NATIONAAL LUCHT- EN RUIMTEVAARTLABORATORIUM

NATIONAL AEROSPACE LABORATORY NLR

THE NETHERLANDS

NLR TR 73112 U

STABILITY OF DOUBLE GIMBALLED MOMENTUM WHEELS

BY

G. BOERSMA and F. J. SONNENSCHEIN

DOCUMENT CONTROL SHEET

	ORIGINATOR'S REF.		SECURITY CLASS.
	NLR TR 73112 U		Unclassified

ORIGINATOR National Aerospace Laboratory (NLR)
Amsterdam. The Netherlands.

TITLE STABILITY OF DOUBLE GIMBALLED MOMENTUM WHEELS.

PRESENTED AT

AUTHORS	DATE	pp	ref
G. Boersma and F.J. Sonnenschein	1-10-1973	35	2

DESCRIPTORS
- Stability analysis
- Hydrodynamic bearings
- Satellite attitude control
- Double gimballed momentum wheels
- Stability criteria

ABSTRACT

Due to the finite stiffness of rotor- and gimbal-suspensions, a double gimballed momentum wheel will oscillate with high natural frequencies. The stability of these oscillations is studied, in order to obtain requirements for the stiffness of the suspensions and/or the gimbal inertia properties. From energy considerations, it is concluded that double gimballed momentum wheels are liable to suffer from instability, if the suspension is such, that the force in a certain direction depends on the displacement in a perpendicular direction. This will occur for hydrodynamic rotor bearings.

For a momentum wheel using these bearings, an equation for the boundary of the stable region is derived and approximate solutions of this equation are obtained. The approximate solutions are compared with numerical solutions - obtained using a digital computer - for some typical configurations.

NLR TR 73112 U

STABILITY OF DOUBLE GIMBALLED MOMENTUM WHEELS

by

G. Boersma and F.J. Sonnenschein

SUMMARY

Due to the finite stiffness of rotor- and gimbal-suspensions, a double gimballed momentum wheel will oscillate with high natural frequencies. The stability of these oscillations is studied, in order to obtain requirements for the stiffness of the suspensions and/or the gimbal inertia properties. From energy considerations, it is concluded that double gimballed momentum wheels are liable to suffer from instability, if the suspension is such, that the force in a certain direction depends on the displacement in a perpendicular direction. This will occur for hydrodynamic rotor bearings.

For a momentum wheel using these bearings, an equation for the boundary of the stable region is derived and approximate solutions of this equation are obtained. The approximate solutions are compared with numerical solutions - obtained using a digital computer - for some typical configurations.

The study was performed under contract with the Netherlands Agency for Aerospace Programs (NIVR).

Division: Spaceflight	Completed : 1-10-1973
Prepared: GB/FJS	Ordernumber: 101.334
Approved: PK	Typ. : MKV

CONTENTS

6 tables

9 figures

LIST OF SYMBOLS

Symbol	Subscripts	
a	-	$2\omega/\Omega$
$\bar{a}$	R,i,o	position vector of hinge centres of rotor, inner- and outer gimbal
a	x,y,z	linear accelerations of satellite
a,b,c,d	11,12,..88	matrix elements - values are given in table 4
A,C,D	12,32,..77	matrix elements in equation (38)
b	x,y,z	components of the position vector of reference point
E	T,R,D	translation, rotation and deformation energy
f	Rx,ix,ox	stiffness of the suspension of rotor, inner- and outer gimbal in x-direction - similar for y- and z-directions
f	Rxz	transverse stiffness of rotor bearing $f_{Rxz} = -\frac{\partial F_{1x}}{\partial \xi_{Rzb}} = \frac{\partial F_{1z}}{\partial \xi_{Rxb}}$
f	tot	total translational stiffness
f	rot	rotational stiffness
f	x,z	defined in equation (39)
f^1	any	non-dimensional value of f (table 5)
F,T	1x,1y,1z	components of force and torque on rotor by inner gimbal
F,T	2x,2y,2z	components of force and torque on inner gimbal by outer gimbal
F,T	3x,3y,3z	components of force and torque on outer gimbal by satellite
F,T	dx,dy,dz	components of disturbance force and torque on satellite

Symbol	Subscripts	
g	R1,R2,R3	shaft bending coefficients
g^1	R1,R2,R3	non-dimensional value of g (table 5)
h	R	angular momentum of rotor about rotor axis
h	Rn	nominal value of h_R
H	R,i,o	steering torques
I	x,y,z	moments of inertia of satellite
I,J,K	R	moments of inertia of rotor about rotor hinge centre
I,J,K	i	moments of inertia of rotor + inner gimbal about inner gimbal hinge centre
I,J,K	o	moments of inertia of rotor + gimbals about outer gimbal hinge centre
I^1,J^1,K^1	R,i,o	moments of inertia of rotor, inner- and outer gimbal about their centres of mass respectively
ℓ	i,o	distance between hinge centres
L	R,i,o	distance between pairs of bearings or springs
m	-	satellite mass
m	R,i,o	mass of rotor, rotor + inner gimbal and rotor + gimbals respectively
m^1	i,o	mass of inner- and outer gimbal
p,q,r	-	angular rates of satellite
s	R	static moment of rotor about rotor hinge centre
s	i	static moment of rotor + inner gimbal about inner gimbal hinge centre
s	o	static moment of rotor + gimbals about outer gimbal hinge centre
T	fR,fi,fo	Coulomb friction torque

Symbol	Subscripts	
T	δ_i, δ_o	spring constants
T	η_R	viscous friction coefficient
T	1x etc	see under F
$\bar{x}$	R,i,o	position vectors of centres of mass
y	R,i	y component of $\bar{x}_R - \bar{a}_R$ and $\bar{x}_i - \bar{a}_i$ respectively
α, β, γ		non-dimensional moments of inertia (table 5)
δ	i,o	gimbals angles
ε, ξ	Rx,ix,ox	x-component of angular and linear displacements due to compliance in rotor, inner gimbal and outer gimbal suspensions - similar for y- and z-directions
ε, ξ	Rxb,Rzb	displacements due to bearing compliance
ε, ξ	Rxs,Rzs	displacements due to shaft compliance
η	any	damping coefficients
λ, Λ		non-dimensional lengths (table 5)
μ		non-dimensional mass (table 5)
σ		non-dimensional static moment (table 5)
τ		non-dimensional spring constant (table 5)
ω		lowest natural frequency at boundary of stable region
Ω		rotor angular velocity

1 INTRODUCTION

Gimballed momentum wheels present a promising system for attitude stabilization and control of satellites. The study of such systems is very difficult if all effects, such as sensor characteristics, steering laws, finite stiffness, gimbal masses etc., have to be taken into account.

Therefore, it is customary to simplify a double gimballed momentum wheel for system studies by neglecting the gimbal masses and assuming that the suspension stiffness is infinite. This simplification is justified if, and only if, the natural frequencies due to finite gimbal masses and finite stiffness of the suspensions are well above the system frequencies and if the motions represented by these natural frequencies are stable. These conditions will be fulfilled for a high stiffness of the suspensions and for low gimbal inertias.

The purpose of this study is to obtain requirements for the stiffness of the suspensions and for the gimbal inertias from stability considerations. As starting point a complete set of linearized equations is used, but neglecting the steering moments which will have a very low frequency compared to the natural frequencies of the double gimballed momentum wheel.

As the number of parameters in the system is rather high, a study by parameter variation would be very labourious. Therefore a different approach is used. Firstly the energy of the system is considered, to obtain insight in the question what kind of systems are liable to suffer from instability and in the instability mode. This insight is used to derive a general equation for the borderline of the stable region. As this equation still contains a large number of parameters, approximate solutions, depending on a limited number of parameters, are looked for. These approximate solutions are obtained using different sets of approximations, and they are tested by comparison with numerical solutions of the equation for some typical configurations. These solutions are obtained using a digital computer.

2 EQUATIONS OF MOTION

The equations of motion for a satellite equiped with a double gimballed momentum wheel are derived in reference 1. These equations are given in table 1 for a coordinate system defined in figure 1. In table 1 the suspension forces and torques are denoted by F_{1x}, T_{1x}, etc.; they are dependent on the method of suspension and will be considered in chapter 3. The inertia forces and torques are given on the right hand side of the equations (a) to (r) and are independent of the suspension method.

For stability considerations some simplifying assumptions can be introduced in the inertia forces and torques:

- The mass and the moments of inertia of the satellite will be large compared to the mass and the moments of inertia of the wheel and the gimbals. Therefore the motions of the satellite can be neglected, i.e. $p = q = r = 0$ and $a_x = a_y = a_z = 0$.
- The deviations from the nominal conditions remain small, permitting linearization of the equations of motion.

A third simplification stems from a restriction of the allowable suspension configurations: The double gimballed momentum wheel must be tested on the ground, and therefore gravitation forces must have only minor effects on the behaviour. This means that the centre of mass of the rotor and inner gimbal must be located on the inner gimbal axis ($s_i = 0$) and that the centre of mass of the rotor, inner and outer gimbal must be located on the outer gimbal axis ($s_o = 0$). In practice, this involves also that the outer and inner gimbal axis intersect ($l_o = 0$). These simplifications are incorporated in table 2.

3 HINGE EQUATIONS

3.1 Rotor suspension

For the rotor suspension ball bearings and hydrodynamic bearings are considered. The torques about the rotor axis will consist of the steering torque and the friction torques (viscous and Coulomb friction). Therefore the general form for the torque about the rotor axis is:

$$T_{1y} = H_R - T_{fR} \operatorname{sign} \dot{h}_R - T_{\eta R} \dot{h}_R$$

For stability considerations the low frequency parts of this equation can be neglected, resulting in:

$$T_{1y} = -T_{\eta R} (\dot{h}_R - \dot{h}_{Rn}) \tag{1}$$

For small displacements, the force on the rotor along the rotor axis will be proportional to the axial displacement in the bearing system and its time derivative:

$$F_{1y} = - f_{Ry} \xi_{Ry} - \eta_{Ry} \dot{\xi}_{Ry} \tag{2}$$

For ball bearing suspensions, the forces along the x and z axis will have a form similar to equation (2). For hydrodynamic suspension, these forces will also have a term proportional to the displacement perpendicular to these forces and to the rotor axis (see Ref. 2) i.e.

$$F_{1x} = -f_{Rx} \xi_{Rxb} - \eta_{Rx} \dot{\xi}_{Rxb} - f_{Rxz} \xi_{Rzb} \tag{3}$$

$$F_{1z} = - f_{Rx} \xi_{Rzb} - \eta_{Rx} \dot{\xi}_{Rzb} + f_{Rxz} \xi_{Rxb} \tag{4}$$

In equations (3) and (4) the same coefficients are used; this is valid for rotary symmetric bearings. The displacements bear the subscript b (for bearing) to distinguish the displacements in the bearing from the total displacements. This distinction is required when the structural displacements outside the bearings must be taken into account.

From the equations (3) and (4) expressions for the torques can be derived under the assumption that the bearing system consists of two identical bearings at a distance L_R:

$$T_{1x} = -\frac{L_R^2}{4}\left[f_{Rx}\,\varepsilon_{Rxb} + \eta_{Rx}\,\dot{\varepsilon}_{Rxb} + f_{Rxz}\,\varepsilon_{Rzb}\right] \tag{5}$$

$$T_{1z} = -\frac{L_R^2}{4}\left[f_{Rx}\,\varepsilon_{Rzb} + \eta_{Rx}\,\dot{\varepsilon}_{Rzb} - f_{Rxz}\,\varepsilon_{Rxb}\right] \tag{6}$$

For the structural displacements outside the bearing, rotor shaft bending without damping and rotary symmetry about the y-axis, but no symmetry about the x and z axes is assumed (see Fig. 2). Denoting the forces in x direction on the inner gimbal at the two bearings by K_1 and K_2 and the structural displacements of the bearings by x_1 and x_2, the relations between these quantities are:

$$K_1 = a_1 x_1 + a_2 x_2 \tag{7}$$

$$K_2 = a_2 x_1 + a_3 x_2 \tag{8}$$

The force and torque on the rotor and the structural displacement and rotation of the rotor are given by the following equations as can be seen from figure 2.

$$F_{1x} = -(K_1 + K_2) \qquad T_{1z} = \frac{L_R}{2}(K_1 - K_2)$$

$$\xi_{Rxs} = \frac{1}{2}(x_1 + x_2) \qquad \varepsilon_{Rzs} = -\frac{1}{L_R}(x_1 - x_2)$$

Elimination of K_1, K_2, x_1 and x_2 results in:

$$F_{1x} = -g_{R1}\,\xi_{Rxs} + g_{R2}\,\frac{L_R}{2}\,\varepsilon_{Rzs} \tag{9}$$

$$T_{1z} = +g_{R2}\,\frac{L_R}{2}\,\xi_{Rxs} - g_{R3}\,\frac{L_R^2}{4}\,\varepsilon_{Rzs} \tag{10}$$

with

$$\left.\begin{aligned} g_{R1} &= a_1 + 2a_2 + a_3 \\ g_{R2} &= a_1 - a_3 \\ g_{R3} &= a_1 - 2a_2 + a_3 \end{aligned}\right\} \tag{11}$$

In the same way one can obtain :

$$F_{1z} = -g_{R1}\,\xi_{Rzs} - g_{R2}\frac{L_R}{2}\,\varepsilon_{Rxs} \tag{12}$$

$$T_{1x} = -g_{R2}\frac{L_R}{2}\,\xi_{Rzs} - g_{R3}\frac{L_R^2}{4}\,\varepsilon_{Rxs} \tag{13}$$

Equations (1) to (6) are incorporated in table 2 (right hand side of equations (a) to (c) and (j) to (l). A combination of the equations (3) to (6) with (9), (10), (12) and (13) results in the equations (s) to (v) in table 2.

It can be noted, that the linear and angular displacements of the rotor with respect to the inner gimbal ξ_{Rx} and ε_{Rz} consist of two parts, one due to bearing compliance and the other due to shaft compliance. As a result, the number of independent variables increases with four. The required additional equations are the equations (s) to (v) in table 2.

3.2 Gimbal suspension

For the gimbal suspensions ball bearings and springs are considered. For ball bearing suspension with torque steering, the torques T_{2z} and T_{3y} consists of the steering torques and the friction torques (mainly Coulomb friction). For stability considerations the (low frequency) steering torques can be neglected, and the equations are:

$$T_{2z} = -T_{fi}\,\text{sign}\,\dot{\delta}_i \qquad T_{3x} = -T_{fo}\,\text{sign}\,\dot{\delta}_o$$

For ball bearing suspension with gimbal position steering, and for spring suspension, the torques will be proportional to the gimbal deflection, with a high proportionality constant in the first case and a low proportionality constant in the second case. The general form of the equations becomes:

$$T_{2z} = -T_{fi}\,\text{sign}\,\dot{\delta}_i - T_{\delta_i}\,\delta_i \tag{14}$$

$$T_{3x} = -T_{fo}\,\text{sign}\,\dot{\delta}_o - T_{\delta_o}\,\delta_o \tag{15}$$

The forces F_{2z} and F_{3x} will be proportional to the axial displacement in the bearing systems and the derivatives (the latter for ball bearings only), giving as general formulae

$$F_{2z} = - f_{iz}\, \xi_{iz} - \eta_{iz}\, \dot{\xi}_{iz} \qquad (16)$$

$$F_{3x} = - f_{ox}\, \xi_{ox} - \eta_{ox}\, \dot{\xi}_{ox} \qquad (17)$$

For the gimbal suspension the structural displacements can be neglected; the general form of the equations for the forces perpendicular to the hinge axis is similar to equation (3) omitting the last term and the subscript b. For spring suspension the terms with η can be neglected. The general form of the equations for the torques perpendicular to the hinge axis is similar to equation (5) with the same omissions.

The resulting equations and the equations (14) to (17) are incorporated in table 2 (right hand side of equations (d) to (i) and (m) to (r)).

With this result, the equations of motion are complete: the number of equations is equal to the number of independent variables.

4 ENERGY CONSIDERATIONS

4.1 Possibility of instability

The energy of a double gimballed momentum wheel consists of kinetic energy and deformation energy. The kinetic energy can be separated into translational energy and rotational energy.

