COMMENTS ON "A SIMPLE EXACT METHOD OF (3-D) MIGRATION THEORY"
BY H. JAKUBOWICZ AND S. LEVIN—
A NOTE IN FAVOUR OF "THE MYTH OF NONSEPARABILITY OF THE (3-D) MIGRATION OPERATOR"*

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INTRODUCTION

The migration of 3-D seismic data volumes presents formidable problems for both data handling and storage requirements in the computer. There are enormous computational advantages to be obtained if the full 3-D one-pass migration can be decomposed into two cascaded 2-D migrations, and this is of course what is often done in practice. For example, Gibson, Larner and Levin (1983) present such a scheme which they describe as ‘exact’ (p. 1)—that is, the two-step approach should provide exactly the same results as the one-step full 3-D migration. The theoretical justification for their claim is provided by the companion paper of Jakubowicz and Levin (1983), which Stolt (1984) has acclaimed as "a fine paper which should put to rest the myth of nonseparability of the migration operator".

The paper of Jakubowicz and Levin (1983) is thus very important: it provides the theoretical basis for the practice of migrating data volumes by two successive passes through the 2-D migration process. If the paper is sound, there is no need to worry about the complexity introduced by allowing the upcoming waves to have curvature out of the plane of the seismic section; this can be accommodated with the second pass of the 2-D migration, orthogonal to the first. If the paper is not sound, then the myth of the nonseparability of the migration operator must hold some credibility. We should then worry about the validity of the two-step approach.

Jakubowicz and Levin prove their result twice, once in the $x$, $\omega$-domain and again in the $k$, $\omega$-domain. This double proof is not necessary, but is done to demonstrate that, as they say, "a result obtained through consideration of a particular domain must be valid for all domains because, for our constant-velocity model, the

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migration equations are all equivalent" (p. 41). We therefore consider both proofs, beginning with the \( k, \omega \)-proof.

In the following discussion we number our equations sequentially giving the corresponding equations of Jakubowicz and Levin in square brackets. For example, our (1) is the same as Jakubowicz and Levin [2.10].

**The \( k, \omega \)-Proof**

Jakubowicz and Levin begin by describing the one-step full 3-D migration, using the exploding reflector model. The data \( p(x, y, z = 0, t) \) are triple Fourier transformed to \( \tilde{p}(k_x, k_y, z = 0, \omega) \) by integrals over \( x, y \) and \( t \) from \(-\infty\) to \(+\infty\). The 3-D migration equation becomes

\[
p(x_0, y_0, z_0, t = 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(ik_x x_0) \exp(ik_y y_0) \exp(ik_z z_0)
\]

\[
\times \tilde{p}(k_x, k_y, z = 0, \omega) \, dk_x \, dk_y \, d\omega,
\]

[2.10]

with

\[
k_z = \left[ \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \right]^{1/2}, \quad \text{Re} \{ k_z \} \geq 0, \quad \text{Im} \{ k_z \} \geq 0,
\]

[2.5]

where \( \text{Re} \{ \ldots \} \) and \( \text{Im} \{ \ldots \} \) mean the real and imaginary part of \( \{ \ldots \} \), respectively. These conditions, plus the complex time factor \( \exp(+i\omega t) \) for the Fourier back transformation to the time domain, imply that

1. for real \( k_z \) we are dealing with downward extrapolation of upward traveling propagating energy, and
2. for imaginary \( k_z \) we are dealing with downward extrapolation of downward traveling evanescent energy.

Stolt (1984) rightfully objects to the last case, stating that we should be dealing with downward extrapolation of upward traveling evanescent energy instead. This means that \( \text{Im} \{ k_z \} \) should be negative. Stolt remarks that this is a minor point, but we consider it to be very important and discuss it later on.

