NATIONAL RESEARCH COUNCIL OF CANADA

AERONAUTICAL REPORT

LR - 260

APPROXIMATE METHOD FOR DETERMINING THE POTENTIAL FLOW ABOUT AN ARBITRARY AEROFOIL SECTION IN A TWO - DIMENSIONAL FINITE STREAM WITH PARTICULAR REFERENCE TO LARGE STREAM DEFLECTIONS

BY

D. G. GOULD

NATIONAL AERONAUTICAL ESTABLISHMENT

OTTAWA

AUGUST 1959

NATIONAL RESEARCH LABORATORIES

Ottawa, Canada

REPORT

National Aeronautical Establishment

Flight Research

Pages - Preface	- 5	Report:	LR-260
Text	- 11	Date:	August 1959
App.	- 17	Lab. Order:	NAE 671
Figures	- 8	File:	M49-7-27

For: Internal

Subject: APPROXIMATE METHOD FOR DETERMINING THE POTENTIAL FLOW ABOUT AN ARBITRARY AEROFOIL SECTION IN A TWO-DIMENSIONAL FINITE STREAM WITH PARTICULAR REFERENCE TO LARGE STREAM DEFLECTIONS

Prepared by: D.G. Gould

Submitted by: A.D. Wood
Head
Flight Research Section

Approved by: F.R. Thurston
Director

SUMMARY

The flow about and on the surface of an aerofoil section in a finite stream and that about and on the surface of the same aerofoil section in an infinite symmetrical cascade are compared. It is shown that, for most cases of practical interest, the differences are small for stream deflection angles up to 90 degrees. Thus, the approximate flow about an arbitrary aerofoil section in a two-dimensional finite stream may be obtained by using the methods existing for the determination of the flow about an arbitrary section in a cascade. The procedure, as applied to the finite stream approximation, is given using the interference method of cascade theory.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

a	=	One fourth of the reference chord of arbitrary aerofoil section.
C_{l_∞}	=	Lift coefficient for isolated aerofoil section in a cascade.
C_{R_c}	=	Resultant force coefficient on a single aerofoil section in a cascade.
ΔC_l	=	$C_{l_\infty} - C_{R_c}$
d	=	Height of finite stream.
h	=	Spacing of aerofoil sections in cascade.
q_∞	=	Velocity at infinity in a cascade or in a finite stream.
R_c	=	Resultant force on a single aerofoil section in a cascade.
R_f	=	Resultant force on an aerofoil section in a finite stream.
W_A	=	Additional complex potential function representing the net change of the flow potential due to the presence of other aerofoil sections in a cascade (see Section 3.1).
W_d	=	Disturbance complex potential function (see Section 3.1).
W_c	=	Compensating complex potential function (see Section 3.1).
W_T	=	Total complex potential function representing flow about an aerofoil section in a cascade (see Section 3.1).
W_Γ	=	Circulation complex potential function (see Section 3.1).
W_∞	=	Complex potential function about aerofoil section isolated from the cascade (see Section 3.1).

LIST OF SYMBOLS (cont'd.)

β = Angle measured from incoming stream direction to reference chord direction of arbitrary aerofoil section.

γ = Angle measured from the vector mean of the velocity far in front of and far behind an aerofoil section in a finite stream to the reference chord direction of the aerofoil section.

Γ = Circulation about an aerofoil section.

ε = Function defining the geometric mapping of an arbitrary aerofoil section into a circle (see Section 3.2).

ε_T = Value of ε at the trailing edge of the aerofoil section.

Θ = Change in flow direction from minus to positive infinity for flow through an infinite cascade or for flow of a finite stream over an aerofoil section.

ρ = Density of fluid.

ϕ = Polar angle of circle into which aerofoil section is mapped.

Φ = Velocity potential.

ψ = Function defining geometric mapping of an arbitrary aerofoil section into a circle.

Ψ = Stream function.

APPROXIMATE METHOD FOR DETERMINING THE POTENTIAL FLOW ABOUT AN ARBITRARY AEROFOIL SECTION IN A TWO-DIMENSIONAL FINITE STREAM WITH PARTICULAR REFERENCE TO LARGE STREAM DEFLECTIONS

1.0 INTRODUCTION

Methods of determining the plane flow of an incompressible, irrotational fluid about an arbitrary body are well known for the case of the plane flow extending to infinity in all directions (see for example, Ref. 1). The boundary conditions that must be satisfied by the potential function for this case are that the velocity at the boundary of the body is tangential to the surface, and the velocity at infinity (in any direction) is equal in both magnitude and direction. A more complicated problem arises if the flow is finite in extent, bounded by free streamlines. The boundary condition of equal magnitude and direction of the velocity at infinity is replaced by the condition that the pressure and hence the velocity is constant along the free boundaries. The boundary condition determines the position of the free boundaries which are not known a priori.

Helmholtz, Kirchoff and others developed general methods based on conformal mapping for dealing with plane flows with free surfaces. The methods are not readily adaptable for determining the flow of a plane finite stream about arbitrary bodies with circulation. Prandtl, Sasaki and Woods (Ref. 2, 3 and 4) treated the effects of a finite stream on the aerodynamic characteristics of aerofoil sections for cases where the deflection of the stream was small. The flow of a finite stream past a point vortex without any restriction on the deflection angle was determined by Simmons (Ref. 5).

In the following paragraphs, the flow on the surface of an aerofoil section in a finite stream and that on the same aerofoil section in an infinite cascade are compared for the purpose of showing that, in most practical instances, the differences are small. Since methods for the determination of the flow about arbitrary aerofoil sections in a cascade exist, it is possible to determine the approximate flow about an arbitrary aerofoil section in a finite stream by using these procedures. The procedure is applicable to highly cambered sections giving large stream deflections.

2.0 COMPARISON OF FLOW THROUGH A CASCADE OF AEROFOIL SECTIONS AND THE FLOW OF A FINITE STREAM ABOUT AN AEROFOIL SECTION

2.1 Cascade Flow

The superposition of an infinite uniform stream and an infinite column of vortices and doublets, with the centres of the vortices and doublets coincident at points of singularities spaced a distance h apart along an axis perpendicular to the direction of the uniform stream, represents the flow through an infinite column of closed oval-shaped bodies surrounding the points of singularity. A suitable transformation transforms the oval-shaped bodies into aerofoil sections and the flow through an infinite cascade* of two-dimensional aerofoil sections is obtained. An example for a particular case is shown in Figure 1. The details of the method by which this particular cascade was obtained are given in Appendix A.

The circulation about each of the aerofoil sections in the cascade is given as a function of the cascade spacing, h, and the flow deflection angle θ, by

$$\Gamma = 2h\, q_{\infty} \sin \theta/2 \tag{1}$$

The resultant aerodynamic force on each aerofoil acts in a direction normal to the vector mean of the velocities far in front of and far behind the cascade (that is, in the cascade direction) and is given by

$$R_c = \rho\Gamma\, q_{\infty} \cos \theta/2 \tag{2}$$

or

$$R_c = 2\rho d\, q_{\infty}^2 \sin \theta/2 \tag{3}$$

---

* The cascade in this analysis is one in which the flow at infinity downstream of the cascade is a mirror image of the flow at infinity upstream of the cascade with respect to the cascade direction.

where $$d = h \cos \theta/2 \tag{4}$$

is the distance measured normal to q_∞, separating two streamlines that are spaced a distance h apart in the direction of the cascade.

2.2 Finite Stream Flow

The analysis of the details of the flow around an aerofoil section in which the flow is finite in extent is complicated by the fact that the position of the free boundaries along which the pressure and velocity must be constant are not known a priori. The conditions at infinity in the finite stream can, however, be determined from momentum considerations without a detailed knowledge of the flow in the vicinity of the aerofoil section. The resultant aerodynamic force on an aerofoil section in a finite stream of height d, magnitude of velocity at infinity of q_∞, and stream deflection of θ is then, (see Fig. 2)

$$R_f = 2\rho d\ q_\infty^2 \sin \theta/2 \tag{5}$$

This force acts normal to the vector mean of the velocities far in front of and far behind the aerofoil as in the case of the cascade and its magnitude is the same as that for an aerofoil section in a cascade for the same stream deflection and for the distance d in the cascade (see Fig. 1) equal to the height of the finite stream.

If one of the aerofoil sections, for example the section GH in Figure 1, is considered as isolated from the cascade, the resultant aerodynamic force on the section is then equal in both magnitude and direction to that of an aerofoil section in a finite stream of height d and stream deflection of θ. Unfortunately, there are no streamlines in the cascade such as AEC or BFD (Fig. 1) along which the velocity and hence the pressure is constant. If the aerofoil section is located somewhat centrally with respect to the streamlines through E and F, however, the variation in velocity along the streamlines is not large, even when the stream deflection is large. The velocity distribution, expressed as a fraction of q_∞, along such streamlines is given in Figure 4 for three example profiles designed to give large stream deflections in a cascade.

Small local distortions of the streamlines AEC and BFD have a powerful effect on the velocity along the streamlines. It would be expected, however, that these distortions would have a smaller effect on the velocity at the aerofoil surface. If the magnitude of the effect on the velocity at the aerofoil surface is small, the cascade solution represents a good approximation to the flow in a finite stream. It has been demonstrated for a particular case, with a large stream deflection, that the effect of these distortions of the streamlines AEC and BFD on the flow at the aerofoil surface is negligible. The effect was determined using a numerical procedure, the details of which are given in Appendix B.

The streamline distortion required to make the velocity constant along the bounding streamlines for the particular case is shown in Figure 3. The streamlines for the aerofoil section in an infinite cascade are shown by the broken lines. The particular cascade and aerofoil section (the same as that shown in Figure 1) was designed to give a flow deflection of 70 degrees. The solid lines represent the position of the streamlines that is required to make the velocity along the bounding streamlines constant to within the accuracy of the numerical procedure used (about one percent of the velocity at infinity). To this order of accuracy it was not possible to detect any change in the velocity on the aerofoil surface. The change in circulation was determined by finding the difference in arc lengths of the bounding distorted streamlines and the bounding streamlines in the cascade solution. The change in circulation determined in this manner was less than one half of one percent of the circulation around one section in the cascade.