The translational energy of the system is equal to:

$$E_T = \sum_{R,i,o} \frac{1}{2} m_j^1 \; V_j^2 \quad \text{or:}$$

$$E_T = \frac{1}{2} m_o^1 \left[\dot{\bar{\xi}}_o + \dot{\bar{\varepsilon}}_o X (\bar{x}_o - \bar{a}_o)\right]^2 + \frac{1}{2} m_i^1 \left[\dot{\bar{\xi}}_o + \dot{\bar{\xi}}_i + \dot{\bar{\varepsilon}}_o X (\bar{x}_i - \bar{a}_o) + \right.$$

$$\left. + \dot{\bar{\varepsilon}}_i X (\bar{x}_i - \bar{a}_i)\right]^2 + \frac{1}{2} m_R \left[\dot{\bar{\xi}}_o + \dot{\bar{\xi}}_i + \dot{\bar{\xi}}_R + \dot{\bar{\varepsilon}}_o X (\bar{x}_R - \bar{a}_o) + \dot{\bar{\varepsilon}}_i \right.$$

$$\left. X (\bar{x}_R - \bar{a}_i) + \dot{\bar{\varepsilon}}_R X (\bar{x}_R - \bar{a}_R)\right]^2$$

Introduction of $\bar{x}_o = \bar{a}_o = \bar{a}_i$ and $\bar{x}_R - \bar{a}_R = y_R \bar{1}_y$, $\bar{x}_i - \bar{a}_i = y_i \bar{1}_y$ and $\bar{a}_R - \bar{a}_i = l_i \bar{1}_y$ (with $\bar{1}_y$ the unity vector in y-direction) gives:

$$E_{Tx} = \frac{1}{2} m_o^1 \dot{\xi}_{ox}^2 + \frac{1}{2} m_i^1 \left[\dot{\xi}_{ox} + \dot{\xi}_{ix} - y_i (\dot{\varepsilon}_{oz} + \dot{\delta}_i)\right]^2$$

$$+ \frac{1}{2} m_R \left[\dot{\xi}_{ox} + \dot{\xi}_{ix} + \dot{\xi}_{Rx} - (y_R + l_i)(\dot{\varepsilon}_{oz} + \dot{\delta}_i) - y_R \dot{\varepsilon}_{Rz}\right]^2 \quad (18)$$

$$E_{Ty} = \frac{1}{2} m_o^1 \dot{\xi}_{oy}^2 + \frac{1}{2} m_i^1 \left[\dot{\xi}_{oy} + \dot{\xi}_{iy}\right]^2 + \frac{1}{2} m_R \left[\dot{\xi}_{oy} + \dot{\xi}_{iy} + \dot{\xi}_{Ry}\right]^2 \quad (19)$$

$$E_{Tz} = \frac{1}{2} m_o^1 \dot{\xi}_{oz}^2 + \frac{1}{2} m_i^1 \left[\dot{\xi}_{oz} + \dot{\xi}_{iz} + y_i (\dot{\delta}_o + \dot{\varepsilon}_{ix})\right]^2$$

$$+ \frac{1}{2} m_R \left[\dot{\xi}_{oz} + \dot{\xi}_{iz} + \dot{\xi}_{Rz} + (y_R + l_i)(\dot{\delta}_o + \dot{\varepsilon}_{ix}) + y_R \dot{\varepsilon}_{Rx}\right]^2 \quad (20)$$

The rotational energy about the three axes is:

$$E_{Rx} = \frac{1}{2} I_o^1 \dot{\delta}_o^2 + \frac{1}{2} I_i^1 (\dot{\delta}_o + \dot{\varepsilon}_{ix})^2 + \frac{1}{2} I_R^1 (\dot{\delta}_o + \dot{\varepsilon}_{ix} + \dot{\varepsilon}_{Rx})^2 \quad (21)$$

$$E_{Ry} = \frac{1}{2} J_o^1 \dot{\varepsilon}_{oy}^2 + \frac{1}{2} J_i^1 (\dot{\varepsilon}_{oy} + \dot{\varepsilon}_{iy})^2 + \frac{1}{2} J_R^1 (\dot{\varepsilon}_{oy} + \dot{\varepsilon}_{iy} + \dot{\delta}_R)^2 \quad (22)$$

$$E_{Rz} = \frac{1}{2} K_o^1 \dot{\varepsilon}_{oz}^2 + \frac{1}{2} K_i^1 (\dot{\varepsilon}_{oz} + \dot{\delta}_i)^2 + \frac{1}{2} K_R^1 (\dot{\varepsilon}_{oz} + \dot{\delta}_i + \dot{\varepsilon}_{Rz})^2 \quad (23)$$

For the assumed linear relation between forces and displacements in the suspension, the total deformation energy is equal to:

$$E_D = \sum \frac{1}{2} \overline{\text{force}} \cdot \overline{\text{displacement}}$$

The deformation energy cannot be divided into translational and rotational energy, for F_{1x} is dependent on the angular displacement ε_{Rzs} and T_{1z} is dependent on the linear displacement ξ_{Rxs} (equations (9) and (10)). Therefore, the deformation energy is divided into four parts:

E_{Dx} - deformation energy for motions along x axis and about z axis.

E_{Dz} - deformation energy for motions along z axis and about x axis.

E_{Dy} - deformation energy for motions along y axis.

E_{DyR} - deformation energy for motions about y axis.

The hinge equations (chapter 3) result in:

$$E_{Dx} = \frac{1}{2} f_{ox} \xi_{ox}^2 + \frac{1}{2} f_{ix} \xi_{ix}^2 + \frac{1}{2} f_{Rx} \xi_{Rxb}^2 + \frac{1}{2} g_{R1} \xi_{Rxs}^2 + \frac{L_R^2}{8} \times$$

$$g_{R3} \varepsilon_{Rzs}^2 - \frac{L_R}{2} g_{R2} \xi_{Rxs} \varepsilon_{Rzs} + \frac{1}{2} T_{\delta i} \delta_i^2 + \frac{L_R^2}{8} f_{Rx} \varepsilon_{Rzb}^2 +$$

$$+ \frac{L_o^2}{8} f_{oy} \varepsilon_{oz}^2 \qquad (24)$$

$$E_{Dz} = \frac{1}{2} f_{oy} \xi_{oz}^2 + \frac{1}{2} f_{iz} \xi_{iz}^2 + \frac{1}{2} f_{Rx} \xi_{Rzb}^2 + \frac{1}{2} g_{R1} \xi_{Rzs}^2 + \frac{L_R^2}{8} \times$$

$$g_{R3} \varepsilon_{Rxs}^2 + \frac{L_R}{2} g_{R2} \xi_{Rzs} \varepsilon_{Rxs} + \frac{1}{2} T_{\delta o} \delta_o^2 + \frac{L_i^2}{8} f_{ix} \varepsilon_{ix}^2 +$$

$$+ \frac{L_R^2}{8} f_{Rx} \varepsilon_{Rxb}^2 \qquad (25)$$

$$E_{Dy} = \frac{1}{2} f_{oy} \xi_{oy}^2 + \frac{1}{2} f_{ix} \xi_{iy}^2 + \frac{1}{2} f_{Ry} \xi_{Ry}^2 \qquad (26)$$

$$E_{DyR} = \frac{L_o^2}{8} f_{oy} \varepsilon_{oy}^2 + \frac{L_i^2}{8} f_{ix} \varepsilon_{iy}^2 \qquad (27)$$

Sommation of equations (18), (23) and (24) and differentation with respect to tine, results (after recombination of terms) in:

$$\frac{d}{dt}\left[E_{Tx} + E_{Rz} + E_{Dx}\right] =$$

$$+ \dot{\xi}_{ox}\left[m_o \ddot{\xi}_{ox} + m_i \ddot{\xi}_{ix} + m_R \ddot{\xi}_{Rx} - s_R \ddot{\varepsilon}_{Rz} + f_{ox} \xi_{ox}\right] +$$

$$+ \dot{\xi}_{ix}\left[m_i \ddot{\xi}_{ox} + m_i \ddot{\xi}_{ix} + m_R \ddot{\xi}_{Rx} - s_R \ddot{\varepsilon}_{Rz} + f_{ix} \xi_{ix}\right] +$$

$$+ \dot{\xi}_{Rx}\left[m_R \ddot{\xi}_{ox} + m_R \ddot{\xi}_{ix} + m_R \ddot{\xi}_{Rx} - (s_R + m_R \ell_i)(\ddot{\varepsilon}_{oz} + \ddot{\delta}_i) - s_R \ddot{\varepsilon}_{Rz} + \right.$$

$$\left. + f_{Rx} \xi_{Rxb}\right] +$$

$$+ \dot{\varepsilon}_{oz}\left[-(s_R + m_R \ell_i) \ddot{\xi}_{Rx} + K_o \ddot{\varepsilon}_{oz} + K_i \ddot{\delta}_i + (K_R + s_R \ell_i) \ddot{\varepsilon}_{Rz} + \frac{L_o^2}{4} f_{oy} \varepsilon_{oz}\right] +$$

$$+ \dot{\delta}_i \left[-(s_R + m_R \ell_i) \ddot{\xi}_{Rx} + K_i \ddot{\varepsilon}_{oz} + K_i \ddot{\delta}_i + (K_R + s_R \ell_i) \ddot{\varepsilon}_{Rz} + T_{\delta i} \delta_i\right] +$$

$$+ \dot{\varepsilon}_{Rz}\left[-s_R (\ddot{\xi}_{ox} + \ddot{\xi}_{ix} + \ddot{\xi}_{Rx}) + (K_R + s_R \ell_i)(\ddot{\varepsilon}_{oz} + \ddot{\delta}_i) + K_R \ddot{\varepsilon}_{Rz} + \frac{L_R^2}{4} f_{Rx} \varepsilon_{Rzb}\right] +$$

$$+ \dot{\xi}_{Rxs}\left[-f_{Rx} \xi_{Rxb} + g_{R1} \xi_{Rxs} - \frac{L_R}{2} g_{R2} \varepsilon_{Rzs}\right] +$$

$$+ \dot{\varepsilon}_{Rzs} \frac{L_R}{2}\left[-g_{R2} \xi_{Rxs} - \frac{L_R}{2} f_{Rx} \varepsilon_{Rzb} + \frac{L_R}{2} g_{R3} \varepsilon_{Rzs}\right] \qquad (28)$$

Substitution of equations (a), (d), (g), (l), (o), (r), (s) and (v) of table 2 resultsin:

$$\frac{d}{dt}\left[E_{Tx} + E_{Rz} + E_{Dx}\right] = - \eta_{ox}\,\dot{\xi}^2_{ox} - \eta_{ix}\,\dot{\xi}^2_{ix} - \eta_{Rx}\,\dot{\xi}_{Rxb} +$$

$$- \eta_{oy}\frac{L_o^2}{4}\,\dot{\varepsilon}^2_{oz} - T_{fi}\,\dot{\delta}_i\,\text{sign}\,\dot{\delta}_i - \eta_{Rx}\frac{L_R^2}{4}\,\dot{\varepsilon}^2_{Rzb} - f_{Rxz}\,(\xi_{Rzb}\,\dot{\xi}_{Rxb} +$$

$$- \frac{L_R^2}{4}\,\varepsilon_{Rxb}\,\dot{\varepsilon}_{Rzb}) - h_{Rn}\,(\dot{\delta}_o + \dot{\varepsilon}_{ix} + \dot{\varepsilon}_{Rx})(\dot{\varepsilon}_{oz} + \dot{\delta}_i + \dot{\varepsilon}_{Rz}) \qquad (29)$$

In the same way can be derived:

$$\frac{d}{dt}\left[E_{Tz} + E_{Rx} + E_{Dz}\right] = - \eta_{oy}\,\dot{\xi}^2_{oz} - \eta_{iz}\,\dot{\xi}^2_{iz} - \eta_{Rx}\,\dot{\xi}^2_{Rzb} +$$

$$- T_{fi}\,\dot{\delta}_o\,\text{sign}\,\dot{\delta}_o - \eta_{ix}\frac{L_i^2}{4}\,\dot{\varepsilon}^2_{ix} - \eta_{Rx}\frac{L_R^2}{4}\,\dot{\varepsilon}^2_{Rxb} + f_{Rxz}\,(\xi_{Rxb}\,\dot{\xi}_{Rzb} +$$

$$- \frac{L_R}{4}\,\varepsilon_{Rzb}\,\dot{\varepsilon}_{Rxb}) + h_{Rn}\,(\dot{\delta}_o + \dot{\varepsilon}_{ix} + \dot{\varepsilon}_{Rx})(\dot{\varepsilon}_{oz} + \dot{\delta}_i + \dot{\varepsilon}_{Rz}) \qquad (30)$$

$$\frac{d}{dt}\left[E_{Ty} + E_{Dy}\right] = - \eta_{oy}\,\dot{\xi}^2_{oy} - \eta_{ix}\,\dot{\xi}^2_{iy} - \eta_{Ry}\,\dot{\xi}^2_{Ry} \qquad (31)$$

$$\frac{d}{dt}\left[E_{Ry} + E_{DyR}\right] = - \frac{L_o^2}{4}\,\eta_{oy}\,\dot{\varepsilon}^2_{oy} - \frac{L_i^2}{4}\,\eta_{ix}\,\dot{\varepsilon}^2_{iy} - \frac{T_{\eta R}}{J_R}\,h_R\,(h_R - h_{Rn}) \qquad (32)$$

From these results some conclusions can be drawn:

<u>a</u> The equations (29) and (30) contain three different types of terms:

- friction terms, which are always negative (energy dissipation) and have a favourable effect on stability (the first six terms)

- gyroscopic coupling terms, which have no net effect on the total energy and cannot cause instability (the last terms)
- terms with f_{Rxz} which are present for hydrodynamic bearings and which are the only terms that can cause instability.

b Equation (31) contains only friction terms, therefore the motion along the y axis is always stable.

c Equation (32) also contains only friction terms. The motion about the y axis is always stable; the equilibrium position is $h_R = h_{Rn}$.

Apparently, the most critical suspension method consists of rotor bearings with large transverse stiffness f_{Rxz} e.g. hydrodynamic bearings and gimbal suspension with a very low damping. Such a system will be analyzed; only the motions along and about the x and z axes require further consideration.

4.2 Instability mode

Neglecting the damping in the gimbal suspensions, the energy equation for motions along and about the x- and z axes can be obtained by addition of equations (29) and (30):

$$\frac{dE}{dt} = -\eta_{Rx}\left[\dot{\xi}^2_{Rxb} + \dot{\xi}^2_{Rzb} + \frac{L_R^2}{4}(\dot{\varepsilon}^2_{Rzb} + \dot{\varepsilon}^2_{Rxb})\right] +$$

$$+ f_{Rxz}\left[\xi_{Rxb}\,\dot{\xi}_{Rzb} - \xi_{Rzb}\,\dot{\xi}_{Rxb} + \frac{L_R^2}{4}(\varepsilon_{Rxb}\,\dot{\varepsilon}_{Rzb} - \varepsilon_{Rzb}\,\dot{\varepsilon}_{Rxb})\right] \quad (33)$$

The condition for stable motion is $\frac{dE}{dt} < 0$; therefore the borderline of the stable region can be defined as $\frac{dE}{dt} = 0$.

As can be seen from equations (24) and (25), the energy will always increase with an increase of the displacements ξ and ε. Hence the suspension system will always be statically stable. Consequently only dynamic instability can occur. On the borderline of the stable region, the motion will be dynamically indifferent, i.e. the motion can be described by:

$$\xi_{Rxb} = A_1 \sin(\omega t + \varphi_1) \qquad \varepsilon_{Rxb} = \frac{2A_3}{L_R} \sin(\omega t + \varphi_3)$$

$$\xi_{Rzb} = A_2 \sin(\omega t + \varphi_2) \qquad \varepsilon_{Rzb} = \frac{2A_4}{L_R} \sin(\omega t + \varphi_4)$$

Substitution in equation (33) results in:

$$\frac{dE}{dt} = -\eta_{Rx}\omega^2 \left[A_1^2 \cos^2(\omega t + \varphi_1) + A_2^2 \cos^2(\omega t + \varphi_2) + A_3^2 \cos^2(\omega t + \varphi_3) + A_4^2 \cos^2(\omega t + \varphi_4)\right] + f_{Rxz}\omega \left[A_1A_2 \sin(\varphi_1 - \varphi_2) + A_3A_4 \sin(\varphi_3 - \varphi_4)\right]$$

Integration over one period and division by π results in:

$$\frac{\Delta E}{\pi} = -\eta_{Rx}\omega^2 \left[A_1^2 + A_2^2 + A_3^2 + A_4^2\right] + 2f_{Rxz}\omega \left[A_1A_2 \sin(\varphi_1 - \varphi_2) + A_3A_4 \sin(\varphi_3 - \varphi_4)\right]$$

From this equation it can be seen that for high frequencies ΔE will be negative and therefore the motion stable. Instability will occur at frequencies at which A_1 to A_4 and φ_1 to φ_4 can be chosen in such a way that ΔE is positive. Therefore, the borderline of instability will be given by the highest frequency for which ΔE can be zero, or:

$$\omega = \frac{2f_{Rxz}}{\eta_{Rx}} \frac{A_1A_2 \sin(\varphi_1 - \varphi_2) + A_3A_4 \sin(\varphi_3 - \varphi_4)}{A_1^2 + A_2^2 + A_3^2 + A_4^2} \qquad (34)$$

with the conditions $\frac{\partial \omega}{\partial A_i} = 0$ $(i = 1,2,3,4)$ $\frac{\partial \omega}{\partial(\varphi_1 - \varphi_2)} = 0$ and $\frac{\partial \omega}{\partial(\varphi_3 - \varphi_4)} = 0$ and the second derivatives of ω negative.

These conditions lead to:

$$\sin(\varphi_1 - \varphi_2) = \sin(\varphi_3 - \varphi_4) = 1, \; A_1 = A_2 \text{ and } A_3 = A_4$$

or with substitution in equation (34):

$$\omega = \frac{f_{Rxz}}{\eta_{Rx}} \qquad (35)$$

This result is obtained under the assumption, that the coefficients A_1 to A_4 and φ_1 to φ_4 can be chosen arbitrarily. In general this will not be the case, and the undamped frequency will be somewhat less than given in equation (35), i.e.:

$$\omega = \frac{af_{Rxz}}{\eta_{Rx}} \text{ with } a \leqq 1 \qquad (36)$$

Equations (35) and (36) are valid for all methods of suspension, provided the rotor suspension can be written as in equations (3) and (4) and the damping of the gimbal suspensions can be neglected.

For a double gimballed momentum wheel with high reliability, hydrodynamic bearings for the rotor and springsuspensions for the gimbals are promising. If this suspension method is used, the damping in the gimbal suspensions can indeed be neglected. The coefficients f_{Rxz} and η_{Rx} depend on the bearings and on the static load on the bearings. A good hydrodynamic bearing is probably the grooved bearing with length-diameter ratio of unity (ref. 2). The static load on the bearings can be neglected under zero-g-conditions. For this case reference 2 gives the following coefficients:

$$f_{Rxz} / f_{Rx} = 1.5 \text{ and } \eta_{Rx} / f_{Rx} = 3/\Omega \qquad (37)$$

in which Ω is the angular rate of the rotor.