Let us, for the moment, follow the argument of Jakubowicz and Levin. After the description of the 3-D migration they describe the two-step 2-D migration. The first step is a double Fourier transform of \( p(x, y, z = 0, t) \) to \( \tilde{p}(k_x, k_y, z = 0, \omega) \) by integrals over \( x \) and \( t \). This is followed by a 2-D operation integrating over \( k_x \) and \( \omega \) to create \( p'(x_0, y, z'_0, t = 0) \). In this intermediate result the prime indicates that this is not the real pressure wave field (it can easily be checked that \( p'(x_0, y, z'_0, t = 0) \) satisfies the 2-D homogeneous wave equation). The coordinate \( z'_0 \) is now related to an intermediate time parameter \( t' \) by \( t' = z'_0/c \).

The second step then consists of a 2-D Fourier transform of \( p'(x_0, y, ct', t = 0) \) to \( \tilde{p}(x_0, k_x, \zeta, t = 0) \) by integrals over \( y \) from \(-\infty\) to \(+\infty\) and over \( t' \) from \( 0 \) to \(+\infty\).

By restricting the range of the second integral to positive values we enforce the representation \( \tilde{p}(x_0, k_x, \zeta, t = 0) \) to be in accordance with the remark made by Jakubowicz and Levin with respect to [3.7], that \( p'(x_0, y, ct', t = 0) \) is zero for \( t' < 0 \).
Next a 2-D operation is performed integrating over \( k_y \) and \( \xi \). The result is an extrapolated imaged wave field

\[
q(x_0, y_0, z_0, t = 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \exp(ik_x x_0) \times \exp(ik_y y_0) \exp(ick_z t) \exp(ikk_x z_0) \times \exp(-i\xi t) \tilde{\Phi}(k_x, k_y, z = 0, \omega) \, dt \, dk_x \, dk_y \, d\omega \, d\xi, \tag{3}
\]

in which the wavenumbers \( k'_x \) and \( k'_z \) are given by

\[
k'_x = [\omega^2/c^2 - k_x^2]^{1/2}, \quad \text{Re} \{k'_x\} \geq 0, \quad \text{Im} \{k'_x\} \geq 0, \tag{4}
\]

and

\[
k'_z = [\xi^2/c^2 - k_z^2]^{1/2}, \quad \text{Re} \{k'_z\} \geq 0, \quad \text{Im} \{k'_z\} \geq 0. \tag{5}
\]

The conditions for the real and imaginary parts are the same as for \( k_x \). Our (3) is the same as [3.28] of Jakubowicz and Levin, except that we have restricted the integration over \( t' \) to positive values of \( t' \) only.

The crux of the \( k, \omega \)-proof is to show that the quintuple integral of (3) is exactly the same as the triple integral of [2.10], and, consequently

\[
q(x_0, y_0, z_0, t = 0) = p(x_0, y_0, z_0, t = 0) \quad \text{for all} \quad x_0, \ y_0 \quad \text{and positive} \ z_0. \tag{6}
\]

Jakubowicz and Levin split the integral into two parts: one part for \( k'_x \) positive real and one part for \( k'_x \) positive imaginary. For the part where \( k'_x \) is real the integral is simply the Fourier transform of a unit-step function. Using the relation (Bracewell 1978, p. 394)

\[
\int_{0}^{\infty} \exp\left[-i\xi - c k'_x t\right] \, dt = \frac{1}{i(\xi - c k'_x)} + \pi \delta(\xi - c k'_x), \tag{7}
\]

we see that when \( k'_x \) is real in (3), we obtain as a |subresult|

\[
q_1(x_0, y_0, z_0, t = 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} \exp(ik_x x_0) \times \exp(ik_y y_0) \tilde{\tilde{\Phi}}(k_x, k_y, z = 0, \omega) \times \int_{-\infty}^{+\infty} \exp(ikk_x z_0) \, dk_y \, dk_y \, d\omega \nonumber \]

\[
+ \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} \exp(ik_x x_0) \times \exp(ik_y y_0) \exp(ikk_x z_0) \Phi(k_x, k_z, z = 0, \omega) \, dk_x \, dk_y \, d\omega, \tag{8}
\]

which is not the same as [2.10].
The next step is to consider the contributions of the part where \( k_x \) is imaginary. This part is discussed by Jakubowicz and Levin in appendix C. The exploding reflector model demands that there are no contributions for \( t' < 0 \). Therefore the unrestricted quintuple integral of equation [C-1] should be replaced by the following restricted integral:

