Market risk calculations in stock- and bond prices

a garch-copula approach

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Abstract

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by Hendrik Pries

The financial crisis of 2008-2009 has led to more strict regulatory supervisory on banks and insurance companies, focusing on better (market) risk models. The linear correlation models did not foresee the extreme losses in asset values, because they were not able to forecast high volatile markets in which the dependence between financial assets seemed to increase. Research on copula theory, a tool to model more advanced dependence structures, predominately analyses the effect of the copula on highly dependent stock indexes, often using one underlying simulation model. This study compares two slightly dependent equity and bond prices for four different combinations of univariate simulation models, including Black-Scholes, Hull-White, GARCH and different residual models, three different copula models (Gaussian, Student \(t\) and flipped Gumbel) and two calibration methods. The goal is to measure the effect of different copula dependence models in economic scenario generators in combination with different simulation models and compare results in terms of accuracy, stability, resilience and complexity.

The main result is that due to the little dependence the impact of the copula model is limited and dependence in the copulas is small compared to the estimations based on stressed markets. Hence, for these low dependent portfolios the copulas do not have much added value.

Remarkable is that the dependence implied by the copula can strongly depend on the underlying model. The estimated dependence parameters of the copulas are lower for models using a GARCH volatility model. This can be explained by the non independent identically distributed residuals in the model without GARCH, i.e. the high volatile market periods lead to volatility clustering in the residuals. If this volatility clustering is not captured by the model, it can lead to amplification of the dependence due to misfitting of the univariate simulation models.

The differences in risk estimations are mainly caused by the choice of simulation models. The GARCH volatility model leads to an increase in the calculated market risk at a horizon of one year. The choice of residual model has large impact on the risk calculations in one day and (depending on the strength of the dependence) can have both negative and positive impact on the risk calculations in one year and the choice of simulation models should therefore be chosen with care.

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Abstract

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| **Market risk** | The effect of movements in equity prices and short rates on the 99.5%–value at risk and 97.5%–expected shortfall of an asset portfolio. |
| **Dependence structure** | (Non-linear) association between different Brownian motions in the univariate simulation models. |
| **Economic Scenario Generator** | The combination of a univariate stock price simulation model, univariate short rate simulation model, copula dependence structure and calibration method. |
| **Stability** | The sensitivity of the parameter calibration to changes in the data used for the calibration thereby leading to comparable market risk measurements for small changes in the calibration data. |
| **Stress testing** | The effect of shocks in the calibration on the calculation of market risk. |
| **Short rate** | The (continuously compounded, annualized) interest rate at which an entity can borrow money for an infinitesimally short period of time at time $t$. |
| **Residuals** | Given the model, and a set of calibrated parameters for which the variance of the stochastic component in the simulation model is minimal, historical realizations of this stochastic variable are obtained. These stochastic values leading to the historically observed price process are called the standardized residuals, or simply residuals. |
List of Abbreviations

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<td>AD test</td>
<td>Anderson-Darling test</td>
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<td>BS</td>
<td>Black Scholes</td>
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<td>cdf</td>
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<td>GARCH model</td>
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<td>GPD</td>
<td>Generalised Pareto Distribution</td>
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<td>HW</td>
<td>Hull White</td>
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<td>IFM</td>
<td>Inference Function for Margins</td>
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<tr>
<td>i.i.d.</td>
<td>independent and identically distributed</td>
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<td>KS test</td>
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The goal of this thesis is to compare the effect of dependence structures in Economic Scenario Generators (ESGs) on the measuring of market risk of an asset portfolio. Market risk models did not foresee the extreme losses in asset values during the financial crisis of 2008-2009, because they were not able to forecast high volatile markets in which the dependence between financial assets seemed to increase and hence led to more extreme losses than financial models had estimated as worst case scenarios.

We perform stochastic simulations in ESGs consisting of a stock price model and a short rate model, based on Stoxx 50 index prices and 3-month German bond rates. 10-years bond prices are calculated based on the short rate simulations and together with the stock prices an asset portfolio is constructed on which the market risk is calculated using a Monte Carlo approach. The dependence between these simulated stock prices and short rates is modelled with copulas, instead of the often used Pearson correlation. Copulas are a flexible way of modelling dependence, separating the dependence structure from the univariate simulation models and, contrary to Pearson correlation, allowing non-linear relations between the underlying processes. Hence, copulas should be more suited to estimate extreme losses which were observed historically. The question then is whether these copula based models lead to better market risk estimations, and what is the effect of the underlying simulation models in the ESGs on the dependence implied by the copula. This leads to the research question:

Do ESGs using more advanced copula dependence structures lead to more accurate, stable, resilient and less complex estimations of market risks and should they be preferred to Pearson correlation models which only model the linear association between variables? And what is the effect of the underlying simulation models in the ESGs on the dependence implied by a copula?

Four different combinations of univariate stock price and short rate models are compared (A–D). Stock prices are simulated using a log normal Black-Scholes model (A), combined with a generalized autoregressive conditional heteroskedasticity (GARCH) volatility model with normal (B) or Student’s $t$ (C and D) distributed residuals. The short rates are simulated using a Hull-White model (A), combined with a GARCH(1,1) volatility model with normal (B), Student’s $t$ (C) or Pareto tailed normal (D) residuals. The dependence between these processes is modelled with three different copulas; the Gaussian, Student $t$ and flipped Gumbel copula. Two calibration methods are compared, namely the Inference Functions for Margins method and the Canonical Maximum Likelihood method, both two step methods separating the calibration of the univariate simulation models from the calibration of the copulas.

The performance of the ESGs and value at risk and expected shortfall calculations is analysed in terms of accuracy, stability, resilience (effect of stress testing) and complexity. Due to the little dependence between the stock prices and short rates however, the effect of the copulas and
calibration methods on the risk calculations is very small and for all three copulas the market risk estimations are comparable.

The four different combinations of univariate models however do have a large impact on the market risk calculations and show large differences mainly in the risk estimations one year from now.

The main result is that due to the little dependence the impact of the copula model is limited and dependence in the copulas is small compared to the estimations based on stressed markets. Furthermore, it is remarkable is that the dependence implied by the copula can strongly depend on the underlying model. The estimated dependence parameters are lower for models using a GARCH volatility model. This effect is explained by the non independent identically distributed residuals in the model without GARCH, i.e. the high volatile market periods lead to volatility clustering in the residuals. If this volatility clustering occurs simultaneously in both historical data sets and if it is not captured by the model, it leads to amplification of the dependence effect.
1.1 Thesis outline

The thesis is outlined as follows (see Figure 1.1): Chapter 2 describes which data sets are used. The Stoxx 50 index, 3-month German bond yield and German bond swap curve are introduced and some statistics of the data are given. Chapter 3 introduces the log normal Black-Scholes stock price simulation model, the Hull-White short rate simulation model, the GARCH volatility model and three residual distribution models; the normal, Student’s $t$ and Pareto tailed normal. Chapter 4 explains how dependence between short rates and stock prices is modelled with copulas, including different methods to calibrate copulas, methods to compute empirical copulas and to simulate from copulas. Chapter 5 introduces measures to assess the performance of the different dependence models in calculating market risks. The performance is measured in terms of accuracy, stability, complexity and resilience, and also model risk is covered. Chapter 6 shows the results and Chapter 7 gives a conclusion and short discussion of the results. Finally, Chapter 8 provides recommendations for improvements and further research.

**Figure 1.1: Overview of thesis**
1.2 State of the art

This section summarizes some prominent articles about the topic of copulas and gives some insights in articles which are extensively used in this thesis.

Already back in 1959 Abe Sklar published one of the most important theorems about copulas, thereby providing the theoretical basis for the use of copulas. Sklar states that every multivariate distribution can be split in two marginal distribution combined with a copula model. In other words, each existing model can also be written in terms of a copula notation! In 1999 Roger B. Nelsen wrote the first edition of the book *An Introduction to Copulas* and the second edition appeared in 2007. Nelsen gives a useful overview of the basic theory about copulas, theorems behind copulas and a lot of different examples of copulas. Another more general overview, clearly explaining concepts of copula theory, is the master thesis written by Jeroen de Kort at Delft University of Technology in 2007.

Aas wrote in 2004 a survey of four copulas about modelling the dependence structure of financial assets. He describes the different correlation measures and the tail dependence coefficients to determine the dependence, and analyses the theoretical differences of the copulas. Others describe the effect of different copulas on real data examples. In 2005, Kole et all describe how the accuracy of copula models can be tested based on four different test statistics which are modifications of the univariate Kolmogorov-Smirnov and Anderson-Darling tests. They use the Gaussian, Student $t$ and Gumbel copula (the same copulas used here), but compare the results for dependence between stock, bond and real estate indices. Ngoga Kirabo Bob wrote a master thesis about combining the copulas with a GARCH model with an extreme value residual model, which is also analysed here. He compares five different copulas including the three copulas studied here, with a historical simulation approach and a variance-covariance method as a reference. The study is performed on a portfolio consisting of stock indices from Germany, Spain, Italy and France. Three different backtests are used to backtest the value at risk of the portfolio values historically by using a rolling time window. The data seems to be highly dependent if we look at the timeseries plot. However, it seems illogical to me that the calibrations of the Gumbel and Frank copula in this report are in fact equal to the independence copula and the correlation matrices (average correlation of 5%) also do not seem to match the historical observed behavior. Hence, his study is very interesting but some further research could be promising. Other articles testing the combination of GARCH volatility models with copulas often compare highly dependent stock indices, where they mostly do not analyse the effect of the choice of univariate simulation models and copula combined. This is the area in which our focus lies: the combined behavior of copulas and underlying models.

Thomas Mikosch wrote in 2006 an article *Copulas: tales and facts* in which he wants to address the shortcomings of copula models. From 2003 until the financial crisis in 2008-2009, the popularity of copulas rose enormous, however he argues most just use the concept without discussing the pros and cons. He summarizes with a list of seven remarks about copulas to indicate his concerns about the extensive blind use of copulas.

Below a short summary of some relevant articles mentioned above is given.

1.2.1 Testing copulas to model financial dependence - Kole et al. *(2005)*

In the article, the authors suggest four tests to determine which copula should be used to model financial dependence. The different tests they use are modifications of the Kolmogorov-Smirnov and Anderson-Darling tests, and the copulas they use are the Gaussian, Student $t$ and Gumbel copula. The data which is used in the article consists of daily returns between January 1, 1999 to December 17, 2014, from Standard & Poor’s 500 Composite Index (stocks), JP Morgan’s US Government Bond Index (bonds) and the NAREIT All Index (real estate). The used approach is as follows.
1. In the estimation step, the parameters of the marginal distributions and those of the copula are estimated, using the one-step or two-step method. The one-step method uses maximum likelihood to jointly estimate the parameters of the marginal models and the copula. The two-step method which is called IFM (Inference Functions for Margins, Joe and Xu, 1997), first estimates the marginal parameters individually and then estimates the copula parameters with the marginal distribution parameters treated as given. This IFM method is less efficient, but computationally more attractive and allows larger flexibility in the estimation techniques for the marginal models.

2. In the evaluation step, the fit of the copula is evaluated using four distance measures. Those measures are the Kolmogorov-Smirnov test (mean and max absolute error between empirical copula and hypothesised copula) and the Anderson-Darling test (which is sort of the scaled version of the KS test, which gives a higher weight to the tails and therefore catches tail dependence better).

\[
D_{KS} = \max_t |F_E(x_t) - F_H(x_t)|; \\
D_{KS} = \int_x |F_E(x_t) - F_H(x_t)| dF_H(x); \\
D_{AD} = \max_t \frac{|F_E(x_t) - F_H(x_t)|}{\sqrt{F_H(x_t)(1 - F_H(x_t))}}; \\
D_{AD} = \int_x \frac{|F_E(x_t) - F_H(x_t)|}{\sqrt{F_H(x_t)(1 - F_H(x_t))}} dF_H(x).
\] (1.1)

The cumulative distribution function of elliptical distributions are generally not available in closed form, and calculation of the hypothesized probabilities will be computationally demanding if the number of dimensions increases. If the copula belongs to the elliptical family, they propose a transformation based on the property that the density functions of elliptical distributions are constant on ellipsoids.

3. In the simulation step, the authors test whether the distance measures provide evidence for rejecting the fit of the copula. Therefore they construct the distribution of the distances under the null hypothesis. For simulation, they generate a random sample of size \(T\) from the copula with parameters \(\hat{\theta}\). For each simulation they calculate the new estimate for \(\theta\). Combined, the simulations result in a distribution of random variables corresponding to \(d_{KS}, d_{KS}, d_{AD}\) and \(d_{AD}\).

4. Finally, in the test step they use the distribution that results from the simulation step to calculate the \(p\)-value. \(p\)-values below a certain threshold lead to rejection of the fit of the copula on that sample.

Three copulas are tested: the Gaussian, Student \(t\) and Gumbel copula. A detailed inspection of the tails reveals that the Student \(t\) copula accurately captures the risk of joint downside movements, while it is underestimated by the Gaussian and overestimated by the Gumbel copula.

1.2.2 Copulas and tail dependence - Kort (2007b)

This paper is the literature study of De Kort, on which his thesis report follows. The main research question of this study is: How to incorporate tail dependence in the pricing of hybrid products?

Sections 1 and 2 explain what copulas are in a mathematical way, using \(H\)-volumes, definitions of grounded, 2-increasing, sub-copulas, Sklar’s theorem, Fréchet-Hoeffding bounds, survival copulas and multivariate copulas. In Section 3 it is described what kind of dependence is captured by copulas. This, among other things, includes measures of concordance like Kendall’s tau, Spearman’s rho and tail dependence.
Well-known parametric families of copulas

Next, Section 4 summarizes the properties of a number of well-known parametric families of copulas, such as the Fréchet family. This copulas consist of linear combinations of the product copula $C^1(u, v)$ and the Fréchet-Hoeffding upper and lower bounds $C^{-}(u, v)$ and $C^{+}(u, v)$, where the product copula models independence, whereas the Fréchet-Hoeffding upper and lower bounds 'add' positive and negative dependence respectively. The weight of the lower bound is $p \in [0, 1]$, the weight of the upper bound is $q \in [0, 1]$ and the weight of the independence copula is then $1 - p - q$. Spearman’s rho is $\rho_{SC} = q - p$ for this class, thereby confirming the intuition of this dependence construction.

The elliptical distributions, whose density function (if it exists) equals

$$f(x) = \left| \sum -\frac{1}{2} x' \Sigma^{-1} x \right| \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} e^{-\frac{1}{2} x' \Sigma x}, \quad x \in \mathbb{R}^n,$$

where $\Sigma$ (dispersion) is a symmetric positive semi-definite matrix, $\mu \in \mathbb{R}^n$ (location) and $g$ (density generator) is a $[0, \infty) \to [0, \infty)$ function. Taking $g(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} x^2)$ yields the Gaussian distribution and $g(x) = (1 + \frac{x^2}{2})^{-\frac{1}{2}}$ leads to a Student’s $t$ distribution with $\nu$ degrees of freedom. Those elliptic distributions are upper and lower tail dependent (symmetric, so both tails or no tail) if the tail of their density generator is a regularly varying function with $\alpha < -n/2$. A function is called regularly varying with index $\alpha$ if for every $t > 0$: \[\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^\alpha.\] I.e. regularly function have tails that behave like power functions.

The bivariate Gaussian and Student $t$ copula are examples of elliptical distributions. The Gaussian copula is defined as

$$C^{G\alpha}(u, v) = \Phi_p(\Phi^{-1}(u), \Phi^{-1}(v)),$$

where

$$\Phi_p(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)} \right) ds dt,$$

and $\Phi$ denotes the standard normal cumulative distribution function.

The univariate Student $t$ copula $t_{\nu}$ with $\nu$ degrees of freedom is defined as:

$$t_{\nu}(x) = \int_{-\infty}^{x} \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu} \Gamma(\nu/2)} \left(1 + \frac{s^2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+1}{2}} ds,$$

where $\Gamma$ is Euler function. Let $t_{\rho, \nu, p} \in [0, 1]$ denote the bivariate distribution corresponding to $t_{\nu}$:

$$t_{\rho, \nu} = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{2\pi \sqrt{1-\rho^2}} \left(1 + \frac{s^2 + t^2 - 2\rho st}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{4}} ds dt$$

The bivariate Student $t$ copula $T_{\rho, \nu}$ is defined as

$$T_{\rho, \nu} = t_{\rho, \nu}(t_{\nu}^{-1}(u), t_{\nu}^{-1}(v)),$$

The generator for the Student $t$ copula is regularly varying:

$$\lim_{x \to \infty} \frac{g(tx)}{g(x)} = \lim_{x \to \infty} \left(1 + \frac{tx}{\nu}\right)^{-\frac{\nu+2}{4}} \left(1 + \frac{x}{\nu}\right)^{-\frac{\nu+2}{4}} = \lim_{x \to \infty} \left(\frac{\nu+x}{\nu+tx}\right)^{\frac{\nu+2}{4}} = t^{\frac{\nu+2}{4}}. \quad (1.6)$$

If follows that the Student $t$ distribution has tail dependence for all $\nu > 0$.

The Archimedean copulas exist in one- and two-parameter and multi-variate form. We have the Gumbel (-Hougaard) $\phi(u) = (-\log(u))^\theta$, $\theta \in [1, \infty)$, or the Clayton, Gumbel-Barnett and Frank copula. The Fréchet-Hoeffding lower bound $C^-$ is Archimedean ($\phi(u) = 1 - u$), whereas the Fréchet-Hoeffding upper bound is not. To see this, note that $\phi^{-1}$ is strictly decreasing on $[0, \phi(0)]$. Clearly, $2\phi(u) > \phi(u)$, so we have for the diagonal section of an Archimedean copula
Chapter 1. Introduction

that:
\[ C^A(u,u) = \phi^{-1}(2\phi(u)) < \phi^{-1}(\phi(u)) = u. \] (1.7)

As \( C^+(u,u) = u \), inequality 1.7 implies that \( C^+ \) is not Archimedean.

For the two-parameter families of the Archimedean copulas we have that as \( \phi \) is a strict generator, then also \( \phi(t^\alpha) \) (interior power family) and \( \phi(t)^\beta \) (exterior power family) are strict generators for \( \alpha \in (0,1] \) and \( \beta \geq 1 \). When \( \phi \) is twice differentiable, then the interior power family is a strict generator for all \( \alpha > 0 \). We can construct the two-parameter families by taking
\[ \phi_{\alpha,\beta} = [\phi(t^\alpha)]^\beta \] (1.8)
as the generator function.

Extensions of Archimedean copulas can be made by noticing that Archimedean copulas are associative:
\[ C(u,v,w) = \phi^{-1}(\phi(u) + \phi(v) + \phi(w)) \]
\[ = \phi^{-1}(\phi \phi^{-1}(\phi(u) + \phi(v)) + \phi(w)) \]
\[ = C_1(C_2(u,v),w). \]

We can then choose different copulas \( C_1 \) and \( C_2 \) to improve the fit of the total copula.

**Calibration of copulas from market data**

In Chapter 5, some calibration techniques are outlined. The author looks at the Maximum Likelihood method, the IFM method and the CML method.

The Maximum likelihood method. Let \( X_i \sim F_i \), \( 1 \leq i \leq n \), be random variables with joint distribution \( H \). From the multi-D Sklar’s theorem we know there exists a copula \( C \) such that
\[ H(u_1,...,u_n) = C(F_1(u_1),...,F_n(u_n)). \]

Differentiation with respect to \( u_1, u_2, ..., u_n \) sequentially yields the canonical representation
\[ h(u_1,...,u_n) = c(F_1(u_1),...,F_n(u_n))\prod_{i=1}^n f_i(u_i), \] (1.9)
where \( c \) is the copula density.