Substitution in equation (36) leads to the conclusion, that, on the borderline of stability, an undamped mode exists with a frequency equal to (or somewhat less than) half of the frequency of the rotor, the socalled "half-frequency whirl". For the remaining part of the study the values given in equation (37) are used.

5 THE STABLE REGION

5.1 The precession mode

In the previous section it is found that a double gimballed momentum wheel is stable, if all possible motions have a frequency above a certain limit value. Therefore, a stable region can only exist if no motions of a low frequency can occur. However, it can be shown that in all cases a motion with a very low frequency will appear among the possible natural motions. This can be inferred from a simplified analysis of the equations of table 2. Introduction of an infinite stiffness for the rotor bearings and for the suspension springs (except $T_{\delta i}$ and $T_{\delta o}$) leads to the following result for the motions along and about the x and z axes:

$$\xi_{Rx} = \xi_{Rz} = \xi_{ix} = \xi_{iz} = \xi_{ox} = \xi_{oz} = \varepsilon_{Rx} = \varepsilon_{Rz} = \varepsilon_{ix} = \varepsilon_{oz} = 0$$

$$K_i \ddot{\delta}_i + h_{Rn} \dot{\delta}_o + T_{\delta i} \delta_i = 0$$

$$I_o \ddot{\delta}_o - h_{Rn} \dot{\delta}_i + T_{\delta o} \delta_o = 0$$

This means, that in the chosen approximation, all frequencies are infinite with two exceptions. These frequencies are about $h_{Rn} / \sqrt{K_i I_o}$ and about $\sqrt{T_{\delta i} T_{\delta o}} / h_{Rn}$. The first of these frequencies will be larger than half the rotor frequency, as K_i and I_o will be smaller than $2J_R$, but the second will certainly be smaller than $\Omega/2$.

This last frequency presents the precession mode of the double gimballed momentum wheel. For infinite bearing and gimbal stiffness the motion in undamped, for finite stiffness the motion will be unstable, although the rate of growth will be extremely small.

It is clear that a stable region can only exist if the precession mode can be damped effectively. Damping can be obtained by means of control torques, provided that the frequency of the precession mode is not too high for the sensors and the torquers.

5.2 General equation for the borderline

The equations of motion given in table 2 can be Laplace transformed and written in matrix form. The result is given in table 3.

The characteristic equation of the system described by the equations of motion is obtained by putting the determinant of the matrix in table 3 equal to zero. After some recasting, the characteristic equation becomes:

$$\begin{vmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{31} & a_{32} & 0 & a_{34} & 0 & 0 & 0 & a_{38} & 0 & b_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{43} & a_{44} & 0 & 0 & 0 & a_{48} & 0 & 0 & b_{43} & b_{44} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{55} & a_{56} & a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{65} & a_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{74} & a_{75} & a_{76} & 0 & a_{78} & 0 & 0 & 0 & 0 & 0 & b_{76} & 0 & 0 \\
0 & 0 & 0 & a_{84} & 0 & 0 & a_{87} & a_{88} & 0 & 0 & 0 & 0 & 0 & 0 & b_{87} & b_{88} \\
0 & 0 & c_{13} & 0 & 0 & 0 & 0 & 0 & d_{11} & d_{12} & d_{13} & 0 & 0 & d_{16} & d_{17} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{21} & d_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{32} & c_{33} & 0 & 0 & 0 & 0 & 0 & d_{31} & d_{32} & 0 & d_{34} & 0 & 0 & 0 & d_{38} \\
0 & 0 & c_{43} & c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & d_{43} & d_{44} & 0 & 0 & 0 & d_{48} \\
0 & 0 & 0 & 0 & 0 & 0 & c_{57} & 0 & 0 & d_{52} & d_{53} & 0 & d_{55} & d_{56} & d_{57} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{65} & d_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{76} & c_{77} & 0 & 0 & 0 & 0 & d_{74} & d_{75} & d_{76} & 0 & d_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & c_{87} & c_{88} & 0 & 0 & 0 & d_{84} & 0 & 0 & d_{87} & d_{88}
\end{vmatrix} = 0$$

The elements of this determinant are given in table 4 (first two columns). Due to the small number of non-zero elements in some rows, the order of the determinant can be reduced easily, leading to:

$$a_{21}\,a_{65}\,d_{21}\,d_{65}\,A_{12}\,A_{56}\begin{vmatrix} A_{33} & a_{34} & 0 & a_{38} & b_{32} & 0 & 0 & 0 & 0 & 0 \\ a_{43} & a_{44} & 0 & a_{48} & 0 & b_{43} & b_{44} & 0 & 0 & 0 \\ 0 & a_{74} & A_{77} & a_{78} & 0 & 0 & 0 & b_{76} & 0 & 0 \\ 0 & a_{84} & a_{87} & a_{88} & 0 & 0 & 0 & 0 & b_{87} & b_{88} \\ c_{13} & 0 & 0 & 0 & D_{12} & d_{13} & 0 & d_{16} & d_{17} & 0 \\ C_{33} & 0 & 0 & 0 & D_{32} & 0 & d_{34} & 0 & 0 & d_{38} \\ c_{43} & c_{44} & 0 & 0 & 0 & d_{43} & d_{44} & 0 & 0 & d_{48} \\ 0 & 0 & c_{57} & 0 & d_{52} & d_{53} & 0 & D_{56} & d_{57} & 0 \\ 0 & 0 & C_{77} & 0 & 0 & 0 & d_{74} & D_{76} & 0 & d_{78} \\ 0 & 0 & c_{87} & c_{88} & 0 & 0 & d_{84} & 0 & d_{87} & d_{88} \end{vmatrix} = 0 \quad (38)$$

with:

$$A_{12} = a_{12} - a_{11}\,a_{22}/a_{21} \qquad C_{33} = c_{33} - c_{32}\,a_{13}/A_{12}$$

$$A_{32} = a_{32} - a_{31}\,a_{22}/a_{21} \qquad C_{77} = c_{77} - c_{76}\,a_{57}/A_{56}$$

$$A_{33} = -A_{32}\,a_{13}/A_{12} \qquad D_{12} = d_{12} - d_{11}\,d_{22}/d_{21}$$

$$A_{56} = a_{56} - a_{55}\,a_{66}/a_{65} \qquad D_{32} = d_{32} - d_{31}\,d_{22}/d_{21}$$

$$A_{76} = a_{76} - a_{75}\,a_{66}/a_{65} \qquad D_{56} = d_{56} - d_{55}\,d_{66}/d_{65}$$

$$A_{77} = -A_{76}\,a_{57}/A_{56} \qquad D_{76} = d_{76} - d_{75}\,d_{66}/d_{65}$$

In section 4.2 it is found, that on the borderline of stability, an undamped oscillation with a frequency equal to or somewhat less than half the rotor frequency occurs. Therefore, the general form of the equation for the borderline of stability is equation (38) with substitution of $s = ja\,\Omega/2$ in the elements of the matrix.

For simplification, non-dimensional parameters are introduced, these parameters are given in table 5 and substituted in the elements of the determinant (table 4). In the last column of table 4 the equations (37) are also substituted.

The non-dimensional form of the equation for the borderline is equation (38) with the non-dimensional values of the elements from table 4.

6 APPROXIMATED STABLE REGION

6.1 Equations for the borderline

A double gimballed momentum wheel can be considered as a system consisting of several masses connected by springs. According to section 4.2 instability will occur if the lowest frequency of the system (except the precession mode) is lower than a limit frequency, which is equal to, or somewhat lower than, half the frequency of the rotor. The lowest frequency in the system (except the precession mode) will largely be determined by the largest mass in the system i.e. the rotor mass, provided that the rotational stiffness of the gimbals is sufficient. The latter condition will be analyzed in section 6.4.

Therefore it can be expected that a realistic approximation of the stability region can be found by neglecting the gimbal masses and moments of inertia. Furthermore, the spring constants $T_{\delta i}$ and $T_{\delta o}$, which will be very small compared to the other spring constants can be neglected. These approximations can be written in the non-dimensional form as:

$$\mu_o = \mu_i = 0$$

$$\sigma = -\lambda$$

$$\alpha_i = \beta_i = \alpha_o = \beta_o = \alpha$$

$$\alpha_R = \beta_R = \alpha + \frac{\lambda^2}{\gamma}$$

$$\tau_o = \tau_i = 0$$

With these approximations and introduction of the abbreviations

$$f_x = \frac{f_{ox}^{1} f_{ix}^{1}}{f_{ox}^{1} + f_{ix}^{1}} \qquad f_z = \frac{f_{oy}^{1} f_{iz}^{1}}{f_{oy}^{1} + f_{iz}^{1}} \tag{39}$$

equation (38) can be written as:

$$\begin{vmatrix}
\mu f_x & -(1+\frac{3}{2}ja) & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
g^1_{R1}(f_x-\mu) & -(1+\frac{3}{2}ja+g^1_{R1}) & 0 & \frac{3}{2} & 0 & \lambda g^1_{R1}+g^1_{R2} & -g^1_{R2} & 0 & 0 & 0 \\
0 & \frac{3}{2} & \mu f_z & (1+\frac{3}{2}ja) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{2} & g^1_{R1}(f_z-\mu) & (1+\frac{3}{2}ja+g^1_{R1}) & 0 & 0 & 0 & 0 & \lambda g^1_{R1}+g^1_{R2} & g^1_{R2} \\
0 & 0 & 0 & 0 & \mu\gamma\alpha & \mu\gamma\alpha & 0 & 2j\mu\gamma/a & 2j\mu\gamma/a & 0 \\
\lambda\mu f_x & 0 & 0 & 0 & 0 & 0 & (1+\frac{3}{2}ja) & 0 & 0 & -\frac{3}{2} \\
-g^1_{R2}(f_x-\mu) & g^1_{R2} & 0 & 0 & 0 & -(\lambda g^1_{R2}+g^1_{R3}) & (1+\frac{3}{2}ja+g^1_{R3}) & 0 & 0 & -\frac{3}{2} \\
0 & 0 & 0 & 0 & 2j\mu\gamma/a & 2j\mu\gamma/a & 0 & -\mu\gamma\alpha & -\mu\gamma\alpha & 0 \\
0 & 0 & -\lambda\mu f_z & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & (1+\frac{3}{2}ja) \\
0 & 0 & g^1_{R2}(f_z-\mu) & +g^1_{R2} & 0 & 0 & \frac{3}{2} & 0 & \lambda g^1_{R2}+g^1_{R3} & (1+\frac{3}{2}ja+g^1_{R3})
\end{vmatrix} = 0$$

Subtraction of row 1 from row 2; row 3 from row 4; λ times row 1 from row 6; row 6 from row 7; $-\lambda$ times row 3 from row 9 and row 9 from row 10, followed by subtraction of λ times column 7 from column 2 and λ times column 10 from column 4 results in:

$$\begin{vmatrix}
\mu f_x & -(1+\frac{3}{2}ja) & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
g^1_{R1}(f_x-\mu)-\mu f_x & \lambda g^1_{R2}-g^1_{R1} & 0 & 0 & 0 & \lambda g^1_{R1}+g^1_{R2} & -g^1_{R2} & 0 & 0 & 0 \\
0 & \frac{3}{2} & \mu f_z & (1+\frac{3}{2}ja) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g^1_{R1}(f_z-\mu)-\mu f_z & g^1_{R1}-\lambda g^1_{R2} & 0 & 0 & 0 & 0 & \lambda g^1_{R1}+g^1_{R2} & g^1_{R2} \\
0 & 0 & 0 & 0 & \alpha & \alpha & 0 & 2j/a & 2j/a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (1+\frac{3}{2}ja) & 0 & 0 & -\frac{3}{2} \\
-g^1_{R2}(f_x-\mu)-\lambda\mu f_x & g^1_{R2}-\lambda g^1_{R3} & 0 & 0 & 0 & -(\lambda g^1_{R2}+g^1_{R3}) & g^1_{R3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2j/a & 2j/a & 0 & -\alpha & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & (1+\frac{3}{2}ja) \\
0 & 0 & g^1_{R2}(f_z-\mu)+\lambda\mu f_z & g^1_{R2}-\lambda g^1_{R3} & 0 & 0 & 0 & 0 & \lambda g^1_{R2}+g^1_{R3} & g^1_{R3}
\end{vmatrix} = 0$$

<u>or:</u>

$$(\alpha^2a^2-4)\begin{vmatrix}
\mu f_x & -(1+\frac{3}{2}ja) & 0 & \frac{3}{2} & 0 & 0 \\
g^1_{R1}(f_x-\mu)-\mu f_x & \lambda g^1_{R2}-g^1_{R1} & 0 & 0 & \lambda g^1_{R1}+g^1_{R2} & 0 \\
0 & \frac{3}{2} & \mu f_z & (1+\frac{3}{2}ja) & 0 & 0 \\
0 & 0 & g^1_{R1}(f_z-\mu)-\mu f_z & g^1_{R1}-\lambda g^1_{R2} & 0 & \lambda g^1_{R1}+g^1_{R2} \\
+g^1_{R2}(f_x-\mu)+\lambda\mu f_x & -g^1_{R2}+\lambda g^1_{R3} & 0 & 0 & +(\lambda g^1_{R2}+g^1_{R3}) & 0 \\
0 & 0 & g^1_{R2}(f_z-\mu)+\lambda\mu f_z & g^1_{R2}-\lambda g^1_{R3} & 0 & \lambda g^1_{R2}+g^1_{R3}
\end{vmatrix} = 0$$

<u>or:</u>

$$(\alpha^2a^2-4)\begin{vmatrix}
\mu f_x & -(1+\frac{3}{2}ja) & 0 \\
g^1_{R1}(f_x-\mu)-\mu f_x & \lambda g^1_{R2}-g^1_{R1} & \lambda g^1_{R1}+g^1_{R2} \\
g^1_{R2}(f_x-\mu)+\lambda\mu f_x & \lambda g^1_{R3}-g^1_{R2} & \lambda g^1_{R2}+g^1_{R3}
\end{vmatrix} \times \begin{vmatrix}
\mu f_z & -(1+\frac{3}{2}ja) & 0 \\
g^1_{R1}(f_z-\mu)-\mu f_z & \lambda g^1_{R2}-g^1_{R1} & \lambda g^1_{R1}+g^1_{R2} \\
g^1_{R2}(f_z-\mu)+\lambda\mu f_z & \lambda g^1_{R3}-g^1_{R2} & \lambda g^1_{R2}+g^1_{R3}
\end{vmatrix} +$$

$$\frac{9}{4}(\alpha^2a^2-4)\begin{vmatrix}
g^1_{R1}(f_x-\mu)-\mu f_x & \lambda g^1_{R1}+g^1_{R2} \\
g^1_{R2}(f_x-\mu)+\lambda\mu f_x & \lambda g^1_{R2}+g^1_{R3}
\end{vmatrix} \times \begin{vmatrix}
g^1_{R1}(f_z-\mu)-\mu f_z & \lambda g^1_{R1}+g^1_{R2} \\
g^1_{R2}(f_z-\mu)+\lambda\mu f_z & \lambda g^1_{R2}+g^1_{R3}
\end{vmatrix} = 0$$

<u>or:</u>

$$(\alpha^2a^2-4)\left[\mu f_x A + (1+\tfrac{3}{2}ja)B_x\right]\left[\mu f_z A + (1+\tfrac{3}{2}ja)B_z\right] + \frac{9}{4}(\alpha^2a^2-4)B_xB_z = 0 \qquad (40)$$

<u>with:</u>

$$A = (\lambda g^1_{R2}-g^1_{R1})(\lambda g^1_{R2}+g^1_{R3})-(\lambda g^1_{R3}-g^1_{R2})(\lambda g^1_{R1}+g^1_{R2}) = (1+\lambda^2)({g^1_{R2}}^2-g^1_{R1}g^1_{R3}) \qquad (41)$$

$$B_x = \left[g^1_{R1}(f_x-\mu)-\mu f_x\right](\lambda g^1_{R2}+g^1_{R3})-\left[g^1_{R2}(f_x-\mu)+\mu\lambda f_x\right](\lambda g^1_{R1}+g^1_{R2}) = (f_x-\mu)(g^1_{R1}g^1_{R3}-{g^1_{R2}}^2)-\mu f_x(\lambda^2g^1_{R1}+2\lambda g^1_{R2}+g^1_{R3}) \qquad (42)$$

$$B_z = (f_z-\mu)(g^1_{R1}g^1_{R3}-{g^1_{R2}}^2)-\mu f_z(\lambda^2g^1_{R1}+2\lambda g^1_{R2}+g^1_{R3}) \qquad (43)$$

Both the real and the imaginary part of equation (40) must be zero, thus:

$$\left[\mu^2 f_x f_z A^2 + \mu A (f_x B_z + f_z B_x) + (1-\frac{9}{4} a^2+\frac{9}{4}) B_x B_z\right](\alpha^2 a^2-4) = 0 \quad (44)$$

$$\left[\mu A (f_x B_z + f_z B_x) + 2 B_x B_z\right](\alpha^2 a^2-4) = 0 \quad (45)$$

These equations define the borderline of the approximate stable region.

6.2 Stable region for symmetric suspension

For symmetric suspension $f_x=f_z$ and $B_x=B_z$, therefore the equations (44) and (45) become:

$$\left[\mu^2 f_x^2 A^2 + 2 \mu f_x A B_x + (1-\frac{9}{4} a^2+\frac{9}{4}) B_x^2\right](\alpha^2 a^2-4) = 0 \quad (46)$$

$$\left[\mu f_x A B_x + B_x^2\right](\alpha^2 a^2-4) = 0 \quad (47)$$

The equations (46) and (47) are satisfied for three sets of parameters:

<u>a</u> $\alpha^2 a^2-4 = 0$. For realistic momentum wheels this condition will not be met as it implies that the moments of inertia of the wheel about the x- and z-axes are larger than, or equal to, twice the moment of inertia about the y-axis (see table 5).