The streamlines AEC and BFD that were chosen in the cascade solution to represent the bounding streamlines in the finite stream were those that passed through the points on the cascade axis E and F (Fig. 1) at which the velocity was equal in magnitude to the velocity at infinity. This choice for the bounding streamlines was made so that the leading edge of the aerofoil section was located centrally with respect to the incoming finite stream. Other cascade streamlines could have been chosen such that the leading edge of the aerofoil would be located in different positions relative to the incoming streamlines. The variation of the magnitude of the velocity along these streamlines would generally be larger than along the particular ones chosen above and the cascade approximation to the finite stream flow would not be as good.

The length of the mean camber line of the aerofoil section in the above example was approximately equal to the distance d between the two parallel streamlines AEC and BDF. If the aerofoil section were larger relative to this distance (corresponding to an increase in the solidity of the cascade), the change in velocity along the streamlines AEC and BFD would increase and the finite stream flow would differ from the cascade flow more than for the example chosen.

The variation of the velocity along the streamlines AEC and BFD for two sections designed to give flow deflections of 90 degrees in cascade were compared with the velocity distribution along the streamlines AEC and BFD in the above example (70 degrees flow deflection) in Figure 4. The differences are not large so that one would expect the cascade solution would give a good approximation for flow deflection angles up to at least 90 degrees, if the aerofoil size were not too large relative to the finite stream height and if the aerofoil section were located somewhat centrally in the finite stream.

3.0 APPROXIMATE COMPUTATION OF FINITE STREAM FLOW ABOUT AN ARBITRARY AEROFOIL SECTION

It was demonstrated in Section 2 that, if the leading edge of the aerofoil section in a finite stream is located fairly centrally with respect to the incoming stream, and if the length of the mean camber line of the section is of the same order as the height of the incoming stream, the flow on the aerofoil surface in a finite stream is closely approximated by the flow on the surface of the same aerofoil section in a cascade. Under these restrictions, therefore, it is possible to use cascade theory to approximate the flow about an arbitrary aerofoil section in a finite stream.

A number of procedures are available for computing the flow about an arbitrary aerofoil section in a cascade. The choice of method depends primarily on the computing facilities available. For the purposes of illustrating the procedure as applied to the finite stream approximation, the interference method, described in Reference 6, was chosen.

3.1 Interference Method for Determining the Flow about an Arbitrary Aerofoil Section in a Cascade (Ref. 6)

A brief resume of the procedure for determining the flow about an arbitrary aerofoil in a cascade using the interference method is given in this section so that the computation method for the finite stream approximation can be outlined. An iteration procedure is required.

(i) The aerofoil section used in the cascade is treated as an isolated aerofoil in an infinite stream (the infinite stream flow is the vector mean of the flow far in front of and far behind the cascade) and the potential flow function, W_∞, is obtained. Since the aerofoil section is taken as the zero streamline in this flow, $W_\infty = \Phi_\infty$.

(ii) The disturbance potential, along the aerofoil boundary, $W_d = \Phi_d + i\,\Psi_d$, caused by the presence of the external aerofoils in the cascade, is determined. For this calculation the flow about each of the external aerofoils is taken to be that computed in (i) above in the first step of the iteration.

(iii) A compensating flow function (which may only have singularities within the central aerofoil), $W_c = \Phi_c + i\,\Psi_c$, is computed in order to maintain the aerofoil section a streamline in the presence of the disturbance flow. It is determined by the condition that on the boundary its stream function, Ψ_c, must be equal and opposite to the disturbance stream function Ψ_d.

(iv) A circulation potential, W_Γ, is computed in order that the Kutta-Joukowski condition is still satisfied at the trailing edge of the aerofoil.

(v) The sum, $W_d + W_c + W_\Gamma$, represents the net change of the flow potential due to the presence of the external aerofoils, and is designated W_A, the additional flow function. The additional flow function, W_A, is added to the original flow function, W_∞, of the aerofoil section isolated in an infinite stream. This new flow function is called W_T and replaces W_∞ in part (ii) of the computation procedure for the second step of the iteration. The procedure is repeated until two successive values of W_T agree to the desired accuracy.

3.2 Application to the Finite Stream Approximation

In the finite stream, the orientation, β, and length of the reference chord of the aerofoil section, 4a, is given relative to the incoming stream (see Fig. 2). The flow deflection produced by the aerofoil section is not known in advance, however, and so the direction of the infinite stream flow is not known for the determination of the flow function, W_∞, described above (Section 3.1). The function, W_∞, depends on, first the geometric mapping function that transforms the aerofoil into a circle, and secondly on the incidence of the reference chord of the aerofoil to the free stream direction. Fortunately, the geometric mapping function is independent of the reference chord incidence. A numerical solution (the numerical solution of an integral equation by iteration, see for example, References 1 and 7), is required to determine the geometric mapping function. The velocity on the aerofoil surface and hence W_∞ can easily be determined by the following relation for any incidence once the geometric mapping function is obtained (see Equation XII of Reference 1).

$$\left|\frac{V}{U}\right| = \left|[\sin(\gamma + \phi) + \sin(\gamma + \varepsilon_T)] \cdot F(\phi)\right| \qquad (6)$$

where γ is the incidence of the reference chord to the direction of the free stream of velocity U, ϕ is the polar angle of the circle obtained by the geometric mapping, ε_T and the function $F(\phi)$ are determined from the mapping function and are independent of γ. Similarly the lift coefficient of the isolated aerofoil in the infinite stream of velocity U is given as a function of the incidence by (see equation VII of Reference 1).

$$C_{l_\infty} = \frac{l_\infty}{\rho/2 \; U^2 \; (4a)} = 2\pi \, e^{\psi_o} \sin(\gamma + \varepsilon_T) \qquad (7)$$

where ψ_o is a constant determined by the mapping function and is independent of γ.

If the resultant aerodynamic force coefficient acting on an aerofoil section in a cascade is defined as

$$C_{R_c} = \frac{R_c}{\rho/2 \; U^2 \cdot 4a} \qquad (8)$$

where $U = q_\infty \cos \theta/2$, the vector mean of the velocity far in front of and far behind the cascade, and use is made of equation (3) and the fact that $\beta = \theta/2 + \gamma$, then C_{R_c} can be written as

$$C_{R_c} = 4\left(\frac{d}{4a}\right) \frac{\sin(\beta - \gamma)}{\cos^2(\beta - \gamma)} \qquad (9)$$

Since d, 4a and β are given, C_{R_c} is obtained as a function of γ.

The value of C_{l_∞} given by equation (7) represents the value of C_{R_c} in the cascade only if the cascade spacing is sufficiently large that the interference of the other aerofoils in the cascade is negligible. If a quantity, ΔC_l, is defined, that represents the reduction in force coefficient on the aerofoil section due to the interference of the other aerofoil sections in the cascade, then

$$C_{R_c} = C_{l_\infty} - \Delta C_l \qquad (10)$$

The quantity ΔC_l cannot be determined in advance so that an iteration procedure is required. Fortunately, the iteration converges rapidly and it is usually only necessary to repeat the process once to give good accuracy.

If the orientation of the reference chord to the incoming finite stream, β, the length of the reference chord, $4a$, and the height of the incoming finite stream, d, are given, the procedure to determine the approximate performance of an arbitrary aerofoil section in a finite stream is therefore as follows:

(i) Determine the geometric mapping function that transforms the aerofoil section into a circle by using the procedure given in Reference 1.

(ii) Assume $\Delta C_l = 0$ and determine the value of γ which makes C_{l_∞} given by equation (7) and C_{R_c} given by equation (9) equal.

(iii) Compute the velocity distribution on the surface of the aerofoil using equation (6) where $F(\phi)$ is obtained from the geometric mapping function obtained above (see equation XII of Reference 1).

(iv) From the velocity distribution obtained in (iii) the value of W_∞ is obtained and the procedure outlined in Section 3.1 and Reference 6 is followed to determine the interference effects of the other aerofoils in the cascade. From this result the first value of ΔC_l can be obtained and a new value of γ can be determined by using equations (7), (9) and (10).

(v) Repeat steps (iii) and (iv) to determine the second value of ΔC_l. This procedure is repeated until two successive values of ΔC_l agree to within the desired accuracy. Usually, the first value of ΔC_l obtained by this procedure gives good accuracy.

A sample calculation is given in Appendix C to illustrate in detail the procedure and the rapidity of the convergence of the iteration.

4.0 CONCLUSION

A procedure has been outlined for the approximate calculation of the flow about arbitrary aerofoil sections in two-dimensional finite streams. For most practical applications the method should give adequate accuracy for stream deflections up to 90 degrees.

5.0 REFERENCES

1. Theodorsen, T. — Theory of Wing Sections of Arbitrary Shape. NACA Report No. 411, October 1931.

2. Glauert, H. — Wind Tunnel Interference on Wings, Bodies and Airscrews. ARC R & M 1566, 1933.

3. Sasaki, T. — On the Effect of the Wall of a Wind Tunnel upon the Lift Coefficient of a Model. Report of the Aeronautical Research Institute, Tokyo Imperial University, No. 46, Vol. IV, 5, December 1928.

4. Woods, L. — On the Deflexion of Jets by Aerofoils. Quarterly Journal of Mechanics and Applied Mathematics, Vol. XI, Part I, February 1958, p. 24 - 38.

5. Simmons, N. — Free Stream-line Flow past a Vortex. The Quarterly Journal of Mathematics, Vol. X, 1939, p. 283.

6. Katzoff, S., Finn, R.S., Laurence, J.C. — Interference Method for Obtaining the Potential Flow past an Arbitrary Cascade of Airfoils. NACA Report No. 879, 1947.

