A FINITE STRAIN SHELL MODEL
FOR THE ANALYSIS OF
MODERATELY THICK-WALLED TUBES
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FOR THE ANALYSIS OF
MODERATELY THICK-WALLED TUBES

PROEFSCHRIFT

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aan de Technische Universiteit Delft,
on gezag van de Rector Magnificus Prof. ir. K.F. Wakker,
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aan Karin en Maarten

aan mijn ouders
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Summary

Thesis: A finite strain shell model for the analysis of moderately thick-walled tubes

This thesis deals with a theory for the simulation of observed load/deformation behaviour of straight tubes and pipe bends. The model is capable to handle thin-walled as well as moderately thick-walled pipes with arbitrary shapes of the cross-sections and/or arbitrary wall-thicknesses.

The formulation of the problem starts out from Kirchhoff-Love assumptions. A deviation from these assumptions occurs when finite membrane strains are admitted (whereas the bending strains are assumed to remain small). For this purpose a finite thickness-stretching is introduced via an independent function. The description of shear deformation is not included.

In the model, a distinction is made between two types of deformation: so-called "beam deformation" and "shell deformation". The "beam deformation" gives the deformation of the tube according to Bernoulli's beam theory. The "shell deformation" constitutes additional displacements and offers the description of effects such as ovalisation and necking.

The "beam deformation" is captured by a uniform axial stretching (first degree-of-freedom) and a uniform curvature (second degree-of-freedom). The unknown fields in the "shell deformation" are discretised by means of Fourier series expansions. The participation factors of the Fourier functions serve as degrees-of-freedom in the model.

The constitutive relations employed are based on a 3D (finite strain) elasto-plastic model, using a Von Mises yield criterion, an associative flow rule and isotropic hardening. The application of this 3D stress-update procedure to the stress and strain measures in the shell formulation is such that any 3D constitutive model may be applied.

A sequence of combinations of basic loads may be imposed to the tube. In the model, the following basic loads are defined: internal/external pressure, forces and bending moments at both end-sections of the tube.

The load/deformation response is traced by means of an incremental method. For this purpose, the following continuation methods are implemented: incrementation of one of the load parameters, incrementation of one of the degrees-of-freedom and a linearised version of the arc-length method.
A special-purpose computer program based on the theory as presented, called TUBEHAVE, is developed. The results obtained with this package compare quite well with the analytical / numerical and/or the experimental work presented by others. Furthermore, the examples show that the application of Fourier series for the discretisation of the various fields leads to an adequate description of the deformation behaviour. Finally, it is concluded that, for the presented analyses, the strain measures defined in the model suffice for an accurate description of the deformation of thin-walled tubes as well as moderately thick-walled tubes (at least up to the maximum loading).

Author: G. van den Berg
Chapter 1 - Introduction

1.1 - Introduction

Tubes are widely used in oil and gas industry, both in onshore and offshore applications. A typical example of a tube is a pipeline, which is used for the transportation of oil and/or gas. More applications of tubes are, amongst many others, the structural members in offshore platforms, the casing of bore-holes and the piping in the processing industry (refinery works). The model which has been developed is mainly focused on the assessment of the strength of pipelines, but has potential application in all other mentioned fields.

Pipelines constitute a cost-effective means of transporting and distributing liquids and gases. Since the early seventies, more and more offshore pipelines are utilised, instead of the use of tankers. Especially in the North Sea and the Gulf of Mexico, where more production platforms are clustered together, the use of pipelines is preferable. In the seventies, the offshore exploration and production of oil and gas was limited to shallow waters, say 50-100 m deep. At the time being, feasibility studies have shown that the technology to explore, drill and produce oil and gas in water depths to 2000 m, and perhaps beyond, can be developed at acceptable costs.

Pipelines are designed on the basis of careful considerations from both an economic and a safety point of view. For example, in the case of deep water pipelines the design is directed to withstand the high ambient pressures. Besides, the laying process in deep water induces a severe loading condition. Safety requires the use of tubes with a low radius to wall-thickness ratio and (if possible) the use of higher strength materials. Economics brings into account the additional cost of the pipe material and the higher installation costs and consequently dictates a very careful use of safety factors. Therefore appropriate design tolerances need to be used. The assessment of safety factors and design tolerances require the use of an accurate method for predicting the strength of pipes under several loading conditions. Also the assessment of the remaining strength of existing (and thus possibly damaged) pipelines calls for an accurate analysis tool.

This thesis is concerned with theoretical foundations for such a tool that would allow an accurate (in the sense of comparisons with experimental data) simulation of observed deformation behaviour of tubes under loadings such as internal/external pressure, axial loading, bending, thermal loading, local forces, etc. The formulation is such that the
ultimate load of the tube under a given loading history as well as the load/deformation response shortly after the ultimate load can be assessed with the model. Also the sensitivity of the ultimate load with respect to imperfections superposed on the idealised tube configuration can be determined.

The basic theory underlying such a tool is presented in this thesis. The implementation of the model will be limited to the following load cases: internal/external pressure, axial loading and bending in one plane. However it is anticipated that these loading cases will generate all the essential types of deformation and therefore will yield an adequate insight into the performance of the model. The addition of other loading cases should be a relatively straightforward process.

It is an option to analyse these type of problems employing a standard finite element package. The required effort may then be minimised by developing a user-friendly input/output handling interface which tailors the standard package to the deformation of tubes only. However the development of such a special-purpose tool is less attractive in view of the requirement of a separate license for every individual user of the model, which may be cost prohibitive. Besides the fact that the tool should be easily portable, it should also enable in-house development of dedicated applications. From the above reasons it is considered more attractive to start a separate development. The details of this special-purpose tool are given in this thesis.

1.2 - Basic philosophy

The main objective of the model is to determine the safety of tubular structures with respect to loadings such as internal/external pressure, axial loading, bending and thermal loading (either combined or sequentially). To this end the ultimate load which the tube is capable to withstand should be evaluated accurately. Safety margins are then determined via a comparison of the ultimate load with the actual load on the pipe. Thus the ultimate load is the most important quantity for the assessment of safety margins of tubular structures in applications where the loading is given. The stability behaviour of the structure after the ultimate load is reached is also important in view of imperfection sensitivity studies.

Here a distinction will be made between two types of deformation modes for tubular structures: distortion in the shape of the cross-sections as well as distortion in the wall-thickness of the tube. It is observed that upon loading initially both types of deformation modes occur. Both types of deformation may lead to an ultimate load, either separately or combined. Usually localisation phenomena will only occur after a load maximum has been reached. Localisation phenomena will be distinguished in localisation in bending
deformations and localisation in membrane deformations. As a consequence of kinematic assumptions which will be introduced, the model will be capable to describe these localisation phenomena as long as for membrane localisation the variation in thickness remains rather smooth and for bending localisation the changes in curvature remain rather small. The description of fracture of the material is not included.

1.2.1 - Shell assumptions

A structure is called a shell structure if one of the dimensions is much smaller than the other two dimensions. In the case of a tube this would require that the wall-thickness of the tube (say t) is smaller than the smallest principal radius of curvature (say R) and a characteristic length over the tube (say L). For typical applications of pipes in the oil-industry the ratio R/t ranges between 5-50, which is considered "moderately thick-walled".

Two options for the description of the deformation behaviour of such moderately thick-walled pipes have been considered. The first option is to use a fully three-dimensional (3D) formulation, where no assumptions are made about the effect through-the-thickness. The three-dimensional nature of the governing equations requires a large amount of unknowns for a discretisation. The number of unknowns might be reduced by introducing less parameters for the discretisation in the thickness direction than in the in-plane directions. However this may lead to an ill-conditioned set of equations, which has been reported by AHMAD, IRONS & ZIENKIEWICZ (1970) in the context of a finite element discretisation. The second option for the description of the deformation behaviour of moderately thick-walled pipes is to make approximate kinematic assumptions about the effect through-the-thickness. In this manner, starting out from the general 3D-description, a two-dimensional (2D) description is obtained. Such a 2D-description will be referred to as a "shell theory". The approximation introduces certain small errors in the results, but leads to considerably less unknowns in the discretisation. It is decided to employ the second option.

From observations of failure mechanisms of tubes in practical situations, it is concluded that a certain class of failures is caused by the buckling phenomenon. For the buckling sensitive cases (e.g. collapse under external pressure and wrinkling under axial compression) the ultimate load occurs at small membrane strains and small bending strains. For the remaining class (such as burst under internal pressure) it is observed that the ultimate load is reached at large membrane strains. The formulation presented in this thesis allows for large membrane strains, whereas the bending strains are required to remain small. The complexity of the description is further reduced by exclusion of shear deformation.

Work on shell theories can be traced back to the nineteenth century. Since then, the theory of shells has been one of the challenges in solid mechanics and consequently the
sophistication of shell theories has been largely extended, see e.g. the work of SANDERS (1963) and KOITER (1966). In the sequel, their work will be referred to as the "classical theory of thin shells". This theory is based on Kirchhoff-Love or equivalent assumptions and assumes a vanishing through-the-thickness stress component. Furthermore the classical theory of thin shells requires \( t \ll R \) and is limited to the small strain regime, although large rotations and large displacements are allowed for. Some recent work on shell theories is due to FOX (1990) and SIMO et al. (1989.\textcircled{1}, 1989.\textcircled{2}, 1990.\textcircled{1}, 1990.\textcircled{2} and 1992). The theory presented in those articles also includes large membrane strains.

Work on numerical simulations of the deformation of shells may be subdivided into finite difference formulations and finite element formulations. Focusing on finite element formulations, two approaches are frequently adopted. On the one hand, implementations of elements which are directly based on shell theory exist. Since the pioneering work of GALLAGHER (1973), a large amount of work on this approach is done. Some recent work is carried out by CRISFIELD (1992), BOUT (1992) and VAN KEULEN (1993). On the other hand, the degenerated-solid approach exists. This approach was first introduced by AHMAD, IRONS & ZIENKIEWICZ (1970) for the linear analysis of moderately thick-walled shells. This concept has been further worked out by HUGHES & LIU (1981.\textcircled{1} and 1981.\textcircled{2}) to the non-linear regime.

Even for the assessment of the deformation behaviour of moderately thick-walled tubes the classical theory of thin shells has still been applied successfully, see for example YEH & KYRIAKIDES (1986), CORONA & KYRIAKIDES (1988) and KYRIAKIDES & JU (1992.\textcircled{1} and 1992.\textcircled{2}). As mentioned, the classical theory of thin shells is restricted to small membrane strains and small bending strains and assumes no change in thickness. However, the assessment of the maximum load behaviour of a tube under, for example, internal pressure requires a more accurate description allowing for finite membrane strains. In the case of finite membrane strains also changes in the wall-thickness of the shell should be included in the formulation. In the sequel changes in the wall-thickness of the shell will be referred to as "thickness stretching".

For inclusion of thickness stretching, the following methods have been considered:

- apply an assumed through-the-thickness stress which varies linearly between the surface tractions and eliminate the through-the-thickness strain from the constitutive relations. The through-the-thickness strain is calculated separately.
- assume the material to be incompressible, which yields a direct relation between the through-the-thickness strain and the in-plane strains.
- introduce the through-the-thickness strain as an independent variable.
The first options involves non-trivial computational effort for the elimination of the through-the-thickness strain and consequently for the determining of the thickness. The assumption in the second option (incompressibility) is valid for the plastic part of the deformation of steels, but leads to significant errors for elastic deformation. Above problems are prevented when an independent variable is introduced (the third option) such that it automatically determines the through-the-thickness strain. A drawback of this option is an extra variable featuring in the governing equations.

On the basis of the conceptual simplicity it is decided to choose for the extra independent variable. As an approximation the through-the-thickness strain is assumed to be constant over the thickness and consequently it is only a function of the mid-surface coordinates. The introduction of such a thickness function is also done by GREEN, NAGHDI & WENNER (1971) and more recently by SIMO et al.(1990.2).

1.2.2 - Decomposition of the deformation

The basic idea behind the tool is to use the model for a more detailed analysis of a characteristic section of a total tube configuration. The idea is that in practice, first a global piping-analysis of a total tube configuration under a given loading is performed. For the global piping-analysis, the total tube configuration is modelled with a set of Bernoulli-type "pipe elements". (Bernoulli's beam theory assumes cross-sections originally in a plane perpendicular to the line-of-centroids of the beam to remain in the plane perpendicular to the line-of-centroids at any stage of deformation, thereby excluding shear and warping of the cross-sections.) The model with pipe elements offers a description of the overall behaviour of the tube. The influence of ovalisation of the cross-sections and the influence of internal/external pressure is usually handled via certain correction factors, leading to an appropriate approximation as long as no local effects such as buckling and localisation occur.

From the piping analysis, sections will be selected for more detailed description. The idea is to apply the current model for this purpose. A consequence then is that the current model should be compatible with the pipe elements at both the end cross-sections. This means that a relation should exist between parameters determining the position, the orientation and the shape of the end cross-sections in the pipe element on the one hand and corresponding parameters in the current model on the other hand. In order to achieve a model that meets these compatibility requirements, a decomposition of the deformation into the following two parts is attractive.

The first part of the decomposition gives the description of the deformation of the tube according to Bernoulli's beam theory. In the sequel this part will be referred to as
"beam deformation". The second part constitutes additional displacements and offers the description of effects such as ovalisation, buckling and necking. This part of the tube deformation will be referred to as "shell deformation".

The beam deformation in the current model is compatible with the deformation in the pipe elements since both are based on Bernoulli's beam theory. In this thesis, the implementation of the beam deformation will be restricted to a constant stretch and a constant curvature in one plane. Since in a global analysis pipe elements usually have a non-constant curvature, an extension of the "beam deformation" would be required. The shell deformation in the current model is defined such that the end cross-sections remain plane (no warping), which is essential in view of compatibility requirements. However, changes in the shape of the cross-sections during deformation in the end-planes are allowed for. The relation between parameters determining the shape of the end cross-sections in the current model and corresponding parameters in the pipe elements have not been investigated. Thus the implementation status of the model is not sufficient for incorporation in a piping analysis yet. Nevertheless, the basic idea of decomposition of the deformation is already implemented.

1.2.3 - Discretisation of the unknown fields

The problem of the deformation of shells is governed by a set of non-linear equations. In general, closed form solutions of the problem do not exist. Therefore the unknown fields are approximated. In doing so, the unknown fields are replaced by assumed fields which in turn are determined by a finite set of parameters (the so-called degrees-of-freedom). The problem is then translated into solving a finite set of algebraic equations. Frequently used approximations of the unknown fields are:

- piece-wise continuous functions defined on sub-domains.
- continuous functions defined on the total domain.

A finite element approximation belongs to the first type. A number of commercial finite element packages is available. However, as already explained in section 1.1, employing a standard finite element package is considered not attractive. It is also possible to develop a special-purpose finite element program and to tailor it to pipe deformation problems. However, in many practical cases a finite element model requires a large amount of degrees-of-freedom for an appropriate discretisation, although the required computational effort per degree-of-freedom to obtain the governing system of algebraic equations is usually small. To solve the relatively large system of non-linear equations takes rather large computational effort. To reduce this effort, the vector of unknown parameters
may be limited by application of a "reduced basis method", which assumes the vector of unknowns to be expressed in terms of orthogonal vectors. BESSELING (1975) gives an application of this method in terms of eigen modes. The advantage of the reduced basis method is the smaller set of non-linear equations to solve. A disadvantage of this method is that the evaluation of eigen modes requires substantial computational effort.

For a discretisation based on the second type mentioned above, several options exist. One of such options is to employ a Fourier series approximation. These are the exact solutions for the eigen modes in cases with a homogeneous (pre-buckling) stress state. For many practical cases, the application of Fourier series leads to a relatively small set of equations, although the required computational effort per unknown parameter to obtain this set is larger compared to the effort required per degree-of-freedom in a finite element discretisation. However, solving the set of equations requires less computational effort compared to a finite element discretisation. It is decided to discretise the unknown fields by means of a Fourier series approximation.

1.2.4 - Constitutive description of the material

It is discussed in section 1.2.1 that approximate kinematic assumptions for the deformation behaviour through-the-thickness of the shell will be made. As a consequence, the general three-dimensional continuum description changes into a shell theory giving rise to the definition of membrane strains and bending strains. Energetically conjugate to these strains, stress measures will be introduced which will be referred to as "stress resultants". Therefore a constitutive description should be formulated in terms of these stress and strain measures.

Two options for the constitutive description are considered:

- Apply a model directly formulated in terms of the measures for the membrane strains and the bending strains on the one hand and the tangential stress resultants and the tangential stress couples on the other hand.

- Apply a three-dimensional model where the 3D strains are determined by the membrane strains and the bending strains from the shell formulation and integrate the obtained 3D stress components through-the-thickness into stress resultants.

The models proposed by Ilyushin and Ivanov (see the review article of EIDSHEIM & LARSEN (1982)) are based on the first option. These models are limited to the small strain regime, which is not acceptable for our application. If the second option is used, all (finite strain elastic/plastic) constitutive models which are available may be applied. The numerical integration through-the-thickness will require higher computational effort and also the required data storage will be larger.
An accurate description of the hardening behaviour of the tube material is essential. Within the class of materials which are of interest, isotropic hardening is sufficient as long as cyclic loading is excluded. A fit on the curve obtained from the uni-axial tensile test of the material serves as input for the model. This is handled either as a power-law fit or as a table containing data points.

Application of a general 3D constitutive model in conjunction with the employed shell formulation requires special care. This is due to the fact that the through-the-thickness strain is assumed to be constant over the thickness. It has been observed by BÜCHTER, RAMM & ROEHL (1994) that this restraining of the thickness-stretch leads to an overly stiff behaviour in bending problems. A method to overcome this problem will be developed such that it is valid for any 3D constitutive model.

1.3 - Notation

In this thesis, the dyadic notation for tensors and vectors is used. The components of tensors and vectors are indicated by Greek or Latin indices. Greek indices range from 1 to 2, whereas Latin indices range from 1 to 3. The summation convention is applied for repeated indices, unless explicitly mentioned otherwise.

Bold-faced letters (e.g. \( S \)) denote either a vector or a tensor. In order to overcome confusion, a distinction is made between a vector and a tensor by means of an arrow above the symbol. Thus, \( \textbf{a} \) denotes a vector and \( \mathbf{S} \) denotes a tensor. Scalars (like for example the tensor components \( S^{ij} \)) are denoted with normal-faced letters.

Throughout the thesis, we make use of two distinct interpretations of vectors. A space vector is a quantity that is characterised by both magnitude and direction. By definition, a space vector can be decomposed into components relative to pre-defined base-vectors. For example in \( \tilde{\mathbf{S}} = S^i \tilde{\mathbf{g}}_i \), where \( S^i \) are the components and \( \{ \tilde{\mathbf{g}}_i \} \) the base vectors. Use will also be made of so-called column vectors, which are defined as columns containing a set of components. For a clear distinction, column vectors are provided with a tilde, for example in \( \tilde{\mathbf{S}} \). If the word vector (without adjective) is used in this thesis, then the space vector is meant.

The distinction between quantities in the undeformed and the deformed configuration is made via small and capital letters. Small letters will refer to the quantity in the undeformed state, whereas capital letters refer to the quantity in the deformed state.

An upper index \( ^o \) indicates that the concerned quantity depends on the mid-surface coordinates only (not \( \xi \)-dependent, where \( \xi \) is the through-the-thickness coordinate, see section 2.1). The upper index \( ^o \) is added to the symbol only in those cases where a
distinction is required from the quantity which is \( \xi \)-dependent and the associated mid-surface quantity.

Moreover, the following special symbols and notations are introduced:

- The notation \( \{ \vec{g}_j \} \) indicates the frame of base vectors constituted by \( \vec{g}_1, \vec{g}_2 \) and \( \vec{g}_3 \). Lower indices indicate covariant base-vectors.

- The notation \( \delta^i_j \) denotes the Kronecker delta symbol, which is defined as

\[
\delta^i_j = \begin{cases} 
1 & , i = j \\
0 & , i \neq j 
\end{cases} \quad .(1.3.1)
\]

- Contravariant base-vectors, denoted with \( \{ \vec{g}^i \} \), are obtained from covariant base-vectors according to an orthogonality condition; i.e.

\[
\vec{g}^i \cdot \vec{g}_j = \delta^i_j \quad .(1.3.2)
\]

Note that a dot between two vectors indicates the inner product between both vectors.

- The metric tensor associated with the bases \( \{ \vec{g}_j \} \) and \( \{ \vec{g}^i \} \) is defined by the covariant and/or contravariant components

\[
g_{ij} = g_{ji} = \vec{g}_i \cdot \vec{g}_j \quad \text{and} \quad g^{ij} = g^{ji} = \vec{g}^i \cdot \vec{g}^j \quad .(1.3.3)
\]

- The notation \( \vec{a} \otimes \vec{b} \) denotes a dyad, which can be seen as a linear transformation with the following properties for pre- and post-multiplication:

\[
\begin{align*}
\left( \vec{a} \otimes \vec{b} \right) \cdot \vec{x} &= \left( \vec{b} \cdot \vec{x} \right) \vec{a} \\
\vec{x} \cdot \left( \vec{a} \otimes \vec{b} \right) &= \left( \vec{x} \cdot \vec{a} \right) \vec{b}
\end{align*} \quad .(1.3.4)
\]

Observe that in general the transformation \( \vec{a} \otimes \vec{b} \neq \vec{b} \otimes \vec{a} \).

- The single-dot and the double-dot products between two dyads are defined according to

\[
\begin{align*}
\left( \vec{a} \otimes \vec{b} \right) \cdot \left( \vec{c} \otimes \vec{d} \right) &= \left( \vec{b} \cdot \vec{c} \right) \left( \vec{a} \otimes \vec{d} \right) \\
\left( \vec{a} \otimes \vec{b} \right) : \left( \vec{c} \otimes \vec{d} \right) &= \left( \vec{b} \cdot \vec{c} \right) \left( \vec{a} \cdot \vec{d} \right)
\end{align*} \quad .(1.3.5)
\]

- A second-order tensor is interpreted as a linear combination of dyads, for example,

\[
S = S^i_{\ j} \vec{g}_i \otimes \vec{g}_j \quad .(1.3.6.a)
\]
In this expression, $S^i$ denote the contravariant components of $S$. Introducing a vector $ar{g}^i$ as $\bar{g}^i = S^i \bar{g}_i$, we can rewrite

$$S = S^i \bar{g}_i \otimes \bar{g}_j \equiv \bar{g}^i \otimes \bar{g}_j \quad \Rightarrow \quad \bar{g}^i = S \cdot \bar{g}^j \quad .(1.3.6.b)$$

The latter relation follows from post-multiplication of the first expression with $\{ \bar{g}^i \}$ and realising that (1.3.2) holds. Expression (1.3.6) demonstrates the physical meaning of a tensor, namely as the mapping between two sets of vectors.

- The inverse of a second-order tensor is physically interpreted as the inverse of the mapping associated with the tensor. Thus, from (1.3.6), we obtain

$$\bar{g}^j = S^{-1} \cdot \bar{g}^j \quad .(1.3.7)$$

- A fourth-order tensor is introduced as a generalisation of second-order tensors. If we denote two second order tensors with $A$ and $B$, then a fourth-order tensor $C$ is introduced by $C = A \otimes B$, where the following property holds

$$Y = C : X = (A \otimes B) : X \equiv A (B : X) \quad ,(1.3.8)$$

namely the double-dot product between a fourth-order and a second-order tensor. In (1.3.8), $X$ and $Y$ are second-order tensors.

- The components of a fourth-order tensor, say $C = C^{ijkl} \bar{g}_i \otimes \bar{g}_j \otimes \bar{g}_k \otimes \bar{g}_l$, are interpreted as

$$C = (A^{il} \bar{g}_i \otimes \bar{g}_l) \otimes (B^{jk} \bar{g}_k \otimes \bar{g}_j) \quad \text{with} \quad C^{ijkl} \equiv A^{il} B^{jk} \quad .(1.3.9.a)$$

The double-dot product of a fourth-order tensor (say, $C$ defined above) and a second-order tensor (say, $S = S_{mn} \bar{g}^m \otimes \bar{g}^n$) in component-form reads

$$C : S = (C^{ijkl} \bar{g}_i \otimes \bar{g}_j \otimes \bar{g}_k \otimes \bar{g}_l) : (S_{mn} \bar{g}^m \otimes \bar{g}^n) \equiv C^{ijkl} S_{lk} \bar{g}_l \otimes \bar{g}_j \quad ,(1.3.9.b)$$

which follows from (1.3.8).

- The index-raising and index-lowering procedure is based on the concept of decomposition of covariant base vectors on contravariant base vectors; i.e.

$$\bar{g}_i = (\bar{g}_i \cdot \bar{g}^j) \bar{g}^j = \delta_{ij} \bar{g}^j.$$ 

Provided the components of $S$ are defined according to $S = S^i \bar{g}_i \otimes \bar{g}_j$, associated components of $S$ read

$$S^{jk} = S^i g_{ik}, \quad S^{i}_{jk} = S^i g_{jk}, \quad S_{kl} = S^i g_{ik} g_{jl} \quad .(1.3.10.a)$$
which are results for the index-lowering procedure. The index-raising procedure works similarly and gives (for example)

\[ S^i = S_k^j g^{ki} = S^i_k g^{ki} = S_k^i g^{ki} g^{ij} \]  

(1.3.10.b)

- The transpose of a dyad is defined as

\[ (\mathbf{a} \otimes \mathbf{b})^T = (\mathbf{b} \otimes \mathbf{a}) \]  

(1.3.11)

- The transpose of a second order tensor follows from the definition of the transpose of a dyad. In the case of \( \mathbf{S} \) in (1.3.6), we have

\[ \mathbf{S}^T = \mathbf{g}_j \otimes \mathbf{g}^j = \mathbf{g}_j \otimes (S^i_\mathbf{l}) = S^i_\mathbf{g}_j \otimes \mathbf{g}_i \]  

(1.3.12)

- The inverse of the transpose of \( \mathbf{S} \) is denoted with \( \mathbf{S}^{-T} \) and reads

\[ \mathbf{S}^{-T} = \mathbf{g}_j \otimes \mathbf{g}^j \]  

(1.3.13)

which is equivalent to the transpose of the inverse in (1.3.7).