<u>b</u> $A = B_x = 0$. This condition cannot occur, as $g^1_{R1} g^1_{R3}$ is always larger than $g^{1\,2}_{R2}$ (see equations (11) and (41)).

<u>c</u> $\mu f_x A + B_x = 0$ and $a = 1$. These conditions give the borderline of the stable region. With expressions (41) and (42) the stability criterion becomes:

$$\frac{1}{\mu} > 1 + \lambda^2 + \frac{1}{f_x} + \frac{\lambda^2 g^1_{R1} + 2\lambda g^1_{R2} + g^1_{R3}}{g^1_{R1} g^1_{R3} - g^{1\,2}_{R2}} \quad (48)$$

or in dimensional form:

$$\frac{4}{\Omega^2 m_R} > \frac{1+4\ell_i^2/L_R^2}{f_{Rx}} + \frac{1}{f_{ox}} + \frac{1}{f_{ix}} + \frac{4\ell_i^2\, g_{R1} + 4\ell_i L_R\, g_{R2} + L_R^2\, g_{R3}}{(g_{R1} g_{R3} - g_{R2}{}^2) L_R^2} \quad (49)$$

To gain physical insight in this equation, the total stiffness of the system is calculated. The total stiffness (f_{tot}) is defined as the static force on the rotor in x-direction which gives a unit displacement of the rotor in x-direction, if the displacement in z-direction is zero.

Assuming a static force F on the rotor, the suspension forces and torques are:

$$F_{1x} = F_{2x} = F_{3x} = -F \qquad T_{1z} = -\ell_i F$$

The total displacement of the rotor and the total stiffness are

$$\xi_{tot} = \xi_{ox} + \xi_{ix} + \xi_{Rx} + \ell_i \varepsilon_{Rz} \qquad f_{tot} = \frac{F}{\xi_{tot}} \tag{50}$$

The equations (a) (d) (g) (l) (s) and (v) of table 2 result in:

$$\xi_{ox} = \frac{F}{f_{ox}} \qquad \xi_{ix} = \frac{F}{f_{ix}}$$

$$\xi_{Rx} = \frac{F}{f_{Rx}} + \frac{(L_R g_{R3} + 2\ell_i g_{R2})F}{(g_{R1} g_{R3} - g_{R2}^2)\, L_R}$$

$$\varepsilon_{Rz} = \frac{4\ell_i F}{f_{Rx} L_R^2} + \frac{(2L_R g_{R2} + 4\ell_i g_{R1})F}{(g_{R1} g_{R3} - g_{R2}^2) L_R^2}$$

Substitution in equation (50) gives:

$$\frac{1}{f_{tot}} = \frac{1}{f_{ox}} + \frac{1}{f_{ix}} + \frac{1 + 4\ell_i^2 / L_R^2}{f_{Rx}} + \frac{L_R^2 g_{R3} + 4L_R \ell_i g_{R2} + 4\ell_i^2 g_{R1}}{L_R^2 (g_{R1} g_{R3} - g_{R2}^2)} \tag{51}$$

Comparison with equation (49) shows that the stability criterion for symmetric suspension can be written as:

$$\frac{4}{\Omega^2 m_R} > \frac{1}{f_{tot}} \quad \text{or} \quad \Omega^2 m_R < 4\, f_{tot} \tag{52}$$

6.3 Stable region for unsymmetric suspension

For unsymmetric suspension the borderline of the stable region is also given in the equations (44) and (45). The only realistic solution is again that the term in square brackets in equation (45) is zero with non-zero values of A, B_x and B_z. With equations (41) to (43) the borderline of the stable region is given by:

$$\frac{1}{\mu} = \frac{1}{2}\left[\frac{1}{f_x} + \frac{1}{f_z} + 1 + \lambda^2 + \sqrt{\left(\frac{1}{f_x} - \frac{1}{f_z}\right)^2 + (1+\lambda^2)^2}\right] + \frac{\lambda^2 g^1_{R1} + 2\lambda g^1_{R2} + g^1_{R3}}{g^1_{R1} g^1_{R3} - g^{1\,2}_{R2}} \tag{53}$$

Introduction of the non-dimensional total stiffness in x- and z-directions:

$$f^1_{tot\ x} = \frac{f_{tot\ x}}{f_{Rx}} \qquad f^1_{tot\ z} = \frac{f_{tot\ z}}{f_{Rx}} \tag{54}$$

with $f_{tot\ x}$ given in equation (51) and $f_{tot\ z}$ the same expression for the z-direction, gives for the stability criterion:

$$\frac{1}{\mu} > \frac{1}{2}\left[\frac{1}{f^1_{tot\ x}} + \frac{1}{f^1_{tot\ z}} - (1+\lambda^2) + \sqrt{\left(\frac{1}{f^1_{tot\ x}} - \frac{1}{f^1_{tot\ z}}\right)^2 + (1+\lambda^2)^2}\right] \tag{55}$$

In dimensional form, denoting $f_{tot\ x}$ by $f_{tot\ max}$ if $f_{tot\ x} > f_{tot\ z}$ and by $f_{tot\ min}$ if $f_{tot\ x} < f_{tot\ z}$ and vice versa, this equation becomes:

$$\frac{8\, f_{tot\ min}}{m_R \Omega^2 a^2} > 1 + \frac{f_{tot\ min}}{f_{tot\ max}} - \left(1 + \frac{4\ell_i^2}{L_R^2}\right)\frac{f_{tot\ min}}{f_{Rx}} + \sqrt{\left(1 - \frac{f_{tot\ min}}{f_{tot\ max}}\right)^2 + \left(1 + \frac{4\ell_i^2}{L_R^2}\right)^2 \frac{f^2_{tot\ min}}{f^2_{Rx}}} \tag{56}$$

The value of a^2 can be obtained from equation (44); the term in square brackets must be equal to zero. With equations (41) to (43) and (53) the result is:

$$a^2 = 1 - \frac{\frac{4}{9}\left(\frac{1}{f_x} - \frac{1}{f_z}\right)^2}{\left[(1+\lambda^2) + \sqrt{\left(\frac{1}{f_x} - \frac{1}{f_z}\right)^2 + (1+\lambda^2)^2}\right]^2} \qquad (57)$$

In dimensional form this becomes:

$$a^2 = 1 - \frac{\frac{4}{9}\left(1 - \frac{f_{tot\ min}}{f_{tot\ max}}\right)^2}{\left[\left(1+\frac{4\ell_i^2}{L_R^2}\right)\frac{f_{tot\ min}}{f_{Rx}} + \sqrt{\left(1 - \frac{f_{tot\ min}}{f_{tot\ max}}\right)^2 + \left(1+\frac{4\ell_i^2}{L_R^2}\right)^2 \frac{f_{tot\ min}^2}{f_{Rx}^2}}\right]^2}$$

(58)

Equations (56) and (58) form the stability criterion for unsymmetric suspensions. This criterion is given in the form of a graph in figure 3. For high values of $\left(1+\frac{4\ell_i^2}{L_R^2}\right)\frac{f_{tot\ min}}{f_{Rx}}$ the gimbal suspensions are stiff compared to the rotor suspension, which has equal stiffness for the x- and z-directions. Therefore the ratio of $f_{tot\ min}$ and $f_{tot\ max}$ will be near unity for this case. The dashed curve gives the limiting value for infinite stiffness of the gimbal suspensions in one direction.

In most practical cases the difference between $f_{tot\ min}$ and $f_{tot\ max}$ will be small and the simple stability criterion

$$m_R \Omega^2 < 4\, f_{tot\ min} \qquad (59)$$

can be used.

6.4 Gimbal rotational instability

The mass and the moments of inertia of the gimbals are small compared to the mass and the moments of inertia of the rotor. Therefore it can be expected that a realistic approximation of the gimbal rotation instability can be obtained from a model in which the rotor is assumed to be fixed. The mass of the gimbals is not important for this rotational mode and will be neglected, as will be the small spring constants $T_{\delta i}$ and $T_{\delta o}$.

Therefore a solution of the equations of motion is derived for:

$$\mu \rightarrow \infty \qquad \mu_o \mu \rightarrow 0 \qquad \mu_i \mu \rightarrow 0 \qquad \mu\gamma \rightarrow \infty$$

$$\mu\gamma\alpha,\ \mu\gamma\beta \text{ and } \mu\lambda^2 \text{ finite} \quad \mu\lambda + \sigma\mu = 0 \qquad \tau_i = \tau_o = 0$$

Introduction of these approximations in the elements of equation (38) leads, after some recasting, to the following result:

$$d_{21} d_{65} \begin{vmatrix} f_x & 1+\frac{3}{2}ja & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ g^1_{R1}+f_x & -g^1_{R1} & -(\lambda g^1_{R1}+g^1_{R2}) & g^1_{R2} & 0 & 0 & 0 & 0 \\ \lambda f_x & 0 & D_{32} & 1+\frac{3}{2}ja & 0 & 0 & 0 & \frac{3}{2} \\ \lambda f_x - g^1_{R2} & +g^1_{R2} & \lambda g^1_{R2}+g^1_{R3}+D_{32} & -g^1_{R3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & f_z & 1+\frac{3}{2}ja & 0 & 0 \\ 0 & 0 & 0 & 0 & g^1_{R1}+f_z & -g^1_{R1} & -(\lambda g^1_{R1}+g^1_{R2}) & g^1_{R2} \\ 0 & 0 & 0 & -\frac{3}{2} & \lambda f_z & 0 & D_{76} & 1+\frac{3}{2}ja \\ 0 & 0 & 0 & 0 & \lambda f_z - g^1_{R2} & +g^1_{R2} & \lambda g^1_{R2}+g^1_{R3}+D_{76} & -g^1_{R3} \end{vmatrix} = 0 \quad (60)$$

Only for the symmetric case ($f_z = f_x$ and $D_{76} = D_{32}$) the solution of this equation was calculated. In this case a=1 and equation (60) can be written as:

$$d_{21} d_{65} \left[E^2 + \frac{9}{4} E^2_{12} + \frac{9}{4} E^2_{34} + \frac{9}{2} E_{14} E_{32} + \frac{81}{16} e^2_{12,34} \right] = 0 \quad (61)$$

in which: E = the determinant of the upper left 4x4 matrix in equation (60).

E_{ij} = the minor of the ij^{th} element of E

$e_{ij,kl}$ = the common minor of E_{ij} and E_{kl}

Denoting the corresponding determinant of the same matrix but omitting the imaginary part of the elements by the index R, equation (61) can be written as:

$$d_{21}d_{65}\left[E_R^2 - \frac{9}{2}(e_{12,34}E_R + E_{R12}E_{R34} - E_{14}E_{32}) - 3\ j\ E_R(E_{R12} + E_{R34})\right] = 0 \tag{62}$$

Now $e_{12,34}E_R - E_{R12}E_{R34} + E_{14}E_{32} = e_{12,34}f_x(D_{32}e_{11,33} - e_{11,34}) +$

$- e_{12,34}E_{R12} + e_{12,34}E_{R12} \quad - f_x e_{11,34}(-\lambda f_x e_{12,31} + D_{32}e_{12,33} - e_{12,34}) +$

$+ f_x e_{11,32}(-\lambda f_x e_{11,34} - D_{32}\ e_{13,34}) =$

$f_x D_{32}(e_{12,34}e_{11,33} - e_{12,33}e_{11,34} - e_{13,34}e_{11,32})$ or denoting the elements of E_R by $h_{i,j}$:

$$f_x D_{32}\left[(h_{21}h_{43} - h_{23}h_{41})(h_{22}h_{44} - h_{24}h_{42}) - (h_{21}h_{44} - h_{24}h_{41})(h_{22}h_{43} - h_{23}h_{42}) + \right.$$

$$\left. - (h_{21}h_{42} - h_{22}h_{41})(h_{23}h_{44} - h_{24}h_{43})\right] = 0$$

Therefore equation (62) can be written as:

$$d_{21}d_{65}E_R\left[E_R - 9\ e_{12,34} - 3\ j\ (E_{R12} + E_{R34})\right] = 0 \tag{63}$$

The real and the imaginary part of the term between square brackets cannot be equal to zero simultaneously (this would lead to contraditions) thus the boundary of the stable region is given by one of the equations:

$$d_{21} = 0 \quad d_{65} = 0 \quad \text{or } E_R = 0$$

$E_R = 0$ can be written as a condition for D_{32} or D_{76}, viz.:

$$D_{32} = - \frac{f_x g^1_{R3} + (f_x+1)(g^1_{R1}g^1_{R3} - g^{1\,2}_{R2}) + 2\lambda f_x g^1_{R2} + \lambda^2 f_x (g^1_{R1} + g^1_{R1}g^1_{R3} - g^{1\,2}_{R2})}{f_x(1+g^1_{R3}) + (f_x+1)(g^1_{R1} + g^1_{R1}g^1_{R3} - g^{1\,2}_{R2})} =$$

$$= - \frac{4\ f_{rot\ z}}{L_R^2\ f_{Rx}} \tag{64}$$

$$D_{76} = -\frac{f_z g^1_{R3} + (f_z+1)(g^1_{R1}g^1_{R3} - g^{1\,2}_{R2}) + 2\lambda f_z g^1_{R2} + \lambda^2 f_z (g^1_{R1} + g^1_{R1}g^1_{R3} - g^{1\,2}_{R2})}{f_z(1+g^1_{R3}) + (f_z+1)(g^1_{R1} + g^1_{R1}g^1_{R3} - g^{1\,2}_{R2})} =$$

$$= -\frac{4\, f_{rot\ x}}{L_R^2\, f_{Rx}} \qquad (65)$$

It can be shown, that the quantities $f_{rot\ z}$ and $f_{rot\ x}$, which are defined in equations (64) and (65), are equal to the total stiffness of the inner gimbal for rotations about the z- and x-axes respectively, for a fixed rotor position.

The stability conditions for rotations about the z-axis follow from

$$d_{21}\left(D_{32} + \frac{4\, f_{rot\ z}}{L_R^2\, f_{Rx}}\right) = 0 \quad \text{or}$$

$$\left[\Lambda_o^2 f^1_{oy} - \mu\gamma\,(\alpha_o - \alpha_i)\right]\left[\frac{4\, f_{rot\ z}}{L_R^2\, f_{Rx}} - \mu(\gamma\alpha_i - \gamma\alpha_R - \sigma\lambda)\right] = 0$$

Noting that with the used approximations $\mu(\gamma\alpha_i - \gamma\alpha_R - \sigma\lambda)$ x $L_R^2\, f_{Rx} = \Omega^2 K_i^1$, these stability conditions become in dimensional form:

$$\Omega^2 K_o^1 < L_o^2\, f_{oy} \qquad (66)$$

$$\Omega^2 K_i^1 < 4\, f_{rot\ z} \qquad (67)$$

For rotations about the x-axis, the boundary equation becomes:

$$d_{65}\left(D_{76} + \frac{4\, f_{rot\ x}}{L_R^2\, f_{Rx}}\right) = 0 \quad \text{or}$$

$$\left[\Lambda_i^2\, f^1_{ix} - \mu\gamma(\beta_o - \beta_i)\right]\left[\frac{4\, f_{rot\ x}}{L_R^2\, f_{Rx}} - \mu(\gamma\beta_i - \gamma\beta_R - \sigma\lambda) + \Lambda_i^2\, f^1_{ix} + \right.$$

$$\left. - \frac{\Lambda_i^4\, f^{1\,2}_{ix}}{\Lambda_i^2\, f^1_{ix} - \mu\gamma(\beta_o - \beta_i)}\right] = 0$$

or in dimensional form:

$$(L_i^2 f_{ix} - I_o^1 \Omega^2)(4 f_{rot\ x} + L_i^2 f_{ix} - I_i^1 \Omega^2) - L_i^4 f_{ix}^2 = 0$$

leading to the stability condition:

$$\Omega^2 < \frac{L_i^2 f_{ix}}{2 I_o^1} + \frac{L_i^2 f_{ix}}{2 I_i^1} + \frac{2 f_{rot\ x}}{I_i^1} - \sqrt{\left[\frac{L_i^2 f_{ix}}{2 I_o^1} + \frac{L_i^2 f_{ix}}{2 I_i^1} + \frac{2 f_{rot\ x}}{I_i^1}\right]^2 - \frac{4 f_{rot\ x} L_i^2 f_{ix}}{I_o^1 I_i^1}} \qquad (68)$$

The complete set of approximate stability conditions is given by the equations (59) and (66) to (68). Equations (66) to (68) can be important if the gimbals have high moments of inertia and/or a low rotational stiffness.

7 STABLE REGION FOR SOME TYPICAL CONFIGURATIONS

7.1 Choice of configurations

In chapter 6 an approximate stable region is determined, which is valid for all possible configurations. Due to the large number of parameters, a general check on the accuracy of the approximations is hardly possible. Therefore this check is only performed for some typical configurations.

The chosen configurations are sketched in figure 4. In configuration I the gimbals are around the rotor and the suspension centres coincide. In configuration II, the gimbals are placed inside the rotor; the suspension centres coincide. In configuration III, the gimbals are also placed inside the rotor; the suspension centre of the rotor does not coincide with the suspension centres of the gimbals.

For configurations II and III, two cases are considered, viz. the case in which the rotor centre of mass coincides with the suspension centre of the gimbals (configurations IIa and IIIa) and the case where some distance in y-direction between these two centres exists (configurations IIb and IIIb). For all configurations the centre of mass of rotor + inner gimbal coincides with the suspension centres of the gimbals.