The maximum likelihood implies choosing \( C \) and \( F_1,...,F_n \) such that the probability of observing the data set is maximal. The possible choices for the copula and the margins are (both) unlimited. Therefore we restrict ourselves to certain classes of functions (copulas), parametrized by some vector \( \theta \in \Theta \subset \mathbb{R}^n \). We should thus find \( \theta \in \Theta \) that maximizes the likelihood
\[ l(\theta) := \prod_{i=1}^T \left( c(F_1(x_{1i}),...,F_n(x_{ni});\theta)\prod_{i=1}^n f_i(u_{id};\theta) \right). \]

This \( \theta \) also maximizes the log-likelihood, which is often computationally more convenient. If the derivative of \( l(\theta) \) exists, then the solutions of
\[ \frac{\partial l(\theta)}{\partial \theta} = 0 \]
are possible candidates for the maximum likelihood estimator \( \theta_{MLE} \).

The Inference Functions for Margins (IFM) method by Joe and Xu, proposes estimation of \( \theta \) in two steps: first the margins’ parameters and then the copulas’. This significantly reduces the computational cost of finding the optimal set. The IFM estimator is consistent (can be shown), though not exactly the same as the MLE.
In the Canonical maximum Likelihood (CML) method, first the margins are estimated using empirical distributions $\hat{F}_1, \ldots, \hat{F}_n$. Then the copula parameters are estimated using an ML approach:

$$\theta_{\text{CML}} := \arg\max_{\theta} \sum_{t=1}^{T} \log c(\hat{F}_1(x_{1t}), \ldots, \hat{F}_n(x_{nt}); \theta).$$

Option pricing with copulas

Section 6 finally describes how prices of hybrid contracts (e.g. digital options, best-of contracts) can be expressed in terms of copulas.

1.2.3 Modelling the dependence structure of financial assets: A survey of four copulas - Aas (2004)

In Chapter 1 the author states that only for elliptical distributions such as the Gaussian or Student’s $t$, the linear correlation coefficient is a meaningful measure of dependence. Outside the world of elliptical distributions, however, the use of the linear correlation coefficient as a measure of dependence may induce misleading conclusions.

In Chapter 2 the (very short) definition of the copula is introduced, as being the joint distribution function of uniform margins.

Examples of copulas (implicit & explicit)

In Chapter 3, based on whether they are implicit or explicit, some examples of copulas are introduced. It is common to represent a bivariate copula by its distribution function

$$C(u, v) = P(U \leq u, V \leq v) = \int_{-\infty}^{u} \int_{-\infty}^{v} c(s, t)dsdt,$$

where $c(s, t)$ is the density of the copula. The article concentrates on two parametric families of copulas; the copulas of normal mixture distributions (implicit) and Archimedean copulas (explicit). For the so-called implicit copulas, the double integral at the right-hand side of Equation 1.10 is implied by a well-known bivariate distribution function, while the latter are explicit copulas, for which this integral has a simple closed form.

The Gaussian copula is an implicit copula, given by

$$C_{\rho}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp \left\{ -\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right\} dx dy,$$

where $\rho$ is the parameter of the copula, and $\Phi^{-1}(\cdot)$ is the inverse of the standard univariate Gaussian distribution function.

The other implicit copula the article considers is the Student $t$ copula, which allows for fat tails

$$C_{\rho,\nu}(u, v) = \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{ 1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1-\rho^2)} \right\}^{-(\nu+2)/2} dx dy,$$

with $\rho$ and $\nu$ the parameters of the copula, $t_{\nu}^{-1}$ the inverse of the standard univariate student-t-distribution with $\nu$ degrees of freedom, expectation 0 and variance $\frac{\nu}{\nu-2}$. With increasing degree of freedom $\nu$, the tendency to exhibit extreme co-movements decreases.
The explicit copulas which are mentioned are the Clayton copula and the Gumbel copula. The Clayton copula allows also for asymmetries in the extreme tail events, exhibiting greater dependence in the negative tail than in the positive tail and is given by

$$C_\delta(u, v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta},$$

where $0 < \delta < \infty$ is a parameter controlling the dependence. When $\delta \to \infty$ we have perfect dependence, when $\delta \to 0$ we have independence.

The Gumbel copula is also asymmetric, but it exhibits greater dependence in the positive tail than in the negative tail, it is given by

$$C_\delta(u, v) = \exp\left(-[(-\log u)^\delta + (-\log v)^\delta]^{1/\delta}\right),$$

with $0 < \delta \leq 1$ again a parameter controlling the dependence. Here $\delta \to 0$ implies perfect dependence, while $\delta = 1$ implies independence.

**Dependence measures**

In Chapter 4 the dependence measures Kendall’s tau and Spearman’s rho, and tail dependence are explained. Kendall’s tau $\rho_T$ can be related to the dependence parameter $\delta$. For the Clayton copula, this relation is given by

$$\rho_T(X, Y) = \frac{\delta}{\delta + 2} \text{ Clayton}$$

$$\rho_T(X, Y) = 1 - \frac{1}{\delta} \text{ Gumbel}$$

For the Gaussian and Student $t$, Kendall’s tau must be estimated empirically.

Spearman’s rank correlation coefficient (or Spearman’s rho) is (for two variables $X$ and $Y$) is given by

$$\rho_S(X, Y) = 12 \int_0^1 \int_0^1 C(u, v) dudv - 3$$

$$= \rho_P(F(X), F(Y)).$$

For the Gaussian and Student $t$ copulas, the relationship between the linear Pearson correlation coefficient $\rho_P$ and Spearman’s rho $\rho_S$ is

$$\rho_P(X, Y) = 2 \sin\left(\frac{\pi}{6} \rho_S\right).$$

Pearson correlation only measures linear dependence, whereas both $\rho_T(X, Y)$ and $\rho_S(X, Y)$ may be considered as measures of the degree of monotonic dependence between $X$ and $Y$. According to Embrechts et al. (1999), one should look more detailed knowledge of the risk management problem, like using tail dependence instead of summarizing dependence with a ‘dangerous single number’ (like linear or rank correlation).

The coefficient for (upper) tail dependence is

$$\lambda_u(X, Y) = \lim_{\alpha \to 1} P(Y > F_Y^{-1}(\alpha)|X > F_X^{-1}(\alpha)),$$

and $\lambda_u(X, Y) = \lambda_l(X, Y)$ holds for elliptical distributions, when $\lambda_u(X, Y) > 0$ we have positive tail dependence (heavy tails), when $\lambda_u(X, Y) = 0$ we have no tail dependence. Asymptotic independence should be considered as the ‘weakest dependence which can be quantified by the coefficient of tail dependence’.
For Gaussian copulas we have that the upper and lower tail dependence coefficients are zero. For the Student \( t \) copula we find
\[
\lambda_l(X, Y) = \lambda_u(X, Y) = 2t_{\nu+1} \left( -\sqrt{\nu + 1} \frac{1 - \rho}{1 + \rho} \right).
\]

Surprisingly the Student \( t \) copula given asymptotic dependence in the tail, even when \( \rho \) is negative or zero.

The Clayton copula is lower tail dependent, with
\[
\lambda_u(X, Y) = 0,
\lambda_l(X, Y) = 2^{1/\delta}.
\]

The Gumbel copula is upper tail dependent, with
\[
\lambda_l(X, Y) = 0,
\lambda_u(X, Y) = 2 - 2^{1/\delta}.
\]

The problem is that the tail dependence coefficient is not easily found for an empirical data set, and there is not yet any reliable estimator to estimate the coefficient of tail dependence, mainly due to the curse of dimensionality and drastic decrease of the number of realisations as \( \alpha \) becomes close to one.

**Estimation of copula parameters**

In Chapter 5, the estimation of copula parameters is explained. There are mainly two ways of doing this; a fully parametric method or a semi parametric method (IFM, Joe, 1997), and for this second method we can choose to use the parametric univariate cumulative distribution functions, or the empirical ones (Demarta and McNeil, 2004). In this method, after having obtained the pseudo-samples, they use maximum likelihood to estimate the copula parameters.

For the Gaussian (one parameter, namely the correlation matrix \( R \)) copula, we can simply use maximum likelihood, by maximizing the likelihood for all possible \( R \), given the sample. For low dimensions this is feasible, but it is very slow in high dimensions. They suggest to use a maximization over the set of all covariance matrices, because this problem has an analytical solution:
\[
\hat{\sum} = \frac{1}{N} \sum_{i=1}^{n} \left( \Phi^{-1}(U_{i,1}), \ldots, \Phi^{-1}(U_{i,d}) \right)^t \left( \Phi^{-1}(U_{i,1}), \ldots, \Phi^{-1}(U_{i,d}) \right).
\]

This matrix is likely to be close to being a correlation matrix, but it does not necessarily have ones at the diagonal. We can force this by
\[
\hat{R} = \Delta^{-1} \hat{\sum} \Delta^{-1},
\]
where \( \Delta \) is a diagonal matrix containing the diagonal elements of \( \hat{\sum} \).

For the Student \( t \) copula, the estimation of parameters is not that easy, because we require numerical optimisation of the log-likelihood function. We have to maximize over the correlation matrix \( R \) and over the degrees of freedom \( \nu \) simultaneously.

Two simpler approaches are suggested. Mashal and Zeevi (2002) and Demarta and McNeil (2004), use a two-stage procedure in which \( R \) is estimated first using Kendall’s tau, and then the pseudo-likelihood function is maximized with respect to \( \nu \). Bouyé et al. (2000, p. 41-42) do the opposite and propose an algorithm to estimate \( \hat{R} \), when \( \nu \) is known (however they do not say anything about how \( \nu \) is determined). The authors show that the maximum likelihood...
estimate of the correlation matrix then is given by
\[ \hat{R}_{m+1} = \frac{1}{n} \left( \nu + \frac{2}{\nu} \right) \sum_{i=1}^{n} \frac{X_i X_i'}{1 + \frac{1}{\nu} X_i' \hat{R}_{m}^{-1} X_i}. \]

This equation is iterated until convergence is obtained. Bouyé et al. (2000) use the maximum likelihood estimate of the correlation matrix for the Gaussian copula as the starting value. According to Mashal and Zeevi (2002), this procedure is computationally intensive and suffers from numerical stability problems arising from the inversion of close to singular matrices. Moreover, it still remains to find an appropriate value for \( \nu \). They therefore state that the first method given in this section should be used.

For the Clayton copula you could just find \( \delta \) by numerical maximization of the log-likelihood, or in the bivariate case a simpler approach can be used. This approach uses the relationship between (the empirically found) Kendall’s tau and the copula parameter \( \hat{\delta} = \frac{2 \hat{\rho}_\tau}{1 - \hat{\rho}_\tau} \).

The parameter \( \delta \) for the Gumbel copula is again estimated using maximum likelihood, see Bouyé et al. (2000, p. 56), but a simpler approach is to use the relationship between Kendall’s tau and \( \delta \), as we did for the Clayton copula.

For the choice of the type of copula, the authors state that, while for the 1-D case a lot of well-known distribution independent Goodness of Fit statistics are available (Kolmogorov-Smirnov, Anderson-Darling), it is more difficult to build distribution independent GOF tests in the multi-D framework. Alternatively they choose the copula that minimizes the distance to the empirical copula of the data (Romano, 2002). The best copula \( C \) can be chosen as the one that minimizes the distance
\[ d(C, \hat{C}) = \sqrt{\sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} \left( C\left(\frac{i_1}{n}, \cdots, \frac{i_d}{n}\right) - \hat{C}\left(\frac{i_1}{n}, \cdots, \frac{i_d}{n}\right)\right)^2}. \]

Finally, a more informal test involves comparing the plot of the estimated parametric copula with the one of a copula function estimated non-parametrically via kernel methods (Fermanian and Scaillet, 2002).

### Simulating from copulas

In Chapter 6 the authors explain how we can simulate from the different copulas.

For the implicit Gaussian copula, the easiest way (given the correlation matrix \( R \)) is to simulate \( X \sim \mathcal{N}_d(0, R) \) (Embrechts et al., 2003), and then using the cumulative distribution function to find \( U = (\Phi(X_1), \ldots, \Phi(X_d)) \).

For the implicit Student \( t \) copula, we simulate \( X \sim t_d(\nu, 0, R) \) and again we set \( U = (t_\nu(X_1), \ldots, t_\nu(X_d)) \).

For the explicit copulas some more work has to be done. If we write the copula as
\[ C(u_1, \ldots, u_d) = \phi^{-1}(\phi(u_1) + \ldots + \phi(u_d)), \]
then we can simulate as follows (Marshall and Olkin, 1988). First simulate a variate \( X \) with distribution function \( G \) such that the Laplace transform of \( G \) is the inverse of the generator. Then simulate \( d \) independent variates \( V_1, \ldots, V_d \), and return \( U = (\phi^{-1}(-\log(V_1)/X), \ldots, \phi^{-1}(-\log(V_d)/X)). \)

Hence for Clayton we have that \( \phi(t) = (t^{-\delta} - 1) \) and \( \phi^{-1}(t) = (t + 1)^{-1/\delta} \).
Chapter 1. Introduction

The inverse of the generator is equal to the Laplace transform \( \hat{G}(t) = \int_0^\infty e^{-tx}dG(x), \ t \geq 0 \) of a Gamma variate \( X \sim Ga(1/\delta, 1) \) (Frees and Valdez, 1998). And we can simulate a gamma variate \( X \sim Ga(1/\delta, 1) \), then simulate \( d \) independent standard uniforms \( V_1, ..., V_d \) and we return
\[ U = (((1 - \log V_1)^{-1/\delta}, ..., (1 - \log V_d)^{-1/\delta}). \]

For the Gumbel copula we have that
\[ \phi(t) = (- \log(t))^\delta \text{ and } \phi^{-1}(t) = \exp(-t^{1/\delta}). \]

The inverse is equal to the Laplace transform of a positive stable variate \( X \sim St(1/\delta, 1, \gamma, 0) \), where \( \gamma = (\cos(\pi/2\delta))^{\delta} \) and \( \delta > 1 \) (Frees and Valdez, 1998). The article uses the parametrisation and simulation algorithm proposed by Nolan (2005).

They also give an example of fitting different copulas on a data set and simulating data from this copulas. Unfortunately the article does not compare or explain the results, the only ’output’ is the plot of the copula simulations.

1.2.4 Copulas: tales and facts - Mikosch (2006)

Mikosch describes the explosion of activity about copulas starting around 2003. He is however critical about the ease of people accepting and using the concept of copulas extensively without really understanding the pros and cons of the concept.

He summarizes his concerns as follows:

- There is no particular advantage of using copulas when dealing with multivariate distributions. Instead one can and should use any multivariate distribution which is suited to the problem at hand and which can be treated by statistical techniques.

- The marginal distributions and the copula of a multivariate distribution are inextricably linked. The main selling point of the copula technology | separation of the copula (dependence function) from the marginal distributions | leads to a biased view of stochastic dependence, in particular when one fits a model to the data.

- Various copula models (Archimedean, t-, Gaussian, elliptical, extreme value) are mostly chosen because they are mathematically convenient; the rationale for their applications is murky.

- Copulas are considered as an alternative to Gaussian models in a non-Gaussian world. Since copulas generate any distribution the class is too big to be understood and to be useful.

- There is little statistical theory for copulas. Sensitivity studies of estimation procedures and goodness-of-fit tests for copulas are unknown. It is unclear whether a good fit of the copula of the data yields a good fit to the distribution of the data.

- Copulas do not contribute to a better understanding of multivariate extremes.

- Copulas do not fit into the existing framework of stochastic processes and time series analysis; they are essentially static models and are not useful for modelling dependence through time.
Market risk is the risk of losses in the value of an asset portfolio (consisting of equity and bonds) due to movements in market prices and is estimated with Monte Carlo simulations in ESGs. A simplified portfolio of an insurance company is mimicked, consisting of 70% government bonds together with 30% equity. The government bonds are represented by 10-years German bonds and the equity by the Euro Stoxx 50 index. The choice of the ratio between these assets is based on the asset portfolios of Delta Lloyd, Achmea and Nationale Nederlanden, which are three large insurance companies in the Netherlands, this choice is further explained in Section 2.2.

The stock price and short rate simulation models are calibrated historically, based on the Euro Stoxx 50 index prices (1) and the 3-month German bond yield (2) respectively. In the calibration of the short rate process the German interest rate term structure is used, consisting of zero-coupon German government bonds (3). 10-years bond prices are then obtained by transforming short rate simulations to 10-years bond values, using the interest rate term structure.

As a remark, market behaviour might change, just as dependence between risk drivers. In the historical calibrations we assume that recent historical market behaviour is representative for future behaviour of market movements, this is however a debatable assumption.

These three data sets are discussed in more detail in Section 2.1. Section 2.2 explains how to construct the forward curve based on the yield curve and Section 2.3 constructs the asset portfolio from the data.

### 2.1 Data description

The historical time series (1, 2) and yield curve (3) are used to calibrate the stock price and short rate simulation models in the ESGs, which will be introduced in Chapter 3. The data consist of daily historical stock prices and 3-month German bond yields between January 2010 and December 2015, because taking data from 2009 and before would result in incorporating a structural break (during the financial credit crisis in 2008-2009, a sharp drop in interest rates from around 4-5% to 0-1% within a very short time was observed). Table 2.1–2.3 give some background information of the data sets, Table 2.4 gives descriptive statistics of the data and Figure 2.1 visualises the data sets.
### Chapter 2. Data

**Table 2.1: Euro Stoxx 50 index**

<table>
<thead>
<tr>
<th>Description</th>
<th>The Euro Stoxx 50 Index covers 50 stocks from 12 Eurozone countries: Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, the Netherlands, Portugal and Spain.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>Daily last index prices</td>
</tr>
<tr>
<td>Date range</td>
<td>01-01-2010 to 11-12-2015</td>
</tr>
<tr>
<td>Use</td>
<td>Analyse the historical evolution of the Euro Stoxx 50 Index prices</td>
</tr>
<tr>
<td>Remark</td>
<td>Some observations are removed, such that only days on which both the Euro Stoxx 50 as the German bond yields have historical values are used.</td>
</tr>
</tbody>
</table>

**Table 2.2: 3-month German bond yield**

<table>
<thead>
<tr>
<th>Description</th>
<th>The Germany 3 month bond yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>Daily bond yields</td>
</tr>
<tr>
<td>Date range</td>
<td>01-01-2010 to 11-12-2015</td>
</tr>
<tr>
<td>Use</td>
<td>Analyse the historical evolution of the 3-month bond yields</td>
</tr>
<tr>
<td>Remark</td>
<td>Some observations are removed, such that only days on which both the Euro Stoxx 50 as the German bond yields have historical values are used.</td>
</tr>
</tbody>
</table>

**Table 2.3: German bond yield curve**

<table>
<thead>
<tr>
<th>Description</th>
<th>The yield curve for German bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>Yields curve</td>
</tr>
<tr>
<td>Date range</td>
<td>11-12-2015</td>
</tr>
<tr>
<td>Use</td>
<td>Calibration of the term structure for the short rate model</td>
</tr>
<tr>
<td>Remark</td>
<td>The curve is constructed based on yields for different maturities (1m, 3m, 6m, 9m, 1y, 2y, ..., 10y, 15y, 20y, 30y) in which cubic spline interpolation is used to construct values in between these maturities.</td>
</tr>
</tbody>
</table>
Table 2.4: Descriptive statistics

<table>
<thead>
<tr>
<th></th>
<th>quantile 0.5%</th>
<th>quantile 99.5%</th>
<th>mean</th>
<th>median</th>
<th>first value</th>
<th>last value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euro Stoxx 50 Index</td>
<td>2 082</td>
<td>3 584</td>
<td>2 858</td>
<td>2 832</td>
<td>3 018</td>
<td>3 203</td>
</tr>
<tr>
<td>3-m German bond yield</td>
<td>-0.38%</td>
<td>1.07%</td>
<td>0.11%</td>
<td>0.03%</td>
<td>0.32%</td>
<td>-0.42%</td>
</tr>
</tbody>
</table>

The Euro Stoxx 50 Index prices in Figure 2.1a fluctuate around a level of 3000. The 3—month German bond yields in Figure 2.1b fluctuate between −0.5% and 1.2%. Due to the negative rates, a simple returns or log return model will not work, because in a simple return model a price of zero is an absorbing state; i.e. the price process is (theoretically) continuous and if at a certain point the price becomes zero, an increase or decrease of $x\%$ will always result in a new value of the price remaining zero. Hence, the model for the short rates should be able to deal with negative rates. Figure 2.1c gives the (zero coupon) yield curve for German bonds and Figure 2.1d shows the forward curve based on this yield curve. The construction of this curve is explained in Section 2.2.
2.2 Construction of current forward curve

The forward curve, which is part of the Hull-White term-structure model, is constructed based on the (current) German zero coupon bond yield curve. The German bond (zero) yield curve \( f(0,t) \) shown in Figure 2.1c is first used to construct discount factors (or bond prices) \( P(t) \),

\[
P(t) = \exp(-f(0,t) \cdot t).
\]

This yield curve \( f(0,t) \) is then interpolated by using cubic spline. The instantaneous forward rates \( f(t,t) = r(t) \) are constructed using the formula,

\[
r(t_i) \approx \log \left( \frac{P(t_{i-1})}{P(t_i)} \right) / \Delta t,
\]

in which \( P(t) \) are the discount values based on the interpolated yields. The resulting forward curve is shown in Figure 2.1d.