- The notation \( \mathbf{1} \) denotes the (second order) unit tensor, which is defined as the mapping of a vector onto itself. For example,

\[ \mathbf{S}^\mathbf{1} = \mathbf{1} \cdot \mathbf{S} \]  

(1.3.14)

Based on (1.3.6) and (1.3.7), it is easily verified that the relation \( \mathbf{S}^{-1} \cdot \mathbf{S} = \mathbf{1} \) holds.

- The notation \( \| \mathbf{S} \| \) denotes the norm of the second-order tensor:

\[ \| \mathbf{S} \| = \| S^i_\mathbf{j} \otimes \mathbf{g}^j \| = \sqrt{\mathbf{S} : \mathbf{S}} = \sqrt{S^i_\mathbf{g}_j S^{kl} (\mathbf{g}_j \cdot \mathbf{g}_k)(\mathbf{g}_i \cdot \mathbf{g}_i)} \]  

(1.3.15)

which has been worked out after realising that (1.3.5) holds.

- The norm of a vector \( \mathbf{A} \) is defined as

\[ \text{say } \mathbf{A} = A^i \mathbf{g}_i \Rightarrow \| \mathbf{A} \| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A^i A^j (\mathbf{g}_i \cdot \mathbf{g}_j)} \]  

(1.3.16)

- The norm of a column vector is defined by

\[ \text{say } \mathbf{S}^T = [S_1 \ldots S_{\text{NCOMP}}] \Rightarrow \| \mathbf{S} \| = \sqrt{\sum_{i=1}^{\text{NCOMP}} (S_i)^2} \]  

(1.3.17)
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- The transpose of a column vector indicates a change from a column-vector into a row-vector, and vice versa.

- The notation $O_{NROW \times NCOL}$ indicates the zero-matrix with NROW rows and NCOL columns. With $O_{NCOMP}$ we denote the zero-vector (interpreted as a so-called column vector) with NCOMP components.
Chapter 2 - Shell formulation

2.1 - Introduction

Considering the deformation of tubes as used in practice (in the oil-industry), it is important to observe that the tubes are moderately thick-walled. This is mathematically stated as follows. Let $t$ represent the wall-thickness of the tube and $R$ a characteristic radius of curvature, then the ratio $t / R$ typically ranges between $1/5 - 1/50$.

As mentioned before, a certain class of failures of tubes is caused by the buckling phenomenon. For these cases the ultimate load occurs at relatively small membrane strains and small bending strains. For other cases such as burst under internal pressure, it is observed that the ultimate load is reached at relatively large membrane strains (say, membrane strains up to 10% - 15%). The model focusses on these types of applications. Therefore the formulation will allow for large membrane strains whereas the bending strains will be required to remain small. The current model is not dedicated to the description of final fracture of the material.

The "classical theory of thin shells", as discussed by SANDERS (1963) and KOITER (1966), is based on Kirchhoff-Love or equivalent assumptions and assumes a vanishing through-the-thickness stress component. Furthermore the classical theory of thin shells requires $t << R$ and is limited to the small strain regime. In view of the $t / R$ ratios mentioned above, the requirement $t << R$ is not satisfied. The limitation to the small strain regime is not acceptable in view of the observations of failures of tubes mentioned above.

It is decided to extend the classical theory to large membrane strains and consequently also to thickness-stretching. As already discussed, several options for inclusion of thickness-stretching in the formulation exist. On the basis of the conceptual simplicity it is decided to introduce an extra independent variable. As an approximation, the thickness-stretching will be assumed to be constant over the thickness.

Justified by the above observations on the range of applicability of the tool, the kinematic assumptions in the model are as follows:

(i) Material points which lie on one and the same normal to the mid-surface in the reference configuration remain on one and the same normal at the same point of the
mid-surface during deformation. (Equivalent to the Kirchhoff-Love assumption in the "classical theory of thin shells").

(II) The thickness-stretching is constant through-the-thickness.

Based on the first assumption, the well-known variables from the classical theory such as mid-surface position vector, membrane and bending deformations, tangential stress resultants and tangential stress couples, etc. will be involved in the theory. The three components of the mid-surface position vector will be considered as fundamental unknowns in the formulation. Besides, based on assumption (II), a fourth fundamental unknown will feature in the theory. This unknown will be introduced as the so-called thickness function. All these fundamental unknowns are assumed to depend on the mid-surface coordinates only.

Equilibrium will be formulated in terms of the principle of virtual work consisting of internal virtual work and external virtual work. In this chapter attention is focussed to the internal virtual work expression. (The expression for the external virtual work is given in Chapter 5.) As a starting point for the evaluation of the internal virtual work, a general expression valid for a three-dimensional continuum is used. This expression is formulated in terms of a volume integral. Incorporation of the kinematic assumptions into this integral makes it possible to carry out a simplification, leading to a surface integral expression, by performing the through-the-thickness integral first. This essentially leads to the definition of stress resultants and stress couples. In the present chapter this procedure is worked out.

2.2 - Kinematic description

The kinematic assumptions are based on the assumption that material points which lie on a normal to the mid-surface in the reference configuration remain on the normal at the same point of the mid-surface during deformation, see assumption (I) in section 2.1. Moreover, following from assumption (II), a displacement of the material points along the normal is allowed for. This relative displacement component captures the change in thickness of the shell. In the present section, expressions for the position vector, the base vectors and the infinitesimal volume and mid-surface elements, both in the reference and the deformed configuration will be derived. These quantities play an important role in the discussion in subsequent sections.

Material points in the shell are parameterized by a set of material convected coordinates \( (\xi^1, \xi^2, \xi) \), see figure 2.1. The coordinates \( \xi^1 \) and \( \xi^2 \) determine the position at the mid-surface of the shell and \( \xi \) is a metric "through-the-thickness coordinate", which
ranges between $-\frac{1}{2} \rightarrow \frac{1}{2}$ and is positive in the direction of the normal $\boldsymbol{n}$. At the mid-surface $\xi = 0$.

\[ \tilde{\varphi}(\xi^1, \xi^2) = \tilde{\varphi}^0(\xi^1, \xi^2) + \xi \lambda(\xi^1, \xi^2) \boldsymbol{n}(\xi^1, \xi^2) \]

Figure 2.1 - The shell configuration in the reference situation.

A reference configuration (i.e. the undeformed state) of the shell is completely defined by the following functions:

- the position vector of the mid-surface $\tilde{\varphi}^0 = \tilde{\varphi}^0(\xi^1, \xi^2)$;

- the unit normal vector perpendicular to the mid-surface of the shell, denoted with $\boldsymbol{n} = \boldsymbol{n}(\xi^1, \xi^2)$. (According to its definition, $\boldsymbol{n}$ is completely defined by the position vector of the mid-surface, see (2.2.5).)

- the thickness function $\lambda = \lambda(\xi^1, \xi^2)$.

The position vector to a material point in the reference state, denoted with the vector $\tilde{\varphi}$, is given by

\[ \tilde{\varphi}(\xi^1, \xi^2, \xi) = \tilde{\varphi}^0(\xi^1, \xi^2) + \xi \lambda(\xi^1, \xi^2) \boldsymbol{n}(\xi^1, \xi^2) \quad (2.2.1) \]

As a result of the definition of the through-the-thickness coordinate, $|\xi\lambda|$ is the distance of the material point to the mid-surface while $\lambda$ represents the thickness.
Covariant base vectors in the reference configuration follow from partial derivatives of the position vector \( \overline{\Phi} \) with respect to the material coordinates. Introducing the notation \( X_\alpha \) as the partial derivative of the function \( X \) with respect to \( \xi^\alpha \) and \( X_3 \) as the partial derivative of \( X \) with respect to the coordinate \( \xi_3 \), (2.2.1) gives

\[
\begin{align*}
\vec{e}_\alpha &= \frac{\partial \overline{\Phi}}{\partial \xi^\alpha} = \overline{\Phi}_\alpha = \overline{\Phi}_\alpha^0 + \xi (\lambda \overline{n})_\alpha \\
\vec{e}_3 &= \frac{\partial \overline{\Phi}}{\partial \xi_3} = \overline{\Phi}_3 = \lambda \overline{n}
\end{align*}
\]

The reciprocal or contravariant base vectors \( \{ \vec{e}^I \} \) are determined from (2.2.2) according to the orthogonality condition (1.3.2). The definition of the metric tensor associated with the bases \( \{ \vec{e}_I \} \) and \( \{ \vec{e}^J \} \), given in (1.3.3), is repeated:

\[
\begin{align*}
g_{ij} &= g_{ij} = \vec{e}_i \cdot \vec{e}_j, \quad g^{ij} = g^{ij} = \vec{e}^i \cdot \vec{e}^j 
\end{align*}
\]

which are the covariant and contravariant metric coefficients, respectively.

Also, a set of covariant mid-surface base vectors, denoted with \( \{ \vec{\alpha}_\alpha \} \), is introduced. These will be defined from the base vectors \( \{ \vec{e}_\alpha \} \) by setting \( \xi = 0 \). Accordingly, from (2.2.2),

\[
\vec{\alpha}_\alpha = \overline{\Phi}_0^\alpha
\]

These vectors are also depicted in figure 2.1. Contravariant mid-surface base vectors, which are denoted with \( \{ \vec{\alpha}^\beta \} \), are obtained according to (1.3.2). Using the above introduced notation, the unit vector perpendicular to the mid-surface of the shell is written as

\[
\overline{n} = \frac{\vec{\alpha}_1 \times \vec{\alpha}_2}{\| \vec{\alpha}_1 \times \vec{\alpha}_2 \|}
\]

For later reference we introduce the infinitesimal volume element in the reference state, denoted by \( d(\text{vol}) \). In terms of infinitesimal changes in material coordinates

\[
d(\text{vol}) = \sqrt{g} \ d\xi^1 \ d\xi^2 \ d\xi^3 \quad \text{with} \quad \sqrt{g} = (\vec{e}_1 \times \vec{e}_2) \cdot \vec{e}_3 > 0
\]

The requirement of linearly independent base vectors \( \{ \vec{e}_I \} \) is expressed by the last statement in (2.2.6). Moreover, this statement assures a right-handed basis \( \{ \vec{e}_I \} \). In close relation to the above, we also give the expression for an infinitesimal mid-surface element in the reference state:

\[
d(\text{mid}) = \sqrt{a} \ d\xi^1 \ d\xi^2 \quad \text{where} \quad \sqrt{a} = \| \vec{\alpha}_1 \times \vec{\alpha}_2 \| 
\]
Expressions (2.2.6) and (2.2.7) can be found in many textbooks on continuum mechanics, e.g. GREEN & ZERNA (1960).

Within the limitation to the class of problems to be analysed, the kinematic assumptions (I) and (II) are adopted. As a consequence, the configuration in the deformed state is defined by a position vector to the mid-surface, a unit normal vector and a thickness function in a similar way as in (2.2.1). Note that the position vector in the reference configuration is arbitrary, whereas the position vector in the deformed configuration embeds the assumed kinematics. Consequently, the position vector to the deformed configuration reads

\[
\vec{\Phi}(\xi^1, \xi^2, \xi^3) = \vec{\Phi}^o(\xi^1, \xi^2) + \xi^3 \Lambda(\xi^1, \xi^2) \vec{N}(\xi^1, \xi^2) \quad (2.2.8)
\]

Here, the three functions in the deformed configuration are denoted with \( \vec{\Phi}^o \), the mid-surface position vector, \( \vec{N} \), the unit normal vector, and \( \Lambda \), the thickness function. From the definition of the \( \xi \)-dependency in \( \vec{\Phi} \) in (2.2.8), it is clear that equidistant material points along a normal remain equidistant, although the thickness itself may change.

Following from (2.2.8), which is similar to (2.2.1), the quantities mentioned in (2.2.2)-(2.2.7), but now in the deformed configuration, are derived. According to the notation convention, the quantities in the deformed configuration are denoted with capital letters. We have

\[
\begin{align}
\text{(2.2.2)} & \implies \begin{cases} 
\vec{G}_x = \vec{\Phi}_x = \vec{\Phi}^o + \xi \Lambda \vec{N} \\
\vec{G}_a = \vec{\Phi}_a = \Lambda \vec{N}
\end{cases} & ; (2.2.9) \\
\text{(2.2.3)} & \implies G^x = G^a \equiv \vec{G}_i \cdot \vec{G}_j, \quad G^{ij} = G^{ji} \equiv \vec{G}^i \cdot \vec{G}^j & ; (2.2.10) \\
\text{(2.2.4)} & \implies \vec{A}_a = \vec{\Phi}^o & ; (2.2.11) \\
\text{(2.2.5)} & \implies \vec{N} = \frac{\vec{A}_1 \times \vec{A}_2}{|\vec{A}_1 \times \vec{A}_2|} & ; (2.2.12) \\
\text{(2.2.6)} & \implies d(\text{VOL}) = \sqrt{G} \, d\xi^1 \, d\xi^2 \, d\xi^3, \quad \sqrt{\Lambda} = (\vec{G}_1 \times \vec{G}_2) \cdot \vec{G}_3 > 0 & ; (2.2.13) \\
\text{(2.2.7)} & \implies d(\text{MID}) = \sqrt{\Lambda} \, d\xi^1 \, d\xi^2, \quad \sqrt{A} = \|\vec{A}_1 \times \vec{A}_2\| & . (2.2.14)
\end{align}
\]

The kinematic assumptions embedded in (2.2.9), together with the choice (2.2.1), establishes the total deformation behaviour in the model.
2.3 - Deformation gradient tensor and unit tensor

The deformation gradient tensor is defined as the mapping of an infinitesimal material line segment from the reference state (denoted with \( d\overline{\varphi} \)) into the deformed state (denoted with \( d\overline{\Phi} \)). Using the definitions in (2.2.2) and (2.2.9), it follows that

\[
\begin{align*}
    d\overline{\varphi} &= \overline{g}_a \, d\xi^a + \overline{g}_a \, d\xi^a \\
    d\overline{\Phi} &= \overline{G}_a \, d\xi^a + \overline{G}_a \, d\xi^a
\end{align*}
\]

(2.3.1)

For any infinitesimal line segments \( d\overline{\varphi} \) and \( d\overline{\Phi} \), the deformation gradient tensor \( \mathbf{F} \) is defined as

\[
d\overline{\Phi} = \mathbf{F} \cdot d\overline{\varphi}
\]

(2.3.2)

By introduction of (2.3.1) and requiring that (2.3.2) holds for every value of \((d\xi^a, d\xi^a)\), we find

\[
\overline{G}_i = \mathbf{F} \cdot \overline{g}_i.
\]

Making use of the notation with dyads, as discussed in section 1.3, the following useful expressions may be derived:

\[
\mathbf{F} = \overline{G}_i \otimes \overline{g}_i \quad \Rightarrow \quad \begin{align*}
    \mathbf{F}^T &= \overline{g}_i \otimes \overline{G}_i \\
    \mathbf{F}^{-1} &= \overline{g}_i \otimes \overline{G}_i \\
    \mathbf{F}^{-T} &= \overline{G}_i \otimes \overline{g}_i
\end{align*}
\]

(2.3.3)

For usage in section 2.4, two different component expressions for the unit tensor \( \mathbf{1} \) as defined in (1.3.14) are given. In this definition, the unit tensor is defined as the mapping of a vector onto itself. Therefore, both

\[
\overline{g}_i = \mathbf{1} \cdot \overline{g}_i \quad \text{and} \quad \overline{G}_i = \mathbf{1} \cdot \overline{G}_i
\]

are valid expressions.

Thus, from (1.3.6), it follows that

\[
\begin{align*}
    \mathbf{1} &= \overline{g}_i \otimes \overline{g}^i = (g_{ij}) \, \overline{g}_j \otimes \overline{g}^i \\
    \mathbf{1} &= \overline{G}_i \otimes \overline{G}^i = (G_{ij}) \, \overline{G}_j \otimes \overline{G}^i
\end{align*}
\]

(2.3.4)

In (2.3.4), the index-raising procedure from (1.3.10.b) is applied. Note that the components of the unit tensor \( \mathbf{1} \) in both expressions (2.3.4) are not identical. This is caused by the fact that the components are defined on different sets of base vectors.
2.4 - Strain fields

As mentioned above, the definition of the deformation gradient tensor is based on the movement of an infinitesimal line segment of the material in the reference state into the deformed state. Based on \( \mathbf{d} \overline{\mathbf{e}} \) (the infinitesimal line segment in the reference state) and \( \mathbf{d} \overline{\mathbf{e}} \) (the same line segment, but now in the deformed state), the scalar \( (\mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} - \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}}) \), which is the square of the length of the line segment in the deformed state minus the square of the length in the reference state, is considered. From this scalar, strain tensors are derived.

The general expressions for the Lagrangian and the Eulerian strain tensors can be found in many text-books. In order to make this thesis self-contained, the general expressions are presented first. After that, the expressions following from the assumed kinematics are given. In these expressions, the dependencies of the components of both strain tensors on the through-the-thickness coordinate \( \xi \) will be clearly separated.

- The *Lagrangian strain tensor*, in our case denoted with \( \mathbf{e} \), is defined according to:

\[
( \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} - \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} ) = 2 \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{e} \cdot \mathbf{d} \overline{\mathbf{e}} \tag{2.4.1}
\]

In view of (2.3.2) and the definition of the unit tensor \( \mathbf{1} \) in (1.3.14), the scalar at the left-hand side of (2.4.1) is written as

\[
( \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} - \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} ) = \mathbf{d} \overline{\mathbf{e}}^T \cdot (\mathbf{F}^T \cdot \mathbf{F}) \cdot \mathbf{d} \overline{\mathbf{e}} - \mathbf{d} \overline{\mathbf{e}}^T \cdot (\mathbf{1}) \cdot \mathbf{d} \overline{\mathbf{e}} \tag{2.4.1.a}
\]

which yields the following expression for the Lagrangian strain tensor:

\[
\mathbf{e} = \frac{1}{2} (\mathbf{C} - \mathbf{1}) \quad , \quad \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \tag{2.4.2}
\]

The tensor \( \mathbf{C} \) is often referred to as the right Cauchy-Green tensor.

- The *Eulerian strain tensor*, denoted with \( \mathbf{E} \), is defined as follows

\[
( \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} - \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} ) = 2 \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{E} \cdot \mathbf{d} \overline{\mathbf{e}} \tag{2.4.3}
\]

The scalar at the left-hand side of (2.4.3) is (due to expressions in (2.3.3)) written as

\[
( \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} - \mathbf{d} \overline{\mathbf{e}}^T \cdot \mathbf{d} \overline{\mathbf{e}} ) = \mathbf{d} \overline{\mathbf{e}}^T \cdot (\mathbf{1}) \cdot \mathbf{d} \overline{\mathbf{e}} - \mathbf{d} \overline{\mathbf{e}}^T \cdot (\mathbf{F}^T \cdot \mathbf{F}^{-1}) \cdot \mathbf{d} \overline{\mathbf{e}} \tag{2.4.3.a}
\]

which yields

\[
\mathbf{E} = \frac{1}{2} (\mathbf{1} - \mathbf{B}^{-1}) \quad , \quad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T = \mathbf{C}^T \tag{2.4.4}
\]
The tensor \( \mathbf{B} \) is called the left Cauchy-Green tensor.

Similar expressions as (2.4.1)-(2.4.4) are also given by BESSELING & VAN DER GIESSEN (1994). It is easily verified from (2.4.2) and (2.4.4) that the following relation between the Lagrangian and Eulerian strain tensors holds:

\[
\mathbf{e} = \mathbf{F}^T \cdot \mathbf{E} \cdot \mathbf{F} \quad (2.4.5)
\]

The components of the Lagrangian strain tensor \( \mathbf{e} \) with respect to the covariant base vectors in the reference configuration and the components of the Eulerian strain tensor \( \mathbf{E} \) with respect to the covariant base vectors in the deformed configuration are identical. This is shown as follows. Successive substitution of (2.3.3) and the first expression in (2.3.4) into (2.4.2) gives

\[
\mathbf{e} = \frac{1}{2} \left\{ (\mathbf{\bar{g}}^i \otimes \mathbf{\bar{g}}_i) \cdot (\mathbf{\bar{g}}_j \otimes \mathbf{\bar{g}}^j) - 1 \right\} = \\
= \frac{1}{2} \left\{ (\mathbf{\bar{G}}_i \cdot \mathbf{\bar{G}}_j) (\mathbf{\bar{g}}^i \otimes \mathbf{\bar{g}}^j) - (\mathbf{\bar{G}}_i \cdot \mathbf{\bar{g}}_j) (\mathbf{\bar{g}}^i \otimes \mathbf{\bar{g}}^j) \right\} = \\
= \frac{1}{2} \left\{ G_{ij} - g_{ij} \right\} \mathbf{\bar{g}}^i \otimes \mathbf{\bar{g}}^j = \\
= \eta_{ij} \mathbf{\bar{g}}^i \otimes \mathbf{\bar{g}}^j \quad (2.4.6)
\]

In a similar fashion, substitution of (2.3.3) and the second expression in (2.3.4) into (2.4.4) yields

\[
\mathbf{E} = \frac{1}{2} \left\{ 1 - (\mathbf{\bar{G}}^i \otimes \mathbf{\bar{G}}_i) \cdot (\mathbf{\bar{G}}_j \otimes \mathbf{\bar{G}}^j) \right\} = \\
= \frac{1}{2} \left\{ (\mathbf{\bar{G}}_i \cdot \mathbf{\bar{G}}_j) (\mathbf{\bar{G}}^i \otimes \mathbf{\bar{G}}^j) - (\mathbf{\bar{G}}_i \cdot \mathbf{\bar{G}}_j) (\mathbf{\bar{G}}^i \otimes \mathbf{\bar{G}}^j) \right\} = \\
= \frac{1}{2} \left\{ G_{ij} - g_{ij} \right\} \mathbf{\bar{G}}^i \otimes \mathbf{\bar{G}}^j = \\
= \eta_{ij} \mathbf{\bar{G}}^i \otimes \mathbf{\bar{G}}^j \quad (2.4.7)
\]

Clearly, the components stated in (2.4.6) and (2.4.7) are identical. As a consequence of the symmetry of \( g_{ij} \) and \( G_{ij} \), see (2.2.3) and (2.2.10), the components \( \eta_{ij} \) are symmetric.

The metric tensor components in (2.4.6) and (2.4.7) are worked out further by substitution of the expressions for the base vectors in the reference configuration and in the deformed configuration as given in (2.2.2) and (2.2.9), respectively. A clear separation of the \( \xi \)-dependency yields
\[ g_{ab} = \bar{a}_a \cdot \bar{a}_b + \xi \left[ \bar{a}_a \cdot (\lambda \bar{n})_b + \bar{a}_b \cdot (\lambda \bar{n})_a \right] + \xi^2 \left[ (\lambda \bar{n})_a \cdot (\lambda \bar{n})_b \right] \]
\[ g_{a3} = \bar{a}_a \cdot (\lambda \bar{n}) + \xi \left[ (\lambda \bar{n})_a \cdot (\lambda \bar{n}) \right] \]
\[ g_{33} = (\lambda \bar{n}) \cdot (\lambda \bar{n}) \] (2.4.8.a)

\[ G_{ab} = \bar{A}_a \cdot \bar{A}_b + \xi \left[ \bar{A}_a \cdot (\Lambda \bar{N})_b + \bar{A}_b \cdot (\Lambda \bar{N})_a \right] + \xi^2 \left[ (\Lambda \bar{N})_a \cdot (\Lambda \bar{N})_b \right] \]
\[ G_{a3} = \bar{A}_a \cdot (\Lambda \bar{N}) + \xi \left[ (\Lambda \bar{N})_a \cdot (\Lambda \bar{N}) \right] \]
\[ G_{33} = (\Lambda \bar{N}) \cdot (\Lambda \bar{N}) \] (2.4.8.b)

The above expressions are simplified by realizing the definitions of \( \bar{n} \) and \( \bar{N} \), being unit normals to the mid-surface. Thus
\[ \bar{n} \cdot \bar{n} = 1, \quad \bar{N} \cdot \bar{N} = 1; \quad \bar{a}_a \cdot \bar{n} = 0, \quad \bar{A}_a \cdot \bar{N} = 0. \]

Upon differentiation with respect to the mid-surface coordinates it follows that
\[ \bar{n} \cdot \bar{n}_a = 0, \quad \bar{N} \cdot \bar{N}_a = 0; \quad \bar{a}_a \cdot \bar{n} + \bar{a}_a \cdot \bar{N}_a = 0, \quad \bar{A}_a \cdot \bar{N} + \bar{A}_a \cdot \bar{N}_a = 0. \]

The latter two relations are equivalent to Weingarten's formulæ, see for example KOITER (1966), usually written as
\[ \bar{n}_a = - (\bar{a}_a \cdot \bar{n}) \bar{a}^a = - b_{ab} \bar{a}^a \] and similarly \( \bar{N}_a = - B_{ab} \bar{A}^a \).