7. Streeter, V.L. Fluid Dynamics. McGraw-Hill Book Company Inc., 1948.

8. Naiman, I. Numerical Evaluation by Harmonic Analysis of the ε-Function of the Theodorsen Arbitrary-Airfoil Potential Theory. NACA ARR No. L5H18, 1945.

/AM

APPENDIX A

ANALYSIS USED TO DETERMINE THE SAMPLE AEROFOIL SECTIONS (FIG. 1 AND 4)

The superposition of an infinite uniform stream and an infinite column of vortices and doublets, with the centres of the vortices and doublets coincident at points of singularities spaced a distance h apart along an axis perpendicular to the direction of the uniform stream, represents the flow through an infinite column of closed oval-shaped bodies surrounding the points of singularities. A suitable transformation transforms the oval-shaped bodies into aerofoil sections and the flow through an infinite cascade of aerofoil sections is obtained. In particular, if the uniform stream is directed along the positive real axis and the vortices and doublets are arranged along the imaginary axis, a special type of cascade termed a symmetrical cascade is obtained. This cascade is one in which the flow at infinity downstream of the cascade is a mirror image of the flow at infinity upstream of the cascade.

A.1 Complex Potential Function

The complex potential for an infinite row of equal strength vortices located at (0, ±ih, ±2ih, ...) along the imaginary axis is (Ref. 7)

$$\chi_V = \frac{i\Gamma}{2\pi}\left[\ln \zeta' + \ln (\zeta' - ih) + \ln (\zeta' + ih) + \ldots\right]$$

$$= \frac{i\Gamma}{2\pi} \ln \sinh \frac{\pi\zeta'}{h} \qquad \text{(A-1)}$$

The complex potential for an infinite row of equal strength doublets with the same centres as the vortices is

$$\chi_D = U\mu^2 \left[\frac{1}{\zeta'} + \frac{1}{\zeta' - ih} + \frac{1}{\zeta' + ih} + \cdots\right]$$

$$= U\mu^2 \frac{d}{d\zeta'}\left[\ln \sinh \frac{\pi\zeta'}{h}\right] \tag{A-2}$$

$$= \frac{U\pi\mu^2}{h} \coth \frac{\pi\zeta'}{h}$$

The desired complex potential for the superposition of a uniform stream and an infinite column of vortices and doublets is, therefore,

$$\chi = \chi_U + \chi_V + \chi_D = U\zeta' + \frac{\pi\mu^2}{h} U \coth \frac{\pi\zeta'}{h} + \frac{i\Gamma}{2\pi} \ln \sinh \frac{\pi\zeta'}{h} \tag{A-3}$$

or
$$X = \frac{\chi}{hU} = \zeta + \frac{k^2}{\pi} \coth \pi\zeta + \frac{iC_\Gamma}{\pi} \ln \sinh \pi\zeta \tag{A-4}$$

where
$$\zeta = \zeta'/h = \xi + i\eta, \; k^2 = \frac{\pi^2\mu^2}{h^2}, \; C_\Gamma = \frac{\Gamma}{2hU} \tag{A-5}$$

The stream function and the velocity potential function are:

$$\frac{\psi}{hU} = \Psi = \eta + \frac{C_\Gamma}{2\pi} \ln (\cosh 2\pi\,\xi - \cos 2\pi\eta)$$

$$- \frac{k^2}{\pi} \frac{\sin 2\pi\eta}{\cosh 2\pi\,\xi - \cos 2\pi\eta} \tag{A-6}$$

and
$$\frac{\phi}{hU} = \Phi = \xi + \frac{k^2}{\pi} \frac{\sinh 2\pi\xi}{\cosh 2\pi\xi - \cos 2\pi\eta}$$
$$- \frac{C_\Gamma}{\pi} \tan^{-1} (\coth \pi\xi \tan \pi\eta) \qquad \text{(A-7)}$$

A.2 Velocity in the ζ Plane

The velocity components $\tau = \tau'/U$ in the ξ direction and $\nu = \nu'/U$ in the η direction, in the ζ plane are:

$$\tau = 1 - \frac{2k^2 (\cosh 2\pi\xi \cos 2\pi\eta - 1)}{(\cosh 2\pi\xi - \cos 2\pi\eta)^2} + \frac{C_\Gamma \sin 2\pi\eta}{\cosh 2\pi\xi - \cos 2\pi\eta} \qquad \text{(A-8)}$$

$$\nu = - \frac{2k^2 \sinh 2\pi\xi \sin 2\pi\eta}{(\cosh 2\pi\xi - \cos 2\pi\eta)^2} - \frac{C_\Gamma \sinh 2\pi\xi}{\cosh 2\pi\xi - \cos 2\pi\eta} \qquad \text{(A-9)}$$

For $\xi \to -\infty$, $\tau \to 1$ and $\nu \to C_\Gamma$
and $\xi \to +\infty$, $\tau \to 1$ and $\nu \to -C_\Gamma$

so that the total deflection of the stream in going from $-\infty$ to ∞ is

$$\theta = 2 \tan^{-1} C_\Gamma \qquad \text{(A-10)}$$

and the magnitude of the velocity at infinity is

$$q_\infty/U = \frac{1}{\cos \theta/2} = \sqrt{1 + C_\Gamma^{\,2}} \qquad \text{(A-11)}$$

A-3 Stagnation Points and Stagnation Streamlines

Stagnation points in the ζ plane (ξ_{S_0}, η_{S_0}) occur when both τ and ν are zero and are found from equations (A-8) and (A-9) to be

$$\eta_{S_0} = - \frac{1}{2\pi} \tan^{-1} C_\Gamma = - \frac{\theta}{4\pi} \qquad \text{(A-12)}$$

$$\xi_{S_0} = \pm \frac{1}{2\pi} \cosh^{-1} \frac{1 + 2k^2}{\sqrt{1 + C_\Gamma^{\,2}}} = \pm \frac{1}{2\pi} \cosh^{-1} [(1 + 2k^2) \cos \theta/2] \qquad \text{(A-13)}$$

The equation for the stagnation streamlines is

$$\begin{aligned} 2\pi \, \Psi_S &= - \theta/2 + [1 + \ln 2k^2 \cos \theta/2] \tan \theta/2 \\ &= 2\pi\eta_S + \tan \theta/2 \ln (\cosh 2\pi\xi_S - \cos 2\pi\eta_S) \\ &\quad - \frac{2k^2 \sin 2\pi\eta_S}{\cosh 2\pi\xi_S - \cos 2\pi\eta_S} \end{aligned} \qquad \text{(A-14)}$$

where ξ_S and η_S are points describing the stagnation streamline. Equation (A-14) defines the closed streamlines surrounding the singularities at $\xi = 0$, $\eta = 0$, ± 1, ± 2 .... that pass through the stagnation points ξ_{S_0}, η_{S_0}. Equation (A-14) can only be solved explicitly for ξ_S or η_S for the case of $\eta_S = 0$ where ξ_S is given by

$$\xi_{S(\eta_S = 0)} = \pm \frac{1}{2\pi} \cosh^{-1} \left[1 + e^{2\pi \Psi_S \cot \theta/2} \right] \quad \text{(A-15)}$$

Equations (A-12), (A-13) and (A-15) give four points on the closed curve defined by equation (A-14) and other values must be obtained by a graphical or numerical procedure. For the purposes of numerical evaluation of the points ξ, η defining any particular streamline, it is convenient to write equation (A-14) in the form:

$$\mu \tan \theta/2 = \frac{2k^2 \sin 2\pi\eta}{e^{\mu}} + 2\pi (\Psi - \eta) \quad \text{(A-16)}$$

where

$$e^{\mu} = \cosh 2\pi\xi - \cos 2\pi\eta \quad \text{(A-17)}$$

A.4 Transformation Equations

It is convenient, for calculation purposes, to perform the following successive transformations, similar in each step to those used in transforming a single circle into a Joukowski type aerofoil section

$$Z''' = \zeta \, e^{i\alpha} \quad \text{(A-18)}$$

$$Z'' = Z''' + m + in \quad \text{(A-19)}$$

where

$$n = -\left(\eta_{S_0} \cos \alpha + \left|\xi_{S_0}\right| \sin \alpha\right) \quad \text{(A-20)}$$

$$Z' = Z'' + b^2 \left(\xi_{S_0} \cos\alpha - \eta_{S_0} \sin\alpha + m\right)^2 \left[\frac{1}{Z''} + \frac{1}{Z'' - ie^{i\alpha}} + \frac{1}{Z'' + ie^{i\alpha}} + \ldots\right]$$

$$= Z'' + b^2 \pi e^{-i\alpha} \left(\xi_{S_0} \cos\alpha - \eta_{S_0} \sin\alpha + m\right)^2 \coth \pi Z'' e^{-i\alpha} \tag{A-21}$$

and finally

$$Z = Z' e^{-i\alpha} \tag{A-22}$$

The significance of the transformation equations (A-18) to (A-22) is illustrated in Figure 1. The oval-shaped bodies in the ζ plane are distributed along the imaginary η axis. Equation (A-18) represents a rotation through the angle α where α is analogous to aerofoil incidence in the infinite stream case. The quantity n in equation (A-19), defined in equation (A-20), ensures that the trailing edges of the aerofoil sections correspond to the stagnation points ξ_{S_0}, η_{S_0} in the ζ plane and hence sets the camber of the sections to give the desired flow deflection as a function of α. The magnitude of m in equation (A-19) defines the thickness of the sections. The transformation given by equation (A-21) transforms the oval-shaped bodies into streamlined aerofoil sections and is analogous to the infinite stream transformation $Z' = Z'' + \frac{b^2}{Z''}$.