\[
q_2(x_0, y_0, z_0, t = 0) = \frac{1}{(2\pi)^{5/2}} \iint_{\Omega/\Omega_x < k_x} \exp (ik_x x_0) \times \exp (ik_y y_0) \exp (ik_z z_0) \Phi(k_x, k_y, z = 0, \omega) \times \int_0^\infty \exp (-k_x c t' - i\xi c t') \, dt' \, d\xi \, dk_x \, dk_y \, d\omega
\]

with \( k_x = (k_x^2 - \omega^2/c^2)^{1/2} \). (10)

The relation

\[
\int_0^\infty \exp [-i\xi c t' - k_x^2 c t'] \, dt' = \frac{1}{i\xi + c k_x^2},
\]

leads to

\[
q_2(x_0, y_0, z_0, t = 0) = \frac{1}{(2\pi)^{5/2}} \iint_{\Omega/\Omega_x < k_x} \exp (ik_x x_0) \times \exp (ik_y y_0) \Phi(k_x, k_y, z = 0, \omega) \times \int_0^\infty \exp (ik_z z_0) \, d\xi \, dk_x \, dk_y \, d\omega.
\]

If we can prove that the sum of (8) and (12) equals [3.30] of Jakubowicz and Levin, the final result is sound; that is,

\[
q(x_0, y_0, z_0, t = 0) = q_1(x_0, y_0, z_0, t = 0) + q_2(x_0, y_0, z_0, t = 0) = q(x_0, y_0, z_0, t = 0), \quad \text{for all } x_0, y_0 \text{ and positive } z_0.
\]

This is true for arbitrary \( \Phi \) and for all \( x_0, y_0 \) and positive \( z_0 \) if and only if the following is true:

(i) for \( |\omega/c| < k_x \)

\[
\int_{-\infty}^{\infty} \frac{\exp (ik_z^2 z_0)}{i\xi + c k_x^2} \, d\xi = 2\pi \exp (-k_x^2 z_0) \text{ for all } k_x, k_y \text{ and positive } z_0,
\]

with \( k_x = (k_x^2 + k_y^2 - \omega^2/c^2)^{1/2} \). (15)
(ii) for $|\omega/c| \geq k_x$

$$\int_{-\infty}^{\infty} \frac{\exp \left(ik_x z_0\right)}{i(\xi - c k_x)} \, d\xi = \pi \exp (ik_x z_0) \text{ for all } k_x, k_y \text{ and positive } z_0,$$

(16)

with $k_x = (\omega^2/c^3 - k_2^2 - k_1^2)^{1/2}$.

(17)

To prove case (i) Jakubowicz and Levin apply Cauchy's theorem and perform a contour integration in the upper half of the complex $\xi$ plane. In this respect they mention that the branch cut, imposed by $k_x'$, is of no significance. This is not true and is so important that we introduce a separate section to explain this point.

**Contour integration of (14)**

In order to evaluate the integral in case (i) we have to be consistent with the constraint in (5). The choice in (5) was made to give $k_\nu'$ a unique meaning, necessary to solve our wave problem. On squaring (5) and separating into real and imaginary parts we obtain the following relations between the real and imaginary parts of $\xi$ and $k_\nu'$:

$$\left(\text{Re} \{k_x'\}\right)^2 - \left(\text{Im} \{k_x'\}\right)^2 = \frac{(\text{Re} \{\xi\})^2}{c^2} - \frac{(\text{Im} \{\xi\})^2}{c^2} - k_\nu^2,$$

(18)

$$\text{Re} \{k_x'\} \text{ Im} \{k_x'\} = \text{Re} \{\xi\} \text{ Im} \{\xi\}.$$

(19)

Now we have to make a definite choice in order to define our Riemann sheet. Once we have fixed it we have to stay with our choice to be consistent. We choose to fix $\text{Im} \{k_x'\} \geq 0$. This choice is sensible because the integrand in (14) is then bounded in the entire $\xi$-plane (of course, other choices are possible, however they would complicate the closure of the integration contour). The resulting relations between the real and imaginary parts of $k_\nu'$ and $\xi$ are as shown in fig. 1. We see that the real part of $k_x'$ changes sign from quadrant to quadrant according to (19).