The above expressions are used for the evaluation of (2.4.8). Due to this, expressions (2.4.8) may be rewritten into
\[ g_{ab} = a_{ab} - 2 (\xi \lambda) b_{ab} + (\xi \lambda)^2 \left\{ c_{ab} + \frac{\lambda_a \lambda_b}{\lambda^2} \right\} \]
\[ g_{a3} = (\xi \lambda) \Lambda_a \]
\[ g_{33} = \lambda^2 \] (2.4.9.a)

\[ G_{ab} = A_{ab} - 2 (\xi \Lambda) B_{ab} + (\xi \Lambda)^2 \left\{ C_{ab} + \frac{\Lambda_a \Lambda_b}{\Lambda^2} \right\} \]
\[ G_{a3} = (\xi \Lambda) \Lambda_a \]
\[ G_{33} = \Lambda^2 \] (2.4.9.b)

with

...
where $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $c_{\alpha\beta}$ represent the covariant components of the first, second and third fundamental tensors of the mid-surface in the reference state respectively and likewise $A_{\alpha\beta}$, $B_{\alpha\beta}$ and $C_{\alpha\beta}$ in the deformed state. The definitions in (2.4.10) are also used in the "classical theory of thin shells", see for example KOITER (1966).

In general, the strain components $\eta_{ij}$ could arbitrarily be developed into a Taylor series expansion in the through-the-thickness coordinate $\xi$ according to

$$\eta_{ij} = \eta_{ij}^{(0)} + (\xi \lambda) \eta_{ij}^{(1)} + (\xi \lambda)^2 \eta_{ij}^{(2)} + \ldots$$

(2.4.11)

According to the definition in (2.4.6) for $\eta_{ij}$ and the expressions (2.4.9) for the metric tensor components, only a few terms from this Taylor expansion are retained. Substitution of (2.4.9) and (2.4.10) into the definition of $\eta_{ij}$ in (2.4.6) yields

$$\eta_{ij} = \left\{ \begin{array}{l}
\eta_{ij}^{(0)} = e_{ij}^{\alpha\beta} - (\xi \lambda) \rho_{ij}^{\alpha\beta} + \frac{1}{2}(\xi \lambda)^2 \mu_{ij}^{\alpha\beta} \\
\eta_{i3} = (\xi \lambda) \chi_{i3} = \eta_{i3} \\
\eta_{33} = \chi_3
\end{array} \right. ,$$

(2.4.12.a)

where

$$\begin{align*}
e_{ij}^{\alpha\beta} &= \frac{1}{2} \left( A_{ij} - a_{ij} \right), & \rho_{ij}^{\alpha\beta} &= A_{ij}B_{ij} - b_{ij} \\
\mu_{ij}^{\alpha\beta} &= \left( \frac{A}{\lambda} \right)^2 C_{ij} - c_{ij}^\alpha + \frac{1}{\lambda^2} \left( \Lambda_{i\alpha} \Lambda_{j\beta} - \Lambda_{j\alpha} \Lambda_{i\beta} \right) \\
\chi_3 &= \frac{1}{2} \left( \Lambda^2 - (\lambda)^2 \right)
\end{align*}$$

(2.4.12.b)

The components $e_{ij}$ and $\rho_{ij}$ are the measures for the membrane and bending strains. The difference of $\rho_{ij}$ and $\mu_{ij}$ with the measures used in the "classical theory of thin shells" is a scaling with the thickness in both the reference and the deformed configuration. Note that when the thickness is not changing (thus $\Lambda = \lambda$), the measures $\rho_{ij}$ and $\mu_{ij}$ are identical to those defined for the classical theory of thin shells. Component $\chi_3$ embodies the thickness-stretching and is additional in comparison with the classical theory.

The formulation is based on the Kirchhoff-Love assumption, i.e. (I) on page 13/14. As a consequence, constant (in $\xi$) shear deformation is excluded, leading to vanishing terms $\eta_{i3}^{(0)}$. However, due to the introduction of the thickness function, a linear (in $\xi$) shear deformation is retained, leading to the contribution of $\eta_{i3}^{(1)} = \chi_3$. In view of the fact that the
influence of the constant (in $\xi$) terms $\eta_{a3}^{(0)}$ vanish, it is decided to neglect the influence of the linear (in $\xi$) terms $\eta_{a3}^{(1)}$.

It is assumed that the variation of the thickness over the mid-surface (or, the smoothness) in the reference as well as in the deformed configuration is small such that the influence of the underlined terms in (2.4.12.b) can be neglected. In addition, as usual in the "classical theory of thin shells", the influence of the quadratic (in $\xi$) terms $\mu_{a3}^{\alpha}$ in the expression for $\eta_{a3}$ will be dropped, leading to relative errors of order $(t/R)$ in the bending deformation.

In conclusion, based on the above mentioned assumptions, the following (relevant) strain measures are arrived at:

\[
\begin{align*}
&\quad \quad \quad \quad \eta_{a3} = \varepsilon_{a3} - (\xi \lambda) \rho_{a3}^* \\
&\eta_{a3} = 0 \\
&\eta_{33} = \chi_3
\end{align*}
\]

with
\[
\begin{align*}
\varepsilon_{a3} &= \frac{1}{2} \left( \bar{A}_a \cdot \bar{A}_b - \bar{a}_a \cdot \bar{a}_b \right) \\
\rho_{a3}^* &= \frac{\lambda}{\chi} \left( \bar{A}_{a,a} \cdot \bar{N} - \bar{a}_{a,b} \cdot \bar{n} \right) \tag{2.4.13} \\
\chi_3 &= \frac{1}{2} \left( (\lambda)^2 - (\lambda)^2 \right)
\end{align*}
\]

The strain components defined above are determined by the mid-surface position vector (together with the first and second derivatives with respect to the material coordinates) and the thickness function, both in the reference and the deformed configuration. By making assumptions for these, restrictions to certain types of configurations are involved. In Chapter 4, dealing with the description of tubular deformation, the specific choice for the mid-surface position vector and the thickness function adequate for the analysis of tubes will be discussed.

### 2.5 - Internal virtual work

The principle of virtual work will be adopted for the description of equilibrium. This principle states that, given equilibrium, the work of the internal stresses under an arbitrary kinematically admissible virtual displacement field is equal to the virtual work done by the external loading on the system. The principle of virtual work is introduced by means of

\[
\delta W = \delta W_{\text{int}} - \delta W_{\text{ext}} = 0 \tag{2.5.1}
\]

where $\delta W_{\text{int}}$ is the internal virtual work and $\delta W_{\text{ext}}$ the external virtual work. This paragraph is focusing on $\delta W_{\text{int}}$, whereas $\delta W_{\text{ext}}$ will be dealt with in Chapter 5.

Some appropriate formulations for the internal virtual work are
\[ \delta W_{el} = \int \sigma : \delta \mathbf{E} \, d(\text{VOL}) = \int \tau : \delta \mathbf{E} \, d(\text{vol}) = \int \mathbf{S} : \delta \mathbf{e} \, d(\text{vol}) \quad (2.5.2) \]

In the formulation, the latter integral is used. In this expression, the infinitesimal volume element \(d(\text{vol})\) is defined in (2.2.6). The variations of the Lagrangian strain tensor \(\mathbf{e}\) are obtained from (2.4.6) and read
\[ \delta \mathbf{e} = \delta (\mathbf{e}) = \delta \eta_{ij} \mathbf{G}^i \otimes \mathbf{G}^j \quad (2.5.3) \]

where the components \(\delta \eta_{ij}\) are derived from (2.4.13); i.e.
\[
\begin{align*}
\delta \eta_{\alpha \beta} &= \delta e_{\alpha \beta} - (\xi \lambda) \delta \rho_{\alpha \beta} \\
\delta \eta_{33} &= \delta \chi_3 \quad (2.5.4.a)
\end{align*}
\]

where
\[
\begin{align*}
\delta e_{\alpha \beta} &= \frac{1}{2} ( \delta \mathbf{A}_\alpha \cdot \mathbf{A}_\beta + \mathbf{A}_\alpha \cdot \delta \mathbf{A}_\beta ) \\
\delta \rho_{\alpha \beta} &= \frac{\delta \lambda}{\lambda} \mathbf{A}_{\alpha \beta} \cdot \mathbf{N} + \frac{\lambda}{\delta \lambda} \mathbf{A}_{\alpha \beta} \cdot \delta \mathbf{N} \\
\delta \chi_3 &= \Lambda \delta \lambda
\end{align*}
\quad (2.5.4.b)
\]

Quantity \(\delta \mathbf{E}\) as used in (2.5.2) is defined by
\[ \delta \mathbf{E} = \delta \eta_{ij} \mathbf{G}^i \otimes \mathbf{G}^j \quad (2.5.5) \]

Note that this quantity is not equal to the variation of the Eulerian strain tensor, thus \(\delta \mathbf{E} \neq \delta (\mathbf{E})\). It is easily verified that a similar relation as (2.4.5) also holds for \(\delta \mathbf{e}\) and \(\delta \mathbf{E}\); i.e.
\[ \delta \mathbf{e} = F^T \cdot \delta \mathbf{E} \cdot F \quad (2.5.6) \]

The tensors \(\sigma\), \(\tau\) and \(\mathbf{S}\) in (2.5.2) are the Cauchy stress tensor, the Kirchhoff stress tensor and the 2\textsuperscript{nd} Piola-Kirchhoff stress tensor, respectively. The mutual relations between these tensors are obtained from (2.5.2) with the aid of (2.2.6), (2.2.13) and (2.5.6) resulting in
\[ \mathbf{S} = F^{-1} \cdot \tau \cdot F^{-T} \quad \text{and} \quad \tau = \frac{\sqrt{G}}{\sqrt{g}} \sigma \quad (2.5.7) \]

These expressions are also given in BESSELING & VAN DER GIESSEN (1994). If the components of the Kirchhoff stress tensor \(\tau\) with respect to the deformed base vectors are defined by
\[ \tau = \tau^{ij} \mathbf{G}_i \otimes \mathbf{G}_j \quad (2.5.8.a) \]
then the first relation in (2.5.7), together with expressions for the deformation gradient tensor in (2.3.3) clearly reveals that the components of the 2nd Piola-Kirchhoff stress tensor \( \mathbf{S} \) with respect to the reference base vectors \( \{ \mathbf{g}_i \} \) are identical. The link of the components of the Kirchhoff stress tensor to the Cauchy stress components is given by the second relation in (2.5.7). Thus,

\[
\begin{align*}
\mathbf{S} &= \tau_{ij} \mathbf{g}_i \otimes \mathbf{g}_j \\
\mathbf{S} &= \sigma_{ij} \mathbf{g}_i \otimes \mathbf{g}_j
\end{align*}
\]

with \( \tau_{ij} = \frac{\sqrt{G}}{\sqrt{g}} \sigma_{ij} \) .

(2.5.8.b)

In the sequel, the expression in (2.5.2) based on the 2nd Piola-Kirchhoff stress tensor \( \mathbf{S} \) and the variation of the Lagrangian strain tensor \( \delta \mathbf{e} \) is further worked out. To this end, expression (2.5.3) and the expression for \( \mathbf{S} \) in (2.5.8.b) are substituted in (2.5.2); it yields

\[
\delta W_{int} = \int (\tau_{ij} \mathbf{g}_i \otimes \mathbf{g}_j) : (\delta \eta_{kl} \mathbf{g}_k \otimes \mathbf{g}_l) \, d(\text{vol}) =
\]

\[
\int \tau_{ij} \delta \eta_{ij} \, d(\text{vol}) = \iint \tau_{ij} \delta \eta_{ij} \sqrt{g} \, d\xi^1 \, d\xi^2 \, d\xi^3 =
\]

\[
\iint \{ \tau^{ab} \delta e_{ab} - (\xi \lambda) \tau^{ab} \delta p_{ab}^* + \tau^{33} \delta \chi_3 \} \sqrt{g} \, d\xi^1 \, d\xi^2 \, d\xi^3
\]

where the definition of the double-dot product between two dyads (from (1.3.5)), the expression for \( d(\text{vol}) \) (from (2.2.6)) and finally expression (2.5.4) are applied.

The internal virtual work expression (2.5.9) is formulated in terms of a volume integral. By integration through-the-thickness first, an equivalent surface integral expression may be derived. If the following "stress-integrals" are defined:

\[
\begin{align*}
n^{ab} &= \frac{1}{\sqrt{a}} \int \tau^{ab}(\xi) \sqrt{g(\xi)} \, d\xi \\
n^{33} &= \frac{1}{\sqrt{a}} \int \tau^{33}(\xi) \sqrt{g(\xi)} \, d\xi \\
m^{ab} &= \frac{-\lambda}{\sqrt{a}} \int \tau^{ab}(\xi) \xi \sqrt{g(\xi)} \, d\xi
\end{align*}
\]

(2.5.10)

then, the surface integral expression equivalent to the latter expression in (2.5.9) reads

\[
\delta W_{int} = \iint \{ \ n^{ab} \delta e_{ab} + m^{ab} \delta p_{ab}^* + n^{33} \delta \chi_3 \} \sqrt{a} \, d\xi^1 \, d\xi^2
\]

(2.5.11)

The quantities \( n^{ab} \) and \( m^{ab} \) are the so-called tangential stress resultants and tangential stress couples, while \( n^{33} \) will be called the "lateral stress resultant".
As a final step, the variations in (2.5.11) are worked out in terms of the (derivatives of the) mid-surface position vector, the unit normal vector and the thickness function. This clearly shows the variations which should be evaluated in the computational model. Substitution of the variations of the strain measures \( (\varepsilon_{a b}, \rho_{a b}, \chi_3) \) as given in (2.5.4.b) yields

\[
\delta W_m = \int \left\{ n^{ab} \delta A_a + \left( \frac{\Lambda}{\lambda} m^{ab} \mathbf{N} \right) \cdot \delta A_{a,b} + \left( \frac{\Lambda}{\lambda} m^{ab} \mathbf{A}_{a,b} \right) \cdot \delta \mathbf{N} + \right. \\
\left. \left[ n^{33} \Lambda + \frac{1}{\lambda} m^{ab} \mathbf{A}_{a,b} \cdot \mathbf{N} \right] \delta \Lambda \right\} \sqrt{a} \, d\xi^1 \, d\xi^2 
\]  
(2.5.12)

2.6 - Thickness integration

The approach as worked out so far requires the evaluation of the "stress-integrals" (2.5.10). In certain cases, it is possible to carry out this evaluation analytically. In the other cases, numerical integration should be applied. Especially, in the case of inelastic material behaviour, closed-form solutions of the "stress-integrals" cannot straightforwardly be given. Therefore, it is decided to apply numerical integration. In the implementation, numerical integration by virtue of the so-called layered model is applied.

Figure 2.2 serves as an illustration of the layered model. The figure shows a part of the shell through-the-thickness. Also the position vector to a certain point on the mid-surface \( \mathbf{\bar{A}} \) together with the associated normal \( \mathbf{N} \) are drawn. Material points through-the-thickness along \( \mathbf{N} \) are determined by the coordinate \( \xi \). Equidistant points through-the-thickness (so-called sampling points) are denoted with \( \xi_\ell \), where the index \( \ell \) ranges between [1, NL]. The value of NL gives the number of layers, see figure 2.2. The sampling points are situated exactly in the middle of the associated layer \( \ell \).

The underlying assumption in the layered model is that the material properties are constant within a layer and thus the information in sampling point \( \xi_\ell \) is representative for layer \( \ell \). Therefore, an integral of a quantity through-the-thickness is evaluated by a summation of this quantity over the total number of layers, where this quantity is determined by the value at the middle of the layer. Evaluation of the stress integrals (2.5.10) according to the above concept yields

\[
\begin{align*}
\left\{ \begin{array}{l}
n^{ab} = \frac{1}{\sqrt{a}} \frac{1}{\text{NL}} \sum_{\ell=1}^{\text{NL}} t^{ab}(\xi_\ell) \sqrt{g(\xi_\ell)} \\
n^{33} = \frac{1}{\sqrt{a}} \frac{1}{\text{NL}} \sum_{\ell=1}^{\text{NL}} t^{33}(\xi_\ell) \sqrt{g(\xi_\ell)}
\end{array} \right. 
\end{align*}
\]

(2.6.1.a)
\[ m^{\alpha} = \frac{-\lambda}{\sqrt{a}} \frac{1}{NL} \sum_{\xi=1}^{NL} \tau^{\alpha}(\xi) \xi \sqrt{g(\xi)} \]  

(2.6.1.b)

Notice that the factor \(1/\text{NL}\) in (2.6.1) is valid under the assumption of equidistant sampling points. For most applications, setting the number of layers \(\text{NL}\) in (2.6.1) equal to 5 \(\ldots\) 7, sufficiently accurate values of the integrals in (2.5.10) are obtained.

Figure 2.2 - Thickness integration based on the layered approach requires a finite set of equidistant sampling points through-the-thickness.
Chapter 3 - Finite strain constitutive equations

3.1 - Introduction

A mechanics problem is governed by equilibrium equations, compatibility equations and constitutive equations. The equilibrium equations are stated in terms of the external loading on the system and the internal stresses. In the previous chapter, the equilibrium equations have been introduced by virtue of the principle of virtual work. The compatibility equations give the relations between the strains and the displacements. In the previous chapter it is shown that the strain measures are obtained from the position vector in both the reference and the deformed configuration. The constitutive equations relate the stress measures to the strain measures. In the current chapter, attention will be focused to the constitutive equations.

The discussion will be started with general constitutive equations, valid for the description of finite strains in a three-dimensional continuum. Amongst others, the textbook of BESSELING & VAN DER GIESSEN (1994) offers a comprehensive overview on this subject. Although many of the results in the present chapter are stated in this textbook, it is appropriate to have an overview of the theory first.

The finite strain 3D constitutive model employed will be presented in terms of stress rates and strain rates. The discussion of this theory is offered in section 3.2. In section 3.3, this theory will be applied to a model based on the shell kinematic assumptions as discussed in Chapter 2. Finally, the algorithm for integration of the rate-equations over a finite time-step is given in section 3.4.

3.2 - 3D finite strain constitutive model

In this section, the constitutive equations which are valid for the description of finite strains in a three-dimensional solid are worked out. In section 3.2.1, we start with the introduction of the so-called Jaumann-rate of the Kirchhoff stress. Constitutive relations are then formulated by relating this stress-rate to the strain-rate. In section 3.2.2, the elastic stress response of the material is formulated. This will be based on a decomposition of the strain-rate into an elastic and a plastic part.
A concept frequently adopted in elastic/plastic computations is to start with a prediction of the stress state based on elastic stress-strain relations. This elastic prediction is checked against a yield condition. In the implementation, the yield condition is based on the Von Mises equivalent stress. In the case plastic deformation is traced (i.e. if the equivalent elastic predictor stress is larger than the yield stress), associative plastic flow is assumed according to the normality condition as stated by PRAGER (1959). In the model which has been implemented, isotropic hardening of the material is accounted for. The plastic flow rule is discussed in section 3.2.3.

A recapitulation of the results in matrix/vector notation is given in section 3.2.4. Finally, in section 3.2.5, the expression for the continuum elastic/plastic tangent moduli is given.

### 3.2.1 - Objective rate of stress

In the review article by ATLURI (1984), the concept of objective stress-rates is discussed. The basic idea of objectivity is that "under a rigid rotation of the material, an observer co-rotating with the material experiences no change in the stress state". An objective stress-rate is the so-called *Jaumann-rate of the Kirchhoff stress*. The definition of this stress rate is recapitulated in our notation in the present section.

Time-differentiation of a tensor involves both the components of the tensor and the base vectors (if time-dependent). As stated in (2.5.8), the components of the Kirchhoff stress tensor are defined on the deformed base vectors (which are by definition time-dependent). Thus, from the last expression in (2.5.8), it follows

\[ \dot{\tau} = \tau^j \ddot{\mathbf{G}}_i \otimes \ddot{\mathbf{G}}_j + \tau^j \ddot{\mathbf{G}}_i \otimes \dot{\mathbf{G}}_j + \tau^j \dot{\mathbf{G}}_i \otimes \dot{\mathbf{G}}_j \]  \hspace{1cm} \text{(3.2.1)}

In the sequel, a superposed dot will indicate the time-derivative of the concerned quantity. The time-derivatives of the covariant deformed base vectors will be decomposed onto the covariant deformed base vectors according to

\[ \ddot{\mathbf{G}}_j = \mathbf{L} \cdot \ddot{\mathbf{G}}_j \quad \Rightarrow \quad \mathbf{L} = \ddot{\mathbf{G}}_j \otimes \ddot{\mathbf{G}}^j \]  \hspace{1cm} \text{(3.2.2)}

Essentially, above expression gives the definition of the velocity gradient tensor \( \mathbf{L} \). The components of the velocity gradient tensor are obtained from (3.2.2) by pre-multiplication of the first expression with \( \ddot{\mathbf{G}}_i \); the result is

\[ \ddot{\mathbf{G}}_i \cdot \ddot{\mathbf{G}}_j = \ddot{\mathbf{G}}_i \cdot \mathbf{L} \cdot \ddot{\mathbf{G}}_j = L_{ij}, \quad \mathbf{L} = L_{ij} \ddot{\mathbf{G}}^i \otimes \ddot{\mathbf{G}}^j \]  \hspace{1cm} \text{(3.2.3)}
Substitution of (3.2.2) into (3.2.1) yields

\[ \dot{\tau} = \dot{\tau}^e + L \cdot \tau + \tau \cdot L^T \]  \hspace{1cm} (3.2.4)

This time-derivative \( \dot{\tau} \) is the so-called material time-derivative of the Kirchhoff stress tensor \( \tau \). The time-derivative \( \dot{\tau}^e \) is the so-called convected time-derivative, which in component form reads

\[ \dot{\tau}^e = \dot{\gamma}^i \bar{G}_i \otimes \bar{G}_j \]  \hspace{1cm} (3.2.5)

The symmetric and the skew-symmetric parts of the velocity gradient tensor \( L \) are the stretching tensor \( D \) and the spin tensor \( W \), respectively. Accordingly

\[ L = D + W \Rightarrow \begin{cases} D = \frac{1}{2} \{ L + L^T \} = D_{ij} \bar{G}^i \otimes \bar{G}^j \\ W = \frac{1}{2} \{ L - L^T \} = W_{ij} \bar{G}^i \otimes \bar{G}^j \end{cases} \]  \hspace{1cm} (3.2.6)

The covariant components of the stretching tensor \( D \) with respect to the base vectors in the deformed configuration are identical to the time-derivatives of the components of the Lagrangian strain tensor as defined in (2.4.6). This becomes clear from the definition of the components \( D_{ij} \) in (3.2.6), which are worked out with the aid of (3.2.3). It gives

\[ D = \frac{1}{2} \left\{ \left( \bar{G}_i \cdot \dot{\bar{G}}_j \right) + \left( \bar{G}_j \cdot \dot{\bar{G}}_i \right) \right\} \bar{G}^i \otimes \bar{G}^j = \dot{\gamma}_{ij} \bar{G}^i \otimes \bar{G}^j \]  \hspace{1cm} (3.2.7)

In this thesis, the Jaumann rate of the Kirchhoff stress is used as an objective stress rate. This rate is defined by (see ATLURI (1984))

\[ \ddot{\tau} = \dot{\tau} - W \cdot \tau - \tau \cdot W^T \]  \hspace{1cm} (3.2.8)

A relation between the convected rate \( \dot{\tau}^e \) and the Jaumann rate \( \ddot{\tau} \) is obtained by substitution of (3.2.4) into (3.2.8). Also using the definitions in (3.2.6), we find

\[ \ddot{\tau} = \dot{\tau}^e + D \cdot \tau + \tau \cdot D^T \]  \hspace{1cm} (3.2.9)

which in component form reads

\[ \ddot{\tau} = \{ \dot{\gamma}^i + \tau^h D_{hk} G^{ki} + \tau^{ik} D_{ik} G^{hj} \} \bar{G}_i \otimes \bar{G}_j = \dot{\gamma}^i \bar{G}_i \otimes \bar{G}_j \]  \hspace{1cm} (3.2.10)

Note that the components given in (3.2.10) are symmetric with respect to the indices \((i, j)\).
3.2.2 - Decomposition of the strain-rate, elastic constitutive relations

The elastic constitutive equations will be stated in terms of the stretching tensor and the Jaumann-rate of the Kirchhoff stress tensor. In the case of elastic/plastic computations, it is assumed that the stretching tensor $\mathbf{D}$ can be decomposed into an elastic and a plastic part. Denoting the elastic part of the tensor $\mathbf{D}$ with $\mathbf{D}^{(e)}$ and the plastic part with $\mathbf{D}^{(p)}$, we set

$$\mathbf{D} = \mathbf{D}^{(e)} + \mathbf{D}^{(p)}$$

where

$$\begin{cases}
\mathbf{D}^{(e)} = \tilde{\mathbf{h}}^{(e)}_{ij} \tilde{\mathbf{G}}^i \otimes \tilde{\mathbf{G}}^j \\
\mathbf{D}^{(p)} = \tilde{\mathbf{h}}^{(p)}_{ij} \tilde{\mathbf{G}}^i \otimes \tilde{\mathbf{G}}^j
\end{cases} \quad (3.2.11)$$

Elastic constitutive relations are now introduced by

$$\tilde{\mathbf{C}} = \mathbf{C}^{(e)} : \mathbf{D}^{(e)} \quad (3.2.12)$$

where $\mathbf{C}^{(e)}$ is the fourth-order tensor containing the elastic moduli. The expression for the double-dot product of a fourth-order tensor with a second-order tensor is given in (1.3.8) and (1.3.9). The contravariant components of the tensor $\mathbf{C}^{(e)}$ are defined in the deformed state and read

$$\begin{cases}
\mathbf{C}^{(e)} = C^{ijkl}_{(e)} \tilde{\mathbf{G}}^i \otimes \tilde{\mathbf{G}}^j \otimes \tilde{\mathbf{G}}^k \otimes \tilde{\mathbf{G}}^l \\
\text{with } C^{ijkl}_{(e)} = \frac{E}{1+\nu} \left\{ \frac{1}{2} \left( G^{ik} G^{jl} + G^{il} G^{jk} \right) + \frac{\nu}{1-2\nu} G^{il} G^{kl} \right\}
\end{cases} \quad (3.2.13)$$

This is a classical definition and (for example) also stated by BESSELING & VAN DER GIESSEN (1994).