2.3 Composition of a simplified asset portfolio based on insurance companies

Based on a benchmark study, we construct an asset portfolio consisting for 30\% of Euro Stoxx 50 Index and for 70\% of 10–years German bonds. The choice of the underlying processes used for modelling the assets mainly depends on the assets in the asset portfolio of three large insurance companies in the Netherlands (Delta-Lloyd, 2014; Achmea, 2014; Nationale-Nederlanden, 2014). As Table 2.5 shows, for all three portfolios about 70\% consisting of fixed income, providing safety, and about 30\% consisting of equity, real-estate and others which are less safe but provide a better return.

<table>
<thead>
<tr>
<th>Fixed income</th>
<th>Delta Lloyd</th>
<th>Achmea</th>
<th>Nationale Nederlanden</th>
</tr>
</thead>
<tbody>
<tr>
<td>government</td>
<td>69%</td>
<td>48%</td>
<td>48%</td>
</tr>
<tr>
<td>non-government</td>
<td>49%</td>
<td>52%</td>
<td>29%</td>
</tr>
<tr>
<td>Mortgages</td>
<td>20%</td>
<td>3%</td>
<td>15%</td>
</tr>
<tr>
<td>Stocks</td>
<td>5%</td>
<td>3%</td>
<td>3%</td>
</tr>
<tr>
<td>Real-Estate</td>
<td>5%</td>
<td>3%</td>
<td>4%</td>
</tr>
<tr>
<td>Other (derivatives, deposits, etc)</td>
<td>1%</td>
<td>20%</td>
<td>11%</td>
</tr>
</tbody>
</table>

Table 2.5: Insurance company portfolio

Insurance companies often match their liabilities with a long duration with highly rated government bonds. Government bonds (fixed income) from triple-A rated countries are considered risk-free because of the small probability of default, and the price therefore only depends on the risk-free interest rate. The cash flows from these products are fixed at the time of buying, but the market price does change when selling the bond before maturity. As a proxy risk-free rate, the 3–month German government bond rates are modelled, but also the European Overnight Index Average (Eonia) swap rates or for example US bond rates could be used. The fixed income product in this thesis is a 10–years zero coupon bond based on the currently observed German bond yield curve and the 3–month bonds. The Euro Stoxx 50 index, consisting of the largest 50 stocks in the Eurozone, is used as a proxy for the equity in the asset portfolio.
Insurance companies constantly re-balance their bonds portfolio to match the duration of their liabilities. Therefore, we assume that the average time to maturity of the bonds remains 10–year through time. Considering a non-changing portfolio, the time to maturity of our portfolio in one year is only 9 year. However, due to the duration matching which insurance companies often use, the portfolio should have the same time to maturity in one year of 10–years. To be precise, the portfolio should be continuously re-balanced.

For correct re-balancing it would mean that at every timestep (of $\frac{1}{252}$ year) the bonds with maturity of $\frac{9.251}{252}$–years should be sold and with the obtained money a new portfolio of bonds with 10–years maturity should be bought (implying no transaction costs). A slightly different approach is used, which is close to continuously re-balancing of the portfolio.

Bond prices are calculated for each short rate realization using the affine term structure of the Hull-White model (see Section 3.1.2), keeping the maturity of these portfolios constant. This adjustment in bond price movements might result in slightly different bond prices. However, the exact prices are not the focus of this study and we mainly focus on the dependence between the short rate and stock prices, which is not affected by this choice.
Chapter 3
Simulation models

Stock prices simulations are performed with a log normal Black-Scholes model and bond prices are obtained from short rate simulations in the Hull-White model. The stock price and short rate models, consisting of a deterministic and a stochastic component, are calibrated historically using maximum likelihood.

Def. (Residuals). Given the model, and a set of calibrated parameters for which the variance of the stochastic component in the simulation model is minimal, historical realizations of this stochastic variable are obtained. These stochastic values leading to the historically observed price process are called the standardized residuals, or simply residuals. The simulation models hypothesize a certain distribution for these residuals, e.g. a normally distributed residuals. The empirical distribution of the historically observed residuals should coincide with the hypothesized residual distribution, such that the behaviour of the simulated price paths is in line with the historically observed behaviour.

However, matching the empirical residual distribution with a hypothesized residual distribution, which is one of the challenges in this thesis, will prove to be difficult. Inspection of the residuals from the Black-Scholes and Hull-White simulation models (see Chapter 6) shows that volatility is clustered and residuals are not normally distributed. The stock price and short rate model are therefore combined with a generalized autoregressive conditional heteroskedasticity (GARCH) volatility model and three different residual models are tested.

Section 3.1 introduces the Hull-White model for simulation of short rates and Section 3.2 explains the lognormal Black-Scholes stock price model. Section 3.3 introduces the GARCH(1,1) volatility model and Section 3.4 shows three different residual distributions. Finally, Section 3.5 gives different combinations of the models in Sections 3.1–3.4, which together with a copula introduced in Chapter 4 forms an ESG. The ESGs are then compared in Chapter 6 based on performance tests introduced in Chapter 5.

Each section gives the dynamics, describes the calibration of the model parameters based on historical data and explains how the historical residuals are obtained.

3.1 Short rate models

The Hull-White one-factor model is used to simulate short rates. For market risk calculations we are mainly interested in the changes in the stock- and bond price processes, therefore it does not make sense to analyse deterministic short rates, because this would only lead to deterministic prices without any risks. Stochastic short rate simulation models however consist of a deterministic part combined with a stochastic part, leading to a spread of short rate realizations.
Chapter 3. Simulation models

around the forward curve and the most dispersed paths are useful for estimation of market risk.

3.1.1 Hull-White 1-factor

The Hull and White (1990) one-factor model is used to model the (risk-free) short rate. The Hull-White short rates follow the dynamics,

\[ dr(t) = (\theta_r(t) - a_r r(t))dt + \sigma_r dW_r(t). \]  

(3.1)

Following Damiano and Fabio (2001), the model is split in two steps, consisting of a mean reverting Ornstein-Uhlenbeck process \( x(t) \) and a deterministic part \( \alpha_r(t) \).

\[ dx(t) = [-a_r x(t)]dt + \sigma_r dW_r(t), \]

\[ r(t) = x(t) + \alpha_r(t). \]  

(3.2)

It can be proved using the Heath-Jarrow-Morton framework that \( \alpha_r(t) \) has the following form:

\[ \alpha_r(t) = f(0, t) + \frac{\sigma_r^2}{2a_r^2} \left( 1 + \exp(-2a_r t) - 2 \exp(-a_r t) \right), \]  

(3.3)

Here \( f(0, t) \) is the instantaneous forward rate at time 0 with maturity \( t \), i.e.

\[ f(0, t) = -\frac{\partial \log P(0, t)}{\partial t}, \]  

(3.4)

with \( P(0, t) \) the discount factor for maturity \( t \).

The 3-month German bond yield is used as proxy for historic values of the risk-free rate. The German bond forward curve \( f(0, t) \), based on the (zero coupon) yield curve \( P(0, t) \), is the term structure used for the forecast of the risk-free rates. The forward curve is also used to construct a 10–years zero coupon forward curve, using the affine structure of the Hull-White model.

Calibration and obtaining the residuals

In the calibration of the Hull-White process, the mean reversion parameter \( a_r \) is fixed to \( a_r = 0.04 \). Calibration to the historical short rates such that \( \text{std}(W_r \cdot \sigma_r) = 1 \), then leads to a volatility parameter of \( \sigma_r \approx 0.007 \). These parameters \( a_r \) and \( \sigma_r \) result in realistic interest rates in thirty years between -2.5\% and 6.5\% given a 99\% confidence interval (see Figure 3.1). The volatility parameter is usually calibrated to caplets, floorlets and / or swaption prices, we however simplified this approach.

During the calibration, the value of the \( \theta_r(t) \)–term in the historical data is set equal to zero, because of the following reasoning. In Formula 3.3 \( \alpha_r \) depends on the forward rate \( f(0, t) \) and a deterministic term dependent on \( a_r, \sigma_r \) and \( t \). Historical residuals at day \( t_i \) are calculated based on all information until day \( t_{i-1} \). Hence, we use as much information as possible and \( dt = \frac{1}{252} \) for 252 business days. This results in,

\[ \alpha_r(t_i|t_{i-1}) = f(t_{i-1}, t_i|t_{i-1}) + \frac{\sigma_r^2}{2a_r^2} \left( 1 + \exp(-2a_r dt) - 2 \exp(-a_r dt) \right), \]

\[ \approx f(t_{i-1}, t_i|t_{i-1}) + 5 \cdot 10^{-7} \approx f(t_{i-1}, t_i|t_{i-1}). \]  

(3.5)
The last step is made because \(5 \cdot 10^{-7}\) is negligible compared to \(f(t_{i-1}, t_i | t_{i-1})\) which is the average absolute difference between \(r(t_{i-1})\) and \(r(t_i)\) (of approximately \(3.3e^{-4}\)). On yearly basis this sums up to 0.01%. The yearly error is of the same order as the rounding error and hence does not lead to problems. Furthermore because the timesteps are small, 

\[
\alpha_r \approx f(t_{i-1}, t_i | t_{i-1}) \approx f(t_{i-1}, t_{i-1} | t_{i-1}) = r(t_{i-1}), \quad \text{for } dt \rightarrow 0.
\]  

(3.6)

Combined with Formula 3.2, \(r(t_i | t_{i-1})\) in the next timestep this becomes 

\[
r(t_i | t_{i-1}) = x(t_i | x_{i-1}) + r(t_{i-1}).
\]  

(3.7)

Hence, 

\[
\begin{align*}
  r(t_i | t_{i-1}) - r(t_{i-1}) &= (-a_r r(t_i))dt + \sigma_r (W_{r}(t_i) - W_{r}(t_{i-1})), \\
  W_{r}(t_i) &= \frac{r(t_i | t_{i-1}) - r(t_{i-1}) + a_r r(t_i)dt}{\sigma_r \sqrt{dt}}.
\end{align*}
\]  

(3.8)

The result is an expression based on the parameters \(a_r\) and \(\sigma_r\) from which the daily historical residuals on days \(t_i = t_1, ..., t_n\) are estimated. The residuals have two purposes; firstly the residuals are used for calibration of the copula parameter and secondly the empirical residual distribution and hypothesized residual distribution are compared in Chapter 6 using the performance measures introduced in Chapter 5.

### 3.1.2 Forward 10-years bond prices

A straightforward way to obtain 10–years bond prices based on the simulated short rates is to use nested Monte Carlo simulations for each short rate simulation at each timestep. The bond price of a 10-years bond in 1 year from now based on (nested) short rate simulations is given by,
Chapter 3. Simulation models

\[
P(t, T|\mathcal{F}_t) = \mathbb{E}_t^Q \left[ \exp \left( - \int_t^T r(u)du \right) \bigg| r(t) = r_t \right]. \tag{3.9}
\]

However, the computational time for a nested Monte Carlo is very large and besides according to Damiano and Fabio (2001) we can make use of the affine term structure of the Hull-White process to find an analytic solution for each short rate simulation of the 10-years bond values based on the model parameters and forward curve.

Using the affine structure the result is,

\[
P(t, T|\mathcal{F}_t) = A(t, T) \exp \left( - B(t, T) r(t) \right),
\]

where

\[
B(t, T) = \frac{1 - \exp(-a_r(T-t))}{a_r}, \tag{3.10}
\]

\[
A(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( - B(t, T) \frac{\partial \log(P(0, t))}{\partial t} - \frac{\sigma^2_r (\exp(-a_r T) - \exp(-a_r t))^2 (\exp(2a_r t) - 1)}{4a_r^3} \right). \tag{3.11}
\]

Hence, for each short rate realization \( r_i(t) \) a 10-years bond value is calculated by letting time \( t \) increase and let \( T \) be equal to \( T = t + 10 \). Then, the 10-years bond price at every timestep is calculated.

### 3.1.3 Alternatives

Two alternatives for modelling interest rates which are not used in this thesis are the Cox-Ingersol-Ross (CIR) and Vasicek model. Notice that the CIR model cannot produce negative interest rates which are observed in the market without shifting the rates first.

### 3.2 Stock price models

Stock prices are modelled with a Geometric Brownian motion as in the Black-Scholes model, where the drift term is split in a risk-free rate on which a constant risk premium is added. The same Hull-White process as is used for the bond price calculation is used as a proxy for the risk-free rate in the drift term of the stock process.

#### 3.2.1 Log normal Black-Scholes

Stock price models under Black and Scholes (1973) follow a Geometric Brownian Motion with a certain drift and volatility. The drift term in this thesis consists of the risk-free interest rate \( r_{rf}(t) \) on which a constant risk premium \( \mu_S \) is added. The dynamics are,

\[
\frac{dS(t)}{S(t)} = (\mu_S + r_{rf}(t))dt + \sigma_S dW_S(t). \tag{3.12}
\]
In this formula \( S(t) \) is the stock price, \( \mu_S \) is the risk premium, \( r_{rf}(t) \) is the risk free rate following from the Hull-White model, \( dt \) is the stepsize in years, \( \sigma_S \) is the volatility parameter and \( dW_S(t) \) is a standard Brownian.

**Calibration and obtaining the residuals**

First the parameters of the stock process are calibrated historically, then the residuals \( W_S(t_i) \) are obtained from historical stock prices. The residuals have two purposes; firstly the residuals are used for calibration of the copula parameter and secondly the empirical residual distribution and hypothesized residual distribution are compared in Chapter 6 using the performance measures introduced in Chapter 5 to verify the quality of the model.

Formula 3.12 is discretized and rewritten, resulting in an expression for the parameters \( \mu_S \) and \( \sigma_S \) based on all historical timevalues \( t_i = t_1, ..., t_n \). The stepsize \( dt \) between two consecutive residuals is for (business) daily residuals on average \( dt = \frac{1}{252} \).

\[
\mu_S = \frac{\mathbb{E}\left( \frac{S(t_i)}{S(t_{i-1})} - r_{rf}(t_i) \right) dt}{dt}, \quad (3.13)
\]
\[
\sigma_S = \sqrt{\frac{\text{std}\left( \frac{S(t_i)}{S(t_{i-1})} - 1 - r_{rf}(t_i) dt \right)}{\sqrt{dt}}}. \quad (3.14)
\]

Given the parameters \( \mu_S \) and \( \sigma_S \), the (historical) residuals \( W_S(t_i) \) on days \( t_i \in t_1, ..., t_n \) are:

\[
W_S(t_i) = \frac{1}{\sigma_S \sqrt{dt}} \left( \frac{S(t_i)}{S(t_{i-1})} - 1 - (r_{rf}(t_i) + \mu_S) dt \right). \quad (3.15)
\]

### 3.2.2 Alternatives

Heston is an alternative stock price model incorporating stochastic volatility. The model is not used because the residuals in Heston’s volatility process have a lot of extreme values and the second stochastic process (the Heston volatility) would increase the copula dimension to three, both leading to more complexity. See Appendix B.1 for more details.

### 3.3 Volatility modelling

The residuals following from Black-Scholes and Hull-White have volatility clustering (see Chapter 6 for the analysis of the residuals). Volatility models can reproduce the volatility clustering which is observed in the residuals of the stock price or short rate models. The GARCH(1,1) model is used to capture volatility clustering. The word residual is however becoming twofold here, because the volatility model splits this stock price or short rate model (output) residual in a volatility process and an (input) residual, which is free of volatility clustering. From now on the residual of a model is the most simplified residual of that model; a Black-Scholes residual is the residual following from the Black-Scholes model, whereas the Black-Scholes GARCH(1,1) residual is the residual from the GARCH(1,1) process in which no volatility clustering is present.

For the Black-Scholes residuals, the values \( \sigma_S W_S(t_i) \) are input for calibration of the volatility model. For the Hull-White residuals, the values \( \sigma W_r(t_i) \) are input for calibration of the volatility model. The reason for choosing \( \sigma_S W_S(t_i) \) instead of the standardized residuals \( W_S(t_i) \) is
that the total volatility process is modelled with GARCH(1,1) and parameter $\sigma_S$ is therefore redundant. The same holds for the $\sigma_r$ parameter in the Hull-White GARCH(1,1) model.

### 3.3.1 Generalized autoregressive conditional heteroskedasticity model

A GARCH model by Bollerslev (1986) is a discrete volatility model. The volatility clustering which is observed in the market prices, and which is not captured by the Black-Scholes and Hull-White model, is captured with a GARCH volatility model.

Under GARCH(1,1), the output $W(t_i)$ is dependent on a time-dependent standard deviation $\sigma(t_i)$ and a residual $\epsilon(t_i)$. The standard deviation $\sigma(t_i)$ is in his turn dependent on a constant term $\omega$, the previous output $W(t_{i-1})$ and the previous standard deviation $\sigma(t_{i-1})$,

\[
W(t_i) = \sigma(t_i)\epsilon(t_i), \tag{3.16}
\]
\[
\sigma^2(t_i) = \omega + \alpha W^2(t_{i-1}) + \beta \sigma^2(t_{i-1}). \tag{3.17}
\]

**Calibration and obtaining the residuals**

This model is fitted using maximum likelihood estimation with the statistical program R. The i.i.d. historical volatility residuals $\epsilon(t_i)$ of the GARCH(1,1) model on days $t_i \in t_1,\ldots,t_n$ should be i.i.d. distributed. The type of distribution for the residuals is discussed in Section 3.4.

### 3.3.2 Alternatives

Alternative volatility models such as higher order GARCH models, variations on the GARCH model, Heston (Appendix B.1) and regime switching (Appendix B.2) are not used in this thesis.

### 3.4 Residual distribution

Three different types of residual distributions are compared: Gaussian, Student’s t and Gaussian with Pareto tails.

Option pricing models often demand normally (Gaussian) distributed residuals. For the calculation under the real-world measure of market risk, the no-arbitrage condition does not necessarily hold and other residual models which have a better fit are possible.

### 3.4.1 Standard normal residuals

The probability density function of the normal (Gaussian) distribution is given by,

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \tag{3.18}
\]

with mean zero and volatility one. The cumulative distribution function is given by,

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt. \tag{3.19}
\]
3.4.2 Student’s t residuals

Bollerslev (1987) introduced an adjustment of the GARCH model containing Student’s t distributed residuals. This model better fits extremes in the data, but at the cost of an additional parameter.

