Memorandum M-374

PROBABILISTIC METHODS IN THE THEORY OF STRUCTURES
Part A: Structures described by single random variable
First Course for Students of Aeronautical Engineering, Mechanical Engineering, and Mechanics
by
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Delft - The Netherlands
August 1980

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CHAPTER 1 - INTRODUCTION

For adequate description of structural behaviour, probabilistic methods must be resorted to. Properly speaking, an element of probability is embodied even in the deterministic approach, which ostensibly claims to "simplify" the structure by eliminating all aspects of uncertainty. Under it, the external loading and the properties of the structure are represented as though they were fully determined, and available (often highly sophisticated) tools yield, with sufficient accuracy, the strains and stresses in systems of complex configuration. Yet, at the same time, these stresses are compared with allowable ones, obtained by division of their ultimate levels by the so-called factor of safety, so as to yield a level below that of failure - a practice due to recognition of the uncertain - and random - features of the stress distribution in the material. This is how a probabilistic consideration is admitted "via the back door"; indeed the safety factor has often been referred to as the "ignorance factor".

The quality of "randomness" is characteristic both of the loads borne by the structures and of their properties. The latter signifies that no two structures, even if produced by the same manufacturing process, have identical properties. Thin-walled structures are often sensitive to so-called imperfections - deviations from their prescribed geometry - in the sense that the buckling load of an imperfect structure may be much lower than that of its ideal counterpart by several percentage decades. The shape and magnitude of these initial imperfections vary widely from case to case, seeing that they derive from an arbitrary manufacturing process, itself subject (by its very nature) to a large number of random influences. These and other examples clearly indicate that investigation of structural behaviour is impossible without resorting to probabilistic methods.
The need for a probabilistic approach does not obviate the classical treatment of the behaviour of an ideal structure with given properties, subjected to given loading. In fact, the solution to a deterministic problem may very often prove useful in a probabilistic setting. For example, assume that the properties of a structure are fully determined, while the external forces or moments are random. We begin by constructing explicit equations of motion (or equilibrium) in terms of the latter, which are then used as input in determining the probabilistic characteristics of the response (output). Where the exact relationship between input and output is unavailable, or its application proves too cumbersome, statistical simulation (the so-called Monte Carlo Method) is the logical remedy, facilitated by the recent advent of high-speed digital computers. The first step of this method consists in simulating the random variable; the second step is numerical solution of the problem for each realisation of the random variable; the third and last - statistical analysis (computation of the characteristics of the output by averaging over the ensemble). Thus, one of the cornerstones of the Monte Carlo Method is solution of a deterministic problem.

The deterministic and probabilistic approaches to design differ in principle. Deterministic design is based on total "discounting" of the contingency of failure; the designer is trained in the doctrine that with the relevant quantities properly chosen, admissible levels would never be exceeded; it is postulated that, as it were, the structure is immune to failure and will survive indefinitely. This approach dates back to antiquity, when design analysis and control were unknown and everything centred on the personal responsibility of the artisan. Its earliest written record is probably Hammurabi's Code - according to which, if a house collapses and its occupants are killed, the builder is liable to the death penalty. Deterministic design has now reached a very high level of sophistication and modern computation techniques make it possible to determine stresses, strains, displacements in highly complex structures. However, problems of structural design always involve an element of uncertainty, unpredictability or randomness: no matter how much is known about the phenomenon, the behaviour of a structure is incapable of precise prediction. In these circumstances there always exists some likelihood of failure, i.e. of an unfavourable state of the structure setting in. Even with safety factors - empirical reserve margins - failures did and still do occur. There can in principle be no
"never-fail" structure; it is a question only of a higher or lower probability of failure. Accordingly, probabilistic design is concerned with the probability of failures - or, preferably, of non-failure - performance, i.e. the probability of the structure realising the function assigned to it - in other words, with reliability.

The McGraw-Hill Dictionary of Scientific and Technical Terms gives the following definition of this basic concept: "Reliability - the probability that a component part, equipment, or system will satisfactorily perform its intended function under given circumstances, such as environmental conditions, limitations as to operating time, and frequency and thoroughness of maintenance, for a specified period of time". The reliability approach was initiated by Maier and Khzialov and carried on by Freudenthal, Johnson, Pugsley, Rzhanitsyn, Shinozuka, Streketskii, Tye and Weibull. The contribution of Ang and Tong, Benjamin and Cornell, Bolotin, Ferry Borges and Costhaneta, Haugen, Kogan, Lind, Moses, and Murzewski might also be mentioned.

The development of high-power rocket jet engines and supersonic transport since the nineteen-fifties has brought out new problems of mechanical and structural vibration, namely the response of panel-like structures to aerodynamic noise and to a turbulent boundary layer, with the attendant aspects of acoustic fatigue and interior noise - all of which are incapable of deterministic solution. The probabilistic methods for these and other problems are embodied in a new discipline called "Random Vibration", dealt with by numerous research centres and their offshoots which came into being throughout the world in the last twenty years. Of these, the teams of Caughey (Caltech), Crandall (M.I.T.), Lin (Urbana-Champaign) and Shinozuka (Columbia) in the U.S.A., of Bolotin (Moscow Energetic Institute) and Palmov (Leningrad Polytechnic) in the U.S.S.R., of Clarkson (Southampton) and Robson (Glasgow) in the U.K., of Ariaratnam (Waterloo) in Canada, and (last not least) of Parkus (Vienna) in Austria - might be mentioned.

The probabilistic approach proved extremely useful in analysis of flexible buildings subjected to earthquakes (Newmark and Rosenblueth) or wind (Cermak), offshore structures subjected to random wave loading (ROSS conference), ships in rough seas (Ekinov, Price and Bishop), structures undergoing fatigue failure (Bogdanoff, Freudenthal, Gumbel, Payne, Weibull),
structurally inhomogeneous media (Beran, Hashin, Kröner, Lomakin, Shermergor, Volkov), stability of stochastic systems (Khas'minskii, Kozin, Kushner) and many other fascinating problems.

However, despite its fundamental achievements the probabilistic approach to the theory of structures has not yet become a regular part of engineering education.

Present text is intending to remedy this oversight; because of its introductory character, mathematical pedantism is dispensed with, bearing in mind that a student or an engineer possessing the necessary basics will know his way to advanced literature on the specific subject of his interest. Since reliability is, in the first place and above all, a probabilistic concept, its analysis is one form of the theory of probability and random processes. Accordingly, the text comprises those parts of the theory which in my opinion are essential for its actual application. This, however, does not imply an "applied" nature, but rather an attempt to provide the student with some degree of "understanding" of underlying issues. And instead of attempting "territorial" separation of the elements of probability and reliability, they are preferentially presented as a single package, on the grounds that the latter is a natural sequel to the former.

It is my agreeable duty to thank the Department of Aerospace Engineering of Delft University of Technology for invitation to present the series of lectures (from which this text grew up) to students and scientific staff during my sabbatical leave in the academic year 1979/1980. My sincere thanks are due to the Dean, Prof.Ir. Jaap A. van Ghesel Grothe, and to Prof.Dr. Johann Arbocz, Professor of Aircraft Structures, for their constant encouragement and help. Appreciation is expressed to the staff members and the students of the University, and especially to ir. Johannes van Geer and Messrs. Willie Kopmans and Kees Venselaar for able assistance in a number of calculations and for their constructive suggestions on the lecture notes, the forerunner of this text. I am also most indebted to Mr. Eliezer Goldberg of the Technion - Israel Institute of Technology, Haifa, for his kind help in editing the text, to Mrs. Marijke Schillemans for typing the manuscript, and to Mr. Willem Spee for preparing the drawings.

General references*


* See page 5.


* Many highly interesting studies simply could not be mentioned here, since the subject is much too vast. A complete list of references on probabilistic methods in mechanics could fill by itself a hefty volume, and I confined myself mostly to books and reviews, so as to give some idea on what has been done. At the end of each Chapter, a list of Cited References and of Recommended Further Reading is given.


- Scatter of Fatigue Life and Fatigue Strength of Aircraft Structural Materials and parts, Proc. of Intern. Conf. on Fatigue in Aircraft Structures, Columbia University, New York, 1956.
CHAPTER 2 - PROBABILITY AXIOMS

§ 2.1. Random event

We will associate mechanical phenomena with a complex of conditions under which they may proceed, assuming that this complex is realisable (or rather reproducible - at least conceptually) an arbitrarily large number of times in essentially identical circumstances, with an observation or a measurement taken at each such realisation; such a process of observation or measurement will be referred to as a trial or an experiment. In this sense an experiment may consist in checking whether stresses in a structure exceed some specified value, or in determining the profile of imperfections of its surface, or else (in modern supersonic aircraft) in calculating the noise level. We define an event as an outcome, or a collection of outcomes, of a given experiment (a positive or a negative conclusion; readings of the scanning mechanism; the final result of a highly complex calculation). The outcome of a deterministic phenomenon is totally predictable and is, or can be, known in advance: deterministic phenomena are either certain or impossible, depending on whether they, inevitably, do or do not occur in the course of the given experiment.

For example, consider the perfectly elastic beam with symmetric uniform cross-section (section modulus S), subject to given constraints under a given transverse load resulting in a maximal bending moment \( M_{\text{max}} \) ("complex of conditions" - Fig. 2.1a). The maximal bending stress, according to the theory of strength of materials, is then given by:

\[
\sigma_{\text{max}} = \frac{M_{\text{max}}}{S}
\]  

(2.1)

Another example is a perfectly cylindrical shell made of perfectly elastic material with radius \( R \), length \( l \), thickness \( h \), Young's modulus \( E \) and Poisson's ratio \( \nu \), under uniform axial compression with ends simply supported ("complex of conditions" - Fig. 2.1c). For not too short shells, the buckling load is given by:

\[
\sigma_c = \frac{Eh}{R\sqrt{3(1 - \nu^2)}}
\]  

(2.2)
If the conditions specified in both examples are realised, the maximal stress in the beam and the critical (buckling) stress in the shell will be determined by Eqs. (2.1) and (2.2) respectively.

The statement of impossibility of some event under a given complex of conditions reduces readily to one of certainty of the opposite event. An event which is neither certain nor impossible is referred to as random, signifying that it may or may not occur under given identical conditions — in other words, that the outcome of the experiment is not known in advance, before it has taken place.

Consider as an example, in more detail, a cylindrical shell manufactured by electroplating from pure copper, and tested on a controlled end-displacement type compression testing machine ("complex of conditions" — Fig. 2.2) of which a suitable stock may be visualised as available. Due to the very nature of the manufacturing process, each realisation of the shell will have a different initial shape which cannot be predicted in advance. The imperfections (deviations of the initial shape from the ideal circular cylinder), amounting to a fraction of the wall thickness, can be picked up and recorded by the special experimental set-up (Fig. 2.3) developed by Arbocz at the California Institute of Technology. The scanning device, moving in both the axial and circumferential directions, yields a complete surface map of the shell. As illustrated by the examples in Figs. 2.4 and 2.5, the two shells produced by the same manufacturing process — have totally different imperfection profiles, and it is intuitively obvious that even when tested on the same machine, they would have different buckling load. These differ considerably from the classical buckling load of the perfect cylindrical shells as per Eq. (2.2): \(0.736 \sigma_{cl}\) for shell A9, and \(0.673 \sigma_{cl}\) for shell A12*.

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* This effect of small imperfections to reduce the buckling load of a cylindrical shell considerably was pointed out by Koiter in his pioneering Ph.D. thesis: an achievement, in terms of its impact on the course of the general theory of structural stability, that any scientist — be he conceited or modest — should dream of.
§ 2.2. Sample space

Although not absolutely essential, some mathematical preliminaries are given below.

The axiomatic foundation of the theory of probability was laid by Kolmogorov, according to whom the primary notion is not the random event, but so-called sample space. When an experiment or phenomenon gives rise to one of a totality of mutually-exclusive events, we denote it by the Greek letter \( \omega \) and refer to it as elementary event, an elementary outcome, or finally a sample point. The totality of all possible sample points are denoted by \( \Omega \) and referred to as the sample space, of which the sample points are the elements. "A sample point is indivisible, in that it embodies no distinguishable outcomes."

Sample spaces are usually classified according to the number of elements they contain. If such a space contains a countable number, i.e. a finite number or a denumerable infinity - so that its elements can be put in one-to-one correspondence with positive integers - it is referred to as discrete; otherwise, as continuous.

We will claim that event \( A \) is associated with the experiment (or the elementary outcome \( \omega \)) if, according to each elementary outcome \( \omega \), we know precisely that \( A \) does or does not take place. Denote by the same latter \( A \) the totality (or set) of all \( \omega \)'s as a result of which \( A \) takes place. Obviously, \( A \) takes place when and only when one of the \( \omega \)'s does; in other words, instead of speaking of \( A \), we may speak of an event of an elementary outcome \( \omega \) (which belongs to \( A \)) happening. Events are thus simple subsets of the sample space \( \Omega \). A certain or a sure event \( \Omega \), which takes place as a result of any outcome of the experiment under consideration, is formally identified with the whole sample space \( \Omega \), while an impossible event (denoted by \( \emptyset \)) is treated as an empty set, not containing any of the \( \omega \)'s.

If \( \omega \) is an elementary outcome belonging to \( A \), we write \( \omega \in A \); if \( \omega \) is not an element of \( A \), we write \( \omega \notin A \).

Two events \( A \) and \( B \) are said to be equal, \( A = B \), if and only if [denoted iff] every element of \( A \) is also an element of \( B \) and vice versa - every element of
B is also an element of A; that is, iff $A \subseteq B$, $B \subseteq A$. (Read: "A is contained in B" and "B is contained in A".)

Events A and B are referred to as mutually exclusive (or disjoint) if they have no sample points in common, i.e. cannot take place simultaneously.

The union or sum of two events $A_1$ and $A_2$ is defined as an event A signifying realisation of at least one of the events $A_1$, $A_2$:

$$A = A_1 \cup A_2 \quad (A = A_1 + A_2)$$

where $\cup$ is the special symbol of union. That is, A's elements are all the elements of $A_1$, or $A_2$, or both. A union of multiple events $A_1$, $A_2$, ..., is defined in an analogous manner and denoted by $A = \bigcup_k A_k$ or $A = \sum_k A_k$.

Sample spaces and events are conveniently represented by so-called Venn diagrams. The sample space $\Omega$ is represented by a square, whereas the events are represented by a region (or part of one), within a square. Many relationships involving events can be demonstrated by this means (see Fig. 2.6).

The intersection or product of two events $A_1$ and $A_2$ is an event A, signifying realisation of both $A_1$ and $A_2$: $A = A_1 \cap A_2$ (or $A_1A_2$), where $\cap$ is the special symbol of intersection. The product of multiple events $A_1$, $A_2$, ..., is defined in an analogous manner and denoted by $A = \bigcap_k A_k$ (or $A = \Pi_k A_k$).

The difference $\Delta$ of events $A_1$ and $A_2$ is an event signifying realisation of $A_1$ but not of $A_2$: $A = A_1 - A_2$ (or $A = A_1 \setminus A_2$). The complement of an event A with respect to the sample space $\Omega$, denoted by $\overline{A}$ or $A^c$, is an event signifying that A does not take place: $\overline{A} = \Omega \setminus A$ or $\overline{A} = \Omega - A$.

It is readily shown that if:

$$A = A_1 + A_2$$

and if:

$$\overline{A} = \overline{A}_1 \overline{A}_2$$
\[ A = A_1 A_2, \text{ then } \bar{A} = \bar{A}_1 + \bar{A}_2 \]

also, if:

\[ A_1 \subset A_2, \bar{A}_1 \supset \bar{A}_2 \]

These are known as De Morgan's laws, signifying that if there is some link between given events, then the link obtained from the original one by transfer to the complementary events, with formal replacement of the symbols of union \( U \), intersection \( \cap \) and inclusion \( \subset \) by \( \cap \), \( U \) and \( \supset \), respectively are likewise valid.

Observe also that:

\[ \bar{\bar{A}} = \emptyset, \emptyset = \Omega \]

and, therefore, the interlinkage will also be preserved if the following formal substitution is resorted to in addition to the above:

\[ \Omega \rightarrow \emptyset, \emptyset \rightarrow \Omega \]

A collection of events \( A_1, A_2, \ldots, A_n \) is said to partition the sample space \( \Omega \) iff they are pairwise mutually exclusive and their sum equals the sample space, that is:

\[ A_i A_j = \emptyset \text{ for } i \neq j \]

and:

\[ \sum_{i=1}^{n} A_i = \Omega \]

§ 2.3. Probability Axioms

Consider now the discrete sample space \( \Omega \), denoting, as previously the set of all possible outcomes of a random experiment. We now formulate axioms, defining the concept of probability.

**Axiom 1** ("non-negativity" axiom)

To each event \( A \), there can be assigned a non-negative real number \( P(A) \geq 0 \) called its probability.
Axiom 2 ("normalisation" axiom)
The probability of a certain event equals unity:

\[ P(\Omega) = 1 \]

Axiom 3 ("additivity" axiom)
If \( A_1, A_2, A_3, \ldots \) is a countable sequence of mutually exclusive events of \( \Omega \), then:

\[ P(A_1 + A_2 + A_3 + \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots \]

From these axioms, the following conclusions can be drawn immediately:

1. From the obvious equality:

\[ \Omega = \Omega + \emptyset \]

and Axiom 3, we conclude:

\[ P(\Omega) = P(\Omega) + P(\emptyset) \]

so that:

\[ P(\emptyset) = 0 \]

i.e., the probability of the impossible event is zero.

2. For any event \( A \),

\[ P(\bar{A}) = 1 - P(A) \] \hspace{1cm} (2.3)

To prove this, we note that an event and its complement are mutually exclusive:

\[ A \bar{A} = \emptyset \]

and the sum of an event and its complement represents the sample space:

\[ A + \bar{A} = \Omega \]
Then, by Axiom 3 we have:

\[ P(A + \bar{A}) = P(\Omega) \]

Thus on the other hand, by Axiom 3 we have:

\[ P(A + \bar{A}) = P(A) + P(\bar{A}) \]

and on the other hand, by Axiom 2 we have:

\[ P(A + \bar{A}) = P(\Omega) = 1 \]

which together yield the sought eq. (2.3).

3. For any pair of events \( A_1 \) and \( A_2 \) in a sample space \( \Omega \):

\[ P(A_1 - A_2) = P(A_1) - P(A_1 A_2) \]

\[ P(A_2 - A_1) = P(A_2) - P(A_1 A_2) \]

For proof, we note that each of the events \( A_1 \) and \( A_2 \) can be represented as:

\[ A_1 = (A_1 - A_2) + (A_1 A_2) \]

\[ A_2 = (A_2 - A_1) + (A_1 A_2) \]

where the events \( A_1 - A_2 \), \( A_1 A_2 \) and \( A_2 - A_1 \) are mutually exclusive.

Then by Axiom 3 we have:

\[ P(A_1) = P(A_1 - A_2) + P(A_1 A_2) \]

\[ P(A_2) = P(A_2 - A_1) + P(A_1 A_2) \]

We therefore conclude that if \( A_1 \subset A_2 \), then \( A_1 \cap A_2 = A_1 \) and:

\[ P(A_1) = P(A_2) - P(A_2 - A_1) \leq P(A_2) \]

that is, if event \( A_1 \) is contained in \( A_2 \), then:
4. The sum $A_1 + A_2$ of events $A_1$ and $A_2$ can be represented as the sum of the following mutually exclusive events:

$$A_1 + A_2 = (A_1 - A_2) + (A_2 - A_1) + (A_1 A_2)$$

and therefore:

$$P(A_1 + A_2) = P(A_1 - A_2) + P(A_2 - A_1) + P(A_1 A_2)$$

$$= P(A_1) - P(A_1 A_2)$$

$$+ P(A_2) - P(A_1 A_2)$$

$$+ P(A_1 A_2)$$

$$= P(A_1) + P(A_2) - P(A_1 A_2)$$ \hspace{1cm} (2.4)

If $A_1$ and $A_2$ are mutually exclusive events, i.e. $A_1 A_2 = \emptyset$, then $P(A_1 A_2) = 0$ and we are back with Axiom 3.

Due to the non-negativity of $P(A_1 A_2)$, we also conclude from (2.4) that:

$$P(A_1 + A_2) \leq P(A_1) + P(A_2)$$

5. Let $A_1', A_2', ..., A_n'$ be events in sample space $\Omega$. We seek to calculate the probability of their sum. Denote:

$$P_1 = \sum_{i=1}^{n} P(A_i'), \quad P_2 = \sum_{i<j}^{n} P(A_i A_j'), \quad P_3 = \sum_{i<j<k}^{n} P(A_i A_j A_k'), \quad ... \hspace{1cm} (2.5)$$

where $1 \leq i < j < k < ... < n$. The following formula is valid for the probability $P \left( \bigoplus_{i=1}^{n} A_i' \right)$ of the sum $\bigoplus_{i=1}^{n} A_i'$ of events $A_1', A_2', ..., A_n'$:

$$P \left( \bigoplus_{i=1}^{n} A_i' \right) = P_1 - P_2 + P_3 - P_4 + ... + (-1)^{n-1} P_n$$ \hspace{1cm} (2.6)

We will prove this formula by induction. For $n = 2$, it is identical with (2.4). Assume that it is valid for the sum of any $n-1$ events, so that:
\[ P \left( \sum_{i=1}^{n} A_i \right) = \sum_{i=2}^{n} P(A_i) - \sum_{i=1}^{n} \sum_{j=1}^{n} P(A_i A_j) + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} P(A_i A_j A_k) - \ldots \]

We then have:

\[ P \left( \sum_{i=1}^{n} A_i A_j \right) = \sum_{i=2}^{n} P(A_i A_j) - \sum_{i=1}^{n} \sum_{j=2}^{n} P(A_i A_j A_k) + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} P(A_i A_j A_k) - \ldots \]

Now we use (2.4) to yield:

\[ P \left( \sum_{i=1}^{n} A_i \right) = P(A_1) + P \left( \sum_{i=2}^{n} A_i \right) - P \left( \sum_{i=2}^{n} A_i A_1 \right) \]

\[ = \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n} \sum_{j=1}^{n} P(A_i A_j) + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} P(A_i A_j A_k) - \ldots \]

\[ = P_1 - P_2 + P_3 - \ldots , \text{ Q.E.D.} \]

From (2.6) we see that:

\[ P(A_1 + A_2 + \ldots + A_n) \leq P(A_1) + P(A_2) + \ldots + P(A_n) \]

**Example 2.1**

When an ordinary die (a regular hexahedron, i.e. a cube, marked on each face with one to six spots) is thrown – there are six possible outcomes – any one of the six numbered faces may land upwards. The sample space is thus:

\[ \Omega = \{1, 2, 3, 4, 5, 6\} \]

These six outcomes are mutually exclusive since two or more faces cannot turn up simultaneously. The event of 7 as the outcome of a throw is an impossible one: \( \Lambda = \emptyset \); \( P(\Lambda) = 0 \).

**Example 2.2**

If a pair of ordinary dice are thrown, there are thirty six possible outcomes:

\[ \Omega = \{(i, j) : \text{the numbers of spots, } i \text{ and } j \text{ integers, } 1 \text{ to } 6\} \]
Define the following three events: A: "The sum of the spots on the upward-landing faces is an even number", \( \Lambda_1 \): "The outcome of a throw of each die is an even number", \( \Lambda_2 \): "The outcome of a throw of each die is an odd number". A represents the union of mutually exclusive events \( \Lambda_1 \) and \( \Lambda_2 \) (\( A = \Lambda_1 \cup \Lambda_2 \)). \( \Lambda_1 = A - \Lambda_2, \ \Lambda_2 = A - \Lambda_1 \). The complement to A is \( \bar{A} \): "The sum of spots on the upward-landing faces is an odd number". The complement to \( \Lambda_1 \) is \( \bar{\Lambda}_1 \): "At least on one die an odd number will appear on the upward-landing face", and the complement to \( \Lambda_2 \) is \( \bar{\Lambda}_2 \): "At least on one die an even number will appear on the upward-landing face".

Significantly, the probability axioms do not advise us how to assign probabilities to events; they merely impose restrictions on how this can be done. For instance, in the example on throwing a die, the following are two possible ways in which we can assign probabilities:

\[
(1) \quad P(B_1) = P(B_2) = P(B_3) = P(B_4) = P(B_5) = P(B_6) = 1/6
\]

\[
(2) \quad P(B_1) = P(B_3) = P(\bar{B}_3) = 1/5, \quad P(B_2) = P(B_4) = P(B_6) = 2/15
\]

so that all probability axioms are satisfied. Here \( B_1 \) is an event of the number of spots on the upward-landing face equalling 1 (\( i \leq 6 \)).

\[ \text{§ 2.4. Equi-probable events} \]

Denote by \( E_1, E_2, \ldots, E_n \) the elementary outcomes of the sample space \( \Omega \). Assume that they partition the sample space, such that:

\[ \Omega = E_1 + E_2 + \ldots + E_n, \quad \text{and} \quad E_i E_j = \emptyset, \quad (i \neq j) \]

i.e. they "cover" the sample space and are pairwise disjoint.

Consider the particular case where all outcomes are equi-probable, i.e.

\[ P(E_i) = p, \quad i = 1, 2, \ldots, n. \]

Then, according to Axiom 3:

\[ P(E_1 + E_2 + \ldots + E_n) = P(E_1) + P(E_2) + \ldots + P(E_n) = np \]

on the other hand, according to Axiom 2:

\[ P(E_1 + E_2 + \ldots + E_n) = P(\Omega) = 1 \]
Comparison of the last two equalities yields:

\[ p = \frac{1}{n} \]

Consider now an event:

\[ A = E_1 + E_2 + \ldots + E_m, \quad m \leq n \]

then:

\[ P(A) = P(E_1 + E_2 + \ldots + E_m) = \frac{m}{n} \]

Consequently, if a random experiment can result in mutually exclusive and equiprobable outcomes, and if in \( m \) of these outcomes the event \( A \) occurs, then \( P(A) \) is given by the fraction:

\[ P(A) = \frac{m}{n} \quad (2.7) \]

these \( m \) outcomes are "favorable" to \( A \).

The above can not be used as such to define probability, since the procedure would be circular. The classical definition of probability (Laplace, 1812), is, however, very similar to (2.7). It states:

If a random experiment can result in \( n \) mutually exclusive and equally likely outcomes, of which \( m \) are favourable to \( A \), then the probability of \( A \) equals the ratio of these favourables to the total number of outcomes.

This definition reduces the notion of probability to that of "equilikelihood". Axiom 1 is satisfied, since the fraction \( m/n \) cannot be negative, and so are Axiom 2 - since \( n \) outcomes are favorable to the sample space, and \( P(\Omega) = n/n = 1 \) - and Axiom 3: assume that \( m_1 \) elementary events are favorable to an event \( A \), and \( m_2 \) elementary events - an event \( B \). Assume that \( A_1 \) and \( A_2 \) are mutually exclusive, therefore the events \( E_i \) favourable to one of these events are different from those favorable to the other. Thus there are \( m_1 + m_2 \) events \( E_i \) favourable to one of the events \( A \) or \( B \), i.e. favourable to the event \( A_1 + A_2 = A \).
Consequently:

\[
P(A) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A_1) + P(A_2), \quad \text{Q.E.D.}
\]

Eq. (2.7) for the probability of an event composed of equiprobable events, has many useful applications, wherever symmetry considerations are involved. The probability of a homogeneous, balanced die, properly thrown, turning up each face, \( P(E_1) = 1/6 \), are equal. The probability of an "honest" coin, properly tossed, turning up heads or tails is the same, \( P(E_1) = 1/2 \), and the probability of any card being drawn from a properly shuffled deck, equals \( P(E_1) = 1/52 \). In these cases 6, 2 and 52 are, respectively, the numbers of outcomes.

**Example 2.3**

The number of times of a tossed pair of coins turning up heads equals 2, 1 or 0, and we seek the probability of each event. The three events are not equiprobable, although they partition the sample space. Yet, in order to invoke the classical definition of probability the sample space must be represented as the sum of elementary, equiprobable events. This can be done in the following way:

<table>
<thead>
<tr>
<th>First coin</th>
<th>Second coin</th>
</tr>
</thead>
<tbody>
<tr>
<td>heads</td>
<td>heads</td>
</tr>
<tr>
<td>heads</td>
<td>tails</td>
</tr>
<tr>
<td>tails</td>
<td>heads</td>
</tr>
<tr>
<td>tails</td>
<td>tails</td>
</tr>
</tbody>
</table>

The above four outcomes are natural equiprobables, being mutually exclusive and covering the sample space. We are now able to invoke the classical definition of probability, namely:

- Probability of two "heads" equals 1/4
- Probability of one "heads" equals 2/4 = 1/2
- Probability of no "heads" equals 1/4

§ 2.5. Probability and relative frequency

Consider a sequence of \( n \) like experiments in each of which occurrence or
non-occurrence of some event $A$ is recorded. A natural characteristic of $A$ appears to be the relative frequency of its occurrence, defined as the ratio of its occurrences to the total number of trials. Denote by $\hat{P}(A)$ the relative frequency of $A$. We have:

$$\hat{P}(A) = \frac{n(A)}{n} \quad (2.8)$$

where $n(A)$ is the number of occurrences of event $A$ in $n$ trials.

Note that the relative frequency is bounded between zero and unity:

$$0 \leq \frac{n(A)}{n} \leq 1$$

since the number of times $n(A)$ of the event $A$ occurring in $n$ trials is bounded between zero and $n$.

If $A_1$ and $A_2$ are mutually exclusive, and if in $n$ experiments, $A_1$ occurred $n(A_1)$ times and $A_2$ occurred $n(A_2)$ times, then the union $A_1 + A_2$ occurred $n(A_1) + n(A_2)$ times and its relative frequency equals:

$$\hat{P}(A + B) = \frac{1}{n} [n(A_1) + n(A_2)]$$

However, the relative frequencies of $A_1$ and $A_2$ are $n(A_1)/n$ and $n(A_2)/n$, respectively, and the last equation can be rewritten as:

$$\hat{P}(A_1 + A_2) = \hat{P}(A_1) + \hat{P}(A_2)$$

Past experience has shown remarkable conformity, which imparted a deep significance to the probability notion. It turned out that in different series of experiments the corresponding relative frequencies $n(A)/n$ practically coincide at large values of $n$, and are concentrated in the vicinity of some number. For example, if a die is made of a homogeneous material and represents a perfect cube (an "honest" die), then the relative frequencies 1, 2, 3, 4, 5 or 6, turning up oscillate in the vicinity of 1/6. Table 2.1 lists the relative frequencies of a simple toss of a coin turning up heads in experiments (total number 10,000) conducted in discrete series of 100 and 1000 respectively.
Table 2.1

<table>
<thead>
<tr>
<th>Relative frequencies in series of 100 experiments</th>
<th>Relative frequency in series of 1000 experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.54 0.46 0.53 0.55 0.46 0.54 0.41 0.46 0.51 0.53</td>
<td>0.501 0.485 0.509 0.536 0.485 0.500 0.497 0.494 0.484</td>
</tr>
<tr>
<td>0.49 0.52 0.58 0.51 0.51 0.50 0.52 0.50 0.53 0.49</td>
<td></td>
</tr>
<tr>
<td>0.58 0.60 0.54 0.55 0.50 0.48 0.47 0.57 0.52 0.55</td>
<td></td>
</tr>
<tr>
<td>0.48 0.51 0.51 0.49 0.44 0.52 0.50 0.46 0.53 0.41</td>
<td></td>
</tr>
<tr>
<td>0.49 0.50 0.45 0.52 0.52 0.48 0.47 0.47 0.47 0.51</td>
<td></td>
</tr>
<tr>
<td>0.45 0.47 0.41 0.51 0.49 0.59 0.60 0.55 0.53 0.50</td>
<td></td>
</tr>
<tr>
<td>0.53 0.52 0.46 0.52 0.44 0.51 0.48 0.51 0.46 0.54</td>
<td></td>
</tr>
<tr>
<td>0.45 0.47 0.46 0.52 0.47 0.48 0.59 0.57 0.45 0.48</td>
<td></td>
</tr>
<tr>
<td>0.47 0.41 0.51 0.59 0.51 0.52 0.55 0.39 0.41 0.48</td>
<td></td>
</tr>
</tbody>
</table>

It is seen that the relative frequencies \( n(\mathbb{A})/n \) in the "1000" series differ surprisingly little from the probability \( P(\mathbb{A}) = 1/2 \) (the relative frequency in series of 10,000 experiments is 0.4979).

This stability of the relative frequency could be interpreted as a manifestation of an objective property of the random event, namely existence of a definite degree of its possibility.

Formality, Eq. (2.8) should be understood in the following way:

\[
P(\mathbb{A}) = \lim_{n \to \infty} \frac{n(\mathbb{A})}{n} \quad (2.9)
\]

Realisation of an infinite number of trials is only feasible conceptually, whereas in a physical experiment the number \( n \) may be large but always remains finite. Accordingly, definition (2.9) refers to the existence of a limit. For large \( n \), however (2.9) - or more properly (2.8) - may be used as an estimate of probability.