Substitution of (3.2.13) into (3.2.12) together with the definition of the Jaumann-rate of the Kirchhoff stress in (3.2.10) and the elastic part of the stretching tensor in (3.2.11) yields the component form of the elastic constitutive relations as

$$\tilde{\tau}^{ij} + \tau^{ij} D_k G^{ki} + \tau^{ik} D_k G^{kj} = C^{ijkl}_{(e)} \tilde{\mathbf{h}}^{(e)}_{kl} \quad (3.2.14)$$

3.2.3 - Yield function, plastic flow

In this section, a rate-independent, finite strain plasticity theory is considered. This theory accounts for isotropic hardening of the material. As long as cyclic plastic deformation is excluded, isotropic hardening suffices for an accurate constitutive description. The yield function, denoted with $\phi$, is assumed to be determined by a Von-Mises sphere in the
deviatoric stress-space which is allowed to expand. The radius of the Von-Mises sphere is
determined by the hardening function \( \kappa(\bar{\varepsilon}^{(p)}) \). Thus, the yield function is introduced as

\[
\phi(\tau, \bar{\varepsilon}^{(p)}) = 3J_2(\tau') - (\kappa(\bar{\varepsilon}^{(p)}))^2 = 0 \tag{3.2.15}
\]

In the yield function, \( J_2(\tau') \) is the second invariant of the deviatoric Kirchhoff stress-
tensor, given by

\[
J_2(\tau') = \frac{1}{2}(\tau' : \tau') \tag{3.2.16}
\]

The deviatoric Kirchhoff stress tensor \( \tau' \) is obtained from the tensor \( \tau \) by subtraction of the
spherical part:

\[
\tau' = \tau - p_{(e)} \mathbf{1} \quad \text{where} \quad p_{(e)} = \frac{1}{3} \tau : \mathbf{1} \tag{3.2.17}
\]

Using the components with respect to the base vectors in the deformed configuration, expression (3.2.17) is rewritten into

\[
\begin{align*}
\tau' &= \{\tau^j - p_{(e)} G^j\} \bar{G}_i \otimes \bar{G}_j = G_{ik} \{\tau^i - p_{(e)} G^i\} G_{kj} \bar{G}^i \otimes \bar{G}^j \\
\text{where} \quad p_{(e)} &= \frac{1}{3} \tau^i G_{ik} 
\end{align*} \tag{3.2.18}
\]

The hardening function \( \kappa(\bar{\varepsilon}^{(p)}) \) in (3.2.15) determines the current yield stress of the
material. At this stage, no specific choice for the function \( \kappa(\bar{\varepsilon}^{(p)}) \) is made (examples of the
function \( \kappa(\bar{\varepsilon}^{(p)}) \) are offered in Chapter 7). Variable \( \bar{\varepsilon}^{(p)} \) in the hardening function is the
equivalent plastic strain. Based on the plastic part of the stretching tensor, \( \mathbf{D}^{(p)} \), the so-called \textit{equivalent plastic strain-rate} is defined by

\[
\dot{\bar{\varepsilon}}^{(p)} = \sqrt{\frac{2}{3}} (\mathbf{D}^{(p)} : \mathbf{D}^{(p)}) \tag{3.2.19}
\]

Plastic deformation is characterised by a flow rule which is stated in terms of the
plastic part of the stretching tensor, \( \mathbf{D}^{(p)} \). Associative plastic flow according to the so-called
normality condition of Drucker-Prager, see PRAGER (1959), is assumed; in our notation

\[
\mathbf{D}^{(p)} = \dot{\gamma} \left| \frac{\partial \phi}{\partial \tau'} \right| \tag{3.2.20}
\]

where, from the definition of \( \phi \) in (3.2.15) together with (3.2.16), it follows that
\[
\frac{\partial \phi}{\partial \tau'} = 3 \tau' \quad \text{(3.2.21.a)}
\]

The denominator in (3.2.20) is obtained from (3.2.21.a) and (3.2.16):
\[
\left\| \frac{\partial \phi}{\partial \tau'} \right\| = 3 \sqrt{\tau' \cdot \tau'} = 3 \sqrt{2 J_2(\tau')}
\quad \text{(3.2.21.b)}
\]

The scalar \( \gamma \) in (3.2.20) is called the plastic consistency parameter. The relation of the scalar \( \dot{\gamma} \) to the material parameter \( \ddot{e}^{(p)} \) is determined by substitution of (3.2.20) and (3.2.21) into (3.2.19); which gives
\[
\ddot{e}^{(p)} = \sqrt{\frac{2}{3}} \dot{\gamma}
\quad \text{(3.2.22)}
\]

Only in the case that the deformation is fully elastic, the time-rate \( \dot{\gamma} \) vanishes. Whether or not the time-rate \( \dot{\gamma} \) is vanishing is determined upon the standard loading/unloading conditions. These conditions, formulated in Kuhn-Tucker form, read
\[
\begin{cases}
\phi \leq 0 \\
\dot{\gamma} \geq 0 \\
\phi \dot{\gamma} = 0
\end{cases}
\quad \text{(3.2.23)}
\]

The first condition states that the stress situation is confined to the elastic domain (\( \phi < 0 \)) or to the yield function (\( \phi = 0 \)). In the case of plastic flow (\( \dot{\gamma} > 0 \)), the third condition yields \( \phi = 0 \). In the case of elastic deformation (\( \phi < 0 \)), the third condition yields \( \dot{\gamma} = 0 \). Along a process of continuous plastic deformation, the so-called plastic consistency condition,
\[
\phi = 0 \quad \text{and} \quad \dot{\phi} = 0
\quad \text{(3.2.24)}
\]
holds, which has the effect of confining the stress situation to the yield function.

For later usage, some of the above results will be presented in component expressions. In the case of (3.2.20), a component expression is obtained by virtue of (3.2.11), (3.2.18) and (3.2.21). We find
\[
\ddot{\tau}_{ij}^{(p)} = \dot{\gamma} \frac{\tau_{ii}'}{\sqrt{2 J_2(\tau')}}
\quad \text{(3.2.25)}
\]
A component expression of $\dot{\phi} = 0$ is obtained after realising the following:

$$\dot{J}_2 = \tau' : \dot{\tau}' = \tau' : \dot{\tau} = \tau_{ij} \dot{\gamma}^{ij}$$

which is stated by e.g. NEALE (1981). From this expression, $\dot{\phi} = 0$ is written as

$$\dot{\phi} = 3 \dot{J}_2 + \frac{\partial \phi}{\partial \dot{e}_p} \dot{\gamma}^p = 3 \tau_{ij} \dot{\gamma}^{ij} - 2 \kappa \frac{\partial \chi}{\partial \dot{e}_p} \dot{\gamma}^p$$

### 3.2.4 - Matrix/vector notation

For a computational implementation of the constitutive model, it has proven to be convenient to use matrix-vector notation. Using this notation, the equations governing the constitutive behaviour (from the previous sections) are rephrased.

To this end, "column vectors" of the stress and strain components are introduced according to

$$\tilde{\sigma} = \begin{bmatrix} \tau^{11} & \tau^{12} & \tau^{13} & \tau^{22} & \tau^{23} & \tau^{33} \end{bmatrix}^T$$

$$\tilde{\varepsilon} = \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & 2\eta_{22} & 2\eta_{23} & 2\eta_{33} \end{bmatrix}^T$$

Using this notation, we rewrite

$$(3.2.7) \ & (3.2.11) \rightarrow \dot{\tilde{\varepsilon}} = \dot{\tilde{\varepsilon}}^{(e)} + \dot{\tilde{\varepsilon}}^{(p)}$$

$$(3.2.12) \ & (3.2.13) \rightarrow \dot{\tilde{\sigma}} = \tilde{M}^{(e)} \cdot \dot{\tilde{\varepsilon}}^{(e)}$$

$$(3.2.10) \rightarrow \dot{\tilde{\sigma}} = \dot{\tilde{\sigma}} + \tilde{J} \cdot \dot{\tilde{\varepsilon}}$$

$$(3.2.25) \rightarrow \dot{\tilde{\varepsilon}}^{(p)} = \dot{\gamma} \tilde{b}$$

$$(3.2.22) \ & (3.2.27) \rightarrow \dot{\phi} = \tilde{\alpha}^T \cdot \dot{\tilde{\sigma}} + c \dot{\gamma}$$

- The general definition of the matrix $\tilde{M}^{(e)}$ is:
\[
M^{(e)} = \begin{bmatrix}
C_{1111}^{(e)} & C_{1122}^{(e)} & C_{1133}^{(e)} & C_{1112}^{(e)} & C_{1113}^{(e)} & C_{1123}^{(e)} \\
C_{2222}^{(e)} & C_{2233}^{(e)} & C_{2112}^{(e)} & C_{2123}^{(e)} & C_{2213}^{(e)} & C_{2223}^{(e)} \\
C_{3333}^{(e)} & C_{3312}^{(e)} & C_{3212}^{(e)} & C_{3223}^{(e)} & C_{3123}^{(e)} & C_{3223}^{(e)} \\
\text{symm} & C_{1212}^{(e)} & C_{1223}^{(e)} & C_{1123}^{(e)} & C_{1132}^{(e)} & C_{1232}^{(e)} \\
\text{symm} & \text{symm} & C_{1212}^{(e)} & C_{1223}^{(e)} & C_{1123}^{(e)} & C_{1232}^{(e)} \\
\text{symm} & \text{symm} & \text{symm} & C_{1212}^{(e)} & C_{1223}^{(e)} & C_{1123}^{(e)} \\
\end{bmatrix}
\]
\hspace{1cm} (3.2.30)

- The matrix \( J \) is due to the introduction of the Jaumann rate of the stress and in general reads

\[
J = \begin{bmatrix}
J_{1111} & J_{1122} & J_{1133} & J_{1112} & J_{1113} & J_{1123} \\
J_{2222} & J_{2233} & J_{2112} & J_{2213} & J_{2223} & J_{2223} \\
J_{3333} & J_{3312} & J_{3212} & J_{3223} & J_{3313} & J_{3323} \\
\text{symm} & J_{1212} & J_{1223} & J_{1223} & J_{1223} & J_{1223} \\
\text{symm} & \text{symm} & J_{1212} & J_{1223} & J_{1223} & J_{1223} \\
\text{symm} & \text{symm} & \text{symm} & J_{1212} & J_{1223} & J_{1223} \\
\end{bmatrix}
\]
\hspace{1cm} (3.2.31)

where

\[
J_{ijkl} = \frac{1}{2} ( \tau_{ij} G_{kl} + \tau_{ij} G_{kj} + \tau_{ik} G_{jl} + \tau_{ik} G_{jl} )
\]
\hspace{1cm} (3.2.32)

- The "column vectors" \( \tilde{b} \) and \( \tilde{a} \) and the scalar \( c \) read

\[
\tilde{b} = \frac{1}{\sqrt{2J_2'(\tau')}} \begin{bmatrix} \tau_{11}' & \tau_{12}' & \tau_{13}' & 2\tau_{12}' & 2\tau_{13}' & 2\tau_{23}' \end{bmatrix}^T
\]

\[
\tilde{a} = 3 \begin{bmatrix} \tau_{11}' & \tau_{12}' & \tau_{13}' & 2\tau_{12}' & 2\tau_{13}' & 2\tau_{23}' \end{bmatrix}^T
\]
\hspace{1cm} (3.2.33)

\[
c = -2 \sqrt{\frac{2}{3}} \kappa \frac{\partial \kappa}{\partial E^{(p)}}
\]

A simple relation exists between the "column vectors" \( \tilde{a} \) and \( \tilde{b} \), which basically follows from the Von Mises yield criterion (3.2.15) and the associative flow rule (3.2.20), respectively. From the definitions of \( \tilde{b} \) and \( \tilde{a} \) in (3.2.33), it is clear that

\[
\tilde{a} = 3 \sqrt{2J_2'(\tau')} \tilde{b}
\]
\hspace{1cm} (3.2.34)
3.2.5 - 3D continuum elastic/plastic tangent moduli

The aim of the present section is to formulate direct relations between $\dot{\sigma}$ and $\dot{\varepsilon}$. This is achieved as follows. Substitute the first and the fourth equation in (3.2.29) into the second and find

$$\begin{align*}
\ddot{\sigma} &= M^{(e)} \cdot (\ddot{\varepsilon} - \dot{\gamma} \tilde{b}) \quad ,(3.2.35)
\end{align*}$$

which substituted in the fifth equation in (3.2.29) yields

$$\begin{align*}
\ddot{\phi} &= \tilde{a}^T \cdot M^{(e)} \cdot (\ddot{\varepsilon} - \dot{\gamma} \tilde{b}) + c \dot{\gamma} = 0 \\
\iff \dot{\gamma} &= \frac{\tilde{a}^T \cdot M^{(e)}}{\tilde{a}^T \cdot M^{(e)} \cdot \tilde{b} - c} \cdot \dot{\varepsilon} \quad ,(3.2.36)
\end{align*}$$

Then, (3.2.36) substituted into (3.2.35) leads to

$$\begin{align*}
\ddot{\sigma} &= \left\{ M^{(e)} - \frac{M^{(e)} \cdot \tilde{b} \cdot \tilde{a}^T \cdot M^{(e)}}{\tilde{a}^T \cdot M^{(e)} \cdot \tilde{b} - c} \right\} \cdot \dot{\varepsilon} \quad ,(3.2.37)
\end{align*}$$

Finally, introduction of the third equation of (3.2.29) into (3.2.37) yields

$$\begin{align*}
\dot{\sigma} &= M^{(e/p)} \cdot \dot{\varepsilon} \\
\text{where} \quad M^{(e/p)} &= M^{(e)} - \frac{M^{(e)} \cdot \tilde{b} \cdot \tilde{a}^T \cdot M^{(e)}}{\tilde{a}^T \cdot M^{(e)} \cdot \tilde{b} - c} - J \quad ,(3.2.38)
\end{align*}$$

The moduli contained in the matrix $M^{(e/p)}$ are the so-called "3D continuum elastic/plastic tangent moduli". The matrices $M^{(e)}$, $J$ and $\tilde{b} \cdot \tilde{a}^T$, because of (3.2.34), are symmetric. Consequently, matrix $M^{(e/p)}$ is symmetric.

3.3 - Application to strain measures from the shell formulation

In section 3.2, a constitutive model valid for a three-dimensional solid at finite strain is discussed. Finally, in (3.2.38), this model is phrased in terms of the rates $\dot{\sigma}$ and $\dot{\varepsilon}$. In the present section, this model will be applied to the shell formulation as described in Chapter 2. By doing so, a model will become available that is stated in terms of the strain rates $\{ \dot{\varepsilon}_{rs}, \dot{\chi}_3, \dot{\rho}_{rs} \}$ on the one hand and the stress rates $\{ \dot{n}^{ab}, \dot{m}^{ab}, \dot{n}^{ab} \}$ on the other hand.
In this section, use will be made of matrix/vector notation again. Therefore, the following "stress vectors" and "strain vectors" are introduced

\[
\begin{align*}
\mathbf{\tilde{S}}^{(0)} &= \begin{bmatrix} n^{11} & n^{21} & n^{31} \\ n^{21} & n^{22} & n^{23} \\ n^{31} & n^{32} & n^{33} \end{bmatrix} \\
\mathbf{\tilde{S}}^{(0)} &= \begin{bmatrix} m^{11} & m^{21} & m^{31} \\ m^{21} & m^{22} & m^{23} \\ m^{31} & m^{32} & m^{33} \end{bmatrix} \\
\mathbf{\tilde{E}}^{(0)} &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \\
\mathbf{\tilde{E}}^{(0)} &= \begin{bmatrix} \rho_{11} & \rho_{22} & \rho_{33} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{bmatrix}
\end{align*}
\]

(3.3.1.a) (3.3.1.b)

Thus, the objective of the present section is to formulate constitutive relations in these measures. Formally, the relations will be written as

\[
\begin{align*}
\mathbf{\tilde{S}}^{(0)} &= \mathbf{H}^{(0/0)} : \mathbf{\tilde{E}}^{(0)} + \mathbf{H}^{(0/1)} : \mathbf{\tilde{E}}^{(1)} \\
\mathbf{\tilde{E}}^{(0)} &= \mathbf{H}^{(1/0)} : \mathbf{\tilde{E}}^{(0)} + \mathbf{H}^{(1/1)} : \mathbf{\tilde{E}}^{(1)}
\end{align*}
\]

(3.3.2)

The moduli in the matrices \( \mathbf{H}^{(0/0)} \), etc. will be derived from the 3D model presented in the previous section.

Using a 3D constitutive model in conjunction with shell kinematic assumptions is rather complicated. In order to demonstrate the complication, a step back into the 3D continuum description is taken.

As a starting point, the general 3D internal virtual work expression (2.5.2), written in matrix/vector notation

\[
\delta W_{in} = \int \tilde{\mathbf{E}}^T \cdot \delta \tilde{\mathbf{E}} \, d(\text{vol})
\]

(3.3.3)

is considered. Introducing shell kinematics, i.e. essentially approximating the strain field by a Taylor series expansion up to first order in the through-the-thickness coordinate \( \xi \) as

\[
\tilde{\mathbf{E}} = \tilde{\mathbf{E}}^{(0)*} - \xi \lambda \tilde{\mathbf{E}}^{(1)*}
\]

(3.3.4)

which leads to

\[
\delta W_{in} = \int \left\{ \tilde{\mathbf{E}}^{(0)*T} \cdot \delta \tilde{\mathbf{E}}^{(0)*} + \tilde{\mathbf{E}}^{(1)*T} \cdot \delta \tilde{\mathbf{E}}^{(1)*} \right\} \, d(\text{mid})
\]

(3.3.5)

where

\[
\begin{align*}
\tilde{\mathbf{E}}^{(0)*} &= \begin{bmatrix} n^{11} & n^{21} & n^{31} \\ n^{21} & n^{22} & n^{23} \\ n^{31} & n^{32} & n^{33} \end{bmatrix} \\
\tilde{\mathbf{E}}^{(1)*} &= \begin{bmatrix} m^{11} & m^{21} & m^{31} \\ m^{21} & m^{22} & m^{23} \\ m^{31} & m^{32} & m^{33} \end{bmatrix}
\end{align*}
\]

(3.3.6.a)
\[
\begin{align*}
\begin{bmatrix}
\eta^{(0)} & = & \begin{bmatrix} e_{11} & e_{22} & \chi_3 & 2 \epsilon_{12} & \eta^{(0)*}_{13} & \eta^{(0)*}_{13} \\
\eta^{(0)*} & = & \begin{bmatrix} \rho^*_{11} & \rho^*_{22} & \eta^{(0)*}_{13} & 2 \rho^*_{12} & \eta^{(0)*}_{33} & \eta^{(0)*}_{33} 
\end{bmatrix}^T
\end{align*}
\tag{3.3.6b}
\]

The above "vectors" have been augmented with a #-sign in order to make a distinction with corresponding vectors in (3.3.1). The stress resultants and the stress couples in (3.3.6a) are

\[
\begin{align*}
\begin{bmatrix}
n^i & = & \frac{1}{\sqrt{a}} \int \tau^i(\xi) \sqrt{g(\xi)} \, d\xi \\
m^i & = & -\frac{\lambda}{\sqrt{a}} \int \xi \tau^i(\xi) \sqrt{g(\xi)} \, d\xi
\end{bmatrix}
\tag{3.3.7}
\end{align*}
\]

The shell model developed in Chapter 2 does not include any through-the-thickness shear deformation \( \eta_{33} \) and would therefore, for orthotropic material assumptions, not generate a through-the-thickness shear stress \( \tau^{33} \). Thus, \( \eta^{(0)*}_{13} = \eta^{(0)*}_{13} = \eta^{(0)*}_{33} = \eta^{(0)*}_{33} = 0 \) leads to \( n^{33} = m^{33} = 0 \). The internal virtual work expression (3.3.5) then reads

\[
\delta W_{\text{int}} = \int \left[ n^{ab} \delta e_{ab} + m^{ab} \delta \rho^*_{ab} + n^{33} \delta \chi_3 + m^{33} \delta \eta^{(0)*}_{33} \right] d(\text{mid})
\tag{3.3.8}
\]

The first three terms in this expression do also feature in the internal virtual work expression embedding the shell kinematic assumptions (2.5.11). The difference between the expressions (3.3.8) and (2.5.11) is constituted by the term \( m^{33} \delta \eta^{(0)*}_{33} \). The complication of using a 3D constitutive model in conjunction with the shell kinematic assumptions from Chapter 2 is due to this term. Therefore, in the sequel, attention is focused to this term.

Direct application of the 3D constitutive model would in general generate a "stress resultant" \( m^{33} \). The internal virtual work of the shell model (2.5.11) is essentially equivalent to assuming \( \eta^{(0)*}_{13} = 0 \) in (3.3.8) under non-vanishing \( m^{33} \). The restraining of the thickness stretch \( \eta_{33} \) to be constant through-the-thickness has been found to lead to an overly stiff behaviour in bending problems. This has also been observed by BUCHTER, RAMM & ROEHL (1994). Therefore, the restraint in the thickness stretch will be relaxed by augmenting it with the linear-in-\( \xi \) term, thus \( \eta_{33} = \chi_3 - \xi \lambda \eta^{(0)*}_{13} \). The method used has its bearings on the procedure given by DE BORST (1991), where a method is described which enforces, in a computational environment, the plane stress condition while still using a general 3D stress update procedure.

### 3.3.1 - Relaxation of the thickness stretch

As mentioned above, an additional term is introduced in the thickness stretch, thus \( \eta_{33} = \chi_3 - \xi \lambda \eta^{(0)*}_{13} \). It would have been an option to handle the term \( \eta^{(0)*}_{13} \) as an independent
variable in the formulation. This is done in BÜCHTER, RAMM & ROEHL (1994). However, in the present formulation, \( \eta_1^{(0)} \) will be introduced as a dependent variable. An option which satisfactorily resolves the problem of an overly stiff behaviour in bending is to choose \( \eta_3^{(0)} \) such that

\[
\mathbf{m}^{33} = 0 \quad \text{at any stage of deformation} \tag{3.3.9}
\]

Upon this requirement, the term \( \eta_1^{(0)} \) will be expressed in terms of the shell strain measures and thus eliminated from the formulation.

The procedure will be worked out in the context of rate equations. From the requirement (3.3.9), it follows by time-differentiation that

\[
\dot{\mathbf{m}}^{33} = 0 \tag{3.3.10}
\]

Using the strain vectors defined in (3.3.4) and (3.3.6.b) and the general 3D constitutive relations from (3.2.38), it follows that

\[
\dot{\sigma}_K = \mathbf{M}_{33}^{(e/p)} \left\{ \eta_1^{(0)} - \xi \lambda \hat{\eta}_1^{(0)} \right\} = \mathbf{M}_{33}^{(e/p)} \left\{ \eta_1^{(0)} - \xi \lambda \hat{\eta}_1^{(0)} \right\} - \mathbf{M}_{33}^{(e/p)} \left\{ \xi \lambda \hat{\eta}_1^{(0)} \right\} \tag{3.3.11}
\]

where use is made of the fact that \( \eta_1^{(0)} = \eta_1^{(0)} = \eta_1^{(0)} = \eta_1^{(0)} = 0 \). Application of \( K = 3 \) in (3.3.11) yields, according to the numbering convention in (3.2.28), an expression for \( \dot{t}^{33} \). Using this expression together with (3.3.7), (3.3.10) yields

\[
\dot{\mathbf{m}}^{33} = -\frac{\lambda}{\sqrt{a}} \int \frac{1}{\xi} \left\{ M_{33}^{(e/p)} \left( \eta_1^{(0)} - \xi \lambda \hat{\eta}_1^{(0)} \right) - \xi \lambda M_{33}^{(e/p)} \hat{\eta}_3^{(0)} \right\} \sqrt{g(\xi)} d\xi = 0 \tag{3.3.12}
\]

In the various terms in this integral, factors \( \hat{\eta}_1^{(0)} \), \( \hat{\eta}_1^{(0)} \) and \( \hat{\eta}_3^{(0)} \) are independent of \( \xi \). Taking these factors outside the various integrals yields

\[
\dot{\mathbf{m}}^{33} = Q_{31}^{(0)} \hat{\eta}_1^{(0)} + Q_{31}^{(0)} \hat{\eta}_1^{(0)} + Q_{33}^{(0)} \hat{\eta}_3^{(0)} = 0 \tag{3.3.13.a}
\]

where

\[
\begin{align*}
Q_{31}^{(0)} &= -\frac{\lambda}{\sqrt{a}} \int \frac{1}{\xi} M_{33}^{(e/p)}(\xi) \sqrt{g(\xi)} d\xi \\
Q_{31}^{(0)} &= \frac{\lambda}{\sqrt{a}} \int \frac{1}{\xi^2} M_{33}^{(e/p)}(\xi) \sqrt{g(\xi)} d\xi \\
Q_{33}^{(0)} &= \frac{\lambda^2}{\sqrt{a}} \int (\xi)^2 M_{33}^{(e/p)}(\xi) \sqrt{g(\xi)} d\xi
\end{align*}
\tag{3.3.13.b}
\]

Note that (3.3.13.a) is a linear relation between the rate \( \hat{\eta}_3^{(0)} \) and the strain rates \( (\dot{\epsilon}_{ab}, \dot{\chi}_3, \rho_{ab}^\circ) \). Expressing \( \hat{\eta}_3^{(0)} \) in terms of the other strain rates yields
\[
\hat{\gamma}_{13}^{(i)} = -\frac{Q_{13}^{(i)}}{Q_{23}^{(i)}} \hat{\eta}_{1}^{(i)} + \frac{Q_{23}^{(i)}}{Q_{33}^{(i)}} \hat{\eta}_{2}^{(i)} = A_{3L} \hat{\eta}_{1}^{(i)} + B_{3L} \hat{\eta}_{2}^{(i)} \tag{3.3.14}
\]

The term \(\hat{\gamma}_{13}^{(i)}\) is eliminated from the constitutive relations by substitution of relation (3.3.14) into (3.3.11):

\[
\dot{\sigma}_{K} = M_{KL}^{(e/p)} \left( \hat{\eta}_{1}^{(i)} - \xi \lambda \hat{\eta}_{2}^{(i)} \right) - M_{KL}^{(e/p)} \xi \lambda \left( A_{3L} \hat{\eta}_{1}^{(i)} + B_{3L} \hat{\eta}_{2}^{(i)} \right) = \left( M_{KL}^{(e/p)} - M_{KL}^{(e/p)} \xi \lambda A_{3L} \right) \hat{\eta}_{1}^{(i)} - \xi \lambda \left( M_{KL}^{(e/p)} + M_{KL}^{(e/p)} B_{3L} \right) \hat{\eta}_{2}^{(i)} \tag{3.3.15}
\]

which yields the so-called "3D corrected moduli". In the next section the thickness-integration of (3.3.15) by virtue of the layered approach is worked out.