The probability density function of the Student’s t distribution is given by,

\[
f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}},
\]

(3.20)
in which \(\nu\) is the degrees of freedom parameter and \(\Gamma\) is the gamma function. The distribution has mean zero and standard deviation \(\sqrt{\frac{\nu}{\nu-2}}\). The distribution of the stochastic variables, which are input for the Black-Scholes and Hull-White processes, should have a volatility of one, therefore the simulated stochastic variables are corrected for that by dividing by the standard deviation. The cumulative distribution function is given by,

\[
F(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} \int_{-\infty}^{x} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} dt.
\]

(3.21)

3.4.3 Normal residuals with Pareto distributed tails

The combination of GARCH(1,1) residuals for which the body of the residuals are normally distributed and the tails are modelled with a Generalized Pareto distribution (GPD) is justified by Bob (2013), Soltane et al. (2012), and McNeil and Frey (2000). The GPD is a one-sided extreme value distribution which also models extremes better than the normal distribution. The probability density function of the GPD is given by,

\[
f_{GPD}(x) = \begin{cases} \frac{1}{\beta} \left[1 + \xi \frac{x-\mu}{\beta}\right]^{-\frac{1}{\xi}} & \text{for } \xi \neq 0, x > \mu, \\ \frac{1}{\beta} \exp\left[-\frac{x-\mu}{\beta}\right] & \text{for } \xi = 0, x > \mu. \end{cases}
\]

The cumulative distribution function of the GPD is given by,

\[
F_{GPD}(x) = \begin{cases} 1 - \left[1 + \xi \frac{x-\mu}{\beta}\right]^{-\frac{1}{\xi}} & \text{for } \xi \neq 0, x > \mu, \\ 1 - \exp\left[-\frac{x-\mu}{\beta}\right] & \text{for } \xi = 0, x > \mu. \end{cases}
\]

The GPD is often used to model the tails of another distribution. It is specified by three parameters: location \(\mu\), scale \(\beta\), and shape \(\xi\). The GPD is fitted on the positive and on the negative tail separately and it leads to a more accurate fit of the extreme residuals than the normal model. The total distribution is then a combination of a normal and two GPD tails, separated by a lower and upper threshold \(\mu_L\) and \(-\mu_U\) respectively.

However, the upper (lower) GPD is defined for \(x > \mu_U\) (\(x < -\mu_L\)) only, and assigns a value to \(F_{GPD}(x)\) between 0 and 1. Hence, in the combined distribution function \(F_{combined}\), the first part of the distribution is described by a scaled negative GPD, the second part by a normal with zero mean and the third part again by a scaled GPD in which the combined function assigns a cumulative probability between 0 and 1. The empirical quantiles of the upper and lower threshold are indicated with \(q_U\) and \(q_L\) respectively.

Because the normal distribution connecting the two Pareto tails is symmetrical, an easy choice is to also make the threshold symmetrical (\(\mu_U = -\mu_L\)). The standard deviation of the normal part of the distribution is then \(\frac{1}{\pi((1-q_U+q_L)/2)}\), such that the empirically observed quantiles of
the threshold approximately matches the quantiles of the constructed distribution. This results in the following combined distribution,

\[
F_{\text{combined}}(x) = \begin{cases} 
-F_{\text{GPD}}(-x)q_l + q_l & \text{for } x < \mu_l, \\
F_{\text{norm}}(x) & \text{for } -1 < x < 1, \\
F_{\text{GPD}}(x)(1 - q_u) + q_u & \text{for } x > \mu_u,
\end{cases}
\]

where \( F_{\text{GPD}}(x) \) is as defined in Equation 3.4.3 and \( F_{\text{norm}}(x) \) is defined in Equation 3.19 for \( \Phi(x\sigma) \).

For the calibration and to determine whether the GPD distribution is indeed fitting the residuals well, the article of Cirillo (2013) is used. The Peak Over Threshold (POT) method splits the normal body of the data from the Pareto distributed tails. The mean excess plot and zipf plot are used to determine whether the data are Pareto distributed (linear increase in the mean excess plot linear decay in zipf plot indicate that a GPD might be a good choice). The threshold is then found to be the value for which the residuals above (below) this value in the zipf plot have a linear decay (i.e. are Pareto distributed). The corresponding quantiles \( q_l (q_u) \) are calculated as the empirical quantile in the historical residuals which exceeds this threshold. The analysis of these plots is given in Appendix A.2.

### 3.5 Univariate models in economic scenario generator

The dependence modelling in the portfolio is performed for several underlying models. Different combinations of stock-, short rate and volatility models are analysed in ESGs, combined with a copula and calibration method (both introduced in Chapter 4).

Four different univariate model combinations, three different copulas and two different calibration methods are analysed. Therefore, results are found for \( 4 \cdot 3 \cdot 2 = 24 \) ESGs. The final ESGs are given in Section 4.4. The combinations of simulation models used in the ESGs are given in Table 3.1 below.

<table>
<thead>
<tr>
<th>ESG</th>
<th>STK model</th>
<th>SR model</th>
<th>vol model</th>
<th>STK residuals</th>
<th>SR residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>BS</td>
<td>HW</td>
<td>-</td>
<td>normal</td>
<td>normal</td>
</tr>
<tr>
<td>B</td>
<td>BS</td>
<td>HW</td>
<td>GARCH(1,1)</td>
<td>normal</td>
<td>normal</td>
</tr>
<tr>
<td>C</td>
<td>BS</td>
<td>HW</td>
<td>GARCH(1,1)</td>
<td>Student’s t</td>
<td>Student’s t</td>
</tr>
<tr>
<td>D</td>
<td>BS</td>
<td>HW</td>
<td>GARCH(1,1)</td>
<td>Student’s t</td>
<td>Pareto</td>
</tr>
</tbody>
</table>

**Table 3.1**: ESG univariate models overview  
(STK = stock, SR = short rate, vol = volatility)
The Gaussian copula, Student $t$ copula and Gumbel copula are used to model the dependence for each ESG introduced in Section 3.5. Furthermore, the Fréchet bounds which present perfect positive and negative dependence, and the independence copula, are used in the stress testing.

The chapter is structured as follows: Section 4.1 introduces some dependence concepts, which are helpful in the copula theory. Section 4.2 explains copulas in detail, starting with an introduction in Subsection 4.2.1. Subsection 4.2.2 places the different copulas which are treated in a classification structure. Subsection 4.2.3–4.2.8 then explain the treated copula distributions in detail, Subsection 4.2.2 compares three different calibration methods for copulas, from which two methods will be used and Subsection 4.2.10 explains how to simulate from a copula. Section 4.3 shortly gives some alternative dependence models for comparison and Section 4.4 gives the combinations of copula and calibration methods in the ESGs which are compared in the next chapters.

4.1 Dependence concepts

In this section, the often used concept of Pearson correlation is compared to other measures of association, such as measures of concordance of which Kendall’s tau is explained in more detail. Correlation is defined as follows,

**Def. (Correlation).** Pearson correlation describes the linear part of the relation between two variables, i.e. a bivariate statistic that measures the degree of association between two random variables. It is the linear relationship $\rho(x, y) \in [-1, 1]$ between i.i.d. random variables $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in which,

\[
\rho(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{\mathbb{E}[(x - \mu_x)(y - \mu_y)]}{\sigma_x \sigma_y}. \tag{4.1}
\]

One of the shortcomings of Pearson correlation, is that it is sensitive to outliers, which can have a large impact on the calculated correlation coefficient. Furthermore, perfect (positive) dependence between two variables does not always lead to a correlation coefficient of 1; consider for example $X \in [0, 1]$ and $Y = X^2$. Obviously, $X$ and $Y$ are perfectly dependent (if $X$ is known, then $Y$ can be calculated and vise versa). However, the correlation coefficient between both variables will be smaller than one.
To overcome these shortcomings, other concepts have been introduced, such as measures of concordance.

**Def. (Measure of concordance).** Concordance is a measure of association which is invariant under strictly monotone transformations of the random variables, hence it describes the relation between the ranks of the observation pairs.

The kind of association implied by a measure of concordance is different from Pearson correlation. It only looks at the ranks of the observed data, the statistic is indifferent to outliers and indifferent to monotone transformations. One example which is helpful in the context of copula theory is Kendall’s tau, which makes use of the amount of concordant and discordant pairs in the observed data, and is defined as follows,

**Def. (concordance).** Let \((x_i, y_i)\) and \((x_j, y_j)\) \((i \neq j)\) be two observations from the jointly random vectors \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) of continuous random variables, then the two random variables are said to be concordant if \((x_i - x_j)(y_i - y_j) > 0\), and discordant if \((x_i - x_j)(y_i - y_j) < 0\).

(Embrechts et al., 2001)

**Def. (Kendall’s tau).** Kendall’s tau coefficient is a statistic used to measure the association between two jointly random vectors \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\). Kendall’s tau is defined as the difference between the amount of concordant pairs \(C\) and discordant pairs \(D\) divided by \(\binom{n}{2} = \frac{1}{2}n(n-1)\), or the chance of concordance minus the chance of discordance.

\[
\tau = \frac{P[(x_i - x_j)(y_i - y_j) > 0] - P[(x_i - x_j)(y_i - y_j) < 0]}{\binom{n}{2}} = \frac{2(C - D)}{n(n - 1)}. \tag{4.2}
\]

The rank correlation coefficient \(\tau \in [-1, 1]\) implies more agreement between the rankings of the observations for increased values of \(\tau\), i.e.

- \(\tau = -1\) means perfect disagreement between rankings, i.e. the rankings are opposite.
- \(\tau = 0\) means no agreement between rankings, i.e. number of concordant and discordant pairs are the same.
- \(\tau = 1\) means perfect agreement between rankings, i.e. the rankings are the same.

Kendall’s tau only describes the relation between the ranks of the observations, thereby making it possible for a number of copulas to derive a direct relationship between the copula parameter and Kendall’s tau. It is therefore either used directly to find a copula parameter or is used as a starting value in copula calibration processes. Kendall’s tau will also prove to be a good indicator when the copula calibration process does not converge.

### 4.2 Modelling dependence with copulas

Copulas make it possible to separate the marginal distributions from the dependence structure, which makes them a flexible tool in dependence modelling. This section explains copulas in more detail.

Subsection 4.2.1 starts with an introduction in copula theory. Subsection 4.2.2 places the different copulas which are treated in a classification structure. Subsection 4.2.3–4.2.8 then explains the treated copula distributions in detail, Subsection 4.2.9 gives three different calibration methods and Subsection 4.2.10 explains how to simulate from a copula.
4.2.1 An introduction to copulas

Nelsen (2007) describes copulas as multivariate distribution functions whose one-dimensional margins are uniform on the interval $[0, 1]$. In other words, copulas can be used as a multivariate model, allowing every possible marginal distribution and only describing the relationship between the cumulative distribution functions of the marginal distributions. A full mathematical definition of copulas is given in Nelsen (2007) and Kort (2007b), where the formal construction of copulas is based on the concepts of $H$-volumes, 2-increasing, grounded and sub-copulas. Some useful consequences of the definition of the bivariate copula $C(u, v)$ are,

\begin{align*}
0 &\leq C(u, v) \leq 1 \quad \text{for } u, v \in [0, 1], \\
C(0, u) &= C(u, 0) = 0, \quad (4.3) \\
C(1, u) &= C(u, 1) = u. \quad (4.4)
\end{align*}

One of the reasons for copulas being widely applicable in dependence modelling is explained by Sklar’s theorem, which proves that every multivariate distribution can also be written in terms of univariate marginal distributions separated by a copula dependence model.

\textbf{Thm. (Sklar’s theorem).} Sklar (1959) states that for any multivariate distribution function $F$ with marginal distributions $F_1$ and $F_2$, there exists a copula $C$ such that for all $x, y \in \mathbb{R}$,

$$F(x, y) = C(F_1(x), F_2(y)).$$

If $F_1$ and $F_2$ are continuous, then $C$ is unique; otherwise, $C$ is uniquely determined on $\text{range}(F_1) \times \text{range}(F_2)$.

The cumulative distribution function (cdf) of the copula is referred to as copula, whereas the probability density function (pdf) of the copula function is referred to as copula density. The copula density is often more intuitive than the copula itself, because the densities can easily be compared with the independence copula.

To motivate the strength of the copula density, Figure 4.1 shows the copula densities (above) and copulas (below) of the Gaussian copula for three choices of the correlation parameter $\rho$. The height of the density indicates the probability of observing two random variables $u$ and $v$ in a certain quantile of the underlying distributions. For $\rho = 0$ we observe actually the independence copula, in which the density is flat. This means that all observed quantile combinations of $u$ and $v$ have the same probability. For a positive (negative) correlation coefficient $\rho$, this density is higher (lower) in the region where $u$ and $v$ are both large or both small and lower (higher) in the region where one of both variables is large and the other is small. From the cdf of the copula, which look comparable, it is however not easy to evaluate the dependence directly.

4.2.2 Different copula types

Copulas can be classified in many different ways. Aas (2004) splits copulas in explicit and implicit copulas, where an explicit copula has a simple closed form and an implicit copula is implied by (the double integral over) a well-known multivariate distribution function. Nelsen (“Properties and applications of copulas: A brief survey”) separates copulas in one- and two-parameter copulas. Others differentiate for example between normal and extreme value copulas.

This thesis classifies copulas based on Aas (2004), as is shown in Figure 4.2. In the next sections, the different types are explained in detail.
Chapter 4. Modelling dependence

4.2.3 Gaussian copula

The Gaussian copula is an implicit copula with one parameter: the correlation parameter \( \rho \in [-1, 1] \), implied by the multivariate Gaussian distribution function. It is probably the most used copula in risk management and for \( \rho = 0 \) it is equal to the independence copula. The Gaussian copula is given by,

\[
C_\rho(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left( - \frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right) dx dy. \tag{4.6}
\]

Notice that the Gaussian copula with normal marginals \((u = \Phi(x), v = \Phi(y))\) is the same as the multivariate Gaussian distribution. By doing a change of variables we get the integral over \(u\) and \(v\), instead of over \(x\) and \(y\). The result is the double integral over the density of the copula instead of the density of the implied multivariate distribution. For the change of variables, consider the transformation \( u = \Phi(x) \) with \( du = \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{x^2}{2} \right) dx \). Analogous for \( v \) use the transformation \( v = \Phi(y) \) with \( dv = \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{y^2}{2} \right) dy \). This results in,

\[
C_\rho(u^*, v^*) = \int_0^{u^*} \int_0^{v^*} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left( - \frac{\Phi^{-1}(u)^2 - 2\rho \Phi^{-1}(u) \Phi^{-1}(v) + \Phi^{-1}(v)^2}{2(1 - \rho^2)} \right) \\
\cdot \sqrt{2\pi} \exp(-\Phi^{-1}(u)^2) \cdot \sqrt{2\pi} \exp(-\Phi^{-1}(v)^2) du dv, \nonumber
\]

\[
= \int_0^{u^*} \int_0^{v^*} \frac{1}{\sqrt{1 - \rho^2}} \exp \left( - \frac{\rho^2 \Phi^{-1}(u)^2 - 2\rho \Phi^{-1}(u) \Phi^{-1}(v) + \rho^2 \Phi^{-1}(v)^2}{2(1 - \rho^2)} \right) du dv. \tag{4.7}
\]
Hence, the density of the copula is given by,

$$c(u, v) = \frac{1}{\sqrt{1 - \rho^2}} \exp \left( \frac{2\rho \Phi^{-1}(u)\Phi^{-1}(v) - \rho^2(\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2)}{2(1 - \rho^2)} \right).$$  \hspace{1cm} (4.8)

### 4.2.4 Student t-copula

Another implicit copula is the Student $t$-copula, which has two parameters: the correlation parameter $\rho \in [-1, 1]$ and degrees of freedom parameter $\nu > 0$, and is implied by the multivariate Student’s $t$-distribution. Compared to the Gaussian copula, it implies stronger dependence in the tails when $\nu$ is small. For the limit of $\nu \to \infty$ the Student $t$ copula is the same as the Gaussian copula. It is given by,

$$C_{\rho, \nu}(u, v) = \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{1}{2\pi \sqrt{1 - \rho^2}} (1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1 - \rho^2)})^{-\frac{\nu + 1}{2}} dx dy. \hspace{1cm} (4.9)$$

Notice that the Student $t$-copula with Student’s $t$-distributed marginals ($u = t_\nu(x)$, $v = t_\nu(y)$) is the same as the multivariate Student’s $t$-distribution. By doing a change of variables we get the integral over $u$ and $v$, instead of over $x$ and $y$. The result is the double integral over the density of the copula instead of the density of the implied multivariate distribution. For the change of variables, consider the transformation $u = t_\nu(x)$ with $du = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi \nu^{\nu/2}}} (1 + \frac{x^2}{\nu})^{-(\nu+1)/2} dx$. Analogous for $v$ use the transformation $v = t_\nu(y)$ with $dv = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi \nu^{\nu/2}}} (1 + \frac{y^2}{\nu})^{-(\nu+1)/2} dy$. This results in,
\begin{equation}
C_{\rho,\nu}(u,v) = \int_0^u \int_0^v \frac{1}{\pi \sqrt{1-\rho^2}} \left( 1 + \frac{t^{-1}_\nu(u)^2 - 2 \rho t^{-1}_\nu(u) t^{-1}_\nu(v) + t^{-1}_\nu(v)^2}{\nu(1-\rho^2)} \right)^{-\nu \frac{1}{2}} \frac{\pi \nu \Gamma(\nu/2)^2}{\Gamma((\nu+1)/2)^2} \left( 1 + \frac{t^{-1}_\nu(u)^2}{\nu} \right)^{-\nu \frac{1}{2}} \left( 1 + \frac{t^{-1}_\nu(v)^2}{\nu} \right)^{-\nu \frac{1}{2}} \frac{dudv}{(1+\nu \frac{1}{2})}. \tag{4.10}
\end{equation}
Hence, the density of the copula is given by,
\begin{equation}
c(u,v) = \frac{\nu}{2 \sqrt{1-\rho^2}} \left( 1 + \frac{t^{-1}_\nu(u)^2 - 2 \rho t^{-1}_\nu(u) t^{-1}_\nu(v) + t^{-1}_\nu(v)^2}{\nu(1-\rho^2)} \right)^{-\nu \frac{1}{2}} \frac{\Gamma(\nu/2)^2}{\Gamma((\nu+1)/2)^2} \left( 1 + \frac{t^{-1}_\nu(u)^2}{\nu} \right)^{-\nu \frac{1}{2}} \left( 1 + \frac{t^{-1}_\nu(v)^2}{\nu} \right)^{-\nu \frac{1}{2}}. \tag{4.11}
\end{equation}

### 4.2.5 Independence copula

The most basic copula, without any parameter, implies independence between the random variables linked by the copula. The independence copula is also an Archimedean copula (see Section 4.2.8) in which the generator is \( \phi(t) = -\log(t) \). The density of this copula is flat.

\begin{align}
C(u,v) &= u \cdot v, \quad (4.12) \\
c(u,v) &= 1. \quad (4.13)
\end{align}

### 4.2.6 Fréchet–Hoeffding copula bounds

The Fréchet copulas, or Fréchet bounds are the theoretical bounds for the (2-dimensional) copula. Chapter 2.1 of Nelsen (2007) and Section 1.2 of Kort (2007a) show how to derive the lower bound \( C_L(u,v) \) and upper bound \( C_U(u,v) \) which are for the two-dimensional domain equal to,

\begin{align}
C_L(u,v) &= \max(u+v-1,0), \quad (4.14) \\
C_U(u,v) &= \min(u,v). \quad (4.15)
\end{align}

Figure 4.3 gives the copula cdfs. The densities are given by,

\begin{align}
c_L(u,v) &= \delta(u+v-1), \quad (4.16) \\
c_U(u,v) &= \delta(u-v), \quad (4.17)
\end{align}

where \( \delta() \) is the Dirac delta function. All copulas lie within the tetrahedron defined by the region between the two Fréchet bounds in order to be defined properly, i.e.

\begin{equation}
\max(u+v-1,0) \leq C(u,v) \leq \min\{u,v\} \quad (4.18)
\end{equation}

The lower bound describes perfect negative dependence, which means all quantiles of the combined observations \( u \) and \( v \) lie on the line \( v = 1-u \). The upper bound describes perfect positive dependence, which means all quantiles of the combined observations lie on the line \( v = u \).
Hence, if the underlying distributions $x$ and $y$ of the copula are known, then the conditional distributions $y|x$ and $x|y$ are deterministic.