Although Kolmogorov's definition is superior, the relative frequency definition (due to von Mises) is suitable for physical applications and is by no means incompatible with Kolmogorov's axiomatics. In these circumstances, results obtained in terms of relative frequency are often generalised to the appropriate probabilities.
§ 2.6. Conditional probability

When analysing some phenomenon, the observer is often concerned how occurrence of an event \( A \) is influenced by that of another event \( B \). The simplest modes of intercorrelation of such a pair of events are:
(a) occurrence of \( B \) necessarily results in that of \( A \); or, on the contrary,
(b) occurrence of \( B \) eliminates that of \( A \). In the theory of probability, this intercorrelation is characterised by the conditional probability \( P(A|B) \) of event \( A \), it being known that \( B \) (whose own probability is positive) actually took place:

\[
P(A|B) = \frac{P(AB)}{P(B)} \tag{2.10}
\]

We shall illustrate this on the example of an experiment with a finite number of equiprobable outcomes \( \omega \). Let \( n \) be the total number of outcomes, \( n(B) \) - of those favourable to \( B \), \( n(AB) \) - of those favourable to both \( A \) and \( B \). Then:

\[
P(B) = \frac{n(B)}{n}, \quad P(AB) = \frac{n(AB)}{n}
\]

so that the conditional probability equals:

\[
P(A|B) = \frac{n(AB)}{n(B)} \tag{2.11}
\]

where \( n(B) \) is the number of all elementary outcomes \( \omega \) when \( B \) occurs, and \( n(AB) \) - of those favourable to \( A \). Recalling (2.8), equation (2.11) determines the probability of \( A \) under new conditions, which arise when \( B \) occurs.

Conditional probability retains all the features of ordinary probability. Axiom 1 is satisfied in an obvious manner, since for each event \( A \), the nonnegative function \( P(A|B) \) is defined according to (2.10). If \( A \) equals \( B \), then according to the definition:

\[
P(B|B) = \frac{P(B)}{P(B)} = \frac{P(2)}{P(2)} = 1
\]

and, therefore:

\[
0 \leq P(A|B) \leq 1
\]
If occurrence of \( B \) eliminates that of \( A \), then \( P(AB) = 0 \) and therefore \( P(A|B) = 0 \). If occurrence of \( B \) necessarily results in that of \( A \) \( (B \subseteq A) \), then \( AB = B \) and \( P(AB) = P(B) \), which means \( P(A|B) = 1 \). If \( A \) is a union of mutually-exclusive events \( \lambda_1, \lambda_2, \ldots, \lambda_n \), then the product \( AB \) represents a union of mutually-exclusive events \( \lambda_1B, \lambda_2B, \ldots, \lambda_nB \) and according to Axiom 3:

\[
P(AB) = \sum_{k=1}^{n} P(A_kB)
\]

and \( P(A|B) \) equals:

\[
P(A|B) = \frac{P(AB)}{P(B)} = \frac{1}{P(B)} \sum_{k=1}^{n} P(A_kB) = \sum_{k=1}^{n} \frac{P(A_kB)}{P(B)} = \sum_{k=1}^{n} P(A_k|B)
\]

**Example 2.4**

A pair of dice are thrown. What is the probability the sum of spots on the upward-landing faces being 7 (event \( A \)), given that this sum is odd (event \( B \))?

The sample space is composed of 36 outcomes:

\[
\Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}
\]

The number of outcomes favourable to \( A \) is 6, hence the unconditional probability is:

\[
P(A) = \frac{6}{36} = \frac{1}{6}
\]

If \( B \) has taken place, then one of 18 events took place (a "new" sample space with 18 points) and the conditional probability is:

\[
P(A|B) = \frac{6}{18} = \frac{1}{3}
\]
The probability of event B is:

\[ P(B) = \frac{18}{36} = \frac{1}{2} \]

and \( P(A \mid B) \) is also obtained from the general formula (2.10):

\[ P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{6/36}{1/2} = \frac{1}{3} \]

Note that the definition of conditional probability enables us to find the probability of a product. From Eq. (2.10), it follows immediately, that

\[ P(AB) = P(A \mid B) P(B) \quad (2.12) \]

i.e. the probability of product \( AB \) is calculated by constructing the product of the conditional probability, under event \( A \), of event \( B \) occurring, and the (unconditional) probability of event \( B \).

On the other hand,

\[ P(AB) = P(B \mid A) P(A) \]

and

\[ P(AB) = P(A \mid B) P(B) = P(B \mid A) P(A) \quad (2.13) \]

i.e., the probability of the product of two events is calculated by constructing the product of the conditional probability, under one of these events, of another event occurring, and the (unconditional) probability of the latter event. Formula (2.12) is readily extended by induction to \( n \) events \( A_1, A_2, \ldots, A_n \):

\[
P(A_1 A_2 A_3 \ldots A_{n-1} A_n) = P(A_1 \mid A_2 A_3 \ldots A_{n-1}) P(A_2 A_3 \ldots A_n) \\
= P(A_1 \mid A_2 A_3 \ldots A_n) P(A_2 \mid A_3 \ldots A_n) P(A_3 \ldots A_n) \\
= \ldots = \\
= P(A_1 \mid A_2 A_3 \ldots A_n) P(A_2 \mid A_3 \ldots A_n) P(A_3 \mid A_4 \ldots A_n) \ldots P(A_{n-1} \mid A_n) P(A_n) \quad (2.14)
\]
(2.14) is known as the multiplication rule.

§ 2.7. Independent events

Events A and B are called independent, if:

\[ P(A|B) = P(A) \]  

(2.15)
i.e. occurrence of B does not affect the probability of A. Note that mutually exclusive events are dependent, in fact if AB = Ø, then \( P(A|B) = P(Z) = 0 \neq P(A) \) unless A = Ø.

If event A is independent of B, then, according to Eqs. (2.13) and (2.15) we have

\[
P(A) P(B|A) = P(B) P(A|B)
\]

\[ = P(B) P(A) \]

Thus

\[ P(B|A) = P(B) \]

implying that event B is equally independent of A, or in other words, that the property of independence is mutual.

The probability of a product of independent events is readily calculated:

\[ P(A_3) = P(A_1|B) P(C) = P(A) P(B) \]

and often used as a definition of an independence.

n events, \( A_1, A_2, \ldots, A_n \) are independent, iff:

\[ P(A_i A_j) = P(A_i) P(A_j) \text{ for } i \neq j \]

\[ P(A_i A_j A_k) = P(A_i) P(A_j) P(A_k) \text{, } i \neq j, j \neq k, i \neq k \]

\[ \vdots \]

\[ P \left( \bigcap_{i=1}^{n} A_i \right) = \prod_{i=1}^{k} P(A_i) \]  

(2.16)
Thus, pairwise independence is **not** sufficient for the component events to be **individually** independent.

This is illustrated by the following example, due to N.S. Bernstein.

**Example 2.5**
Given events $A$, $B$, $C$ such that:

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

and

$$P(AB) = P(AC) = P(BC) = P(ABC) = \frac{1}{4}$$

These events are pairwise independent:

$$P(AB) = P(A) \cdot P(B), \quad P(AC) = P(A) \cdot P(C), \quad P(BC) = P(B) \cdot P(C)$$

but not **individually** independent, since

$$P(ABC) \neq P(A) \cdot P(B) \cdot P(C).$$

**Note:** $A$, $B$ and $C$ have a "physical meaning" - as in the case of an "honest" tetrahedron, with one face red (event $A$), another green (event $B$), the third blue (event $C$) and the fourth in all three colours.

**Example 2.6**
A device consists of $n$ components (Fig. 2.6a) connected in series, i.e. all components are so interrelated that failure of any one of them implies failure of the entire system; these component failures are taken as independent random events $X_i$. Denote the reliability, i.e. probability of non-failure performance of element $X_i$ by $R_i$; the reliability of the entire system is then:

$$R = P(\bar{X}_1 \bar{X}_2 \ldots \bar{X}_n) = P(\bar{X}_1) \cdot P(\bar{X}_2) \ldots P(\bar{X}_n) = R_1 \cdot R_2 \ldots R_n = \prod_{k=1}^{n} R_k$$

(2.17)
As reliability in principle does not exceed unity, multiplication of the component reliabilities makes for decrease of the overall system reliability. R as the number of components increases; in fact, R cannot exceed the reliability of the weakest component:

\[ R \leq \min (R_1, R_2, \ldots, R_n) \]

**Example 2.7 - Parallel system**

Consider now the same elements as above, this time connected in parallel (see Fig. 2.8b). In this case the system fails when all components fail; the probability of failure equals:

\[ P(A_1 A_2 \ldots A_n) = P(A_1) P(A_2) \ldots P(A_n) = (1 - R_1)(1 - R_2)\ldots(1 - R_n) \]

and the reliability of the entire system is:

\[ R = 1 - (1 - R_1)(1 - R_2)\ldots(1 - R_n) = 1 - \prod_{k=1}^{n} (1 - R_k) \quad (2.18) \]

For \( R_i = 0.9 \) and \( n = 2 \), the reliability of a series system is \( 0.9^2 = 0.81 \), whereas that of a parallel system is \( 1 - (1 - 0.9)^2 = 0.99 \).

**5 2.8. Reliability of statically-determinate truss**

Consider a statically-determinate truss consisting of \( n \) bars under given specified deterministic loading (Fig. 2.9a,b), with failure (overloading beyond the yield-stress level) of a single bar implying failure of the entire truss. Assuming the failures of component bars are random events, the reliability of the truss is given by:

\[ R = 1 - \text{Prob (any bar failing)} \]

\[ = 1 - \text{Prob} \left( \sum_{i=1}^{n} A_i + A_2 + \ldots + A_n \right) \]

\[ = 1 - \sum_{i=1}^{n} P(A_i) + \sum_{i < j}^{n} P(A_i A_j) - \sum_{i < j < k} P(A_i A_j A_k) + \ldots \quad (2.19) \]

Where

\[ P(A_i A_j) = P(A_i | A_j) P(A_j) \]
\[ P(A_1 A_j A_k) = P(A_i | A_j A_k) \cdot P(A_j | A_k) \cdot P(A_k) \quad \text{etc.} \]

For example, for the case \( n = 2 \) (Fig. 2.9c):

\[
R_2 = 1 - \left[ P(A_1) + P(A_2) - P(A_1 A_2) \right] \\
= 1 - \left[ P(A_1) + P(A_2) - P(A_1 | A_2) \cdot P(A_2) \right] \\
= 1 - \left[ P(A_1) + P(A_2) - P(A_1 | A_2) \cdot P(A_2) \right] \\
(2.20)
\]

In order to find the reliability we must know the conditional probability.

For \( n = 3 \) (Fig. 2.9d) we have:

\[
R_3 = 1 - \left[ P(A_1) + P(A_2) + P(A_3) - P(A_1 A_2) - P(A_1 A_3) - P(A_2 A_3) + P(A_1 A_2 A_3) \right] \\
= 1 - \left[ P(A_1) + P(A_2) + P(A_3) - P(A_1 | A_2) \cdot P(A_2) - P(A_1 | A_3) \cdot P(A_3) - \\
- P(A_2 | A_3) \cdot P(A_3) + P(A_1 | A_2 A_3) \cdot P(A_2 | A_3) \cdot P(A_3) \right] \\
(2.21)
\]

In this case we must know the conditional probabilities of failure
\( P(A_1 | A_2) \), \( P(A_1 | A_3) \), \( P(A_2 | A_3) \) and \( P(A_1 | A_2 A_3) \).

Two extreme cases are possible:

1. Bar failures represent independent random variables (the bars can be visualized as manufactured at different plants, using different processes). In this case it suffices to know the reliabilities of individual bars:

\[
P(A_1 A_j) = P(A_1) \cdot P(A_j) = (1 - R_i) (1 - R_j)
\]

\[
P(A_1 A_j A_k) = (1 - R_i) (1 - R_j) (1 - R_k)
\]

and Eq. (2.19) becomes:

\[
R_n = 1 - \sum_{i=1}^{n} (1 - R_i) + \sum_{i<k} (1 - R_i) (1 - R_j) + \cdots
\]

Another way of finding the reliability is:
\[ R_n = P \text{ (no one of bars fail)} = P(\bar{A}_1 \bar{A}_2 ... \bar{A}_n) \]

\[ = P(\bar{A}_1) P(\bar{A}_2) ... P(\bar{A}_n) = R_1 R_2 ... R_n \]

so that in this case the truss behaves like a "series" system (see also Fig. 2.9b).

For \( n = 2 \) we have:

\[ R_2 = 1 - [P(A_1) + P(A_2) - P(A_1) P(A_2)] = \left[ 1 - P(A_1) \right] \left[ 1 - P(A_2) \right] = R_1 R_2 \]

for \( n = 3 \):

\[ R_3 = 1 - [P(A_1) + P(A_2) + P(A_3) - P(A_1) P(A_2) - P(A_1) P(A_3) - P(A_2) P(A_3) + P(A_1) P(A_2) P(A_3)] = \left[ 1 - P(A_1) \right] \left[ 1 - P(A_2) \right] \left[ 1 - P(A_3) \right] = R_1 R_2 R_3 \]

In the particular case where \( R_1 = R_2 = ... = R_n = r \), we have:

\[ R_n = r^n \quad (2.22) \]

(2) Each realisation of the truss, and therefore, of its constituent bars, has the same strength which changes from realisation to realisation (all bars manufactured at the same plant by the same process). Reliability then depends generally on the level of the stress in it. Let the weakest bar be the first one. Then

\[ R = 1 - \text{Prob (weakest bar failing, or any other bar failing)} \]

Since the event "any other bar failing" is contained in the event "weakest bar failing", we have:

\[ P \text{ (weakest bar failing or any other bar failing)} \]

\[ = \text{Prob (weakest bar failing)} \]

and therefore
\[ R = 1 - \text{Prob (weakest bar falling)} = \min (R_1, R_2, \ldots, R_n) \quad (2.23) \]

For \( n = 2 \), we have by Eq. (2.20):

\[ P(A_2 | A_1) = 1 \]

and:

\[ R_2 = 1 - P(A_1) = R_1 \]

For \( n = 3 \), we have:

\[ P(A_1 | A_2) = P(A_1 | A_3) = P(A_1 | A_2 A_3) = 1 \]

and again:

\[ R_3 = 1 - P(A_1) = R_1 \]

If \( R_1 = R_2 = \ldots = R_n = r \) in (2.23):

\[ R = r \quad (2.24) \]

and the reliability of the entire truss equals that of a single bar.

The latter case is illustrated in Figs. 2.9e, 2.9f, where due to equality of the stresses in all bars, equal reliabilities are assigned to them.

\[ \section*{2.9 Overall probability and Bayes formula} \]

Let \( B \) be an event in a sample space partitioned by events \( A_1, A_2, \ldots, A_n \).

Then \( B = B \Omega = B(A_1 + A_2 + \ldots + A_n) \), and

\[ P(B) = \sum_{i=1}^{n} P(B | A_i) \]

Using the multiplication rule for each of \( P(B | A_i) \) we arrive at:

\[ P(B) = \sum_{i=1}^{n} P(B | A_i) \cdot P(A_i) \quad (2.25) \]

known as the formula of overall probability.
Now, provided \( P(B) \neq 0 \), \( P(A_1|B) \) can be expressed as:

\[
P(A_1|B) = \frac{P(A_1B)}{P(B)} = \frac{\sum_{i=1}^{n} P(B|A_i) P(A_i)}{P(B)}
\]

This formula is due to Bayes. The unconditional probabilities \( P(A_i) \) are called \textit{a priori probabilities}, and the conditional ones \( P(A_i|B) \) - \textit{a posteriori probabilities}.

**Example 2.8**

Given two boxes, the first containing a white and b black balls, the second - c white and d black balls. One ball is removed at random from the first box and placed in the second, after which one ball is removed from the latter. What is the probability of this ball being white?

Denote the following events: \( A \) - white ball removed from second box,

\( H_1 \) - white ball placed in second box, \( H_2 \) - black ball placed in second box:

\[
P(H_1) = \frac{a}{a + b}, \quad P(H_2) = \frac{b}{a + b}
\]

\[
P(A|H_1) = \frac{c + 1}{c + d + 1}, \quad P(A|H_2) = \frac{c}{c + d + 1}
\]

According to the formula of overall probability, we have:

\[
P(A) = P(A|H_1) P(H_1) + P(A|H_2) P(H_2)
\]

\[
= \frac{a}{a + b} \frac{c + 1}{c + d + 1} + \frac{b}{a + b} \frac{c}{c + d + 1}
\]

In the particular case of both boxes containing equal numbers of white and black balls (\( c = a, d = b \)) we have:

\[
P(A) = \frac{a}{a + b} \frac{a + 1}{a + b + 1} + \frac{b}{a + b} \frac{a}{a + b + 1} = \frac{a}{a + b}
\]

indicating that the probability of a white ball being removed from the second box is unaffected by adding the ball from the first box.
Example 2.9
The structure under a time-dependent load, \( P = P(t) \), shown in Fig. 2.9c, consists of two bars with reliability [defined as non-failure performance in the time interval \((0, T)\)] \( R_1 \) and \( R_2 \) respectively; non-failure of both is required for non-failure of the structure. The structure was inspected at the end of time interval \( T \) and found to have failed. Find the probability of only the first bar having failed, the second not.

Before the experiment, the four following hypotheses were possible:

- \( H_0 \) - both bars do not fail,
- \( H_1 \) - first bar fails, second does not,
- \( H_2 \) - first bar does not fail, second does,
- \( H_3 \) - both bars fail.

The respective probabilities of these hypotheses are:

\[
P(H_0) = R_1 R_2 \quad , \quad P(H_1) = (1-R_1) R_2 \\
P(H_2) = R_1 (1-R_2) \quad , \quad P(H_3) = (1-R_1) (1-R_2)
\]

Event \( A \) has taken place - the structure failed, hence:

\[
P(A|H_0) = 0 \quad , \quad P(A|H_1) = P(A|H_2) = P(A|H_3) = 1
\]

and Bayes' formula yields:

\[
P(H_1|A) = \frac{(1-R_1) R_2}{(1-R_1) R_2 + (1-R_2) R_1 + (1-R_1) (1-R_2)} = \frac{(1-R_1) R_2}{1 - R_1 R_2}
\]

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PROBLEMS

2.1. Present Venn's diagram for \( \bar{C} \), where \( C = B \setminus A \).

2.2. Verify by means of Venn's diagram that a union and an intersection of random events are distributive, that is

\[
(A \cup B) \cap C = (A \cap C) \cup (B \cap C)
\]

\[
(A \cap B) \cup C = (A \cup C) \cap (B \cup C)
\]

2.3. A telephone relay satellite is known to have five malfunctioning channels out of 500 available. If a customer gets one of the malfunctioning channels on first dialling, what is the probability of his hitting on another malfunctioning channel on dialling again?

2.4. Let \( m \) terms be chosen at random from a lot containing \( n > m \) items of which \( p(m < p < n) \) are defective. Find the probability of all \( m \) items being nondefective. Consider also a particular case:

\[ m = 3, p = 4, n = 5 \]

2.5. A single playing card is picked at random from a well-shuffled ordinary deck of 52. Consider the events: \( A \) - king picked, \( B \) - ace picked, \( C \) - heart picked.
Check whether (1) \( A \) and \( B \), (2) \( A \) and \( C \), (3) \( B \) and \( C \), are dependent or independent.

2.6. A guest is calling on a family with two children. When he arrives the door is opened by one of the children, a boy, who introduces himself as the older of the two. What is the probability of the other child also being a boy?

2.7. A guest is calling on a family with two children. When he arrives the door is opened by one of the children, a boy. What is the probability of the other child also being a boy?

Compare the result with that obtained in the preceding problem.
Explain the discrepancy or agreement between these results.
2.9. A statically-determinate truss consists of n bars, whose failures represent independent random variables with identical probabilities p. Find the permissible value of n such that the probability of failure Q of the entire truss does not exceed some prescribed value q.

2.9. A space shuttle is assigned to visit a TV satellite in a GSO (geosynchronous orbit) with radius 36,000 km (event A₁), and carry out maintenance (event B₁). Alternatively it may, due to failure of one of the engines, go into a LEO (low Earth orbit) with radius 200 km (event A₂) and carry out a crop survey (event B₂). B₁ and B₂ are regarded as successful performance. The third possibility is failure to take off (event A₃). What is the reliability (probability of successful performance) of the space shuttle, if

\[ P(B₁ | A₁) = 0.75, \ P(B₂ | A₂) = 0.85, \ P(A₁) = 0.80, \ P(A₂) = 0.15 \]

2.10. (Birger). During inspection of a gas turbine two symptoms are checked: increase of the engine gas temperature at the turbine outlet by more than 50°C (symptom k₁) and increase of the acceleration time of the R.P.M. from minimum to maximum, by more than 5 seconds (symptom k₂). Assume that for the given type of engine these symptoms are associated either with failure of the fuel flow regulator (state D₁), or with reduction of the radial clearance of the turbine (state D₂). The normal state is denoted by D₃. The following probabilities are given:

\[ P(D₁) = 0.05, \ P(k₁ | D₁) = 0.2, \ P(k₂ | D₁) = 0.3 \]

\[ P(D₂) = 0.15, \ P(k₁ | D₂) = 0.4, \ P(k₂ | D₂) = 0.5 \]

\[ P(D₃) = 0.80, \ P(k₁ | D₃) = 0.0, \ P(k₂ | D₃) = 0.05 \]

Show that:

\[ P(D₁ | k₁k₂) = 0.09, \ P(D₂ | k₁k₂) = 0.91, \ P(D₃ | k₁k₂) = 0.0 \]

\[ P(D₁ | k₁k₂) = 0.12, \ P(D₂ | k₁k₂) = 0.46, \ P(D₃ | k₁k₂) = 0.41 \]
\[ P(D_1 | \bar{k}_1 \bar{k}_2) = 0.03, \quad P(D_2 | \bar{k}_1 \bar{k}_2) = 0.05, \quad P(D_3 | \bar{k}_1 \bar{k}_2) = 0.92 \]

indicating, for example, that in the presence of symptoms \( k_1 \) and \( k_2 \) the probability of reduction of the radial clearance is about ten times that of failure of the fuel flow regulator.
CHAPTER 3 - SINGLE RANDOM VARIABLE

§ 3.1. Random variable

A real random variable \( X(\omega) \), \( \omega \in \Omega \), is a real function mapping a sample space \( \Omega \) into the real line \( \mathbb{R} \), i.e. making a real number \( X(\omega) \) correspond to every outcome \( \omega \) of an experiment such that: (1) the set \( \{ X(\omega) \leq x \} \) is an event for any real number \( x \), and (2) the probabilities \( P \{ X(\omega) = +\infty \} = P \{ X(\omega) = -\infty \} \) are zero.

Random variables are conventionally denoted by Roman capitals. The notations \( X(\omega) \) and \( X \) will hereinafter be used interchangeably, although the former is preferable as an explicit indication of the functional character of the variable.

Example 3.1

Consider a bar of given geometry and strength. The tensile force \( N \) may take on two values: \( n_1 \) (with probability \( p \)) at which the bar fails, and \( n_2 \) (with probability \( 1-p \)) at which it survives; accordingly, \( \Omega = \{ \text{fail, survive} \} \).

Let \( X(\omega) = 1 \) if \( \omega = \text{fail} \), and \( X(\omega) = 0 \) if \( \omega = \text{survive} \); we thus have a real number \( X(\omega) \) corresponding to either outcome of the experiment. We shall show that \( \{ \omega: X(\omega) \leq x \} \) is an event. If \( x < 0 \), \( \{ \omega: X(\omega) \leq x \} = \emptyset \); if \( 0 \leq x < 1 \), \( \{ \omega: X(\omega) \leq x \} = \{ \text{survive} \} \); if \( x \geq 1 \), \( \{ \omega: X(\omega) \leq x \} = \Omega = \{ \text{fail, survive} \} \).

Consequently for each \( x \) the set \( \{ \omega: X(\omega) \leq x \} \) is an event, and \( X(\omega) \) is a random variable.

Example 3.2

Consider a three-dice experiment. The sample space \( \Omega \) contains \( 6^3 = 216 \) points: \( \Omega = \{ (i, j, k); i, j, k = 1, 2, \ldots, 6 \} \).

Let \( X \) denote the sum of spots on the upward-landing faces; so \( X(\omega) = i + j + k \) if \( \omega = (i, j, k) \). \( X \) is a random variable; the codomain of \( X(\omega) \) is an ensemble of positive numbers between 3 and 18.

§ 3.2. Distribution function

The (cumulative) distribution function \( F_X(x) \) of the random variable \( X \) is defined as:

\[
F_X(x) = P(X \leq x) = P \{ \omega \in \Omega: X(\omega) \leq x \} \tag{3.1}
\]

for every real number \( x \).
Example 3.3
Consider the random variable described in Example 3.1. The probability distribution
\[ F_X(x) = 0, \text{ if } x < 0, \text{ since } (X \leq x) \text{ is an impossible event.} \]
\[ F_X(x) = 1 \text{ if } x \geq 1 \text{ since } \{X(\omega) \leq x\} \text{ is a certain event, because:} \]
\[ X(\text{fail}) = 1, \ X(\text{survive}) = 0 < x \]
If \( 0 \leq x < 1 \), then \( \{X(\omega) \leq x\} = \{\text{survive}\}, \) since:
\[ X(\text{fail}) = 1 > x, \ X(\text{survive}) = 0 \leq x \]
and, consequently,
\[ F_X(x) = P(X \leq x) = P(\text{survive}) = 1 - p \]
The distribution function is shown in Fig. 3.1.

Example 3.4
Consider the experiment of tossing a single "honest" coin:
\[ \Omega = \{\text{heads, tails}\} \]
This example can be reduced to the preceding one, by the following formal substitution:
\[ \text{fail} \rightarrow \text{head}, \ \text{survive} \rightarrow \text{tail}, \ p \rightarrow \frac{1}{2} \]

Example 3.5
Consider the experiment of throwing a single "honest" die. Let \( X \) denote the number of spots on the upward-landing face. \( X(\omega_i) = i, \ i = 1, 2, \ldots, 6. \)
\[ P(\omega_i) = \frac{1}{6}. F_X(x) = 0 \text{ for } x < 1, \text{ since the event } \{X < 1\} = \emptyset, \ P(X = P(\emptyset) = 0. \]
For \( 1 \leq x < 2 \) we have:
\[ F_X(x) = P(X \leq x) = P(\omega_1) = \frac{1}{6} \]
For $2 \leq x < 3$, the event $(X \leq x)$ equals $(\omega_1, \omega_2)$, and due to their mutual exclusiveness, we have:

$$F_X(x) = P(\omega_1, \omega_2) = P(\omega_1) + P(\omega_2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Thus:

$$F_X(x) = \begin{cases} 
0 & \text{, for } x < 1 \\
\frac{1}{6} i & \text{, for } i \leq x < i+1, \ i = 1, 2, 3, 4, 5 \\
1 & \text{, for } 6 \leq x
\end{cases}$$

The last equality is readily obtainable by noting that for $6 \leq x$ the event $(X < x)$ equals $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6)$, i.e. represents a certain event. The distribution function is shown in Fig. 3.2; it may also be formulated as:

$$F_X(x) = \frac{1}{6} \sum_{i=1}^{6} U(x - i)$$

where $U(x - i)$ is a unit step function, defined as:

$$U(x - i) = \begin{cases} 
1 & \text{, } x \geq i \\
0 & \text{, otherwise}
\end{cases}$$

(3.2)

This function is shown in Fig. 3.3.

§ 3.3. Properties of distribution function

The distribution function has the following properties:

1. $F_X(-\infty) = 0$.
2. $F_X(+\infty) = 1$.
3. $F_X(x)$ is a monotone nondecreasing function, that is $F_X(a) \leq F_X(b)$ if $a < b$.
4. $F_X(x)$ is continuous on the right, meaning that:

$$\lim_{\varepsilon \to 0} F_X(x + \varepsilon) = F_X(x^+) = F_X(x)$$
All these properties can be shown to hold, using the definition of the distribution function. In fact, \( F_X(-\infty) = P(X \leq -\infty) = P(X < -\infty) = 0 \), the last equality was according to the definition of a random variable; \( F_X(+\infty) = \text{Prob}(X \leq +\infty) = \text{Prob}(\Omega) = 1 \), since the event \( \{X \leq +\infty\} \) is a certain one; for each outcome \( X(\omega) \leq +\infty \).

Let us establish the third property:

\[
\text{Prob}(X \leq b) = \text{Prob}\{(X \leq a) \cup (a < X \leq b)\}
\]

since events \( \{X \leq a\} \) and \( \{a < X \leq b\} \) are mutually exclusive,

Axiom 3 yields:

\[
F_X(b) = \text{Prob}(X \leq a) + \text{Prob}(a < X \leq b)
\]

\[
= F_X(a) + \text{Prob}(a < X \leq b)
\]

and:

\[
F_X(b) - F_X(a) = \text{Prob}(a < X \leq b) \tag{3.3}
\]

Due to Axiom 1, the last quantity is nonnegative. This yields in the desired result:

\[
F_X(b) - F_X(a) \geq 0
\]

The last property follows from the definition of \( F_X(x) \), as \( P(X \leq x) \).

It follows from the properties that the distribution function is bounded between zero and unity.

Rewriting Eq. (3.3) for \( a - \infty \), we have:

\[
F_X(b) - F_X(a - \infty) = \text{Prob}(a - \infty < X \leq b)
\]

Now, considering the limit when \( b \rightarrow a + \infty \), we have:

\[
F_X(a + \infty) - F_X(a - \infty) = \text{Prob}(a - \infty < X \leq a + \infty) = \text{Prob}(X = a)
\]
Since \( F_X(a + \delta) = F_X(a) \), we obtain

\[
F_X(a) = F_X(a - \delta) + \text{Prob}(X = a)
\]

For \( \text{Prob}(X = a) = 0 \), the distribution function turns out to be continuous on the left hence continuous at \( a \). For \( F(X = a) \neq 0 \), we have:

\[
\text{Prob}(X = a) = F_X(a) - F_X(a - \delta)
\]  

(3.4)

That is, the probability of the event of the random variable \( X \) taking on the value \( a \) equals the jump discontinuity of its distribution function at \( a \) (see Fig. 3.4). Since the distribution function is bounded between zero and unity, it can have at most a countable number of jump discontinuities. In fact, it cannot have more than \( 2^1 - 1 \) jumps having values between \( 2^{-1} \) and \( 2^{1-1} \).

A random variable \( X \) is called discrete if the range of \( X \) is countable, i.e. there exists a finite or denumerable set of real numbers \( x_1, x_2, \ldots \), such that \( X \) takes on values only within that set. These values \( x_1, x_2, \ldots \), we call possible values of the discrete random variable \( X \). In order to characterize the discrete random variable completely, it suffices to know the probabilities \( p_i = P(X = x_i) \). The distribution function then becomes

\[
F_X(x) = \sum_{i : x_i \leq x} p_i
\]  

(3.5)

or, using the unit step function defined by Eq. (3.2):

\[
F_X(x) = \sum \ p_i \ \delta(x - x_i)
\]  

for all \( x_i \)

(3.6)

and summation is taken over all indices of possible values of the random variable \( X \). The distribution function of a discrete random variable is discontinuous, jumping by \( F[X = x_i] \) at the discontinuity point \( x_i \) (see e.g. Figs. 3.1 or 3.2).

A random variable \( X \) is called continuous if there exists a non-negative function \( f_X(x) \), such that:

\[ F_X(x) = \int_{-\infty}^{x} f_X(x) \, dx \tag{3.7} \]

for every real number \( x \). The function \( f_X(x) \) is called the *probability density function* of \( X \). If \( F_X(x) \) is absolutely continuous and differentiable at every \( x \), then its derivative equals the probability density function of \( X \):

\[ f_X(x) = \frac{dF_X(x)}{dx} = \lim_{\Delta x \to 0} \frac{F_X(x + \Delta x) - F_X(x)}{\Delta x} \tag{3.8} \]

The properties of the probability density function are:

1. \( f_X(x) \geq 0 \).
2. for every \( x_1 \) and \( x_2 \):

\[ P\{x \in [x_1, x_2]\} = \int_{x_1}^{x_2} f_X(x) \, dx \tag{3.9} \]

that is, the probability of a random variable taking on a value in any interval in its range equals the integral of its probability density over that interval; in particular, if \( f_X(x) \) is continuous in \( x \), then \( P(x \leq X \leq x + \Delta x) \approx f_X(x) \, \Delta x \), i.e. the probability that \( X \), in a small interval containing the value \( x \), is approximately equal to the width of that interval times \( f_X(x) \).

Taking the limit as \( \Delta x \) approaches zero, we have:

\[ \lim_{\Delta x \to 0} P(x \leq X < x + \Delta x) = P(X = x) = 0 \tag{3.10} \]

for any real \( x \). That is, for a continuous random variable, the probability of \( X \) taking on any specified value \( x \) is zero. This conclusion is also obtainable from Eq. (3.4), since the distribution function of a continuous variable is continuous also on the left, \( F_X(a) = F_X(a - \epsilon) \), and \( P(X = a) = 0 \). This implies that at a large number of trials the random variable \( X \) would very seldom take on this value so that the relative frequency of the latter will tend to zero.