Assumed is, that the suspension of inner and outer gimbal is identical i.e.

$$f_{iz} = f_{ox} \qquad f_{oy} = f_{ix}$$

With these assumptions also the following identities are obtained:

$$f_z = f_x \qquad f_{tot\ z} = f_{tot\ x} \qquad f_{rot\ z} = f_{rot\ x}$$

The non-dimensional parameters for the chosen configurations are given in table 6.

7.2 Stable region

The approximate stable region is given by the equations (59) and (66) to (68). For the chosen configurations, the conditions given by equations (66) and (67) are always fulfilled if the condition given by equation (68) is fulfilled. Therefore the approximate stable region is given by the equations (59) and (68); for the chosen configurations this region is given in figures 5 and 6.

From figure 5 it can be concluded, that for configuration I both stability criteria are important, but that the parameter f_{tot}/f_{ix} has hardly any influence on the stable region. For configurations II and III the stable region is given by equation (59). It can be noted that for configurations I and II the parameter $L_R^2\ f_{tot}/4\ f_{rot}$ will be smaller than 1, for configuration III this parameter will be smaller than 0.01 as can be inferred from the equations (51) and (64).

In figures 7 to 9 the approximate stable region is again given, using a different parameter viz. $f_{tot}/f_{Rx}f_x$. This parameter ranges from 0 to 1 for all configurations; it is zero for infinite stiffness of the gimbal suspensions and one for infinite stiffness of the rotor suspension. In figures 7 to 9 points are given for a number of numerical solutions of the exact equation (38); these solutions were obtained using a digital computer.

The differences between the exact and the approximate boundary of the stable region are not large - maximum about 20 percent. Equation (68) seems to be somewhat conservative (Fig. 7), equation (59) somewhat unconservative.

For configuration I (Fig. 7) all exact solutions are located nearly on one line; the differences due to differences in f_{tot}/f_{Rx}, f_{ix}/f_{iz} and τ are very small. The rather low value of $m_R\Omega^2/f_{tot}$ at $f_{tot}/f_{Rx}f_x = 1$ can be explained as follows: For $f_{tot}/f_{Rx}f_x = 1$ the stiffness of the rotor suspension is infinite, therefore the rotor and the inner gimbal must be regarded as one mass. As for configuration I the ratio $m_i^1/m_R = 0.2$ a reduction of about 20 percent in the parameter $m_R\,\Omega^2/f_{tot}$ can be expected.

Figure 8 shows that the differences between configurations IIa and IIb are very small and also that the parameter f_{ix}/f_{iz} has hardly any influence on the boundary of the stable region. The rather low value of $m_R\,\Omega^2/f_{tot}$ at $f_{tot}/f_{Rx}f_x = 1$ can be explained as before - in this case $m_i^1/m_R = 0.1$. For configuration II a remarkable difference between exact and approximate solutions exists for low values of $f_{tot}/f_{Rx}f_x$ and low values of f_{tot}/f_{Rx}. In this case the stiffness is largely determined by the stiffness of the rotor shaft, which is highly unsymmetric. Coupling between translations and rotations will occur, which leads to a reduction in the stability parameter.

Configuration III (Fig. 9) gives the same picture as configurations II except at low values of $f_{tot}/f_{Rx}f_x$. The reduction in the stability parameter is not present in this case, due to the relatively much higher rotational stiffness. However, this does not imply that configuration III is better than configuration II. For the same springs and bearings i,e. the same value of f_{ix}, f_{iz} and f_{Rx} the value of f_{tot} will be less for configuration III, due to the large value of $\lambda = 2\ell_i/L_R$ (see equation (51)).

8 CONCLUSIONS

For the stability of the high frequency motions of a double gimballed momentum wheel, the following conclusions can be drawn:

1 The motions along and about the rotor axis will always be stable as no mechanism exists to increase the energy of the motion.

2 For the motions along and about the other axes, there exists a mechanism to increase the energy of the motion in the rotor suspension if this suspension is such that the force in x-direction depends on the displacement in z-direction and vice versa ($f_{Rxz} \neq 0$).

3 A useful stability criterion is $\omega > f_{Rxz}/\eta_{Rx}$, that means that all natural frequencies of the system must be higher than the ratio of the transverse stiffness f_{Rxz} and the damping coefficient η_{Rx}. This stability criterion is exact for symmetric suspensions, but somewhat conservative for unsymmetric suspensions.

4 For ball bearings ($f_{Rxz} \sim 0$) no instability will occur; for hydrodynamic bearings ($f_{Rxz}/\eta_{Rx} \sim \Omega/2$) the stability criterion becomes $\omega > \Omega/2$, that means that all natural frequencies of the system must be higher than half the rotor frequency.

5 The stability criterion $\omega > \Omega/2$ leads to a criterion for the stiffness of the system viz.: $f_{tot} > m_R \Omega^2/4$ in which m_R is the rotor mass and f_{tot} the total static force on the rotor divided by the displacement of the rotor caused by this force. This criterion is somewhat unconservative; for some cases it will be better to use the total mass of rotor and inner gimbal instead of the mass of the rotor.

6 With some configurations rotational instability can occur, especially if the gimbals have high moments of inertia and/or a low rotational stiffness. The criteria for the rotational stiffness are given in the equations (66) (67) and (68) on pages 30 and 31.

9 REFERENCES

1 Boersma, G.
Sonnenschein, F.J.
Accuracy of attitude control systems for satellites using double gimballed momentum wheels.
NLR TR 73055 C (1973)

2 Reinhoudt, J.P.
On the stability of Rotor-and-bearing systems and on the calculation of sliding bearings
Ph. D. Thesis Techn. University Eindhoven (1972).

Table 1: Equations of motion.

(a) $$F_{1x} = -\ddot{\epsilon}_{oz}\left[s_R+m_R(\ell_i+\ell_o)\right] -\ddot{\delta}_i(s_R+m_R\ell_i) -\ddot{\epsilon}_{Rz}s_R +(a_x+\ddot{\xi}_{ox})m_R+\ddot{\xi}_{ix}m_R +\ddot{\xi}_{Rx}m_R$$

(b) $$F_{1y} = +(a_y+\ddot{\xi}_{oy})m_R+\ddot{\xi}_{iy}m_R +\ddot{\xi}_{Ry}m_R$$

(c) $$F_{1z} = +\ddot{\delta}_o\left[s_R+m_R(\ell_i+\ell_o)\right] +\ddot{\epsilon}_{ix}(s_R+m_R\ell_i)+\ddot{\epsilon}_{Rx}s_R +(a_z+\ddot{\xi}_{oz})m_R+\ddot{\xi}_{iz}m_R +\ddot{\xi}_{Rz}m_R$$

(d) $$F_{2x} = -\ddot{\epsilon}_{oz}(s_i+m_i\ell_o) -\ddot{\delta}_i s_i -\ddot{\epsilon}_{Rz}s_R +(a_x+\ddot{\xi}_{ox})m_i+\ddot{\xi}_{ix}m_i +\ddot{\xi}_{Rx}m_R$$

(e) $$F_{2y} = +(a_y+\ddot{\xi}_{oy})m_i+\ddot{\xi}_{iy}m_i +\ddot{\xi}_{Ry}m_R$$

(f) $$F_{2z} = +\ddot{\delta}_o(s_i+m_i\ell_o) +\ddot{\epsilon}_{ix}s_i +\ddot{\epsilon}_{Rx}s_R +(a_z+\ddot{\xi}_{oz})m_i+\ddot{\xi}_{iz}m_i +\ddot{\xi}_{Rz}m_R$$

(g) $$F_{3x} = -\ddot{\epsilon}_{oz}s_o -\ddot{\delta}_i s_i -\ddot{\epsilon}_{Rz}s_R +(a_x+\ddot{\xi}_{ox})m_o+\ddot{\xi}_{ix}m_i +\ddot{\xi}_{Rx}m_R$$

(h) $$F_{3y} = +(a_y+\ddot{\xi}_{oy})m_o+\ddot{\xi}_{iy}m_i +\ddot{\xi}_{Ry}m_R$$

(i) $$F_{3z} = +\ddot{\delta}_o s_o +\ddot{\epsilon}_{ix}s_i +\ddot{\epsilon}_{Rx}s_R +(a_z+\ddot{\xi}_{oz})m_o+\ddot{\xi}_{iz}m_i +\ddot{\xi}_{Rz}m_R$$

(j) $$T_{1x} = +h_R\left[(q\sin\delta_o-r\cos\delta_o-\dot{\delta}_i-\dot{\epsilon}_{oz}-\dot{\epsilon}_{Rz})\right] +\ddot{\delta}_o\left[I_R+s_R(\ell_i+\ell_o)\right] +\ddot{\epsilon}_{ix}(I_R+s_R\ell_i)+\ddot{\epsilon}_{Rx}I_R +(a_z+\ddot{\xi}_{oz})s_R+\ddot{\xi}_{iz}s_R +\ddot{\xi}_{Rz}s_R$$

(k) $$T_{1y} = \dot{h}_R +\ddot{\epsilon}_{oy}J_R +\ddot{\epsilon}_{iy}J_R$$

(l) $$T_{1z} = +h_R\left[(r\sin\delta_o+q\cos\delta_o)\sin\delta_i+(p+\dot{\delta}_o)\cos\delta_i+\dot{\epsilon}_{ix}+\dot{\epsilon}_{Rx}\right] +\ddot{\epsilon}_{oz}\left[K_R+s_R(\ell_i+\ell_o)\right] +\ddot{\delta}_i(K_R+s_R\ell_i)+\ddot{\epsilon}_{Rz}K_R -(a_x+\ddot{\xi}_{ox})s_R-\ddot{\xi}_{ix}s_R -\ddot{\xi}_{Rx}s_R$$

(m) $$T_{2x} = -\dot{h}_R\sin\delta_i +h_R\left[(q\sin\delta_o-r\cos\delta_o-\dot{\delta}_i)\cos\delta_i-\dot{\epsilon}_{oz}-\dot{\epsilon}_{Rz}\right] +\ddot{\delta}_o(I_i+s_i\ell_o) +\ddot{\epsilon}_{ix}I_i +\ddot{\epsilon}_{Rx}(I_R+s_R\ell_i) +(a_z+\ddot{\xi}_{oz})s_i+\ddot{\xi}_{iz}s_i +\ddot{\xi}_{Rz}(s_R+m_R\ell_i)$$

(n) $$T_{2y} = +\dot{h}_R\cos\delta_i +h_R\left[(q\sin\delta_o-r\cos\delta_o-\dot{\delta}_i)\right]\sin\delta_i +\ddot{\epsilon}_{oy}J_i +\ddot{\epsilon}_{iy}J_i$$

(o) $$T_{2z} = +h_R\left[(r\sin\delta_o+q\cos\delta_o)\sin\delta_i+(p+\dot{\delta}_o)\cos\delta_i+\dot{\epsilon}_{ix}+\dot{\epsilon}_{Rx}\right] +\ddot{\epsilon}_{oz}(K_i+s_i\ell_o) +\ddot{\delta}_i K_i +\ddot{\epsilon}_{Rz}(K_R+s_R\ell_i) -(a_x+\ddot{\xi}_{ox})s_i-\ddot{\xi}_{ix}s_i -\ddot{\xi}_{Rx}(s_R+m_R\ell_i)$$

(p) $$T_{3x} = -\dot{h}_R\sin\delta_i +h_R\left[(q\sin\delta_o-r\cos\delta_o-\dot{\delta}_i)\cos\delta_i-\dot{\epsilon}_{oz}-\dot{\epsilon}_{Rz}\right] +\ddot{\delta}_o I_o +\ddot{\epsilon}_{ix}(I_i+s_i\ell_o)+\ddot{\epsilon}_{Rx}\left[I_R+s_R(\ell_i+\ell_o)\right] +(a_z+\ddot{\xi}_{oz})s_o+\ddot{\xi}_{iz}(s_i+m_i\ell_o)+\ddot{\xi}_{Rz}\left[s_R+m_R(\ell_i+\ell_o)\right]$$

(q) $$T_{3y} = +\dot{h}_R\cos\delta_i\cos\delta_o-h_R\left[(r+\dot{\delta}_i\cos\delta_o)\sin\delta_i+(p+\dot{\delta}_o)\sin\delta_o\cos\delta_i\right] +\ddot{\epsilon}_{oy}J_o +\ddot{\epsilon}_{iy}J_i$$

(r) $$T_{3z} = +\dot{h}_R\cos\delta_i\sin\delta_o+h_R\left[(q-\dot{\delta}_i\sin\delta_o)\sin\delta_i+(p+\dot{\delta}_o)\cos\delta_o\cos\delta_i+\dot{\epsilon}_{ix}+\dot{\epsilon}_{Rx}\right] +\ddot{\epsilon}_{oz}K_o +\ddot{\delta}_i(K_i+s_i\ell_o)+\ddot{\epsilon}_{Rz}\left[K_R+s_R(\ell_i+\ell_o)\right] -(a_x+\ddot{\xi}_{ox})s_o-\ddot{\xi}_{ix}(s_i+m_i\ell_o)-\ddot{\xi}_{Rx}\left[s_R+m_R(\ell_i+\ell_o)\right]$$

(s) $$F_{dx} = F_{3x} + (m-m_o)a_x$$

(t) $$F_{dy} = F_{3y} + (m-m_o)a_y$$

(u) $$F_{dz} = F_{3z} + (m-m_o)a_z$$

(v) $$T_{dx} = T_{3x}+I_x\dot{p}+(I_z-I_y)qr-s_oa_z+b_y(F_{3z}-m_oa_z)-b_z(F_{3y}-m_oa_y)$$

(w) $$T_{dy} = T_{3y}+I_y\dot{q}+(I_x-I_z)rp \quad +b_z(F_{3x}-m_oa_x)-b_x(F_{3z}-m_oa_z)$$

(x) $$T_{dz} = T_{3z}+I_z\dot{r}+(I_y-I_x)pq+s_oa_x+b_x(F_{3y}-m_oa_y)-b_y(F_{3x}-m_oa_x)$$

Table 2: Equations of motion used for stability considerations.