![Diagram of the complex $\xi$-plane with branch cuts; distributions of the sign of $k_x'$ and location of the integration contours.](image)

Fig. 1. The complex $\xi$-plane with branch cuts; distributions of the sign of $k_x'$ and location of the integration contours.
In order to keep \( \text{Im} \{k_\xi^n\} \geq 0 \) in the entire complex \( \xi \)-plane, we have to introduce branch cuts along \( \text{Im} \{k_\xi\} = 0 \) (see fig. 1), i.e., along

\[
\text{Im} \{\xi\} = 0, \quad c^2k_\xi^2 < (\text{Re} \{\xi\})^2 < \infty.
\]

Then, corresponding to our conditions that \( \text{Re} \{k_\xi^n\} \geq 0 \) and \( \text{Im} \{k_\xi^n\} \geq 0 \) in (14), the integration interval can be subdivided into three intervals:

\[
\int_{-\infty}^{\infty} \frac{\exp(ik_\xi^n z_0)}{i\xi + cK_\xi} \, d\xi = \int_{L_1} \ldots \, d\xi + \int_{L_2} \ldots \, d\xi + \int_{L_3} \ldots \, d\xi,
\]

where on \( L_1 - \infty < \text{Re} \{\xi\} \leq -ck_\xi \) and on \( L_3 ck_\xi \leq \text{Re} \{\xi\} < \infty \). Then it follows from fig. 1 that the contour of integration cannot be closed in the upper half plane without crossing the branch cut. Instead, considering the simple closed contour \( C \) in the upper half plane, we arrive at:

\[
\oint_C \frac{\exp(ik_\xi^n z_0)}{i\xi + cK_\xi} \, d\xi = \int_{L_1} \ldots \, d\xi + \int_{L_2} \ldots \, d\xi + \int_{L_3} \ldots \, d\xi + \int_{L_4} \ldots \, d\xi
\]

in which \( R \) tends to infinity. Using the theorem of residues and noting that the contribution of \( L_4 \) vanishes on account of Jordan’s Lemma, we obtain

\[
2\pi \exp(-\bar{k}_\xi z_0) = \int_{L_1} \ldots \, d\xi + \int_{L_2} \ldots \, d\xi + \int_{L_3} \ldots \, d\xi.
\]

Finally, by subtracting (23) from (21), we arrive at

\[
\int_{-\infty}^{\infty} \frac{\exp(ik_\xi^n z_0)}{i\xi + cK_\xi} \, d\xi = 2\pi \exp(-\bar{k}_\xi z_0) + \int_{-\infty}^{\infty} \frac{\exp(ik_\xi^n z_0) - \exp(-ik_\xi^n z_0)}{i\xi + cK_\xi} \, d\xi,
\]

where the integral on the right-hand side follows from \( \int_{L_1} \ldots \, d\xi - \int_{L_3} \ldots \, d\xi \) taking the limit \( \epsilon \downarrow 0 \); then, \( k_\xi^n \) is positive real on \( L_1 \) and is negative real on \( L_3 \).

Returning to our main argument, Jakubowicz and Levin do not have the integral contributions of the branch cut on the right hand side. This means that case (i) is valid if, and only if, this integral vanishes for all positive values of \( z_0 \). A necessary condition for this is that

\[
\int_{0}^{+\infty} \int_{-\infty}^{-ck_\xi^n} \frac{\exp(ik_\xi^n z_0) - \exp(-ik_\xi^n z_0)}{i\xi + cK_\xi} \, d\xi \, dz_0 = 0.
\]

Using again the Fourier transform of the unit-step function we arrive at

\[
2i \int_{-\infty}^{-ck_\xi^n} \frac{1}{K_\xi^2(i\xi + cK_\xi)} \, d\xi = 0
\]

or

\[
2iCK_\xi^n \int_{-\infty}^{-ck_\xi^n} \frac{1}{K_\xi^2(i\xi + cK_\xi)^2} \, d\xi + 2 \int_{-\infty}^{-ck_\xi^n} \frac{\xi}{K_\xi^2(i\xi + cK_\xi)^2} \, d\xi = 0.
\]
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Inspection of the integrals in (27) shows that the real part is always negative, while
the imaginary part is always positive. Therefore (27) is not true. It follows that the
contribution of the branch cut is non-zero for every positive \( z_0 \).