![Fréchet bounds](image)

**Figure 4.3:** Fréchet bounds

### 4.2.7 Archimedean copulas

Archimedean copulas are explicit copulas, which are constructed with a generator $\phi_\theta(t)$. The copulas can have different number of parameters, depending on the generator function. Every generator function for an Archimedean copula is a continuous, decreasing, convex function $\phi_\theta : [0, 1] \to [0, \infty)$ such that $\phi_\theta(1) = 0$. If furthermore $\phi_\theta(0) = \infty$, then the generator is called strict. The pseudo-inverse of $\phi_\theta$ is defined as,

$$\phi_\theta^{-1} = \begin{cases} \phi_\theta^{-1}(u), & 0 \leq u \leq \phi(0), \\ 0, & \phi(0) \leq u \leq \infty \end{cases},$$

which leads in case of a strict generator to $\phi_\theta^{-1} = \phi_\theta^{-1}$. The Archimedean copula $C^A$ is then defined as,

$$C^A(u, v) = \phi_\theta^{-1}((\phi_\theta(u) + \phi_\theta(v))).$$

Many Archimedean copulas are known, for those interested please have a look at Table 4.1 in Nelsen (2007). The Clayton, Gumbel and Frank are widely used Archimedean copulas, of which the Clayton copula is an asymmetric Archimedean copula, exhibiting greater dependence in the negative tail than in the positive. The Frank copula is a symmetric Archimedean copula. The Gumbel copula (a.k.a. Gumbel-Hougaard copula) is an asymmetric Archimedean copula, exhibiting greater dependence in the positive tail than in the negative (see Genest et al., 2011)). The next paragraph goes more deeply in this Gumbel copula, which is used in some of the ESGs.

**Gumbel copula**

The Gumbel copula (a.k.a. Gumbel-Hougaard copula) is an asymmetric (extreme value) Archimedean copula, exhibiting greater dependence in the positive tail than in the negative. The Gumbel copula is given by,

$$C_\theta(u, v) = \exp \left[ - \left( (-\log(u))^\theta + (-\log(v))^\theta \right)^{1/\theta} \right], \quad \theta \in [1, \infty),$$

(4.21)
where the generator function is

\[ \phi_\theta(u) = \left( -\log(u) \right)^\theta. \]

In the historical returns more strongly combined negative returns are present than positive returns, therefore the flipped (or rotated or survival) Gumbel copula is used instead. The flipped Gumbel copula \( C_F \) is computed based on Venter (2002),

\[
C_F = u + v - 1 + C(1 - u, 1 - v), \\
= u + v - 1 + \exp \left[ - \left( -\log(u) \right)^\theta + (-\log(v))^{\frac{1}{\theta}} \right].
\]

(4.22)

Special cases are found for \( \theta = 1 \) (independence copula) and \( \theta = \infty \) (Fréchet upper bound). Furthermore, there is a theoretical relation between Kendall’s tau and the (flipped) Gumbel copula, being \( \tau = 1 - \theta \), which is often used as a starting point in calibrations of the (flipped) Gumbel copula.

### 4.2.8 Empirical copulas

The empirical copula is the multivariate empirical cdf of the historically observed residuals. It is a useful tool as input for different statistical tests for the quality of a copula fit, which compare the differences between the hypothesized copula and the empirical copula. Two alternatives of the empirical copula are presented, which are both used in the performance tests later on.

The Deheuvels (1979) copula is the most used empirical copula and is (in two dimensions) constructed based on a bivariate observed sample \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \). It is presented by Bouyé et al. (2000) as,

\[
C(\frac{i}{n}, \frac{j}{n}) = \frac{1}{n} \cdot \text{(number of pairs } x, y \text{ in the sample with } x < x^{(i)} \text{ and } y < y^{(j)}) , \\
= \frac{1}{n} \sum_{k=1}^{n} 1_{(x_k \leq x^{(i)})}1_{(y_k \leq y^{(j)})} .
\]

(4.23)

In this formula \( x^{(i)} \) and \( y^{(j)} \) denote the \( i^{th} \) and \( j^{th} \) order statistics of the samples \( x \) and \( y \) respectively, i.e. \( x^{(i)} \) is equal to the \( i^{th} \) -smallest value of the sample \( x \) and \( y^{(j)} \) is equal to the \( j^{th} \) -smallest value of the sample \( y \).

In fact, this formulation can be rewritten as a function of the empirical univariate cumulative distribution functions of the observations as follows,

\[
C(\frac{i}{n}, \frac{j}{n}) = \frac{1}{n} \sum_{k=1}^{n} 1_{(F_{\text{emp}}(x_k) \leq \frac{i}{n})}1_{(F_{\text{emp}}(y_k) \leq \frac{j}{n})} ,
\]

(4.24)

in which \( F_{\text{emp}}(\ldots) \) is the empirical univariate cumulative distribution function, defined as the rank number of the observation divided by \( n \).

If however \( F_{\text{emp}}(x_k) \) and \( F_{\text{emp}}(y_k) \) are replaced by the hypothesized residual distributions \( F_1(x_k) \) and \( F_2(y_k) \) respectively, then this results in the empirical copula given the residual distribution \( F_1 \) and \( F_2 \).
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4.2.9 Calibration of copula models

A copula is calibrated based on the standardized residuals of the univariate stock price and short rate models introduced in the previous chapter. According to Berentsen et al. (2014), there are three widely used methods for estimation of the parameters for the copula and residual model distribution. The first method is the maximum likelihood estimator, where the parameters of the copula and residual model are estimated together. The second, computationally more convenient method is to first estimate the residual model parameters using maximum likelihood, and then given the residual model estimate the copula parameter(s) using maximum likelihood estimation. This method, introduced by Joe and Xu (2016), is called the Inference Functions for Margins method (IFM). If no assumption is made about the residual model and the empirical residual distributions are used instead, the Canonical Maximum Likelihood (CML) method is obtained.

Take into consideration that for a good fit of the marginal distributions and a sufficiently large sample size of the historical data, the copula parameter estimation of the CML and the IFM method should be approximately the same. Durrleman et al. (2000) argues that if this is not the case, it could indicate that the residual distributions do not fit the data well. According to Scaillet et al. (2007) without any valuable prior information, non-parametric estimation (CML) should be favoured (above IFM).

Figure 4.4 visualises the different methods and in the next subsections the three methods are explained in detail.

Maximum likelihood estimation

Maximum likelihood (ML) estimates the parameters of the marginal distribution functions and copula jointly, by maximizing the (log-)likelihood. The likelihood is the multiplication of the probability density function of the observed returns at each timestep, given the parameters of the copula and marginal distribution functions, i.e. for two dimensions we get,

\[
L(x|\theta_{C1}, \theta_{C2}, \theta_{F1}, ..., \theta_{F1}, \theta_{F2}, ..., \theta_{F2}) = \prod_{i=1}^{n} f(x_{i1}, x_{i2}|\theta_{C1}, ..., \theta_{F1}).
\]

in which \( f(x_{i1}, x_{i2}) \) is the copula density as a function of the observed residuals \( x_{i1} \) (residuals from stock price model) and \( x_{i2} \) (residuals from short rate model). \( n \) is the number of observations, \( \theta_{C1}, \theta_{C2} \) are the copula parameters (which is only one parameter in the case of the Gaussian or flipped Gumbel copula), \( \theta_{F1}, ..., \theta_{F2} \) are the parameters of the stock price residual model \( F_1 \) and \( \theta_{F1}, ..., \theta_{F2} \) are the parameters of the short rate residual model \( F_2 \). The density of the copula is however often not directly observed, hence it should be derived by taking the derivative of the copula, resulting in,
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\[ L(x|\theta_{C^1}, ..., \theta_{F^2}) = \prod_{i=1}^{n} \frac{\partial^2 C(F_1(x_{i,1}), F_2(x_{i,2}))}{\partial x_{i,1} \partial x_{i,2}}, \quad (4.27) \]

\[ = \prod_{i=1}^{n} \frac{\partial^2 C(F_1(x_{i,1}), F_2(x_{i,2}))}{\partial F_1 \partial F_2} \frac{\partial F_1(x_{i,1})}{\partial x_{i,1}} \frac{\partial F_2(x_{i,2})}{\partial x_{i,2}}, \quad (4.28) \]

\[ = \prod_{i=1}^{n} \frac{\partial^2 C(F_1(x_{i,1}), F_2(x_{i,2}))}{\partial F_1 \partial F_2} f_1(x_{i,1}) f_2(x_{i,2}). \quad (4.29) \]

In this representation, \( |\theta_{C^1}, ..., \theta_{F^2}| \) is omitted in the right side of the formula for better readability of the formulas. Furthermore, \( f_1(x_{i,1}) \) and \( f_2(x_{i,2}) \) are the densities of the stock price residual model and short rate residual model respectively. The log likelihood follows by taking the logarithm,

\[ l(x|\theta_{C^1}, ..., \theta_{F^2}) = \sum_{i=1}^{n} \left[ \log \left[ c(F_1(x_{i,1}), F_2(x_{i,2})) \right] + \log f_1(x_{i,1}) + \log f_2(x_{i,2}) \right]. \quad (4.30) \]

The parameters in the maximum likelihood calibration are then those values for which the log-likelihood is maximized. This is done by taking the derivatives of the log-likelihood with respect to all parameters \( \theta_{C^1}, \theta_{C^2}, \theta_{F_1^1}, ..., \theta_{F_1^2}, \theta_{F_2^1}, ..., \theta_{F_2^2} \) individually, setting these formulas equal to zero and then look for the global maximum.

**Inference Functions for Margins**

The Inference Functions for Margins (IFM) introduced by Joe and Xu (2016) first uses maximum likelihood estimation to calculate the parameters for both residual models. Then treating the residual model parameters as given, it uses maximum likelihood to estimate the copula parameter(s),

\[ l(x|\theta_C) = \sum_{i=1}^{n} \log \left[ c_\theta(F_1(x_{i,1}), F_2(x_{i,2})|\theta_C) \right] \quad (4.31) \]

Now maximize the log-likelihood by solving the following equation and then look for the global maximum,

\[ \frac{\partial l(\theta_C|x)}{\partial \theta_C} = 0. \quad (4.32) \]

**Canonical Maximum Likelihood**

The Canonical Maximum Likelihood (CML) is the same as the IFM method, in which the marginal distribution functions are replaced by their empirical distribution functions. This empirical cumulative distribution function is equal to,

\[ \hat{F}_{\text{emp}}(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{x_i \leq t}, \quad (4.33) \]
CML uses maximum likelihood to estimate the copula parameter(s), as was also done with the IFM method,

\[ l(\theta_C) = \sum_{i=1}^{n} \log \left[ c_\theta \left( \hat{F}_{\text{emp}}(x_{i,1}), \hat{F}_{\text{emp}}(x_{i,2}) \right| \theta_C \right] \]  

(4.34)

Now maximize the log-likelihood by solving the following equation and then look for the global maximum,

\[ \frac{\partial l(\theta_C|x)}{\partial \theta_C} = 0. \]  

(4.35)

4.2.10 Simulation from copulas

The two randomly simulated copula samples should have the distribution which is implied by the copula. For simulation of a univariate distribution, the idea is to simulate a uniform random sample and transform the sample using the inverse cumulative distribution function, such that it is distributed according to the desired distribution. This is however not directly possible for multivariate distributions.

For multivariate cumulative distribution functions, such as a copula, this method is adjusted because the inverse of the cdf has not one, but two output variables and therefore the inverse cannot be taken. The method is explained in most copula books or articles (e.g. Section 2.9 in...
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Nelsen (2007)) and makes use of the fact that a conditional distribution is perpendicular to the distribution on which it is conditioned, i.e.

$$z = C(u|v) \perp v.$$  \hfill (4.36)

This variable $z$ is distributed uniformly in $[0, 1]$, given $v$, and is independent from $v$. Both $z$ and $v$ are uniformly distributed, independent of each other. To find $u$ and $v$, which are distributed according to the copula cdf, take the inverse of this conditional cdf, which (if the inverse exists) results in,

$$u = C^{-1}(z|v).$$  \hfill (4.37)

Hence, if the inverse exists, an expression is found for the variable $u$ based on two uniform random samples $v$ and $z$, such that $u$ and $v$ are distributed according to the copula distribution. The remaining part is then to compute the conditional distribution of $C(u|v)$, which is done as follows,

$$C(u|v) = \frac{P(U < u \cup V = v)}{P(V = v)},$$

$$\approx \frac{P(U < u \cup V < v + dv) - P(U < u \cup V < v)}{P(V < v + dv) - P(V < v)},$$

$$= \frac{C(u, v + dv) - C(u, v)}{dv},$$

$$= \lim_{dv \to 0} \frac{\partial C(u, v)}{\partial v}.$$

Now to construct two uniform vectors $u$ and $v$ distributed according to the copula, start with two independent uniform vectors $u$ and $z$. Based on the relation above and the values of $u$, transform $z$ into vector $v$. Then use the marginal distributions to get the samples from the original distributions.

### 4.3 Alternative dependence models

Copulas are not the only way to model dependence. Also multivariate distributions can be used to directly model the combined observations.

The disadvantage of multivariate distributions is that we cannot easily separately choose the univariate distributions and the type of dependence. For example the bivariate Gaussian is a combination of two normal marginal distributions with correlation parameter $\rho$, but we cannot easily construct a distribution with a correlated Gaussian marginal with a Student’s $t$ marginal for example.

Moreover, from Sklar’s theorem (Theorem 4.2.1) it follows that each multivariate distribution function can be written as two marginal distribution functions and a copula, which means that using copulas does not exclude any multivariate distribution function, but does add much flexibility.

Another method, which also gives some flexibility in modelling dependence, is regime switching. The correlation in the tails of the distributions could for example be modelled apart from the correlation in the body of the data with this method.

These alternative methods are not used in the thesis, but are mainly given to remind that copulas are not the only possible way to model dependence.
4.4 Copula models in economic scenario generators

The ESGs introduced in Section 3.5 are extended with three different copulas and two calibration methods. The Gaussian copula, Student’s t copula and flipped Gumbel copula are analysed for all four combinations of univariate simulation models.

The Gaussian copula is the most used copula and is therefore a logical choice for comparing the results. The Student t copula is a symmetric implicit copula based on the multivariate Student’s t distribution, enabling fatter tails in all quadrants, thereby making it interesting for risk analysis. The flipped Gumbel copula is an asymmetric explicit Archimedean copula allowing more dependence in the negative tail than in the positive tail. This behaviour is in line with historical observations in which more combined extreme negative residuals occur than extreme positive combined movements.

The Inference Functions for Margins and the Canonical Maximum Likelihood method are used for calibration of the ESGs. Maximum likelihood is not used because of computational complexity.

The final ESGs consist of one of the univariate model combinations indicated by \( x \in \{A, B, C, D\} \) in Table 3.1, combined with a number 1, 2, ... or 6 given in Table 4.1. This number indicates which copula and calibration method is used. The results of the performance measures (Chapter 5) are given for each ESG in Chapter 6.

<table>
<thead>
<tr>
<th>ESG</th>
<th>copula</th>
<th>calibration method</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>Gaussian</td>
<td>CML</td>
</tr>
<tr>
<td>x2</td>
<td>Gaussian</td>
<td>IFM</td>
</tr>
<tr>
<td>x3</td>
<td>Student t</td>
<td>CML</td>
</tr>
<tr>
<td>x4</td>
<td>Student t</td>
<td>IFM</td>
</tr>
<tr>
<td>x5</td>
<td>Flipped Gumbel</td>
<td>CML</td>
</tr>
<tr>
<td>x6</td>
<td>Flipped Gumbel</td>
<td>IFM</td>
</tr>
</tbody>
</table>

Table 4.1: ESGs overview, with \( x \in \{A, B, C, D\} \) indicating the univariate simulation models of Section 3.5.
Chapter 5

Performance measures

This chapter describes how to compute market risk using value at risk and expected shortfall and how to measure the performance of the ESGs and their ability to calculate market risk. Different performance measures are introduced to test the performance in terms of accuracy, stability, resilience (by comparing stress test results) and complexity, and also model risk is covered.

The market risk is quantified using the 1-day and 1-year 99.5% value at risk, as well as the 1-day and 1-year 97.5% expected shortfall.

Section 5.1 introduces the market risk measures, Section 5.2 describes the performance measures for the market risk calculations for the univariate simulation models used in the ESGs and Section 5.3 describes the performance measures for the market risk calculations in the constructed portfolio and for the combination of marginal distributions, copula and calibration type together in the ESGs.

5.1 Market risk: value at risk & expected shortfall

Def. (Market risk). Market risk is the effect of movements in equity prices and short rates on the 99.5%—value at risk and 97.5%—expected shortfall of an asset portfolio.

The goal of the dependence modelling in this thesis is to decide whether copulas are a good alternative (compared to Pearson correlation) for the calculation of market risk. Market risk is quantified with the 1-day and 1-year 99.5% value at risk (VaR) and 97.5% expected shortfall (ES), and is calculated for the stock prices and short rates separately and for the combined portfolio value. The definitions of these measures is given below.

Measure def. (Value at risk). The 1-day (1-year) \( VaR_{0.995} \) is a market risk measure. It estimates how much a set of investments might lose with probability 0.5% within 1-day (1-year), given normal market conditions.

Measure def. (Expected shortfall). The 1-day (1-year) \( ES_{0.975} \) is a market risk measure. It estimates the average loss of a set of investments, given that we are in the worst 2.5% percentage of losses within 1-day (1-year), given normal market conditions.

The bond value in the portfolio is negatively dependent on the short rates, and the risks for the short rate are therefore measured in the right tail. The market risk measures are calculated
using Monte Carlo simulations. The number of timesteps in our simulations is 252 per year and the number of paths is 10,000.

Furthermore, for visualisation of these results, a forecast of these 99% VaR confidence bounds (i.e. the 0.5% upper and lower bounds) for one year for the stock price, short rate and bond price is given, as well as for the portfolio value. These plots are based on the simulations performed in the ESGs.

5.1.1 Justification of market risk measures

Justification for the risk measures is based on the Solvency II regulatory capital requirements for insurance companies. Insurance companies (under Solvency II) need to calculate a 1-year horizon 99.5% VaR. Banks (under Basel II / 2.5) need to calculate a 10-day 99% VaR. Furthermore, for banks also the 10-day 99% stressed VaR was introduced in Basel 2.5:

“In addition to the 10-day VaR requirements, the 2009 amendments require banks to calculate a ‘stressed VaR’ measure. The stressed VaR is intended to replicate a VaR calculation that would be generated on the bank’s current portfolio if the relevant market factors were experiencing a period of stress. It should be based on the 10-day, 99th percentile, one-tailed confidence interval VaR measure, with model inputs calibrated to historical data from a continuous 12-month period of significant financial stress. The introduction of stressed VaR is intended, in part, to dampen the cyclicality of the VaR measure and to mitigate the problem of market stresses falling out of the data period used to calibrate the VaR after some time.”

For banks capital requirement regulations are changing. The Basel Committee on Banking Supervision (BCBS) initiated a ‘fundamental review of the trading book’ (FRTB) regime. The BCBS published the first consultative paper in May 2012, in which the BCBS has confirmed its intention to pursue two key reforms, from which one is the ‘move from VaR to ES’. The reason is that a number of weaknesses have been identified with using VaR for determining regulatory capital requirements, including its inability to capture ‘tail risk’. ES measures the riskiness of a position by considering both the size and the likelihood of losses above a certain confidence level. The BCBS has agreed to use a 97.5% ES for the internal models-based approach and has also used that approach to calibrate capital requirements under the revised market risk standardised approach.

It is expected that also Solvency will move from VaR to ES, because of the weaknesses mentioned by the BCBS and because the VaR is not a coherent measure of risk (for example adding two risky portfolios could increase the total risk, whereas a coherent measure, see Artzner et al. (1999), would ensure that the total risk is non-increasing).

5.2 Univariate model performance measures

This section introduces the risk measures for the analysis of the univariate model simulations individually, excluding the combined behaviour. An overview of all univariate measures used is given below.

- Accuracy
  - Ljung-Box auto-correlation test on (squared) residuals
  - Kolmogorov-Smirnov goodness-of-fit test
  - Anderson-Darling goodness-of-fit test
  - 1-day (10-day) one-sided binomial VaR backtest
  - 1-day (10-day) two-sided Kupiec VaR backtest
- Stability
5.2.1 Accuracy

**Def.** (Accuracy). **Accuracy is the agreement in distribution between historical stock prices (or short rates) and the model distribution.**

More precisely, the accuracy of the simulation models introduced in Chapter 3 is the agreement between the distribution of the historical residuals and the theoretical residual model (see Definition 3).

The agreement in distribution is measured in terms of auto-correlation, matching quantiles of the historical residuals and model distribution and VaR backtesting.