(a)	$F_{1x} =$		$-\ddot{\varepsilon}_{oz}(s_R+m_R\ell_i)$	$-\ddot{\delta}_i(s_R+m_R\ell_i)$	$-\ddot{\varepsilon}_{Rz}s_R$	$+\ddot{\xi}_{ox}m_R+\ddot{\xi}_{ix}m_R+\ddot{\xi}_{Rx}m_R$	$=-f_{Rx}\xi_{Rxb}-\eta_{Rx}\dot{\xi}_{Rxb}-f_{Rxz}\xi_{Rzb}$
(b)	$F_{1y} =$					$+\ddot{\xi}_{oy}m_R+\ddot{\xi}_{iy}m_R+\ddot{\xi}_{Ry}m_R$	$=-f_{Ry}\xi_{Ry}-\eta_{Ry}\dot{\xi}_{Ry}$
(c)	$F_{1z} =$		$+\ddot{\delta}_o(s_R+m_R\ell_i)$	$+\ddot{\varepsilon}_{ix}(s_R+m_R\ell_i)$	$+\ddot{\varepsilon}_{Rx}s_R$	$+\ddot{\xi}_{oz}m_R+\ddot{\xi}_{iz}m_R+\ddot{\xi}_{Rz}m_R$	$=-f_{Rx}\xi_{Rzb}-\eta_{Rx}\dot{\xi}_{Rzb}+f_{Rxz}\xi_{Rxb}$
(d)	$F_{2x} =$				$-\ddot{\varepsilon}_{Rz}s_R$	$+\ddot{\xi}_{ox}m_i+\ddot{\xi}_{ix}m_i+\ddot{\xi}_{Rx}m_R$	$=-f_{ix}\xi_{ix}-\eta_{ix}\dot{\xi}_{ix}$
(e)	$F_{2y} =$					$+\ddot{\xi}_{oy}m_i+\ddot{\xi}_{iy}m_i+\ddot{\xi}_{Ry}m_R$	$=-f_{ix}\xi_{iy}-\eta_{ix}\dot{\xi}_{iy}$
(f)	$F_{2z} =$				$+\ddot{\varepsilon}_{Rx}s_R$	$+\ddot{\xi}_{oz}m_i+\ddot{\xi}_{iz}m_i+\ddot{\xi}_{Rz}m_R$	$=-f_{iz}\xi_{iz}-\eta_{iz}\dot{\xi}_{iz}$
(g)	$F_{3x} =$				$-\ddot{\varepsilon}_{Rz}s_R$	$+\ddot{\xi}_{ox}m_o+\ddot{\xi}_{ix}m_i+\ddot{\xi}_{Rx}m_R$	$=-f_{ox}\xi_{ox}-\eta_{ox}\dot{\xi}_{ox}$
(h)	$F_{3y} =$					$+\ddot{\xi}_{oy}m_o+\ddot{\xi}_{iy}m_i+\ddot{\xi}_{Ry}m_R$	$=-f_{oy}\xi_{oy}-\eta_{oy}\dot{\xi}_{oy}$
(i)	$F_{3z} =$				$+\ddot{\varepsilon}_{Rx}s_R$	$+\ddot{\xi}_{oz}m_o+\ddot{\xi}_{iz}m_i+\ddot{\xi}_{Rz}m_R$	$=-f_{oy}\xi_{oz}-\eta_{oy}\dot{\xi}_{oz}$
(j)	$T_{1x} =$	$-h_{Rn}(\dot{\delta}_i+\dot{\varepsilon}_{oz}+\dot{\varepsilon}_{Rz})$	$+\ddot{\delta}_o(I_R+s_R\ell_i)$	$+\ddot{\varepsilon}_{ix}(I_R+s_R\ell_i)$	$+\ddot{\varepsilon}_{Rx}I_R$	$+\ddot{\xi}_{oz}s_R+\ddot{\xi}_{iz}s_R+\ddot{\xi}_{Rz}s_R$	$=-\frac{1}{4}L_R^2\left[f_{Rx}\varepsilon_{Rxb}+\eta_{Rx}\dot{\varepsilon}_{Rxb}+f_{Rxz}\varepsilon_{Rzb}\right]$
(k)	$T_{1y} = \dot{h}_R$		$+\ddot{\varepsilon}_{oy}J_R$	$+\ddot{\varepsilon}_{iy}J_R$			$= -T_{\eta R}(h_R-h_{Rn})$
(l)	$T_{1z} =$	$+h_{Rn}(\dot{\delta}_o+\dot{\varepsilon}_{ix}+\dot{\varepsilon}_{Rx})$	$+\ddot{\varepsilon}_{oz}(K_R+s_R\ell_i)$	$+\ddot{\delta}_i(K_R+s_R\ell_i)$	$+\ddot{\varepsilon}_{Rz}K_R$	$-\ddot{\xi}_{ox}s_R-\ddot{\xi}_{ix}s_R-\ddot{\xi}_{Rx}s_R$	$=-\frac{1}{4}L_R^2\left[f_{Rx}\varepsilon_{Rzb}+\eta_{Rx}\dot{\varepsilon}_{Rzb}-f_{Rxz}\varepsilon_{Rxb}\right]$
(m)	$T_{2x} =$	$-h_{Rn}(\dot{\delta}_i+\dot{\varepsilon}_{oz}+\dot{\varepsilon}_{Rz})$	$+\ddot{\delta}_oI_i$	$+\ddot{\varepsilon}_{ix}I_i$	$+\ddot{\varepsilon}_{Rx}(I_R+s_R\ell_i)$	$+\ddot{\xi}_{Rz}(s_R+m_R\ell_i)$	$=-\frac{1}{4}L_i^2\left[f_{ix}\varepsilon_{ix}+\eta_{ix}\dot{\varepsilon}_{ix}\right]$
(n)	$T_{2y} = \dot{h}_R$		$+\ddot{\varepsilon}_{oy}J_i$	$+\ddot{\varepsilon}_{iy}J_i$			$=-\frac{1}{4}L_i^2\left[f_{ix}\varepsilon_{iy}+\eta_{ix}\dot{\varepsilon}_{iy}\right]$
(o)	$T_{2z} =$	$+h_{Rn}(\dot{\delta}_o+\dot{\varepsilon}_{ix}+\dot{\varepsilon}_{Rx})$	$+\ddot{\varepsilon}_{oz}K_i$	$+\ddot{\delta}_iK_i$	$+\ddot{\varepsilon}_{Rz}(K_R+s_R\ell_i)$	$-\ddot{\xi}_{Rx}(s_R+m_R\ell_i)$	$=-T_{fi}\ \mathrm{sign}\ \dot{\delta}_i-T_{\delta i}\delta_i$
(p)	$T_{3x} =$	$-h_{Rn}(\dot{\delta}_i+\dot{\varepsilon}_{oz}+\dot{\varepsilon}_{Rz})$	$+\ddot{\delta}_oI_o$	$+\ddot{\varepsilon}_{ix}I_i$	$+\ddot{\varepsilon}_{Rx}(I_R+s_R\ell_i)$	$+\ddot{\xi}_{Rz}(s_R+m_R\ell_i)$	$=-T_{fo}\ \mathrm{sign}\ \dot{\delta}_o-T_{\delta o}\delta_o$
(q)	$T_{3y} = \dot{h}_R$		$+\ddot{\varepsilon}_{oy}J_o$	$+\ddot{\varepsilon}_{iy}J_i$			$=-\frac{1}{4}L_o^2\left[f_{oy}\varepsilon_{oy}+\eta_{oy}\dot{\varepsilon}_{oy}\right]$
(r)	$T_{3z} =$	$+h_{Rn}(\dot{\delta}_o+\dot{\varepsilon}_{ix}+\dot{\varepsilon}_{Rx})$	$+\ddot{\varepsilon}_{oz}K_o$	$+\ddot{\delta}_iK_i$	$+\ddot{\varepsilon}_{Rz}(K_R+s_R\ell_i)$	$-\ddot{\xi}_{Rx}(s_R+m_R\ell_i)$	$=-\frac{1}{4}L_o^2\left[f_{oy}\varepsilon_{oz}+\eta_{oy}\dot{\varepsilon}_{oz}\right]$
(s)					$-\varepsilon_{Rz}\frac{L_R}{2}g_{R2}$	$+\xi_{Rx}g_{R1}$	$=(f_{Rx}+g_{R1})\xi_{Rxb}+\eta_{Rx}\dot{\xi}_{Rxb}+f_{Rxz}\xi_{Rzb}-g_{R2}\frac{L_R}{2}\varepsilon_{Rzb}$
(t)					$+\varepsilon_{Rx}\frac{L_R}{2}g_{R2}$	$+\xi_{Rz}g_{R1}$	$=(f_{Rx}+g_{R1})\xi_{Rzb}+\eta_{Rx}\dot{\xi}_{Rzb}-f_{Rxz}\xi_{Rxb}+g_{R2}\frac{L_R}{2}\varepsilon_{Rxb}$
(u)					$+\varepsilon_{Rx}\frac{L_R}{2}g_{R3}$	$+\xi_{Rz}g_{R2}$	$=+\frac{1}{2}L_R\left[(f_{Rx}+g_{R3})\varepsilon_{Rxb}+\eta_{Rx}\dot{\varepsilon}_{Rxb}+f_{Rxz}\varepsilon_{Rzb}\right]+g_{R2}\xi_{Rzb}$
(v)					$+\varepsilon_{Rz}\frac{L_R}{2}g_{R3}$	$-\xi_{Rx}g_{R2}$	$=+\frac{1}{2}L_R\left[(f_{Rx}+g_{R3})\varepsilon_{Rzb}+\eta_{Rx}\dot{\varepsilon}_{Rzb}-f_{Rxz}\varepsilon_{Rxb}\right]-g_{R2}\xi_{Rxb}$

Table 3: Laplace transformed equations of motion in matrix form

$s^2m_o+f_{ox}$	s^2m_i	s^2m_R	0	0	0	0	0	0	0	$-s^2s_R$	0	0	0	0	0	ξ_{ox}
s^2m_i	$s^2m_i+f_{ix}$	s^2m_R	0	0	0	0	0	0	0	$-s^2s_R$	0	0	0	0	0	ξ_{ix}
s^2m_R	s^2m_R	s^2m_R	$s\eta_{Rx}+f_{Rx}$	0	0	0	f_{Rxz}	$-s^2(s_R+m_R\ell_i)$	$-s^2(s_R+m_R\ell_i)$	$-s^2s_R$	0	0	0	0	0	ξ_{Rx}
0	0	g_{R1}	$-s\eta_{Rx}-f_{Rx}-g_{R1}$	0	0	0	$-f_{Rxz}$	0	0	$-\frac{L_R}{2}g_{R2}$	$\frac{L_R}{2}g_{R2}$	0	0	0	0	ξ_{Rxb}
0	0	0	0	$s^2m_o+f_{oy}$	s^2m_i	s^2m_R	0	0	0	0	0	0	0	s^2s_R	0	ξ_{oz}
0	0	0	0	s^2m_i	$s^2m_i+f_{iz}$	s^2m_R	0	0	0	0	0	0	0	s^2s_R	0	ξ_{iz}
0	0	0	$-f_{Rxz}$	s^2m_R	s^2m_R	s^2m_R	$s\eta_{Rx}+f_{Rx}$	0	0	0	0	$s^2(s_R+m_R\ell_i)$	$s^2(s_R+m_R\ell_i)$	s^2s_R	0	ξ_{Rz}
0	0	0	$+f_{Rxz}$	0	0	g_{R1}	$-s\eta_{Rx}-f_{Rx}-g_{R1}$	0	0	0	0	0	0	$\frac{L_R}{2}g_{R2}$	$-\frac{L_R}{2}g_{R2}$	ξ_{Rzb}
0	0	$-s^2(s_R+m_R\ell_i)$	0	0	0	0	0	$s^2K_o+\frac{L_o^2}{4}f_{oy}$	s^2K_i	$s^2(K_R+s_R\ell_i)$	0	sh_{Rn}	sh_{Rn}	sh_{Rn}	0	ϵ_{oz}
0	0	$-s^2(s_R+m_R\ell_i)$	0	0	0	0	0	s^2K_i	$s^2K_i+T_{\delta i}$	$s^2(K_R+s_R\ell_i)$	0	sh_{Rn}	sh_{Rn}	sh_{Rn}	0	δ_i
$-s^2s_R$	$-s^2s_R$	$-s^2s_R$	0	0	0	0	0	$s^2(K_R+s_R\ell_i)$	$s^2(K_R+s_R\ell_i)$	s^2K_R	$(s\eta_{Rx}+f_{Rx})\frac{L_R^2}{4}$	sh_{Rn}	sh_{Rn}	sh_{Rn}	$-f_{Rxz}\frac{L_R^2}{4}$	ϵ_{Rz}
0	0	$-g_{R2}\frac{L_R}{2}$	$g_{R2}\frac{L_R}{2}$	0	0	0	0	0	0	$g_{R3}\frac{L_R^2}{4}$	$-(s\eta_{Rx}+f_{Rx}+g_{R3})\frac{L_R^2}{4}$	0	0	0	$f_{Rxz}\frac{L_R^2}{4}$	ϵ_{Rzb}
0	0	0	0	0	0	$s^2(s_R+m_R\ell_i)$	0	$-sh_{Rn}$	$-sh_{Rn}$	$-sh_{Rn}$	0	$s^2I_o+T_{\delta o}$	s^2I_i	$s^2(I_R+s_R\ell_i)$	0	δ_o
0	0	0	0	0	0	$s^2(s_R+m_R\ell_i)$	0	$-sh_{Rn}$	$-sh_{Rn}$	$-sh_{Rn}$	0	s^2I_i	$s^2I_i+\frac{L_i^2}{4}f_{ix}$	$s^2(I_R+s_R\ell_i)$	0	ϵ_{ix}
0	0	0	0	s^2s_R	s^2s_R	s^2s_R	0	$-sh_{Rn}$	$-sh_{Rn}$	$-sh_{Rn}$	$f_{Rxz}\frac{L_R^2}{4}$	$s^2(I_R+s_R\ell_i)$	$s^2(I_R+s_R\ell_i)$	s^2I_R	$(s\eta_{Rx}+f_{Rx})\frac{L_R^2}{4}$	ϵ_{Rx}
0	0	0	0	0	0	$g_{R2}\frac{L_R}{2}$	$-g_{R2}\frac{L_R}{2}$	0	0	0	$-f_{Rxz}\frac{L_R^2}{4}$	0	0	$g_{R3}\frac{L_R^2}{4}$	$-(s\eta_{Rx}+f_{Rx}+g_{R3})\frac{L_R^2}{4}$	ϵ_{Rxb}

Table 4: The elements of the determinant of chapter 5

element	value	non-dimensional value with $s = ja\,\Omega/2$ (see table 5)
a_{11}	$s^2 m_o^1 + f_{ox}$	$f_{ox}^{\ 1} - \mu_o \mu$
$a_{12}=a_{56}$	$s^2 m_i$	$-(1+\mu_i)\mu$
$a_{13}=a_{57}$	$s^2 m_R$	$-\mu$
a_{21}	$s^2 m_o^1 + f_{ox} + f_{ix}$	$f_{ox}^{\ 1} + f_{ix}^{\ 1} - \mu_o \mu$
$a_{22}=a_{31}$	$-f_{ix}$	$-f_{ix}^{\ 1}$
a_{32}	$s^2 m_i^1 + f_{ix}$	$f_{ix}^{\ 1} - \mu_i \mu$
$a_{34}=a_{78}$	$-(s\eta_{Rx} + f_{Rx})$	$-(1+\frac{3}{2}ja)$
$a_{38}=a_{48}=-a_{74}=-a_{84}$	$-f_{Rxz}$	$-\frac{3}{2}$
$a_{43}=a_{87}$	g_{R1}	g_{R1}^1
$a_{44}=a_{88}$	$-(s\eta_{Rx} + f_{Rx} + g_{R1})$	$-(1+\frac{3}{2}ja + g_{R1}^1)$
a_{55}	$s^2 m_o^1 + f_{oy}$	$f_{oy}^{\ 1} - \mu_o \mu$
a_{65}	$s^2 m_o^1 + f_{oy} + f_{iz}$	$f_{oy}^{\ 1} + f_{iz}^{\ 1} - \mu_o \mu$
$a_{66}=a_{75}$	$-f_{iz}$	$-f_{iz}^{\ 1}$
a_{76}	$s^2 m_i^1 + f_{iz}$	$f_{iz}^{\ 1} - \mu_i \mu$
$b_{32}=-b_{76}=-c_{13}=c_{57}$	$s^2(s_R + m_R \ell i)$	$-\mu(\sigma+\lambda)$
$b_{43}=-b_{87}$	$\frac{s_R}{m_R} g_{R1} - \frac{L_R}{2} g_{R2}$	$\sigma g_{R1}^1 - g_{R2}^1$
$b_{44}=-b_{88}=-c_{43}=c_{44}=c_{87}=-c_{88}$	$\frac{L_R}{2} g_{R2}$	g_{R2}^1
$c_{32}=-c_{76}$	$s^2 s_R$	$-\sigma\mu$
$c_{33}=-c_{77}$	$-s^2 m_R \ell_i$	$\lambda\mu$
d_{11}	$s^2(K_o - K_i) + \frac{L_o^2}{4} f_{oy}$	$-\mu\gamma(\alpha_o - \alpha_i) + \Lambda_o^2 f_{oy}^{\ 1}$
d_{12}	$s^2 K_i$	$-\mu\gamma\alpha_i$
d_{13}	$s^2(K_R - \frac{s_R^2}{m_R})$	$-\mu(\gamma\alpha_R - \sigma^2)$
$d_{16}=d_{17}=-d_{52}=-d_{53}$	$s h_{Rn}$	$2j\mu\gamma/a$
d_{21}	$s^2(K_o - K_i) + \frac{L_o^2}{4} f_{oy} + T_{\delta i}$	$-\mu\gamma(\alpha_o - \alpha_i) + \Lambda_o^2 f_{oy}^{\ 1} + \tau_i$
$d_{22}=d_{31}$	$-T_{\delta i}$	$-\tau_i$
d_{32}	$s^2(K_i - K_R - s_R \ell_i) + T_{\delta i}$	$-\mu(\gamma\alpha_i - \gamma\alpha_R - \sigma\lambda) + \tau_i$
$d_{34}=d_{78}$	$-(s\eta_{Rx} + f_{Rx})\frac{L_R^2}{4}$	$-(1+\frac{3}{2}ja)$
$d_{38}=d_{48}=-d_{74}=-d_{84}$	$\frac{L_R^2}{4} f_{Rxz}$	$+\frac{3}{2}$
$d_{43}=d_{87}$	$-\frac{s_R}{m_R}\frac{L_R}{2} g_{R2} + \frac{L_R^2}{4} g_{R3}$	$-\sigma g_{R2}^1 + g_{R3}^1$
$d_{44}=d_{88}$	$-(s\eta_{Rx} + f_{Rx} + g_{R3})\frac{L_R^2}{4}$	$-(1+\frac{3}{2}ja + g_{R3}^1)$
d_{55}	$s^2(I_o - I_i) + T_{\delta o}$	$-\mu\gamma(\beta_o - \beta_i) + \tau_o$
d_{56}	$s^2 I_i$	$-\mu\gamma\beta_i$
d_{57}	$s^2(I_R - \frac{s_R^2}{m_R})$	$-\mu(\gamma\beta_R - \sigma^2)$
d_{65}	$s^2(I_o - I_i) + T_{\delta o} + \frac{L_i^2}{4} f_{ix}$	$-\mu\gamma(\beta_o - \beta_i) + \tau_o + \Lambda_i^2 f_{ix}^{\ 1}$
$d_{66}=d_{75}$	$-\frac{L_i^2}{4} f_{ix}$	$-\Lambda_i^2 f_{ix}^{\ 1}$
d_{76}	$s^2(I_i - I_R - s_R \ell_i) + \frac{L_i^2}{4} f_{ix}$	$-\mu(\gamma\beta_i - \gamma\beta_R - \sigma\lambda) + \Lambda_i^2 f_{ix}^{\ 1}$

Table 5: Non dimensional parameters

$f^1_{ij} = f_{ij}/f_{Rx}$

$g^1_{Rn} = g_{Rn}/f_{Rx}$

$\lambda = 2\ell_i/L_R$

$\Lambda_j = L_j/L_R$

$\tau_j = 4\ T_{\delta j}/L_R^2\ f_{Rx}$

$\mu = m_R a^2\ \Omega^2/4\ f_{Rx}$

$\mu_j = m^1_j/m_R$

$\sigma = 2\ s_R/m_R L_R$

$\gamma = 4\ J_R/L_R^2 m_R$

$\alpha_j = K_j/J_R$

$\beta_j = I_j/J_R$

Table 6: Non-dimensional parameters for the chosen configurations

Configuration	I	IIa	IIb	IIIa	IIIb
g_{R2}/g_{R1}	0	-1.75	-1.75	+1.75	+1.75
g_{R3}/g_{R1}	1	3.5	3.5	3.5	3.5
γ	10	50	50	50	50
λ	0	0	0	5	5
σ	0	0	-0.1	-5	-5.1
$\mu_i=\mu_o$	0.2	0.1	0.1	0.1	0.1
$\Lambda_i=\Lambda_o$	4	2	2	2	2
$\alpha_R=\beta_R$	0.51	0.52	0.52	1.02	1.04
α_i	0.71	0.53	0.53	0.53	0.53
β_i	0.53	0.53	0.53	0.53	0.53
α_o	0.91	0.54	0.54	0.54	0.54
β_o	0.73	0.54	0.54	0.54	0.54