A.5 Velocity in the Cascade Plane

The non-dimensional velocity components in the cascade or Z plane, $u = u'/U$, $v = v'/U$ are given by

$$u = R\left[\frac{dX}{d\zeta} \cdot \frac{\sinh^2 \pi e^{-i\alpha} \cdot Z''}{\sinh^2 \pi e^{-i\alpha} Z'' - \pi^2 b^2 e^{-2i\alpha}(\xi_{S_0} \cos\alpha - \eta_{S_0} \sin\alpha + m)^2}\right] \tag{A-23}$$

$$v = I\left[\frac{dX}{d\zeta}\cdot\frac{\sinh^2 \pi e^{-i\alpha}\cdot Z''}{\sinh^2\pi e^{-i\alpha}Z'' - \pi^2 b^2 e^{-2i\alpha}(\xi_{S_0}\cos\alpha - \eta_{S_0}\sin\alpha + m)^2}\right] \tag{A-24}$$

and as

$$x \to +\infty \qquad u \to \tau$$
$$x \to -\infty \qquad v \to \nu$$

so that the velocities at infinity in the two planes are the same in both magnitude and direction.

A.6 Example Profiles

Three sample profiles were determined using the above procedures. The values of the arbitrary constants were taken as $m = -0.01200$, $b = 0.98000$ and $k^2 = 0.39478$. The values of the flow deflection angle θ and of the angle α for the three profiles were:

$$\begin{array}{lll} \text{(i)} & \theta = 70° & \alpha = 10° \\ \text{(ii)} & \theta = 90° & \alpha = 10° \\ \text{(iii)} & \theta = 90° & \alpha = 0° \end{array} \tag{A-26}$$

The profiles and the pressure distributions plotted normal to a chord line parallel to the vector mean of the velocity far in front of and far behind the cascade are given in Figures 5, 6, and 7. The pressure coefficient in these plots is defined as:

$$C_P = 1 - (u^2 + v^2)\cos^2\theta/2 = 1 - \frac{(u'^2 + v'^2)}{q_\infty^2} \tag{A-27}$$

The streamlines AEC and BFD (see Fig. 1) that were chosen arbitrarily to represent the bounding streamlines for the finite stream approximation are such that the velocity at the points E and F is equal to q_{∞}. The velocity distributions along the streamlines AEC or BFD for the three sample profiles are shown in Figure 4.

APPENDIX B

NUMERICAL ANALYSIS TO DETERMINE THE REQUIRED STREAMLINE DISTORTION TO GIVE CONSTANT VELOCITY ALONG THE BOUNDING STREAMLINES

The sample profile chosen is that shown in Figure 3 (profile (i) in Appendix A), designed to give a flow deflection of 70 degrees when in a symmetrical cascade. The relaxation method was used in making the numerical computations. The initial grid spacing was taken to be 1/10 of the finite stream height at infinity, d. The value of the stream function was computed at each nodal point in the grid to an accuracy of 1/5 of one percent of the change in the stream function value from the lower to the upper bounding streamline ($\Psi_L = 0$ and $\Psi_U = 1000$ in Figure 3). The bounding streamlines were distorted such that the distance between the modified streamlines and the original $\Psi = 100$ and $\Psi = 900$ streamlines was 1/10 everywhere and the new position of the $\Psi = 100$ and $\Psi = 900$ streamlines was determined. The above procedure was repeated and it was found that the new positions of the $\Psi = 100$ and $\Psi = 900$ streamlines had not changed appreciably from the previous positions. The grid spacing was reduced to d/40 and final small modifications to the bounding streamlines were made to ensure the velocity was constant to within one percent along them.

The modified streamlines are compared with those given by the cascade solution in Figure 3. The streamline distortion in the vicinity of the aerofoil was not detectable within the accuracy of the numerical solution. Since, in the cascade, the streamlines AEC and BFD are parallel and the velocity distribution along them is the same, the contribution of the integral of the velocity along these streamlines to the circulation about each aerofoil section is zero and the circulation is simply:

$$\Gamma = 2d\, q_\infty \tan \theta/2 \qquad \text{(B-1)}$$

In the case of the finite stream, since the bounding streamlines are distorted, they are no longer necessarily equal in length. The change in circulation in this case can be measured, then, by determining the difference in arc length of the upper and lower bounding streamlines, say, Δs, and the change in circulation is:

$$\Delta\Gamma = q_{\infty} \Delta s \qquad \text{(B-2)}$$

If $\frac{\Delta\Gamma}{\Gamma}$ is large, the flow deflection angle θ must be changed. In the above example, $\Delta\Gamma$ determined in this manner was less than 1/2 percent of the value of Γ in the cascade, so that the change in θ would be less than 0.3 degree.

APPENDIX C

SAMPLE CALCULATION OF APPROXIMATE FLOW ABOUT AN ARBITRARY AEROFOIL IN A FINITE STREAM

For the purposes of the sample calculation, the aerofoil section (i) of Appendix A was considered as the arbitrary aerofoil. The reference chord of this aerofoil was taken as 4a, its orientation to the incoming stream, β was 46.85 degrees and the ratio of the reference chord length 4a to the slipstream height was 0.72356. The particular reference chord chosen was arbitrary. Its choice does, however, influence the length of the calculations required to obtain the geometric transformation of the aerofoil into a circle. It is shown in Figure 2, along with the orientation of the aerofoil section to the incoming stream.

C.1 Geometric Transformation of Aerofoil into a Circle*

The Theodorsen method (Ref. 1) for obtaining the potential flow about aerofoil sections of arbitrary shape was used. Points on the aerofoil surface are defined by

$$\zeta = x + iy \tag{C-1}$$

The aerofoil contour is then transformed into a near cycle in the Z' plane where ζ and Z' are related by

$$\zeta = Z' + \frac{a^2}{Z'} \tag{C-2}$$

The near circle in the Z' plane is given by

---

* The notation in this Appendix corresponds to that used in Reference 1. In particular, θ, ζ, and Z' are different from those defined in the list of symbols and in the main text of this report.

$$Z' = a\, e^{\psi + i\theta} \tag{C-3}$$

Using (C-2) and (C-3), ψ and θ are determined from

$$2 \sinh^2 \psi = - p + \sqrt{p^2 + (y/a)^2} \tag{C-4}$$

$$2 \sin^2 \theta = p + \sqrt{p^2 + (y/a)^2} \tag{C-5}$$

where
$$p = 1 - \left(\frac{x}{2a}\right)^2 - \left(\frac{y}{2a}\right)^2 \tag{C-6}$$

The near circle in the Z' plane is then transformed into a circle

$$Z = a\, e^{\psi_o + i\phi} \tag{C-7}$$

in the Z plane using the general transformation

$$Z' = Z\, e^{\sum_n (A_n + iB_n) \cdot 1/Z^n} \tag{C-8}$$

or
$$Z' = Z\, e^{\psi - \psi_o + i(\theta - \phi)} \tag{C-9}$$

Equating equations (C-8) and (C-9), $\psi - \psi_o$ and $\theta - \phi$ are found to be given by:

$$\psi - \psi_o = \sum_n \left[\frac{A_n}{r^n} \cos n\phi + \frac{B_n}{r^n} \sin n\phi \right] \tag{C-10}$$

$$\Theta - \phi = \sum_n \left[\frac{B_n}{r^n} \cos n\phi - \frac{A_n}{r^n} \sin n\phi \right] \tag{C-11}$$

where

$$\frac{A_n}{r^n} = \frac{1}{\pi} \int_0^{2\pi} \psi \cos n\phi \, d\phi \tag{C-12}$$

$$\frac{B_n}{r^n} = \frac{1}{\pi} \int_0^{2\pi} \psi \sin n\phi \, d\phi \tag{C-13}$$

$$\psi_o = \frac{1}{2\pi} \int_0^{2\pi} \psi d\phi \tag{C-14}$$

Eliminating the coefficients A_n/r^n and B_n/r^n from (C-11) by using (C-13) and C-14), the expression for $\varepsilon = -(\Theta - \phi)$ can be written as

$$\varepsilon(\phi_c) = -\frac{1}{2\pi} \int_0^{2\pi} \psi \cot \left(\frac{\phi - \phi_c}{2}\right) d\phi \tag{C-15}$$

Since ψ is given as a function of $\Theta = -\varepsilon + \phi$, it is necessary to use an iteration procedure to determine ε and ψ_o by first assuming ε as zero, computing $\varepsilon^{(1)}$ and then determing ψ as a function of $\phi^{(1)} = \Theta + \varepsilon^{(1)}$. A second approximation to $\varepsilon = \varepsilon^{(2)}$

is then obtained from equation (C-15). The procedure is repeated until two successive values of ε agree to within the required accuracy. A procedure for the numerical evaluation of the ε function (equation C-15) using harmonic analysis is given in Reference 8. The quantities ε and ψ_o define the point to point transformation of the aerofoil in the ζ plane to the circle $a\, e^{\psi_o + i\phi}$ in the Z plane.

The calculations for this sample aerofoil were made with an electronic digital computer. An eighty point formula (see Ref. 8) was used in the determination of ε and six iteration steps were required to determine the value of ε to an accuracy of 0.01 radian at the value of ϕ corresponding to the trailing edge of the aerofoil. The final computed curves of ψ and ε as functions of θ are shown in Figure 8.

C.2 Lift Coefficient of the Isolated Aerofoil and Velocity on the Aerofoil Surface

The lift coefficient of the isolated aerofoil section in an infinite stream is (from equation VII of Reference 1):

$$C_{l_\infty} = \frac{l}{\rho/2\ U^2\ (4a)} = 2\pi\, e^{\psi_o} \sin(\gamma + \varepsilon_T) \qquad \text{(C-16)}$$

where γ is the incidence of the reference chord to the free stream direction and ε_T is the value of ε at the trailing edge of the aerofoil.

The velocity on the surface of the aerofoil is

$$\left|\frac{v}{U}\right| = \left|\ [\sin(\gamma + \phi) + \sin(\gamma + \varepsilon_T)] \cdot F(\theta)\ \right| \qquad \text{(C-17)}$$

where $$F(\theta) = \frac{(1 + d\varepsilon/d\theta)\, e^{\psi_o}}{\sqrt{(\sinh^2 \psi + \sin^2 \theta)\,[1 + (d\psi/d\theta)^2]}} \qquad \text{(C-18)}$$

Since $F(\theta)$ and ε_T are independent of the incidence, γ, they can be computed as functions of θ or ϕ and it is then a simple computation to obtain the corresponding values of v/U for any γ.