3. Since \( F_X(\pm \infty) = 1 \), we get from Eq. (3.7):

\[ \int_{-\infty}^{\infty} f_X(x) \, dx = 1 \tag{3.11} \]
We formally define the probability density function of a discrete random variable with a distribution function as per (3.6):

\[ f_x(x) = \sum p_i \delta(x - x_i) \]  

(3.12)

summation is again taken over all indices of possible values of the random variable \( x \), \( \delta(x - x_i) \) is Dirac's delta function, so that:

\[ U(x - x_i) = \int_{-\infty}^{x} \delta(x - x_i) \, dx \]  

(3.13)

The basic property of the Dirac delta function is:

\[ \int_{-\infty}^{\infty} \varphi(x) \delta(x - a) \, dx = \varphi(a) \]  

(3.14)

for any continuous function \( \varphi(x) \). The integral in (3.14) "screens" out as it were, the value of function \( \varphi \) at the argument \( a \); \( \delta \) is a functional or a generalised function. With this function \( a \) we are already familiar from the course of Mechanics of Solids (see, e.g., Crandall et al or Popov).

\[ \delta(x - a) = (x - a)^{-1}, \quad U(x - a) = (x - a)^0 \]  

(3.15)

As a "refresher" on its use, consider the beam in Fig. 3.6 with \( n \) concentrated forces acting on it. The shear force will then be:

\[ V_y(x) = p_A (x - a)^0 - p_1 (x - x_1)^0 - p_2 (x - x_2)^0 - \ldots - p_n (x - x_n)^0 \]

and the distributed force:

\[ p_y(x) = \frac{dv}{dx} = -R_A (x - a)^{-1} + p_1 (x - x_1)^{-1} + p_2 (x - x_2)^{-1} + \ldots + p_n (x - x_n)^{-1} \]

where jumps in the graph of \( V_y(x) \) equal the values of the concentrated forces.

Analogically, the jumps in the distribution function of a discrete random variable equal the probability of its taking on the specific value of \( x_i \).
where the jump takes place. The analogy is obviously incomplete.

Jumps in forces can be either positive or negative (p_i \geq 0), whereas jumps in the probability distribution function are exclusively positive (p_i > 0) and \( F_X(x) \) is a non-decreasing function. Fig. 3.6 shows a probability density function of Example 3.5.

A random variable \( X \) is called mixed if it is neither purely discrete nor purely continuous, in other words - if it possesses the properties of both discreteness and continuity: it contains jumps, equal \( p_1, p_2, \ldots, p_n \), respectively, at points \( x_1, x_2, \ldots, x_n \), but is continuous between them. Subtracting then from the distribution function the sum:

\[
\sum_{i=1}^{n} p_i \, U(x - x_i)
\]

we obtain a continuous function. We can thus express the probability distribution function of a mixed random variable as:

\[
F_X(x) = F^*(x) + \sum_{i=1}^{n} p_i \, U(x - x_i)
\] (3.16)

where \( F^*(x) \) is a continuous function, and \( p_i \) \((i = 1, 2, \ldots, n)\) are the jumps of \( F_X(x) \) at points \( x_i \) \((i = 1, 2, \ldots, n)\). If \( F^*(x) \) is absolutely continuous and differentiable at every \( x \), then the probability density function of \( X \) may be expressed:

\[
f_X(x) = f^*(x) + \sum_{i=1}^{n} p_i \, \delta(x - x_i)
\] (3.17)

where:

\[
f^*(x) = [F^*(x)]'
\]

Unlike a continuous random variable, a mixed random variable has a countable number of possible values which it takes on with nonzero probability.

An "analogue" of the probability density of a mixed random variable is represented by the following example. The beam, simply supported at its ends, is subjected to a concentrated moment \( m \) (see Fig. 3.7).

The bending moment will then be written as:
\[ M_z(x) = -\frac{m}{L} x + m \langle x - a \rangle^0 \]

and the shear force as:

\[ V_y(x) = +\frac{m}{L} - m \langle x - a \rangle^{-1} \]

This expression comprises a continuous part \( m/L \) and a singular, discontinuous, part, \( -m \langle x - a \rangle^{-1} \), whose probability counterparts are obviously positive.

For clearer understanding of the notions of "discrete", "continuous" and "mixed" distributions, the following analogy is useful. Let us visualise a string with a mass distribution such that the entire mass equals unity, and its density - the probability density function. The discrete case correspond to the entire mass being lumped at certain points \( x_1, x_2, \ldots, x_n \); the continuous case - to a distributed (uniformly, or otherwise) mass without concentrations; the mixed case - to a combination of continuities and concentrations. This analogy will also be useful in further analysis of random variables. We actually assumed that the distribution function of a continuous random variable has only a countable number of points for which the derivative does not exist. At this points we are assigning any positive value to \( f(x) \) so that it becomes defined for all points \( x \).

\[ \text{§ 3.4. Mathematical expectation} \]

While the distribution function provides a complete characterisation of the random variable, it is sometimes possible to make do with a simpler, albeit incomplete, characterisation based on few numbers. Suppose that we are concerned with a series of trials whose possible outcomes are \( x_i^* \), \( i = 1, 2, \ldots, n \); the simplest characteristic of the discrete random variable \( X \) in the given series would be the arithmetical mean:

\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i^* \quad (3.18) \]

If some of the \( n_i \) values of \( x_1^*, x_2^*, \ldots, x_n^* \) taken on by the random variable \( X \) in \( n \) experiments coincide, the coefficient of the common \( x_i^* \) value will be \( n_i \). Denoting the possible values by \( x_1, x_2, \ldots, x_m \), and their relative frequencies by \( n_i/n \), Eq. (3.18) becomes:

\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{m} n_i x_i \quad (3.19) \]
We regard a discrete random variable $X$ as having a mathematical expectation, if:

$$\sum_{i=-\infty}^{\infty} |x_i| P(X = x_i) < \infty$$

(3.20)

The expectation being given by the expression:

$$E(X) = \sum_{i=-\infty}^{\infty} x_i P(X = x_i)$$

(3.21)

with summation over a finite or countable number of possible values $x_i$, $-\infty < x_i < \infty$.

If $\varphi(X)$ is a deterministic function of a random variable, then:

$$E[\varphi(X)] = \sum_{i=-\infty}^{\infty} \varphi(x_i) P(X = x_i)$$

(3.22)

Since the random variable $Y = \varphi(X)$ takes on only values $y_i = \varphi(x_i)$, where $x_i$ are the possible values of $X$, we have:

$$P(Y = y) = \sum_{x_i : \varphi(x_i) = y} P(X = x_i)$$

so that:

$$E(Y) = \sum_{j=-\infty}^{\infty} y_j P(Y = y_j)$$

$$= \sum_{j=-\infty}^{\infty} y_j \sum_{x_i : \varphi(x_i) = y_j} P(X = x_i) = \sum_{i=-\infty}^{\infty} \varphi(x_i) P(X = x_i)$$

Example 3.6

A random variable $X$ takes on values $i$ ($i = 2, 3, \ldots$) with probabilities $1/i^2$. Since:

$$\sum_{i=2}^{\infty} \frac{1}{i^2} = \sum_{i=2}^{\infty} \frac{1}{i} + \infty$$

this random variable has no mathematical expectation.
Example 3.7
Consider the experiment of throwing an "honest" die, \(X\) denoting the number of spots on the upward-landing face. We have:

\[
E[X] = \sum_{i=1}^{6} i \cdot P(X = i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3 \frac{1}{2}
\]

Since the result does not coincide with one of the possible values of \(X\), it is seen that the mathematical expectation is not a value to be "expected", but rather an average or a mean, as it is actually referred to sometimes.

Note that in the example of a string of unity mass with concentrations, \(E(X)\) coincides numerically with the coordinate of the centre of gravity (or centroid).

Let \(X\) be a continuous random variable with probability density function \(f_X(x)\). We will say that \(X\) has a mathematical expectation

\[
E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx \tag{3.23}
\]

if

\[
\int_{-\infty}^{\infty} |x| \cdot f_X(x) \, dx < \infty \tag{3.24}
\]

For constant \(c\), the following properties the mathematical expectation:

1. \(E(c) = c\) \tag{3.25}

2. \(E(cX) = cE(X)\) \tag{3.26}

are readily established from the definition (3.23).

Example 3.8
If the probability density function of a continuous random variable \(X\) is given by:

\[
f_X(x) = \frac{a}{\pi \left(a^2 + x^2\right)}, (-\infty < x < \infty)
\]
we say that $X$ has a Cauchy distribution.

The distribution function $F_X(x)$ is:

$$F_X(x) = \frac{a}{\pi} \frac{x}{a^2 + x^2} = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{a}$$

$X$ has no mathematical expectation; in fact the integral

$$\int_{-\infty}^{\infty} \frac{a}{\pi} \frac{|x|}{a^2 + x^2} \, dx$$

does not exist.

Example 3.9

If the probability density function of a continuous random variable $X$ is given by:

$$f_X(x) = \frac{a}{2} e^{-a|x|}$$

we say that $X$ has a Laplace distribution.

$$F_X(x) = \begin{cases} 
\frac{1}{2} e^{ax}, & -\infty < x < 0 \\
1 - \frac{1}{2} e^{-ax}, & 0 \leq x < \infty 
\end{cases}$$

and its mathematical expectation is:

$$E(X) = \int_{-\infty}^{\infty} \frac{a}{2} x e^{-a|x|} \, dx = 0$$

In complete analogy with (3.22), the mathematical expectation of $Y = \varphi(X)$ is calculated as follows:

$$E[\varphi(X)] = \int_{-\infty}^{\infty} \varphi(x) f_X(x) \, dx$$

(3.27)

of that integral is absolutely convergent, i.e., if:
\[ \int_{-\infty}^{\infty} |\varphi(x)| f_X(x) \, dx \]  

is finite.

§ 3.5. Moments of random variable; Variance

A particular case of Eq. (3.27) is \( \varphi(x) = x^k \) where \( k \) is zero or a positive integer. For \( k = 0 \) we have unity, for \( k = 1 \) the mathematical expectation, for \( k > 1 \) - the so-called \( k \)-th moment:

\[ m_k = E(x^k) = \int_{-\infty}^{\infty} x^k f_X(x) \, dx \]  

The mathematical expectation is thus a first moment of a random variable. The mathematical expectation of \( [x - E(x)]^k \) is defined as the \( k \)-th central moment:

\[ \mu_k = E\{[x - E(x)]^k\} = \int_{-\infty}^{\infty} [x - E(x)]^k f_X(x) \, dx \]  

Obviously, the zero-th central moment equals unity, the first central moment - zero. The second central moment of a random variable is called variance (provided the integral in (3.30) is absolutely convergent) and denoted by \( \text{Var}(x) \):

\[ \text{Var}(x) = E\{[x - E(x)]^2\} = \int_{-\infty}^{\infty} [x - E(x)]^2 f_X(x) \, dx \]  

The mathematical expectation is a measure of the "average" of the values taken on by the random variable, whereas the variance is one of spread. The mean is the center of gravity of density, variance is the moment of inertia of the same density about an axis through the center of gravity.

For a discrete random variable, variance is defined by:

\[ \text{Var}(x) = \sum_{i=-\infty}^{\infty} [x_i - E(x)]^2 p(X = x_i) \]  

if the sum in the right number of (3.32) is finite.
It follows from the definitions (3.31) and (3.32) that:

\[ \text{Var}(X) = \mu_2 = \mathbb{E}\{(X - \mathbb{E}(X))^2\} = m_2 - 2[\mathbb{E}(X)]^2 + [\mathbb{E}(X)]^2 = m_2 - [\mathbb{E}(X)]^2 = m_2 - m_1^2 \]

so that:

\[ \text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \] (3.33)

This is equivalent to the parallel-axis theorem or Steiner's theorem in Statics, which relates the respective moments of inertia of a body about an arbitrary axis and about the central axis parallel to it.

The standard deviation of \( X \) is defined as \( \sqrt{\text{Var}(X)} \), and denoted by \( \sigma_X \):

\[ \sigma_X = \sqrt{\text{Var}(X)} \] (3.34)

For constant \( c \) the following properties of variance:

1. \( \text{Var}(c) = 0 \) \hspace{1cm} (3.35)

2. \( \text{Var}(cX) = c^2 \text{Var}(X) \) \hspace{1cm} (3.36)

are readily established from the definition (3.31).

**Example 3.10**

Consider the experiment of throwing an "honest" die, \( X \) denoting the number of spots on the upward-landing face:

\[ \text{Var}(X) = \sum_{i=1}^{6} \left( X_i - \mathbb{E}(X) \right)^2 p(X=X_i) = (1 - 3.5)^2 \cdot \frac{1}{6} + (2 - 3.5)^2 \cdot \frac{1}{6} + \]

\[ + (3 - 3.5)^2 \cdot \frac{1}{6} + (4 - 3.5)^2 \cdot \frac{1}{6} + (5 - 3.5)^2 \cdot \frac{1}{6} + \]

\[ + (6 - 3.5)^2 \cdot \frac{1}{6} = \frac{11}{12} \]

**Example 3.11**

It is readily shown that for a random variable with Cauchy distribution, no variance is defined (see also Example 3.8).
Example 3.12
The variance of a random variable \( X \) with Laplace distribution is (see Example 3.9):

\[
\text{Var}(X) = \int_{-\infty}^{\infty} \frac{a}{2} x^2 e^{-a|x|} \, dx = \frac{2}{a^2}
\]

Example 3.13
If the probability density function of a continuous random variable \( X \) is given by:

\[
f_X(x) = \begin{cases} 
\frac{1}{b-a}, & \text{for } a \leq x \leq b \\
0, & \text{otherwise}
\end{cases}
\]

that is, it represents a rectangular "pulse", we say that \( X \) is uniformly distributed in the interval \((a, b)\).

Using the definition of a distribution function (3.7), we obtain:

\[
P_X(x) = \begin{cases} 
0, & x < a \\
\frac{x-a}{b-a}, & a \leq x < b \\
1, & b \leq x
\end{cases}
\]

(see Fig. 3.8), the \( k \)-th moment is:

\[
m_k = \frac{1}{b-a} \int_a^b x^k \, dx = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}
\]

In particular, the mean equals:

\[
E(X) = m_1 = \frac{a + b}{2}
\]

the \( k \)-th central moment:

\[
\mu_k = \frac{1}{b-a} \int_a^b \left( x - \frac{a+b}{2} \right)^k \, dx = \begin{cases} 
0, & \text{for odd } k \\
\frac{(b-a)^k}{2^k(k+1)}, & \text{for even } k
\end{cases}
\]
and the variance $\text{Var}(X)$:

$$\text{Var}(X) = \mu_x^2 = \frac{(b - a)^2}{12}$$

This value equals numerically the central moment of inertia of a rectangular cross section with width $1/(b - a)$ and depth $b - a$.

The absolute $k$-th moment is defined by:

$$\tilde{m}_k = E(|X|^k) = \int_{-\infty}^{\infty} |x|^k f_X(x) \, dx \quad (3.37)$$

and the absolute central $k$-th moment by:

$$\tilde{\mu}_k = E[(X - E(X))^k] = \int_{-\infty}^{\infty} |x - E(X)|^k f_X(x) \, dx \quad (3.38)$$

The generalised $k$-th moment are defined, respectively by:

$$\bar{m}_k = E[(X - a)^k] \quad \bar{\mu}_k = E[|X - a|^k] \quad (3.39)$$

The median of a continuous random variable $X$, denoted by $\text{med}(X)$, is defined as the smallest root of the equation:

$$F_X[\text{med}(X)] = 0.5$$

Note, that the curves $y = F_X(x)$ and $y = 0.5$ may have more than one common point, the minimal of them being the median. The median has the following property:

If the absolute moment $E(|X - a|)$ of a continuous random variable $X$ is treated as a function of $a$, $X$ has a minimum at $a = \text{med}(X)$.

This property follows immediately from the following:

$$E(|X - a|) =
\begin{cases}
E(|X - \text{med}(X)|) + 2 \int_{\text{med}(X)}^{a} (a-x) f_X(x) \, dx, & \text{for } a > \text{med}(X) \\
E(|X - \text{med}(X)|) + 2 \int_{a}^{\text{med}(X)} (x-a) f_X(x) \, dx, & \text{for } a < \text{med}(X)
\end{cases}$$
The $q$-th quantile of a continuous random variable $X$ is defined as a smallest root of the equation:

$$F_X(x_q) = q, \quad \text{for } 0 < q < 1$$

Obviously, the median is the 0.5-th quantile.

The point at which $f_x(x)$ attains maximum is called mode.

Consider now the third central moment:

$$\mu_3 = \int_{-\infty}^{\infty} [x - E(X)]^3 f_x(x) \, dx$$

$$= \int_{-\infty}^{\infty} y^3 f_x(y + E(X)) \, dy$$

$$= \int_{0}^{\infty} y^3 \{f_x[E(x) + y] - f_x[E(x) - y]\} \, dy$$

where:

$$y = x - E(X)$$

If the probability density $f_x(x)$ is symmetric about $E(X)$, then:

$$f_x[E(x) + y] = f_x[E(x) - y]$$

and, consequently:

$$\mu_3 = 0$$

Therefore, if the probability density function is symmetric about the mean, the third central moment vanishes. The opposite is not always valid: for the probability distribution to be symmetric about the mean, all odd central moments must vanish. Despite this, the third moment $\mu_3$ is sometimes regarded as a measure of asymmetry or skewness. The coefficient of skewness being defined as the ratio $\gamma_1 = \mu_3/\mu_2^{3/2} = \mu_3/\sigma^3$. Fig. 3.9 shows examples
of probability densities with positive, zero and negative coefficients of skewness respectively.

The fourth central moment $\mu_4$ is used to estimate the steepness of the peak of the probability density near its center. The value:

$$\gamma_2 = \frac{\mu_4}{\sigma_X^4} - 3$$

is called the coefficient of excess, or kurtosis. $\gamma_2$ is constructed so as to equal zero for the very important normal probability density curve (see § 4.1.0). Probability density curves which are flatter at the peak than the normal have negative kurtosis, while those which are steeper than the latter have positive kurtosis. Fig. 3.10 shows an example of a normal distribution curve ($\gamma_2 = 0$) and probability densities with positive and negative kurtosis, respectively.

§ 3.6. Characteristic function

The characteristic function of a random variable $X$, denoted by $M_x(\theta)$, is defined as the mathematical expectation of the complex variable $e^{i\theta X}$, treated as a function of $\theta$:

$$M_x(\theta) = E[e^{i\theta X}] = \int_{-\infty}^{\infty} e^{i\theta x} f_x(x) \, dx$$

(3.40)

in other words, the characteristic function is the Fourier transform of the probability density function. Since $|e^{i\theta x}| = 1$ for every real $\theta$ and $x$, due to a property of the probability density function (3.11), the characteristic function does not exceed unity in its absolute value and equals unity at $\theta = 0$. Since $f_x(x)$ is nonnegative, the probability density function can be found as the inverse Fourier transform of the characteristic function:

$$f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_x(\theta) e^{-i\theta x} \, d\theta$$

(3.41)

Example 3.14

The characteristic function of a random variable, distributed uniformly in the interval $(a, b)$, equals:
\[ M_X(\theta) = \frac{1}{b-a} \int_a^b e^{i\theta x} \, dx = \frac{1}{i\theta (b-a)} (e^{i\theta b} - e^{i\theta a}) \]

Example 3.15
The characteristic function of a random variable with Laplace distribution, equals:

\[ M_X(\theta) = \frac{a}{2} \int_{-\infty}^{\infty} e^{i\theta x - a|x|} \, dx = \frac{a^2}{a^2 + \theta^2} \]

Example 3.16
The characteristic function \( M_X(\theta) \) of a random variable \( Y = aX + b \), where \( a \) and \( b \) are real numbers and \( X \) is a continuous random variable, is found as:

\[ M_Y(\theta) = M_X(a\theta) e^{ib\theta} \quad (3.42) \]

since:

\[ M_Y(\theta) = E(e^{i\theta Y}) = E[e^{i\theta(aX+b)}] = E[e^{ia\theta X} e^{i\theta b}] = M_X(a\theta) e^{ib\theta} \]

Here property (3.35) of the mathematical expectation was used.

If a random variable \( X \) has an absolute moment of \( k \)-th order, its characteristic function is differentiable \( k \) times. Conversely, the value of the \( k \)-th derivative of the characteristic function \( M_X(\theta) \) at \( \theta = 0 \) determines the \( k \)-th moments \( m_k \) of a random variable \( X \). Indeed, after \( k \)-fold differentiation of \( M_X(\theta) \) we obtain:

\[ \frac{d^k M_X(\theta)}{d\theta^k} = i^k \int_{-\infty}^{\infty} x^k e^{i\theta x} f_X(x) \, dx \quad (3.42) \]

This derivative can be estimated by its absolute value:

\[ \left| \frac{d^k M_X(\theta)}{d\theta^k} \right| = \left| \int_{-\infty}^{\infty} x^k e^{i\theta x} f_X(x) \, dx \right| \leq \int_{-\infty}^{\infty} x^k e^{i\theta x} f_X(x) \, dx = \int_{-\infty}^{\infty} |x|^k f_X(x) \, dx \]
Since the latter integral is finite by assumption, the derivative $d^k M_X(\theta)/d\theta^k$ exists.

Equation (3.42) yields:

$$
\left[ \frac{d^k M_X(\theta)}{d\theta^k} \right]_{\theta=0} = i^k \int_{-\infty}^{\infty} x^k f_X(x) \, dx = i^k m_k
$$

Hence

$$
m_k = \frac{1}{i^k} \left[ \frac{d^k M_X(\theta)}{d\theta^k} \right]_{\theta=0}
$$

The mean of a random variable equals then:

$$
m_1 = E(X) = \frac{1}{i} \left[ \frac{d M_X(\theta)}{d\theta} \right]_{\theta=0}
$$

(3.44)

If all derivatives of the characteristic function $M_X(\theta)$ exist at $\theta = 0$, it can be represented by a Maclaurin-series expansion:

$$
M_X(\theta) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{d^k M_X(\theta)}{d\theta^k} \right]_{\theta=0} \theta^k
$$

(3.45)

or, with Eq. (3.43) taken into account, we obtain:

$$
M_X(\theta) = 1 + \sum_{k=1}^{\infty} \frac{(i\theta)^k}{k!} m_k
$$

The principal value of the logarithm of a characteristic function is called a log-characteristic function:

$$
\psi_X(\theta) = \ln M_X(\theta)
$$

(3.46)

Let us differentiate $\psi_X(\theta)$ and put $\theta = 0$. We have:
\[
\frac{d\psi_X(0)}{d\theta} \bigg|_{\theta=0} = \frac{1}{M_X(0)} \frac{dM_X(\theta)}{d\theta} \bigg|_{\theta=0} = im_1 = iE(X)
\]

\[
\frac{d^2 \psi_X(\theta)}{d\theta^2} \bigg|_{\theta=0} = \frac{1}{M_X(0)} \left\{ \frac{d^2 M_X(\theta)}{d\theta^2} - \left[ \frac{dM_X(\theta)}{d\theta} \right]^2 \right\} \bigg|_{\theta=0}
= -m_2 + n_1^2 = -\text{Var}(X)
\]

We used here the fact \( M_X(0) = 1 \) and Eq. (3.43). As a result we obtain:

\[
E(X) = -i \left[ \frac{d\psi_X(\theta)}{d\theta} \right]_{\theta=0} \tag{3.47}
\]

\[
\text{Var}(X) = - \left[ \frac{d^2 \psi_X(\theta)}{d\theta^2} \right]_{\theta=0} = - \left[ \frac{d^2 \ln M_X(\theta)}{d\theta^2} \right]_{\theta=0} \tag{3.48}
\]

Further differentiation of \( \psi_X(\theta) \) yields:

\[
\left[ \frac{d^3 \psi_X(\theta)}{d\theta^3} \right]_{\theta=0} = -(m_3 - 3m_1 m_2 + 2m_1^3) / 1^3
\]

\[
\left[ \frac{d^4 \psi_X(\theta)}{d\theta^4} \right]_{\theta=0} = m_4 - 4m_1 m_3 - 3m_2^2 + 12m_1^2 m_2 - 6m_1^4
\]

For the coefficients of skewness and kurtosis we obtain, respectively:

\[
\gamma_1X = \psi_X^{III}(0) / [\psi_X^{II}(0)]^{3/2} \tag{3.49}
\]

\[
\gamma_2X = \psi_X^{IV}(0) / [\psi_X^{II}(0)]^2 \tag{3.50}
\]

The value \( i^k \frac{d^k \psi_X(\theta)}{d\theta^k} \) is called the \( k \)-th cumulative or semi-invariant of \( X \). With semi-invariants known the various moments of a random variable are readily obtainable. It should be noted that moments (if they exist) are uniquely determined through the probability density function (or the characteristic function, or the log-characteristic function). Accordingly, the question arises whether the set of moments determines uniquely the probability density function of a random variable, and the answer is generally "no"; this is known as the problem of moments and will not be discussed here (on this subject the reader may consult, for example, Kendall and Stuart).
§ 3.7. Conditional probability distribution density functions and mathematical expectation

First, we recall Eq. (2.10), which states that if \( B \) is an event with nonzero probability, then the conditional probability of an event \( A \), knowing that \( B \) has taken place, is:

\[
P(A|B) = \frac{P(AB)}{P(B)}
\]

The conditional distribution function of the random variable \( X \) under condition \( B \) is defined as the probability of the event \( \{X \leq x\} \):

\[
F_X(x|B) = P(X \leq x|B) = \frac{P(X \leq x, B)}{P(B)}
\]  \hspace{1cm} (3.51)

where \( \{X \leq x, B\} \) is the product of events \( \{X \leq x\} \) and \( B \).

All properties of the unconditional distribution functions are preserved in the conditional distribution function:

\[
F_X(\pm \infty|B) = 1, \quad F_X(-\infty|B) = 0
\]

and

\[
P(x_1 < X \leq x_2|B) = F(x_2|B) - F(x_1|B)
\]  \hspace{1cm} (3.52)

If \( X \) is a continuous random variable, we will define the conditional probability density function \( f_X(x|B) \) as the derivative of \( F_X(x|B) \):

\[
f_X(x|B) = \frac{df}{dx}(x|B) = \lim_{\Delta x \to 0} \frac{P(x \leq X \leq x + \Delta x|B)}{\Delta x}
\]

If events \( B_1, B_2, \ldots, B_n \) partition the sample space \( \Omega \) we have by (2.25):

\[
P(X \leq x) = P(X \leq x|B_1)P(B_1) + P(X \leq x|B_2)P(B_2) + \cdots + P(X \leq x|B_n)P(B_n)
\]

In view of the definition (3.51), we arrive at:

\[
F_X(x) = \sum_{i=1}^{n} F_X(x|B_i)P(B_i)
\]  \hspace{1cm} (3.53)
Example 3.17

Assume that the event B is the following:

\[ \{B\} = \{X \leq a\}, \quad P(X \leq a) = F_X(a) \neq 0 \]

We seek the conditional distribution function \( F_X(\cdot | X \leq a) \):

\[ F_X(x | X \leq a) = P(X \leq x | X \leq a) \]

Note that if \( x \geq a \), then the event \( (X \leq x | X \leq a) \) is a certain one and

\[ F_X(x | X \leq a) = 1 \]

If \( x < a \), then

\[ F_X(x | X \leq a) = F_X(x | X \leq a) = \frac{P((X \leq x), (X \leq a))}{P(X \leq a)} = \frac{P(X \leq x)}{P(X \leq a)} = \frac{F_X(x)}{F_X(a)} \]

The conditional probability density function is found by differentiating the latter equation:

\[ f_X(x | X \leq a) = \begin{cases} \frac{f_X(x)}{F_X(a)}, & x < a \\ 0, & x \geq a \end{cases} \]

As an example, consider the random variable distributed uniformly in the interval \((0,2a)\). Then:

\[ F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2a}, & 0 \leq x < 2a \\ 1, & 2a \leq x \end{cases} \]

The conditional probability distribution is (see Fig. 3.11):
\[ F_X(x | X \leq a) = \begin{cases} 
0, & x < 0 \\
\frac{x}{a}, & 0 \leq x < a \\
1, & a \leq x 
\end{cases} \]

since \( F_X(a) = 1/2. \)

The conditional mathematical expectation of a random variable \( X \), under condition \( B \), is defined as:

\[ E(X | B) = \int_{-\infty}^{\infty} x f_X(x | B) \, dx \quad (3.54) \]

where \( f_X(x | B) \) is the conditional probability density function.

Differentiating Eq. (3.53) and taking (3.54) into account, we obtain the relation between the unconditional mathematical expectation \( E(X) \) and the conditional ones \( E(X | B_i) \) under conditions \( B_i \), respectively, random variables partitioning \( \Omega \):

\[ E(X) = \sum_{i=1}^{n} E(X | B_i) P(B_i) \quad (3.55) \]

§ 3.8. Inequalities of Bienaymé and Tchebycheff

Let \( Y \) be a random variable taking on only non-negative values. Then the following inequality is valid, provided \( \alpha \) is a positive constant:

\[ P(Y \geq \alpha) \leq \frac{E(Y)}{\alpha} \quad (3.56) \]

Indeed, if \( f_Y(y) \) is a probability density function of a continuous random variable

\[ E(Y) = \int_{-\infty}^{\infty} y f_Y(y) \, dy = \int_{0}^{\alpha} y f_Y(y) \, dy + \int_{\alpha}^{\infty} y f_Y(y) \, dy \]

\[ \geq \int_{\alpha}^{\infty} y f_Y(y) \, dy \geq \alpha \int_{\alpha}^{\infty} f_Y(y) \, dy = \alpha P(Y \geq \alpha) \]

which was to be proved, here the fact was used that \( f_Y(y) \) vanishes for negative values of \( y \).
Consider now the random variable \( Y = |X-a|^n \), where \( n \) is a positive integer. This random variable takes on only non-negative values irrespective of the sign of \( X \). Therefore for \( a = \epsilon^n (\epsilon > 0) \):

\[
P(|X - a|^n \leq \epsilon^n) \leq \frac{E[|X - a|^n]}{\epsilon^n}
\]

that is:

\[
P(|X - a| \leq \epsilon) \leq \frac{E[|X - a|^n]}{\epsilon^n}
\]

This inequality, named after Bienaymé, signifies that:

\[
P(|X - a| \geq \epsilon) \leq \min_n \frac{E[|X - a|^n]}{\epsilon^n}
\]

If a random variable has a finite variance \( \sigma_X^2 \), then we may put in the inequality \( a = E(X) \), \( n = 2 \) and \( \epsilon = k \sigma_X \) to get:

\[
P(|X - E(X)| \geq k \sigma_X) \leq \frac{1}{k^2}
\]

This inequality, named after Tchebycheff, signifies also that:

\[
P(|X - E(X)| < k \sigma_X) \geq 1 - \frac{1}{k^2}
\]

For example, for \( k = 2 \) we obtain:

\[
P \{ E(X) - 2\sigma_X < X < E(X) + 2\sigma_X \} \geq \frac{3}{4}
\]

for any random variable \( X \) with finite variance. For \( k = 3 \):

\[
P \{ E(X) - 3\sigma_X < X < E(X) + 3\sigma_X \} \geq \frac{8}{9}
\]

for any random variable \( X \) with finite variance, the latter inequality signifies that the probability of \( X \) falling within three standard deviations of its mean is at least \( 8/9 \). This bound is independent of the distribution of \( X \), provided it has a finite variance.

**Cited references**


Recommended Further Reading


PROBLEMS

3.1. A spring-mass system is subjected to harmonic sinusoidal excitation with specified frequency $\omega$. Suppose the spring coefficient $k$ is a random variable with given probability distribution $F_k(k)$, where the mass $m$ is a specified quantity. What is the probability of no resonance occurring in a system picked up at random?

3.2. A random variable $X$ is said to have a **triangular distribution**, if its density is:

$$f_X(x) = \begin{cases} A \left(1 - \frac{|x|}{a}\right), & \text{for } -a \leq x \leq a \\ 0, & \text{otherwise} \end{cases}$$

Find:

(a) the value of $A$,

(b) the mean value, $E(X)$,

(c) the median, $\text{med}(X)$,

(d) the coefficient of variation, $\gamma_X = \sigma_X / E(X)$,

(e) the 0.9-th quantile, $x_{0.9}$.

3.3. A random variable $X$ is said to have a **beta-distribution**, if its density is:

$$f_X(x) = \begin{cases} A x^{\alpha-1} (1 - x)^{\beta-1}, & \text{for } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find:

(a) the value of $A$,

(b) the distribution function associated with this probability density,

(c) the probability of $X$ taking on values less than 0.1,

(d) the values of $\alpha$ and $\beta$ for which a uniformly distributed random variable is obtained from $X$. 
3.4. The hazard function \( h(t) \) is defined as the conditional instantaneous failure rate, that is, \( h(t) \, dt \) is the probability of the system failing in the time interval \((t, t+dt)\), given that the system has not failed prior to time \( t \). It may be interpreted as the rate at which the system population still under test at time \( t \) is failing. Let \( T \) denote the random time of failure with density \( f_T(t) \), such that \( f_T(t) \, dt \) represents the probability of the system failing in the interval \((t, t+dt)\).