### 3.3.2 - Layered approach

In the previous section, stress vectors \(\mathbf{\hat{S}}^{(o)}\) and \(\mathbf{\hat{S}}^{(i)}\) have been introduced. The components in these vectors are determined by (3.3.7). An expression of (3.3.7) in "column vector" notation may be obtained by using (3.2.28). In rate form, the result is

\[
\begin{align*}
\dot{\mathbf{\hat{S}}}^{(o)}_{K} & = \frac{1}{\sqrt{a}} \int \dot{\sigma}_{K}(\xi) \sqrt{g(\xi)} \, d\xi \\
\dot{\mathbf{\hat{S}}}^{(i)}_{K} & = -\frac{\lambda}{\sqrt{a}} \int \xi \dot{\sigma}_{K}(\xi) \sqrt{g(\xi)} \, d\xi \tag{3.3.16.a}
\end{align*}
\]

The integrals through-the-thickness are approximated by means of the layered model, see section 2.6. Therefore,

\[
\begin{align*}
\dot{\mathbf{\hat{S}}}^{(o)}_{K} & = \frac{1}{\sqrt{a}} \frac{1}{NL} \sum_{\ell=1}^{NL} \dot{\sigma}_{K}(\xi_{\ell}) \sqrt{g(\xi_{\ell})} \\
\dot{\mathbf{\hat{S}}}^{(i)}_{K} & = -\frac{\lambda}{\sqrt{a}} \frac{1}{NL} \sum_{\ell=1}^{L} \xi_{\ell} \dot{\sigma}_{K}(\xi_{\ell}) \sqrt{g(\xi_{\ell})} \tag{3.3.16.b}
\end{align*}
\]

Substitution of (3.3.15) in (3.3.16.b) yields

\[
\begin{align*}
\dot{\mathbf{\hat{S}}}^{(o)}_{K} & = H_{KL}^{(o/o)} \mathbf{\hat{S}}_{KL}^{(o)} + H_{KL}^{(o/i)} \mathbf{\hat{S}}_{KL}^{(i)} \\
\dot{\mathbf{\hat{S}}}^{(i)}_{K} & = H_{KL}^{(i/o)} \mathbf{\hat{S}}_{KL}^{(o)} + H_{KL}^{(i/i)} \mathbf{\hat{S}}_{KL}^{(i)} \tag{3.3.17}
\end{align*}
\]

where

\[
\begin{align*}
H_{KL}^{(o/o)} & = Q_{KL}^{(0)} + Q_{KL}^{(1)} A_{3L} \\
H_{KL}^{(o/i)} & = Q_{KL}^{(0)} + Q_{KL}^{(1)} A_{3L} \\
H_{KL}^{(i/o)} & = Q_{KL}^{(0)} + Q_{KL}^{(1)} A_{3L} \\
H_{KL}^{(i/i)} & = Q_{KL}^{(0)} + Q_{KL}^{(1)} A_{3L} \tag{3.3.18}
\end{align*}
\]
The 0th, 1st and 2nd moments of the constitutive moduli $\mathbf{M}^{(e/p)}_{KL}$ are obtained according to

\[
\begin{align*}
Q^{(0)}_{KL} &= \frac{1}{a} \sqrt{\frac{1}{NL}} \sum_{\xi} M^{e/p}_{KL}(\xi) \sqrt{g(\xi)} \\
Q^{(1)}_{KL} &= -\frac{\lambda}{a} \sqrt{\frac{1}{NL}} \sum_{\xi} \xi M^{e/p}_{KL}(\xi) \sqrt{g(\xi)} \\
Q^{(2)}_{KL} &= \frac{\lambda^2}{a} \sqrt{\frac{1}{NL}} \sum_{\xi} (\xi)^2 M^{e/p}_{KL}(\xi) \sqrt{g(\xi)} 
\end{align*}
\]

(3.3.19)

The symmetry of the constitutive relations (3.3.17) becomes clear by considering the four sub-matrices in detail. Using (3.3.14) in (3.3.18) reveals that

\[
\begin{align*}
H^{(0/0)}_{KL} &= Q^{(0)}_{KL} - Q^{(1)}_{KL} Q^{(3)}_{KL} Q^{(1)}_{KL} = Q^{(0)}_{KL} - Q^{(3)}_{KL} Q^{(3)}_{KL} = H^{(0/0)}_{KL} \\
H^{(1/1)}_{KL} &= Q^{(1)}_{KL} - Q^{(3)}_{KL} Q^{(1)}_{KL} Q^{(1)}_{KL} = Q^{(1)}_{KL} - Q^{(3)}_{KL} Q^{(3)}_{KL} = H^{(1/1)}_{KL} \\
H^{(2/2)}_{KL} &= Q^{(2)}_{KL} - Q^{(3)}_{KL} Q^{(2)}_{KL} Q^{(3)}_{KL} = Q^{(2)}_{KL} - Q^{(3)}_{KL} Q^{(3)}_{KL} = H^{(2/2)}_{KL}
\end{align*}
\]

(3.3.20)

which follow from the symmetries of the matrices $Q^{(0)}_{KL}$, $Q^{(1)}_{KL}$ and $Q^{(2)}_{KL}$ in itself.

A demonstration of the above constitutive equations for a linear elastic case is given in Appendix A. It is shown that, due to the elimination of the term $\tilde{\eta}^{(1)*}$, the bending stiffness is reduced, whereas the membrane stiffness is not changed. The moduli in the bending stiffness are identical to those used in the "classical theory of thin shells", see e.g. KOITER (1966). The moduli in the membrane stiffness are general moduli. Finally it is shown that if no lateral loading is applied (in which case $\sigma_3 = \tilde{\sigma}_a = 0$), then the membrane moduli change into the same moduli used in the "classical theory".

### 3.4 - Integration of the rate-equations

Due to non-linearities, an incremental method will be applied for the simulation of the deformation behaviour. Within the increments, a Newton-Raphson iteration scheme is employed (this procedure is discussed in Chapter 6). At each Newton-Raphson iteration, an update of a (finite) increment of strain is established. From this (finite) increment of strain, a (finite) increment of stress together with the stiffness moduli for that updated stress/strain state should be calculated. This is done with a finite difference scheme. The employed scheme is based on the rate-equations from the previous sections.
Focusing on the adopted shell formulation, the aim of the present section is to establish an algorithm to calculate the stress increments \((\Delta \tilde{\sigma}^{(o)}, \Delta \tilde{\sigma}^{(p)})\) from the strain increments \((\Delta \tilde{\varepsilon}^{(o)}, \Delta \tilde{\varepsilon}^{(p)})\). An option to achieve this is to apply a finite difference scheme directly to (3.3.17). This would result in an algorithm which is specific for the shell formulation employed. However, we will describe a method which uses a general 3D constitutive algorithm. This algorithm will be made compatible with the adopted shell formulation by enforcing a condition similar to (3.3.10) separately. The present approach has the advantage that any 3D constitutive model may be applied.

Many schemes for 3D stress-update procedures have been proposed in the literature. For a review, see the article of ORTIZ & POPOV (1985). The 3D stress-update procedure employed here is a so-called "implicit, backward Euler difference scheme" and has been implemented by LEROY (1991). The general features of this procedure will be presented in section 3.4.1. The application of this model to the shell formulation is discussed in section 3.4.2.

### 3.4.1 - Stress-update for the 3D model

Applied to a certain finite time step, the implicit, backward Euler difference scheme works as follows. Suppose that the stress and strain situations are known at the time \((t)\). Further suppose the increment \(\Delta \tilde{\varepsilon}\), as the total strain increment from the time \(t\) to \((t + \Delta t)\), to be given. Application of the difference scheme to the set of rate-equations (3.2.29.a)-(3.2.29.d) results in

\[
\begin{align*}
(a) & \quad \Delta \tilde{\varepsilon} = \Delta \tilde{\varepsilon}^{(o)} + \Delta \tilde{\varepsilon}^{(p)} \\
(b) \ & \& (c) & \quad \Delta \tilde{\sigma} + J_{t+\Delta t} \cdot \Delta \tilde{\varepsilon} = M_{t+\Delta t}^{(e)} \cdot \Delta \tilde{\varepsilon}^{(e)} \\
(d) & \quad \Delta \tilde{\varepsilon}^{(p)} = \Delta \gamma \tilde{b}_{t+\Delta t} \\
\phi_{t+\Delta t} & = \phi\left(\tilde{\sigma}_{t+\Delta t}, \tilde{\varepsilon}^{(p)}_{t+\Delta t}\right) = 0 \\
\text{with} & \quad \tilde{\varepsilon}^{(p)}_{t+\Delta t} = \tilde{\varepsilon}^{(p)}_{t} + \sqrt{2/3} \Delta \gamma
\end{align*}
\]

(3.4.1.a)

(3.4.1.b)

The latter expressions follow from (3.2.15) and (3.2.22). The subscripts as used in (3.4.1) indicate whether the concerned quantities are taken at the start of the increment \((t)\) or at the end of the increment \((t + \Delta t)\).

Note that the geometric information at the end of the increment is known since the strain increment \(\Delta \tilde{\varepsilon}\) is given. Therefore, the matrix \(M^{(e)}\) at \((t + \Delta t)\) is determined. The stress information at the end of the increment is not known beforehand. However, for the
evaluation of the $\tilde{B}$-vector at $(t + \Delta t)$, the geometric information and the stress information at the end of the increment are required. This is essentially why the difference scheme is called implicit. Finally, it is noted that for the evaluation of the matrix $J_{t/t+\Delta t}$, the stress information at $(t)$ and the geometric information at $(t + \Delta t)$ are used.

Solving (3.4.1) in terms of $\tilde{\sigma}_{t+\Delta t}$ requires special care. This is due to the fact that it is not known in advance whether the deformation is elastic and/or plastic. In the case of elastic deformation, $\Delta \gamma$ will be zero. In the case of plastic deformation, the value $\Delta \gamma$ needs to be determined.

As advocated by ORTIZ & SIMO (1986), an effective procedure for solving (3.4.1) is to compute an "elastic trial state of stress" first. If the trial stress state lies outside the yield surface, then as a second step, one defines the state of stress at the time $(t + \Delta t)$ by means of the shortest projection of the trial state onto the yield surface. For the Von Mises yield criterion this is effectively the "radial return method".

The elastic (trial) state of stress is obtained by 'freezing' the plastic flow during the time-interval $[t, t + \Delta t]$. Accordingly, we set $\Delta \gamma = 0$ in (3.4.1) and obtain

\[
\begin{align*}
\Delta \tilde{e}_{\text{trial}}^{(p)} &= \tilde{\sigma}_6 \\
\Delta \tilde{\varepsilon}_{\text{trial}}^{(e)} &= \Delta \tilde{\varepsilon} \\
\tilde{\sigma}_{\text{trial}} &= \tilde{\sigma}_t + \left( M_{t+\Delta t}^{(e)} - J_{t/t+\Delta t} \right) \cdot \Delta \tilde{\varepsilon}
\end{align*}
\] (3.4.2.a)

\[
\phi_{\text{trial}} = \phi\left( \tilde{\sigma}_{\text{trial}}, \tilde{\varepsilon}_{\text{trial}}^{(p)} \right) \quad \text{with} \quad \tilde{\varepsilon}_{\text{trial}}^{(p)} = \tilde{\varepsilon}_t^{(p)}
\] (3.4.2.b)

The decision whether these (trial)-stresses are the correct $(t + \Delta t)$-stresses depends on the check against the yield condition. A return mapping to the yield surface is required in the case $\phi_{\text{trial}} > 0$ (i.e. plastic deformation is traced). In the case $\phi_{\text{trial}} \leq 0$ (i.e. purely elastic deformation), then the (trial)-stresses are indeed the correct $(t + \Delta t)$-stresses.

In the case that plastic deformation is traced, a return to the yield surface (i.e. the shortest projection) is required. For this return mapping, the relation between the stress and strain quantities in the (trial) and the $(t + \Delta t)$ state should be specified. By comparison of (3.4.1.a) with (3.4.2.a), it follows that

\[
\begin{align*}
\Delta \tilde{e}_{t+\Delta t}^{(p)} &= \Delta \tilde{\varepsilon}_{\text{trial}}^{(p)} + \Delta \gamma \tilde{B}_{t+\Delta t} \\
\Delta \tilde{\varepsilon}_{t+\Delta t}^{(e)} &= \Delta \tilde{\varepsilon}_{\text{trial}}^{(e)} - \Delta \tilde{\varepsilon}_{t+\Delta t}^{(p)} \\
\tilde{\sigma}_{t+\Delta t} &= \tilde{\sigma}_{\text{trial}} - M_{t+\Delta t}^{(e)} \cdot \Delta \tilde{\varepsilon}_{t+\Delta t}^{(p)}
\end{align*}
\] (3.4.3.a)
\[
\begin{align*}
\{ \phi_{t+\Delta t} &= \phi\left( \tilde{\sigma}_{t+\Delta t}, \tilde{\varepsilon}_{t+\Delta t}^{(p)} \right) = 0 \\
\text{with } \tilde{\varepsilon}_{t+\Delta t}^{(p)} &= \tilde{\varepsilon}_{\text{total}}^{(p)} + \sqrt{2/3} \Delta \gamma 
\end{align*}
.(3.4.3.b)
\]

where the latter expression follows from (3.4.1.b) and (3.4.2.b). Substitution of (3.4.3.a) into (3.4.3.b) essentially leads to a non-linear relation in \( \Delta \gamma \). This expression is solved by "local" (which means at the level of the integration points) Newton-Raphson iteration. The obtained solution for \( \Delta \gamma \) is used in (3.4.3.a) and determines the \( (t+\Delta t) \) state.

In the early years of computational mechanics, it was observed that a full Newton-Raphson iteration scheme applied to plastic deformation problems did not always result in a quadratic rate of asymptotic convergence. The reason for this loss of quadratic convergence was first addressed by NAGTEGAAL (1982). He showed the difference between the derivatives of the stresses with respect to the strains based on the rate-equations and the derivatives based on the stress-update procedure. He pointed out that proper (quadratic) convergence behaviour is obtained by employing the moduli which follow from "consistent linearisation of the stress-update". Since he introduced this concept in the context of a linear isotropic hardening rule, a large amount of effort in computational mechanics has been spent on consistent linearisation methods.

In order to show the concept of consistent linearisation, the stress-update is first reformulated. To this end, the equations in (3.4.1.a) are combined, leading to

\[
\tilde{\sigma}_{t+\Delta t} = \tilde{\sigma}_t + M_{t+\Delta t}^{(e)} \cdot \left( \Delta \tilde{\varepsilon} - \Delta \gamma \tilde{\mathbf{b}}_{t+\Delta t} \right) - J_{t+\Delta t} \cdot \Delta \tilde{\varepsilon} 
\]  
.(3.4.4)

From this expression, the so-called "3D consistent moduli" are obtained according to

\[
\hat{M}_{KL,t+\Delta t} = \frac{\partial \sigma^K}{\partial \varepsilon_L} \bigg|_{t+\Delta t} 
\]  
.(3.4.5)

These moduli are used in the model, instead of the "3D continuum elastic/plastic tangent moduli" \( M_{KL,t+\Delta t}^{(e/p)} \).

It is essential to realise that both the matrices \( M^{(e)} \) and \( J \) and the vector \( \tilde{\mathbf{b}} \) depend on the strain situation at the time \( (t+\Delta t) \). Therefore, the linearisation with respect to \( \tilde{\varepsilon} \) in (3.4.5) also involves \( M^{(e)} \), \( J \) and \( \tilde{\mathbf{b}} \) at \( (t+\Delta t) \). If the linearisations of these quantities are left out of consideration, then the "3D continuum elastic/plastic tangent moduli", from (3.2.38), are recovered (\( \hat{M} = M^{(e/p)} \)).
3.4.2 - Strategy for the stress-update in the 2D model

As mentioned, the objective of the present chapter is to establish an algorithm to calculate the stress increments \((\Delta \tilde{\sigma}^{(o)}, \Delta \tilde{\sigma}^{(n)})\) from the strain increments \((\Delta \tilde{\eta}^{(o)}, \Delta \tilde{\eta}^{(n)})\). This algorithm will be based on a general 3D stress-update procedure, such as the procedure presented in section 3.4.1.

The stress vector \(\tilde{\sigma}_{i+\Delta t}\) and the 3D consistent moduli \(\hat{M}_{KL,i+\Delta t}\) are obtained from a 3D stress-update procedure. Integration through-the-thickness of the stresses and the moduli will then lead to

\[
\begin{align*}
\tilde{\sigma}_{i+\Delta t}^{(o)} &= \frac{1}{\sqrt{a}} \int \tilde{\sigma}_{i+\Delta t} \sqrt{g} \, d\xi \\
\tilde{\sigma}_{i+\Delta t}^{(n)} &= \frac{-\lambda}{\sqrt{a}} \int \xi \tilde{\sigma}_{i+\Delta t} \sqrt{g} \, d\xi \\
\hat{\sigma}_{KL,i+\Delta t}^{(o)} &= \frac{1}{\sqrt{a}} \int \hat{M}_{KL,i+\Delta t} \sqrt{g} \, d\xi \\
\hat{\sigma}_{KL,i+\Delta t}^{(n)} &= \frac{-\lambda}{\sqrt{a}} \int \xi \hat{M}_{KL,i+\Delta t} \sqrt{g} \, d\xi \\
\hat{\sigma}_{KL,i+\Delta t}^{(n)} &= \frac{\lambda^2}{\sqrt{a}} \left( \xi \right)^2 \hat{M}_{KL,i+\Delta t} \sqrt{g} \, d\xi 
\end{align*}
\]

\text{(3.4.6.a)}

\text{(3.4.6.b)}

In general, a 3D stress-update procedure will result in \(s^{(l)}_{3,i+\Delta t} \equiv m^{33}_{i+\Delta t} \neq 0\), which is in contradiction with (3.3.9). In the present section, an algorithm is described which, although using a general 3D stress-update procedure, satisfies (3.3.9).

The input of a general 3D stress-update procedure is the strain increment \(\Delta \tilde{E}\), which is determined by the increments \((\Delta \tilde{\eta}^{(o)}, \Delta \tilde{\eta}^{(n)}, \Delta \eta^{(n)}_{3})\). The increments \((\Delta \tilde{\eta}^{(o)}, \Delta \tilde{\eta}^{(n)})\) follow from the available geometric information, whereas the evaluation of the increment \(\Delta \eta^{(n)}_{3}\) is more complicated. The idea is to choose \(\Delta \eta^{(n)}_{3}\) such that (3.3.9) is satisfied; i.e.

\[
s^{(l)}_{3,i+\Delta t} = m^{33}_{i+\Delta t} = \frac{-\lambda}{\sqrt{a}} \int \xi \sigma_{3,i+\Delta t} \sqrt{g} \, d\xi = 0
\]

\text{(3.4.7)}

This is achieved as follows.

In general terms, (3.4.7) reads

\[
m^{33}_{i+\Delta t} = z \left( \tilde{\eta}^{(o)}_{i+\Delta t}, \tilde{\eta}^{(n)}_{i+\Delta t}, \eta^{(n)}_{3,i+\Delta t} \right) = 0
\]

\text{(3.4.8)}
The strain component η^{(i)}_{3,i+1} following from (3.4.8) is obtained by application of a Newton-Raphson iteration procedure. A typical iteration (the number of this iteration is denoted with the subscript \( (n) \), where the value of the iteration counter is initially 1) of this procedure reads

\[
\begin{align*}
\frac{\partial z}{\partial \eta^{(i)}_{3}} \bigg|_{i+\Delta t} \quad \Delta_{(n)} \eta^{(i)}_{3} &= - z_{i+\Delta t} \\
\text{with} \quad \eta^{(i)}_{3,(n+1)} &= \eta^{(i)}_{3,(n)} + \Delta_{(n)} \eta^{(i)}_{3} 
\end{align*}
\]  

(3.4.9)

The derivative of the function \( Z \) with respect to \( \eta^{(i)}_{3} \) at the time \( (t + \Delta t) \) follows from (3.4.6); i.e.

\[
\frac{\partial z}{\partial \eta^{(i)}_{3}} \bigg|_{i+\Delta t} = \frac{\partial}{\partial \eta^{(i)}_{3}} \left( -\frac{\lambda}{\sqrt{\alpha}} \int \xi \sigma_{3} \sqrt{g} \, d\xi \right) \bigg|_{i+\Delta t} =
\]

\[
= -\frac{\lambda}{\sqrt{\alpha}} \int \xi \frac{\partial \sigma_{3}}{\partial \eta^{(i)}_{3}} \bigg|_{i+\Delta t} \sqrt{g} \, d\xi =
\]

\[
= -\frac{\lambda}{\sqrt{\alpha}} \int \xi \frac{\partial \sigma_{3}}{\partial \varepsilon_{K}} \bigg|_{i+\Delta t} \frac{\partial \varepsilon_{K}}{\partial \eta^{(i)}_{3}} \bigg|_{i+\Delta t} \sqrt{g} \, d\xi
\]  

(3.4.10.a)

Next, it is observed that the term \( \eta^{(i)}_{3} \) features in \( \varepsilon_{3} \) only; i.e. \( \varepsilon_{3} = \eta^{(i)}_{3} - \xi \lambda \eta^{(i)}_{3} \). If also (3.4.5) is used, (3.4.10.a) becomes

\[
\frac{\partial z}{\partial \eta^{(i)}_{3}} \bigg|_{i+\Delta t} = -\frac{\lambda}{\sqrt{\alpha}} \int \xi \frac{\partial \sigma_{3}}{\partial \varepsilon_{3}} \bigg|_{i+\Delta t} \frac{\partial \varepsilon_{3}}{\partial \eta^{(i)}_{3}} \bigg|_{i+\Delta t} \sqrt{g} \, d\xi =
\]

\[
= -\frac{\lambda}{\sqrt{\alpha}} \int \xi \hat{M}_{33,i+\Delta t} \big(-\xi \lambda\big) \sqrt{g} \, d\xi = \hat{Q}^{(i)}_{33,i+\Delta t}
\]  

(3.4.10.b)

The last statement in (3.4.10.b) follows from (3.4.6.b).

At the start of the iteration procedure, \( \Delta \eta^{(i)}_{3} \) is unknown but will be estimated on the basis of an elastic prediction as follows. For the elastic prediction, the rate-equation (3.3.13.a) is written in a finite difference scheme according to

\[
Q^{(I),\text{elastic}}_{3L,i+\Delta t} \Delta \eta^{(I)}_{3} + Q^{(II),\text{elastic}}_{3L,i+\Delta t} \Delta \eta^{(II)}_{3} + Q^{(III),\text{elastic}}_{33,i+\Delta t} \Delta_{\text{elastic}} \eta^{(III)}_{3} = 0
\]  

(3.4.11)

where the moduli \( Q^{(I),\text{elastic}}_{3L,i+\Delta t} \) and \( Q^{(II),\text{elastic}}_{3L,i+\Delta t} \) follow from (3.3.13.b) by incorporating the elastic material assumption. The subscript \( i+\Delta t \) in the \( Q \)-matrices indicates that the geometric information is calculated at the \( (t + \Delta t) \)-state.
In the model, the elastic prediction $\Delta_{(\text{elastic})}\eta_{3}^{(0)}$ is approximated. For this purpose, it is assumed that the Jacobian $\sqrt{g}$ and the contravariant components of the metric tensor ($G^i_j$) are no functions of $\xi$ and $G^{ii}(\xi) = G^i_i(\xi = 0) = G^i_i^0$. In this case, we obtain from (3.4.11) together with (3.3.13b), (3.2.30) and (3.2.13)

$$\Delta_{(\text{elastic})}\eta_{3}^{(0)} = \frac{-\nu}{1-\nu} \frac{G_{33}^{11,0}}{G_{33}^{11,0}} \Delta \eta_{1}^{(0)} + \frac{-\nu}{1-\nu} \frac{G_{33}^{22,0}}{G_{33}^{11,0}} \Delta \eta_{2}^{(0)}$$  \hspace{1cm} (3.4.12)

The metric tensor components $G^{ii}$ are available from the geometric information. In the elastic prediction in (3.4.12), it is assumed that the influence of the Jaumann derivative may be neglected, which is reasonable as long as $\gamma^{33}_{i+\Delta t} << E G^{33}_{i+\Delta t}$ ($E$ is Young's modulus).