C.3 Determination of Approximate Flow about the Aerofoil Section in Cascade

The resultant force coefficient on an aerofoil section in a finite stream, or in an infinite cascade, from the change of momentum in the stream far in front of and far behind the section is

$$C_{R_c} = C_{R_f} = \frac{R_c}{\rho/2\; q_\infty^2 \cos^2 \theta/2 \cdot (4a)} = 4\left(\frac{d}{4a}\right) \frac{\sin \theta/2}{\cos^2 \theta/2} \qquad \text{(C-19)}$$

and, since for a symmetrical cascade, (see Fig. 1 and 2)

$$\theta/2 = \beta - \gamma \qquad \text{(C-20)}$$

$$C_{R_c} = 4\left(\frac{d}{4a}\right) \frac{\sin(\beta - \gamma)}{\cos^2(\beta - \gamma)} \qquad \text{(C-21)}$$

The interference of the other aerofoil sections in a cascade on a particular section reduces the lift on this particular section from the lift it would give if it were isolated in an infinite stream with the flow velocity of the infinite stream equal to the vector mean of the velocity far in front of and far behind the cascade. If this reduction in lift is given by ΔC_1, the force coefficients given in equations (C-16) and (C-21) are related by:

$$C_{R_c} = C_{1_\infty} - \Delta C_1 \qquad \text{(C-22)}$$

A number of procedures are available for the determination of the value of ΔC_1. The method used in this analysis is known as "the interference method" and is described fully in Reference 6. In order to use the methods for determing ΔC_1 directly it is necessary to be given the value of γ, or $\theta/2$ from which γ can be determined. In the present application, neither γ or $\theta/2$ are given but only the value of β. An iteration procedure is thus required. The procedure used was:

(i) Assume $\Delta C_1 = 0$, and determine $\gamma^{(1)}$ by equating equations (C-16) and (C-21). For this case $\gamma^{(1)} = 10.7$ degrees.

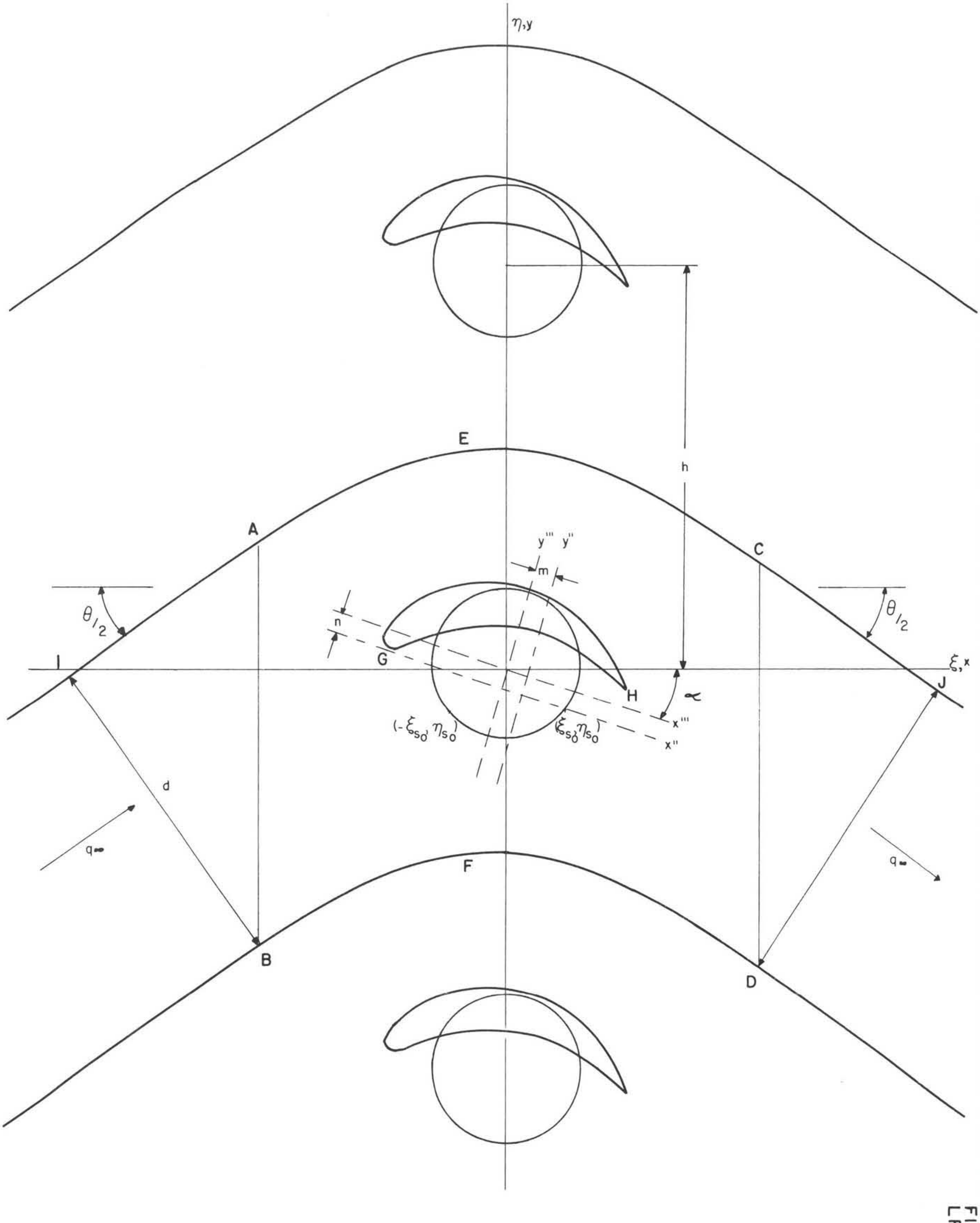
(ii) Compute the velocity distribution on the isolated aerofoil section using equation (C-17), taking $\gamma = \gamma^{(1)}$.

(iii) The cascade spacing, $h^{(1)}$, is then given by

$$h^{(1)} = \frac{d}{\cos\,\theta/2} = \frac{d}{\cos\,(\beta - \gamma^{(1)})} \qquad \text{(C-23)}$$

(iv) The procedure given in Reference 6 was then used to determine the value of $\Delta C_1^{(1)}$ for $\gamma = \gamma^{(1)}$. The calculations were made on an electronic digital computer using the contour integral method. Two iterations were required to determine $\Delta C_1^{(1)}$ to an accuracy of 0.02 (corresponding to an accuracy of about 0.2 degree in the value of γ or $\theta/2$).

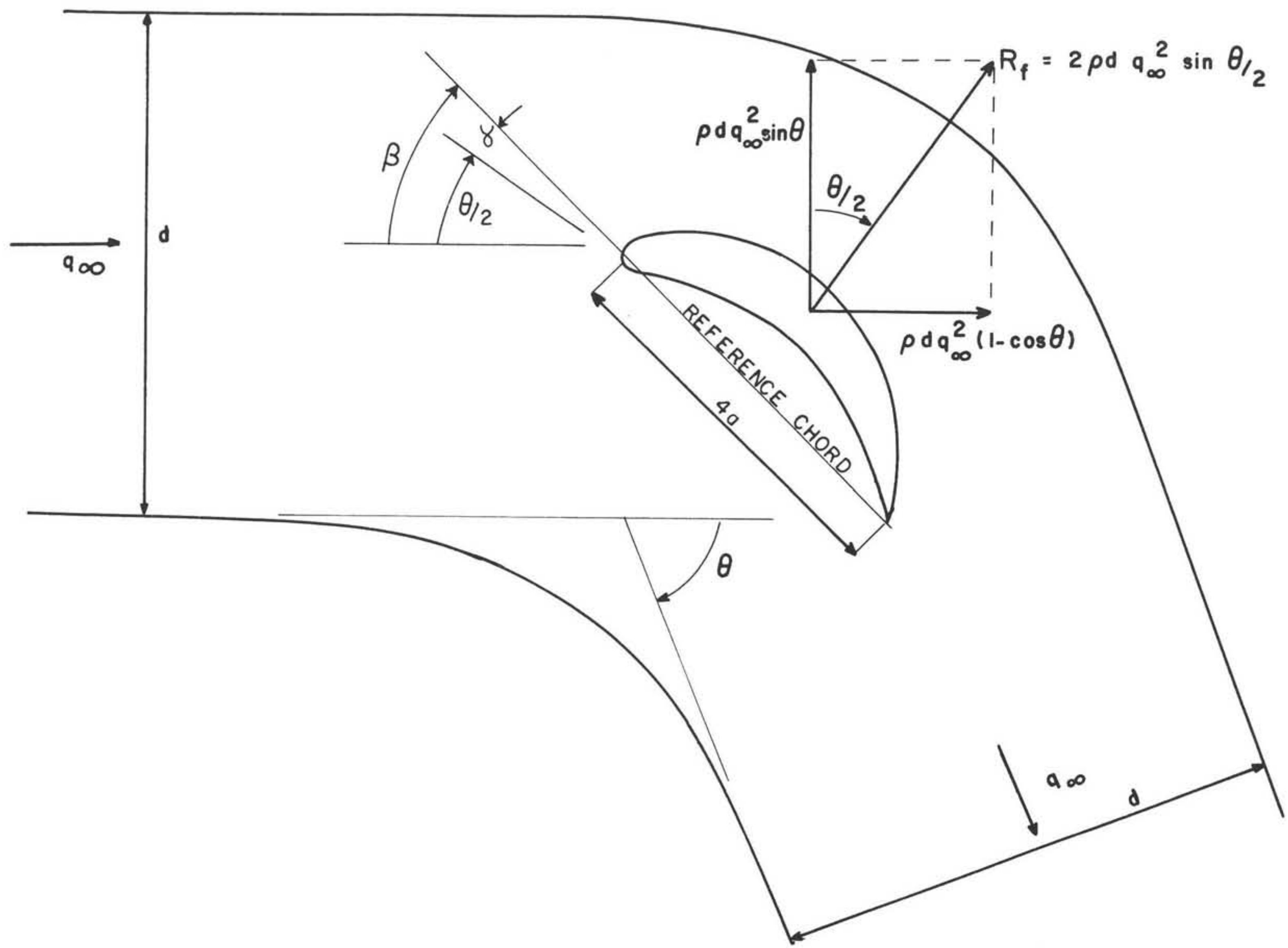
(v) Using this value of $\Delta C_1^{(1)}$, the second approximation to γ, $\gamma = \gamma^{(2)}$ was obtained using equations (C-16), (C-21), and (C-22). $\gamma^{(2)}$ was found to be 12.4 degrees. The correct value of γ is 11.85 degrees, so that the second step in the iteration gave an answer accurate to 0.55 degree in γ or 1.10 degrees in θ.