(a) Show that:

\[
h(t) = f_T(t \mid T \geq t)
\]

(b) Show that the hazard function at time \( t \) equals the probability density of failure divided by the reliability \( R(t) \), the probability of the system surviving up to \( t \):

\[
h(t) = \frac{f_T(t)}{R(t)}, \quad f_T(t) = F_T'(t), \quad R(t) = 1 - F_T(t)
\]

(c) Verify that:

\[
R(t) = [1 - F_T(0)] \exp \left[ - \int_0^t h(t) \, dt \right]
\]

\[
f_T(t) = [1 - F_T(0)] h(t) \exp \left[ - \int_0^t h(t) \, dt \right]
\]

where \( F_T(0) \) is the probability of failure at \( t = 0 \).

Remark: Since \( 0 \leq R \leq 1 \), \( h(t) \geq f_T(t) \). By analogy, the probability of a specimen subjected to a fatigue test with sufficiently high amplitude of the retested load, fracturing between \( 10^9 \) cycles and \( 10^9 + 10 \) cycles [corresponding to \( f_T(t) \, dt \)] is very small. The probability of fracture in the same interval, provided the specimen survived up to \( 10^9 \) cycles [corresponding to \( h(t) \, dt \)] is much higher.

3.5. Find the probability density of the time of failure \( f_T(t) \), if the hazard function is constant \( h(t) = a \), and the probability of failure is zero, \( F_T(0) = 0 \). Show that \( a \) is the reciprocal of the mean time of failure, \( E(T) \).
3.6. Find the conditional probability of a system failing in the time interval \((t_1, t_2)\), assuming that it did not fail prior to time \(t_1\); \(h(t) = a\).

3.7. Find the conditional probability of a system surviving in the time interval \((t_1, t_2)\) assuming that it survived up to time \(t_1\) [probability of prolongation by an additional time interval \(\Delta t = t_2 - t_1\)]; \(h(t) = a\).

3.8. Verify that if \(X\) is uniformly distributed in the interval \((a, b)\), the probability of \(X \leq a + p(b - a)\), where \(0 < p < 1\), equals \(p\).

3.9. Following the steps used in the text, prove the inequalities of Bienaymé and Tchebycheff for a discrete random variable.

3.10. If \(X\) is uniformly distributed in the interval \([8, 12]\). Calculate the following probability:

\[ P\{E(X) - \sigma_X < X < E(X) + \sigma_X\} \]

and compare it with the upper bound furnished by Tchebycheff's inequality.
CHAPTER 4 - EXAMPLES OF PROBABILITY - DISTRIBUTION AND DENSITY FUNCTIONS

FUNCTIONS OF SINGLE RANDOM VARIABLE

In this chapter we present some widely used discrete and continuous probability-distribution and density functions.

§ 4.1. Causal Distribution

In order to represent the constant \( c \) probabilistically, we use a causally-distributed random variable in the following probability density function (Fig. 4.1):

\[
f_X(x) = \delta(x - c)
\]

(4.1)

that is, the random variable \( X \) takes on the value \( c \) with probability unity. The distribution function is readily obtained by integration of (4.1):

\[
F_X(x) = U(x - c)
\]

The mathematical expectation is obviously:

\[
E(X) = c
\]

and the variance:

\[
\text{Var}(X) = 0
\]

The characteristic function is:

\[
M_X(\theta) = E(e^{i\theta X}) = e^{ic\theta}
\]

§ 4.2. Discrete Uniform Distribution

A random variable \( X \) has a discrete uniform distribution, if its probability density function reads (Fig. 4.2):

\[
f_X(x) = \frac{1}{n} \sum_{i=1}^{n} \delta(x - c_i)
\]

(4.2)
that is, \( X \) can take on only values \( c_1, c_2, \ldots, c_n \) each with probability \( 1/n \). The distribution function is:

\[
F_X(x) = \frac{1}{n} \sum_{i=1}^{n} U(x - c_i)
\]

The mathematical expectation is:

\[
E(X) = \sum_{i=1}^{n} c_i P(X = c_i)
\]

For the particular case \( c_i = i \), we get for the mathematical expectation:

\[
E(X) = \sum_{i=1}^{n} i \frac{1}{n} = \frac{n + 1}{2}
\]

and for the variance:

\[
\text{Var}(X) = E(X^2) - [E(X)]^2 = \sum_{i=1}^{n} i^2 \frac{1}{n} - \left( \frac{n + 1}{2} \right)^2
\]

\[
= \frac{n(n+1)(2n+1)}{6n} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}
\]

§ 4.3. Binomial or Bernoulli Distribution

Independent trials, each of which involves an event \( A \), occurring with positive probability \( p = P(A) \), are called **Bernoulli trials**. The event itself is referred to as a "success" and the complementary event \( \bar{A} \), likewise occurring in each of the trials with probability \( q = 1 - p \) as a "failure", i.e. \( p = P(\text{success}) \), \( q = P(\text{failure}) \).

If \( n \) trials are considered, then each elementary outcome \( \omega \) can be described by a mixed sequence of "successes" and "failures", for example \( (s, s, f, f, f, s, \ldots, f, s) \). The probability \( P(\omega) \) of each elementary outcome \( \omega \), at which there are exactly \( m \) successes and \( n - m \) failures, is given - in view of the mutual independence of the outcomes - by:

\[
P(\omega) = p^m q^{n-m}
\]

As can be seen, the elementary outcomes are equiprobable if \( p = q = 1/2 \).

Consider now the random variable \( X \), denoting the number of successes in \( n \) Bernoulli trials. \( X(\omega) = m \), if an elementary outcome indicates exactly \( m \) successes. The number of different outcomes \( \omega \), resulting in \( m \) successes in any sequence, equals that of combinations of \( m \) "s"'s and \( n-m \) "f"'s which in turn equals that of combinations of \( m \) objects drawn from an ensemble of \( n \) objects:
\[ \binom{n}{m} = \frac{n!}{m!(m-n)!} \]

All these outcomes have the same probability \( P(\omega) \), hence the event \( (X = m) \) has the probability:

\[ P(X = m) = \binom{n}{m} p^m q^{n-m} \quad , \quad m = 0, 1, 2, \ldots, n \quad (4.3) \]

The probability density function of \( X \) is

\[ f_X(x) = \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} \delta(x - m) \quad (4.4) \]

Note that since all possible mutually exclusive outcomes of \( n \) trials consist in occurrence of a success 0 times, once, twice, \ldots, \( n \) times, it is obvious that:

\[ \sum_{m=0}^{n} P(X = m) = 1 \]

which also follows from the equality:

\[ \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} = (p + q)^n = 1 \]

The mathematical expectation of \( X \) equals:

\[ E(X) = \sum_{m=0}^{n} m \binom{n}{m} p^m q^{n-m} = \sum_{m=1}^{n} m \binom{n}{m} p^m q^{n-m} \]

Since, however:

\[ \binom{n}{m} = \frac{(n-1)}{m-1} \binom{n}{m} \]

we have:

\[ E(X) = n p \sum_{m=1}^{n} \binom{n-1}{m-1} p^{m-1} q^{n-1-(m-1)} \]

\[ = n p \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{n-1-k} = n p \]

The variance is

\[ \text{Var}(X) = E(X^2) - [E(X)]^2 \]
so that

\[ E(X^2) = \sum_{m=1}^{n} m^2 \binom{n}{m} p^m q^{n-m} = n p \sum_{m=1}^{n} \binom{n-1}{m-1} p^{m-1} q^{n-1-(m-1)} \]

\[ = n p \sum_{k=0}^{n-1} (k+1) \binom{n-1}{k} p^k q^{n-k-1} \]

\[ = n p \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{n-k-1} + n p \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{n-k-1} \]

\[ = n p \left[ (n-1) p \right] + n p \quad (4.5) \]

It was taken into account that the first sum in Eq. (4.5) represents the mathematical expectation of \( X \) in \( n - 1 \) trials, and the second sum equals unity as the probability of a certain event. As a result,

\[ \text{Var}(X) = n p (n - 1) p + np - (np)^2 = np(1 - p) = npq. \]

**Example 4.1**

A device consists of 5 components the reliability of which equals \( p \) and the failures of which are mutually independent. Then the probability of none of the components failing equals \( P_0 = p^5 \), that of at least one of the components failing equals \( 1 - p^5 \); that of exactly one component failing \( - P_1 = \binom{5}{1} p^4 q = 5pq \); that of two components failing \( - P_2 = \binom{5}{2} p^3 q^2 = 10 p^3 q^2 \); that of at least two components failing \( 1 - P_0 - P_1 = 1 - p^5 - 5pq \).

**§ 4.4. Poisson Distribution**

A random variable \( X \) is said to have a Poisson distribution if it takes on values 0, 1, 2, ..., with probabilities:

\[ P(X = m) = \frac{a^m}{m!} e^{-a}, \quad m = 0, 1, ... \quad (4.6) \]

\( a \) being a positive constant.

The probability density function is
\[ f_X(x) = e^{-a} \sum_{m=0}^{\infty} \frac{a^m}{m!} \delta(x - m) \]

The characteristic and the log-characteristic functions are, respectively:

\[ M_X(\theta) = E(e^{i\theta X}) = e^{-a} \sum_{m=0}^{\infty} \frac{(ae^{i\theta})^m}{m!} = e^{-a} \sum_{m=0}^{\infty} \frac{(ae \cdot e^{i\theta})^m}{m!} = e^{-a} e^{ae e^{i\theta}} = e^{ae^{i\theta} - a} \]

\[ \psi_X(\theta) = ae^{i\theta} - a \]

The mathematical expectation and variance are obtainable for example, from eqs. (3.47) and (3.48) respectively:

\[ \mathbb{E}(X) = -i \psi_X'(0) = a \]

\[ \text{Var}(X) = -\psi_X''(0) = a \]

The Poisson distribution may be regarded as a limiting case of the binomial one, where the number of trials is large and the probability of success very small but the mean number of successes \( a = np \) not too small - in which case it can be shown that:

\[ \binom{n}{m} p^m q^{n-m} \frac{a^m}{m!} e^{-a} \]

Indeed, as is known:

\[ \lim_{n \to \infty} \left( 1 - \frac{a}{n} \right)^n = e^{-a} \]

and since \( p = \frac{a}{n} \), we obtain from Eq. (4.3):

\[ p(X = 0) = q^n = (1 - p)^n = \left( 1 - \frac{a}{n} \right)^n \approx e^{-a} \]

and moreover:

\[ \frac{p(X = m)}{p(X = m - 1)} = \frac{mp - (m - 1)p}{mq} \approx \frac{a}{m}, \quad m = 1, 2, \ldots \]

when \( n \to \infty \). Therefore
\[ P(X = 1) \sim \frac{a}{1} \quad P(X = 0) \sim \frac{a}{1} e^{-a} \]
\[ P(X = 2) \sim \frac{a}{2} \quad P(X = 1) \sim \frac{a^2}{1,2} e^{-a} \]
\[ \quad \ldots \]
\[ P(X = m) \sim \frac{a}{m} \quad P(X = m - 1) \sim \frac{a^k}{k!} e^{-a} \]

which is the Poisson distribution.

§ 4.5. Rayleigh Distribution

A continuous random variable \( X \) is said to have a Rayleigh distribution, if its probability density function is given by (Fig. 4.3):

\[
f_X(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\frac{x}{a^2} e^{-x^2/2a^2} & \text{for } x \geq 0
\end{cases}
\]  

(4.7)

with parameter \( a^2 \).

The distribution function is:

\[ F_X(x) = [1 - e^{-x^2/2a^2}] U(x) \]

The mathematical expectation and the variance are, respectively:

\[ E(X) = \frac{\sqrt{\pi} a}{\sqrt{2}} \approx 1.25 a \quad \text{Var}(X) = \frac{4 - \pi}{2} a^2 \approx 0.43 a^2 \]

The Rayleigh distribution is an example of a nonsymmetric one. The third moment \( m_3 \) equals:

\[ m_3 = \frac{1}{a^2} \int_0^\infty x^4 e^{-x^2/2a^2} \, dx = 3a^3 \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \]

and the third central moment equals:

\[ \mu_3 = 3a^3 \left( \frac{\pi}{2} \right)^{\frac{3}{2}} - 3a \left( \frac{\pi}{2} \right)^{\frac{3}{2}} 2a^2 + 2a^3 \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \]

\[ = (\pi - 3) \left( \frac{\pi}{2} \right)^{\frac{3}{2}} a^3 \]

the coefficient of skewness being:
\[ Y_{1x} = \frac{2}{4} \left( \frac{\pi}{4-\pi} \right)^{1/2} \approx 0.63 \]

§ 4.6. Exponential Distribution

A random variable \( X \) is said to have an exponential distribution, if its density \( f_X(x) \) is given by

\[
f_X(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\alpha e^{-\alpha x} & \text{for } x \geq 0
\end{cases}
\]  

(4.8)

the distribution function being

\[ F_X(x) = [1 - e^{-\alpha x}] U(x) \]

and the characteristic function

\[ M_X(\theta) = \frac{\alpha}{\alpha - i\theta} \]

The mathematical expectation and the variance are, respectively:

\[ E(X) = \frac{1}{\alpha}, \quad \text{Var}(X) = \frac{1}{\alpha^2} \]

§ 4.7. \( \chi^2 \) (chi-square) Distribution with \( m \) Degrees of Freedom

A random variable is said to have this distribution if its density is given by

\[
f_X(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\frac{e^{-x/2} x^{\lambda - 1}}{2^{\lambda} \Gamma(\lambda)} & \text{for } x \geq 0, \quad (\lambda = m/2)
\end{cases}
\]  

(4.9)

where \( m \) is the so-called number of degrees of freedom; and \( \Gamma(x) \) = a gamma function defined as:

\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \quad \text{for } x > 0
\]

For \( x > 1 \) we obtain, after integration by parts,

\[
\Gamma(x) = (x - 1) \, \Gamma(x - 1)
\]
moreover, $\Gamma(1) = 1$; hence, for $x$ a positive integer, we have:

$$\Gamma(x) = (x - 1)!$$

The mathematical expectation and the variance are, respectively,

$$E(X) = m, \quad \text{Var}(X) = 2m$$

§ 4.8. Gamma Distribution

A random variable $X$ is said to have a gamma distribution if its density is given by (Fig. 4.4):

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{\Gamma(\alpha + 1)} \frac{\alpha}{\beta} e^{-x/\beta} & \text{for } x \geq 0 \end{cases}$$

(4.10)

where $\alpha > -1$, $\beta > 0$ are constants. For positive $\alpha$ (4.10) takes form for $x \geq 0$

$$f_X(x) = \frac{x^{\alpha} e^{-x/\beta}}{\Gamma(\alpha + 1)} U(x)$$

The characteristic function reads:

$$M_X(\theta) = \frac{1}{(1 - i\theta \beta \alpha + 1)}$$

The mathematical expectation and the variance are, respectively

$$E(X) = (\alpha + 1)\beta, \quad \text{Var}(X) = (\alpha + 1) \beta^2$$

Note that for $\alpha = 0$, a gamma distribution reduces to an exponential one with $\lambda = 1/\beta$; for $\beta = 2, \alpha = \frac{m}{2}$ it reduces to a $\chi^2$ one with $m$ degrees of freedom.

§ 4.9. Weibull Distribution

A random variable $X$ is said to have a Weibull distribution, if its density is given by

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{\alpha \beta}{\alpha} \left(\frac{x}{\beta}\right)^{\beta - 1} e^{-\alpha x^\beta} & \text{for } x \geq 0 \end{cases}$$

(4.11)
where \( \alpha \) and \( \beta \) are positive constants; the mathematical expectation and the variance are, respectively,

\[
E(X) = \frac{1}{\alpha^{1/\beta}} \Gamma \left( 1 + \frac{1}{\beta} \right), \quad \text{Var}(X) = \frac{1}{\alpha^{2/\beta}} \left[ \Gamma \left( 1 + \frac{2}{\beta} \right) - \Gamma^2 \left( 1 + \frac{1}{\beta} \right) \right]
\]

Note that for \( \beta = 1 \), the Weibull distribution reduces to an exponential one, and for \( \beta = 2 \), \( \alpha = 1/2a^2 \) to a Rayleigh one.

\[ § 4.10. \text{Normal or Gaussian Distribution} \]

A random variable \( X \) is said to have a normal or Gaussian distribution if its density is given by:

\[
f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-a}{\sigma} \right)^2}, \quad \text{for } -\infty < x < \infty \quad (4.12)
\]

The relevant distribution function (inexpressible in terms of elementary functions - polynomial, trigonometric, or exponential) is:

\[
P_X(x) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{x} \exp \left\{ -\frac{1}{2} \left( \frac{x-a}{\sigma} \right)^2 \right\} \, dx
\]

\[
= \frac{1}{2} + \text{erf} \left( \frac{x-a}{\sigma \sqrt{2}} \right); \quad P_X(a) = \frac{1}{2} \quad (4.13)
\]

where \( \text{erf}(x) \) is the so-called error function, defined as:

\[
\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-y^2/2} \, dy \quad (4.14)
\]

Note that this function is odd:

\[
\text{erf}(-x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{-x} e^{-y^2/2} \, dy
\]

\[
= -\frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-z^2/2} \, dz = -\text{erf}(x)
\]
In view of the familiar integral
\[ \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{\alpha}} \]  
(4.15)

it can be shown that:
\[ \text{erf}(\infty) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y^2/2} dy = \frac{1}{2} \]

Comprehensive tables of erf(x) are readily available; some values are listed in Appendix B. Note that often (also in computer subroutines) the error function is defined in a different way, namely
\[ \text{erf}^*(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\xi^2} d\xi \]  
(4.16)

The connection between erf(x) and erf^*(x) is as follows:
\[ \text{erf}(x) = \frac{1}{2} \text{erf}^* \left( \frac{x}{\sqrt{2}} \right) \]  
(4.17)

A normal distribution is often denoted by \( N(a, \sigma_x^2) \) and the parameters \( a \) and \( \sigma_x^2 \) can be shown to represent the mathematical expectation and variance of \( X \). Indeed:
\[ E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \frac{1}{\sigma_X \sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp \left[ -\frac{1}{2} \left( \frac{x-a}{\sigma_X} \right)^2 \right] dx \]

or, introducing a new variable \( \xi = (x-a)/\sigma_X \),
\[ E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_X (\xi + a) \exp \left( -\frac{1}{2} \xi^2 \right) d\xi \]
\[ = \frac{\sigma_X}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi \exp \left( -\frac{1}{2} \xi^2 \right) d\xi = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \xi^2 \right) d\xi \]

the first integral vanishes (the integrand being odd) and the second equals \( \sqrt{2\pi} \) by Eq. (4.15). Thus:
\[ E(X) = a \]  
(4.18)

This also follows immediately from the fact that a normal density (4.12) is symmetric about \( x = a \). For the variance we obtain:
\[ \text{Var}(X) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi} \sigma_X} \exp \left[ -\frac{(x-a)^2}{2\sigma_X^2} \right] dx = \frac{\sigma_X^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi^2 \exp \left( -\frac{1}{2} \xi^2 \right) d\xi = \sigma_X^2 \]  
(4.19)
thus justifying use of the symbol $\sigma_X^2$ for this parameter. The graph of a normal density function is shown in Fig. 4.5a. The curve is symmetrical about $a$ and descends steeply as $(x - a)$ increases. At $x = a$ the curve has a maximum equal to $1/\sqrt{2\pi}$; this ordinate increases as $\sigma_X$ decreases.

A normal random variable with zero mean and unity variance $N(0,1)$ is called \textbf{standard or normalised}, and its density and distribution function are, respectively:

$$
\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \quad P(x) = \int_{-\infty}^{x} \phi(x) \, dx = \frac{1}{2} + \text{erf}(x)
$$

(4.20)

For $P(x)$ following approximate formula holds (see Abramowitz and Stegun):

$$
P(x) = 1 - \phi(x) \sum_{i=1}^{5} b_i t^i, \quad t = \frac{1}{1 + px},
$$

$p = 0.2316419, \quad b_1 = 0.319381530, \quad b_2 = -0.356563782, \quad b_3 = 1.781477937, \quad b_4 = -1.821255978, \quad b_5 = 1.330274429

The central $k$-th moment of a normally distributed random variable, is given by:

$$
\mu_k = \int_{-\infty}^{\infty} (x - a)^k f_X(x) \, dx = \sigma_X^k \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \xi^2\right) \, d\xi
$$

Integration by parts yields:

$$
\mu_k = (k - 1) \sigma_X^2 \left[ \sigma_X^{k-2} \int_{-\infty}^{\infty} \xi^{k-2} \exp\left(-\frac{1}{2} \xi^2\right) \, d\xi \right]
$$

or

$$
\mu_k = (k - 1) \sigma_X^2 m_{k-2}
$$

Since

$$
\mu_1 = 0 \quad \text{and} \quad m_2 = \sigma_X^2
$$

all odd central moments vanish, while the even ones equal:

$$
\mu_4 = 3\sigma_X^4 = 3\mu_2^2
$$

$$
\mu_6 = 15\sigma_X^6
$$

$$
\vdots
$$

$$
\mu_{2k} = \binom{2k - 1}{k} \sigma_X^{2k}, \quad (k = 1, 2, \ldots)
$$

(4.21)
where

\[(2k - 1) !! = 1 \cdot 3 \cdot 5 \cdots (2k - 1)\]

The absolute k-th moments \(\tilde{m}_k\) as per (3.37), equal

\[
\tilde{m}_k = E(|X|^k) = \begin{cases} 
(2r - 1) !! \sigma_X^{2r}, & \text{for } k = 2r \\
\left(\frac{2}{\pi}\right)^{r} \cdot \Gamma(r) \cdot \sigma_X^{2r+1}, & \text{for } k = 2r + 1 
\end{cases}
\]

Note that for a normal random variable \(u_X\) and \(\tilde{m}_k\) are functions of the variance.

The characteristic function of a normally distributed random variable is

\[
M_X(\theta) = \frac{1}{\sigma_X \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[i\theta x - \frac{(x-a)^2}{2\sigma_X^2}\right] \, dx \\
= e^{i\theta a} \frac{1}{\sigma_X \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[i\theta t - \frac{t^2}{2\sigma_X^2}\right] \, dt
\]

Recalling the equality (see Appendix C)

\[
\int_{-\infty}^{\infty} e^{at-\frac{1}{2}a^2t^2} \, dt = \frac{\sqrt{\pi}}{a} e^{\frac{1}{4}a^2} 
\]

we have

\[
M_X(\theta) = \exp \left(i\theta a - \frac{1}{2} \sigma_X^2 \theta^2\right); \quad \psi_X(\theta) = i\theta a - \frac{1}{2} \sigma_X^2 \theta^2 \quad (4.23)
\]

This formula can also be used for the moments of \(X\).

From (4.13) it follows that

\[
P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1) = \text{erf} \left(\frac{x_2 - a}{\sigma_X}\right) - \text{erf} \left(\frac{x_1 - a}{\sigma_X}\right) \quad (4.24)
\]

In the particular case where the interval \((x_1, x_2)\) is symmetric about the mathematical expectation, \(x_1\) and \(x_2\) may be represented as:
\[ x_1 = a + \varepsilon, \quad x_2 = a - \varepsilon \]

where \( \varepsilon > 0 \), and (4.24) takes the form:

\[ P(|X - a| < \varepsilon) = \text{erf}\left(\frac{\varepsilon}{\sigma_X}\right) - \text{erf}\left(-\frac{\varepsilon}{\sigma_X}\right) \]

or, in view of (4.15),

\[ P(|X - a| < \varepsilon) = 2 \text{erf}\left(\frac{\varepsilon}{\sigma_X}\right) \quad (4.24) \]

In particular,

\[ P(|X - a| < k\sigma_X) = 2 \text{erf}\,(k) \]

and (see Appendix B)

\[ P(|X - a| < \sigma_X) = 2 \text{erf}\,(1) = 2 \times 0.34134 = 0.68268 \]

\[ P(|X - a| < 2\sigma_X) = 2 \text{erf}\,(2) = 2 \times 0.47725 = 0.95450 \quad (4.25) \]

\[ P(|X - a| < 3\sigma_X) = 2 \text{erf}\,(3) = 2 \times 0.49865 = 0.99730 \]

\[ P(|X - a| < 4\sigma_X) = 2 \text{erf}\,(4) = 2 \times 0.499968 = 0.999936 \]

etc. Accordingly, the probability of \( X \) falling outside the interval

\( (a - \sigma_X, a + \sigma_X) \) is 0.31732, outside \((a - 2\sigma_X, a + 2\sigma_X)\) -0.0455, outside

\((a - 3\sigma_X, a + 3\sigma_X)\) - 0.0027 and outside \((a - 4\sigma_X, a + 4\sigma_X)\) - 0.000064.

It is thus seen that the probability of large deviations from \( a \) decreases steeply as \( k \) increases and that for "practical" purposes "possible" values are confined to the range \((a - 3\sigma_X, a + 3\sigma_X)\) (Fig. 4.6).

\section*{§ 4.11. Truncated Normal Distribution}

A random variable is said to have the above distribution, if its density reads

\[ f_X(x) = \begin{cases} \frac{A}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x - x_0)^2}{2\sigma^2}\right\}, & \text{if } x_1 \leq x \leq x_2 \\ 0, & \text{otherwise} \end{cases} \quad (4.26) \]
i.e. is completely similar to its normal counterpart except for being confirmed to the interval \([x_1, x_2]\) (Fig. 4.7). \(\Lambda\) is found from the requirement (3.11)

\[
\Lambda = \left[ \text{erf} \left( \frac{x_2 - x_0}{\sigma} \right) - \text{erf} \left( \frac{x_1 - x_0}{\sigma} \right) \right]^{-1}
\]  
(4.27)

Hence, the distribution function is given by

\[
F_X(x) = \begin{cases} 
0 & , \quad \text{if } -\infty < x < x_1 \\
\frac{\text{erf} \left( \frac{x - x_0}{\sigma} \right) - \text{erf} \left( \frac{x_1 - x_0}{\sigma} \right)}{\text{erf} \left( \frac{x_2 - x_0}{\sigma} \right) - \text{erf} \left( \frac{x_1 - x_0}{\sigma} \right)} & , \quad x_1 \leq x < x_2 \\
1 & , \quad x_2 \leq x < \infty
\end{cases}
\]  
(4.28)

Also

\[
E(X) = x_0 + B\sigma
\]

\[
B = \frac{1}{\Lambda} \left\{ \exp \left[ -\frac{(x_1 - x_0)^2}{2\sigma^2} \right] - \exp \left[ -\frac{(x_2 - x_0)^2}{2\sigma^2} \right] \right\}
\]  
(4.29)

\[
\frac{\text{Var}(X)}{\sigma^2} = 1 - B^2 - \frac{1}{\Lambda} \left\{ \frac{(x_2 - x_0)}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{(x_2 - x_0)^2}{2\sigma^2} \right] \right. \\
\left. - \frac{(x_1 - x_0)}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{(x_1 - x_0)^2}{2\sigma^2} \right] \right\}
\]

If \(x_1 \to -\infty\) and \(x_2 \to +\infty\), then \(A \to 1\) and \(X\) becomes a normally distributed random variable with \(E(X) = x_0\) and \(\text{Var}(X) = \sigma^2\).

A normal variable is said to have a **symmetrical truncated distribution** if in (4.26) we have

\[
x_1 = x_0 - k\sigma, \quad x_2 = x_0 + k\sigma
\]

A then becomes
\[ A = [2 \operatorname{erf}(k)]^{-1} \]  

(4.30)

From Appendix B we have for \( k = 2 \operatorname{erf}(2) = 0.47725 \), and \( A \) differs only slightly from unity.

§ 4.12. Function of random variable

We now consider mapping of a random variable into another random variable by means of some deterministic relationship.

Let \( X \) be a random variable. This signifies that to each experimental outcome \( \omega \) we assign a real number \( X(\omega) \); the set \( \{X \leq x\} \) is an event for any real number \( x \), and the probabilities are

\[ P(X = +\infty) = P(X = -\infty) = 0 \]

Suppose that a real function \( \varphi(x) \) of a real variable \( x \) is given. We construct the function

\[ Y = \varphi(x) \]

(4.31)

Suppose that the set \( \{Y \leq y\} \) is an event for any real number \( y \), and the probabilities are

\[ P(Y = +\infty) = P(Y = -\infty) = 0 \]

To every outcome \( \omega \) of the experiment we assign the real number \( \varphi[X(\omega)] \).

Then \( Y = \varphi(x) \) so defined is a random variable.

§ 4.13. Moments of function of random variable

According to the definitions

\[ m_k(Y) = E(Y^k) \]

(4.32)

\[ \mu_k(Y) = E[(Y - E(Y))^k] \]

(4.33)
Substituting (4.31) in (4.33) and (4.31), we obtain

\[
m_k(Y) = E[\varphi(X)]
\]

(4.34)

\[
u_k(Y) = E(\varphi(X) - E[\varphi(X)])^k
\]

(4.35)

and, therefore, for a continuously distributed \(X\):

\[
m_k(Y) = \int_{-\infty}^{\infty} [\varphi(x)]^k f_X(x) \, dx
\]

(4.36)

\[
u_k(Y) = \int_{-\infty}^{\infty} \{\varphi(x) - E[\varphi(x)]\}^k f_X(x) \, dx
\]

(4.37)

As is seen from (4.32)-(4.37), the moments of a function of a random variable can be determined directly through \(f_X(x)\) without recourse to \(F_Y(y)\). For the mathematical expectation and the variance of \(Y\) we obtain, respectively:

\[
E(Y) = \int_{-\infty}^{\infty} \varphi(x) f_X(x) \, dx
\]

(4.38)

\[
\text{Var}(Y) = \int_{-\infty}^{\infty} \varphi^2(x) f_X(x) \, dx - \left[ \int_{-\infty}^{\infty} \varphi(x) f_X(x) \, dx \right]^2
\]

(4.39)

§ 4.14. Distribution and density functions of a function of a random variable (special case)

We seek the distribution function \(F_Y(y)\) of \(Y\). By definition,

\[
F_Y(y) = P(Y \leq y) = P[\varphi(X) \leq y]
\]

(4.40)

Consider first the case where \(Y = \varphi(x)\) is a strictly monotone increasing function (Fig. 4.8). A straight line parallel to the abscissa axis intersects the graph of \(\varphi(x)\) at a point with abscissa depending on \(y\), and the interval where the inequality

\[Y = \varphi(x) \leq y\]
holds is marked off; \( x = \psi(y) \) is then a unique inverse function:

\[
F_Y(y) = \int_{-\infty}^{\psi(y)} f_X(x) \, dx = F_X[\psi(y)]
\]  

(4.41)

The probability density function \( f_Y(y) \) of \( Y \) is obtained as:

\[
f_Y(y) = \frac{dF_Y(y)}{dy} = f_X[\psi(y)] \frac{d\psi(y)}{dy}
\]  

(4.42)

Next, the case where \( \varphi(x) \) is a strictly monotone decreasing function (Fig. 4.9). Here

\[
F_Y(y) = \int_{\psi(y)}^{\infty} f_X(x) \, dx = 1 - F_X[\psi(y)]
\]  

(4.43)

and the probability density function is

\[
f_Y(y) = \frac{dF_Y(y)}{dy} = -f_X[\psi(y)] \frac{d\psi(y)}{dy}
\]  

(4.44)

Since in the latter case \( \psi(y) < 0 \), both cases can be unified:

\[
f_Y(y) = f_X[\psi(y)] \left| \frac{d\psi(y)}{dy} \right|
\]  

(4.45)

§ 4.15. Linear function of random variable

If the random variable \( Y \) is a linear function of \( X \):

\[ Y = \alpha X + \beta \]

and the probability density function \( f_X(x) \) of \( X \) is known, then

\[ y = \varphi(x) = \alpha x + \beta \]

\[ x = \psi(y) = (y - \beta)/\alpha \quad \psi'(y) = 1/\alpha \]

Substituting these in (4.45), we obtain
\[ f_Y(y) = \frac{1}{|\alpha|} f_X\left(\frac{y - \beta}{\alpha}\right) \]  

(4.46)

Let now \( \alpha = \frac{1}{\sigma_X} \) and \( \beta = -\frac{E(X)}{\sigma_X} \), where \( E(X) \) is the mathematical expectation of \( X \) and \( \sigma_X \) its standard deviation; then

\[ y = \frac{X - E(X)}{\sigma_X} \]

is the normalised random variable, and

\[ f_Y(y) = \sigma_X f_X[\sigma_X y + E(X)] \]  

(4.47)

Consider two special cases.

**Example 4.2**

Let \( X \) be uniformly distributed in the interval \((x_1, x_2)\), that is

\[ f_X(x) = \begin{cases} 
\frac{1}{x_2 - x_1} & \text{, for } x_1 < x < x_2 \\
0 & \text{, otherwise}
\end{cases} \]

Then we have

\[ f_Y(y) = \begin{cases} 
\frac{1}{|\alpha|} \frac{1}{x_2 - x_1} & \text{, for } x_1 < \frac{y - \beta}{\alpha} < x_2 \\
0 & \text{, otherwise}
\end{cases} \]  

(4.48)

In other words, \( Y \) is likewise uniformly distributed in the interval \((\alpha x_1 + \beta, \alpha x_2 + \beta)\) for \( \alpha > 0 \), and in the interval \((\alpha x_2 + \beta, \alpha x_1 + \beta)\) for \( \alpha < 0 \).