The iteration procedure discussed above requires additional computational effort, since for every iteration a stress-update for every integration point through-the-thickness should be evaluated. It is decided to combine this local iteration procedure with the global (at structural level) Newton-Raphson process, thus reducing computational effort. It has been experienced that the combination of the local iteration procedure with the global (at structural level) Newton-Raphson iteration process yields satisfactory results in the sense that convergence of the local iteration process is of the same order as convergence at the structural level.

In relation to the stress-update procedure discussed above, a consistent linearisation will be worked out. This essentially leads to the so-called "2D consistent moduli", which are defined as follows

$$\left. \frac{\partial \hat{s}_{R}^{(0)}}{\partial \eta_{L}^{(0)}} \right|_{i+\Delta t} = \hat{H}_{R,1+\Delta t}^{(0)/(0)} \quad \text{and} \quad \left. \frac{\partial \hat{s}_{R}^{(0)}}{\partial \eta_{L}^{(1)}} \right|_{i+\Delta t} = \hat{H}_{R,1+\Delta t}^{(1)/(0)}$$ \hspace{1cm} (3.4.13)

The "2D consistent moduli" are worked out by realising that the strains $\eta_{L}^{(0)}$ and $\eta_{L}^{(1)}$ are related to $\eta_{3}^{(0)}$ by means of $m^{33}$, see (3.4.8). This means that $s_{R}^{(0)} = y_{R}^{(0)}(\eta_{L}^{(0)}, \eta_{L}^{(1)}, \eta_{3}^{(0)})$ and $s_{R}^{(1)} = y_{R}^{(1)}(\eta_{L}^{(0)}, \eta_{L}^{(1)}), \eta_{3}^{(0)}, \eta_{3}^{(1)})$, where $\eta_{3}^{(0)} = \eta_{3}^{(0)}(\eta_{L}^{(0)}, \eta_{L}^{(1)}). These relations should be incorporated in (3.4.13).

In the derivation of the "2D consistent moduli" use will be made of the following identities (which are obtained by employing similar procedures as in (3.4.10.a) and (3.4.10.b)):

$$\left. \frac{\partial \hat{z}}{\partial \eta_{L}^{(0)}} \right|_{i+\Delta t} = \hat{G}_{31,1+\Delta t}^{(0)} \quad , \quad \left. \frac{\partial \hat{z}}{\partial \eta_{L}^{(1)}} \right|_{i+\Delta t} = \hat{G}_{31,1+\Delta t}^{(1)} \quad \text{and} \quad \left. \frac{\partial \hat{z}}{\partial \eta_{3}^{(0)}} \right|_{i+\Delta t} = \hat{G}_{33,1+\Delta t}^{(0)}$$
\[
\frac{\partial y_{K}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} = \hat{Q}_{KL,t+\Delta t}^{(0)} \quad \frac{\partial y_{K}^{(1)}}{\partial \eta_{L}^{(1)}_{t+\Delta t}} = \hat{Q}_{KL,t+\Delta t}^{(1)} \quad \frac{\partial y_{K}^{(0)}}{\partial \eta_{13}^{(0)}_{t+\Delta t}} = \hat{Q}_{KL,t+\Delta t}^{(0)}
\] (3.4.14)

\[
\frac{\partial y_{K}^{(1)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} = \hat{Q}_{KL,t+\Delta t}^{(1)} \quad \frac{\partial y_{K}^{(1)}}{\partial \eta_{L}^{(1)}_{t+\Delta t}} = \hat{Q}_{KL,t+\Delta t}^{(1)} \quad \frac{\partial y_{K}^{(1)}}{\partial \eta_{13}^{(1)}_{t+\Delta t}} = \hat{Q}_{KL,t+\Delta t}^{(1)}
\]

If attention is focused to the first expression in (3.4.13), we have

\[
\frac{\partial s_{K}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} = \frac{\partial y_{K}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \quad + \quad \frac{\partial y_{K}^{(0)}}{\partial \eta_{L}^{(1)}_{t+\Delta t}} \quad \frac{\partial \eta_{13}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}}
\] (3.4.15)

The linearisations of \(y_{K}^{(0)}\) with respect to \(\eta_{L}^{(0)}\) and \(\eta_{13}^{(0)}\) are given in (3.4.14). The derivative of \(\eta_{13}^{(1)}\) with respect to \(\eta_{L}^{(0)}\) is obtained from the differential of (3.4.8) at the \((t+\Delta t)\)-state, we find

\[
d m^{3}_{t+\Delta t} = \left. \frac{\partial z}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} \left. d \eta_{L}^{(0)} \right|_{t+\Delta t} + \left. \frac{\partial z}{\partial \eta_{13}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} \left. d \eta_{13}^{(0)} \right|_{t+\Delta t} + \left. \frac{\partial z}{\partial \eta_{13}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} \left( \left. \frac{\partial \eta_{13}^{(0)}_{t+\Delta t}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} \left. d \eta_{L}^{(0)} \right|_{t+\Delta t} + \left. \frac{\partial \eta_{13}^{(0)}_{t+\Delta t}}{\partial \eta_{13}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} \left. d \eta_{13}^{(0)} \right|_{t+\Delta t} \right) = 0
\] (3.4.16)

Using the results from (3.4.14) and the observation that (3.4.16) should hold for every value of \(\eta_{L}^{(0)}\) and \(\eta_{13}^{(0)}\) gives

\[
\left. \frac{\partial \eta_{13}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} = \frac{\hat{Q}_{3L,t+\Delta t}^{(0)}}{\hat{Q}_{33,t+\Delta t}^{(0)}} \quad \text{and} \quad \left. \frac{\partial \eta_{13}^{(0)}}{\partial \eta_{13}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} = \frac{\hat{Q}_{3L,t+\Delta t}^{(0)}}{\hat{Q}_{33,t+\Delta t}^{(0)}}
\] (3.4.17)

Substitution of the above mentioned linearisations in (3.4.14) yields

\[
\left. \frac{\partial s_{K}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} = \hat{Q}_{KL,t+\Delta t}^{(0)} - \hat{Q}_{KL,t+\Delta t}^{(0)} \left. \frac{1}{\hat{Q}_{33,t+\Delta t}^{(0)}} \right|_{t+\Delta t} \hat{Q}_{3L,t+\Delta t}^{(0)} = \hat{H}_{KL,t+\Delta t}^{(0)}
\] (3.4.18.a)

In a similar fashion, the other three expressions in (3.4.13) yield

\[
\left. \frac{\partial s_{K}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} = \hat{Q}_{KL,t+\Delta t}^{(0)} - \hat{Q}_{KL,t+\Delta t}^{(0)} \left. \frac{1}{\hat{Q}_{33,t+\Delta t}^{(0)}} \right|_{t+\Delta t} \hat{Q}_{3L,t+\Delta t}^{(0)} = \hat{H}_{KL,t+\Delta t}^{(0)}
\] (3.4.18.b)

\[
\left. \frac{\partial s_{K}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} = \hat{Q}_{KL,t+\Delta t}^{(0)} - \hat{Q}_{KL,t+\Delta t}^{(0)} \left. \frac{1}{\hat{Q}_{33,t+\Delta t}^{(0)}} \right|_{t+\Delta t} \hat{Q}_{3L,t+\Delta t}^{(0)} = \hat{H}_{KL,t+\Delta t}^{(0)}
\] (3.4.18.c)

\[
\left. \frac{\partial s_{K}^{(0)}}{\partial \eta_{L}^{(0)}_{t+\Delta t}} \right|_{t+\Delta t} = \hat{Q}_{KL,t+\Delta t}^{(0)} - \hat{Q}_{KL,t+\Delta t}^{(0)} \left. \frac{1}{\hat{Q}_{33,t+\Delta t}^{(0)}} \right|_{t+\Delta t} \hat{Q}_{3L,t+\Delta t}^{(0)} = \hat{H}_{KL,t+\Delta t}^{(0)}
\] (3.4.18.d)
As a recapitulation of the present section, the implication of the evaluation of $\eta_{3}^{(\text{I})}$ in the stress-update procedure is shown in the chart on the next page. In this chart, the subscript $(\text{itr})$ denotes the current number of the global (at structural level) Newton-Raphson iteration. It is assumed that the strain increments $(\Delta_{(\text{itr})}^{(\text{I})} \sigma_{3}^{(\text{I})}, \Delta_{(\text{itr})}^{(\text{I})} \sigma_{3}^{(\text{I})})$ are available from the geometric information.

(1) At the start of the increment $(\text{itr}) = (0)$, calculate $\Delta_{(\text{itr})}^{(\text{I})} \sigma_{3}^{(\text{I})}$ from the elastic prediction in (3.4.12).

(2) Obtain stress increments together with the "3D consistent elastic/plastic tangent moduli" from the 3D stress-update procedure. Thus,

$$(\Delta_{(\text{itr})}^{(\text{I})} \sigma^{(0)}_{3}, \Delta_{(\text{itr})}^{(\text{I})} \sigma^{(I)}_{3}, \Delta_{(\text{itr})}^{(\text{I})} \sigma^{(I)}_{3}) \Rightarrow (\sigma_{3}(\text{itr}), \tilde{M}_{(\text{itr})})$$

(3) Integrate $(\sigma_{3}(\text{itr}), \tilde{M}_{(\text{itr})})$ from (2) through-the-thickness and obtain the stress-resultants from (3.4.6.a):

$$(\sigma_{3}(\text{itr})) \Rightarrow (s_{3}(\text{itr}), \tilde{s}_{3}(\text{itr}))$$

and the $\tilde{Q}$-matrices from (3.4.6.b):

$$(\tilde{M}_{(\text{itr})}) \Rightarrow (\tilde{Q}_{KL}(\text{itr}), \tilde{q}_{KL}(\text{itr}), \tilde{q}_{KL}(\text{itr}))$$

From the $\tilde{Q}$-matrices, the "2D consistent moduli" follow from (3.4.18). Thus,

$$(\tilde{Q}_{KL}(\text{itr}), \tilde{q}_{KL}(\text{itr}), \tilde{q}_{KL}(\text{itr})) \Rightarrow (\hat{H}_{KL}(\text{itr}), \hat{H}_{KL}(\text{itr}), \hat{H}_{KL}(\text{itr}))$$

(4) Compute the value of the increment $\Delta_{(\text{itr})}^{(\text{I})} \sigma_{3}^{(\text{I})}$. Thus,

$$\Delta_{(\text{itr})}^{(\text{I})} \sigma_{3}^{(\text{I})} := \Delta_{(\text{itr})-1}^{(\text{I})} \sigma_{3}^{(\text{I})} - \frac{s_{3}^{(\text{itr})}}{\tilde{Q}_{33}(\text{itr})}$$

Observe that in (2) the increment $\Delta \sigma_{3}^{(\text{I})}$ from the previous iteration is used. The value of this increment is updated afterwards, see (4).
Chapter 4 - Description of tube deformation

4.1 - Introduction

In the shell formulation as presented in Chapter 2, the position vector to the mid-surface of the shell and the thickness function, which are denoted with $\vec{d}^0$ and $\lambda$ in the reference configuration and with $\vec{d}^0$ and $\Lambda$ in the deformed configuration, have been introduced. The reference configuration will be considered as the stress-free situation. In the present chapter, the functions $(\vec{d}^0, \lambda)$ and $(\vec{d}^0, \Lambda)$ will be discretised in such a way that an adequate description of tube deformation is obtained.

As already discussed in section 1.2.2, the deformation is decomposed into two separate contributions. These contributions are the so-called "beam deformation" and "shell deformation". The beam deformation will be defined according to Bernoulli’s beam theory. The shell deformation offers the description of effects such as ovalisation, buckling and necking.

The decomposition, in terms of the position vector to the mid-surface in both the reference and the deformed configuration, reads

\[
\begin{align*}
\vec{d}^0 &= \vec{d}^0 \,^R + \vec{d}^0 \,^S \\
\vec{d}^0 &= \vec{d}^0 \,^R + \vec{d}^0 \,^S
\end{align*}
\]

(4.1.1)

Here, the superscript $^R$ refers to the "beam deformation" and the superscript $^S$ to the "shell deformation". Recall the convention that the superscript $^0$ indicates the value of the quantities at the mid-surface ($\xi = 0$).

The decomposition of the thickness functions $\lambda$ and $\Lambda$ is as follows. For the "beam deformation", the thickness is assumed to be constant at any stage of deformation. Thickness stretching and the dependency of the thickness function on the mid-surface coordinates $\xi^a$ are considered to be captured in the "shell deformation".

The decomposition for the deformed case is illustrated in figure 4.1. For one typical material point, the decomposition in terms of the position vector $\vec{d}^0$ into $\vec{d}^0 \,^R$ and $\vec{d}^0 \,^S$ is shown. The quantities appearing in this figure, so far as these were not yet defined, will be discussed in the next sections.
The formulation of the beam deformation will be based on a finite strain generalisation of the Bernoulli beam theory. This is discussed in section 4.2. The idea is to start with a general (finite strain) beam theory and to simplify systematically. In this way, future developments where certain simplifying assumptions are avoided might be easier accomplished.

It is discussed in section 1.2.3 that Fourier series will be used for the discretisation of the shell deformation. More details of this discretisation are presented in sections 4.3 and 4.4.

4.2 - Kinematics of a finite strain beam formulation

The presentation of the finite strain generalisation of the Bernoulli beam theory is based on SIMO (1985) and SIMO & VU-QUOC (1986). In these articles, a finite strain beam formulation is derived from the continuum mechanics equations by incorporating beam-like kinematic assumptions. Their model accounts for three-dimensional bending, torsion, axial stretching and shearing of the cross-sections of the beam. In the present section, a recapitulation of the theory is given first. The simplifying assumptions which are applied will be explicitly mentioned.
The reference configuration is introduced as a curved beam. The centroids of the cross-sections of the beam are connected by a curve, the so-called "line-of-centroids". The cross-sections are taken perpendicular to the line-of-centroids. Accordingly, the following objects of the beam in the reference configuration are introduced:

- the line-of-centroids as the vector field \( \bar{\phi}^c \), in general stated by

\[
\bar{\phi}^c = \bar{\phi}^c(\xi^1), \text{ where the centroidal coordinate } \xi^1 \in \left[ -\ell/2, \ell/2 \right].
\]  

(4.2.1)

By definition, \( \ell \) is the length of the line-of-centroids in the reference configuration. The centroidal coordinate \( \xi^1 \) measures the true length along the line-of-centroids. Halfway the line-of-centroids, we take \( \xi^1 = 0 \).

- the "cross-section orthonormal basis" \( \{ \bar{\mathbf{e}}_k \} \). This basis is related to the (fixed) Cartesian basis by means of a rotation tensor; i.e.

\[
\bar{\mathbf{e}}_k(\xi^1) = \Theta(\xi^1) \cdot \mathbf{e}_k
\]  

(4.2.2)

The explicit choice for the components of \( \Theta \) follows.

The cross-section base vector \( \bar{\mathbf{e}}_1(\xi^1) \) is taken as the normal to the cross-section at \( \xi^1 \). Since the cross-sections are taken perpendicular to the line-of-centroids, \( \bar{\mathbf{e}}_1 \) is determined by the tangent to the line-of-centroids; i.e.

\[
\bar{\mathbf{e}}_1(\xi^1) = \bar{\phi}^c_1(\xi^1)
\]  

(4.2.3)

Thus, the cross-section at \( \xi^1 \) is defined by the plane through \( \bar{\phi}^c(\xi^1) \) with the normal \( \bar{\mathbf{e}}_1(\xi^1) \).

An illustration of the objects "line-of-centroids", "cross-section" and the "cross-section orthonormal basis" \( \{ \bar{\mathbf{e}}_k \} \) is given in figure 4.2.

In the reference configuration, circular cross-sections are taken, where the shape is independent of \( \xi^1 \). This is no limitation to the model, since at this stage the beam deformation is considered only. For example, initially ovalised cross-sections will be incorporated in \( \bar{\phi}^{b,0} \). This will become clear in the sequel of this chapter.

For the definition of the shape of the cross-sections, the cross-section coordinates \( x_2(\xi^2,\xi) \) and \( x_3(\xi^2,\xi) \) are introduced. These coordinates are measured along the cross-section base vectors \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \), respectively. The material coordinate \( \xi^2 \) is a circumferential coordinate and is considered to be the mid-surface coordinate from the shell formulation, whereas the coordinate \( \xi \) is considered to be the through-the-thickness coordinate from the shell formulation. Circular cross-sections are determined by
\[
\begin{aligned}
  x_2 &= (R + \xi t) \sin(\psi_2) \\
  x_3 &= (R + \xi t) \cos(\psi_2)
\end{aligned}
\]

with
\[
\begin{aligned}
  \psi_2 &= \frac{\xi^2}{R} \in [0, 2\pi] \\
  \xi &= [-\frac{1}{2}, \frac{1}{2}]
\end{aligned}
\] (4.2.4)

An illustration of the definitions in (4.2.4) is given in figure 4.3. The scalar $R$ in (4.2.4) denotes a characteristic radius of the tube, whereas $t$ denotes a characteristic thickness.

**Figure 4.2** - The objects "line-of-centroids", "cross-section" and the "cross-section orthonormal basis" \( \{ \mathbf{e}_k \} \).

**Figure 4.3** - The cross-section coordinates \( x_2(\xi^2, \xi) \) and \( x_3(\xi^2, \xi) \).
The position vector of the material points of the beam in the reference configuration is taken as follows:

$$\bar{\mathbf{p}}^c(\xi^1) = \bar{\mathbf{p}}^c(\xi^1) + x_2(\xi^2, \xi^1) \bar{\mathbf{e}}_2(\xi^1) + x_3(\xi^2, \xi^1) \bar{\mathbf{e}}_3(\xi^1)$$  \hspace{1cm} (4.2.5)

From (4.2.5) together with (4.2.2), (4.2.3) and (4.2.4), the base vectors in the reference beam configuration are found to be

$$\begin{align*}
\bar{\mathbf{e}}_1^b &= \bar{\mathbf{e}}_1^c + x_2 \bar{\mathbf{e}}_{2,1} + x_3 \bar{\mathbf{e}}_{3,1} = \bar{\mathbf{e}}_1 + \bar{\boldsymbol{\vartheta}}_1 \cdot \left( x_2 \bar{\mathbf{I}}_2 + x_3 \bar{\mathbf{I}}_3 \right) = \\
&= \bar{\mathbf{e}}_1 + \omega^b \cdot (x_2 \bar{\mathbf{e}}_2 + x_3 \bar{\mathbf{e}}_3) \quad \text{where } \omega^b = \bar{\boldsymbol{\vartheta}}_1 \cdot \bar{\boldsymbol{\vartheta}}^{-1} \\
\bar{\mathbf{e}}_2^b &= x_{2,2} \bar{\mathbf{e}}_2 + x_{3,2} \bar{\mathbf{e}}_3 \\
\bar{\mathbf{e}}_3^b &= x_{2,3} \bar{\mathbf{e}}_2 + x_{3,3} \bar{\mathbf{e}}_3
\end{align*}$$  \hspace{1cm} (4.2.6)

The tensor $\bar{\boldsymbol{\vartheta}}$ as defined in (4.2.2) is a rotation tensor and thus $\bar{\boldsymbol{\vartheta}} \cdot \bar{\boldsymbol{\vartheta}}^T = \mathbf{I}$, from which it follows that $\bar{\boldsymbol{\vartheta}}^{-1} = \bar{\boldsymbol{\vartheta}}^T$. Differentiation of $\bar{\boldsymbol{\vartheta}} \cdot \bar{\boldsymbol{\vartheta}}^T = \mathbf{I}$ with respect to $\xi^1$ yields

$$\bar{\boldsymbol{\vartheta}}_1 \cdot \bar{\boldsymbol{\vartheta}}^T + \bar{\boldsymbol{\vartheta}} \cdot \bar{\boldsymbol{\vartheta}}_1^T = 0$$  \hspace{1cm} \text{and thus }  \bar{\boldsymbol{\vartheta}}_1 \cdot \bar{\boldsymbol{\vartheta}}^T = -\left( \bar{\boldsymbol{\vartheta}}_1 \cdot \bar{\boldsymbol{\vartheta}}^T \right)^T.$$

From this observation, it is concluded that the tensor $\omega^b = \bar{\boldsymbol{\vartheta}}_1 \cdot \bar{\boldsymbol{\vartheta}}^T = \bar{\boldsymbol{\vartheta}}_1 \cdot \bar{\boldsymbol{\vartheta}}^{-1}$ is a skew-symmetric tensor.

The kinematic assumptions are such that

- **beam (1):** cross-sections initially plane and perpendicular to the line-of-centroids, remain plane and perpendicular to the line-of-centroids at any stage of deformation.
- **beam (2):** the cross-sections do not deform.

Thus, in the deformed beam configuration similar definitions as in the reference beam configuration are valid. Therefore, the objects defined for the beam in the reference configuration are re-introduced, but in this case for the deformed beam:

- $\bar{\mathbf{F}}^c = \bar{\mathbf{F}}^c(\xi^1)$ as the line-of-centroids in the deformed beam configuration.
- the cross-section orthonormal basis in the deformed state $\{ \bar{\mathbf{E}}_k \}$, which is related to the basis by means of a rotation tensor; i.e.

$$\bar{\mathbf{E}}_k(\xi^1) = \Theta(\xi^1) \cdot \bar{\mathbf{I}}_k$$  \hspace{1cm} (4.2.7)

The explicit choice for the components of $\Theta$ follows.
The cross-section base vector $\overline{E}_1(\xi^1)$ is taken as the normal to the cross-section at $\xi^1$. Since shear deformation is excluded, see "beam (1)" , the vector $\overline{E}_1$ is determined by the tangent to the line-of-centroids. Thus,

$$\overline{E}_1(\xi^1) = \frac{\overline{G}_c(\xi^1)}{\xi^B(\xi^1)}$$

where $\xi^B(\xi^1) = \|\overline{G}_c(\xi^1)\|$.  

(4.2.8)

The three orthonormal bases $\{\overline{E}_k\}$, $\{\overline{C}_k\}$ and $\{\overline{C}_k\}$ for a certain cross-section are shown in figure 4.4. Also the position vectors $\overline{q}_C$ and $\overline{q}_C$ are shown. In the figure it is illustrated that the tangents to the line-of-centroids in both the reference and the deformed configuration are parallel to $\overline{e}_1$ and $\overline{E}_1$, respectively.

![Figure 4.4 - The orthonormal bases \(\{\overline{E}_k\}\), \(\{\overline{C}_k\}\) and \(\{\overline{C}_k\}\).](image)

It has been assumed before that, at any stage of deformation of the beam, the shape of the cross-sections remains unchanged, see "beam (2)". (Again, this is no limitation to the model, for possible changes in the shape of cross-sections will be incorporated in the shell deformation.) Thus, the cross-section coordinates in the reference and in the deformed beam configuration are identical: $X_2 = x_2$ and $X_3 = x_3$. Consequently, the position vector to the deformed beam reads

$$\overline{q}_B(\xi^a, \xi) = \overline{q}_C(\xi^1) + x_2(\xi^2, \xi) \overline{E}_2(\xi^1) + x_3(\xi^2, \xi) \overline{E}_3(\xi^1)$$

(4.2.9)

From this, the base vectors in the deformed configuration are derived in a similar fashion as for the base vectors in the reference configuration, see (4.2.5) and (4.2.6). Application of a similar derivation yields
\[
\begin{align*}
\bar{G}_1^b &= \zeta^b \bar{E}_1 + \Omega^b \cdot (x_2 \bar{E}_2 + x_3 \bar{E}_3) \\
\bar{G}_2^b &= x_{2,2} \bar{E}_2 + x_{3,2} \bar{E}_3 \\
\bar{G}_3^b &= x_{2,3} \bar{E}_2 + x_{3,3} \bar{E}_3
\end{align*}
\]

where \( \Omega^b = \Theta^b \cdot \Theta^T \)

\( (4.2.10) \)

Above expression is rewritten after the following observations. Firstly, observe that, due to
\((4.2.9), x_2 \bar{E}_2 + x_3 \bar{E}_3 = \bar{\Phi}^b - \bar{\Phi}^c \). Secondly, it appears convenient to work with the so-called "axial vector" \( \bar{\Omega}^b \) associated with the skew-symmetric tensor \( \Omega^b \). The axial vector is defined as follows. Given any vector \( \bar{h} \), the axial vector \( \bar{\Omega}^b \) related to the skew-symmetric tensor \( \Omega^b \) is defined by \( \Omega^b \cdot \bar{h} = \bar{\Phi}^b \times \bar{h} \). If, in the third place, expressions \((4.2.7)\) and \((4.2.2)\) are used in the expressions of \( \bar{G}_2^b \) and \( \bar{G}_3^b \), we find

\[
\begin{align*}
\bar{G}_1^b &= \zeta^b \bar{E}_1 + \bar{\Omega}^b \times (\bar{\Phi}^b - \bar{\Phi}^c) \\
\bar{G}_2^b &= \Theta \cdot \Theta^T \cdot \bar{G}_2^b \\
\bar{G}_3^b &= \Theta \cdot \Theta^T \cdot \bar{G}_3^b
\end{align*}
\]

\( (4.2.11) \)

The base vectors \( \{ \bar{G}_1^b \} \) and \( \{ \bar{G}_b^b \} \) for \( \xi = 0 \) will be used in the definition of the components of the vectors \( \bar{\Phi}^{b,0} \) and \( \bar{\Phi}^{s,0} \), respectively. These definitions follow in \((4.3.1)\) together with \((4.3.2)\) and \((4.3.3)\).

### 4.2.1 - Simplifying assumptions

In addition to the assumptions "beam (1)" and "beam (2)" as stated previously, the following basic idea is incorporated in the model: the beam is a part of a torus at any stage of deformation. The shape of the cross-sections of the torus is already defined, see \((4.2.4)\).