POTENTIAL FLOW THROUGH AN INFINITE SYMMETRICAL CASCADE

$\theta = 70°$

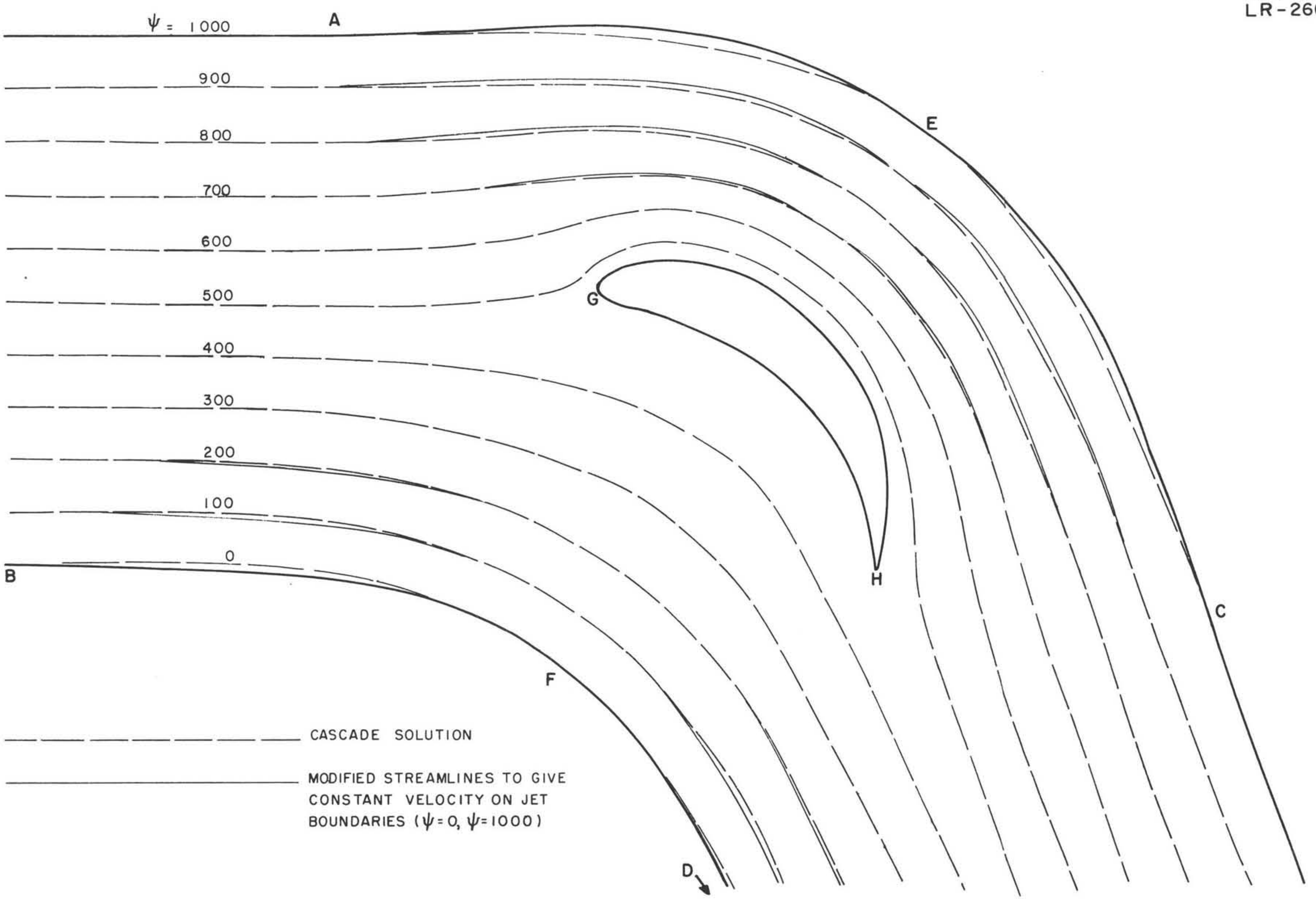
FIG. 1
LR-260



AERODYNAMIC FORCES ACTING ON AN AEROFOIL SECTION IN A FINITE STREAM

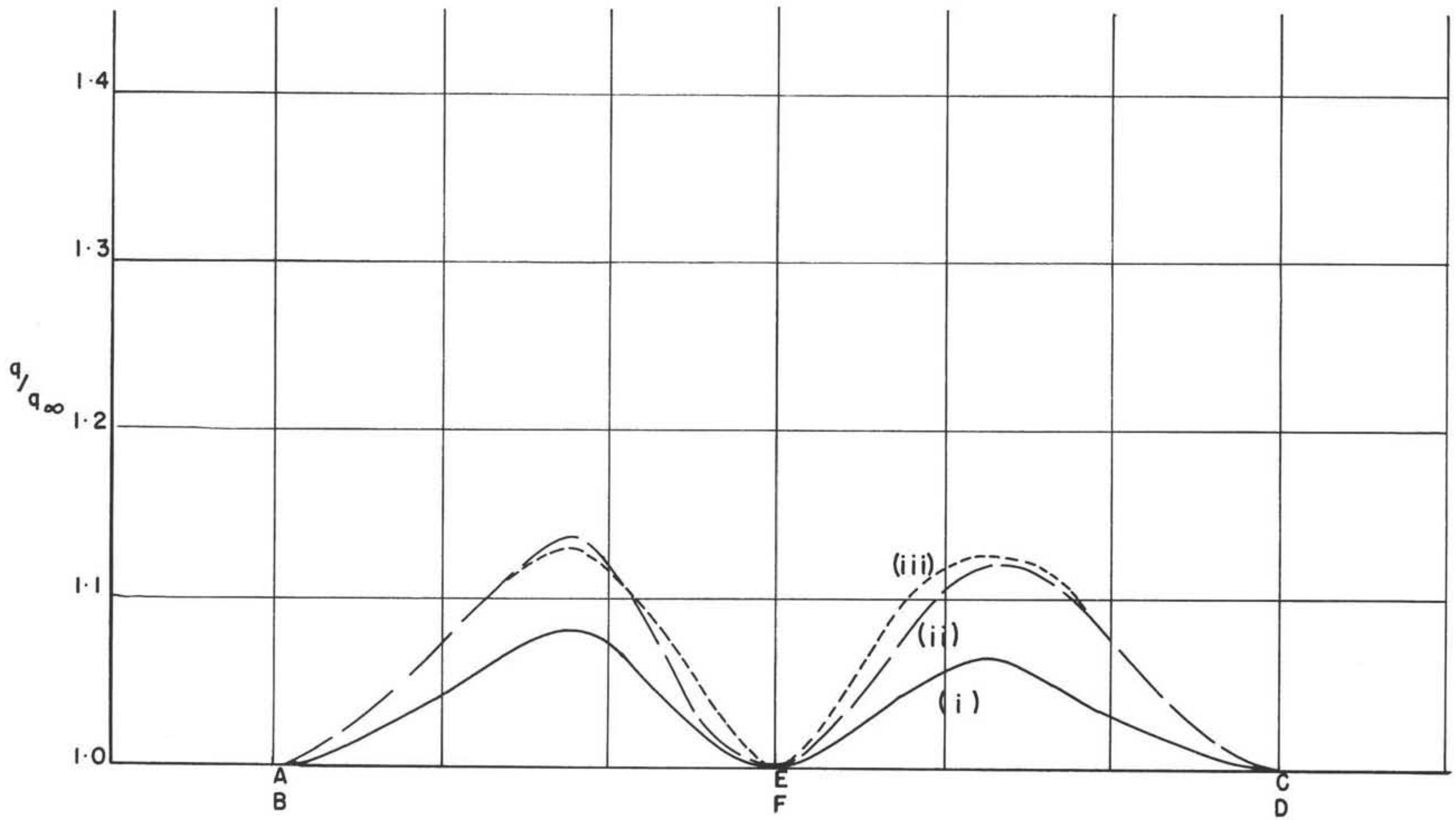
FIG. 2
LR-260

FIG. 3
LR-260



COMPARISON OF STREAMLINE PATTERNS OF AEROFOIL SECTION IN FINITE STREAM AND IN A SYMMETRICAL CASCADE

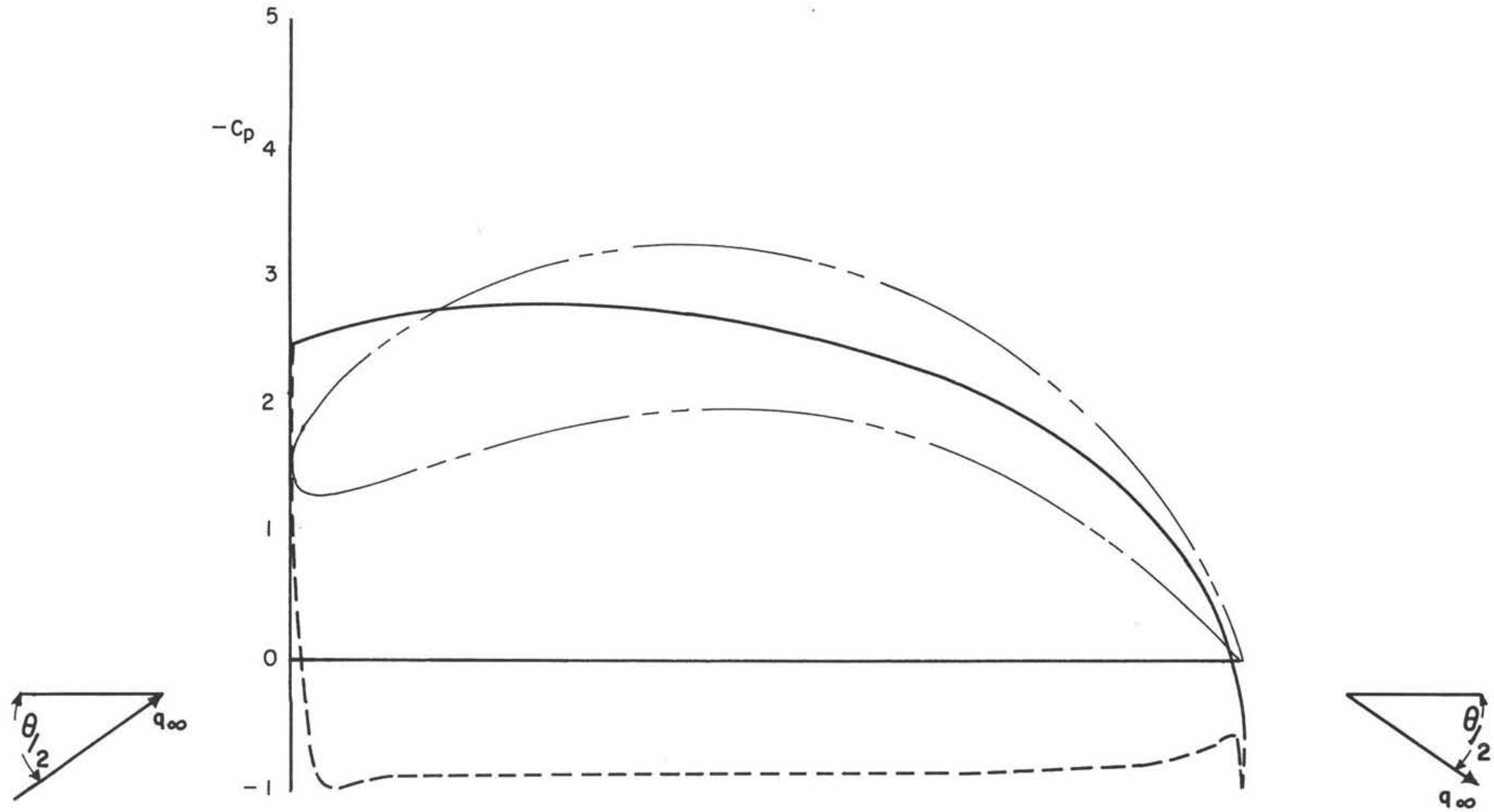