The auto-correlation plot is used to visualise the auto-correlation in the historical (squared) residuals. It is assumed that if most lags lie between the 95% confidence interval bounds, no significant auto-correlation is present. The Ljung-Box test is used to formally check the amount of auto-correlation in the historical (squared) residuals.

**Measure def.** (Ljung-Box test). The Ljung-Box test is used to formally test whether a no autocorrelation null hypothesis should be rejected using a significance level $\alpha = 5\%$. When the $p$-value following from Ljung-Box is smaller than $\alpha$, the null hypothesis should be rejected (for that significance level). If $p$ is larger than $\alpha$ no conclusion can be drawn.

The agreement between the empirical residual quantiles and the theoretical quantiles of the residual model distribution is visualised with the quantile-quantile (qq) plot. If the observations lay approximately on a straight line in this plot, then the (historical and theoretical) distributions are close to each other. The Kolmogorov-Smirnov test and Anderson-Darling test are used to formally test whether the quantiles agree. The definitions of these two measures are given below.

**Measure def.** (Kolmogorov-Smirnov test). The Kolmogorov-Smirnov (KS) test is used to formally test whether the maximum distance between the empirical distribution and the theoretical hypothesized distribution can be explained by the randomness of the simulation or whether the hypothesized distribution should be rejected for a significance level $\alpha = 5\%$. If the $p$-value following from the KS test is smaller than $\alpha$ the null hypothesis should be rejected.

The measure is based on the following distance,

$$D_{KS} = \sup_x |F_E(x) - F_H(x)|,$$

and is distribution free; i.e. given the value $D_{KS}$, the $p$-value does not depend on the underlying distribution anymore and is given by critical value tables for the KS test.

**Measure def.** (Anderson-Darling test). The Anderson and Darling (1952) (AD) test does the same as the KS test, but gives higher priority to the tails than to the body of the data by applying a scaling.

The measure is based on the following distance,

$$D_{AD} = n \int_{-\infty}^{\infty} (F_E(x) - F_H(x))^2 \frac{dF(x)}{F_H(x)(1 - F_H(x))},$$

(5.2)
and the test statistic $A^2$ is given by,

$$A^2 = -n - S,$$  \hspace{1cm} (5.3)

where $n$ is the number of observations and

$$S = \sum_{i=1}^{n} \frac{2i - 1}{n} \left[ \ln(F(Y_i)) + \ln(1 - F(Y_{n+1-i})) \right],$$  \hspace{1cm} (5.4)

with $F$ the cumulative distribution function of the tested distribution. In contrary to the KS-test, the AD-test is distribution dependent; i.e. given the value $A$, the $p$–value has to be determined based on the underlying distribution which is done based on simulations (i.e. bootstrapping) or based on a $p$–value table for known distributions.

The accuracy of the tails of the theoretical residual distribution is measured with value at risk back-testing. The historical VaR is visualised, i.e. the historically observed residuals of the Black-Scholes and Hull-White models are plotted, in which the historical 99\% confidence bounds (i.e. 0.5\% upper and lower bounds) are given for the four different models $A – D$. Two tests described in Chapter 8.6 of Hull (2012) are used for back-testing the VaR, namely the one-sided binomial and two-sided Kupiec (1995) test. The tests are performed on the univariate model residuals for the 1–day and 10–day 99.5\%–VaR as introduced in Definition 5.1. The definitions of the tests are given below.

**Measure def. (One-sided binomial test).** The one-sided binomial test determines whether the VaR is underestimated or not. Days when the actual change exceeds the VaR at a confidence level $\alpha$ are referred to as exceptions. The one-sided binomial test is used to compute the probability of $m$ exceptions in the data,

$$\sum_{k=0}^{m} \frac{n!}{k!(n-k)!} \alpha^k (1 - \alpha)^{n-k}. $$  \hspace{1cm} (5.5)

The null-hypothesis is rejected for values of this probability below the confidence level of $\alpha = 5\%$.

**Measure def. (Two-sided Kupiec test).** The two-sided Kupiec test states that if the probability of an exception under the VaR model is $p$ and $m$ exceptions are observed in $n$ trials, then

$$-2 \ln[(1-p)^{n-m}(p)^m] + 2 \ln[(1-m/n)^{n-m}(m/n)^m] $$  \hspace{1cm} (5.6)

should have a chi-square distribution with one degree of freedom. The value of the statistic is high for very high or very low numbers of exceptions. The chi-square distribution exceeds the value 3.84 with probability 5\%, therefore the VaR of the model should be rejected based on a confidence level of 5\% whenever the expression above is greater than 3.84.

The historical number of exceptions for the 1–day tests is the number of standardized residuals exceeding the theoretical 99.5\% quantile of the underlying distribution (i.e. normal, student’s t or pareto tailed).

The historical number of exceptions for the 10–day tests is more difficult to calculate, especially for the Student’s $t$ and Pareto distributed residuals, because the sum of 10 Student’s $t$ or Pareto distributed residuals does not have the same distribution. Simulations are used to overcome this problem. For each 10th timestep, the model is used to simulate 10–day forward stock prices and short rates, and based on these simulations a VaR is constructed at these timesteps. For each of the simulations it is checked whether the actual 10–day future stock prices or short rates are smaller or larger than the simulated 10–day VaR. The number of exceedances is then compared with the expected number of exceedances, which is equal to 0.005 times the total number of
10–day observations. Hence, one tenth of 1509 points in the historical observations are checked for their 0.5% quantile, and therefore on average only 0.75 exceptions are expected.

5.2.2 Stability

**Def. (Stability).** *Stability is the sensitivity of the model parameters and market risk calculations to changes in data used for calibration, and the ability to minimize the effect of outliers in the data.*

Stability is measured with relative measures, i.e. no rejection criteria are used. Stability is visualised by plotting the calibrated parameter values for different calibration time-windows. A moving time-window of 6-month length is used, and a decreasing time-window starting with all data from 01-01-2010 to 11-12-2015. On the x-axis the starting date of the historical sample is increased in steps of 20 days and the last date of the sample is fixed. The y-axis gives the calibrated model parameters for all models $A – D$.

If the resulting parameter plots are approximately constant over time, no drift seems to be present and no jumps occur, the parameter is said to be stable. The measure is relative compared to the other parameters, where the model is said to be more stable if the parameters of a model seem to be more flat than for other models.

5.3 Economic scenario generator performance measures

This section describes how to measure the performance of the ESGs and their ability to calculate market risk in the constructed asset portfolio. Different performance measures are introduced to test the performance in terms of accuracy, stability, resilience (by comparing stress test results) and complexity, and also model risk is covered.

The set-up of this section is the same as for Section 5.2. Some definitions introduced in the univariate case are comparable to the definitions used in this section and therefore only the adjustments are mentioned instead of restating the definitions.

An overview of the analyses treated in Section 5.3.1–5.3.5 is given below.

- **Accuracy**
  - Adjusted Kolmogorov-Smirnov goodness-of-fit test for copulas
  - Adjusted Anderson-Darling goodness-of-fit test for copulas

- **Stability**
  - Moving time-window parameter plot for copula parameter
  - Decreasing time-window parameter plot for copula parameter
  - Effect of outliers on copula calibration

- **Resilience: effect of stress tests on model and market risk calculations**
  - Change copula to independence copula
  - Change copula to Fréchet upper bound
  - Change copula to Fréchet lower bound
  - Shock volatility parameter of stock / short rate process
  - Calibrate model based on stress scenario 2008-2009

- **Complexity**
  - Number of parameters
  - Calibration method
Chapter 5. Performance measures

5.3.1 Accuracy

Based on the definition from Section 5.2.2, the accuracy is the agreement between the combined distribution of historical stock prices and short rates and the model simulation.

Adjusted versions of the Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) test are used as accuracy measures. These adjustments to the KS and AD tests are introduced by Kole et al. (2005) and compare the empirical copula $C_E(x, y)$ (Equation 4.24) of the historical residuals $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ with the calibrated hypothetical copula $C_H(u, v)$ in the points $u = (1/n, 2/n, ..., 1)$ and $v = (1/n, 2/n, ..., 1)$. The method is used for the copulas that are calibrated with the CML method, because the calibration minimizes (using maximum likelihood) the distance between the empirical DeHeuvels copula and the theoretical copula, and it hence makes sense to test for these differences.

For the copula models calibrated with the IFM method a small adjustment is used. The IFM calibration compares (using maximum likelihood) the cdf values resulting from the univariate model calibrations with the copula model, hence it compares in fact not the DeHeuvels empirical copula, but uses Equation 4.25. Hence, it makes sense to test for the differences between the copula observations and the empirical copula given the residual distributions.

Two of the tests statistics shown in the article are used,

$$D_{KS} = \max_{(u,v)} |C_E(u, v) - C_H(u, v)|,$$

$$D_{AD} = \max_{(u,v)} \frac{|C_E(u, v) - C_H(u, v)|}{\sqrt{C_H(u, v)(1 - C_H(u, v))}}.$$

The KS test statistic is the maximum distance between the empirical and theoretical copula. The AD test statistic is the weighted KS test statistics in which the tails get more weight than the bulk of the data and should therefore be more useful for risk calculations in the tails of the data.

In contrary to the one-dimensional case, both the adjusted KS and AD tests are distribution dependent (because of the dependence structure between the variables) and the $p-$values are both determined with simulations (i.e. bootstrapping).

5.3.2 Stability

For the performance of the ESGs, the copula parameter stability is visualised and compared to other copula parameters. The tests are performed for a moving and for a decreasing time-window in the same way as in Section 5.2.2.

If the resulting parameter plots on the y-axis are approximately constant over time, no drift seems to be present and no jumps occur, the ESG is said to be stable. The measure is relative compared to the other ESGs, where the model is said to be more stable if the graphs seems to be more flat than for the other ESGs.

Furthermore, the theoretical stability of and the effect of outliers on the Gaussian copula is analysed.
Sensitivity analysis of Gaussian copula

The theoretical stability of the Gaussian copula parameters on the probability density function of the copula is investigated by computing the derivative of the copula density with respect to the copula parameter. For most explicit copulas the result is complex, because of three partial derivatives in the computation,

$$\frac{\partial c(u,v)}{\partial \theta} = \frac{\partial^3 C(u,v)}{\partial u \partial v \partial \theta},$$  \hspace{1cm} (5.9)

which leads to an equation of multiple pages for most explicit copulas. For this reason the theoretical stability is not shown for explicit copulas. For implicit copulas however, the derivation is more easy because $c(u,v)$ is given explicitly and only one derivative has to be computed.

Figure 5.1 shows the result for the Gaussian copula, for different values of $\rho$. For $(u, v, \rho)$ close to one of the corners $(1, -1, -1), (-1, 1, -1), (1, 1, 1)$ or $(-1, -1, 1)$, the size of the derivatives rapidly increases. Hence, small changes in the copula parameter have mainly influence on the density of the copula in the corners. Moreover, the closer the copula parameter is to its maximum (or minimum), the more influence it has on the density in the corners.

Figure 5.1: Derivative of Gaussian copula with respect to $\rho$, for several values of $\rho$.

The effect of outliers on the calibrated parameter value depends on the calibration method. For IFM calibration of the Gaussian copula with normal marginal distributions, the calibration comes down to calculation of the correlation between the observations $x$ and $y$. Consider the following: Suppose that, in a calibration of the parameter based on $n$ historical observations $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, one observation is changed into an outlier $(x_i, y_i) = (p, q)$. Suppose all observations are standard normally distributed (i.e. $N(0, 1)$) except for the outlier $(p, q)$. Then the parameter $\rho$ is,
\[ \rho = \frac{E[x \cdot y] - E[x]E[y]}{\sigma(x)\sigma(y)}, \]
\[ = \frac{(p \cdot q) + \rho \cdot n - \frac{1}{n}(p \cdot q)}{\sqrt{n - 1 + p^2} \cdot \sqrt{n - 1 + q^2}}, \]
\[ \approx \frac{p \cdot q + \rho \cdot n}{\sqrt{n + p^2} \cdot \sqrt{n + q^2}} \text{ for large } n. \quad (5.10) \]

For ‘symmetrical’ outliers \((p, p)\) this expression simplifies to,
\[ \rho \approx \frac{p^2 + \rho \cdot n}{n + p^2}. \quad (5.11) \]

Hence, outliers with \(\frac{p^2}{n} \neq 0\) and / or \(\frac{q^2}{n} \neq 0\) outliers have significant influence on the Gaussian copula with normal marginal distributions.

For symmetric outliers with \(p^2 \gg n\) then \(\rho \approx 1\). Asymmetric outliers with \(p \gg \sqrt{n}\) and \(q \ll -\sqrt{n}\) lead to \(\rho \approx -1\). Asymmetric outliers where only \(p^2 \gg n\), but \(|q| \approx 1\) lead to \(\rho \approx 0\).

The impact of outliers on the parameters is larger for IFM calibration than for CML calibration. The outliers have impact on the univariate model calibration, but also lead to problems in the calibration of the copula. If the CML method is used for calibration, the impact on the copula parameter is smaller because the CML method uses only the rank as input for calibration and not the actual value of the observations. Hence, extreme values are just handled as the largest (smallest) observation, but not as being an outlier and thus the CML calibration is more robust than IFM.

### 5.3.3 Resilience: stress testing

**Def.** (Stress testing). Stress testing is computing what would happen in the model under certain extreme scenarios. We look at the increase in market risk based on shocks in the parameters and copula, and the effect on the calibration.

The resilience of the models is measured with stress-testing, by re-calculating the 1-year 99.5% VaR and 97.5% ES for the stressed scenarios. The stress test results are analysed to determine whether the model still performs well under extreme circumstances. The following shocks are defined,

1. Change the copula to the independence copula \(C(u, v) = u \cdot v\).
2. Change copula to Fréchet upper bound
3. Change copula to Fréchet lower bound
4. Shock volatility parameter of stock price and short rate process with a factor 1.5.
5. Determine the parameters based on a stress period in 2008 - 2009 and use this as stress scenario. The resulting 1-year 99% VaR is the stressed VaR.

Based on the most realistic scenario (i.e. scenario 5), a judgement is made whether the performance is relatively good +, medium ++ or bad −.

### 5.3.4 Complexity

**Def.** (Complexity). Complexity is the degree in which a model can produce unexpected errors, due to the combined behavior of different sub-models.
Changing the dependence structure influences the complexity and dimensionality of the model. The amount of calibrated parameters determines part of the complexity, because extra parameters can lead to over-parametrization. The choice of the IFM method in the calibration also adds complexity to the model, because errors made in the calibration of the underlying univariate models are transferred to the calibration of the copula models (i.e. the input for the copula calibration should be uniform distributed, whereas the residuals following from the IFM calibration are most times not really uniform distributed). For CML calibration these errors are reduced due to the fact that calibration is based on the ranks of the data instead of the actual cdf values, hence the errors resulting from the univariate models are sort of masked.

Based on the following three aspects a ranking from \(-\) to \(+\) is discussed for the complexity,

- Number of parameters
- Calibration method
- Overall complexity

5.3.5 Model risk

**Def. (Model risk).** Model risk consists of limitations of the model, and users of the model who may not completely understand its assumptions and limitations, thereby limiting the usefulness and application of the model itself.

This section is mainly intended to raise some questions which a user of the proposed models should be aware of and should take into consideration. No foolproof solutions to this risks are given, but awareness is the first step. Hence, no specific measures are given and, except for the overall analysis here, no analysis of the different models is presented in Chapter 6.

Model risk due to limitations occurs in every model, because by definition a model is a simplification of the real-world. The limitations can occur due to a wrong choice of model, wrong assumptions in the model, due to a lack of data for calibration or due to changes in market behaviour, such as structural breaks after which historical based models may not be useful anymore.

Model risk due to wrong use of a model often occurs because of not taking into account certain boundaries of the model in which it behaves well, and not taking into account that the outcomes of the model might mis-estimate risks due to the limitations. One example of model risk in copula models was given an article which is written as a reaction on the financial crisis of 2008-2009 and it is given in the paragraph below. Other limitations are presented in the subsequent paragraph.

**Recipe for disaster: The formula that killed Wall-Street**

In the period before the crisis, the Gaussian copula was used to calculate the combined risks in investments of banks. The Gaussian copula model combined all investments in one model and allowed combined risks for a large part to be cancelled out due to diversification (and no perfect correlation). However, one of the limitation of the model was that it did not take systemic risk explicitly into account. For example, a crash in the housing markets in 2008-2009 led to a drop in almost all housing prices simultaneously (which can be seen as a strong tail correlation effect), an event which was almost impossible from the Gaussian copula perspective.

On the 22nd of February 2009 an article appeared (*Recipe for Disaster: The Formula that Killed Wall Street. 2009*), in which the Gaussian copula is blamed to be one of the main causes of the financial crisis. Salmon blames the inventor of the copula-based risk model, David X. Li, and states:
Li’s Gaussian copula formula will go down in history as instrumental in causing the unfathomable losses that brought the world financial system to its knees.

This example clearly shows what the impact of model risk can be, and it shows that one should never blindly believe the outcome of a model. This specific model, the Gaussian copula, is indeed not perfect, but should also not be blamed to be the cause of the crisis. The model should be a tool to support risk calculations, but the limitations have to be taken into account.

**Limitations of the ESG models**

The main limitations of simulation models (for stock prices and short rates) in general is that the real-world scenarios are not based on Brownian motions, but on supply and demand. Macroeconomic factors are not always included in ESG models and the fact that some observations are driven by decisions of some large institutions is also not always taken into account, such as central banks lowering the short rates to historical low levels. Not taking into account these systemic factors can in times of stress lead to underestimation of risks.

Other problems of the ESGs are that calibration is performed historically and the copulas are static. Nothing guarantees that behaviour of short rates or stock prices in the past years are representative for future behaviour of these risk drivers. Market behaviour might change, just as dependence between risk drivers.

This thesis is mainly aimed at choosing the best possible model. Fitting a copula always gives some parameter (if the calibration process converges), but it does not guarantee that the distribution of the copula is close to the true (or historical) distribution. If for example historical observations are asymmetrically distributed, but a symmetric copula is chosen as a model, the model can of course never find a good fit for the historical data. The same holds for example for the Gumbel copula which can only model positive dependence, hence calibration of the model for negative dependent data leads to either the independence copula or the calibration algorithm does not converge at all. The empirical distribution of the historical residuals should therefore always be compared closely with the calibrated model to minimize the risks involved in the choice of the model.

A copula model can be calibrated in different ways. The three methods introduced in Section 4.2.2 give different results and therefore if not one method is preferred above the others, this could lead to (slightly) different calculations of market risks. The question then is: which one is the right risk, and the truth is that the calculated risks are all not the truth but just an estimation based on the assumption of a certain model. Furthermore, as is explained in Section 5.2.2 the effect of outliers on the calibration of copulas can be significant. The calibration for different methods (CML / IFM) should therefore be compared and should lead to comparable results.

A lack of data can result in a bad copula choice, because historically it cannot be determined what the empirical dependence structure looked like. Lack of data can also result in a poor calibration for the same reason. In the end this means that the model might not be reliable for risk calculations.

The combination of GARCH models and copulas makes it more easy to use the favorable aspects of both kind of models. However, the reason for using copulas at first hand was to increase the dependence between the random variables in the tails of the distribution. However, if we are in high volatile markets due to the GARCH model in which we expect more extreme high and low realizations of the short rate and stock price simultaneously, the returns are not more dependent than in low volatile periods. This suggest that the combination of GARCH and copulas is not ideal, which is also seen in the calibrations of the stress test based on 2008-2009; i.e. we observed that during this period the dependence parameter observed in the copula calibrations was much stronger than in the quiet periods, but this is not reflected in our model.
The use of heavy tailed distributions such as the Student’s \textit{t} and Pareto tailed normal, lead for daily returns to more extreme estimations of the VaR and ES. However, as observed in the results, it can lead to limited estimations of market risk on yearly basis. This behaviour might not be in line with the historically observed behaviour.