**Example 4.3**

If \( X \) is \( N(a, \sigma_X^2) \), then

\[ f_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} \exp \left[ - \frac{(x - a)^2}{2\sigma_X^2} \right] \]

and then
\[ f_Y(y) = \frac{1}{\sigma_X |\alpha| \sqrt{2\pi}} \exp \left\{ - \frac{(y - \beta)/\alpha - a)^2}{2\sigma_X^2} \right\} \]

or

\[ f_Y(y) = \frac{1}{\sigma_X |\alpha| \sqrt{2\pi}} \exp \left\{ - \frac{(y - (\alpha a + \beta))^2}{2\sigma_X^2} \right\} \quad (4.49) \]

stating that a linear function of a normally distributed random variable is likewise normal, \( N(\alpha a + \beta, \sigma_X^2 a^2) \).

§ 4.16. Exponent and logarithm of random variable

Let now

\[ Y = e^X \quad (4.50) \]

The function \( y = e^X \) is strictly monotone increasing, and we have

\[ F_Y(y) = F_X[\ln y] = F_X(\ln y) \ U(y) \]

\[ f_Y(y) = \frac{1}{y} f_X(\ln y) \ U(y) + F_X(\ln y) \ \delta(y) \quad (4.51) \]

In particular, when \( X \) is \( N(a, \sigma_X^2) \), we obtain

\[ f_Y(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \frac{1}{y\sigma_X \sqrt{2\pi}} \exp \left[ - \frac{(\ln y - a)^2}{2\sigma_X^2} \right] & \text{for } y > 0 \end{cases} \quad (4.52) \]

The random variable \( Y \) is said to have a logarithmic-normal, or for brevity, log-normal distribution. We can show that the probability of \( Y \) falling within an interval \([y_1, y_2]\) is \((y_1, y_2 \geq 0)\):

\[ P(y_1 \leq Y \leq y_2) = \text{erf} \left( \frac{\ln y_2 - a}{\sigma_X} \right) - \text{erf} \left( \frac{\ln y_1 - a}{\sigma_X} \right) \quad (4.53) \]

In fact,

\[ P(y_1 \leq Y \leq y_2) = \int_{y_1}^{y_2} \frac{1}{y\sigma_X \sqrt{2\pi}} \exp \left[ - \frac{(\ln y - a)^2}{2\sigma_X^2} \right] \, dy \]
Substituting in the integral
\[ \frac{\ln y - a}{\sigma_x} = z \]
we obtain
\[ P(y_1 \leq Y \leq y_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\ln \frac{y_2 - a}{\sigma_x}} \exp\left(-\frac{z^2}{2}\right) dz \]
\[ \ln \frac{y_1 - a}{\sigma_x} \]
which immediately leads to the desired result (4.53).

We now seek the mean \( E(Y) \) and the variance \( \text{Var}(Y) \), using Eqs. (4.38) and (4.39):
\[ E(Y) = \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\infty} e^x \exp\left(-\frac{(x - a)^2}{2\sigma_x^2}\right) dx = \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\frac{x - (x - a)^2}{2\sigma_x^2}\right] dx \]
Which, in view of Eq. (4.), transforms into
\[ E(Y) = e^{a + \frac{1}{2} \sigma_x^2} \]  
(4.54)

In order to find the variance, we first calculate in Eq. (4.39)
\[ \int_{-\infty}^{\infty} \psi^2(x) f_X(x) \, dx = \int_{-\infty}^{\infty} \frac{e^{2x}}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{(x - a)^2}{2\sigma_x^2}\right] e^{-a} = e^{2(a+\sigma_x^2)} \]
whence,
\[ \text{Var}(Y) = e^{2a+\sigma_x^2} \quad \sigma_x^2 \]
(4.55)

If instead of (4.50) we have
\[ Y = 10^X \]
(4.52) becomes
\[ f_y(y) = \frac{\log_{10} e}{y} f_{\log_{10} y}(y) U(y) \]  

(4.56)

where \( \log_{10} e \approx 0.4343 \).

Consider now the Napierian logarithm of a random variable

\[ U = \log_e V \]

where \( V \) is a positive-valued random variable with probability density \( f_V(v) \). Then, since the function \( u = \log_e v \) is strictly monotone increasing, we have

\[ f_U(u) = e^u f_V(e^u) \]

In particular, if \( V \) has a log-normal distribution as per (4.52) then \( U \) has a normal distribution \( N(a, \sigma_X^2) \), as anticipated.

§ 4.17. Distribution and density functions of a function of a random variable (general case)

Consider now the case when \( y = \varphi(x) \) is not a monotone function (see Fig. 4.10). On the abscissa axis, the interval where

\[ Y = \varphi(x) \leq y \]  

(4.57)

is marked off. The number of intervals where (4.57) is satisfied depends on \( y \). Let this number be \( n \); the condition (4.57) is then equivalent to the event of \( X \) taking on values in one of the intervals \([\psi_1^-(y), \psi_1^+(y)]\). Thus

\[ F_Y(y) = P[\varphi(x) \leq y] = P[[\psi_1^-(y) \leq x \leq \psi_1^+(y)] U[\psi_2^-(y) \leq x \leq \psi_2^+(y)] U \ldots \]

\[ \ldots U[\psi_n^-(y) \leq x \leq \psi_n^+(y)]} \]  

(4.58)

Since the random events in (4.58) are mutually exclusive, we may write:

\[ F_Y(y) = \sum_{i=1}^{n} P[\psi_i^-(y) \leq x \leq \psi_i^+(y)] = \sum_{i=1}^{n} \int_{\psi_i^-(y)}^{\psi_i^+(y)} f_X(x) \, dx \]
\[
F_Y(y) = \sum_{i=1}^{n} \left\{ F_X[\psi_i^+(y)] - F_X[\psi_i^-(y)] \right\}
\]

(4.59)

the probability density function is:

\[
f_Y(y) = \frac{dF_Y(y)}{dy} = \sum_{i=1}^{n} \left\{ f_X[\psi_i^+(y)] \frac{d\psi_i^+(y)}{dy} - f_X[\psi_i^-(y)] \frac{d\psi_i^-(y)}{dy} \right\}
\]

(4.60)

Eqs. (4.59) and (4.60) represent respectively, in general terms, the distribution and probability density function of a function of a random variable.

For a strictly monotone increasing function (Fig. 4.8) \( n = 1 \), the interval being \( \psi_1^-(y) = -\infty, \psi_1^+(y) = \psi(y) = \psi^{-1}(y) \) - the inverse of \( \psi \), and the equation reduces to (4.42); for a strictly monotone decreasing function (Fig. 4.9) again \( n = 1 \), the interval this time being \( \psi_1^+(y) = +\infty, \psi_1^-(y) = \psi^{-1}(y) \), and the equation reduces to (4.44).

Consider now the case where \( \psi(x) \) is a constant in some interval \( (x_1, x_2) \) (Fig. 4.11). Comparison of Figs. 4.11b and 4.11c shows that:

\[
F_Y(y) \bigg|_{y=c+0} - F_Y(y) \bigg|_{y=c-0} = \text{Prob}(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x) \, dx
\]

and, therefore, the probability distribution function \( F_Y(y) \) has a jump discontinuity at \( c \).

If the range of \( y = \psi(x) \) is not the whole real line but only part of it, say the interval \( (x_1, x_2) \), we put \( f_Y(y) = 0 \) outside that interval.

Example 4.4

Let

\[
y = x^2
\]

where \( X \) has a Rayleigh distribution, with
\[ f_X(x) = \frac{x}{a^2} \exp \left( -\frac{x^2}{2a^2} \right) U(x) \]

The function \( y = x^2 \) \((0 \leq x < \infty)\) is strictly monotone increasing, and (4.42) may be applied; the inverse function is \( \psi(y) = \sqrt{y} \), and we have

\[ f_Y(y) = \frac{1}{2a^2} \exp \left( -\frac{\sqrt{y}}{2a^2} \right) \]

i.e. an exponentially distributed random variable.

**Example 4.5**

Consider once again the transformation (Fig. 4.12)

\[ y = x^2 \]

where \( X \) can now take on both positive and negative values. In this case \( Y \) cannot take on negative values, so:

\[ F_Y(y) = 0, \quad f_Y(y) = 0 \quad \text{for} \quad y < 0 \]

For \( y \geq 0 \) there is a single interval, where \( x^2 \leq y \) with

\[ \psi_1(y) = -\sqrt{y}, \quad \psi_1^+(y) = \sqrt{y} \]

And in accordance with (4.59) and (4.60) we write for \( y > 0 \)

\[ F_Y(y) = \left[ F_X(\sqrt{y}) - F_X(-\sqrt{y}) \right] \quad (4.61) \]

\[ f_Y(y) = \frac{1}{2\sqrt{y}} \left[ f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right] \quad (4.62) \]

and \( F_Y(y) = 0, \ f_Y(y) = 0 \) for \( y \leq 0 \).

If, for example, \( X \) is a standard normal variable \( N(0, 1) \) (4.20), we find

\[ f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} \quad \text{for} \quad y > 0 \]

which represents one-degree-of-freedom chi-square probability density (4.9), with \( \lambda = 1/2 \) \((\Gamma(1/2) = \sqrt{\pi})\).

**Example 4.6**

Consider now the function (Fig. 4.13)
\[ Y = |x| \]

In this case we again have a single interval where \(|x| \leq y\) is satisfied with

\[
\psi_1^-(y) = -y, \quad \psi_2^+(y) = y
\]

As in the preceding example,

\[ F_Y(y) = 0, \quad f_Y(y) = 0 \quad \text{for} \quad y < 0 \]

Combining (4.59) and (4.60), we obtain:

\[
F_Y(y) = F_X(y) - F_X(-y) \quad (4.63)
\]

\[
f_Y(y) = f_X(y) + f_X(-y) \quad (4.64)
\]

If \(X \sim N(a, \frac{q^2}{X})\) we obtain for \(F_Y(y)\) and \(f_Y(y)\), respectively,

\[
f_Y(y) = \frac{1}{\sqrt{2\pi}q_X} \left\{ \exp \left[ -\frac{(y-a)^2}{2q_X^2} \right] + \exp \left[ -\frac{(y+a)^2}{2q_X^2} \right] \right\} \quad (4.65)
\]

\[
F_Y(y) = \left[ \text{erf} \left( \frac{y-a}{q_X} \right) + \text{erf} \left( \frac{y+a}{q_X} \right) \right] U(y) \quad (4.66)
\]

for \(y > 0\) and \(f_Y(y) = 0\).

For \(a = 0\) we obtain

\[
f_Y(y) = \begin{cases} 
\frac{1}{\sqrt{2\pi}q_X} \exp \left( -\frac{y^2}{2q_X^2} \right), & \text{for} \quad y \leq 0 \\
0, & \text{for} \quad y > 0
\end{cases} \quad (4.67)
\]

\[
F_Y(y) = 2 \text{erf} \left( \frac{y}{q_X} \right) U(y) \quad (4.68)
\]

These functions are shown in Fig. 4.14.

Sometimes a random variable \(Y\) with density (4.67) is said to have a one-sided normal distribution with parameter \(q_X\). Note that \(q_X\) does not exceed the standard deviation of \(Y\).
Example 4.7
A random variable $X$ is uniformly distributed in the interval $[0, 1]$. The random variable $Y$ is a strictly monotone increasing function of $X$: $Y = \varphi(X)$. We are interested in $F_Y(y)$ and $f_Y(y)$.

We have

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad (4.69)$$

Let $\varphi(y)$ be the inverse function. Then

$$f_Y(y) = f_X(\varphi(y)) \varphi'(y) = \psi'(y)$$

which yields

$$F_Y(y) = \psi(y) \quad (4.70)$$

i.e. the desired distribution function of $Y$ equals the inverse of $\varphi$.
By this means, different random variables may be derived from an uniformly distributed one. Consider, as an illustration, determination of an exponentially distributed random variable as a transformation of a uniformly-distributed one:

$$f_Y(y) = \begin{cases} 0, & \text{for } y < 0 \\ ae^{-ay}, & \text{for } y \geq 0 \end{cases}$$

We have then to put

$$Y = \varphi(X)$$

where $\varphi$ is an inverse $F_Y(y) = 1 - e^{-ay}$:

$$x = 1 - e^{-ay}$$

and, therefore

$$Y = -\frac{1}{a} \ln (1 - X), \quad 0 \leq X \leq 1 \quad (4.71)$$
In complete analogy, consider the following problem. Let $Y = \phi(X)$ with $\phi$ being strictly monotone increasing with $f_X(x)$ and $f_Y(y)$ given, we seek $\phi$.

We write in accordance with (4.41),

$$F_X(x) = \int_{-\infty}^{0} f_Y(y) \, dy = F_Y[\phi(x)]$$

$$\phi(x) = F_Y^{-1}[F_X(x)]$$

where $F_Y^{-1}$ is the inverse of $F_Y$. Thus

$$Y = F_Y^{-1}[F_X(x)] \quad (4.72)$$

Cited References


Recommended Further Reading


PROBLEMS

4.1. A dyke is designed with 5 m freeboard above the mean sea level. The probability of its being topped by waves in one year is 0.005. What is the probability of waves exceeding 5 m within 200 years?

4.2. Using eq. (4.23) for the characteristic function of a normally-distributed random variable, show that:

\[ E(X) = a, \quad \text{Var}(X) = \sigma_x^2 \]

4.3. Suppose that the duration of successful performance (lifetime, in years) of a piece of equipment is normally distributed with a mean of 8 years. What is the largest value its standard deviation may have if the operator requires at least 95% of the population to have lifetimes exceeding 6 years? Find the probability of a piece of equipment turning out faulty at delivery.

4.4. A system is activated at \( t = 0 \); its time of failure is the random variable \( T \) with distribution function \( F_T(t) \) and density \( f_T(t) \). Denote by \( h(t) \) the hazard function (see Prob. 3.4).
   (a) Check in Prob. 3.4 whether \( T \) has an exponential distribution.
   (b) Show that if \( h(t) = \alpha t \), \( T \) has a Rayleigh distribution.
   (c) Show that if \( h(t) = \alpha t^{-\beta-1} \), \( T \) has a Weibull distribution.
   (d) Find \( f_T(t) \) if \( h(t) = \alpha t e^{\gamma t} \).

Remark: The resulting \( f_T(t) \) is a so-called an extreme-value density function. For an interesting failure model leading to the extreme-value distribution, see paper by Epstein.

4.5. A system consisting of \( n \) elements in parallel with equal reliabilities \( R(t) \), fails only when all elements fail simultaneously.
   (a) Show that the reliability \( R_n(t) \) of such a system is

\[ R_n(t) = [1 - R(t)]^n \]

(b) Find \( R_n(t) \) when the conditional failure rate of each element is constant, \( h(t) = a \).
(c) Find the allowable operation time \( t_x \) of the system, such that
\[ R(t_x) = r \] where \( r \) is a required reliability, for \( n = 1, 2, 3 \).

(d) Show that the mean life time of the system is
\[ E(T) = \frac{1}{a} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) \]
and interpret the result for \( n \to \infty \).

4.6. As above, but with the elements arranged in series instead of in parallel.

4.7. Suppose that the reliabilities of \( n \) individual elements are identical and given by
\[ R(t) = \frac{t}{T} \text{ for } 0 \leq t \leq T \]
Show that the mean life time of the system is
\[ E(T) = \begin{cases} \frac{r}{(n + 1)} & \text{for elements in series} \\ \frac{rn}{(n + 1)} & \text{for elements in parallel} \end{cases} \]
Interpret the result for \( n \to \infty \).

4.8. The random variables \( X \) and \( Y \) are linked the following functional relationship
\[ Y = \begin{cases} -a^3 & \text{if } x < -a \\ x^3 & \text{if } -a \leq x \leq a \\ a^3 & \text{if } x > a \end{cases} \]
\( X \) has a uniform distribution in the interval \((-a, a)\).
Find \( F_Y(y) \) and \( f_Y(y) \), and present them graphically.

4.9. As above, but \( X \) has a uniform distribution in the interval \((-2a, 2a)\).
Show that under these conditions \( Y \) is a mixed random variable.

4.10. \( X \) has a Cauchy distribution. Show that \( Y = 1/X \) also has a Cauchy distribution.
CHAPTER 5 - RELIABILITY OF STRUCTURES

DESCRIBED BY SINGLE RANDOM VARIABLE

The scope of the knowledge acquired by us so far suffices for reliability analysis (recalling that reliability is the probability of non-failure performance) of the simplest structures, described by a single random variable. We are concerned with generally-differing structures called upon to realise different functional assignments, and will not consider non-functional ones; in other words, reliability is associated with a purpose-exploitation of the structure in accordance with defined goals. The acceptability criterion consists in the reliability exceeding some special level.

We now proceed to concrete examples of structural reliability.

§ 5.1. Bar under random tensile force

Consider a bar of constant cross-sectional area \( a \), under a tensile force \( N \) which is a random variable with the distribution function \( F_N(n) \), \( n \geq 0 \). The conventional strength requirement is for the normal stress \( \Sigma \), with values \( \sigma \), to be less than the allowable stress \( \sigma_{\text{allow}} \):

\[
\Sigma = \frac{N}{a} \leq \sigma_{\text{allow}}
\]

We simplify the analysis by assuming both \( a \) and \( \sigma_{\text{allow}} \) to be deterministic quantities. The reliability \( R \) will then be defined as the probability of event (5.1):

\[
R = P(\Sigma \leq \sigma_{\text{allow}}) = P(N/a \leq \sigma_{\text{allow}})
\]

or

\[
R = F_N(\sigma_{\text{allow}} a)
\]

i.e., the reliability of a bar equals the distribution function of the tensile force \( N \) at the \( \sigma_{\text{allow}} \) level.

If, for example, \( N \) has a uniform distribution
\[ f_N(n) = \begin{cases} \frac{1}{n_2 - n_1}, & \text{if } n_1 \leq n \leq n_2 \\ 0, & \text{otherwise} \end{cases} \] (5.4)

The \( R \) is:

\[ R = \begin{cases} 0, & \text{for } \sigma_{\text{allow}} a < n_1 \\ \frac{\sigma_{\text{allow}} a - n_1}{n_2 - n_1}, & \text{for } n_1 \leq \sigma_{\text{allow}} a < n_2 \\ 1, & \text{for } n_2 \leq \sigma_{\text{allow}} a \end{cases} \] (5.5)

These results are readily understood: if the minimal possible value \( n_1 \) of the random tensile force exceeds its maximal deterministically-allowable counterpart \( \sigma_{\text{allow}} a \), failure is a certain event. In that case the unreliability of the structure

\[ Q = 1 - R \] (5.6)

defined as the probability of failure, is unity and its reliability zero. If, however, the maximal possible value \( n_2 \) of the force does not exceed its maximal deterministically allowable counterpart, the structure never fails, its unreliability is zero and its reliability unity. These situations are illustrated in Fig. 5.2, where the shaded areas represent the reliability of the tensile member.

With \( R \) known we can solve the following two problems, noting that the structure functions acceptably if the reliability exceeds (or equals) a specified probability \( r \):

\[ R \geq r, \quad 0 < r \leq 1 \] (5.7)

The first problem is concerned with checking whether this inequality is satisfied; the second - with actual designing of the bar. Suppose we have to choose a deterministic quantity \( \Lambda \) so that Eq. (5.7) is satisfied. The value of \( a \) which satisfies the equality

\[ R = r \] (5.8)
is called the minimal permissible value.

Comparison of Eqs. (5.3) and (5.8) immediately indicates that \( \sigma_{\text{allow}} A \) is the \( r \)-th quantile of \( N \), and that the required area is obtained simply by dividing this \( r \)-th quantile by \( \sigma_{\text{allow}} \).

Once the reliability is known, the problem of checking the strength becomes trivial. We now proceed to the design problem. Note that by choosing

\[
a \geq \frac{n_2}{\sigma_{\text{allow}}} \equiv a_{\text{worst}}
\]

(5.9)

we have \( R = 1 \), and inequality (5.7) is satisfied for any \( r \).

This design is according to the maximal possible value \( n_2 \) of the tensile force \( N \): the "worst" case consideration. Concentrate now on the "short-of-the-worst" case:

\[
n_1 \leq \sigma_{\text{allow}} a \leq n_2
\]

Eq. (5.8) reads then

\[
\frac{\sigma_{\text{allow}} a - n_1}{n_2 - n_1} = r
\]

yielding the required area:

\[
a_{\text{req}} = \frac{r(n_2 - n_1) + n_1}{\sigma_{\text{allow}}} = \frac{n_2 - (n_2 - n_1)(1 - r)}{\sigma_{\text{allow}}} = a_{\text{worst}} - \frac{n_2 - n_1}{\sigma_{\text{allow}}}(1 - r)
\]

(5.10)

Comparison of Eqs. (5.9) and (5.10) shows that the required area is less than that corresponding to the "worst" case. The gain in area, however, may be very small for high values of \( r \):

\[
a_{\text{worst}} - a_{\text{req}} \leq a_{\text{worst}} (1 - r)
\]

equality is achieved only if \( n_1 = 0 \). If, for example, \( r = 0.99 \), then the gain in area is only one percent of \( a_{\text{worst}} \).

\( a_{\text{req}} \) may be expressed in terms of the mean \( E(N) \) and the standard deviation \( \sigma_n = \sqrt{\text{Var}(N)} \)
\[ E(N) = \frac{1}{2} (n_1 + n_2), \quad \sigma_N = \frac{1}{\sqrt{12}} (n_2 - n_1) \]
to yield
\[ a_{\text{req}} = \frac{E(N) + (r - 0.5) \sqrt{12} \sigma_N}{\sigma_{\text{allow}}} \] (5.11)
If \( n_2 + n_1 = n, \sigma_N = 0, \) and \( N \) tends to become a random variable with causal
distribution, it takes on the value \( n \) with probability 1, we return to the
deterministic design load
\[ a = \frac{E(N)}{\sigma_{\text{allow}}} = \frac{n}{\sigma_{\text{allow}}} \] (5.12)
otherwise Eq. (5.11) has to be used for the sought area. Eq. (5.11) indicates
that the "design" according to mean load is unconservative if \( r > 0.5. \) It
may only be used if \( \sigma_N << E(N). \)

Thus the deterministic design represents a particular case of the probabilistic one.

Note that Eq. (5.11) tells us, that when a reliability as low as \( r = 0.5 \)
is required, the mean load-based design may be used, as then the second
term in Eq. (5.11) vanishes and \( a_{\text{req}} = E(N)/\sigma_{\text{allow}}. \) This, however, implies,
in accordance with the statistical interpretation of probability, that
nearly half the ensemble of structures so designed will fail (which is quite a lot!).
Note also that Eq. (5.3) is valid for all positive-valued distributions of \( N, \)
since then the associated deterministic strength requirements is (5.2). If,
for example, \( N \) has an exponential distribution (§ 4.6.):
\[ F_N(n) = \{1 - \exp[-n/E(N)]\} U(n) \]
then
\[ R = 1 - \exp[-\sigma_{\text{allow}} a/E(N)] \] (5.13)
and the required area is
\[ a_{\text{req}} = \frac{E(N)}{\sigma_{\text{allow}}} \ln \frac{1}{1 - r} \] (5.14)
Thus, if the required reliability is 0.99, then the required area is \( \ln 100 = \)
4.605 times that calculated according to the mean. Note that now the
calculation according to the "worst" case is ruled out: it would yield an
infinite \(a_{\text{req}}\). The determination of \(a_{\text{req}}\) is illustrated in Fig. 5.3.

Consider now the case where the force \(N\) can take on negative values as well. Assume also that the bar is not slender, so that there is no possibility of buckling failure. Then the strength requirement is written as

\[
|\sigma| = \frac{|N|}{a} < \sigma_{\text{allow}}
\]

The reliability is given by

\[
P\left(\frac{|N|}{a} < \sigma_{\text{allow}}\right) = P\left(|N| < \sigma_{\text{allow}}a\right)
\]

\[
= P\left(-\sigma_{\text{allow}}a < N < \sigma_{\text{allow}}a\right)
\]

\[
= F_N(\sigma_{\text{allow}}a) - F_N(-\sigma_{\text{allow}}a)
\]

and the strength requirement becomes

\[
F_N(\sigma_{\text{allow}}a) - F_N(-\sigma_{\text{allow}}a) > r
\]

where the equality yields the required area \(a\).

Note that in this case also the design according to the "worst" case ensures any level of reliability. Indeed, if \(n_2 > -n_1\) (Fig. 5.4a), we may choose

\(a = n_2/\sigma_{\text{allow}}\) yielding

\[
F_N(\sigma_{\text{allow}}a) = 1, \quad F_N(-\sigma_{\text{allow}}a) = 0
\]

and \(R = 1\). If, however, \(n_2 < -n_1\), we may choose \(a = -n_1/\sigma_{\text{allow}}\) with

\[
F_N(-\sigma_{\text{allow}}a) = F_N(n_1) = 0, \quad F_N(\sigma_{\text{allow}}a) = F_N(-n_1) = 1
\]

and still \(R = 1\). If \(n_2 = -n_1\) (Fig. 5.4c), choosing \(a = n_2/\sigma_{\text{allow}}\) again yields \(R = 1\).

If, however, the required reliability is not too high, the bar will be "overdesigned". For \(n_2 = -n_1\), we have from (5.17):

\[
F_N(\sigma_{\text{allow}}a) - F_N(-\sigma_{\text{allow}}a) = r
\]
but

\[ F_N(-\sigma_{\text{allow}} a) = 1 - F_N(\sigma_{\text{allow}} a) \]

yielding,

\[ F_N(\sigma_{\text{allow}} a) = \frac{1}{2} (1 + r) \]

or

\[ \frac{\sigma_{\text{allow}} a + n_2}{2n_2} = \frac{1}{2} (1 + r) \quad \sigma_{\text{allow}} a = n_2 r \quad a_{\text{req}} = n_2 r / \sigma_{\text{allow}} \]

and

\[ a_{\text{req}} = \frac{n_2 r}{\sigma_{\text{allow}}} = a_{\text{worst}} r \]

For \( r = 0.9 \) the required area is only 0.9 \( a_{\text{worst}} \); however, for \( r = 0.99 \), it is 0.99 \( a_{\text{worst}} \), i.e. \( a_{\text{worst}} \) may be used as the required area.

Consider now the case where \( N \) has a symmetrical truncated normal distribution with zero mean, in the interval \((n_1, n_2)\) (§ 4.11.)

\[ n_1 = -k\bar{\sigma}, \quad n_2 = k\bar{\sigma} \]

where \( \bar{\sigma} \) is a parameter of the distribution, and

\[
F_N(n) = \begin{cases} 
0 & , \quad -\infty < n < n_1 \\
\frac{\text{erf}(n/\bar{\sigma}) + \text{erf}(k)}{2 \text{erf}(k)} & , \quad n_1 \leq n \leq n_2 \\
1 & , \quad n_2 < n 
\end{cases}
\]

Eq. (5.17) takes then the form for \( a_{\text{req}} \):

\[
\text{erf} \left( \frac{\sigma_{\text{allow}} a}{\bar{\sigma}} \right) = r \text{erf}(k)
\]

\[ (5.18) \]

For \( k = 3 \), we have
\[
\text{erf} \left( \frac{\sigma_{\text{allow}}}{\sigma_{\text{req}}} \right) = 0.49865 \ r
\]

(5.19)

for \( r = 0.9 \), \( \text{erf}(\sigma_{\text{allow}} / \sigma_{\text{req}}) = 0.448785 \), and \( \sigma_{\text{req}} = 1.63 \frac{\sigma}{\sigma_{\text{allow}}} \).

for \( r = 0.99 \), \( \text{erf}(\sigma_{\text{allow}} / \sigma_{\text{req}}) = 0.4936635 \), and \( \sigma_{\text{req}} = 2.49 \frac{\sigma}{\sigma_{\text{allow}}} \).

Note that calculation according to the "worst" load \( n_2 = 3\sigma \), yields \( \sigma_{\text{req}} = 3\sigma / \sigma_{\text{allow}} \) implying 84.05% and 20.48% overdesign, for \( r = 0.9 \) and \( r = 0.99 \) respectively.

Note that if we assume a normal distribution of the load with zero mean and standard deviation \( \sigma \), we obtain, instead of (5.19),

\[
\text{erf} \left( \frac{\sigma_{\text{allow}}}{\sigma_{\text{req}}} \right) = 0.5 \ r
\]

(5.20)

which is practically coincident with the result obtained from that associated with a truncated normal distribution.

Calculation according to "three standard deviations", i.e. choice of \( \sigma_{\text{req}} = \frac{3\sigma}{\sigma_{\text{allow}}} \)

corresponds to a reliability \( R = 2 \ 	ext{erf}(3) = 0.9973 \); calculation according to "two standard deviations" is equivalent to choice of \( \sigma_{\text{req}} = \frac{2\sigma}{\sigma_{\text{allow}}} \)

yielding \( R = 2 \ 	ext{erf}(2) = 0.9545 \).

Note that calculation of the required area for a bar under an exponentially distributed tensile force according to the "mean plus \( \alpha \) times standard deviation" yields, according to (5.13),

\[
R = 1 - e^{-\left(1+\alpha\right)}
\]

so that in order to achieve the required reliability \( r = 0.9973 \), we have to use now: \( \alpha = 4.9145 \) instead of \( \alpha = 3 \) for the normally distributed tensile force.

For the case where the bar is slender and buckling in compression is possible, the strength requirement (5.15) reads
\[-\frac{\pi^2 EI}{4L^2} < N < \sigma_{\text{allow a}}\]

where the left member represents the buckling load of a clamped-free bar, as shown in Fig. 5.1. Reliability is determined as the probability of the above random event:

\[R = P \left(-\frac{\pi^2 EI}{4L^2} < N < \sigma_{\text{allow a}}\right)\]

\[= F_N(\sigma_{\text{allow a}}) - F_N\left(-\frac{\pi^2 EI}{4L^2}\right)\]

which may be used either for determining the reliability at the desired level, or for designing the bar with that level.

Another example of structural elements in tension is shown in Fig. 5.5. A rigid block constrained by three bars under a random force \(N\) with distribution function \(F_N(n)\). Since two bars suffice, this is a statically-indeterminate system.

The force equilibrium equations read:

\[F_y = N_1 + N_2 + N_3 - N = 0\]

\[M_1(z) = N_2 c + N_3 2c - N(1.5 c) = 0\]

Geometric compatibility indicates

\[\frac{\delta_1 - \delta_3}{\delta_2 - \delta_3} = \frac{2c}{c} = 2\]

which yields the third equation,

\[N_1 - 2N_2 + N_3 = 0\]

Solving the three equations, we find

\[N_1 = \frac{N}{12}, \quad N_2 = \frac{N}{3}, \quad N_3 = \frac{7}{12} N\]

By the strength requirement,
\[ R = P \left( \left( \frac{1}{12} \frac{N}{a} \leq \sigma_{\text{allow}} \right) \cap \left( \frac{1}{3} \frac{N}{a} \leq \sigma_{\text{allow}} \right) \cap \left( \frac{7}{12} \frac{N}{a} \leq \sigma_{\text{allow}} \right) \right) \geq r \]

Denote the events

\[ \{B_1\} = \left\{ \frac{1}{12} \frac{N}{a} \leq \sigma_{\text{allow}} \right\}, \quad \{B_2\} = \left\{ \frac{1}{3} \frac{N}{a} \leq \sigma_{\text{allow}} \right\}, \quad \{B_3\} = \left\{ \frac{7}{12} \frac{N}{a} \leq \sigma_{\text{allow}} \right\} \]

so that

\[ R = P(B_1 \cap B_2 \cap B_3) \]

However,

\[ B_3 \subset B_2 \subset B_1 \]

that is, \( B_3 \) implies both \( B_2 \) and \( B_1 \), while \( B_2 \) implies \( B_1 \). Therefore

\[ B_1 \cap B_2 \cap B_3 = B_3 \]

\[ R = P(B_3) = P \left( \frac{7}{12} \frac{N}{a} \leq \sigma_{\text{allow}} \right) = P \left( \frac{12}{7} \sigma_{\text{allow}} \frac{N}{a} \right) \geq r \]

§ 5.2. Bar with random strength

Consider now a bar under a deterministic load \( n \), and having a random strength with given continuous probability distribution \( F_{\Sigma \text{allow}}(\sigma_{\text{allow}}) \).

We confine ourselves here to the case where \( \Sigma_{\text{allow}} \) takes on only positive values, and moreover \( n > 0 \). Then reliability is

\[ R = P \left( \frac{n}{a} \leq \Sigma_{\text{allow}} \right) = 1 - F_{\Sigma_{\text{allow}}} \left( \frac{n}{a} \right) \]

(5.21)

Assuming an exponential distribution for \( \Sigma_{\text{allow}} \),

\[ F_{\Sigma_{\text{allow}}} (\sigma_{\text{allow}}) = 1 - \exp \left[ - \frac{\sigma_{\text{allow}}}{E(\Sigma_{\text{allow}})} \right] \]

we obtain immediately

\[ R = \exp \left[ - \frac{n}{E(\Sigma_{\text{allow}}) a} \right] \]
The reliability $R$ is given by the shaded area in Fig. 5.6.