(PROFILE (i) IN APPENDIX A)



VELOCITY DISTRIBUTION ALONG STREAMLINES AEC (OR BFD)
FOR THREE SAMPLE PROFILES IN AN INFINITE CASCADE

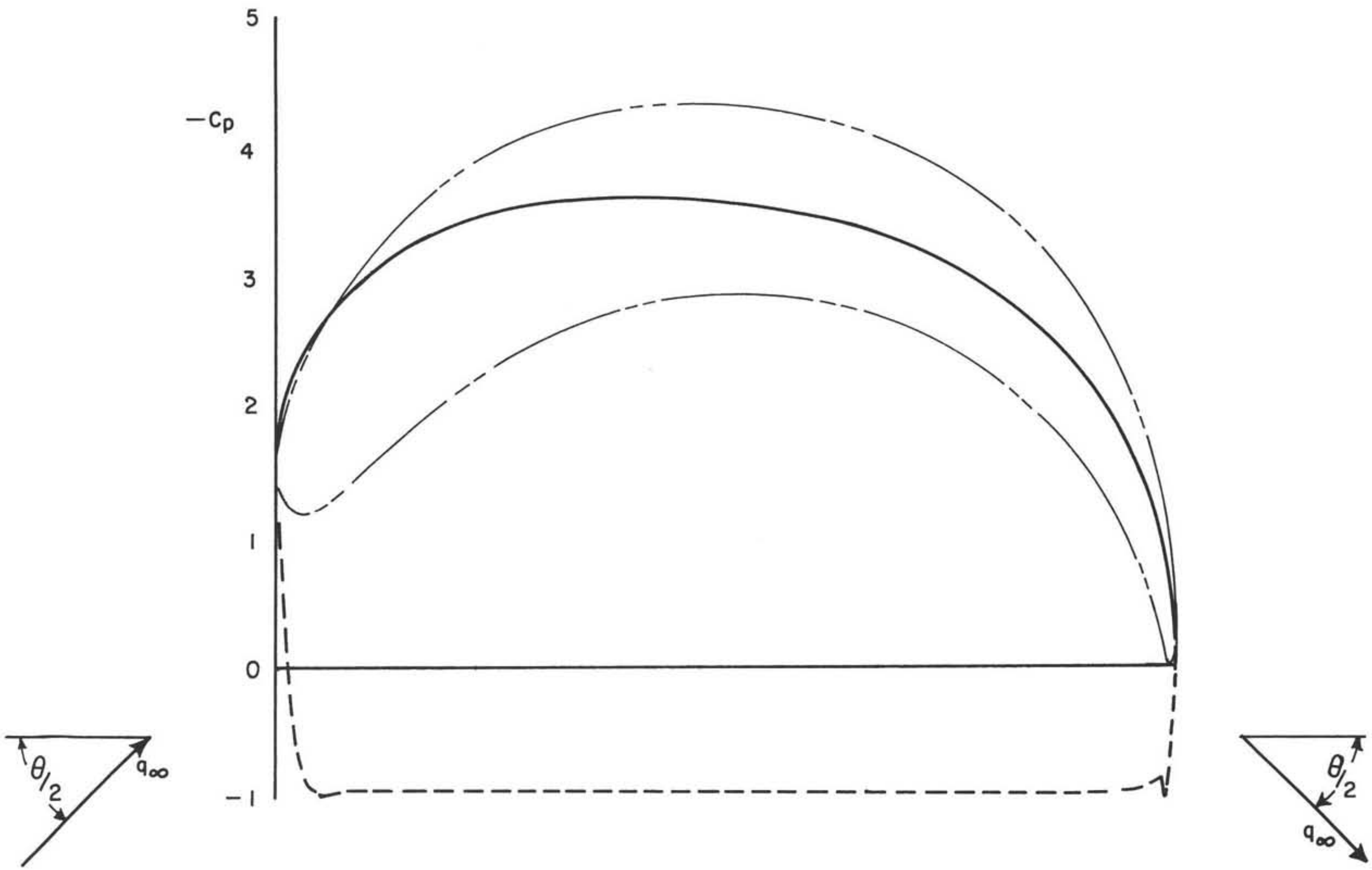
(SEE APPENDIX A)

FIG. 4
LR-260



PRESSURE DISTRIBUTION ON PROFILE (i)
$\theta = 70°$, $\alpha = 10°$ (SEE APPENDIX A)

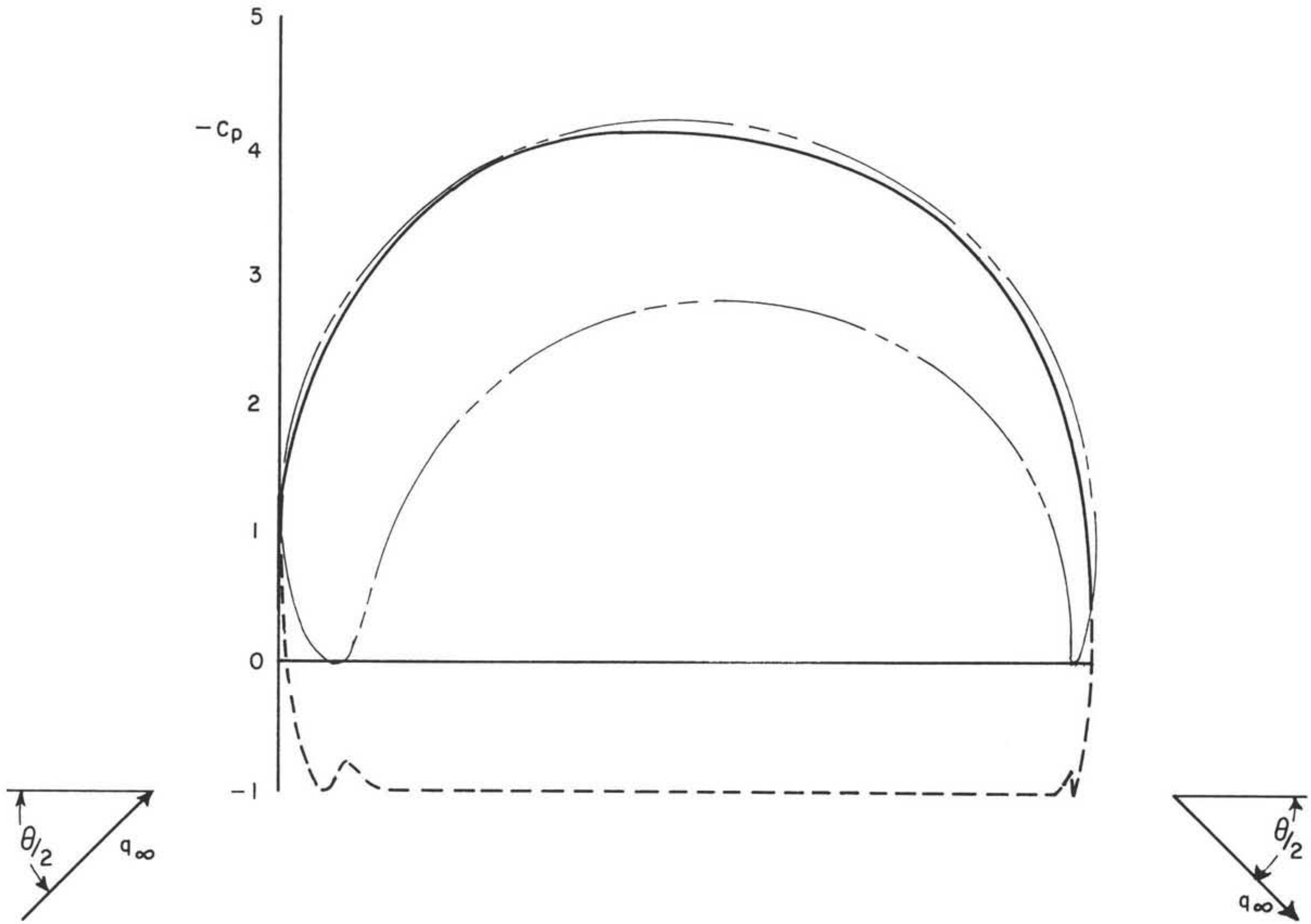
FIG. 5
LR-260



PRESSURE DISTRIBUTION ON PROFILE (ii)