\section*{5.4 Combining risk measures}

Combining the measures from Section 5.2 and 5.3 leads to a ranking of the performance of the market risk models. The results are valued $-, +-$ or $+$, in which $+$ means that the model performs better than the others, but it does not necessarily mean the models performs good. Combining the results leads to an overall ranking of the methods for calculation of market risk.
Chapter 6

Results

Results are produced for all univariate model combinations $A - D$ introduced in Section 3.5 and copula models 1 – 6 from Section 4.4. The performance measures are all introduced in Chapter 5. Most results are obtained by calculations using the statistical program $R$, and some tests are performed with Matlab.

The results section is divided in three sections. Section 6.1 describes the calibrations for the different ESGs models. Then based on the performance measures, Section 6.2 first describes the obtained market risk calculations, Section 6.3 describes the univariate model performance and Section 6.4 describes the ESG performance on portfolio value level.

6.1 Calibration results

Figure A.3 in Appendix A gives an overview of the residual analysis of the Euro Stoxx 50 index for the different univariate models $A - D$. In the left column, the auto-correlation function (ACF) plot for the residuals and squared residuals of the Black-Scholes (BS) model are given. The amount of significant lags in the squared residuals ACF indicate that the volatility of the stock prices are clustered, therefore the GARCH(1,1) model is included in the middle column. The auto-correlation is now indeed removed from the squared residuals, hence we conclude that the GARCH process was a good choice. The quantile-quantile (qq) plots show that Student’s $t$ distributed residuals better fit the tails of the distribution that the normal distribution does. The third column increases the GARCH order, but does not lead to significant improvements of the fit and is therefore not used. Hence, from this analysis we expect that model $C$ or $D$ will give the most accurate results.

Figure A.4 gives the results for the 3–month German bond yield. The Hull-White residuals are analysed in the left column. The ACF plot of the squared residuals indicates that the volatility of the short rates are also clustered, hence the GARCH(1,1) model is used again (middle column). Fitting the residuals of this GARCH model is however difficult. The qq plot comparing the residual quantiles with the quantiles of a normal distribution indicates that the residuals are more heavy tailed than the normal distribution. Combined calibration of the GARCH(1,1) model and Student’s $t$ distributed residuals leads to an unstable model (i.e. the persistence $\alpha + \beta > 1$), it is therefore chosen to first calibrate the GARCH model with normal residuals and then fit a Student’s $t$ distribution on the residuals. The Student’s $t$ distribution fits the heavy tails better, but the result is still not close to a straight line (i.e. the distributions still do not match). The last qq plot compares the quantiles of the Pareto tailed normal distribution with
the residuals. The results are comparable in the tails, but the kink in the plot observed around zero is smaller for the Student’s $t$ distribution.\(^1\)

The calibration of (constant) parameters in the ESGs (based on the data in Chapter 2) is given in Table 6.1, with the standard deviation given between brackets.

Remarkable is that $\omega_r$ in models $B - D$, the constant part in the GARCH volatility model of the Hull-White residuals, is not significant and hence is left out of the model. Furthermore, in short rate simulation model $D$, the significance of the $\xi$ parameters is doubtful, especially in the negative tail. However, from the analysis of the tails in Figure A.1 in Appendix A.2 the quantile distributions of the tails seem fine.

The results for the copula calibrations are given in Table 6.2. The first remarkable observation is that the dependence implied by the copulas based on model $A$ are all higher than the estimations following from model $B - D$. An explanation for this observation are the high volatile market periods, leading to volatility clustering in both residual models (i.e. stock prices and short rates) at the same time. These combined high- and low volatile periods, and the absence of a volatility model to compensate for the behaviour in the residuals in model $A$, lead to amplification of the dependent behaviour.

Also notice that the CML calibration results for models $B - D$ in the short rate model are identical. The reason is that all three calibrations are performed on the same residuals of the GARCH model and hence, the CML observations are identical. Also remarkable is that most of the calibrated copula parameters (based on 1509 residuals) are not significantly different from zero. The correlation coefficients in the Gaussian and Student $t$ copula should be at least 1.96 times the standard deviation in order to be significantly different from zero given a confidence interval of 95%. Furthermore, the flipped Gumbel copula with IFM calibration in model $B$ has a parameter of 1. This parameter value corresponds with the independence copula. The finding of this calibration can not be explained by Kendall’s tau (being 0.02), which would imply that the copula parameter is around 1.020. As a last remark, also the degrees of freedom parameters of the Student $t$ copulas have a high standard deviation. This is mainly due to the fact that for large values of $\nu$, the copulas are close to the normal copula which is the same as a Student $t$ copula with a $\nu$ parameter of infinity.

### 6.2 Market risk: value at risk & expected shortfall

The market risk in terms of value at risk and expected shortfall for a 1-day and 1-year horizon is given in Table 6.3 for the actual values of the stock prices, short rates and bond prices. However, because these values depend on the current stock price or short rate and on the current residual values and volatility, these results are only valid at 10-12-2015 (the discount curve of this day is used (see Chapter 2)) and cannot be used for different points in time due to the time-dependent GARCH models.

Furthermore, the VaR forecasts are also visualised in Figure 6.1 for the univariate models, and in Figure 6.2 for the portfolio value.

#### 6.2.1 Univariate model market risk

The VaR and ES results for the stock price models are as expected, $C$ and $D$ give the most extreme results on both daily and yearly basis. The stock price models for $C$ and $D$ are the same and on daily basis the extreme returns are caused by the heavy tailed Student’s $t$ distribution. The 1-year VaR and ES results are again the most extreme for model $C$ and $D$, caused by the combination of the GARCH model (high volatile periods make large drops in stock prices more

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\(^1\)Adding a moving average fit before applying the GARCH fit does improve the results of the Ljung-Box test and qq plot, but if we want to study the individual effects of all models, it would double the amount of analysed model combinations. Hence, it is not included in this study.
likely to occur) and heavy tailed Student’s t distribution. The results for model \( B \) are some less extreme due to the normally distributed random variables, but it is more extreme than model \( A \), which has no volatility model.

The findings for the short rates in the table seem somewhat adverse. For the 1–day VaR and ES, model \( B \) estimates less risk than the three other models (i.e. the VaR and ES differ less from the mean than for the other models). This is caused by the combination of the GARCH model forecasting (on average) a low volatility for the next day, hence the risk of model \( B \) is lower than model \( A \), which both have a normally distributed residual model. Model \( C \) and \( D \) however lead to more extreme observations due to the heavy tailed distributions. For the 1–year forecast model \( B \) has the most extreme estimation for the VaR and ES, which is different from what was observed in the stock price model. The reason that model \( C \) leads to estimations of even less risk than model \( A \) and also model \( D \) does not estimate extreme risk lies probably in the combination of the GARCH model with extremely heavy tailed distributions. This hypothesis is tested by first changing the residual model in short rate model \( A \) to Student’s t or Pareto tailed normal, with the same parameters as for model \( C/D \). These changes lead to a VaR of 0.011 and 0.011 respectively. Hence, the first conclusion is that summation over 252 timesteps of (extremely) heavy tail distributed residuals does lead to approximately the same result as normally distributed residuals (for which a VaR of 0.0127 was found). The combination of distributions \( C \) and \( D \) with a GARCH model however lead to lower risk calculations than GARCH combined with normally distributed residuals. The reason might be that due to the less dense bulk of the distributions, the high volatility realizations in the GARCH model revert faster back to the mean. The effect (of lower estimations of risk) becomes stronger if the degrees of freedom parameter is lowered in model \( C \) (e.g. using 2.1 degrees of freedom leads to a VaR of −0.002 in model \( C \)), confirming our hypothesis.

The difference between VaR and ES are for all ESGs comparable, for both time horizons of 1–day and 1–year, where the VaR results in some higher risk than the ES.

The 0.5% and 99.5% VaR confidence bounds forecasts of the stock price, short rate and bond price in one year (based on 10.000 Monte Carlo simulations) observed at 11-12-2015 are visualised in Figure 6.1.

### 6.2.2 Multivariate model market risk

The market risk calculations for the multivariate models in Table 6.4, which are computed with Monte Carlo using 10.000 simulations, show that the influence of the copula type is quite low. The results for the portfolio VaR and ES for 1–day ahead no not vary much. For the 1–year results some more fluctuations are observed, mainly between the different univariate models \( A – D \). The choice of the copula model and calibration methods seem however to have little impact on the outcomes of the market risk calculations. The reason for these small differences lies in the (very) weak dependence in the copula models (the correlation parameters are all between 0.029 and 0.055). Increasing the dependence (by comparing the value of the copula parameters for the CML and IFM method), gives a risk neutralizing effect, i.e. market risk calculation decrease. The effect of stronger dependence becomes visible in the stress tests.

The forecast results of the VaR and ES in the portfolio are visualised in Figure 6.2. The results for the different copula models are comparable, as already mentioned.

### 6.3 Univariate model performance

This section gives the results of the performance measures and plots introduced in Section 5.2, for the combinations \( A, B, C \) and \( D \) of univariate models. Table 6.3 shows the \( p \)–values and values of all test outcomes for all univariate models. The \( p \)–values are rejected for \( p < 0.05 \) (i.e. a 95% confidence interval). The results are discussing in more detail in the following subsections.
6.3.1 Accuracy

Model A consists of a log normal Black-Scholes stock model and a Hull-White short rate model. The volatility clustering in both sets of residuals is confirmed by the Ljung-Box test on the squared residuals. Furthermore, the Kolmogorov-Smirnov (KS) and Anderson Darling (AD) tests lead to rejection of the residual model distributions. This is in line with the binomial and Kupiec (back-)tests both leading to rejection of the number of value at risk exceedances, and it is also what was expected based on the quantile-quantile plots in which heavy tails are observed.

Model B improves the results of model A with a GARCH(1,1) volatility model for the stock- and short rate processes. The model is indeed improved, because the volatility clustering is mostly removed and the model cannot be rejected anymore based on auto-correlation in the squared residuals. Also the fit of the normal model on the residuals improved; i.e. the $p-$values of the KS and AD tests of the stock model increased, however the stock model is still rejected based on the $p-$values because of fat tails. For the short rate residuals, the fit improved a little, but the tails are more heavy than those of the normal distribution. As expected, the binomial and Kupiec tests both lead to rejection for single (1−day) residuals, however for the 10−days test results the number of observations in the 0.5% lowest quantiles is not rejected (probably due to the effects of the GARCH(1,1) model). The results for the 10−day Kupiec test of 'NaN' (Not a Number) arise due to the fact that historically no exceptions are observed and hence the $p-$value (which should be around 1) cannot be computed. However, this is interpreted as a no rejection, because no observed exceptions is reasonable if 0.75 exceptions are expected to occur. Model C uses Student’s $t$ distributed residuals, which are heavy tailed and better fit the observed heavy tails in both return sets. For the Black-Scholes GARCH(1,1) Student’s $t$ distributed residuals, the model cannot be rejected based on the KS and AD tests, hence the $t$ distribution is indeed a good choice for the stock residual model. For the Student’s $t$ distributed short rate residuals in the Hull-White GARCH(1,1) model, the distribution is however still strongly rejected based on the KS and AD tests. The quantile-quantile plot shows that the fit is improved, however the heavy tails are not fully captured and around zero a bump is observed. The 1−day binomial and Kupiec results are however not rejected anymore as was expected. Model D combines the stock model of model C with a short rate residual model which is normally distributed with Pareto tails. However the results are comparable with the Student’s $t$ distributed residuals; i.e. the model is still rejected based on the distribution of the residuals, probably caused by the bump in the quantile-quantile plot around zero.

For visualisation of the 1−day binomial and Kupiec backtests, the historical observed 1−day VaR based on the models is given in Figure 6.3. The black color are the actual residuals from Black-Scholes and Hull-White processes respectively, and the different colors indicate the 0.5% and 99.5% confidence bounds on which the rejections are based.

6.3.2 Stability

The stability of the univariate model parameters is visualised by performing the calibration of the model for a moving time-window and a decreasing time-window, as explained in Section 5.2.2.

Theoretically, the GARCH(1,1) models should be stable if the persistence (i.e. $\alpha + \beta$) is smaller than one, which is indeed the case for the BS-GARCH models. Remarkable is that the combined calibration for the GARCH(1,1) model with Student’s $t$ residuals distribution for the short rate model, lead to unstable results of the model (i.e. $\alpha_r = 1$ almost everywhere, and hence the persistence is larger than one). For this reason, the calibration of the parameters $\alpha_r$, $\beta_r$, and $\omega_r$ is based on the calibration with normal residuals, and only $\nu_r$ is based on the combined calibration, which gave $df = 2.45$ for the total calibration. A direct fit of the Student’s $t$ distribution on the residuals would lead to $df \approx 1.1$, leading to such extreme results that a Monte Carlo approach does not converge (i.e. for $df < 2$ the variance of the residuals does not exist). It does indicate however that the short rate model is definitely not perfect.
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Given that the GARCH models are theoretically stable, it is remarkable that the simulations of all three GARCH models for the short rate process leads to some realizations of extreme (unrealistic) volatility. The average volatility is around 0.7%, but in the 60,000 simulated paths for all three models $B - D$ the maximum observed volatility was above 100%, the volatilities do however return to the mean and therefore give no reasons for concerns.

Moving time-window

The moving time window has a length of 6 months and is performed for 01-01-2010 to 01-07-2010 (in the graphs the dates indicate the starting date of the time-window). Each time the time window is shifted 20 working days in the future and the model is re-calibrated. The results of these calibrations are visualised in Figure 6.4.

It is remarkable that the volatility parameters from the Black-Scholes and Hull-White process in model $A$ follow the same trend. The correlation between relative increments of both volatility processes is around 22%, even though the correlation between the original processes itself is only 5.5%.

The calibration of the mean parameter $\mu$ in the Black-Scholes model seems unstable when using a time-window of 6 months, because the trend moves between $-60\%$ to $60\%$, whereas the parameter for the calibration using all historical stock prices is $3.34\%$. Hence, if the trend is randomly varying and the time window is increased from 6 months to 6 years, then due to the law of large numbers the $\mu$ parameter will be between $-\frac{1}{\sqrt{12}}60$ and $\frac{1}{\sqrt{12}}60$, i.e. $[-17\%, 17\%]$. This would imply that the parameter of $\mu = 0.0334$ is the effect of noise and not a ‘stable’ parameter as is intended.

The calibration of $\alpha$ in the Black-Scholes GARCH(1,1) model is relatively stable: it is mainly between 0 and 0.2. For the combined calibration with Student’s $t$ residuals the results are approximately the same. The $\alpha$ parameter for the Hull-White GARCH(1,1) model is less stable and fluctuates between its maxima zero and one. The calibration combined with Student’s $t$ residuals is comparable.

Remarkable is that the combined calibration for the GARCH(1,1) model with Student’s $t$ residuals distribution for the short rate model, leads to unstable results of the GARCH model (i.e. $\alpha_r = 1$ almost everywhere, and hence the persistence is larger than one). For this reason, the calibration of the parameters $\alpha_r$, $\beta_r$ and $\omega_r$ is based on the calibration with normal residuals, and only $\nu_r$ is based on the combined calibration, as we already mentioned before.

The calibration of $\beta$ is for the Black-Scholes GARCH(1,1) with normal and with Student’s $t$ residuals comparable. Most calibrations lead to values around 0.8 – 0.9 (except some calibrations which did not converge and got value 0), comparable with the calibration of of the total model. For the Hull-White GARCH(1,1) calibration of $\beta$, the results are less stable, and $\beta$ fluctuates between 0 – 1.

The calibrated value of $\omega$ is for both models very small; for the Black-Scholes GARCH(1,1) model the value is negligible, for the Hull-White GARCH(1,1) model the value is mostly negligible and sometimes just a few percent. This suggests that the parameter is insignificant and will therefore be left out of the total model.

The degrees of freedom parameter for the Black-Scholes GARCH(1,1) model with Student’s $t$ residuals is (due to the calibration type) capped to 10, but the results fluctuate between 4 – 10 and are not (very) stable in the moving time-window. For the Hull-White GARCH(1,1) model the results are mainly around 2.5 which is also the value of the total calibration. This value seems more stable.
Decreasing time-window

The decreasing time-window calibrations start with the calibration using every historical time point between 01-01-2010 and 11-12-2015. In the graphs the dates indicate the starting date of the time-window, while the end date of the time frame is fixed at 11-12-2015. Each time the starting date of the time window is shifted 20 working days in the future and the model is re-calibrated. The results of these calibrations are visualised in Figure 6.5.

The parameters visualised in the decreasing time-window should in case of stability have a parameter value which fluctuates a little on the right of the graph (in which only the last half year is used for the calibration) and converges to some value on the left side of the graph (in which five years of historical data is used). Most conclusions are in line with the results of the moving time-window.

The volatility parameters are both not really converging to one certain value, however the fluctuations are small (Black-Scholes volatility is between 0.19 – 0.26 and Hull-White volatility is between 0.004 – 0.008. The observation that the volatilities move together is in line with the moving time-window and is clearly visible.

The $\mu$ parameter of the Black-Scholes model is rather unstable; with all historical stock prices a value of 3.34% is found, but removing the first two years of data increases the parameter to approximately 11%. This is again in line with the moving time-window results.

The GARCH(1,1) parameters $\alpha_S$ of the stock process converges quite well to a value around 0.9, while for the short rate process $\alpha_r$ fluctuates between 0.4 – 0.6, but does not really seem to converge to one specific value, especially due to the period 2011 – 2012 in which the process seems to jump. The combined calibration of the GARCH(1,1) with Student’s $t$ residuals results in $\alpha_r$ being constantly one, which was already observed in the moving time-window.

The $\beta$ parameter for the Black-Scholes GARCH(1,1) process converges for both residual models to around 0.9 without many fluctuations and thereby seems to be stable. The $\beta$ for the Hull-White GARCH(1,1) process seems to have a little downward trend when increasing the starting date of the time-window, but nevertheless fluctuates just a little.

The GARCH(1,1) $\omega$ parameters for both the stock price and short rate models seem to converge to a negligible small number.

6.4 Economic scenario generator performance

Table 6.4 shows the $p$-values and values of test outcomes for all copula models and the market risks based on the portfolio value simulations. The $p$-values are rejected for $p < 0.05$ (i.e. a 5% confidence interval). The 1–d sd and 1–y sd indicate the 1–day and 1–year standard deviation between the different realizations of portfolio values and give some indications about the spread of the portfolio realizations.

6.4.1 Accuracy

The adjusted Kolmogorov-Smirnov and adjusted Anderson-Darling tests indicate that all copula models using IFM calibration should be rejected, but all copula models using CML calibration are not rejected.

An explanation for this rejections is found by inspecting Figure 6.6. On the left, the cdf values of the combined residuals are given. On the right, empirical cdf values of the combined residuals are given. Notice that the amount of cdf observations (left) for the short rate process of approximately 0.5 is disproportionally large compared to a uniform distribution (this indicates that the short rate model is not perfect). By taking the scaled ranks as is done in the CML method (right), a much more uniform pattern is found. These observations are (empirically) translated
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to a multivariate distribution function (or copula), as in shown in Figure 6.7. The right graph is in fact the DeHeuvels copula (see Equation 4.24) on which the CML calibration is based and the left graph shows the empirical copula on which the IFM calibration method is based (see Equation 4.25). The graphs on the right are much closer to a copula than the left ones (e.g. the left figures do not satisfy Equation 4.5) and hence the observed rejections for the IFM calibrated copulas seem reasonable.

The significance of the copula parameters is low. In Table 6.2, the standard deviations of the calibrated copula parameters show that most of the copula calibrations are not significant, and most results are almost equal to the independence copula. This could also be one of the reasons that none of the three different copula types is rejected (for the CML calibration), i.e. the observed residuals do not significantly differ from independence, are thus all three comparable and lead to approximately the same probability of rejection.

6.4.2 Stability

Figure 6.8 shows the stability of the calibrated copula parameters for a moving time-window; i.e. the calibration is based on half a year of historical data. Then the time-window is shifted 20 days and the calibration is redone. This procedure is approximately the same as for the stability plots of the univariate models. Figure 6.9 gives the test results for a decreasing time-window, starting with all historical data and each time removing the first 20 dates and recalibrate the model. Figure 6.10 gives the results for the degrees of freedom parameter \( \nu \) for both time-windows.