The assumptions following from this basic idea are:

| \begin{tabular}{|l|}
| beam (3): the line-of-centroids is situated in one plane at any stage of deformation. |
| beam (4): the curvature of the line-of-centroids is uniform (independent of \( \xi^1 \)) at any stage of deformation and thus fully determined by the angle between both end sections. |
| beam (5): the stretching of the line-of-centroids is uniform (independent of \( \xi^1 \)) at any stage of deformation. |
| beam (6): torsion is excluded. |
| \end{tabular} |

Moreover, it is assumed that
For the interpretation of the above assumptions, we start to consider the length of the line-of-centroids measured between both end cross-sections of the beam. In the reference configuration this length will be denoted with $l^C$ and the length in the deformed configuration will be denoted with $L^C$. These follow from

$$l^C = \int_{\xi_1=-\zeta_2}^{\zeta_2} \lVert \Phi_0^1 \rVert d\xi_1 \quad \text{and} \quad L^C = \int_{\xi_1=-\zeta_2}^{\zeta_2} \lVert \Phi_0^2 \rVert d\xi_1$$ \hspace{1cm} (4.2.12)

which are determined by virtue of (4.2.3) and (4.2.8). It is easily verified that $l^C = \ell$. Since the stretching along the line-of-centroids is assumed to be uniform ($\xi^B$ is independent of $\xi^1$)

$$L^C = \xi^B \ell$$ \hspace{1cm} (4.2.13)

In the sequence, the factor $\xi^B$ will be considered as a degree-of-freedom.

The curvature of the line-of-centroids is considered next. According to the above mentioned "beam (3)", the line-of-centroids is situated in one plane. This plane is taken to be the $\mathbf{I}_1-\mathbf{I}_2$-plane. According to "beam (4)", the curvature of the line-of-centroids is fully determined by the angle measured between both end-sections of the beam. In the reference configuration, this angle is denoted with $\Theta^B$, whereas the angle in the deformed beam is denoted with $\Theta^B$. This is illustrated in figure 4.5.

![Diagram showing the line-of-centroids in both the reference and the deformed state.](image)

*Figure 4.5 - The line-of-centroids in both the reference and the deformed state.*
For the reference configuration, the parameter
\[ \vartheta(\xi^1) \] is introduced as the angle from the vector \( \mathbf{I}_1 \) to the vector \( \mathbf{e}_1(\xi^1) \).

Analogously, for the deformed configuration, the parameter
\[ \Theta(\xi^1) \] is introduced as the angle from the vector \( \mathbf{I}_1 \) to the \( \mathbf{E}_1(\xi^1) \).

These parameters are given by
\[
\vartheta(\xi^1) = \frac{\xi^1}{\ell} \vartheta^B \quad \text{and} \quad \Theta(\xi^1) = \Theta^B + \frac{\xi^1}{\ell} \Theta^B \quad (4.2.14)
\]

The radius of curvature of the line-of-centroids in the reference configuration is denoted with \( 1/\kappa^B \). In the deformed configuration, the radius of curvature of the line-of-centroids reads \( 1/K^B \). These radii follow from
\[
\frac{1}{\kappa^B} \vartheta^B = \ell \quad \text{and} \quad \frac{1}{K^B} \Theta^B = \zeta^B \ell \quad (4.2.15)
\]

In the sequence, the angle \( \Theta^B \) will be considered as a degree-of-freedom.

The relations between the bases \( \{ \mathbf{I}_k \}, \{ \mathbf{e}_k \} \) and \( \{ \mathbf{E}_k \} \) are captured by
\[
\begin{align*}
\mathbf{e}_1 &= \cos \vartheta \mathbf{I}_1 - \sin \vartheta \mathbf{I}_2 \\
\mathbf{e}_2 &= \sin \vartheta \mathbf{I}_1 + \cos \vartheta \mathbf{I}_2 \\
\mathbf{e}_3 &= \mathbf{I}_3
\end{align*}
\quad \text{and} \quad
\begin{align*}
\mathbf{E}_1 &= \cos \Theta \mathbf{I}_1 - \sin \Theta \mathbf{I}_2 \\
\mathbf{E}_2 &= \sin \Theta \mathbf{I}_1 + \cos \Theta \mathbf{I}_2 \\
\mathbf{E}_3 &= \mathbf{I}_3
\end{align*} \quad (4.2.16)
\]

From these expressions, the components of the rotation tensors \( \vartheta(\xi^1) \) and \( \Theta(\xi^1) \) (introduced in (4.2.2) and (4.2.7), respectively) are determined as follows. The components of both rotation tensors are defined by means of
\[
\vartheta = \vartheta_{kl} \mathbf{I}^k \otimes \mathbf{I}^l \quad \text{and} \quad \Theta = \Theta_{kl} \mathbf{I}^k \otimes \mathbf{I}^l.
\]

Thus, according to (4.2.2),
\[
\mathbf{e}_m = \vartheta \cdot \mathbf{I}_m = \vartheta_{km} \mathbf{I}^k \quad \text{and} \quad \mathbf{E}_m = \Theta \cdot \mathbf{I}_m = \Theta_{km} \mathbf{I}^k.
\]

From the above expressions, together with (4.2.16) and the fact that \( \mathbf{I}_k = \mathbf{I}^k \), the components \( \vartheta_{km} \) and \( \Theta_{km} \) follow.

The axial vectors \( \mathbf{e}^B \) and \( \mathbf{E}^B \) are found by considering their definitions. Focusing on the reference configuration, we have
\[
(\vartheta \cdot \vartheta^T) \cdot (h^k \mathbf{e}_k) \equiv \mathbf{e}^B \times (h^k \mathbf{e}_k).
\]
This definition is conveniently worked out after realising that \( \mathbf{e}_{k,1} = (\vartheta \cdot \vartheta^T) \cdot \mathbf{e}_k \), which are readily obtained from (4.2.14) - (4.2.16); i.e., \( \mathbf{e}_{1,1} = -\kappa^B \mathbf{e}_2 \) and \( \mathbf{e}_{2,1} = \kappa^B \mathbf{e}_1 \). Using
these expressions, the axial vector $\vec{\omega}^B$ is derived. A similar procedure is employed for the derivation of $\vec{\Omega}^B$. The results are

$$
\begin{align*}
\vec{\omega}^B &= -\kappa^B \vec{e}_a = -\kappa^B \vec{I}_a \\
\vec{\Omega}^B &= -\gamma^B K^B \vec{E}_a = -\gamma^B K^B \vec{I}_a
\end{align*}
$$

(4.2.17)

The explicit expressions for the position vector to the line-of-centroids in the reference configuration and in the deformed configuration read

$$
\begin{align*}
\vec{x}^C(\xi^1) &= \frac{1}{\kappa^B} \left\{ \sin \theta \vec{I}_1 - (1 - \cos \theta) \vec{I}_2 \right\} \\
\vec{x}^C(\xi^1) &= \vec{x}^C_0 + \frac{1}{K^B} \left\{ \sin \theta \vec{I}_1 - (1 - \cos \theta) \vec{I}_2 \right\}
\end{align*}
$$

(4.2.18)

From these expressions, together with (4.2.4), (4.2.5), (4.2.9) and (4.2.16), the position vector to the beam in the reference configuration and the position vector to the beam in the deformed configuration (both in components on the Cartesian basis) are obtained:

$$
\begin{align*}
\vec{x}^B &= \frac{1}{\kappa^B} \left\{ \sin \theta \vec{I}_1 - (1 - \cos \theta) \vec{I}_2 \right\} + \\
&\quad + (R + \xi t) \left\{ \sin(\psi_2) \left\{ \sin \theta \vec{I}_1 + \cos \theta \vec{I}_2 \right\} + \cos(\psi_2) \vec{I}_a \right\} \quad \text{(4.2.19.a)}
\end{align*}
$$

$$
\begin{align*}
\vec{x}^B &= \vec{x}^C_0 + \frac{1}{K^B} \left\{ \sin \theta \vec{I}_1 - (1 - \cos \theta) \vec{I}_2 \right\} + \\
&\quad + (R + \xi t) \left\{ \sin(\psi_2) \left\{ \sin \theta \vec{I}_1 + \cos \theta \vec{I}_2 \right\} + \cos(\psi_2) \vec{I}_a \right\} \quad \text{(4.2.19.b)}
\end{align*}
$$

Rigid-body translations and rotations are eliminated by taking $\vec{x}^C_0 = \vec{x}$ and $\Omega^B = 0$.

### 4.3 - "Shell deformation"

Resuming the considerations so far, the mid-surface position vector is decomposed into the following two parts (see (4.1.1)):

1. the so-called "beam deformation". This is discussed in section 4.2. In this case, the configuration of the tube is essentially a part of a torus.
2. the so-called "shell deformation". This part accommodates the deviations from the toroidal tube.

In the present section, the "shell deformation" is considered in more detail.
The components of the vectors \( \vec{\sigma}^{b,0} \) and \( \vec{\Gamma}^{b,0} \), governing the shell deformation, are defined as follows

\[
\begin{align*}
\vec{\sigma}^{b,0} &= u^s \vec{b}_1 + v^s \vec{b}_2 + w^s \vec{b}_3 \\
\vec{\Gamma}^{b,0} &= U^s \vec{B}_1 + V^s \vec{B}_2 + W^s \vec{B}_3
\end{align*}
\]  

(4.3.1)

where the vectors \{ \vec{b}_1 \} and \{ \vec{B}_1 \}, called the "beam-like base vectors", are defined as the partial derivatives of \( \vec{\sigma}^b \) and \( \vec{\Gamma}^b \) with respect to the material coordinates (at \( \xi = 0 \)). From (4.2.6) together with (4.2.3), (4.2.4) and (4.2.16):

\[
\begin{align*}
\vec{b}_1 &= \vec{\sigma}^{b,0}_1 = \vec{e}_1 + ( -\kappa^b \vec{e}_3 ) \times \left( R \sin(\psi_2) \vec{e}_2 + R \cos(\psi_2) \vec{e}_3 \right) = \\
&= \{ 1 + \kappa^b R \sin(\psi_2) \} \vec{e}_1 \equiv s^b(\psi_2) \vec{e}_1 \\
\vec{b}_2 &= \vec{\sigma}^{b,0}_2 = \cos(\psi_2) \vec{e}_2 - \sin(\psi_2) \vec{e}_3 \\
\vec{b}_3 &= \vec{\sigma}^{b,0}_3 / t = \sin(\psi_2) \vec{e}_2 + \cos(\psi_2) \vec{e}_3
\end{align*}
\]  

(4.3.2)

and similarly, from (4.2.10) together with (4.2.8), (4.2.4) and (4.2.16):

\[
\begin{align*}
\vec{B}_1 &= \vec{\Gamma}^{b,0}_1 = \zeta^b \vec{E}_1 + ( -\zeta^b K^b \vec{E}_3 ) \times \left( R \sin(\psi_2) \vec{E}_2 + R \cos(\psi_2) \vec{E}_3 \right) = \\
&= \zeta^b \{ 1 + K^b R \sin(\psi_2) \} \vec{E}_1 \equiv S^b(\psi_2) \vec{E}_1 \\
\vec{B}_2 &= \vec{\Gamma}^{b,0}_2 = \cos(\psi_2) \vec{E}_2 - \sin(\psi_2) \vec{E}_3 \\
\vec{B}_3 &= \vec{\Gamma}^{b,0}_3 / t = \sin(\psi_2) \vec{E}_2 + \cos(\psi_2) \vec{E}_3
\end{align*}
\]  

(4.3.3)

Note that in the definitions of the base vectors \( \vec{b}_3 \) and \( \vec{B}_3 \), a normalisation with respect to the characteristic thickness \( t \) has been carried out.

Due to the definitions (4.3.2) and (4.3.3), adequate directions for the components \( (u^s, v^s, w^s) \) and \( (U^s, V^s, W^s) \) are chosen: they represent the axial, circumferential and the radial deviations from the toroidal configuration.

Driven by the fact that only a section of the length of a total tube is analysed, a limitation is placed on the mid-surface position vector (at any stage of deformation) such that both end-sections of the tube remain plane. In the definitions of the beam deformation in section 4.2, the requirement of plane end-sections is fulfilled. However, for the definition of the shell deformation, this requirement should be enforced explicitly. Therefore, it is required that

shell (1): warping at both end-sections is excluded.
In addition, it is decided that

shell (2): the orientation of the planes of the end-sections are determined by the beam deformation only. However, the position of the material points in the end planes are determined by both the beam deformation and the shell deformation.

Figure 4.6 serves as an illustration for the plane of an end-section, in this case for the deformed configuration.

![Figure 4.6 - The influence of the "shell deformation" in the plane of an end-section.](image)

In figure 4.6, the vector \( \mathbf{\Phi^c} \) points to the line-of-centroids at the concerned end-section. The vector \( \mathbf{\Phi^{B,0}} \) is the position vector to the mid-surface according to the beam deformation. The mid-surface position vector to the deformed tube is denoted by \( \mathbf{\Phi^0} \). For later reference, the mean position vector to the material points on the mid-surface of the cross-sections is introduced as

\[
\mathbf{\Phi^m}(\xi^1) = \frac{1}{2\pi R} \int \mathbf{\Phi^0}(\xi^1, \xi^2) \, d\xi^2
\]  

(4.3.4)

Use of the mean position vector \( \mathbf{\Phi^m}(\xi^1) \) is made in the definition of loading conditions which are applied at both end-sections of the tube, see Chapter 5.

The mutual orientation of the planes of the end-sections is determined by the beam deformation only (i.e. by \( \Theta^B \)). Therefore, conditions "shell (1)" and "shell (2)" require, at \( \xi^1 = \pm \frac{1}{2} \ell \), the vector \( \mathbf{\Phi} - \mathbf{\Phi}^B \) ( = \( \mathbf{\Phi}^0 \) ) to remain in these end planes during shell
deformation. Since the orientation of an end plane is defined by the normal $\mathbf{E}_1$, it follows that the $\mathbf{E}_1$-component of the vector $\mathbf{\bar{G}} - \mathbf{\bar{G}}^o$ should vanish at $\xi^1 = \pm \frac{1}{2} \ell$; i.e.,

$$
(\mathbf{\bar{G}} - \mathbf{\bar{G}}^o) \cdot \mathbf{E}_1 = 0 \quad \text{at} \quad \xi^1 = \pm \frac{1}{2} \ell
$$

(4.3.5)

This expression should hold for every value of the material coordinates $\xi^2$ and $\xi^3$.

Due to the fact that the cross-section coordinates $x_2(\xi^0, \xi^3)$ and $x_3(\xi^0, \xi^3)$ are linear-in-$\xi$, relation $\mathbf{\bar{G}}^o = \mathbf{\bar{G}}^{o,0} + \xi t \mathbf{\bar{B}}_3$ holds (see the definitions in (4.2.4), (4.2.9) and (4.3.3)). This relation together with (4.1.1) yields $\mathbf{\bar{G}}^o = \mathbf{\bar{G}}^o - \mathbf{\bar{G}}^{o,0} + \xi t \mathbf{\bar{B}}_3$. If also use is made of the expression $\mathbf{\bar{G}} = \mathbf{\bar{G}}^o + \xi (\mathbf{\bar{A}} \mathbf{\bar{N}} - t \mathbf{\bar{B}}_3)$, valid within the shell theory as presented in Chapter 2, expression (4.3.5) can be rewritten into

$$
(\mathbf{\bar{G}}^{o,0} + \xi \left( \mathbf{\bar{A}} \mathbf{\bar{N}} - t \mathbf{\bar{B}}_3 \right) ) \cdot \mathbf{E}_1 = 0 \quad \text{at} \quad \xi^1 = \pm \frac{1}{2} \ell
$$

(4.3.6)

Following from the observation that (4.3.6) should hold for every value of $\xi$, two separate requirements are obtained: $\mathbf{\bar{G}}^{o,0} \cdot \mathbf{E}_1 = 0$ and $(\mathbf{\bar{A}} \mathbf{\bar{N}} - t \mathbf{\bar{B}}_3) \cdot \mathbf{E}_1 = 0$. From the definition in (4.3.3), it is concluded that $\mathbf{\bar{B}}_2 \cdot \mathbf{E}_1 = 0$ as well as $\mathbf{\bar{B}}_3 \cdot \mathbf{E}_1 = 0$ hold. Thus, (4.3.6) leads to

$$
\begin{align*}
\begin{cases}
S^B(\psi_2) U^s = 0 & \text{at} \quad \xi^1 = \pm \frac{1}{2} \ell \\
\mathbf{\bar{N}} \cdot \mathbf{\bar{B}}_2 & = 0
\end{cases}
\end{align*}
$$

(4.3.7)

where the first expression follows from the definition of $\mathbf{\bar{G}}^{o,0}$ in (4.3.1).

Focusing to the second expression in (4.3.7), it is observed that the direction of $\mathbf{\bar{N}}$ is given by the direction of the vector $(\mathbf{\bar{A}}_1 \times \mathbf{\bar{A}}_2)$, see (2.2.12). This vector is worked out from

$$
\mathbf{\bar{A}}_a = \mathbf{\bar{G}}^o_a = \mathbf{\bar{B}}_a + \mathbf{\bar{G}}^{o,0}_a,
$$

which follow from (4.1.1).

Using (4.3.1) and (4.3.3), the vectors $\mathbf{\bar{A}}_1$ and $\mathbf{\bar{A}}_2$ are written in terms of components on the $(\mathbf{\bar{B}}_a)$-basis. With the aid of these expressions for the mid-surface base-vectors and the first expression in (4.3.7), the second expression in (4.3.7) leads to

$$
V^s_{i2} \left( -\frac{V^s}{R} + W^s_{12} \right) - W^s_i \left( 1 + V^s_{22} + \frac{W^s}{R} \right) = 0 \quad \text{at} \quad \xi^1 = \pm \frac{1}{2} \ell
$$

(4.3.8)

The first expression in (4.3.7) and the constraint in (4.3.8) are fulfilled by setting

$$
U^s = 0 \quad , \quad V^s_{i2} = 0 \quad \text{and} \quad W^s_i = 0 \quad \text{at} \quad \xi^1 = \pm \frac{1}{2} \ell
$$

(4.3.9)

The discretisation of the shell deformation, as introduced in the next section, is such that (4.3.9) is fulfilled.
4.3.1 - Discretisation

The components \((u^s, v^s, w^s)\) and \((U^s, V^s, W^s)\) as defined in (4.3.1) will be developed in terms of Fourier series. (This choice has been advocated in section 1.2.3.) The objective of the present section is to come to expressions which satisfy the requirements "shell (1)" and "shell (2)". In addition, the following condition should be satisfied:

**shell (3):** the displacement modes which are added by means of the shell deformation should be independent of those incorporated in the beam deformation.

In the sequel, attention is focused to the components of the shell deformation in the deformed configuration: \((U^s, V^s, W^s)\). The expressions for the components in the reference configuration may be obtained via a similar treatment.

The components \((U^s, V^s, W^s)\) are developed in Fourier series expansions as follows

\[
\begin{align*}
U^s(\psi_1, \psi_2) & = U^1(\psi_1) U^2(\psi_2) \\
V^s(\psi_1, \psi_2) & = V^1(\psi_1) V^2(\psi_2) \quad \text{with} \quad \psi_1 = \pi \frac{L_1}{\ell} \\
W^s(\psi_1, \psi_2) & = W^1(\psi_1) W^2(\psi_2) \quad \psi_2 = \frac{2}{R}
\end{align*}
\]

where

\[
\begin{align*}
U^1(\psi_1) & = \sum_{m = 1}^{\infty} U^1_{\sin, m} \sin(m\psi_1) + \sum_{m = 0}^{\infty} U^1_{\cos, m} \cos(m\psi_1) \\
V^1(\psi_1) & = \sum_{m = 1}^{\infty} V^1_{\sin, m} \sin(m\psi_1) + \sum_{m = 0}^{\infty} V^1_{\cos, m} \cos(m\psi_1) \quad (4.3.11.a) \\
W^1(\psi_1) & = \sum_{m = 1}^{\infty} W^1_{\sin, m} \sin(m\psi_1) + \sum_{m = 0}^{\infty} W^1_{\cos, m} \cos(m\psi_1) \\
U^2(\psi_2) & = \sum_{n = 1}^{\infty} U^2_{\sin, n} \sin(n\psi_2) + \sum_{n = 0}^{\infty} U^2_{\cos, n} \cos(n\psi_2) \\
V^2(\psi_2) & = \sum_{n = 1}^{\infty} V^2_{\sin, n} \sin(n\psi_2) + \sum_{n = 0}^{\infty} V^2_{\cos, n} \cos(n\psi_2) \quad (4.3.11.b) \\
W^2(\psi_2) & = \sum_{n = 1}^{\infty} W^2_{\sin, n} \sin(n\psi_2) + \sum_{n = 0}^{\infty} W^2_{\cos, n} \cos(n\psi_2)
\end{align*}
\]

From the general definitions in (4.3.11.a), the expressions which fulfil the requirements formulated in (4.3.9) are obtained by skipping the odd sine terms and the even
cosine terms in $U^1(\psi_1)$, as well as the even sine terms and the odd cosine terms in both $V^1(\psi_1)$ and $W^1(\psi_1)$. Thus, the following terms are retained

\[
\begin{align*}
U^1(\psi_1) &= U^1_{\cos,0} \cos(\psi_1) + U^1_{\sin,2} \sin(2\psi_1) + \ldots \\
V^1(\psi_1) &= V^1_{\cos,0} + V^1_{\sin,1} \sin(\psi_1) + V^1_{\cos,2} \cos(2\psi_1) + \ldots \\
W^1(\psi_1) &= W^1_{\cos,0} + W^1_{\sin,1} \sin(\psi_1) + W^1_{\cos,2} \cos(2\psi_1) + \ldots 
\end{align*}
\]

(4.3.12.a)

The class of problems to be analysed will be further limited due to the assumption:

\[
\text{shell (4): symmetry with respect to } \psi_2 = \pi/2.
\]

This means that, from the complete Fourier series expansions in (4.3.11.b), the even sine terms and the odd cosine terms in both $U^2(\psi_2)$ and $W^2(\psi_2)$, as well as the odd sine terms and the even cosine terms in $V^2(\psi_2)$ are skipped. Thus, the following terms in $\psi_2$ are retained only:

\[
\begin{align*}
U^2(\psi_2) &= U^2_{\cos,0} + U^2_{\sin,1} \sin(\psi_2) + U^2_{\cos,2} \cos(2\psi_2) + \ldots \\
V^2(\psi_2) &= V^2_{\cos,1} \cos(\psi_2) + V^2_{\sin,2} \sin(2\psi_2) + \ldots \\
W^2(\psi_2) &= W^2_{\cos,0} + W^2_{\sin,1} \sin(\psi_2) + W^2_{\cos,2} \cos(2\psi_2) + \ldots 
\end{align*}
\]

(4.3.12.b)

At this stage, the components ($U^8, V^8, W^8$) are completely determined by substitution of (4.3.12) into (4.3.10). The actual degrees-of-freedom which determine the "shell deformation" are introduced as the multiplications of the participation factors in (4.3.12.a) and (4.3.12.b). This yields

\[
U^8 = \{U_{10} + U_{11} \sin(\psi_2) + U_{12} \cos(2\psi_2) + \ldots \} \cos(\psi_1) + \\
+ \{U_{20} + U_{21} \sin(\psi_2) + \ldots \} \sin(2\psi_1) + \\
+ \{U_{30} + \ldots \} \cos(3\psi_1) + \ldots \ldots \\
V^8 = \{V_{01} \cos(\psi_2) + V_{02} \sin(2\psi_2) + V_{03} \cos(3\psi_2) + \ldots \} + \\
+ \{V_{11} \cos(\psi_2) + V_{12} \sin(2\psi_2) + \ldots \} \sin(\psi_1) + \\
+ \{V_{21} \cos(\psi_2) + \ldots \} \cos(2\psi_1) + \ldots \ldots \\
W^8 = \{W_{00} + W_{01} \sin(\psi_2) + W_{02} \cos(2\psi_2) + \ldots \} + \\
+ \{W_{10} + W_{11} \sin(\psi_2) + \ldots \} \sin(\psi_1) + \\
+ \{W_{20} + \ldots \} \cos(2\psi_1) + \ldots \ldots 
\]
Next, requirement "shell (3)" is checked. This is carried out by checking whether $\Phi^{b.0}$ does not contain any deformation mode which is already incorporated in $\Phi^{b.0}$. For this reason, the general expressions for both these vectors are given first. Written in components on the \{ $\tilde{E}_k$ \}-basis, the vector $\Phi^{b.0}$ is obtained from (4.2.19) and (4.2.16); i.e.

$$\Phi^{b.0} \equiv \frac{\sin \Theta}{KB} \tilde{E}_1 + \left\{ 1 - \frac{\cos \Theta}{KB} + R \sin(\psi_2) \right\} \tilde{E}_2 + R \cos(\psi_2) \tilde{E}_3 \quad (4.3.14.a)$$

The vector $\Phi^{b.0}$ follows from (4.3.1) and (4.3.3). Thus,

$$\Phi^{b.0} \equiv U^s S^b(\psi_2) \tilde{E}_1 + \left\{ V^s \cos(\psi_2) + W^s \sin(\psi_2) \right\} \tilde{E}_2 +$$

$$+ \left\{ - V^s \sin(\psi_2) + W^s \cos(\psi_2) \right\} \tilde{E}_3 \quad (4.3.14.b)$$

The underlined terms in (4.3.14.a) capture the deformation modes incorporated in the beam deformation. Insight into the deformation modes incorporated in the shell deformation is obtained by substitution of (4.3.13) into (4.3.14.b).