$\theta = 90°$, $\alpha = 10°$ (SEE APPENDIX A)

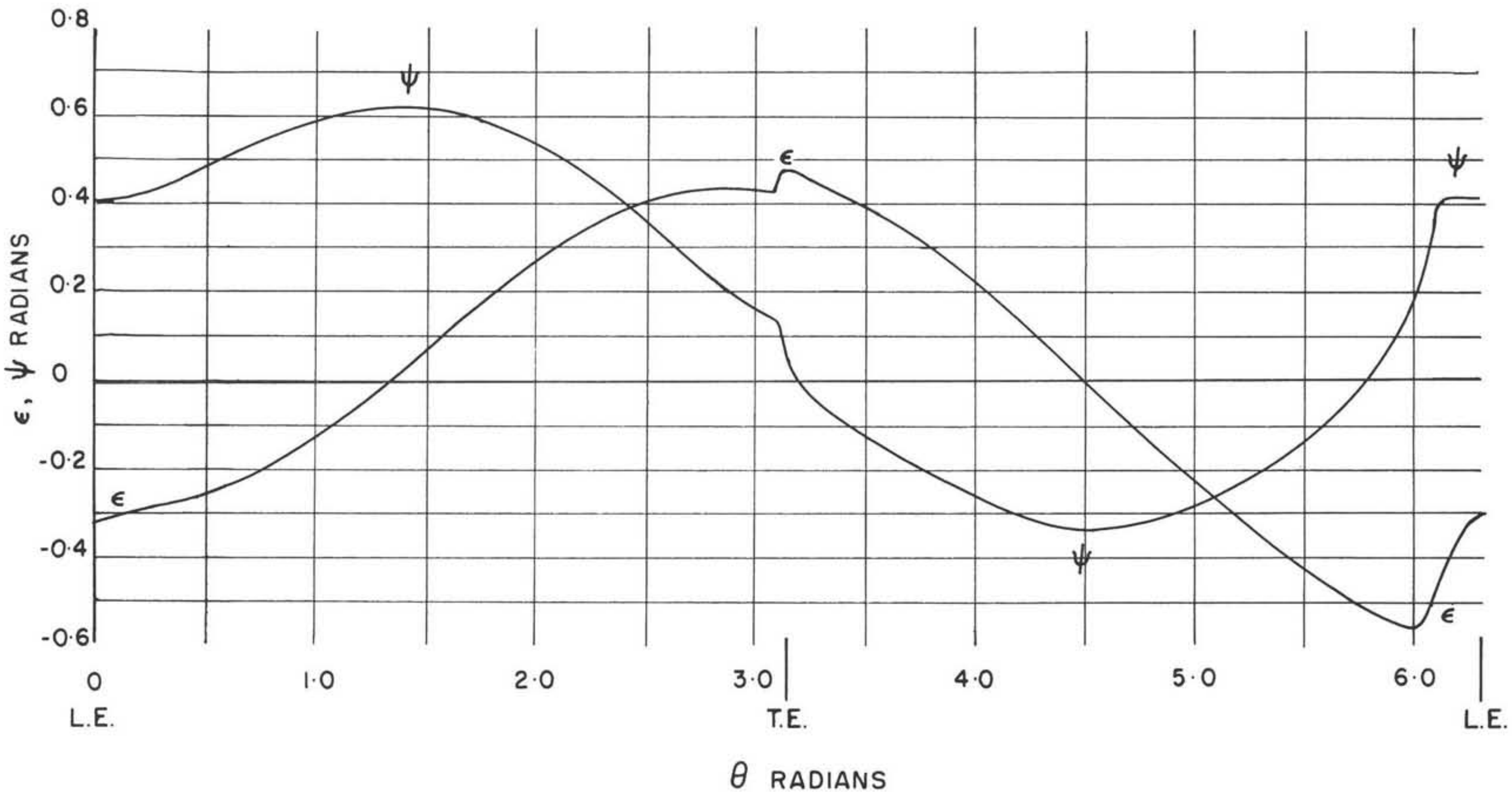
FIG. 6
LR-230



PRESSURE DISTRIBUTION ON PROFILE (iii)

$\theta = 90°$, $\alpha = 0°$ (SEE APPENDIX A)

FIG. 7
LR-260



VARIATION OF ψ AND ϵ WITH θ FOR PROFILE (i)

(SEE APPENDIX C)

FIG. 8
LR-260

NRC LR-260
National Research Council, Canada. National Aeronautical Establishment

APPROXIMATE METHOD FOR DETERMINING THE POTENTIAL FLOW ABOUT AN ARBITRARY AEROFOIL SECTION IN A TWO-DIMENSIONAL FINITE STREAM WITH PARTICULAR REFERENCE TO LARGE STREAM DEFLECTIONS
D.G. Gould. August 1959. 33 p. + 8 figs.

The flow about and on the surface of an aerofoil section in a finite stream and that about and on the surface of the same aerofoil section in an infinite symmetrical cascade are compared. It is shown that, for most cases of practical interest, the differences are small for stream deflection angles up to 90 degrees. Thus, the approximate flow about an arbitrary aerofoil section in a two-dimensional finite stream may be obtained by using the methods existing for the determination of the flow about an arbitrary section in a cascade. The procedure, as applied to the finite stream approximation, is given using the interference method of cascade theory.

UNCLASSIFIED

1. Aerofoils - Aerodynamic characteristics
2. Cascades

I. Gould, D.G.
II. NRC LR-260

NRC LR-260
National Research Council, Canada. National Aeronautical Establishment

APPROXIMATE METHOD FOR DETERMINING THE POTENTIAL FLOW ABOUT AN ARBITRARY AEROFOIL SECTION IN A TWO-DIMENSIONAL FINITE STREAM WITH PARTICULAR REFERENCE TO LARGE STREAM DEFLECTIONS
D.G. Gould. August 1959. 33 p. + 8 figs.

The flow about and on the surface of an aerofoil section in a finite stream and that about and on the surface of the same aerofoil section in an infinite symmetrical cascade are compared. It is shown that, for most cases of practical interest, the differences are small for stream deflection angles up to 90 degrees. Thus, the approximate flow about an arbitrary aerofoil section in a two-dimensional finite stream may be obtained by using the methods existing for the determination of the flow about an arbitrary section in a cascade. The procedure, as applied to the finite stream approximation, is given using the interference method of cascade theory.

UNCLASSIFIED

1. Aerofoils - Aerodynamic characteristics
2. Cascades

I. Gould, D.G.
II. NRC LR-260

NRC LR-260
National Research Council, Canada. National Aeronautical Establishment

APPROXIMATE METHOD FOR DETERMINING THE POTENTIAL FLOW ABOUT AN ARBITRARY AEROFOIL SECTION IN A TWO-DIMENSIONAL FINITE STREAM WITH PARTICULAR REFERENCE TO LARGE STREAM DEFLECTIONS
D.G. Gould. August 1959. 33 p. + 8 figs.

The flow about and on the surface of an aerofoil section in a finite stream and that about and on the surface of the same aerofoil section in an infinite symmetrical cascade are compared. It is shown that, for most cases of practical interest, the differences are small for stream deflection angles up to 90 degrees. Thus, the approximate flow about an arbitrary aerofoil section in a two-dimensional finite stream may be obtained by using the methods existing for the determination of the flow about an arbitrary section in a cascade. The procedure, as applied to the finite stream approximation, is given using the interference method of cascade theory.

UNCLASSIFIED

1. Aerofoils - Aerodynamic characteristics
2. Cascades

I. Gould, D.G.
II. NRC LR-260

NRC LR-260
National Research Council, Canada. National Aeronautical Establishment

APPROXIMATE METHOD FOR DETERMINING THE POTENTIAL FLOW ABOUT AN ARBITRARY AEROFOIL SECTION IN A TWO-DIMENSIONAL FINITE STREAM WITH PARTICULAR REFERENCE TO LARGE STREAM DEFLECTIONS
D.G. Gould. August 1959. 33 p. + 8 figs.

The flow about and on the surface of an aerofoil section in a finite stream and that about and on the surface of the same aerofoil section in an infinite symmetrical cascade are compared. It is shown that, for most cases of practical interest, the differences are small for stream deflection angles up to 90 degrees. Thus, the approximate flow about an arbitrary aerofoil section in a two-dimensional finite stream may be obtained by using the methods existing for the determination of the flow about an arbitrary section in a cascade. The procedure, as applied to the finite stream approximation, is given using the interference method of cascade theory.

UNCLASSIFIED

1. Aerofoils - Aerodynamic characteristics
2. Cascades

I. Gould, D.G.
II. NRC LR-260