From the moving time-window results, it is clear that the correlation coefficient \( \rho \) fluctuates a lot over time, but follows the same trend for both calibration methods and both copulas (Gaussian and Student \( t \)). Below the range of \( \rho \) is given, just as the mean absolute differences of the four realizations of \( \rho \) with their trend (average).

<table>
<thead>
<tr>
<th>moving window</th>
<th>Model A</th>
<th>Model B</th>
<th>Model C</th>
<th>Model D</th>
</tr>
</thead>
<tbody>
<tr>
<td>range ( \rho )</td>
<td>[-20%, 35%]</td>
<td>[-14%, 32%]</td>
<td>[-21%, 33%]</td>
<td>[-18%, 20%]</td>
</tr>
<tr>
<td>mean abs diff with trend ( \rho )</td>
<td>1.4%</td>
<td>1.2%</td>
<td>0.9%</td>
<td>4.1%</td>
</tr>
<tr>
<td>range ( \theta )</td>
<td>[1, 1.29]</td>
<td>[1, 1.25]</td>
<td>[1, 1.26]</td>
<td>[1, 1.14]</td>
</tr>
<tr>
<td>mean abs diff ( \theta )</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

The \( \theta \) parameter in the flipped Gumbel copula is also comparable for both calibration methods. Remarkable is that \( \theta \) sometimes becomes 1, which indicates the independence copula, and is most of the times explicable by a negative Kendall’s tau correlation coefficient between the residuals. This is not in line with our expectations, but it indicates that the dependence between the stock price and short rate returns is smaller than expected.

The decreasing time-windows for models A and B show a downward trend (from approximately 5\% to -5\%), which indicates that the dependence was mainly positive in the first years (2010-2012) and mainly negative during the last years (2014-2015). This is confirmed when the moving time-window is inspected more precisely, in which the correlation coefficient \( \rho \) is on average higher in the first few years. For the \( \theta \) parameter in the flipped Gumbel copula, the same effect is observed.

6.4.3 Resilience: stress testing

The stress test results are shown in Table 6.6. The first stress test scenario consists of the independence copula, for which the VaR and ES are computed. The results are in line with the VaR and ES following from the other copulas, which indicates that the dependence between the residuals from the stock price and short rate process are not as strong as was expected.

The second stress test scenario is the Fréchet upper bound copula which is in fact perfect positive dependence. At each timestep the quantile of the residual in the stock price model is the
same as the quantile of the residual in the short rate model. Due to this perfect dependence, the processes either both increase or both decrease. Because an increase in the short rate always leads to a decrease in the bond price (and vice versa), the value of the stock price and bond price move in the opposite direction and lead to less fluctuations. This is confirmed by the results from the second shock, in which the VaR and ES are much closer to one than for the fitted copulas.

The third stress test scenario shows the opposite behaviour, because the Fréchet lower bound is in fact perfect negative dependence, hence leading to combined movements of the stock- and bond price, which leads to more extreme scenarios and risks.

The fourth stress test scenario shocks the volatilities of both processes, by multiplying the stochastic variables with a factor $1.5$ before entering the Black-Scholes and Hull-White models. These shocks are performed after the GARCH(1,1) simulations and not in the stochastic variables entering these GARCH models, because this would lead to explosion of the GARCH(1,1) processes.

The fifth stress scenario (the stressed VaR) is based on the calibration in a period of stress. The stressed period is the financial crisis of 2008-2009, in which the data between 13-08-2008 to 01-01-2010 is chosen. The resulting parameters for the log-normal Black-Scholes model are $\mu = -0.0343$, $\sigma = 0.39$, indicating a negative slope of the stock prices and higher volatility in the stock prices. For the short rate process a volatility parameter $\sigma = 0.018$ is found, more than twice as high as in the calibration in the period 2010-2015. The calibration of the copulas in the stressed market also lead to higher dependence parameters (see Table 6.5). This stress scenario leads to the most extreme risk estimations, i.e. $99\%$-VaR estimations of $16-35\%$ and $97.5\%$-ES estimations of $18-45\%$. The main differences are also in these scenarios not caused by the choice of the copula, but mainly by the choice of the underlying models.

### 6.4.4 Complexity

The number of parameters for model $A$ is four (two for the stock, two for the short rates). Model $B$ has seven parameters as $\omega_r$ is not significant and therefore omitted (four stock parameters, three short rate parameters). Model $C$ has nine parameters (as two degrees of freedom parameters are added) and model $D$ has even fourteen parameters. The Gaussian, Student $t$ and flipped Gumbel copula have one, two and one parameter respectively. However it should be noted that $\alpha_r$ which is present in all four models is fixed to $\alpha_r = 0.04$, just like the $\mu_{\text{left}}^r$ and $\mu_{\text{right}}^r$ parameters in the Pareto calibration of model $D$ and these values do not lead to (optimization) problems in the calibration process.

The overall complexity of model $A$ is the lowest, followed by model $B$ and models $C$ and $D$ are the most complex. The reason for more complexity in models $C$ and $D$ is that the heavy tailed residual models are not just introducing large returns, but also affect the GARCH models. The heavy tailed stochastic variables can increase the simulated GARCH volatility more rapid than normally distributed stochastic variables (with parameter $\alpha$), whereas the volatility does not decrease much more rapid. This leads to somewhat unexpected behavior in the risk calculations and hence more complexity.

### 6.5 Combining risk measures

Due to the fact that the results differ insignificantly little for the three different copula models (Gaussian, Student $t$ and flipped Gumbel copula), it is difficult to determine which of these models performs better and as shown in the first stress test even the independence copula gives comparable results. In the ranking of the copulas this leads to similar results for all three copula types.
The market risk calculations and performance measure results however do vary a lot for the four different underlying model combinations and copula calibration methods, and also do lead to different rankings of the models.

The performance of the models in terms of accuracy, is determined by the different accuracy tests. For the stock price model A seven out of eight accuracy tests are rejected, for model B four tests are rejected and for model C/D no accuracy tests are rejected. For the short rate process six tests failed for model A, five tests failed for model B and three tests failed for model C and D. The adjusted KS-test for the copula models failed for every IFM calibration and is not rejected by every CML calibration, whereas the adjusted AD-test is only not rejected for the CML calibrations of the flipped Gumbel copula. This leads to a ranking based on accuracy of $-$ for model A, $+$ for model B and $+$ for model C and D.

The performance of the models in terms of stability, is determined by the different stability plot analyses. In this univariate analysis, for model A the stock price and short rate model do not follow the trend which is clearly present in the volatility of both processes. This has a big impact on the risk calculations and leads to lower ranking of model A. Models C and D lead to some extreme realizations in the short rates of the Monte Carlo simulations, due to the strongly heavy tailed residual distributions. The effect on the VaR calculations is small (because the 0.5% worst case scenarios are not included in the VaR), but can have negative effect on the ES calculations. This leads to a ranking based on stability of $-$ for model A, $+$ for model C and D and $+$ for model B.

The performance of the models in terms of resilience, is determined by the different stress tests. In the ranking, the effect of the calibration based on the stressed period 2008-2009 weighs heavily. Model A is for example not resilient in terms of volatility regimes. The observed volatility parameter during stress of 0.39 for the stock model and 0.018 for the short rate model imply that model A leads to underestimation of market risk in stressed periods. The copula calibrations in the stressed period imply stronger dependence for all four models, than the original copulas (the range of the correlation parameters in the Gaussian and Student $t$ copulas increased from $[3.1\%, 4.3\%]$ to $[7.6\%, 14.6\%]$, and the range of the $\theta$ parameter in the CML calibrated flipped Gumbel copula increased from $[1.000, 1.037]$ to $[1.054, 1.128]$). From these results we conclude that all copula models are not resilient in times of stress.

The performance of the models in terms of complexity is ranked based on the number of parameters, calibration method and overall complexity (less complex models are more desirable (+) and very complex models are less desirable (−)). The number of parameters is the smallest for model A and the largest for model D. As explained in Section 6.4.4, the CML calibration method is considered less complex than the IFM calibration method, and the overall complexity of models C and D is the highest, followed by model B and model A. The combined results are given in Table 6.7.
### Univariate simulation model

<table>
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<tr>
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<th>D</th>
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### Short rate model

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<td>(0.1043)</td>
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</table>

* Minus signs indicate unused parameters in model.
** Minus sign between brackets means the parameter is fixed by hand.

**Table 6.1:** Univariate model calibrations in ESGs
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### Table 6.2: ESG calibration copula

<table>
<thead>
<tr>
<th>Copula Calibration</th>
<th>Model A</th>
<th>Univariate simulation model</th>
<th>Model B</th>
<th>Model C</th>
<th>Model D</th>
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<td>Par₂</td>
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<td>Par₂</td>
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<td>(0.026)</td>
<td>0.031</td>
<td>(0.027)</td>
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<td>Student t²</td>
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<td>IFM</td>
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</table>

¹ Std. deviation based on 1,000 simulations.
² Std. deviation based on 100 simulations.
³ Std. deviation based on 500 simulations. If Kendall’s τ < 0, a parameter of 1 is given.
(A) stock model 99.5% VaR confidence bounds forecasts

(b) short rate model 99.5% VaR confidence bounds forecasts

(c) bond price 99.5% VaR confidence bounds forecasts

Figure 6.1: Confidence interval forecasts
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Figure 6.2: Portfolio 99% confidence interval forecasts

(A) Model A 99.5% portfolio VaR confidence bounds forecasts
(B) Model B 99.5% portfolio VaR confidence bounds forecasts
(C) Model C 99.5% portfolio VaR confidence bounds forecasts
(D) Model D 99.5% portfolio VaR confidence bounds forecasts
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Univariate simulation model

<table>
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Stock prices

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Short Rates

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Bond prices

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* Rejected with a significance level of 5%
1 Initial stock price is 3203 euro at 11-12-2015
2 Initial short rate is -0.42% at 11-12-2015
3 Initial bond price is 0.9519 euro at 11-12-2015
4 Based on 60,000 simulations
5 Due to one extreme simulation this value is blown up.

Table 6.3: Univariate model test results
Chapter 6. Results

(A) Stock model 99% VaR confidence bounds historically

(B) Short rate model 99% VaR confidence bounds historically

Figure 6.3: Historical 1–day 0.5% and 99.5% confidence bounds
Chapter 6. Results

- BS volatility (l) & HW volatility (r)
- BS mean (l)
- BS GARCH alpha & HW GARCH alpha
- BS GARCH T alpha (l) & HW GARCH T alpha (r)
- BS GARCH beta & HW GARCH beta
- BS GARCH T beta & HW GARCH T beta
- BS GARCH omega (l) & HW GARCH omega (r)
- BS GARCH T omega (l) & HW GARCH T omega (r)
- BS GARCH T df (l) & HW GARCH T df (r)

Figure 6.4: Moving window (6 months) parameter calibration
Chapter 6. Results

Figure 6.5: Decreasing window parameter calibration
Chapter 6. Results

### Table 6.4: Multivariate model test results

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### Table 6.5: ESG copula calibrations under stressed markets (2008-2009)

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Chapter 6. Results

(A) Empirical copula observations model A

(B) Empirical pseudo copula observations model A

(C) Empirical copula observations model B

(D) Empirical pseudo copula observations model B

(E) Empirical observations model C

(F) Empirical pseudo copula observations model C

(G) Empirical copula observations model D

(H) Empirical pseudo copula observations model D

Figure 6.6: Empirical copula density visualisation
Chapter 6. Results

(A) Empirical copula cdf model A

(b) Empirical pseudo copula cdf model A

(C) Empirical copula cdf model B

(D) Empirical pseudo copula cdf model B

(E) Empirical copula cdf model C

(F) Empirical pseudo copula cdf model C

(G) Empirical copula cdf model D

(h) Empirical pseudo copula cdf model D

Figure 6.7: Empirical copula cdf visualisation
Chapter 6. Results

Figure 6.8: Moving window (6 months) parameter calibration copulas
Chapter 6. Results

(A) Copula A parameter rho

(B) Copula A parameter theta

(C) Copula B parameter rho

(D) Copula B parameter theta

(E) Copula C parameter rho

(F) Copula C parameter theta

(G) Copula D parameter rho

(H) Copula D parameter theta

Figure 6.9: Decreasing window parameter calibration copulas
Chapter 6. Results

Figure 6.10: Moving window (mw) & Decreasing window (dw) calibrations of $\nu$ parameter in Student $t$ copula
### Chapter 6. Results

#### Table 6.6: Stress test results providing 99.5% VaR and 97.5% ES

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<th>Model C</th>
<th>Model D</th>
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1: Independence copula.
2: Fréchet upper bound.
3: Fréchet lower bound.
4: Volatility processes times 1.5 for CML calibration.

* 99% (stressed) VaR instead of 99.5% VaR.

#### Table 6.7: Economic scenario generator overview results

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*Note: + indicates better performance.*
The effect of copulas on an ESG for the calculation of market risk in an asset portfolio is analysed, consisting of equity and fixed income. The equity is based on the Euro Stoxx 50 index and fixed income products are represented by 10–years German bonds. Short rates are simulated with the Hull-White model based on the 3–month bonds and the affine term structure of the model is used to relate these short rates to 10–years bond prices.

Different economic scenario generators are compared, for four different combinations of univariate simulation models, including Black-Scholes, Hull-White, GARCH and different residual models, three different copula models (Gaussian, Student $t$ and flipped Gumbel) and two calibration methods.

The goal is to measure the effect of these model combinations in economic scenario generators on the value at risk and expected shortfall calculations, in terms of accuracy, stability, resilience (effect of stress testing) and complexity.

The main findings are,

- Due to the little dependence between the stock prices and short rates, the effect of the copulas and calibration methods on the risk calculations is small and for all three copulas the market risk estimations are comparable. Hence, for these low dependent portfolios, the copula models have hence not much added value.

- The four different combinations of univariate models do have a large impact on the market risk calculations. The GARCH volatility model leads to an increase in the calculated market risk at a horizon of one year. The choice of residual model has large impact on the risk calculations in one day and (depending on the strength of the dependence) can have both negative and positive impact on the risk calculations in one year and the choice of simulation models should therefore be chosen with care.

- Based on the short summary in Table 6.7, model $B – D$ should be preferred to $A$. Model $B$ gives moderate results for the risk calculations one day ahead, but performs in line with models $C$ and $D$ for the 1–year risk calculations. Models $C$ and $D$ are more complex, but give more accurate results than the other models.

- The dependence in the copulas is small compared to the estimations of dependence based on historically stressed markets, implying that the static dependence of the copula could be unfavourable.

- The dependence implied by the copula strongly depends on the underlying models and is lower when a GARCH volatility model is used. The effect is explained by the non i.i.d. residuals in the model without GARCH, i.e. the high volatile periods in the market lead to
simultaneously volatility clustering in both residual sets. If this clustering is not captured by the model, it leads to amplification of the dependence effect due to misfitting of the univariate simulation models.

- The combination of GARCH models and copulas makes it more easy to use the favorable aspects of both kind of models. However, the reason for using copulas at first hand was to increase the dependence between the random variables in the tails of the distribution. However, if we are in high volatile markets due to the GARCH model in which we expect more extreme high and low realizations of the short rate and stock price simultaneously, the returns are not more dependent than in low volatile periods. This suggest that the combination of GARCH and copulas is not ideal, which is also seen in the calibrations of the stress test based on 2008-2009; i.e. we observed that during this period the dependence parameter observed in the copula calibrations was much stronger than in the quiet periods, but this is not reflected in our model.
Chapter 8

Recommendations

Based on the thesis results, I would like to give four suggestions for further research,

- A shortcoming of the used models is that only the dependence between the residuals is modelled, whereas it could also be interesting to model the dependence between the volatility. After all, we observed in the moving- and decreasing time-windows that the volatilities estimated (for the stock price and short rate model) were highly dependent. In the simulations these volatilities are only indirectly dependent, due to the effect of the dependent stochastic variables on the volatility forecasts. This effect is however very small and not in line with the historically observed (stronger) dependence. The volatility could be included for example in the GARCH processes with a joint component.

- The stress test results which used the historical data from the crisis to calibrate the model, resulted in copula parameters which implicated higher dependence. The static copulas do not adjust for the stressed scenarios. This indicates that market risk calculations based on more flexible dynamic copula models are better able to model risks during stressed periods.

- Portfolios of financial institutions do most times consist of more than two asset types. The effect of copulas in multiple dimensions is not the same as for two dimensions. The analysis performed in this thesis could be extended to higher dimensions to test the combined results for the univariate model and copulas. The results of these tests are more useful to be used in practice.

- It could also be interesting to test a regime switching model, which switches from ‘quiet’ to ‘stress’ based on the observed volatilities. The high and low volatile periods could then also use two different copulas, hence allowing more flexibility for stressed periods.
Appendix A

Univariate model analysis

A.1 Analysis of Hull-White and Black-Scholes residuals

Figure A.3 gives the different plots used for the analysis of the Euro Stoxx 50 index prices and Figure A.4 gives the plots used for the analysis of the German bond yield.

A.2 Tail analysis: Generalized Pareto Distribution

The residuals of the Hull-White GARCH model are heavy tailed, therefore an analysis is performed to check for Pareto distributed tails. Figure A.1 shows the quantile-quantile plots of the positive and negative tails of the distribution, compared to a Pareto distribution with $\xi = 0.26$ and $\xi = 0.18$ respectively.

For further analysis of the threshold value, the zipf plot and mean excess plots are used, given in Figure A.2. In the zipf plot, a linear decay is found after a threshold of 1 (for both tails).

The Hull-White-Garch(1,1) residuals in model $D$ are fitted with an normal distribution with Pareto distributed tails. The threshold of 1 from the zipf plot, gives 148 observations for $x > 1$ (i.e. 9.8% of the data is in the upper tail) and 141 observations for $x < -1$ (i.e. 9.3% of the data is in the lower tail). From the mean-excess a (linear) increase is found for a threshold of zero (or higher), but the linear increase in the upper 20 points is not very clear (however the reliability of the mean excess plot for the largest few observations is small).
Appendix A. Univariate model analysis

Figure A.1: plots for GPD distribution

(A) qplot for positive tail $\xi = 0.26$

(B) qplot for negative tail $\xi = 0.18$

Figure A.2: zipf and mean excess plots for GPD distribution

(A) zipf plot for positive tail

(B) zipf plot for negative tail

(C) mean excess plot for positive tail

(D) mean excess plot for negative tail
Appendix A. Univariate model analysis

Figure A.3: Stoxx 50 Overview 2010-2015
Figure A.4: German bond 3M Overview 2010-2015
Appendix B

Alternative stock price simulation models

B.1 Heston model

The Heston (1993) model is a stochastic volatility model, where the stock is log normally modelled and the volatility is modelled with a Cox-Ingersoll-Ross mean reversion model. Commonly used methods for calibration of the Heston model make use of variance swaps or optimization such that the squared differences between vanilla option market prices and that of the model are minimised over the parameter space. Historical calibration is preferred for consistency, because also the copula calibrations are performed historically.

Based on Körkkö (2013), the Heston model is calibrated historically using the stock prices only, by dividing the historic time-frame into smaller time-frames, on which the volatility is supposed to be constant. The volatility between those constant volatility’s is the vol-vol parameter. Taking time-frames of three months gives 63 time-points to estimate the volatility between 2002 and 2015. A moving window is used to find a volatility at each day. Based on the average volatility we then compute an estimate of the mean reversion parameter. The residuals of the stock price process are much better than those following from Black-Scholes. However, the residuals estimated for the volatility are extremely heavy tailed. Figure B.1 the estimated volatility process and quantile-quantile plot for the volatility residuals is shown, from which the fat tailed behaviour is clear. Increasing the time frames for the volatility estimation does not improve this result, i.e. volatility residuals are still extremely heavy tailed and the stock price residuals get worse. Smaller time frames results in even more heavy tailed variances and hence, we will not continue with the Heston model.

B.2 Regime switching

Regime switching could be used to model variance switching, see for an example Figure B.2.
Appendix B. Alternative stock price simulation models

(A) Estimation of Heston volatility using timeframes of three months

(B) qq plot of volatility returns

**Figure B.1:** Heston fit on stock prices results in extremely heavy tailed volatility distribution.

**Figure B.2:** Possible historical variance regime switches for stock returns. High volatile regimes are indicated with bold underlining.


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