The condition $R = r$ yields the required area:

$$a_{\text{req}} = \frac{n}{E(\Sigma_{\text{allow}})} (\ln \frac{1}{r})^{-1}$$

So, for the desired reliability $r = 0.99$, we have

$$a_{\text{req}} = 99.5 \frac{n}{E(\Sigma_{\text{allow}})}$$

almost a hundred times that obtained from the mean value approach: $n/E(\Sigma_{\text{allow}})$.

The results obtained for a single bar can be generalised in a simple way for the truss system of § 2.8. Consider a statically determinate truss, assuming that the stresses in the bars $\sigma_1, \sigma_2, \ldots, \sigma_n$ are deterministic quantities and that the distribution functions of the allowable stresses $F_{\Sigma_{\text{allow}}}(\sigma_{\text{allow}}(j))$ for each bar $(j = 1, 2, \ldots, n)$ are given. Then $F_{\Sigma_{\text{allow}}}(\sigma_{\text{allow}}(j))$ is the probability of failure of the $j$-th bar, and its reliability is

$$R_j = 1 - F_{\Sigma_{\text{allow}}}(\sigma_{\text{allow}}(j))$$

If bar failures represent independent random events, the reliability of the entire truss is

$$R = \prod_{j=1}^{n} \left[ 1 - F_{\Sigma_{\text{allow}}}(\sigma_{\text{allow}}(j)) \right]$$

If the particular cone where the allowable stresses are distributed identically, this formula becomes

$$R = \prod_{j=1}^{n} \left[ 1 - F_{\Sigma_{\text{allow}}}(\sigma_{\text{allow}}(j)) \right]$$

and in the subcase where the stresses are equal $\sigma_j = \sigma$, $(j = 1, 2, \ldots, n)$
(see Fig. 2.9e), we obtain

\[ R = [1 - F_{\Sigma}^{\text{allow}}(\sigma)]^N \]

instead of Eq. (2.22).

§ 5.3. Bar with random cross-sectional area

Consider now a circular bar with given \( \sigma_{\text{allow}} \) under a given tensile force \( n \), its cross-sectional area \( A \) being a random variable with continuous distribution function \( F_A(a) \), \( a > 0 \). The strength requirement reads:

\[ \Sigma = \frac{n}{A} \lesssim \sigma_{\text{allow}} \]

and the reliability is

\[ R = P \left( A \geq \frac{n}{\sigma_{\text{allow}}} \right) = 1 - F_A \left( \frac{n}{\sigma_{\text{allow}}} \right) \]  \hspace{1cm} (5.22)

The maximal allowable tensile force \( n_{\text{allow}} \) is then determined from the equality

\[ F_A \left( \frac{n_{\text{allow}}}{\sigma_{\text{allow}}} \right) = 1 - r \]

\( n \) being the \( (1 - r) \)-th quantile of the cross-sectional distribution, times \( \sigma_{\text{allow}} \).

Randomness of the cross-sectional area is due to that of its radius \( c \), which has a continuous distribution function \( F_C(c) \); then

\[ \Sigma = \frac{n}{\pi c^2} \lesssim \sigma_{\text{allow}} \hspace{0.5cm} R = F_{\Sigma}(\sigma_{\text{allow}}) \]

The function \( \sigma = n/\pi c^2 \) is strictly monotone decreasing so that, according to Eq. (4.43)

\[ F_{\Sigma}(\sigma) = \int_{\psi(\sigma)}^{\infty} f_C(c) \, dc = 1 - F_C[\psi(\sigma)] \]
where \( \psi(\sigma) = \left( \frac{n}{\pi \sigma} \right)^{\frac{1}{2}} \), the reliability then becomes:

\[
R = 1 - F_C \left( \sqrt{\frac{n}{\pi \sigma_{\text{allow}}} } \right)
\]

and the maximum allowable tensile force is determined as the root of the equation

\[
F_C \left( \sqrt{\frac{n}{\pi \sigma_{\text{allow}}} } \right) = 1 - r
\]

§ 5.4. Beam under random distributed force

We now seek the reliability of a beam of span \( \ell \), with a uniform symmetrical cross-section, simply supported at its ends, under a distributed force \( p(x) = G \phi(x) \), where \( G \) is a continuous random variable with the probability density function \( f_G(g) \), and \( \phi(x) \) is a given (deterministic) function representing the load pattern (Fig. 5.7).

The bending moment is given by

\[
M_z(x) = R_A x + \int_0^x G \phi(u) (x - u) \, du
\]

but \( M_z(\ell) = 0 \), so that

\[
M_z(x) = - \frac{X}{\ell} \int_0^\ell G \phi(u) (\ell - u) \, du + \int_0^x G \phi(u) (x - u) \, du
\]

This expression can be written as

\[
M_z(x) = G \psi(x), \quad \psi(x) = - \frac{X}{\ell} \int_0^\ell \phi(u) (\ell - u) \, du + \int_0^x \phi(u) (x - u) \, du \quad (5.23)
\]

Denote by \( \Sigma' \), the extremal normal stress within the beam cross section \( x \).

The reliability is defined as the probability of the stress \( \Sigma \), not exceeding in absolute value the \( \sigma_{\text{allow}} \) at any section of the beam:

\[
R = P \left\{ |\Sigma'(x)| \leq \sigma_{\text{allow}} \quad 0 \leq x \leq \ell \right\} \quad (5.24)
\]

where
\[ \Sigma'(x) = \frac{M_z(x)}{S} \]

where \( S \) denotes the section modulus. The requirement (5.24) can be replaced by

\[ R = \mathcal{P} \left( |\Sigma| \leq \sigma_{\text{allow}} \right) \quad (5.25) \]

where

\[ \Sigma = \frac{M_z(a)}{S} \]

and in section \( x = a \), the function, \(|\psi(a)|\), has a maximum.

Consider now the case where a stiffness requirement is imposed on the beam: its transverse displacement must not exceed in absolute value some given allowable deflection \( w_{\text{allow}} \). We first find the displacement, by integrating the differential equation

\[ E I_z \frac{d^2 w}{dx^2} = M_z(x) \]

where \( E \) is the modulus of elasticity of the beam material and \( I_z \) is the central moment of inertia of the cross-section about the Oz axis. By virtue of Eq. (5.23), we have:

\[ EIw = G \int \int_0^V \psi(u) \ du \ dv + C_1 x + C_2 \]

The constants are determined by the boundary conditions:

\[ w = 0 \text{ at } x = 0 \text{ and } x = l \]

which finally yield

\[ EIw = G \Psi(x) \]

where

\[ \Psi(x) = \int \int_0^x \psi(u) \ du \ dv - \frac{x}{k} \int \int_0^x \psi(u) \ du \ dv \quad (5.26) \]
Let $|\Psi(x)|$ reach maximum at $x = b$. Denoting the displacement at this section, which is a random variable, by $W$, the reliability is then defined as

$$R = P \left( |W| \leq w_{\text{allow}} \right)$$  \hspace{1cm} (5.27)

Sometimes both stress and stiffness requirements are imposed on the beam. In these circumstances the reliability is

$$R = P(B_1 \cap B_2)$$  \hspace{1cm} (5.28)

where random events $B_1$, $B_2$ are

$$B_1 = \{ |\Sigma| \leq \sigma_{\text{allow}} \}, \quad B_2 = \{ |W| \leq w_{\text{allow}} \}$$  \hspace{1cm} (5.29)

and

$$R = P(B_1) \quad \text{if} \quad B_1 \subseteq B_2$$

$$R = P(B_2) \quad \text{if} \quad B_2 \subseteq B_1$$  \hspace{1cm} (5.30)

$$R = P(B_1) = P(B_2) \quad \text{if} \quad B_1 = B_2$$

Now for some concrete examples.

**Example 5.1**

Let $\varphi(x) = 1$ (Fig. 5.8), then the bending moment reaches its extremum at the middle section, $a = l/2$,

$$M_z \left( \frac{l}{2} \right) = \frac{Gl^2}{8}$$

and the reliability is:

$$R = P \left( \frac{|M|}{M_0} \leq \sigma_{\text{allow}} \right) = F_G \left( \frac{8S \sigma_{\text{allow}}}{l^2} \right) - F_G \left( - \frac{8S \sigma_{\text{allow}}}{l^2} \right)$$  \hspace{1cm} (5.31)

The extremal displacement also occurs at this section, namely

$$W = \frac{5Gl^2}{384EI}$$
and the reliability (5.27) according to the stiffness requirement is:

\[ R = P\left(\frac{5 |G| L^4}{384 EI_Z} \leq w_{allow}\right) = F_G\left(\frac{384 EI_Z w_{allow}}{5L^4}\right) - F_G\left(-\frac{384 EI_Z w_{allow}}{5L^4}\right) \]  (5.32)

If both strength- and stiffness requirements have to be satisfied,

\[ R = P\left(\left|G\right| \leq \frac{8S \sigma_{allow}}{L^2}\right) \cap \left(\left|G\right| \leq \frac{384 EI_Z w_{allow}}{5L^4}\right) \]  (5.33)

and if

\[ \frac{8S \sigma_{allow}}{L^2} < \frac{384 EI_Z w_{allow}}{5L^4} \]  (5.34)

then

\[ \left\{\left|G\right| \leq \frac{8S \sigma_{allow}}{L^2}\right\} \subset \left\{\left|G\right| \leq \frac{384 EI_Z w_{allow}}{5L^4}\right\} \]

and the reliability is as per Eq. (5.31). In the case,

\[ \frac{8S \sigma_{allow}}{L^2} > \frac{384 EI_Z w_{allow}}{5L^4} \]  (5.35)

then

\[ \left\{\left|G\right| \leq \frac{384 EI_Z w_{allow}}{5L^4}\right\} \subset \left\{\left|G\right| \leq \frac{8S \sigma_{allow}}{L^2}\right\} \]

and the reliability is as per Eq. (5.32). If, however

\[ \frac{8S \sigma_{allow}}{L^2} = \frac{384 EI_Z w_{allow}}{5L^4} \]  (5.36)

either equation may be used.

We proceed now the design of the beam. The requirement is as before

\[ R \geq r \]
whereas the equality
\[ R = r \]
yields the required beam dimensions. Let the cross section be circular
with radius \( c \); then
\[ I = \frac{\pi c^4}{4}, \quad S = \frac{I}{c} = \frac{\pi c^3}{4} \]

Determination of \( c \) according to either the strength- or the stiffness
requirement is straightforward. When both are imposed we calculate first
the reliability as a function of \( c \)

\[ R(c) = P \left( \left| G \right| \leq \frac{96\pi EI_{allow}}{5l^4} c^4 \right) \cap \left( \left| G \right| \leq \frac{2\pi \sigma_{allow}}{l^2} c^3 \right) \]
as described above, and then solve the equation \( R(c) = r \), which yields \( c_{req} \).

§ 5.5. Static imperfection-sensitivity of nonlinear model structure

We now proceed to buckling problems, concentrating on the simple model
structure proposed by Budiansky and Hutchinson for illustrating Koiter's
deterministic imperfection-sensitivity notion, such a model of an idealized
column is shown on Fig. 5.9. The three-hinge, rigid-rod system is constrained
laterally by a nonlinear spring with the mass concentrated at the hinge
joining the two rods. We suppose that the restoring force \( F \) is related to the
end shortening (or elongation) \( x \) by

\[ F = k_1 \xi + k_2 \xi^2 + k_3 \xi^3 \quad (5.37) \]

where \( \xi = x/L \). Refraining temporarily from a restriction as to the sign of
\( k_2 \) or \( k_3 \), we assume \( k_1 > 0 \). Equilibrium of the single member of buckling
dictates

\[ \lambda \xi = \frac{1}{2} F \sqrt{1 - \xi^2} = \frac{1}{2} \left( k_1 \xi + k_2 \xi^2 + k_3 \xi^3 \right) \sqrt{1 - \xi^2} \quad (5.38) \]

and for small values of \( \xi \), we obtain the following asymptotic result
\[ \lambda \xi = \lambda_c (\xi + a \xi^2 + b \xi^3 + \ldots) \]  \hspace{1cm} (5.39)

where \( \lambda_c \) is the classical buckling load — that of a linear structure — \[ \lambda_c = \frac{1}{2} k_1 \]  \hspace{1cm} (5.40)

and "a" and "b" are
\[ a = \frac{k_2}{k_1}, \quad b = \frac{k_3}{k_1} - \frac{1}{2} \]  \hspace{1cm} (5.41)

Equation (5.37) represents the axial load-additional displacement relationship. The diagram \( \lambda - \xi \) consists of two branches: the straight line

\[ \xi = 0 \]  \hspace{1cm} (5.42)

and a parabola (if \( b \neq 0 \)) which intersects the \( \lambda \)-axis at the value of the classical buckling load \( \lambda_c \):

\[ \frac{\lambda}{\lambda_c} = b \xi^2 + a \xi + 1 \equiv b \left( \xi + \frac{a}{2b} \right)^2 + 1 - \frac{a^2}{4b} \]  \hspace{1cm} (5.43)

with its vertex at \((-a/2b; 1 - a^2/4b), b \neq 0\). Consequently,

\[ \left( \frac{\lambda}{\lambda_c} \right)_{\text{min}} = 1 - \frac{a^2}{4b}, \quad \text{for } b > 0; \quad \left( \frac{\lambda}{\lambda_c} \right)_{\text{max}} = 1 - \frac{a^2}{4b}, \quad \text{for } b < 0 \]

The structure is designated as nonsymmetric in the general case \( a \neq 0, b \neq 0 \); as symmetric if \( a = 0, b \neq 0 \); and as asymmetric if \( a \neq 0, b = 0 \). In the latter case the parabola degenerates into a straight line.

\[ \lambda/\lambda_c = a \xi + 1 \]  \hspace{1cm} (5.44)

Typical \( \lambda - \xi \) curves are shown in Figs. 5.10-5.13. Note that the curve for \( a = -a_1 \) and specified "b" is the mirror image of its counterpart for \( a = a_1 \) and "b" as before.

We now proceed to the realistic imperfect structure. Assuming that unloaded structure has an initial displacement \( \bar{x} = L \bar{\xi} \), then equilibrium dictates, instead of Eq. (5.38), the following relationship:
\[
\lambda(\xi + \xi) = \frac{1}{2} F \left[ 1 - (\xi + \xi)^2 \right]^2 \left( k_1 \xi + k_2 \xi^2 + k_3 \xi^3 \right) \left[ 1 - (\xi + \xi)^2 \right]^2
\]

where \( \xi \) is an additional displacement and \( \xi + \xi \) a total displacement.

For small values of \( \xi \), we arrive at the following asymptotic result:

\[
\lambda(\xi + \xi) = \lambda_c \left[ \xi + a \xi^2 + b \xi^3 + o(\xi^2 \xi) \right]
\]

Equation (5.45) indicates that \( \xi \) and \( \xi \) have the same sign (i.e. additional displacement \( \xi \) of the system is such that the total displacement \( \xi + \xi \) is increased by its absolute value). Otherwise, the assumption \( \xi \xi < 0 \) would imply \( \lambda < 0 \) for \( 0 < |\xi| < \xi \), i.e. presence of tension, which is contrary to our formulation of the problem. Note also that the graph \( \lambda/\lambda_c \) vs. \( \xi \) for an imperfect structure issues from the origin of the coordinates. Additional zeroes of \( \lambda/\lambda_c \) coincide with the zero points \((-a \pm \sqrt{a^2 - 4b})/2b\) of the parabola (5.43) representing the behaviour of a perfect structure.

We now seek the static buckling load \( \lambda_s \), which is defined as the maximum of \( \lambda \) on the branch of the solution \( \lambda - \xi \) originating at zero load, for specified \( \xi \). We refer to a structure as imperfection-sensitive if an imperfection results in reduced values of the maximum load the structure is able to support; otherwise, we designate the structure imperfection-insensitive.

To be able to conclude whether a structure is sensitive to initial imperfections or not, we have to find whether the first derivative of \( \lambda \) with respect to \( \xi \)

\[
\frac{d\lambda}{d\xi} = \frac{\lambda_c}{(\xi + \xi)^2} \left[ 2b \xi^3 + (a + 3b \xi) \xi^2 + 2a \xi \xi + \xi \right]
\]

has at least one real root. For our purpose, it suffices to examine the numerator

\[
\phi(\xi) = 2b \xi^3 + (a + 3b \xi) \xi^2 + 2a \xi + \xi
\]

The structure buckles if the equation

\[
\phi(\xi) = 0
\]

has at least one real positive root for \( \xi > 0 \), or at least one real negative root for \( \xi < 0 \).

† The point with coordinates \( (\lambda_{\text{max}}, \xi_0) \) where \( \xi_0 \) is the additional displacement corresponding to \( \lambda_{\text{max}} \), is sometimes called the limit point, and \( \lambda_{\text{max}} \) itself the limit load.
In some cases Descartes' rule of signs provides an answer on buckling of the structure. This rule states that if the coefficients $a_0$, $a_1$, $a_2$, ..., $a_n$ of the polynomial

$$
\phi(\xi) = a_0 \xi^n + a_1 \xi^{n-1} + \ldots + a_{n-1} \xi + a_n
$$

have $v$ variations of sign, the number of positive roots of the polynomial equation $\phi(\xi) = 0$ does not exceed $v$ and is of the same parity. The number of negative roots of this equation equals that of positive roots of equation $\phi(-\xi) = 0$.

Consider the case $b < 0$, $\xi > 0$. We wish to know whether there is at least one positive root between those of $\phi(\xi) = 0$. In the subcase $a < 0$, we have

$$
a_0 = 2b < 0, \quad a_1 = a + 3b \xi < 0, \quad a_2 = 2a \xi < 0, \quad a_3 = \xi > 0
$$

so there is a single change in sign, indicating occurrence of buckling.

In the subcase $a > 0$, we have

$$
a_0 = 2b < 0, \quad a_2 = 2a \xi > 0, \quad a_3 > 0
$$

so, irrespective of the sign of $a_1 = a + 3b \xi$, there is again a single change in sign. Thus the structure has a buckling load if $b < 0$, $\xi > 0$.

Consider now the case $b < 0$, $\xi < 0$. The question is now whether $\phi(\xi) = 0$ has at least one negative root. Then

$$
\phi(-\xi) = -2b \xi^3 + (a + 3b \xi) \xi^2 - 2a \xi \xi + \xi
$$

for $a > 0$ we have

$$
a_0 = -2b > 0, \quad a_1 = a + 3b \xi > 0, \quad a_2 = -2a \xi > 0, \quad a_3 = \xi < 0
$$

i.e. a single change in sign; for $a < 0$ we have

$$
a_0 = -2b > 0, \quad a_2 = -2a \xi < 0, \quad a_3 = \xi < 0
$$

and irrespective of the sign of $a_1$ we again have a single change in sign, the conclusion being that for $b < 0$ (irrespective of the signs of $a$ or $\xi$) the
structure carries a finite maximum load. In complete analogy, it can be shown that the structure is imperfection-insensitive for $b > 0$ and $a \xi > 0$. (left as exercise for the reader).

For $b > 0$ and $a \xi < 0$, neither Descartes' rule nor the Rouche-Hurwitz criterion for the number of roots with positive real part (used in conjunction with the fact that $\phi(\xi) = 0$ always has one real root) suffice for a conclusion, this case, however, can be treated by Evans' root-locus method (see e.g. Ogata), frequently used in control theory.

Let us consider first the particular subcase $b > 0$, $a < 0$ and $\xi > 0$. The formal substitution $\xi \rightarrow s$, where $s = \text{Re}(s) + i \text{Im}(s)$ is a complex variable, in Eq. (5.46).

\[ 1 + \xi \psi(s) = 0, \quad \psi(s) = \frac{3bs^2 + 2as + 1}{s^2(2bs + a)} \quad (5.47) \]

We now construct the root-locus plot with $\xi$ varying from zero to infinity (obviously, only $\xi << 1$ has physical significance). For $\xi$ approaching zero, the roots of Eq. (5.47) are the poles of $\psi(s)$, marked with "crosses" (X's):

\[ s_1 = s_2 = 0, \quad s_3 = -\frac{a}{2b} \]

The $\xi \rightarrow \infty$ points of the root loci approach the zeros of $\psi(s)$, marked with "circles" (O's):

\[ s_{1,2} = \frac{1}{3b} \left(-a \pm \sqrt{a^2 - 3b}\right) \]

$\psi(s)$ has three poles: one double at zero, and another at $(-a/2b) > 0$. A root locus issues from each pole as $\xi$ increases above zero; a root locus arrives at each zero of $\psi(s)$ or at infinity, as $\xi$ approaches infinity. For the case $a^2 = 3b$, the two "circles" coincide. As is seen from Fig. 5.14, Eq. (5.47) has two real positive roots, and the structure buckles for any $\xi > 0$. For $a^2 < 3b$ both "circles" are complex (Fig. 5.15); for a certain value of $\xi$ (called the critical one, $\xi_{cr,s}$) a pair of loci break away from the real axis. For $\xi > \xi_{cr,s}$ Eq. (5.47) has no real positive root, and, consequently, the structure is imperfection-insensitive. The breakaway point is found as the root of the equation

\[ \frac{dc}{ds} = 0, \quad C(s) = \frac{s^2(2bs + a)}{3bs^2 + 2as + 1} \]
This equation has only one real root \( s_0 = -a/3b \). The appropriate value of \( \xi \) equals \(-C(s_0)\):

\[
\xi_{cr,s} = -\frac{a}{9b} \frac{1}{3b - a^2}
\]  

(5.48)

Static buckling load associated with \( \xi = s_0 \) and \( \xi = \xi_{cr,s} \) is

\[
\frac{\lambda_s}{\lambda_c} = \frac{a}{3b} \left( \frac{2}{b} \frac{a^2}{b^2} \right)^{-1} \left( \xi_{cr,s} - \frac{a}{3b} \right)^{-1}
\]

For example, for \( b/a^2 = 2/3 \), we have \( \xi_{cr,s} = -1/6a \) and \( \lambda_s/\lambda_c = 1/2 \), and for \( \xi > \xi_{cr,s} \) static buckling does not occur.

For the case \( a^2 > 3b \) (see Figs. 5.16-5.18) there are always two real positive roots to \( \phi(\xi) = 0 \) and the structure buckles.

Consequently, the structure turns out to buckle for \( a^2 > 3b \); in the range \( a^2 < 3b \) the structure buckles if \( \xi < \xi_{cr,s} \).

We next consider the case \( a > 0, b > 0, \pi < 0 \). It is readily shown that the inverse root loci for \(-\pi < \xi < 0 \) are the mirror images of the original root loci for \( 0 \leq \xi \leq \infty \) with respect to the imaginary axis. The system buckles for \( a^2 > 3b \); and also for \( a^2 < 3b \) if \( \xi > \xi_{cr,s} \) as per Eq. (5.48).

For the particular case \( a = 0 \) the structure buckles if \( b < 0 \) (for both \( \xi > 0 \) and \( \xi < 0 \)) and is insensitive if \( b > 0 \).

For \( b = 0 \), the structure buckles if \( a \xi < 0 \) and does not buckle in the opposite case.

As for the structure which buckles, differentiating Eq. (5.45) with respect to \( \xi \) and setting (for \( b \neq 0 \))

\[
\frac{d\lambda}{d\xi} = 0, \quad \lambda = \lambda_s
\]  

(5.49)

we obtain the relation between the buckling load \( \lambda_s \) and initial imperfection amplitude \( \xi \):

\[
\left( 1 - \frac{\lambda_s}{\lambda_c} - \frac{a^2}{3b} \right)^3 = -\frac{27}{4} b \left[ \frac{a}{3b} \left( 1 - \frac{\lambda_s}{\lambda_c} \right) - \frac{2}{27} \frac{a^3}{b^2} + \frac{\lambda_s}{\lambda_c} \xi \right]^2
\]  

(5.50)

For \( b < 0 \) and \( a = 0 \) we get from (5.50):
\[(1 - \frac{\lambda_S}{\lambda_C})^{3/2} - \frac{3\sqrt{3}}{2} |\xi| \sqrt{\frac{\lambda_S}{\lambda_C}} = 0 \] (5.51)

Note that the displacement \(\xi\) corresponding to \(\lambda_S/\lambda_C\) is given by
\[\xi_{1,2} = \frac{1}{3b} \left[-a + \sqrt{a^2 - 3b \left(1 - \frac{\lambda_S}{\lambda_C}\right)}\right] \] (5.52)

where \(\xi_{1,2}\) depend on \(\xi\) via \(\lambda_S/\lambda_C\). From Eq. (5.50) the static buckling load is obtainable, given the initial imperfection \(\xi\). The meaningful root \(\lambda_S/\lambda_C\) of Eq. (5.50) is the greatest of those which meet the requirement \(\xi > 0\).

The case \(b = 0\) has to be considered separately. Eq. (5.46) reduces
\[\phi(\xi) = a \xi^2 + 2a \xi^2 + \xi^2\]

Descartes' rule then immediately yields the conclusion that the structure is imperfection-sensitive if \(a_F < 0\). Eq. (5.49) then leaves us with
\[(1 - \frac{\lambda_S}{\lambda_C})^2 + 4a_F \frac{\lambda_S}{\lambda_C} = 0 \] (5.53)

Equations (5.50), (5.51) or (5.53) permit us now to find the probabilistic characteristics of the random buckling load \(\Lambda_S\) (with possible values \(\lambda_S\)) provided the initial imperfection \(\bar{\xi}\) (with possible values \(\bar{\xi}\)) is a random variable with given probability distribution \(P_{\bar{\xi}}(\xi)\). We seek the reliability of the structure, which is defined in these new circumstances as the probability of the event of the nondimensional buckling load \(\Lambda_S/\lambda_C\) exceeding the given nondimensional load \(\alpha\):
\[R(\alpha) = P\left(\frac{\lambda_S}{\lambda_C} > \alpha\right) \] (5.54)

Consider for example, the symmetric structure \((a = 0)\), and assume that the initial imperfection \(\bar{\xi}\) is normally distributed \(N(m_{\bar{\xi}}, \sigma_{\bar{\xi}}^2)\).

Fig. 5.19a, which shows schematically the solution of the equation (5.51) indicates that \(\lambda_S/\lambda_C\) exceeds \(\alpha\), when \(\bar{\xi}\) falls within the interval \((-\xi', \xi')\) where
\[\xi' = -\frac{2}{3\sqrt{3}} \frac{(1 - \alpha)^{3/2}}{\alpha \sqrt{b}}\]
\[ \frac{1}{\omega_1^2} \frac{d^2 \xi}{dt^2} + \left[ 1 - \frac{\lambda f(t)}{\lambda_c} \right] \xi + a \xi^2 + b \xi^3 = \frac{\lambda f(t)}{\lambda_c} \xi \]  \tag{5.57}

where \( \omega_1 = \sqrt{\kappa_1/ML} \) is the natural frequency of the mass for \( \lambda = 0 \). As an example, we have the load represented by the step function of unit magnitude and unlimited duration, defined in Eq. (3.2)

\[ f(t) = U(t) \]

which vanishes for \( t < 0 \) and equals unity for \( t \geq 0 \). The first integral of Eq. (5.57) subject to the initial conditions

\[ \xi = 0, \quad \frac{d\xi}{dt} = 0 \text{ at } t = 0 \]

(zero displacement and zero velocity \( t = 0 \), i.e. the mass at rest up to that time) is readily found to be

\[ \frac{1}{\omega_1^2} \xi^2 + \left( 1 - \frac{\lambda}{\lambda_c} \right) \xi^2 + \frac{2}{3} a \xi^3 + \frac{1}{2} b \xi^4 = 2 \left( \frac{\lambda}{\lambda_c} \right) \xi \]

and the corresponding integral curves satisfy

\[ \pm \int_0^\xi \left[ 2 \left( \frac{\lambda}{\lambda_c} \right) \xi - \left( 1 - \frac{\lambda}{\lambda_c} \right) \xi^2 - \frac{2}{3} a \xi^3 - \frac{1}{2} b \xi^4 \right]^{-1/2} d\xi = \omega_1 t \]

where the left member may be evaluated in terms of elliptic integrals.
For sufficiently small $\lambda/\lambda_c$ the motion is periodic, with its amplitude satisfying

$$
\left(1 - \frac{\lambda}{\lambda_c}\right) \xi_{\text{max}} + \frac{2}{3} a \xi_{\text{max}}^2 + \frac{1}{2} b \xi_{\text{max}}^3 = 2 \left(\frac{\lambda}{\lambda_c}\right) \xi
$$

(5.58)

The dynamic buckling load $\lambda_d$ is defined as the maximum value of $\lambda$, such that the response $\xi(t)$ remains bounded. At $\lambda_d$ a finite jump in $\xi_{\text{max}}$ is produced by an infinitesimal increase in $\lambda$ (see Fig. 5.25). For all $\lambda \neq \lambda_d$, the response $\xi(t)$ is periodic; as $\lambda$ approaches $\lambda_d$ from below, the period tends to infinity and it takes an infinitely long time for $\xi(t)$ to reach $\xi_{\text{max}}$. The value of $\lambda_d$ occurs at the first maximum of the relation $\lambda$ versus $\xi_{\text{max}}$. Thus the dynamic buckling load is defined by the criterion

$$
\frac{d\lambda}{d\xi_{\text{max}}} = 0, \quad \lambda = \lambda_d
$$

(5.59)

Note that Eq. (5.59) identifies with Eq. (5.45) on the following formal substitution

$$
\xi \rightarrow \xi_{\text{max}}, \quad a \rightarrow \frac{2}{3} a, \quad b \rightarrow \frac{1}{2} b, \quad \xi \rightarrow 2\xi
$$

(5.60)

Analogically, all conclusions in the static case are readily extended to the dynamic one; namely, the structure under a step load buckles for any $b < 0$ (irrespective of the sign of "a" or $\xi$), and does not buckle for $b > 0$ and $a \xi > 0$; for $b > 0$ and $a \xi < 0$ it buckles for any $\xi$, given the following inequality

$$
a^2 > \frac{27}{8} b
$$

obtainable from its static counterpart $a^2 > 3b$ by formal substitution (5.60), and also when

$$
a^2 < \frac{27}{8} b
$$

for $\xi \leq \xi_{cr,d}$, where

$$
\xi_{cr,d} = -\frac{4}{27} \frac{a^3}{b} \frac{1}{(27/6) b - a^2}
$$

The $\xi_{\text{max}}$ value associated with $\xi_{cr,d}$ equals $4a/9b$ and, consequently, the dynamic buckling load is
\[ \frac{\lambda_d}{\lambda_c} = \frac{4}{9} \frac{a^2}{b (61 b - 1) \left(2 \xi_{cr,d} - \frac{4 a}{9 b}\right)^{-1}} \]

At \( \xi = \xi_{cr,d} \) the concept of dynamic buckling is preserved by associating \( \lambda_d \) with the point of inflection in the \( \lambda - \xi_{\text{max}} \) curves according to Budiansky (1965) (Fig. 5.26). Comparison of the latter results with their static counterparts shows that the interval \( 3b < a^2 < 27b/8 \) is characterised by duality: the structure is statically imperfection-sensitive, but dynamically imperfection-insensitive. In the particular case \( b/a^2 = 2/3 \), considered by Budiansky (pp. 95-96), \( \xi_{cr,d} = -8/45a \) and \( \xi_{cr,s} > \xi_{cr,d} \), and consequently, the interval \( \xi_{cr,s} < \xi < \xi_{cr,d} \) is similarly characterised by duality - the reverse of the preceding case.

For \( b/a^2 = 1/3 \) the structure is statically imperfection-sensitive for any \( \xi \), but dynamically imperfection-insensitive, if \( \xi > \xi_{cr,d} = (-32/9a) \).

Finally, the relation between the buckling load \( \lambda_d \) and the initial imperfection \( \xi \) is given by (\( b \neq 0 \))

\[ \left(1 - \frac{\lambda_d}{\lambda_c} - \frac{8}{27 b} a^2 \right)^{3/2} = -\frac{27}{8} b \left[ \frac{4 a}{9 b} \left(1 - \frac{\lambda_d}{\lambda_c}\right) - \frac{64}{729} a^3 \frac{b^2}{2} + 2 \xi \frac{a^4}{\lambda_c^2} \right] \]  
(5.61)

For vanishing \( a \) and \( b < 0 \), we obtain the following relationship

\[ \left(1 - \frac{\lambda_d}{\lambda_c}\right)^{3/2} - \frac{3\sqrt{6}}{2} \xi^2 \frac{\lambda_d}{\lambda_c} = 0 \]  
(5.62)

For the case \( b = 0 \), \( a \neq 0 \) we immediately obtain, substituting (5.60) in (5.53):

\[ \left(1 - \frac{\lambda_d}{\lambda_c}\right)^{3/2} + 16 a \xi \frac{\lambda_d}{\lambda_c} = 0. \]

In the case \( a < 0 \), \( b < 0 \) with \( \xi \) such that \( \lambda_s/\lambda_c < 1 \) and \( \lambda_d/\lambda_c < 1 \), \( \xi \) is readily eliminated by correlating eqs. (5.50) and (5.61) for a given structure with a given imperfection. The result relates \( \lambda_d \) to \( \lambda_s \):

\[ \begin{align*}
\left[ -\frac{4}{27 b} \left(1 - \frac{\lambda_d}{\lambda_c} - \frac{8}{27 b} a^2 \right)^{3/2} - \frac{4 a}{9 b} \left(1 - \frac{\lambda_d}{\lambda_c}\right) + \frac{64}{729} a^3 \frac{b^2}{2} \right] \\
= 2 \frac{\lambda_d}{\lambda_c} \left[ -\frac{4}{27 b} \left(1 - \frac{\lambda_s}{\lambda_c} - \frac{a^2}{3 b} \right)^{3/2} - \frac{a}{3 b} \left(1 - \frac{\lambda_s}{\lambda_c}\right) + \frac{2}{27} \frac{a^3}{b^2} \right] \\
\end{align*} \]  
(5.63)
In this form, $\frac{\lambda_d}{\lambda_s}$ is no longer directly dependent on the imperfection, but via $\frac{\lambda_s}{\lambda_c}$. For vanishing "a" we obtain the expression:

$$\frac{\lambda_d}{\lambda_s} = \frac{\sqrt{2}}{2} \left( \frac{\lambda_c - \lambda_d}{\lambda_c - \lambda_s} \right)^{3/2}$$  \hspace{1cm} (5.64)

Eqs. (5.62) - (5.64) are due to Budiansky and Hutchinson.