Operating in the same way for case (ii) and taking into account that the pole is
now on the contour (meaning that we have to take only half of the residue), we obtain

\[
\int_{-\infty}^{\infty} \frac{\exp (ikz_0)}{i(\xi - ck_0)} \, d\xi = \pi \exp (ikz_0) + \int_{-\infty}^{\infty} \frac{\exp (ikz_0) - \exp (-ikz_0)}{i(\xi - ck_0)} \, d\xi. \tag{28}
\]

Again we have to prove that the integral on the right-hand side vanishes for all
positive values of \( z_0 \), leading to the necessary condition

\[
\int_{0}^{+\infty} \int_{-\infty}^{-\alpha_x} \frac{\exp (ikz_0) - \exp (-ikz_0)}{i(\xi - ck_0)} \, d\xi \, dz_0 = 0 \tag{29}
\]
or

\[
2 \int_{-\infty}^{\infty} \frac{1}{ik_0' (\xi - ck_0')} \, d\xi = 0. \tag{30}
\]

From this we can conclude that (30) is not true, because the left-hand side is
negative real and thus the contribution of the branch cut is not equal to zero for
every positive \( z_0 \).

In conclusion we can say: the 2-D operators as proposed by Jakubowicz and
Levin will never result in a relation that is exactly equivalent to [2.10].

The \( x, \omega \)-Proof

The analysis in the \( x, \omega \)-domain requires the demonstration that [3.12] with \( t' \)
restricted to positive values, i.e.,

\[
q(x_0, y_0, z_0, t = 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \frac{k'z_0 c t'}{4pr'} H_1(kr)H_1(k'r') \exp (-i\xi t')
\]

\[
\times P(x, y, z = 0, \omega) \, dx \, dy \, d\omega \, d\xi \, dt' \tag{31}
\]
can be reduced to

\[
p(x_0, y_0, z_0, t = 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{z_0}{\rho^2} (1 - ik\rho)
\]

\[
\times \exp (ik\rho)P(x, y, z = 0, \omega) \, dt \, dy \, d\omega. \tag{32}
\]

[3.22]
In their demonstration Jakubowicz and Levin evaluate the integral over $\xi$ first:

$$I_1 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \xi \exp \left( -i \xi' c \right) H_1 \left( \frac{\xi'}{c} \right) d\xi. \quad [3.13]$$

$I_1$ is the Fourier transform of the function $\xi H_1(\xi'c)$. From the theory of Fourier transforms it follows that a necessary condition for the existence of $I_1$ is that $\xi H_1(\xi'c)$ is absolutely integrable in the interval from $-\infty$ to $+\infty$ (Bracewell 1978, p. 9). Since $|\xi H_1(\xi'c)|$ is of order $|\xi|^{1/2}$ for $|\xi| \to \infty$ (see Abramowitz and Stegun 1965, 9.2.3 p. 364), it follows that $I_1$ does not exist and represents a divergent integral. Consequently the transition from

$$I_1 = i \left( \frac{2}{\pi} \right)^{1/2} \int_0^{+\infty} \xi \left\{ \cos \left( \xi \xi' \right) Y_1 \left( \frac{\xi'}{c} \right) - \sin \left( \xi \xi' \right) J_1 \left( \frac{\xi'}{c} \right) \right\} d\xi \quad [A-5]$$

to

$$I_1 = i \left( \frac{2}{\pi} \right)^{1/2} \frac{d}{d\xi} \int_0^{+\infty} \left\{ \sin \left( \xi \xi' \right) Y_1 \left( \frac{\xi'}{c} \right) + \cos \left( \xi \xi' \right) J_1 \left( \frac{\xi'}{c} \right) \right\} d\xi \quad [A-6]$$

in appendix A is not permitted. The authors interchange integration and differentiation, but this is only allowed for convergent integrals.

