr>
<td>2</td>
<td>4×10</td>
<td>0.254</td>
<td>0.376</td>
<td>0.329</td>
</tr>
<tr>
<td>3</td>
<td>8×20</td>
<td>0.281</td>
<td>0.385</td>
<td>0.376</td>
</tr>
<tr>
<td>4</td>
<td>16×40</td>
<td>0.301</td>
<td>0.437</td>
<td>0.468</td>
</tr>
</tbody>
</table>

Table 4.5: Channel flows at $Re=1$, lexicographical SCGS, $npre = npost = 1$, W-cycle.
5 Conclusions

An invariant formulation of the incompressible Navier-Stokes equations has been presented in which Christoffel symbols occur. The discretization of the invariant formulation shows good results for many geometries and fairly non-uniform grids. Sometimes with severe jumps in mesh-size a non-physical pressure distribution occurs. Further investigation will be done to handle such situations.

A multigrid solution method has been presented for solving the Navier-Stokes equations. A level-independent convergence rate has been found for the test problems. Reduction factors for rectangular and some more complex geometries do not seem to differ much.

The code is robust, especially for low Reynolds number flows. Then it is insensitive to large variations of the relaxation parameters. The convergence rate slows down with larger relaxation parameters for higher Reynolds numbers (≥ 400). Probably nested iteration (i.e. a better starting solution on the finest grid) will help here.

Further development of the code will include nested iteration, a vectorizable smoother will be implemented, and probably a line smoother to cope with stretched cells. Neumann boundary conditions will also be implemented.
References


Figure 4.1. Streamlines and pressure contours for the skewed (under 63 degrees) driven cavity, $Re = 1,64 \times 64$-grid.

Figure 4.2. Streamlines and pressure contours for the skewed driven cavity, $Re = 100$, $64 \times 64$-grid.
Figure 4.3. Streamlines and pressure contours for the skewed driven cavity, $Re = 400$, $\times 64$-grid.

Figure 4.4. Streamlines and pressure contours for the skewed driven cavity, $Re = 1000$, $\times 64$-grid.
Figure 4.5. Streamlines and pressure contours for the skewed driven cavity under 79 degrees, $Re = 1000$, 128 $\times$ 128-grid.
Figure 4.6a. Flow through an L-shaped pipe, (1) the $8 \times 20$-grid, (2) flow pattern, (3) streamlines, (4) pressure.

Figure 4.6b. Flow through an L-shaped pipe, (1) the $16 \times 40$-grid, (2) flow pattern, (3) streamlines, (4) pressure.
Figure 4.7a. Flow through a nozzle, (1) the 8 x 20-grid, (2) flow pattern, (3) streamlines, (4) pressure.

Figure 4.7b. Flow through a nozzle, (1) the 16 x 40-grid, (2) flow pattern, (3) streamlines, (4) pressure.
Figure 4.8. A geometry to be investigated, (a) the grid, (b) the non-physical pressure.
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