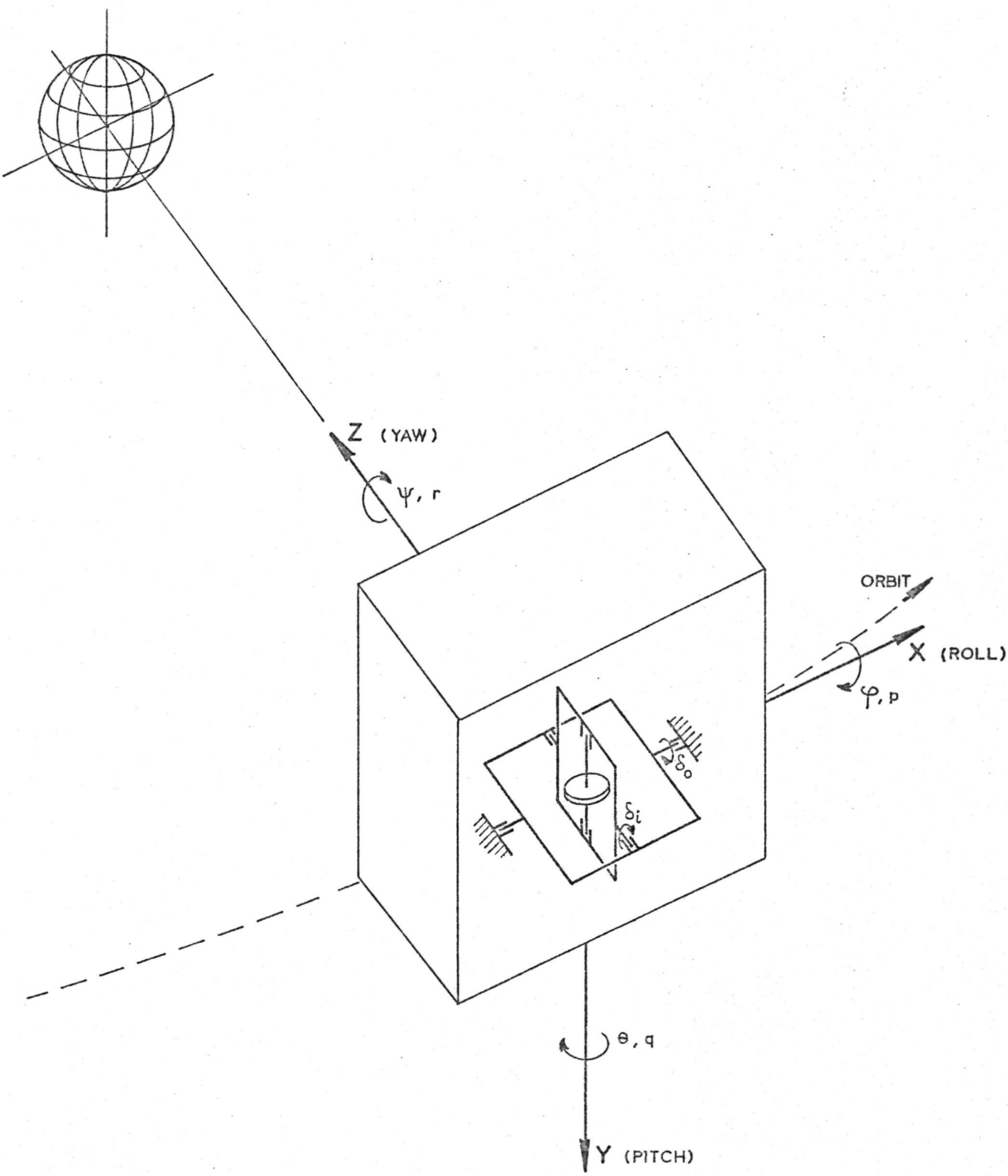


Fig.1 SATELLITE AND GIMBAL ARRANGEMENT.

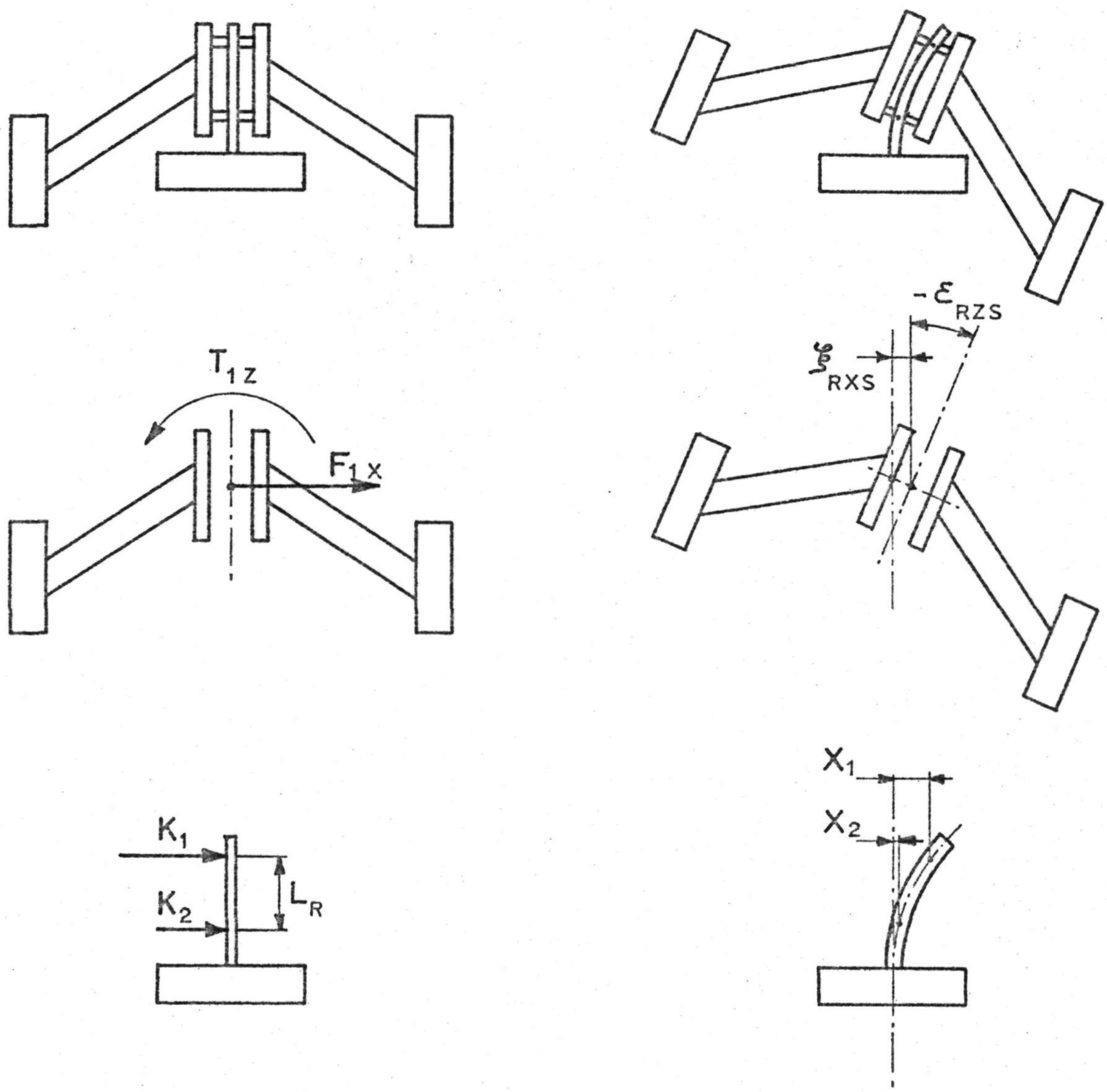


Fig. 2 STRUCTURAL DISPLACEMENTS DUE TO ROTOR SHAFT BENDING.

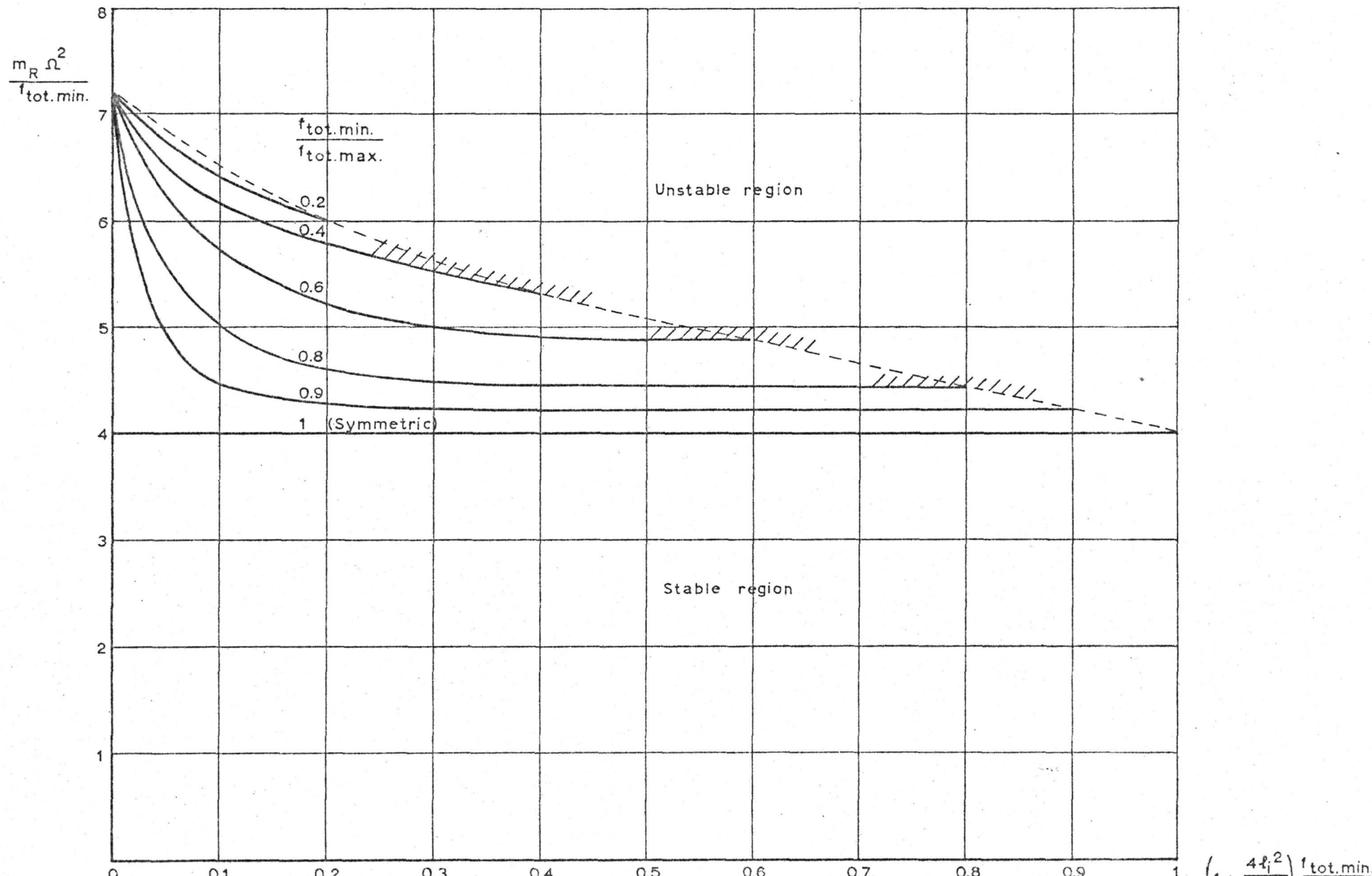
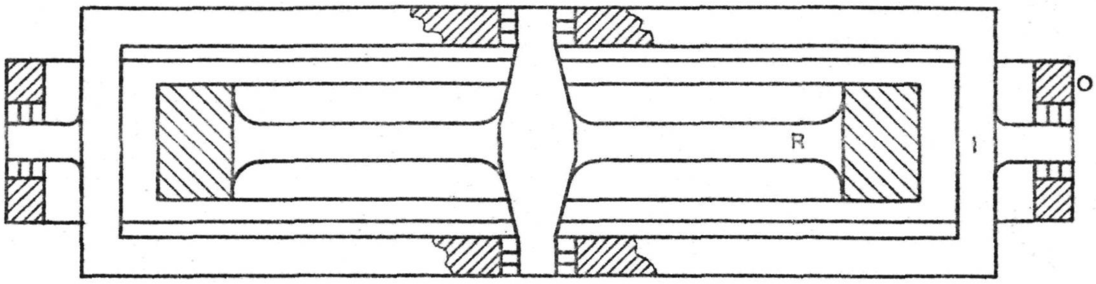
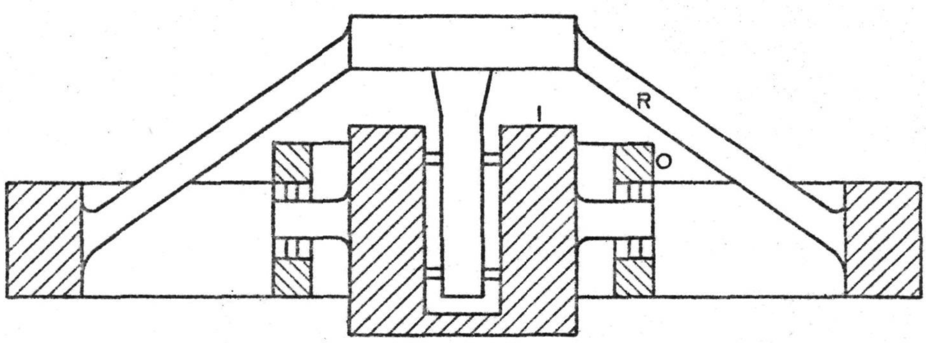


Fig 3 APPROXIMATE STABLE REGION - EFFECT OF UNSYMMETRY

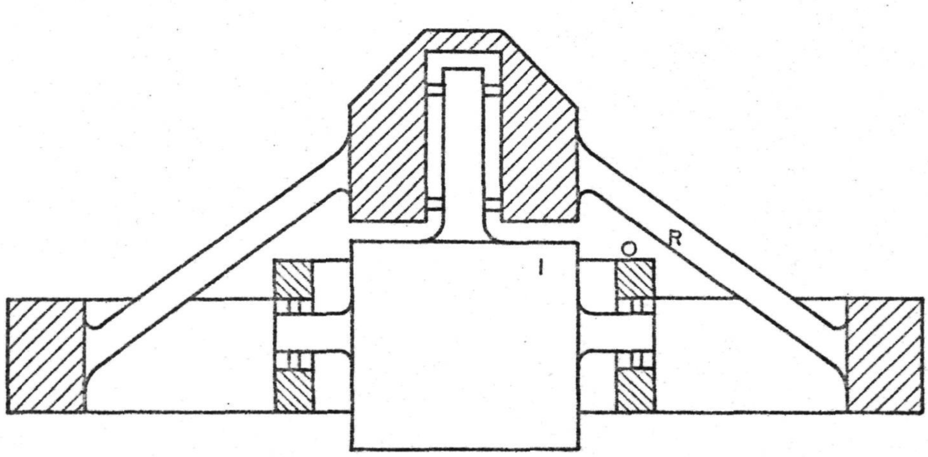
Figure 3



CONFIGURATION I



CONFIGURATION II



CONFIGURATION III

Fig. 4 SOME TYPICAL CONFIGURATIONS

R = ROTOR
I = INNER GIMBAL
O = OUTER GIMBAL

(Outer gimbal axis perpendicular to plane of paper)