In the resulting integral representations $[A-7]$ and $[A-8]$, the boundary depends on the variable $\xi'$. Then the differentiation with respect to $\xi'$ in equation $[A-9]$ and $[A-10]$ must include the contributions of the boundary points, leading to a singular result. Jakubowicz and Levin make the same mistake in appendix B in the evaluation of the integral $I_2$ by stating that

$$\int_\rho^{+\infty} \frac{H_1(\xi r)}{(r^2 - \rho^2)^{1/2}} dr = \frac{1}{\rho} \int_\rho^{+\infty} \frac{H_1(\xi r)}{(r^2 - \rho^2)^{1/2}} dr,$$

which is not true, because the result of the integrand for $r = \rho$ should be included, again leading to singular behaviour.

Jakubowicz and Levin should have evaluated the integral over $\xi'$ first, followed by the evaluation of the integral over $\xi$, corresponding to the procedure in the $k$, $\omega$-domain.

**The Effect of $\text{Im} \{k_z\} \leq 0$**

We now discuss the 'minor point' made by Stolt (1984) and take $\text{Im} \{k_z\} \leq 0$. Consequently also $\text{Im} \{k_z\} \leq 0$ and $\text{Im} \{k_z^*\} \leq 0$. It now follows that a demonstration of the exact equivalence of the two-pass 2-D migration operation and the one-pass 3-D migration operation is out of the question. First of all the representation in the $k$, $\omega$-domain is no longer equivalent to the representation in the $x$, $\omega$-domain. When the negative sign is taken for the imaginary part of $k_z$ and $k_z^*$, the
factors exp (ik'_{t} c't') and exp (ik''_{z} z_{0}) (where t' ≥ 0 and z_{0} ≥ 0) explode as t' → ∞ and as z_{0} → ∞. Therefore, the identification of the Hankel functions in [3.1] and [3.8] is wrong. Thus we restrict our attention to the k, ω-domain. To show the equivalence between the two-pass 2-D and the one-pass 3-D processes, it is essential to evaluate the isolated integrals in [3.28] over t' and ξ. But, because of the exploding nature of these integrals for Im \{k'_{z}\} < 0 and Im \{k''_{z}\} < 0, we can only consider Re \{k'_{z}\} ≥ 0 and Re \{k''_{z}\} ≥ 0. Then, although [3.23] is valid both for the propagating and evanescent parts of the wave field, isolation of the integrals over t' and ξ is only possible for the propagating part of the wave field. Thus a complete demonstration of the equivalence between the two-pass 2-D and one-pass 3-D migration processes is impossible.

Accordingly, we restrict our analysis to the propagating part of the wave field and require that

\[ \tilde{P}(k_{x}, k_{y}, z = 0, \omega) = 0, \text{ for } |\omega/c| < (k'_{z}^{2} + k''_{z}^{2})^{1/2}. \]  

(37)

Then the modified version of (8) is given by

\[
\begin{align*}
q'(x_{0}, y_{0}, z_{0}, t = 0) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} \exp (ik_{x} x_{0}) \exp (ik_{y} y_{0}) \\
&\quad \times \tilde{P}(k_{x}, k_{y}, z = 0, \omega) \int_{|\xi| < c k'_{z}} \frac{\exp (ik''_{z} z_{0})}{i(\xi - c k'_{z})^2} d\xi \; dk_{x} \; dk_{y} \; d\omega \\
&+ \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} \exp (ik_{x} x_{0}) \exp (ik_{y} y_{0}) \\
&\quad \times \exp (ik''_{z} z_{0}) \frac{\tilde{P}(k_{x}, k_{y}, z = 0, \omega)}{2} \; dk_{x} \; dk_{y} \; d\omega.
\end{align*}
\]

(38)