Recapitulating, our structure is statically imperfection-sensitive and dynamically imperfection-insensitive when

$$3b < a^2 < \frac{27}{8} b, \quad \bar{\xi} > \bar{\xi}_{cr,d}$$

and also when

$$a^2 < 3b, \quad \bar{\xi}_{cr,d} < \bar{\xi} < \bar{\xi}_{cr,s}$$

For this case, following Budiansky, the criterion of the dynamic buckling is generalized, the dynamic buckling load being defined as the point of inflection on the $\lambda - \bar{\xi}_{max}$ curve:

$$\frac{d^2\lambda}{d\bar{\xi}_{max}^2} = \frac{\lambda_c}{(\bar{\xi}_{max} + 2\bar{\xi})^3} \left[ b(\bar{\xi}_{max} + 2\bar{\xi})^3 - 8b\bar{\xi}^3 + 16\frac{b}{3} - 6\frac{a^2}{3} - 4\bar{\xi} \right] = 0$$ \hspace{1cm} (5.65)

(see also Fig. 5.21). For $b > 0$, $a < 0$ and $\bar{\xi} > 0$, Descartes' rule of signs indicates a single positive root. Eliminating $\bar{\xi}_{max}$ between Eq. (5.58) and (5.65), we relate the generalized buckling load and the initial imperfection

$$\frac{\lambda_d}{\lambda_c}^d = (d - 2\bar{\xi}) \left[ 1 + \frac{2}{3} a (d - 2\bar{\xi}) + \frac{1}{2} b (d - 2\bar{\xi})^2 \right]$$ \hspace{1cm} (5.66)

where

$$d = \left( \frac{24b\bar{\xi}^3 - 16a\bar{\xi} + 12\bar{\xi}}{3b} \right)^{1/3}$$

We proceed now to the reliability analysis for a symmetric structure, assuming $\bar{X}$ be $N(\mu, \sigma^2 X)$:

$$R(\alpha) = P\left( \frac{\lambda_d}{\lambda_c} > \alpha \right) = P\left( \bar{\xi}' < \bar{X} < \bar{\xi}'' \right)$$ \hspace{1cm} (5.67)
where
\[
\xi^n = \frac{2}{3\sqrt{6}} \frac{(1 - \alpha)^{3/2}}{\alpha^{1/3}}
\]
resulting in
\[
R(\alpha) = \operatorname{erf} \left[ \frac{\xi^n(\alpha) - m_X}{\sigma_X} \right] + \operatorname{erf} \left[ \frac{\xi^n(\alpha) + m_X}{\sigma_X} \right]
\]
(5.68)

Note that the probability densities of static or dynamic buckling loads are obtainable via the reliability function. Indeed, the unreliability at the load level \( \alpha \) is
\[
\Omega(\alpha) = 1 - R(\alpha) = \Gamma \left( \frac{\Lambda_s}{\lambda_c} \leq \alpha \right) = F_{\Lambda_s}(\alpha)
\]
(5.69)

and therefore
\[
\frac{dF_{\Lambda_s} / \lambda_c}{d\alpha} \bigg|_{\alpha = \lambda_s / \lambda_c} = - \frac{dR(\alpha)}{d\alpha} \bigg|_{\alpha = \lambda_s / \lambda_c}
\]
(5.70)

Eq. (5.56) yields
\[
\frac{dF_{\Lambda_s} / \lambda_c}{d\alpha} \bigg|_{\alpha = \lambda_s / \lambda_c} = \frac{1}{9} \sqrt{\frac{6}{\pi(-b)}} \frac{2 + (\lambda_s / \lambda_c)}{\sigma_X(\lambda_s / \lambda_c)^2} \left( 1 - \frac{\lambda_s}{\lambda_c} \right)^{1/2} \exp \left[ - \frac{2(1 - \lambda_s / \lambda_c)^3}{(\lambda_s / \lambda_c)^2(-b)} \right]
\]
(5.71)

(see Fig. 5.27).

With (5.71) available, we are able, if necessary, to calculate the mean buckling load \( \bar{\Lambda}_s \)
\[
\bar{\Lambda}_s = \int_0^\infty \lambda_s f_{\Lambda_s} (\lambda_s) \, d\lambda_s
\]
(5.72)

the variance of the buckling load
\[
\text{Var}(\Lambda_s) = \int_0^\infty \lambda_s^2 f_{\Lambda_s} (\lambda_s) \, d\lambda_s - [\bar{\Lambda}_s]^2
\]
(5.73)

and its higher moments.

In particular, in view of the identity
\begin{equation*}
\frac{f_{\Lambda_s}(\lambda_s)}{\lambda_c} = \frac{1}{\lambda_c} \frac{f_{\Lambda_s/\lambda_c}(\lambda_s/\lambda_c)}{\lambda_c}
\end{equation*}

we obtain for the mean buckling load

\begin{equation*}
E(\Lambda_s) = \int_0^\lambda_s \lambda_s \left[ \frac{1}{\lambda_c} f_{\Lambda_s/\lambda_c}(\lambda_s/\lambda_c) \right] \, d\lambda_s
\end{equation*}

or

\begin{equation*}
E(\Lambda_s/\lambda_c) = \int_0^1 \alpha \cdot f_{\Lambda_s/\lambda_c}(\alpha) \, d\alpha.
\end{equation*}

Further, taking into account Eq. (5.70), we arrive at

\begin{equation*}
E(\Lambda_s/\lambda_c) = \int_0^1 \alpha \left[ - \frac{dR(\alpha)}{d\alpha} \right] \, d\alpha
\end{equation*}

Integrating by parts

\begin{equation*}
E(\Lambda_s/\lambda_c) = -\alpha R(\alpha) \bigg|_0^1 + \int_0^1 R(\alpha) \, d\alpha
\end{equation*}

However,

\begin{equation*}
R(1) = \text{Prob}(\Lambda_s > \lambda_c) = 0,
\end{equation*}

and finally,

\begin{equation*}
E(\Lambda_s/\lambda_c) = \int_0^1 R(\alpha) \, d\alpha
\end{equation*}

so that the nondimensional mean buckling load equals the area under the reliability curve (see Fig. 5.28).

The standard deviation of the nondimensional buckling load

\begin{equation*}
\sigma_{\Lambda_s/\lambda_c} = \sqrt{E[(\Lambda_s/\lambda_c)^2] - [E(\Lambda_s/\lambda_c)]^2}
\end{equation*}

where

\begin{equation*}
E[(\Lambda_s/\lambda_c)^2] = \int_0^1 \alpha^2 \cdot f_{\Lambda_s/\lambda_c}(\alpha) \, d\alpha
\end{equation*}
is shown in Fig. 5.29 $\sigma_{\Delta_s}/\lambda_c$ increases with the standard deviation $\sigma_X$ of the initial imperfections.

With the reliability function available, we can solve the design problem, namely, determine the allowable buckling load from the requirement

$$R(\alpha_{\text{allow}}) = r, \quad \alpha_{\text{allow}} = \frac{\lambda_{\text{allow}}}{\lambda_c}$$

(5.79)

where $r$ is the required reliability. Referring to Eq. (5.56), we obtain the following transcendental equation for $\alpha_{\text{allow}}$:

$$\text{erf} \left[ \frac{\bar{\xi}'(\alpha_{\text{allow}}) - m_X}{\sigma_X} \right] + \text{erf} \left[ \frac{\bar{\xi}'(\alpha_{\text{allow}}) + m_X}{\sigma_X} \right] = r$$

(5.80)

For $m_X = 0$ the above, bearing in mind the expression for $\bar{\xi}'(\alpha_{\text{allow}})$ reduces to

$$\frac{2}{3^{3/2} \alpha_{\text{allow}}} = \text{erf}^{-1} \left(\frac{r}{2}\right)$$

(5.81)

where $\text{erf}^{-1}(\ldots)$ is the inverse of $\text{erf}(\ldots)$, found from the table in Appendix B. For example, for $r = 0.99$ we have $\text{erf}^{-1}(0.495) = 2.757$. The last equation may be rewritten as

$$\sigma_X = \frac{2}{3^{3/2} \alpha_{\text{allow}}} \frac{(1 - \alpha_{\text{allow}})^{3/2}}{\sqrt{-b} \text{erf}^{-1}(r/2)}$$

(5.82)

and $\sigma_X$ treated as a function of $\alpha_{\text{allow}}$. This function is given in Fig. 5.30 as curve 1.

For $m_X \neq 0$ the transcendental equation (5.80) has to be solved numerically. Results are given in Fig. 5.30 as curves 2 and 3. It is seen that the allowable buckling load decreases as the mean imperfection increases, and that for larger values of $\sigma_X$ the mean imperfection becomes less significant and the allowable buckling loads associated with different mean imperfections lie closer together.

Note also that for $\sigma_X = 0$ the allowable buckling load coincides with the mean buckling load irrespective of the required reliability; as is seen from Eq. (5.82), $\sigma_X = 0$ if $\alpha_{\text{design}} = 1$ for any $r$. 
Fig. 5.31 contrasts the design buckling load associated with the required reliability \( r = 0.99 \) and the mean buckling load. As is seen, the allowable buckling loads are much smaller than the mean ones. This implies that the design according to the latter overestimates the load carrying capacity of the structure, as against \( \alpha \) associated with the high reliability.

Significantly, Figs. 5.20 and 5.30 apply for the dynamic case as well, albeit for a different \( b \) — namely, \( b = 1/(-1) = -1 \), according to the analogy in Eq. (5.60). Comparison of the two cases for the same \( b = -1 \) shows (Fig. 5.32) that the dynamic allowable buckling loads are lower than their static counterparts, the analogue of (5.82) being

\[
\sigma_R = \frac{2}{3\sqrt{6}} \frac{(1 - \alpha_{\text{allow}})}{\sigma_{\text{allow}}} \sqrt{-b} \text{ erf}^{-1}(r/2) \tag{5.83}
\]

Comparing the two analogues it is seen that for the loads to be equal, the standard deviation of the initial imperfections in the dynamic case should be \( 1/\sqrt{2} \) that of the static case. In other words, the dynamic allowable buckling load at a specific \( \sigma_R \) is obtainable directly from the static curve by reading the latter at \( \sqrt{2}\sigma_R \).

§ 5.7. Axial impact of a bar with random initial imperfection

We consider now another problem of dynamic buckling — that of the initially imperfect bar under axial impact (Fig. 5.33). In the case under consideration, we use Hoff’s definition, which states that "A structure is in a stable state if admissible finite disturbances of its initial state of static or dynamic equilibrium are followed by displacements whose magnitude remains within allowable bounds during the required lifetime of the structure" (in contrast to the previous paragraph, where boundedness of the displacements was required for the structure to be considered stable). By virtue of Hoff’s definition, we may postulate that a bar with initial imperfections fails (buckles) under axial forces when its dynamic response (deflection, strain or stress) first reaches an upper-bound level \( q^+ \) or a lower-bound level \( q^- \), \( q^+ \) and \( q^- \) being prescribed positive numbers which represent borderlines between stability and buckling (i.e., safety and failure) (Fig. 5.34).

Consider an ensemble of response histories \( y(t) \) in the interval \( 0 \leq t < t^* \), all of them originating under the same initial conditions \( t = 0 \). Let \( R(t; t^*) \)
be the probability of \( y(t) \) remaining in the same domain throughout the interval \((0, t^*)\). Formally, the reliability is then

\[
R(t; t^*) = \text{Prob} \left\{ \left[ y(t) \leq \Omega^+ \right] \cap \left[ y(t) \geq -\Omega^- \right] ; \ 0 \leq t \leq t^* \right\}
\] (5.84)

The probability of failure (unreliability) is

\[
Q(t; t^*) = \text{Prob} \left\{ \left[ y(t) > \Omega^+ \right] \cup \left[ y(t) < -\Omega^- \right] ; \ 0 \leq t \leq t^* \right\}
\] (5.85)

since failure is either the event \( \{ y(t) > \Omega^+ \} \) or \( \{ y(t) < -\Omega^- \} \).

Obviously,

\[
R(t; t^*) = 1 - Q(t; t^*)
\]

which is readily understood, if we recall that in these circumstances reliability is the probability of survival up to time \( t \), or that of "being", whereas unreliability is the probability of failure of "not being"; the random event "to be or not to be" has a unity probability.

For \( t^* \) tending to infinity, the reliability and the unreliability are functions of \( t \) only, and we denote

\[
R(t) = \lim_{t^* \to \infty} R(t; t^*) , \quad Q(t) = \lim_{t^* \to \infty} Q(t; t^*)
\]

We seek probabilistic information on the random time when failure occurs (i.e. the structure buckles) under suitable initial conditions, in terms of \( y(t) \) and perhaps of its derivatives. This problem is very difficult in the general case and is known as the "first-passage" (or "first-exursion") problem. We give here the exact solution to this problem for impact buckling of a bar with random initial imperfections of given shape, with the amplitude as a continuous random variable. We formulate the problem as follows:

Given the probability distribution function of the amplitude of random initial imperfection, find the probability of the time required for the response process to move outside the prescribed safe domain (i.e. the first excursion time) being less than the given time, \( t^* \) being infinity.
Consider first the corresponding deterministic problem. We disregard axial
wave propagation and assume uniform compression throughout the bar, whose
motion obeys the differential equation:

$$\frac{EI}{\lambda^4} \frac{d^4 y}{dx^4} + \frac{P}{\lambda^2} \frac{d^2 y}{dx^2} + \rho A \frac{d^2 y}{dt^2} = -P \frac{d^2 y}{dx^2}$$  \hspace{1cm} (5.86)

Here \( x \) is the axial coordinate, \( t \) - time, \( y(x) \) - the initial imperfection
(a small perturbation to the perfect, straight shape of the bar), \( y(x,t) \) - the additional transverse deflection measured from \( y_o(x) \) \( [y_o(x) + y(x,t)] \) - being the total deflection of the beam axis from the straight line between
the two ends \( x = 0, x = \ell \), \( E \) - Young's modulus, \( I \) - the moment of inertia,
\( \rho \) - mass density, \( A \) - the cross-sectional area, \( P \) - the applied axial load,
\( \xi I \) and \( \rho A \) are taken as constants.

The differential equation (5.86) is supplemented by the boundary conditions:

\[
\begin{align*}
  y(x,t) &= 0, \quad \frac{d^2 y(x,t)}{dx^2} = 0, \quad \text{at } x = 0 \\
y(x,t) &= 0, \quad \frac{d^2 y(x,t)}{dx^2} = 0, \quad \text{at } x = \ell \\
\end{align*}
\]

and the initial conditions:

\[
\begin{align*}
  y(x,t) &= 0, \quad \frac{d y(x,t)}{dt} = 0, \quad \text{at } t = 0 \\
\end{align*}
\]

The shape of the imperfection is taken as:

\[
\tilde{y}(x) = H \sin \frac{\pi x}{\ell} \quad \text{ (5.87)}
\]

We now introduce the nondimensional quantities:

\[
\begin{align*}
  \xi &= \frac{x}{\ell}, \quad \lambda = \omega_1 t, \quad \alpha = \frac{P}{P_c}, \quad u(\xi, \lambda) = \frac{y(x,t)}{\Delta} \\
  \tilde{u}(\xi) &= \frac{\tilde{y}(x)}{\Delta}, \quad \Delta = \frac{H}{\lambda} \\
\end{align*}
\]

Where

\[
\begin{align*}
P_c &= \frac{\pi^2 EI}{\ell^2}, \quad \omega_1 = \left(\frac{\pi}{\ell}\right)^2 \sqrt{EI/\rho A}, \quad \Delta = \sqrt{I/A}
\end{align*}
\]
$P_c$, being the classical (or Euler) buckling load of a perfect bar, $\omega_1$ - its fundamental natural frequency in the absence of axial compression, $A$ - its radius of gyration, $\alpha$ - the nondimensional applied force, $u(\xi, \tau)$, $\tilde{u}(\xi)$ - the nondimensional additional and initial displacements, respectively, $G$ - the nondimensional amplitude of the initial imperfection, $\lambda$ - the non-dimensional time.

Eqs. (5.76) and (5.77) become, respectively,

$$\frac{\partial^4 u}{\partial \xi^4} + \pi^2 \alpha \frac{\partial^2 u}{\partial \xi^2} + \pi^4 \frac{\partial^2 u}{\partial \lambda^2} = -\pi^2 \alpha \frac{d^2 \tilde{u}}{d \xi^2}$$  \hspace{1cm} (5.88)

$$\tilde{u}(\xi) = c \sin \pi \xi$$  \hspace{1cm} (5.89)

Eq. (5.88) is based on the conventional strength-of-materials assumptions of uniform geometrical and physical properties, linear elastic behaviour, and small displacement - disregarding rotational and axial inertia and shear effects. It is also assumed that the standard deviation of the amplitude of the initial imperfections is smaller than the radius of gyration. Thus the problem may be treated in a statistically linear setting.

The boundary conditions are satisfied by setting:

$$u(\xi, \lambda) = e(\lambda) \sin \pi \xi$$  \hspace{1cm} (5.90)

Substituting Eqs. (5.89) and (5.90) in (5.88), we obtain the differential equation for $e(\tau)$

$$\frac{d^2 e(\lambda)}{d\tau^2} + (1 - \alpha) e(\lambda) = 0$$  \hspace{1cm} (5.91)

with initial conditions:

$$e(0) = \frac{de(0)}{d\lambda} = 0$$

The solution is given by:

$$e(\lambda) = \begin{cases} \frac{G}{1 - \beta} (\cosh r\lambda - 1), & \beta < 1 \\
\frac{1}{2} \frac{G\lambda^2}{1 - \beta}, & \beta = 1 \\
\frac{G}{1 - \beta} (\cos r\lambda - 1), & \beta > 1, \quad r = \sqrt{1 - \alpha}, \quad \beta = 1/\alpha \end{cases}$$  \hspace{1cm} (5.92)
and the total displacement $v(\xi, \lambda) = V(\lambda) \sin \pi \xi$ by:

$$V(\lambda) = \begin{cases} \frac{G}{1 - \beta} (\cosh r\lambda - \beta), & \beta < 1 \\ \frac{1}{2} G(\lambda^2 + 2), & \beta = 1 \\ \frac{G}{1 - \beta} (\cos r\lambda - \beta), & \beta > 1 \end{cases}$$

(5.93)

where $V(\lambda) = e(\lambda) + G$.

We proceed now to buckling under random imperfections.

Let $G$ be a continuous random variable with probability distribution function $F_G(g)$. For simplicity, we consider the case of symmetrical bounds, $Q^+ = Q^- = c > 0$, failure being thus identified with the total displacement reaching, in absolute value, the critical point $c$.

We seek the probability $\text{Prob}(T \leq t)$ of the first passage time $T$ being equal to or smaller than the given time $t$; or, in nondimensional form, $\text{Prob}(\Lambda \leq \lambda)$ of the nondimensional first passage time $\Lambda = \omega_1 T$ being equal to or smaller than the given nondimensional time $\lambda = \omega_1 t$.

Denoting by $\{L\}$ the contingency of buckling being possible (i.e. in the time interval $(0, \infty)$), $\text{Prob}(L)$ is the probability of failure of the system in the infinite time interval. Note, moreover, that if this probability is zero (i.e. buckling cannot occur) then likewise $\text{Prob}(\Lambda \leq \lambda) = 0$.

This also follows from the formula of overall probability

$$\text{Prob}(\Lambda \leq \lambda) = \text{Prob}(\Lambda \leq \lambda|L) \text{ Prob}(L) + \text{Prob}(\Lambda \leq \lambda|L') \text{ Prob}(L')$$

where $(\Lambda \leq \lambda|L)$ signifies that the first passage time $\Lambda$ is equal to or smaller than the given time $\lambda$; $(\Lambda \leq \lambda|L)$ refers to the contingency that the first passage time $\Lambda$ is equal to or smaller than $\lambda$, provided buckling is possible, whereas $(\Lambda \leq \lambda|L')$ - that the first passage time $\Lambda$ is equal to or smaller than $\lambda$, provided buckling is impossible. However,

$$(\Lambda \leq \lambda|L') = \emptyset, \quad \text{Prob}(\Lambda \leq \lambda|L') = 0$$

and finally we are left with
\[ \text{Prob}(\Lambda \leq \lambda) = \text{Prob}(\Lambda \leq \lambda | L) \text{ Prob}(L) \] (5.94)

Note that for nonzero \text{Prob}(L) and

\[ \text{Prob}(\Lambda \leq \infty | L) = 1 \] (5.95)

the following equality holds

\[ \text{Prob}(\Lambda \leq \lambda) = \text{Prob}(\Lambda) \] (5.96)

If the latter probability is unity [i.e. buckling is certain in the time interval \((0, \infty)\)] then also \text{Prob}(\Lambda \leq \infty) = \text{Prob}(\Lambda < \infty) = 1$. In such a case we denote
\[ \text{Prob}(\Lambda \leq \lambda) = F_{\Lambda}(\lambda), \quad \text{Prob}(\Lambda \leq \lambda|L) = F_{\Lambda}(\lambda|L) \]  
\[ (5.97) \]

where \( F_{\Lambda}(\lambda) \) is the probability distribution function of \( \Lambda \).

Note that \( F_{\Lambda}(0) \) does not necessarily vanish. There could be some nonzero probability of the amplitude of the initial imperfection having already exceeded the critical point. This probability is

\[ F_{\Lambda}(0) = \text{Prob}(\Lambda \leq 0) = \text{Prob}(\Lambda = 0) = \text{Prob}(|G| \geq c) = 1 - F_{G}(c) + F_{G}(c) \]

if \( F_{G}(-c) - F_{G}(c) = 1 \) then \( F_{\Lambda}(0) = 0 \).

We now proceed to calculate \( F_{\Lambda}(\lambda) \) for three different cases, in accordance with Eq. (5.92).

(i) Case 1 (\( \beta < 1 \))
In this case, owing to the exponential growth of \( u(\gamma, \tau) \) in time, \( \{L\} \) is a certain event; the conditional and unconditional probability distribution functions of the first passage time coincide, \( F_{\Lambda}(\lambda) = F_{\Lambda}(\lambda|L) \), and \( \Lambda \) satisfies the equation:

\[ \frac{|G|}{1 - \beta} (\cosh \lambda - \beta) = c \]  
\[ (5.98) \]

the probability distribution function sought being

\[ F_{\Lambda}(\lambda) = \text{Prob}\left\{ G \leq -\frac{c(1 - \beta)}{\cosh(\lambda) - \beta} \cup \left\{ G \geq \frac{c(1 - \beta)}{\cosh(\lambda) - \beta} \right\} \right\} = \]

\[ = 1 - F_{G}\left[\frac{c(1 - \beta)}{\cosh(\lambda) - \beta}\right] + F_{G}\left[-\frac{c(1 - \beta)}{\cosh(\lambda) - \beta}\right] (\lambda \geq 0), \quad F_{\Lambda}(\lambda) = 0, (\lambda < 0). \]  
\[ (5.99) \]

The probability density function is obtained by differentiation:

\[ f_{\Lambda}(\lambda) = \frac{rc(1 - \beta) \sinh(\lambda)}{[\cosh(\lambda) - \beta]^2} \left[ f_{G}\left[\frac{c(1 - \beta)}{\cosh(\lambda) - \beta}\right] + f_{G}\left[-\frac{c(1 - \beta)}{\cosh(\lambda) - \beta}\right]\right] + \]

\[ + F_{\Lambda}(0) \delta(\lambda), \quad (\lambda \geq 0) \]

where \( \delta(\lambda) \) is the Dirac delta function.

When the initial imperfections are symmetrically distributed in the positive
and negative domains, \( F_G(-g) = 1 - F_G(g) \) and, therefore, Eq. (5.99) becomes

\[
F_A(\lambda) = 2 - 2F_G \left[ \frac{c(1 - \beta)}{\cosh(\lambda) - \beta} \right] \quad \text{for} \quad \lambda \geq 0 ; \quad F_A(\lambda) = 0 \quad \text{for} \quad \lambda < 0 \quad (5.100)
\]

\[
f_A(\lambda) = \frac{2c(1 - \beta) \sinh(\lambda)}{[\cosh(\lambda) - \beta]^2} F_G \left[ \frac{c(1 - \beta)}{\cosh(\lambda) - \beta} \right] + F_A(0) \delta(\lambda) , \quad (\lambda \geq 0)
\]

(ii) Case 2 \((\beta = 1)\)

In this case, \( \{L\} \) is again a certain event. The first passage time satisfies the equation

\[
\frac{1}{2} \left| G \right| (\lambda^2 + 2) = c \quad (5.101)
\]

The probability distribution of \( A \) is given by

\[
F_A(\lambda) = \text{Prob} \left[ \left\{ G \geq \frac{2c}{\lambda^2 + 2} \right\} \cup \left\{ G \leq -\frac{2c}{\lambda^2 + 2} \right\} \right] = \quad (5.102)
\]

\[
= 1 - F_G \left[ \frac{2c}{\lambda^2 + 2} \right] + F_G \left[ -\frac{2c}{\lambda^2 + 2} \right] \quad \text{for} \quad \lambda \geq 0 ; \quad F_A(\lambda) = 0 \quad \text{for} \quad \lambda < 0
\]

and the probability density by

\[
f_A(\lambda) = \frac{4ac\lambda}{(\lambda^2 + 2)^2} \left\{ F_G \left[ \frac{2c}{\lambda^2 + 2} \right] + F_G \left[ -\frac{2c}{\lambda^2 + 2} \right] \right\} + F_A(0) \delta(\lambda) , \quad (\lambda \geq 0)
\]

For symmetrically distributed initial imperfections, we have

\[
F_A(\lambda) = 2 - 2F_G \left[ \frac{2c}{\lambda^2 + 2} \right], (\lambda \geq 0) ; \quad F_A(\lambda) = 0 \quad \text{for} \quad \lambda < 0 \quad (5.103)
\]

\[
f_A(\lambda) = \frac{8ac\lambda}{(\lambda^2 + 2)^2} F_G \left[ \frac{2c}{\lambda^2 + 2} \right] + F_A(0) \delta(\lambda) , \quad (\lambda \geq 0)
\]

(iii) Case 3 \((\beta > 1)\)

In this case \( \{L\} \) is no longer a certain event; buckling is possible if

\[
\max_{\tau} \max_{\xi} v(\xi, \tau) = \left| G \right| \frac{\beta + 1}{\beta - 1} \quad (5.104)
\]

reaches the critical point \( c \). That is,
\[
\text{Prob}(L) = \text{Prob}\left\{ |G| \frac{\beta + 1}{\beta - 1} \geq c \right\} = \\
1 - F_G\left[ c \frac{\beta - 1}{\beta + 1} \right] + F_G\left[ - c \frac{\beta - 1}{\beta + 1} \right]
\] (5.105)

Note here, that for \( \beta \gg 1 \), \( \text{Prob}(L) \rightarrow \text{Prob}(\lambda = 0) \), i.e. the probability of the bar buckling at all (i.e., at any time) approaches that of failure at zero time.

The first passage time satisfies the equation

\[
\frac{|G|}{\beta - 1} (\beta - \cos r\lambda) = c
\] (5.106)

and the probability of failure is

\[
\text{Prob}(\lambda \leq \lambda) = \text{Prob}\left\{ G \leq -c \frac{\beta - 1}{\beta - \cos r\lambda} \right\} \cup \left\{ G \geq c \frac{\beta - 1}{\beta - \cos r\lambda} \right\}
\] (5.107)

\[
= 1 - F_G \left[ c \frac{\beta - 1}{\beta - \cos r\lambda} \right] + F_G \left[ - c \frac{\beta - 1}{\beta - \cos r\lambda} \right], \text{ for } \lambda \in [0, \pi/\tau]
\]

\( \text{Prob}(\lambda \leq \lambda) = 0, \text{ for } \lambda < 0 \) and \( \text{Prob}(\lambda \leq \lambda) = \text{Prob}(\lambda \leq \pi/\tau) \) for \( \lambda > \pi/\tau \)

For symmetrically distributed random imperfections, Eq. (5.107) becomes

\[
\text{Prob}(\lambda \leq \lambda) = 2 - 2F_G \left[ \frac{c(\beta - 1)}{\beta - \cos r\lambda} \right], \text{ for } \lambda \geq 0
\] (5.108)

\( \text{Prob}(\lambda \leq \lambda) = 0, \text{ for } \lambda < 0 \).

The conditional probability distribution function is

\[
F_{\lambda}(\lambda|L) = \frac{1 - F_G \left[ c(\beta - 1) \right] + F_G \left[ - c(\beta - 1) \right]}{1 - F_G \left[ \frac{c(\beta - 1)}{\beta + 1} \right] + F_G \left[ - \frac{c(\beta - 1)}{\beta + 1} \right]}, \text{ for } \lambda > 0
\] (5.109)

since \( F_{\lambda}(\lambda) = \text{Prob}(\lambda \leq \lambda, L) = \text{Prob}(\lambda \leq \lambda|L) \text{ Prob}(L) \).

The conditional probability density is

\[
f_{\lambda}(\lambda|L) = \frac{1}{\text{Prob}(L)} \frac{d\text{Prob}(\lambda \leq \lambda)}{d\lambda}
\] (5.110)

Note that as \( \lambda \rightarrow \pi/\tau \), \( F_{\lambda}(\lambda|L) \rightarrow 1 \); this is due to the obvious fact that if
buckling is possible, it must occur in the time interval $[0, \pi/r]$. Note also that in all three cases the functions may be written down in the general form (for $\lambda \geq 0$)

$$\text{Prob}(\Lambda \leq \lambda) = 1 - F_G[\phi(\lambda)] + F_G[-\phi(\lambda)]$$  \hspace{1cm} (5.111)

where

$$\phi(\lambda) = \frac{c(1 - \beta)}{\cosh(r\lambda) - \beta}, \quad \beta < 1; \quad \phi(\lambda) = \frac{2c}{\lambda^2 + 2}, \quad \beta = 1;$$

$$\phi(\lambda) = \frac{c(\beta - 1)}{\beta - \cos r\lambda}, \quad \beta > 1$$  \hspace{1cm} (5.112)

Finally, note that for $F_A(0) = 0$, the last terms in the equation for the density functions have to be omitted.

Consider now some numerical examples. Let $G$ have a uniform distribution in the interval $(q_{\text{min}}, q_{\text{max}})$.

$$F_G(g) = \begin{cases} 
0, & g \leq q_{\text{min}} \\
 \frac{g - q_{\text{min}}}{q_{\text{max}} - q_{\text{min}}}, & q_{\text{min}} < g \leq q_{\text{max}} \\
1, & g > q_{\text{max}} 
\end{cases}$$  \hspace{1cm} (5.113)

For times instances $\lambda$ satisfying the inequality

$$\phi(\lambda) \geq q_{\text{max}} > 0$$  \hspace{1cm} (5.114)

then $F_G[\phi(\lambda)] = 1$ and Eq. (5.101) yields

$$\text{Prob}(\Lambda \leq \lambda) = F_G[-\phi(\lambda)]$$  \hspace{1cm} (5.115)

If moreover $-\phi(\lambda) < q_{\text{min}}$, then $\text{Prob}(\Lambda \leq \lambda) \equiv 0$. For symmetrically distributed random imperfections $q_{\text{min}} = -q_{\text{max}}$, this happens for times $\lambda \leq \lambda^*$ where
\[
\lambda^* = \begin{cases} 
\frac{\sqrt{2c/g_{\text{max}}}}{2}, & \beta = 1 \\
\frac{1}{r^{-1} \cos^{-1} \left[ \beta - c(\beta - 1)/g_{\text{max}} \right]}, & \beta > 1 \text{ and } c < g_{\text{max}} \\
\frac{1}{r^{-1} \cosh^{-1} \left[ (1 - \beta)/g_{\text{max}} \right]}, & \beta < 1
\end{cases}
\] (5.116)

For \(\beta > 1\), the probability of buckling may turn out to be zero. From expression (5.94) it may be seen that this is the case, for example, if

\[
c > g_{\text{max}} \frac{\beta + 1}{\beta - 1} \text{ and } g_{\text{in}} = -g_{\text{max}}
\] (5.117)

then \(\text{Prob}(L) = 0\) and \(\text{Prob}(\Lambda \leq \lambda) = 0\) for any \(\lambda\) - and the bar remains in the safe region throughout.