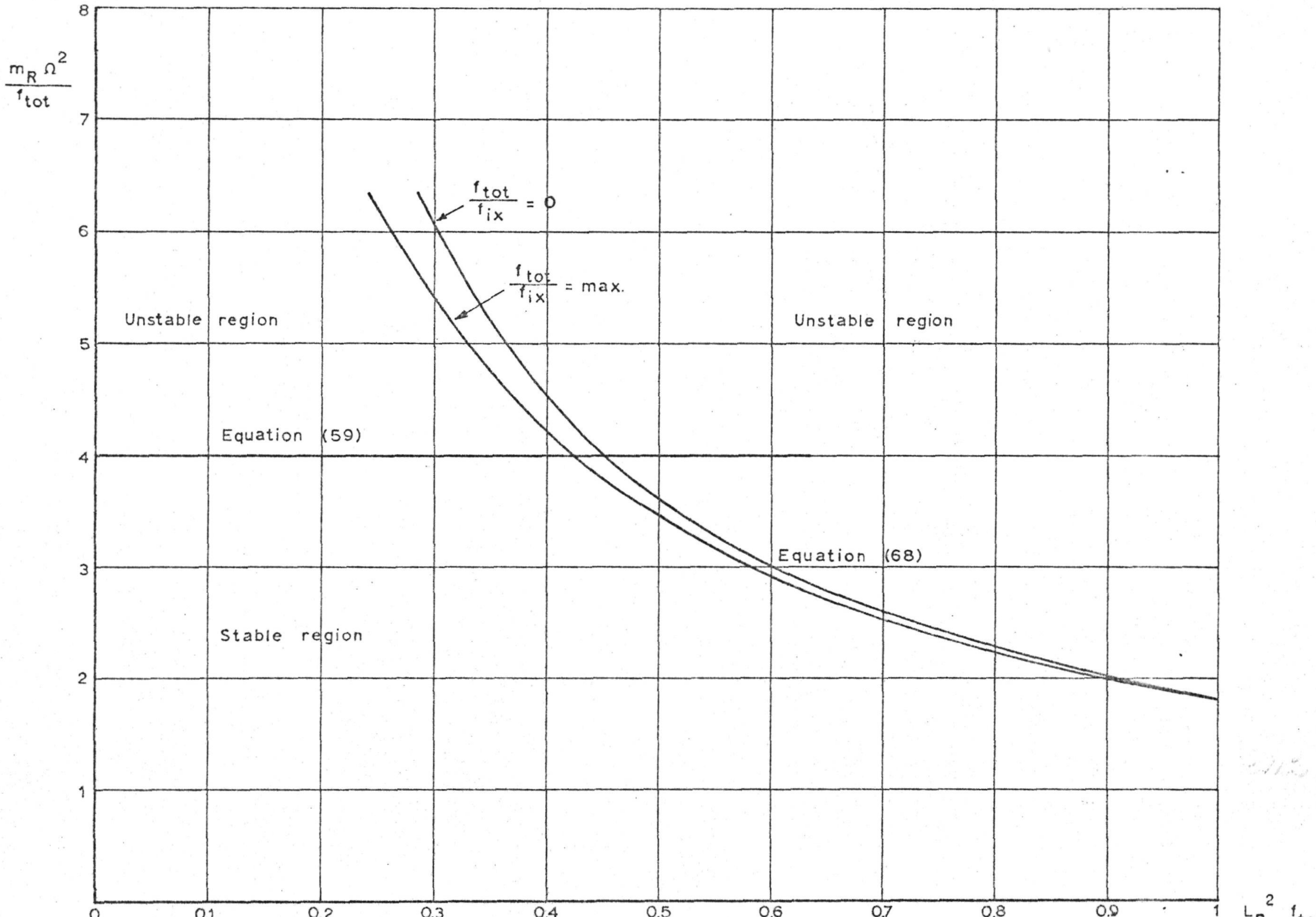


Fig. 5 APPROXIMATE STABLE REGION FOR CONFIGURATION I

Figure 5

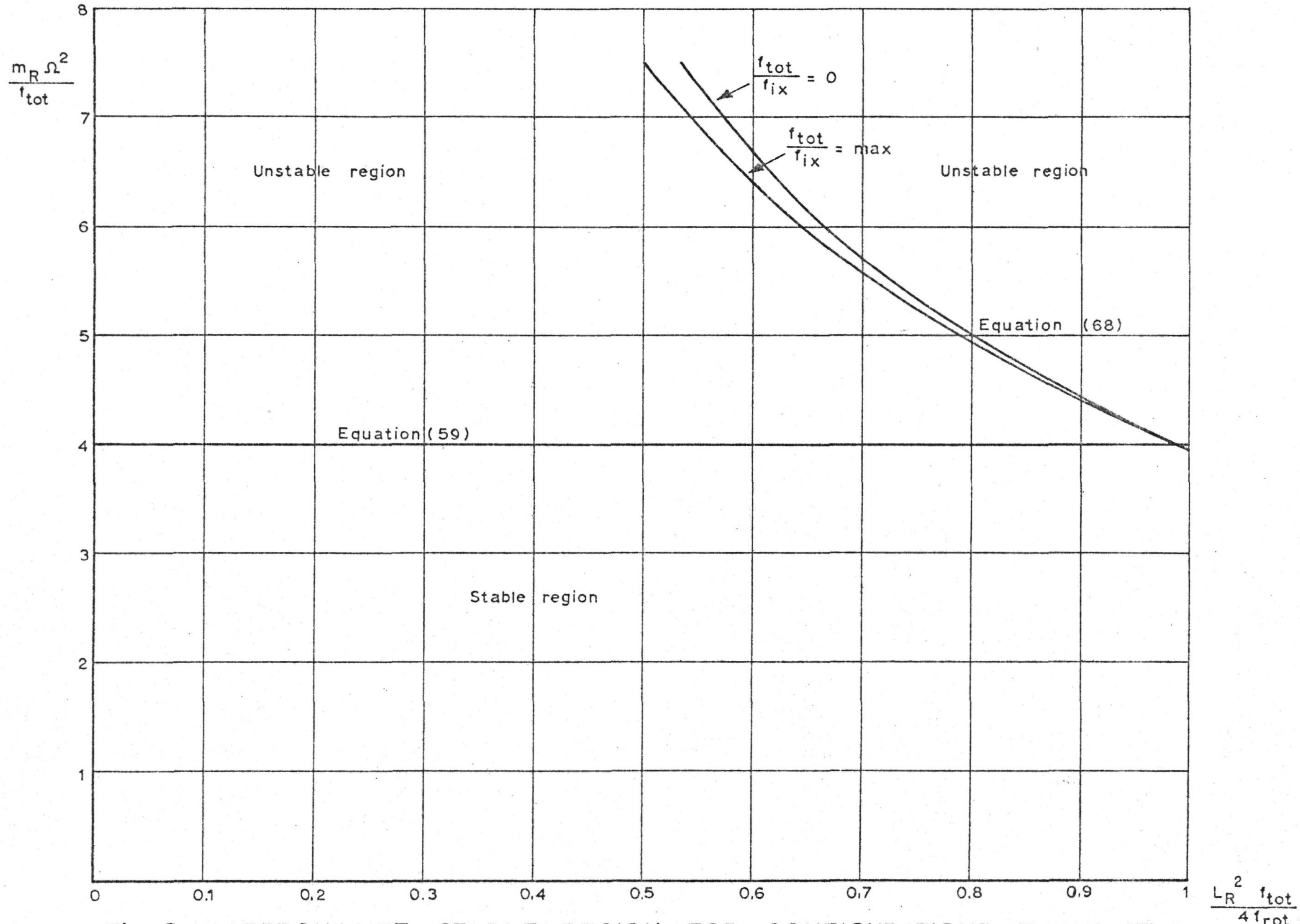


Fig. 6 APPROXIMATE STABLE REGION FOR CONFIGURATIONS II AND III

Figure 6

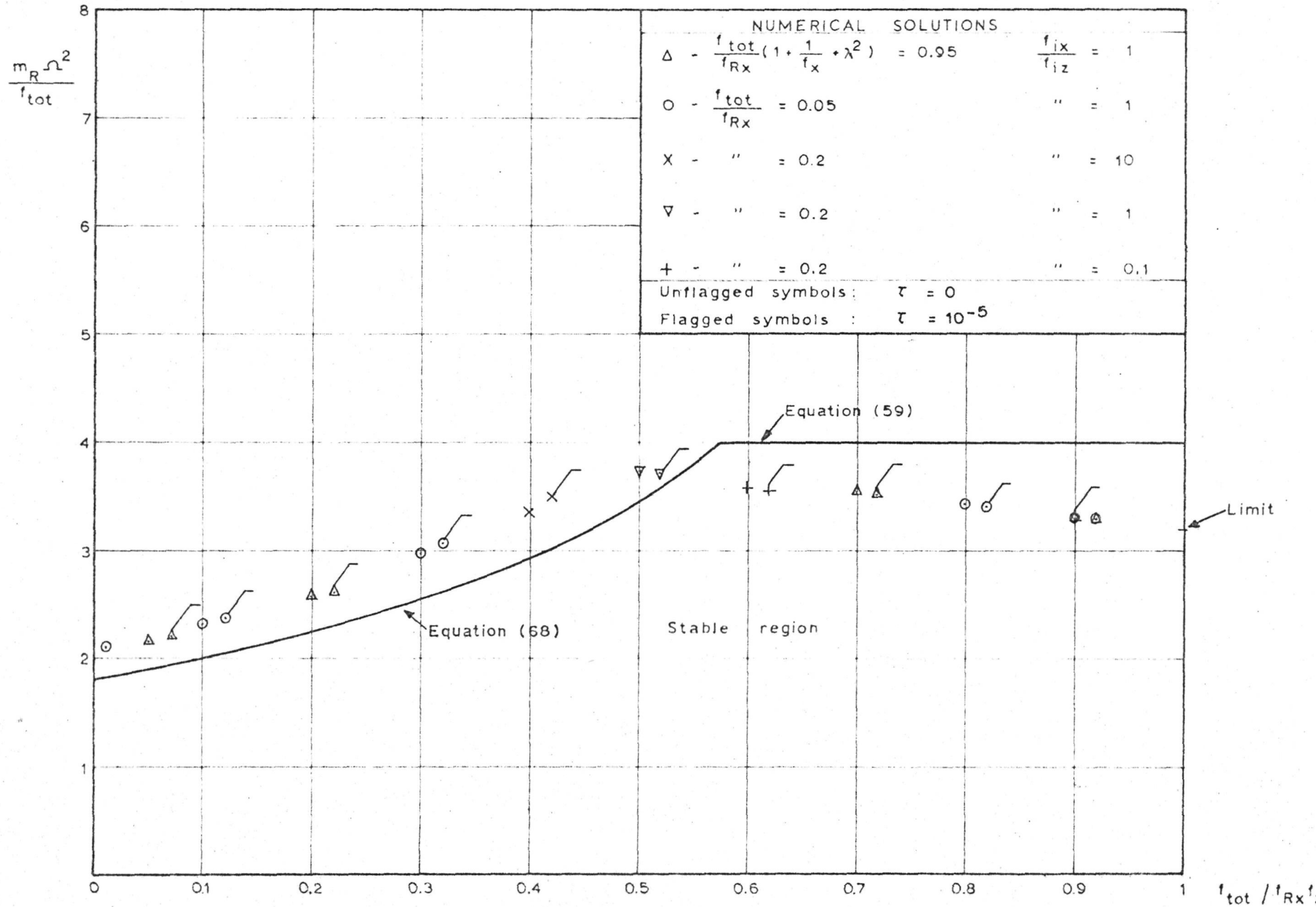


Fig.7 STABLE REGION FOR CONFIGURATION I, COMPARISON BETWEEN NUMERICAL AND APPROXIMATE SOLUTIONS.

Figure 7

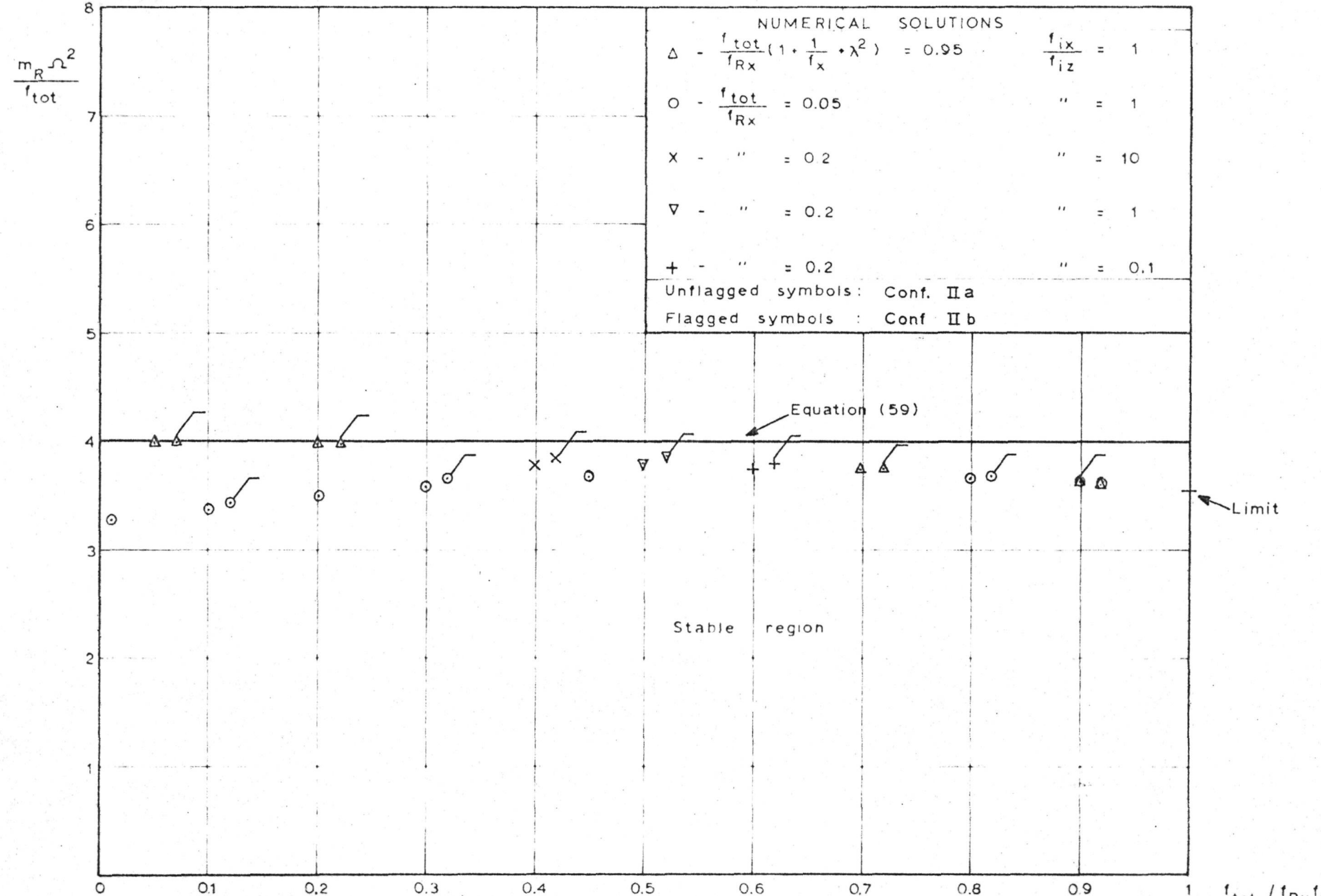


Fig. 8 STABLE REGION FOR CONFIGURATION II, COMPARISON BETWEEN NUMERICAL AND APPROXIMATE SOLUTIONS.

Figure 8

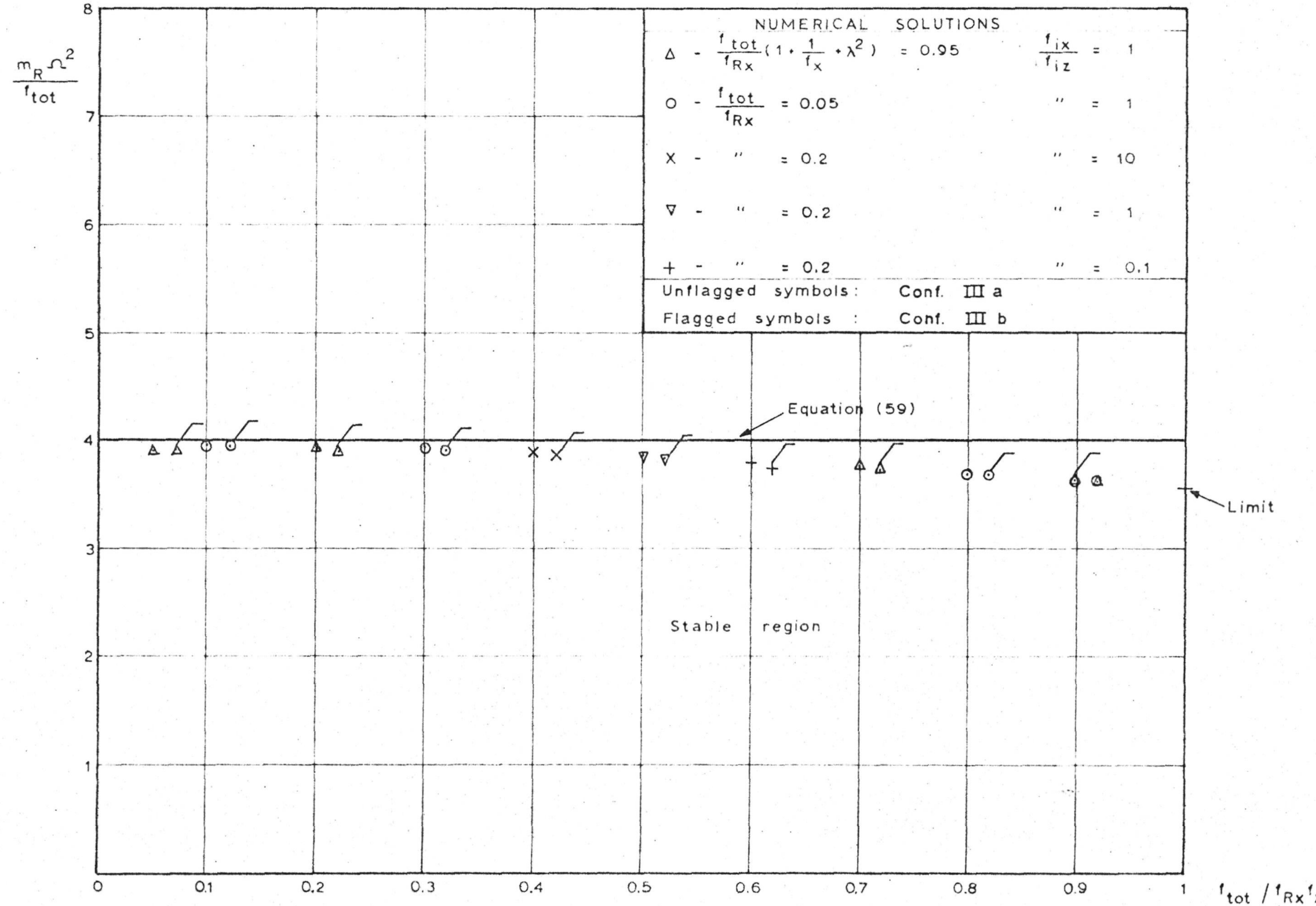


Fig. 9 STABLE REGION FOR CONFIGURATION III, COMPARISON BETWEEN NUMERICAL AND APPROXIMATE SOLUTIONS.

Figure 9