Since |\omega/c| ≥ (k'_{z}^{2} + k''_{z}^{2})^{1/2} and |\xi| ≥ c k'_{z} the simple pole ξ = c k'_{z} always lies on the ξ integration interval. This latter integral can be evaluated as follows:

\[
\begin{align*}
\int_{\xi > c k'_{z}} \frac{\exp (ik''_{z} z_{0})}{i(\xi - c k'_{z})^2} d\xi &= 2c k'_{z} \int_{\xi = c k'_{z}}^{+\infty} \frac{\exp (ik''_{z} z_{0})}{i(\xi^2 - (c k'_{z})^2)} d\xi \\
&= 2c k'_{z} \int_{\xi = c k'_{z}}^{+\infty} \frac{\exp (ik''_{z} z_{0})}{i(\xi^2 - (c k'_{z})^2)} d\xi + 2c k'_{z} \int_{\xi = c k'_{z}}^{+\infty} \frac{\exp (ik''_{z} z_{0})}{i(\xi^2 - (c k'_{z})^2)} d\xi,
\end{align*}
\]

(39)

where \( \tilde{\mathcal{I}} \) denotes the Cauchy principle integral and is defined by

\[
\int_{\xi = c k'_{z}}^{+\infty} \frac{\exp (ik''_{z} z_{0})}{i(\xi^2 - (c k'_{z})^2)} d\xi = \lim_{\varepsilon \downarrow 0} \left( \int_{\xi = c k'_{z}}^{c k'_{z} - \varepsilon} \frac{\exp (ik''_{z} z_{0})}{i(\xi^2 - (c k'_{z})^2)} d\xi + \int_{c k'_{z} + \varepsilon}^{+\infty} \frac{\exp (ik''_{z} z_{0})}{i(\xi^2 - (c k'_{z})^2)} d\xi \right)
\]

(40)
and \( f_\lambda \) denotes the integral over a semi-circle with its center at \( \zeta = c k_s \) with a vanishingly small radius \( \delta \). Because of the condition \( \text{Im} \{ k_s \} \leq 0 \) we are now at the other Riemann sheet and consequently the semi-circle lies in the half-plane \( \text{Im} \{ \zeta \} < 0 \), implying that

\[
2 c k_s \int \frac{\exp(i k_s z_0)}{\sqrt{\zeta^2 - (c k_s)^2}} \, d\zeta = \pi \exp(i k_s z_0).
\]

Finally representation (38) can be written as

\[
q(x_0, y_0, z_0, \tau = 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{|\omega| \geq (k_x^2 + k_y^2)^{1/2}} \exp(i k_x x_0) \exp(i k_y y_0)
\]

\[
\times \bar{P}(k_x, k_y, z = 0, \omega) \int_{c k_s}^{+\infty} \frac{2 c k_s \exp(i k_s z_0)}{\sqrt{\zeta^2 - (c k_s)^2}} \, d\zeta \, dk_x \, dk_y \, d\omega
\]

\[
+ \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \int_{|\omega| \geq (k_x^2 + k_y^2)^{1/2}} \exp(i k_x x_0) \exp(i k_y y_0)
\]

\[
\times \exp(i k_s z_0) \bar{P}(k_x, k_y, z = 0, \omega) \, dk_x \, dk_y \, d\omega.
\]

In general the first integral on the right-hand side will not vanish. Thus, even if we restrict ourselves to the propagating part of the wave field, we conclude that the two-pass 2-D migration process is not equivalent to the one-pass 3-D migration process.

**CONCLUSIONS**

The proofs by Jakubowicz and Levin (1983), that the two-pass 2-D migration process is exactly the equivalent to the one-pass 3-D migration process, are mathematically wrong, even for the propagating part of the wave field alone. Thus the two cascaded operations proposed by Jakubowicz and Levin (1983) and by Gibson, Larner and Levin (1983) do not do the trick and must create errors, which are now presumably of some concern.

It remains an open question whether there exist two cascaded operations that can do the trick. In the mean time it follows that Jakubowicz and Levin (1983) have not put to rest "The myth of non separability of the 3-D migration operator", which may, therefore, be true.

**REFERENCES**