The above cases are illustrated in Figs. 5.35 - 5.37, where \(\text{Prob}(\Lambda \leq \lambda)\) is plotted against \(\lambda\). These figures represent, respectively, the cases where the actual load is greater than, equal to or less than the corresponding classical buckling load of a perfect structure. Particularly, in Fig. 5.35 the actual load was chosen double the classical buckling load and in Fig. 5.37 - half of the latter. For all calculations the random imperfections were chosen with a symmetric distribution \((g_{\text{max}} = -g_{\text{min}})\). The curves marked 1 are all associated with \(g_{\text{max}} = 0.5\), whereas those marked 2 are associated with \(g_{\text{max}} = 0.3\). The nondimensional critical level \(c\) was taken identical for all curves in the above figures, the failure being thus identified with the absolute value of the total displacement reaching four-tenths of the bar's radius of gyration. In the cases where \(g_{\text{max}} > c\) (curves 1) the probability of the bars buckling at \(\Lambda = 0\), \(\text{Prob}(\Lambda = 0)\) differs from zero; in particular \(\text{Prob}(\Lambda = 0) = 0.2\). Thus, under the statistical interpretation of probability, 20 percent of bars from a large population have already buckled(!).

In the cases where \(g_{\text{max}} < c\) (curves 2) \(\text{Prob}(\Lambda \leq \lambda) = 0\), for \(\lambda \leq \lambda^*\), \(\lambda^*\) being \(\cosh^{-1}(7/6) = 0.5697\) for \(P = 2P_c\), \(\sqrt{2/3} = 0.8165\) for \(P = P_c\), \(\sqrt{2} \cos^{-1}(2/3) = 1.1895\) for \(P = 0.5 P_c\); that is below these values of time, we would expect no bar buckles. For \(P < P_c\), \(\text{Prob}(L) \neq 1\): for \(g_{\text{max}} = 0.5\), and \(\alpha = 0.5\) \(\text{Prob}(L)\) equals 0.733; that implies in accordance with the statistical interpretation of probability, that, for example, from a series of 1,000 bars, say, the relative number of unbuckled bars (in the time interval \(0 \leq \lambda < \infty\)) would stabilize near \((1 - 0.733) \times 1000 = 267\). For \(g_{\text{max}} = 0.3\), and \(\alpha = 0.5\), \(\text{Prob}(L)\) equals 0.555, indicating, once more, that the number of unbuckled bars
would now stabilize at \((1 - 0.555) \times 1000 = 445\). For \(q_{\text{max}}\) less than some critical value, found from Eq. (5.117) as

\[
g_{\text{max,cr}} = \frac{c(\beta - 1)}{\beta + 1}
\]

\(\text{Prob}(\Lambda \leq \lambda)\) is identically zero. For \(\alpha = 0.5, g_{\text{max,cr}} = 0.133\), implying that for \(q_{\text{max}} < 0.133\) negligible number of bars would buckle at all in a large sample.

Fig. 5.38 presents \(F_\Lambda\) as a function of \(\lambda\) for three cases: \(P = 2p_c^\prime\), \(P = p_c^\prime, P = 0.5 p_c^\prime\) for specified \(q_{\text{max}}\), respectively. As is seen from the figure, the probability of the time to the first buckling being less than \(\lambda\) increases with the actual load for specified \(\lambda\).

Let us find the mean buckling time, defined as

\[
E(\Lambda) = \int_0^\infty \lambda f_\Lambda(\lambda) \, d\lambda
\]

However, since in general

\[
f_\Lambda(\lambda) = \frac{d}{d\lambda} \, \text{Prob}(\Lambda \leq \lambda) \quad \text{Prob}(\Lambda \leq \lambda) = 1 - \text{Prob}(\Lambda > \lambda) = 1 - R(\lambda)
\]

where \(R(\lambda)\) is the reliability of a structure at time instant \(\lambda\), we have

\[
E(\Lambda) = \int_0^\infty \lambda \frac{d}{d\lambda} \, \text{Prob}(\Lambda \leq \lambda) \, d\lambda = \int_0^\infty \lambda \frac{d}{d\lambda} \, [1 - R(\lambda)] \, d\lambda
\]

Integration by parts, as in Eq. (5.75), yields

\[
E(\Lambda) = -\lambda R(\lambda) \bigg|_0^\infty + \int_0^\infty R(\lambda) \, d\lambda
\]

\[
(5.119)
\]

Let us demonstrate that the first term is zero

\[
\lambda R(\lambda) \bigg|_0^\infty = \lim_{\lambda \to \infty} \lambda R(\lambda) = 0
\]

Indeed, for the random variable \(\Lambda \geq 0\) with finite mathematical expectation, it follows from convergence of the integral

\[
\int_0^\infty \lambda f_\Lambda(\lambda) \, d\lambda
\]

that
\[
\int_{\Lambda} \lambda f_{\Lambda}(\lambda) \, d\lambda = 0 \quad \text{as} \quad M \to \infty
\]

However,
\[
M \int_{\Lambda} f_{\Lambda}(\lambda) \, d\lambda \leq \int_{\Lambda} \lambda f_{\Lambda}(\lambda) \, d\lambda
\]

Therefore,
\[
M [1 - F_{\Lambda}(M)] = M R_{\Lambda}(M) \to 0 \quad \text{as} \quad M \to \infty
\]

Consequently,
\[
\lim_{\lambda \to \infty} \lambda R_{\Lambda}(\lambda) = 0
\]

implying that
\[
E(\Lambda) = \int_{\Lambda} R(\lambda) \, d\lambda = \int_{\Lambda} [1 - F_{\Lambda}(\lambda)] \, d\lambda \tag{5.121}
\]

\(E(\Lambda)\) is represented by the area, shaded in Fig. 5.39.

For \(P < P_c\), however, \(\text{Prob}(\Lambda \leq \infty) = \text{Prob}(L) \neq 0\), as we have seen above.

In other words, in this case
\[
E(\Lambda) \to \infty \tag{5.122}
\]

This result becomes obvious if we recall that for \(P < P_c\) a certain percentage of the bars do not fail, i.e. have an infinite mean buckling time.

With the expressions for the unreliability \(\text{Prob}(\Lambda \leq \lambda)\) to hand, we can pose the problem of determining the allowable operating time \(\lambda_r\) for which the reliability, \(R(\lambda)\) reaches a given level:
\[
R(\lambda_r) = r \tag{5.123}
\]

This allowable operating time is clearly finite, whereas the mean buckling time may be infinite, again demonstrating that reliability based design is superior to mean-behaviour based one (compare with Prob. 4.5).
Cited References


Hoff, N.J., Dynamic Stability of Structures (Keynote Address), in Dynamic Stability of Structures (Herrmann, G., ed.), Pergamon Press, 1965, pp. 7-44.


Recommended Further Reading


Problems

5.1. The loads acting on different machine components often have a chi-square distribution with m degrees of freedom, as was shown by Serensen and Bugloff,

\[ f_N(n) = \frac{e^{-n/2} n^\nu/2}{\Gamma(\nu)} U(n), \quad \nu = \frac{m}{2} \]

Find the reliability of the bar under random tensile force (Table 2 contains the values of \( \chi^2_{\alpha, \nu} \) where

\[ \int_{\chi^2_{\alpha, \nu}}^{\infty} f_N(n) \, dn = \alpha \]

for \( \alpha = 0.995, 0.99, 0.975, 0.95, 0.05, 0.025, 0.01, 0.005 \) and \( \nu = 1, 2, \ldots, 30 \) (see figure)).

5.2. Find the reliabilities of the truss structures shown in Figs. 2.9a, c, d and e, if \( P \) is treated as a random variable with given distribution function \( f_P(p) \).

5.3. The truss shown in the figure carries the random normally distributed load \( P \) with \( E(P) = 10 \) kips, \( \sigma_p = 5 \) kips, \( a = 10' \). Check whether the reliability exceeds the desired done \( r = 0.999 \).
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5.4. A cantilever of rectangular cross section is loaded as shown in the figure. \( g \) is a random variable with given \( F_g(g) \); \( \sigma_y \) - the yield stress in the tensile test of the cantilever material - is given. Use the maximum shear-stress criterion to find the reliability of the cantilever.

On the concrete numerical example, discuss the change of the reliability estimate under the von Mises criterion.

5.5. A beam is loaded as shown in the figure, \( G \) is a random variable with given probability distribution \( F_G(g) \) and \( \alpha \) is a given number. Verify that the extremal bending moment occurs at the section \( x = (5\alpha - 1)/4\alpha \) and equals

\[
M_z = G \alpha^2 \left[ \frac{3+\alpha}{4} \frac{5\alpha-1}{4\alpha} - \frac{1}{2} \left( \frac{5\alpha-1}{4\alpha} \right)^2 - \frac{\alpha-1}{2} \left( \frac{\alpha-1}{4\alpha} \right)^2 \right]
\]

and calculate the reliability.

5.6. Determine the reliability of the cantilever under a given force \( q \) applied at the random distance \( X \) from the clamped edge. \( F_X(x) \) is given. (For a generalization of this problem with both \( X \) and \( q \) random variables, or with random concentrated loads and moments applied at random positions on the beams with different boundary conditions, see paper by Strukla and Stark.)

5.7. A thick-walled cylinder is under external pressure \( P \) with a discrete uniform distribution

\[
P_p(p) = \frac{1}{10} \sum_{i=1}^{10} u(x - p_i)
\]

that is, \( P \) takes on values \( p_o, 2p_o, \ldots, 10p_o \) with constant probability 1/10. For the transverse stresses, the following expressions are valid (see e.g. Timoshenko and Goodier):

\[
\Sigma_r = p \left( \frac{r_o/r_1}{r_o/r_1 - 1} \right)^2, \quad \Sigma_\theta = -p \left( \frac{r_o/r_1}{r_o/r_1 - 1} \right)^2 - 1
\]

where \( r_o \) and \( r_1 \) are the outer and inner radii respectively. Using the von Mises criterion, find \( r_o/r_1 \) such that the desired reliability is not less than 0.99.
5.8. A rectangular plate, simply supported all round, is under a load \( q \) uniform over its surface, with chi-square distribution (§ 4.7.)

\[
f_Q(q) = \frac{e^{-q/2} q^{\nu - 1}}{2^\nu \Gamma(\nu)} \quad \text{for} \quad q > 0, \quad \nu = m/2
\]

The displacement of the plate under a deterministic uniform load \( q \) is (see e.g. Timoshenko and Woinowski-Krieger):

\[
w = \frac{16q}{\pi D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

where \( a \) and \( b \) are the sides of the plate, \( D = E h^3/12(1 - \nu^2) \) is the flexural rigidity, \( E \)-modulus of elasticity, \( \nu \) - Poisson's ratio and \( h \) the thickness of the plate. Find the probability density of the maximum deflection.

5.9. Derive an equation, analogous to (5.63) for an asymmetric structure.

5.10. Find the probability density function of the dynamic buckling loads and find the mean dynamic buckling load for the symmetric structure.

5.11. Find the reliability function of the asymmetric structure if the initial imperfection is normally distributed \( N(0, \sigma^2) \), in the static setting.

5.12. Do the same for the dynamic buckling problem.

5.13. Assume that the initial imperfection has an exponential distribution, with parameter \( E(\lambda) \) given. Find the reliability of the asymmetric structure.

5.14. Plot the nondimensional static buckling \( \lambda_s/\lambda_c \) versus initial imperfection \( \xi \) curve for the nonsymmetric structure, according to Eq. (5.50). Find the reliability at the load level \( \lambda \).

5.15. A rigid weightless bar with a frictionless pin joint at A, constrained by nonlinear springs with \( k > 0, \beta > 0 \), is under an eccentric load \( P \). The equilibrium equation is

\[
P(x + \epsilon) = 2k\beta x \left(1 - \beta x^2/\lambda^2\right)
\]
yielding \( P_c = 2kl \). Find the expression for the maximum force \( P_{\text{max}} \) supported by the bar as function of the eccentricity \( \varepsilon \). Assume eccentricity to be a continuous random variable with probability density \( f_\varepsilon(\varepsilon) \); find the reliability of the structure at load level \( \lambda \).

5.16. Generalize the results of § 5.7 for the case where the load function is a rectangular impulse

\[
P(t) = P[U(t) - U(t - \tau)]
\]

with \( P \) and \( \tau \) given positive quantities.

5.17. Generalize the results of § 5.7 for the case where failure is considered in the finite time interval \( 0 \leq t \leq t^* \) [see Eq. (5.85)].

5.18. Verify, that the buckling time \( \Lambda \) in § 5.7 for \( P < P_c \) does not represent a random variable. Assign an infinite buckling time to the structure which does not buckle. How does Eq. (5.84) change in these circumstances? Show analytically that for \( P < P_c \), \( P(\Lambda) \) approaches infinity.

5.19. Modify the results of § 5.7 for the case where the structure possesses viscous damping.

5.20. Consider the load-bearing capacity of an imperfect bar. As can be seen from the figure, a concentric load \( P \) produces a bending moment \( M_z = -Pw \) and increases the displacement by an amount \( (w - w_1) \). The differential equation of the column is, therefore

\[
EI_z \frac{d^2}{dx^2} (w - w_1) = -Pw
\]

The bar is simply-supported at its ends and has an initial imperfection

\[
w_1 = g \sin \frac{\pi x}{L}
\]

Equation (5.124) becomes, upon substitution of Eq. (5.125)

\[
\frac{d^2 w}{dx^2} + \frac{P}{EI_z} = -a_1 \frac{\pi^2}{L^2} \sin \frac{\pi x}{L}
\]
The solution of this equation is

\[ w = C_1 \sin \sqrt{\frac{P}{EIz}} x + C_2 \cos \sqrt{\frac{P}{EIz}} x + \frac{a_1}{1 - \frac{P}{P_c}} \sin \frac{\pi x}{L} \]  

(5.126)

where \( P_c = \pi^2 \frac{EI}{l^2} \) is the classical or Euler buckling load. The boundary conditions are \( w = 0 \) at \( x = 0, l \) and these yield

\[ C_1 \sin \sqrt{\frac{P}{EIz}} l = 0 \]

\[ C_2 = 0 \]

For \( P < P_c \), both \( C_1 \) and \( C_2 \) must be zero, and the total deflection is represented by the last term in Eq. (5.126):

\[ w = \frac{1}{1 - \frac{P}{P_c}} g \sin \frac{\pi x}{L} \]  

(5.127)

We see that the total deflection becomes increasingly large, as \( P \to P_c \). The normal stresses in the bar are given

\[ \sigma_x = -\frac{P}{A} - \frac{M}{I_z}, \quad M = -Pw \]

Thus the maximum compressive stress takes place at \( x = L/2 \) and is given by

\[ \sigma_{\text{max}} = \frac{P}{A} \left[ 1 + \frac{gA}{S} \frac{1}{1 - \frac{P}{P_c}} \right] \]

(5.128)

where \( S \) is the section modulus \( (Z = I_z/y_{\text{max}}, y_{\text{max}} \) being the distance from the neutral axis to the point of maximum stress). Denote

\[ \frac{P}{A} = \sigma_{\text{av}} \quad \text{and} \quad \frac{P_c}{C} = \frac{\sigma_{\text{av}}}{C} \]

Eq. (5.127) becomes then

\[ \sigma_{\text{max}} = \sigma_{\text{av}} \left[ 1 + \frac{gA}{S} \frac{1}{1 - \sigma_{\text{av}}/\sigma_c} \right] \]

where \( \sigma_c = \pi^2 E/(l/r)^2 \), being the radius of gyration of the cross sectional area of the bar. The load \( P_L \) for which \( \sigma_{\text{max}} \) equals the yield stress \( \sigma_y \), is the limit load for which the column remains elastic. This load results in the average stress \( \sigma_L = P_L/A \), and Eq.
(5.127) becomes

\[ \sigma_y = \sigma_L \left[ 1 + \frac{\sigma_L}{S} \frac{1}{1 - \sigma_L/\sigma_c} \right] \]

where

\[ \left( \frac{\sigma_L}{\sigma_y} \right)^2 = \left[ 1 + \frac{\sigma_c}{\sigma_y} \left( 1 + \frac{gA}{S} \right) \right] \frac{\sigma_L}{\sigma_y} + \frac{\sigma_c}{\sigma_y} = 0 \]

(5.129)

Fig. 5.20c. shows \( \sigma_L/\sigma_y \) as a function of the slenderness ratio \( \lambda/r \).

Treating the initial imperfection amplitude \( G \) as a random variable with gamma distribution (§ 4.8.),

\[ f_G(g) = \frac{1}{\beta^{\alpha+1} \Gamma(\alpha+1)} g^\alpha e^{-g/\beta} U(g) \]

The average limit stress is also a random variable.

(i) Extract \( \sigma_L/\sigma_y \) explicitly from Eq. (5.128) as a function of \( g \)

(ii) Find \( F_G(\sigma_L) \).

(iii) Consider also the special cases of an exponential distribution and of a chi-square distribution with \( m \) degrees of freedom.

Perform the numerical calculations.

5.21. In the preceding problem, assume \( G \) has an exponential distribution.

Find the probability of the maximum total displacement \( w(\lambda/2) \) taking on values in the interval \([0, 2\varepsilon(G)]\). Investigate the behaviour of this probability as \( P \) approaches the classical buckling load \( P_c \).

5.22. Consider now another important case of an initially perfect simply-supported bar under eccentric load with eccentricity \( \epsilon \), as shown in Fig. 5.22a. The differential equation for the bar deflection reads

\[ \frac{EI}{z} \frac{d^2w}{dx^2} + Pw = 0 \]

(5.130)

The solution is

\[ w = C_1 \sin \sqrt{\frac{P}{EI}z} x + C_2 \cos \sqrt{\frac{P}{EI}z} x \]

The integration constants are determined by the boundary conditions \( w = \epsilon \) at \( x = \pm \lambda/2 \), so that
\[ \begin{align*}
C_1 &= 0, \quad C_2 = \varepsilon \sec \left( \sqrt{\frac{P}{EI_z}} \right) \frac{L}{2} \cos \left( \sqrt{\frac{P}{EI_z}} \right) x \\
\end{align*} \]

The maximum displacement is reached at \( x = 0 \)

\[ w_{\text{max}} = \varepsilon \sec \left( \sqrt{\frac{P}{EI_z}} \right) \frac{L}{2} = \varepsilon \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_c}} \right) \quad (5.131) \]

It is seen that the maximum deflection becomes increasingly large as \( P \) approaches \( P_c \). The elastic limit is reached at the most stressed point when

\[ \sigma_y = \frac{P}{A} + \frac{M_c}{I_z} \quad (5.132) \]

or when

\[ \sigma_o = \frac{P}{A} + \frac{P_{EC}}{I_z} \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_c}} \right) \quad (5.133) \]

This equation can be rewritten as

\[ \sigma_y = \sigma_L \left( 1 + \frac{\varepsilon c}{r^2} \sec \frac{\theta}{2} \sqrt{\frac{\sigma_L}{E}} \right) \quad (5.134) \]

where \( \sigma_L = \frac{P}{A} \) the "average" limit stress. Equation (5.134) is referred to as the secant formula, and is usually plotted in the form of \( \sigma_L/\sigma_y \) versus \( \lambda/r \) for a particular material (with \( \sigma_y \) and \( E \)) specified for different values of \( \varepsilon c/r^2 \), as shown on Fig. 5.22b.

Treating now the eccentricity as the random variable \( \varepsilon \) with \( P_E(\varepsilon) \) given as exponentially distributed, the average limit stress also is a random variable. Find \( P_E(\sigma_L) \).

5.23. In his (now historic) Ph.D. thesis in 1945, and in his 1963 paper, Koiter analysed a sufficiently long cylindrical shell with an axisymmetric initial imperfection, under axial load. He chose an initial imperfection function \( w_o(x) \), co-configurational with the axisymmetric buckling mode of a perfect cylindrical shell, as

\[ w_o(x) = gh \sin \frac{\pi i x}{L}, \quad i_c = \frac{L}{\pi} \sqrt{\frac{2c}{Rh}} \quad (5.135) \]

\[ c = \sqrt{3(1 - \nu^2)} \]
where $\mu$ is the nondimensional initial imperfection magnitude, $h$
thickness and $i_C$ the number of half-waves at which the associated
perfect shell buckles, $L$ the shell length, $R$ the shell radius, $h$ the
shell thickness. Using his own general nonlinear theory, he derived
inter alia a relationship between the critical load and the initial
imperfection magnitude, as follows

$$(1 - \lambda)^2 - \frac{3}{2} c \left| g \right| \lambda = 0 \quad (5.136)$$

where

$$\lambda = \frac{P_{bif}}{P_c}, \quad P_c = 2\pi Rh \sigma_c, \quad \sigma_c = \frac{Eh}{Rc}$$

is the nondimensional buckling load, $P_{bif}$ - the buckling load of an imperfect
shell, $P_c$ - the classical buckling load of a perfect shell, $E$ - the modulus
of elasticity, $\nu$ - Poisson's ratio. The buckling load $P_{bif}$ was defined as
that at which the axisymmetric fundamental equilibrium state bifurcates
into a nonsymmetric one. The absolute value of $g$ in Eq. (5.136) stands,
since for a sufficiently long shell the sign of the imperfection is
immaterial: positive and negative initial imperfections with equal absolute
values cause the same reduction on the buckling load.

Eq. (5.136) yields the explicit buckling load-initial imperfection
relationship:

$$\lambda = 1 + \frac{3}{4} |\xi| - \frac{1}{2} \left( 6 |\xi| + \frac{9}{4} |\xi|^2 \right)^{1/2}$$

where $\xi = cg$.

(i) Assume $X$ with possible values $\xi$ to be a normally distributed random
variable

$$N(\bar{\xi}, \sigma^2)$$

(ii) Find the probability density function of $|X|$ (consult Example 4.6.).

(iii) Find the reliability of the shell at the load level $\bar{\lambda}$.

(iv) Assume, after Roorda,
\[ \xi = 0.333 \times 10^{-3} \frac{R}{h} , \quad \phi^2 = 10^{-3} \frac{R}{h} \]

and find the stress level at which the system has a given reliability. Compare your result with Roorda's.

5.24. a) Koiter (1945) also analysed the imperfection sensitivity of a shell with non-axisymmetric, periodic, imperfections.

\[ w_0(x) = gh \left( \cos \frac{1}{L} \cos \frac{1}{2L} \cos \frac{1}{2L} \right) \]  \hspace{1cm} (5.137)

(where \( y \) is the circumferential coordinate, the remaining notation as in the preceding problem) to arrive, instead of Eq. (5.136), at the following equation

\[ (1 - \lambda)^2 + 6c g\lambda = 0 \]  \hspace{1cm} (5.138)

for the nondimensional buckling load

\[ \lambda = \frac{P_{\text{lim}}}{P_c} \]

where \( P_{\text{lim}} \) is the limit load (as in § 5.5.). For the imperfection function (5.138) the limit load exists only at negative values of the imperfection parameter \( g \). For positive \( g \), the origin of the coordinate system may be shifted, and since the shell is sufficiently long, the analysis would be unaffected except that the sign of \( g \) would change to yield

\[ (1 - \lambda)^2 - 6c g\lambda = 0 \]  \hspace{1cm} (5.139)

Combining Eqs. (5.138) and (5.139), we arrive at the final equation

\[ (1 - \lambda)^2 - 6c |g| \lambda = 0 \]

Perform calculations as in Prob. 5.23. Compare the reliabilities of shells with axisymmetric and non-axisymmetric imperfections.

b) Sometimes the imperfection is represented by a local dimple extending over a small region of the shell. A more or less localized imperfection may be represented in the form
\[ w_o(x) = gh \left( \cos \frac{i \pi x}{L} + 4 \cos \frac{i \pi x}{2L} \cos \frac{i \pi y}{2L} \right) \exp \left[ -\frac{1}{2} \frac{\mu^2}{R^2} (x^2 + y^2) \right] \]  

(5.140)

which is the function in Eq. (5.140) multiplied by an exponentially decaying function. For example, at a distance \( x = (4\pi/i_c) \) R or \( y = (4\pi/i_c) \) R, a complete wavelength of the periodic part in Eq. (5.140), the exponential factor reduces to \( \exp(-8\pi^2\mu^2/i_c^2) \). At first approximation, the term \( \mu^2/i_c^2 \) may be neglected against unity. Koiter's analysis (1978) yields then

\[ (1 - \lambda)^2 = -4 \text{cg} \lambda \]  

(5.141)

Assume again \( x = \text{cg} \), with possible values \( \xi = \text{cg} \), to be a normally distributed random variable \( N(0, \sigma^2) \) and find the reliability of the shell. Are the localized imperfections as harmful as the periodic ones?
Fig. 1.1. Section of Hammurabi's stela at high magnification (photo-assembly by courtesy of Prof. J. Kogan).
Fig. 2.1. (a): Elastic beam simply-supported at its ends, (b): Bending moment diagram, (c): Elastic cylindrical shell under uniform axial compression.
Fig. 2.2.: Compression testing machine for buckling tests on thin-walled structures (designed by Prof. W.D. Verduyn, in the Delft University of Technology).
Fig. 2.3. (a): Testing machine and data acquisition equipment, (b): Cylindrical shell testing configuration.
Fig. 2.4. Realisation of the imperfection profile (shell A9).
Fig. 2.5. Realisation of the imperfection profile (shell Al2).
Fig. 2.6. (a): Ω shaded, (b): $A_1$ and $A_2$ are mutually exclusive events, (c): $A_1$ is contained in $A_2$, (d): $A_2$ is contained in $A_1$, (e): $A_1$ and $A_2$ are equal, (f): Sum $A_1 + A_2$ shaded, (g): Sum $A_1 + A_2$ shaded, (h): Product $A_1A_2$ shaded, (i): Product $A_1A_2$ of mutually exclusive events is an impossible event, (j): Difference $A_1 - A_2$ shaded, (k): $A_1$ is a complement of $A_2$. 
Fig. 2.7. Conditional probability: probability of event A, given that the event B has taken place: $P(A|B) = P(AB)/P(B)$. 
Fig. 2.8. (a): Series system, (b): Parallel system.
Fig. 2.9. Truss systems: (a,b): With different unconditional and conditional probabilities of bar failure, (c): n = 2, (d): n = 3, (e,f): Constituent bars have the same probabilities of failure.
Fig. 3.1. Probability distribution function of random variable X in Example 3.1.

Fig. 3.2. Probability distribution function of number of spots on upward-landing face in single "honest" die throwing experiment.

Fig. 3.3. Unit step function.
Fig. 3.4. Probability of random variable X taking on value "a" equal to jump discontinuity of $F_X(x)$ at "a".

Fig. 3.5. Beam under concentrated forces: distributed force diagram representing "analogue" of probability density of discrete random variable.
Fig. 3.6. Probability density function of number of spots on upward-landing face in single "honest" die throwing experiment.

Fig. 3.7. Beam under concentrated moment: shear force diagram representing "analogue" of probability density of mixed random variable.
Fig. 3.8. Probability density function (1) and distributed function (2) of random variable uniformly distributed in interval (a,b).
Fig. 3.9. Examples of probability density functions with positive, zero and negative coefficients of skewness.
Fig. 3.10. Examples of probability density functions with positive, zero and negative coefficients of kurtosis.
Fig. 3.11. (1) Unconditional probability distribution and density functions, (2) conditional probability distribution and density functions.
Fig. 4.1. Probability density of causally distributed random variable represented by Dirac's delta function.

Fig. 4.2. Probability density of random variable with discrete uniform distribution represented by combination of Dirac's delta functions.

Fig. 4.3. Rayleigh probability density function.
Fig. 4.4. Gamma probability density function

1) \( \alpha = -1/2, \beta = 1 \), 2) \( \alpha = 1, \beta = 1/2 \), 3) \( \alpha = 10, \beta = 1/5 \).

Fig. 4.5. Probability density function of normal distribution

with 1) \( a = 0, \sigma_X = 1/2 \), 2) \( a = 0, \sigma_X = 1 \), 3) \( a = 3, \sigma_X = 1 \).
Fig. 4.6. Area under normal probability density curve.

Fig. 4.7. Probability density function of truncated normal distribution.

Fig. 4.8. $\varphi(x)$ - strictly monotone increasing function.
Fig. 4.9. $\varphi(x)$ — strictly monotone decreasing function.

Fig. 4.10. $\varphi(x)$ — non-monotone function.
Fig. 4.11. $\omega(x)$ - constant in some interval $(x_1, x_2)$: probability distribution function $F_Y(y)$ of random variable $Y = \omega(X)$ undergoing jump discontinuity equal to integral of $f_X(x)$ over interval $(x_1, x_2)$. 

Fig. 4.12. Determination of probability density function of square of random variable.

Fig. 4.13. Determination of probability density function of absolute value of random variable.
Fig. 4.14. Probability density (a), and distribution function (b) of square of normally distributed random variable.
Fig. 5.1. Bar under tensile force.
Fig. 5.2. Reliability of bar under random tension having uniform distribution; shaded area represents reliability.
Fig. 5.3. Calculation of required cross-sectional area.

Fig. 5.4. Worst-case design: worst situation governed by deterministic load equal to (a) \( n_2 \), (b) \( n_1 \), (c) either \( n_1 \) or \( n_2 \).
Fig. 5.5. Rigid block A constrained by three bars; 
N - a random variable.

Fig. 5.6. Shaded area equals the reliability of the structure with random strength, subjected to a deterministic load.
Fig. 5.7. Simply-supported beam under random distributed force.

Fig. 5.8. Beam, simply supported at its ends, under random distributed load.
Fig. 5.9. 1) Idealised column, 2) equilibrium of single bar.

Fig. 5.10. Nondimensional load versus additional displacement relationship for symmetric structure.
Fig. 5.11. Nondimensional load versus additional displacement relationship for asymmetric structure.
Fig. 5.12. Nondimensional load-additional displacement curves for nonsymmetric structure (a = -7.5, b = 25).
Fig. 5.13. Nondimensional load-additional displacement curves for nonsymmetric structure (\(a = -1.5\), \(b = -25\)).
Fig. 5.14. Root-locus plot \((a < 0, b < 0, \xi > 0, a^2 = 3b)\).

Fig. 5.15. Root-locus plot \((a < 0, b > 0, \xi > 0, a^2 < 3b)\).

Fig. 5.16. Root-locus plot \((a < 0, b > 0, \xi > 0, a^2 = 4b)\).
Fig. 5.17. Root-locus plot \((a < 0, b > 0, \xi > 0, 3b < a^2 < 4b)\).

Fig. 5.18. Root-locus plot \((a < 0, b > 0, \xi > 0, a^2 > 4b)\).
Fig. 5.19. (a) Buckling-load parameter as function of initial imperfection amplitude, (b) probability density of initial imperfection amplitude (shaded area equals the reliability of the structure at nondimensional load level $\alpha$).
Fig. 5.20. Influence of standard deviation of initial imperfection on reliability - mean imperfection nonzero ($m_x = 0.05, b = -1$).
Fig. 5.21. Influence of standard deviation of initial imperfection on reliability - mean imperfection identically zero (b = -1).
Fig. 5.22: Influence of mean imperfection on reliability ($a_x = 0.1, b = -1$).
Fig. 5.24. Derivation of Eq. (5.57).
Fig. 5.25. Load versus maximum response; $\lambda_d$ is the dynamic buckling load (after Budiansky and Hutchinson).

Fig. 5.26. Generalisation of dynamic buckling criterion (after Budiansky).
Fig. 5.27. Probability densities of nondimensional static buckling loads ($\sigma_{\bar{X}} = 0.1$).
Fig. 5.28. (1) Nondimensional mean buckling load $E(\Lambda_s/\Lambda_c)$ equals to area under realiability curve ($m_\bar{x} = 0.05, \sigma_\bar{x} = 0.1$, $b = -1$).

(2) $E(\Lambda_s/\Lambda_c)$ versus the standard deviation $\sigma_\bar{x}$ of initial imperfections.
Fig. 5.29: Standard deviation of nondimensional buckling load as function of standard deviation of initial imperfection.
Fig. 5.30. Allowable buckling load corresponding to required reliability \( r = 0.99 \) as function of standard deviation of initial imperfections \( (b = -1) \).
Fig. 5.31. Mean buckling load exceeding allowable buckling load associated with high reliability.
Fig. 5.32. Comparison of allowable buckling loads corresponding to required reliability $r = 0.99$ for static and dynamic cases ($b = -1, m_X = 0$).
Fig. 5.33. Imperfect bar under axial impact, represented by unit step function in time, \( P(t) = P_U(t) \).
Fig. 5.34. Illustration of Hoff's definition of stability, (a) $t_1$-time to fail, (b) structure does not fail.
Fig. 5.35. Probability distribution function of first passage time \( (P = 2P_c) \).
Fig. 5.37. $\text{Prob}(\Lambda < \lambda)$ versus $\lambda$, $P = 0.5 \ P_c$. 
Fig. 5.38. Influence of load ratio $P/P_c$ on $\text{Prob}(\Lambda \leq \lambda)$. 
Fig. 5.39. Shaded area equal to mean buckling time of structure ($P = 2P_c$).
Prob. 5.1
Prob. 5.3

Prob. 5.4
Prob. 5.16

Prob. 5.17
Prob. 5.20
(a)

(b)

Prob. 5.22