The deformation mode captured by the first underlined term in (4.3.14.a) is non-vanishing at $\xi_1 = \pm \gamma_2 \ell$. Since, due to (4.3.9), $U^s(\xi_1 = \pm \gamma_2 \ell) = 0$, the shell deformation captured by $U^s$ will never violate "shell (3)". Considering the components $V^s$ and $W^s$, it is concluded that the higher order sin/cos terms will never violate "shell (3)", since such terms do not feature in $\Phi^{b.0}$. Further, it is concluded that special care is required for the terms $V^s = V_{01} \cos(\psi_2)$ and $W^s = W_{01} \sin(\psi_2)$ in $\Phi^{b.0}$. These terms are examined further:

$$\left\{ V_{01} \cos^2(\psi_2) + W_{01} \sin^2(\psi_2) \right\} \tilde{E}_2 +$$

$$+ \left\{ ( - V_{01} + W_{01} ) \sin(\psi_2) \cos(\psi_2) \right\} \tilde{E}_3 =$$

$$= \left\{ \frac{1}{2} ( V_{01} + W_{01} ) + \frac{1}{2} ( V_{01} - W_{01} ) ( \cos^2(\psi_2) - \sin^2(\psi_2) ) \right\} \tilde{E}_2 +$$

$$+ \left\{ ( - V_{01} + W_{01} ) \sin(\psi_2) \cos(\psi_2) \right\} \tilde{E}_3$$

The underlined term in (4.3.15.a) leads to a similar deformation mode as the second underlined term in (4.3.14.a) and is therefore not independent. For this reason, it is required that

$$\frac{1}{2} ( V_{01} + W_{01} ) = 0 \quad (4.3.15.b)$$

which is satisfied by taking
\[ V_{01} = V W_{01} \quad \& \quad W_{01} = -V W_{01} \quad (4.3.15.c) \]

This essentially defines another degree-of-freedom (\( V W_{01} \)).

As a recapitulation of the present chapter, the following summary of the results is given.

**Summary** (for the position vector to the deformed mid-surface)  
(4.3.16)

By definition, see (4.1.1) and (4.3.1),

\[ \vec{\Phi}^o = \vec{\Phi}^{n.0} + U^3 \vec{B}_1 + V^3 \vec{B}_2 + W^3 \vec{B}_3 \]

where the position vector to the mid-surface of the beam follows from (4.2.19); i.e.

\[ \vec{\Phi}^{n.0} = \frac{1}{K^B} \left\{ \sin \Theta \vec{I}_1 - (1 - \cos \Theta) \vec{I}_2 \right\} + \]

\[ + R \left\{ \sin (\psi_2) \left\{ \sin \Theta \vec{I}_1 + \cos \Theta \vec{I}_2 \right\} + \cos (\psi_2) \vec{I}_3 \right\} \]

with \( \Theta = \frac{E}{K} \Theta^B \) and \( \frac{1}{K^B} \Theta^B = \zeta^B \ell \).

The relation of the basis \( \{ \vec{E}_k \} \) with the fixed Cartesian basis \( \{ \vec{I}_k \} \) is stated in (4.2.16):

\[
\begin{align*}
\vec{E}_1 &= \cos \Theta \vec{I}_1 - \sin \Theta \vec{I}_2 \\
\vec{E}_2 &= \sin \Theta \vec{I}_1 + \cos \Theta \vec{I}_2 \\
\vec{E}_3 &= \vec{I}_3
\end{align*}
\]

The basis \( \{ \vec{B}_1 \} \) is given by, see (4.3.3),

\[
\begin{align*}
\vec{B}_1 &= \zeta^B \left\{ 1 + K^B R \sin (\psi_2) \right\} \vec{E}_1 = S^B(\psi_2) \vec{E}_1 \\
\vec{B}_2 &= \cos (\psi_2) \vec{E}_2 - \sin (\psi_2) \vec{E}_3 \\
\vec{B}_3 &= \sin (\psi_2) \vec{E}_2 + \cos (\psi_2) \vec{E}_3
\end{align*}
\]

The components of the "shell deformation" follow from (4.3.13), together with (4.3.15.c); i.e.

(continued on next page)
\[ U^s = \left\{ U_{10} + U_{11} \sin(\psi_2) + U_{12} \cos(2\psi_2) + U_{13} \sin(3\psi_2) + \ldots \right\} \cos(\psi_1) + \]
\[ + \left\{ U_{20} + U_{21} \sin(\psi_2) + U_{22} \cos(2\psi_2) + \ldots \right\} \sin(2\psi_1) + \]
\[ + \left\{ U_{30} + U_{31} \sin(\psi_2) + \ldots \right\} \cos(3\psi_1) + \ldots \ldots \ldots \]
\[ V^s = \left\{ VW_{01} \cos(\psi_2) + VW_{02} \sin(2\psi_2) + VW_{03} \cos(3\psi_2) + VW_{04} \sin(4\psi_2) + \ldots \right\} + \]
\[ + \left\{ VW_{11} \cos(\psi_2) + VW_{12} \sin(2\psi_2) + VW_{13} \cos(3\psi_2) + \ldots \right\} \sin(\psi_1) + \]
\[ + \left\{ VW_{21} \cos(\psi_2) + VW_{22} \sin(2\psi_2) + \ldots \right\} \cos(2\psi_1) + \ldots \ldots \ldots \]
\[ W^s = \left\{ W_{00} - VW_{01} \sin(\psi_2) + W_{02} \cos(2\psi_2) + W_{03} \sin(3\psi_2) \ldots \right\} + \]
\[ + \left\{ W_{10} + W_{11} \sin(\psi_2) + W_{12} \cos(2\psi_2) + \ldots \right\} \sin(\psi_1) + \]
\[ + \left\{ W_{20} + W_{21} \sin(\psi_2) + \ldots \right\} \cos(2\psi_1) + \ldots \ldots \ldots \]

The angles \( \psi_1 \) and \( \psi_2 \) are defined by

\[
\psi_1 = \pi - \frac{\xi_1}{\ell} \quad \text{and} \quad \psi_2 = \frac{\xi_2}{R}.
\]

The components of \( \bar{\psi}^{s,o} \) (being \( u^s, v^s, w^s \)) are also developed in Fourier series expansions. A similar treatment as for the components (\( U^s, V^s, W^s \)) as discussed above is employed for (\( u^s, v^s, w^s \)). Therefore, the position vector to the reference mid-surface \( \bar{\psi} \) is obtained in a similar fashion as explained in the summary (4.3.16) by taking the corresponding quantities in the reference configuration.

In Chapter 2, dealing with the shell formulation, a theory is discussed which makes use of strain measures associated with the deformation of the mid-surface. In section 2.4 (about strain components), it is shown that the components \( \varepsilon_{\alpha\beta} \) and \( \rho_{\alpha\beta}^s \) are determined by the first and second derivatives of the position vector to the mid-surface in both the reference and the deformed configurations. Thus, from the Summary (4.3.16), components \( \varepsilon_{\alpha\beta} \) and \( \rho_{\alpha\beta}^s \) can be derived. At the moment, this derivation will not be discussed, but is postponed until Chapter 6, dealing with the solution strategy. The other strain measure (\( \chi_s \)) follows by comparison of the thickness function in the reference and the deformed configurations. The evaluation of this strain measure is discussed in the next section.

### 4.4 - Thickness function

As mentioned, the model is developed for the detailed analysis of a certain section of a total tube configuration. For reasons of compatibility of this tube model with the remainder of
the total tube, the end-sections are enforced to remain plane. In relation to this, it is required for

\textbf{thickn (1): the thickness to have a vanishing gradient in the centroidal direction at both end-sections.}

Thus,

\[ \Lambda_{1} = 0 \quad \text{at} \quad \xi^{1} = \pm \frac{\ell}{2} \]  \hspace{1cm} (4.4.1)

Based on a similar treatment as for the shell deformation, a Fourier series expansion is applied for the thickness function \( \Lambda \). Thus,

\[ \Lambda(\psi_{1}, \psi_{2}) = \Lambda^{1}(\psi_{1}) \Lambda^{2}(\psi_{2}) \]  \hspace{1cm} (4.4.2)

where

\[
\begin{align*}
\Lambda^{1}(\psi_{1}) &= \sum_{m=1}^{\infty} A_{\sin, m}^{1} \sin(m\psi_{1}) + \sum_{m=0}^{\infty} A_{\cos, m}^{1} \cos(m\psi_{1}) \\
\Lambda^{2}(\psi_{2}) &= \sum_{n=1}^{\infty} A_{\sin, n}^{2} \sin(n\psi_{2}) + \sum_{n=0}^{\infty} A_{\cos, n}^{2} \cos(n\psi_{2})
\end{align*}
\]  \hspace{1cm} (4.4.3)

In the result which satisfies (4.4.1), the even sine terms and the odd cosine terms in \( \Lambda^{1}(\psi_{1}) \), as stated in (4.4.3), should be skipped.

In addition, the deformation is required to be

\textbf{thickn (2): symmetric with respect to } \psi_{2} = \pi / 2. \text{ }

This means that the even sine terms and the odd cosine terms in \( \Lambda^{2}(\psi_{2}) \), as stated in (4.4.3), should be skipped.

Again, the actual degrees-of-freedom are defined as the multiplications of the participation factors of the Fourier series expansions. Thus,

\[ \Lambda = \left\{ \Lambda_{00} + \Lambda_{01} \sin(\psi_{2}) + \Lambda_{02} \cos(2\psi_{2}) + \Lambda_{03} \sin(3\psi_{2}) + \ldots \right\} + \\
+ \left\{ \Lambda_{10} + \Lambda_{11} \sin(\psi_{2}) + \Lambda_{12} \cos(2\psi_{2}) + \ldots \right\} \sin(\psi_{1}) + \\
+ \left\{ \Lambda_{20} + \Lambda_{21} \sin(\psi_{2}) + \ldots \right\} \cos(2\psi_{1}) + \ldots \ldots \ldots \]  \hspace{1cm} (4.4.4)

By employing a similar treatment, the thickness function in the reference configuration \( \lambda(\lambda) \) is obtained. The strain measure \( \chi_{3} \) is obtained from the thickness functions \( \Lambda \) and \( \lambda \) by virtue of (2.4.13).
Chapter 5 - Loading conditions

5.1 - Introduction

In practical situations, the loading history on a tube is a sequence of combinations of basic loading conditions. As an example, the installation of an offshore pipeline followed by the production status is considered. During the installation, a combination of external pressure (due to the water pressure, since the tube is empty at that stage), bending and tension loads is applied. The latter two are induced by the laying process itself. After installation, when the production is started, an internal pressure is build up.

In the model, a distinction is made between "basic loads", a "load case" and the "loading history". The following basic loads are defined:

- pressure at the inner and/or the outer surface
- forces in the \( \vec{t}_1 \)-direction and in the \( \vec{t}_2 \)-direction at the end-sections
- moments in the \( \vec{t}_2 \)-direction at the end-sections

A load case is defined as either a basic load or a combination of basic loads. Furthermore, a sequence of load cases is called a loading history. So, the loading history of the tube is given by a (set of) load case(s), where a load case is defined by a (set of) basic load(s).

In Chapter 2, the principle of virtual work has been introduced, see expression (2.5.1). This expression states that (given equilibrium) the virtual work of the internal stresses under arbitrary kinematically admissible virtual displacements equals the virtual work done by the external loading on the system. Chapter 2 is concerned with the internal virtual work. The present chapter is focusing on external virtual work expressions associated with the basic loads.

The external virtual work, determined by loads applied at the end-sections of the tube and by pressure loading applied at the inner and/or the outer surface of the tube, is written as

\[
\delta W_{\text{ext}} = \delta W_{\text{end}} + \delta W_{\text{surf}} = \sum_{k=1}^{NF} \lambda^{(k)} \delta W^{(k)}_{\text{ext}}, \quad (5.1.1)
\]
where NF is the total number of basic loads defined in the model. The details about $\delta W_{\text{end}}$, $\delta W_{\text{surf}}$, $\lambda^{(k)}$ and $\delta W_{\text{ext}}^{(k)}$ are discussed in sections 5.2 and 5.3. In the model, the external virtual work due to volume forces is left out of consideration.

### 5.2 - End loading

At both the end-sections of the tube, a force vector as well as a moment vector may be applied. In the sequel, the applied force and moment at both end-sections will be denoted with the vectors $\overline{F}(\pm \ell/2)$ and $\overline{M}(\pm \ell/2)$, respectively. It is assumed that the directions of these vectors are constant.

The discussion of the end loading is started with the external virtual work done by the force vectors $\overline{F}(\pm \ell/2)$. This virtual work is determined by

$$
\delta W_{\text{end}} = \overline{F}(-\ell/2) \cdot \delta \overline{\Theta}^M(-\ell/2) + \overline{F}(+\ell/2) \cdot \delta \overline{\Theta}^M(+\ell/2)
$$

(5.2.1)

The components of the vectors $\overline{F}(\pm \ell/2)$ are defined on the fixed $\{\overline{I}_k\}$-basis; i.e.

$$
\overline{F}(\pm \ell/2) = F_{k \ell/2} \overline{I}_k
$$

(5.2.2)

where these components of the force vectors are used for the definition of the load parameters in the model. The virtual displacements $\delta \overline{\Theta}^M$ at $\xi^1 = \pm \ell/2$ follow from the mean position vector. This vector has been defined in Chapter 4. The expression of the vector $\overline{\Theta}^M$ is worked out from (4.3.4) together with the definition for $\overline{\Theta}^0$ in the summary (4.3.16). The result is

$$
\overline{\Theta}^M = \frac{1}{K_B} \left\{ \sin \Theta \overline{I}_1 - (1 - \cos \Theta) \overline{I}_2 \right\} + \\
+ \xi^B \left\{ \left( (U_{10} + \frac{1}{2} K_BRU_{11}) \cos(\psi_1) + (U_{20} + \frac{1}{2} K_BRU_{21}) \sin(2\psi_1) + \right) (U_{30} + \frac{1}{2} K_BRU_{31}) \cos(3\psi_1) + \ldots \right\} \left\{ \cos \Theta \overline{I}_1 - \sin \Theta \overline{I}_2 \right\} + \\
+ \frac{1}{2} \left\{ (V_{11} + W_{11}) \sin(\psi_1) + \right\} \left\{ \sin \Theta \overline{I}_1 + \cos \Theta \overline{I}_2 \right\}
$$

(5.2.3)

The virtual displacements at both the end-sections, following from (5.2.3), are
\[
\delta \tilde{\Theta}^B(\pm \ell/2) = \ell \left\{ \frac{\sin(\pm \Theta^B/2)}{\Theta^B} \tilde{I}_1 - \left( \frac{1 - \cos(\pm \Theta^B/2)}{\Theta^B} \right) \tilde{I}_2 \right\} \delta \zeta^B + \\
+ \left\{ \zeta^B \ell \left[ \frac{1}{2} \Theta^B \cos(\pm \Theta^B/2) - \sin(\pm \Theta^B/2) \right] \tilde{I}_1 + \\
+ \zeta^B \ell \frac{1}{2} \Theta^B \sin(\pm \Theta^B/2) \left( 1 - \cos(\pm \Theta^B/2) \right) \tilde{I}_2 + \\
+ \left\{ \frac{1}{2} \left[ (V_{11} + W_{11}) \sin(\pm \pi/2) + \\
+ (V_{21} + W_{21}) \cos(\pm 2 \pi/2) + \\
+ \ldots \ldots \right] \right\} \left[ \frac{1}{2} \cos(\pm \Theta^B/2) \tilde{I}_1 \mp \frac{1}{2} \sin(\pm \Theta^B/2) \tilde{I}_2 \right] \right\} \delta \Theta^B + \\
+ \frac{1}{2} \left\{ (\delta V_{11} + \delta W_{11}) \sin(\pm \pi/2) + \\
+ (\delta V_{21} + \delta W_{21}) \cos(\pm 2 \pi/2) + \\
+ \ldots \ldots \right\} \left\{ \sin(\pm \Theta^B/2) \tilde{I}_1 + \cos(\pm \Theta^B/2) \tilde{I}_2 \right\}
\]
\]

Since \( \Theta^B \) is in the denominator of some terms in the above expression, special care is required in the case of a (nearly) straight tube. In this case,

\[
\text{for small } \Theta^B \Rightarrow \left\{ \begin{array}{l}
\frac{\sin(\pm \Theta^B/2)}{\Theta^B} \equiv \pm \frac{1}{2} \\
\frac{1 - \cos(\pm \Theta^B/2)}{\Theta^B} \equiv \Theta^B/8 \\
\frac{\pm \frac{1}{2} \Theta^B \cos(\pm \Theta^B/2) - \sin(\pm \Theta^B/2)}{(\Theta^B)^2} \equiv \mp \frac{\Theta^B}{24}
\end{array} \right. \]

In these approximations, the terms quadratic-in-\( \Theta^B \) as well as the higher order terms in \( \Theta^B \) are neglected. In the implementation of the model, the above approximations are taken for \( \Theta^B < 10^{-2} \).

Next, the external virtual work done by the moment vectors \( \tilde{M}(\pm \ell/2) \) is considered. This is determined by

\[
\delta W_{ext} = \tilde{M}(-\ell/2) \cdot \delta \tilde{\Theta}(-\ell/2) + \tilde{M}(+\ell/2) \cdot \delta \tilde{\Theta}(+\ell/2) \]

\( (5.2.6.a) \)
where $\delta \Theta(\pm \ell/2)$ denote the virtual rotations at $\xi^1 = \pm \ell/2$. From the discussion in Chapter 4, it is concluded that the virtual rotation is

$\begin{align*}
+\delta \Theta^B/2 & \text{ in the $\vec{T}_s$-direction for the end plane at $\xi^1 = -\ell/2$ and} \\
-\delta \Theta^B/2 & \text{ in the $\vec{T}_s$-direction for the end plane at $\xi^1 = +\ell/2$.}
\end{align*}$

Thus, the virtual work due to the moment vectors $\mathbf{M}(\pm \ell/2)$ reads

$$\delta W_{\text{end}} = M_{e_s}^3 \frac{\delta \Theta^B}{2} - M_{s2}^3 \frac{\delta \Theta^B}{2}, \quad (5.2.6.b)$$

where $M_{e_s}^3$ are the components of $\mathbf{M}(\pm \ell/2)$ in the $\vec{T}_s$-direction.

In the implementation of the model, dimensionless load parameters are used. These are defined by scaling the components of the force vectors $\mathbf{F}(\pm \ell/2)$ and the moment vectors $\mathbf{M}(\pm \ell/2)$ with the values of $N_{\text{yield}}$ and $M_{\text{yield}}$, respectively, which are defined by

$$N_{\text{yield}} = \sigma_y 2\pi R t \quad \text{and} \quad M_{\text{yield}} = \sigma_y \pi R^2 t \quad (5.2.7)$$

The value $N_{\text{yield}}$ represents the axial force which, in the case of pure axial stretch of a straight thin-walled tube, with a radius $R$ and a thickness $t$, leads to first yielding of the material. The value of $M_{\text{yield}}$ is the bending moment which, in the case of pure bending of the above mentioned tube, leads to first yielding of the material in the tube wall.

Finally, the result is presented in the format of the last expression in (5.1.1):

$$\lambda^{(i)} = \frac{F_{e_s}^i}{N_{\text{yield}}} \quad ; \quad \delta W^{(i)}_{\text{ext}} = N_{\text{yield}} \left( \delta \Theta^B_0(\pm \ell/2) \cdot \vec{T}_s \right), \quad (5.2.8.a)$$

$$\lambda^{(2)} = \frac{F_{s2}^i}{N_{\text{yield}}} \quad ; \quad \delta W^{(2)}_{\text{ext}} = N_{\text{yield}} \left( \delta \Theta^B_0(-\ell/2) \cdot \vec{T}_s \right), \quad (5.2.8.b)$$

$$\lambda^{(3)} = \frac{F_{e_s}^2}{N_{\text{yield}}} \quad ; \quad \delta W^{(3)}_{\text{ext}} = N_{\text{yield}} \left( \delta \Theta^B_0(\pm \ell/2) \cdot \vec{T}_s \right), \quad (5.2.8.c)$$

$$\lambda^{(4)} = \frac{F_{s2}^2}{N_{\text{yield}}} \quad ; \quad \delta W^{(4)}_{\text{ext}} = N_{\text{yield}} \left( \delta \Theta^B_0(-\ell/2) \cdot \vec{T}_s \right), \quad (5.2.8.d)$$

$$\lambda^{(5)} = \frac{M_{e_s}^3}{M_{\text{yield}}} \quad ; \quad \delta W^{(5)}_{\text{ext}} = -M_{\text{yield}} \frac{\delta \Theta^B}{2}, \quad (5.2.8.e)$$

$$\lambda^{(6)} = \frac{M_{s2}^3}{M_{\text{yield}}} \quad ; \quad \delta W^{(6)}_{\text{ext}} = M_{\text{yield}} \frac{\delta \Theta^B}{2}, \quad (5.2.8.f)$$

A load-case should be defined as an equilibrium system of the above defined basic loads. The inclusion of such equilibrium equations in the governing set of equations is discussed in section 6.5.
5.3 - Surface loading

In this section, the surface loading due to internal/external pressure is considered. This section starts with a general discussion of the external virtual work due to pressure loading on a surface. After this, explicit expressions for the load cases implemented in the model are given.

Consider the surface $S$, which is the surface where the pressure loading is applied and an infinitesimal part of $S$ (denoted with $dS$), see figure 5.1. The surface $S$ is taken in the deformed configuration.

![Figure 5.1 - The surface $S$ where the pressure loading is applied, the infinitesimal part $dS$ and the normal to $dS$.](image)

The expression for the infinitesimal surface element $dS$ reads

$$dS = \| \bar{\mathbf{G}}_1 \times \bar{\mathbf{G}}_2 \| \, d\xi^1 \, d\xi^2$$

where $\xi^a$ are the material coordinates as shown in figure 5.1. The vectors $\bar{\mathbf{G}}_\alpha$ are the base vectors in the deformed configuration, given in (2.2.9). These vectors read

$$\bar{\mathbf{G}}_\alpha = \bar{\mathbf{A}}_\alpha + \xi^\alpha \Lambda^{\alpha}_{\beta} \bar{\mathbf{N}}_{,\beta} + \xi^\alpha \Lambda_{,\alpha} \bar{\mathbf{N}}$$

For the internal surface, $\xi = -1/2$, while for the external surface $\xi = 1/2$. The influence of the underlined term is neglected, which is based on a similar smoothness assumption for $\Lambda$ as discussed in section 2.4; i.e. $\Lambda_{,\alpha} \ll 1$.

The vector $\bar{\mathbf{q}}$ is introduced as representing the pressure loading on the surface element $dS$. Physically, the pressure "follows" the surface, which means that the vector $\bar{\mathbf{q}}$
is perpendicular to \( dS \) at any stage of deformation. Denoting the value of the internal pressure with \( p_{\text{int}} \) and the value of the external pressure with \( p_{\text{ext}} \), we have

\[
\bar{q}_{\text{int}} = p_{\text{int}} \frac{\overline{G}_1 \times \overline{G}_2}{\overline{G}_1 \times \overline{G}_2} \bigg|_{\xi = -1/2} ; \quad \bar{q}_{\text{ext}} = -p_{\text{ext}} \frac{\overline{G}_1 \times \overline{G}_2}{\overline{G}_1 \times \overline{G}_2} \bigg|_{\xi = 1/2}
\]

(5.3.3)

The virtual work due to the pressure loading on the internal surface and the external surface is given by

\[
\delta W_{\text{surf}} = \int s \bar{q}_{\text{int}} \cdot \delta \tilde{\Phi} \, dS \bigg|_{\xi = -1/2} + \int s \bar{q}_{\text{ext}} \cdot \delta \tilde{\Phi} \, dS \bigg|_{\xi = 1/2}
\]

(5.3.4)

The virtual displacements \( \delta \tilde{\Phi} \) are derived from (2.2.8); i.e.

\[
\delta \tilde{\Phi} = \delta \tilde{\Phi}^\circ + \xi \left( \delta \Lambda \tilde{N} + \Lambda \delta \tilde{N} \right)
\]

(5.3.5)

Substitution of (5.3.1) and (5.3.3) into (5.3.4) yields

\[
\delta W_{\text{surf}} = \iint p_{\text{int}} \left( \overline{G}_1 \times \overline{G}_2 \right) \cdot \delta \tilde{\Phi} \bigg|_{\xi = -1/2} \, d\xi^1 \, d\xi^2 + \iint p_{\text{ext}} \left( \overline{G}_1 \times \overline{G}_2 \right) \cdot \delta \tilde{\Phi} \bigg|_{\xi = 1/2} \, d\xi^1 \, d\xi^2
\]

(5.3.6)

Dimensionless load parameters are introduced by scaling the value of the internal and external pressure with a reference value \( p_{\text{yield}} \). This value is the internal pressure at which, for a straight thin-walled tube with radius \( R \) and thickness \( t \), yielding of the material starts to occur. Thus,

\[
\lambda^{(7)} = \frac{p_{\text{int}}}{p_{\text{yield}}} \quad , \quad \lambda^{(8)} = \frac{p_{\text{ext}}}{p_{\text{yield}}} \quad \text{where} \quad p_{\text{yield}} = \sigma_y \frac{t}{R}
\]

(5.3.7)

Finally, the external virtual work done by an internal pressure is written in terms of the notation in (5.1.1). Using the definition of \( \lambda^{(7)} \) and \( \lambda^{(8)} \) in (5.3.7), \( \delta W_{\text{ext}}^{(7)} \) and \( \delta W_{\text{ext}}^{(8)} \) are determined by

\[
\delta W_{\text{ext}}^{(7)} = p_{\text{yield}} \iint \left( \overline{G}_1 \times \overline{G}_2 \right) \cdot \delta \tilde{\Phi} \bigg|_{\xi = -1/2} \, d\xi^1 \, d\xi^2
\]

(5.3.8.a)

\[
\delta W_{\text{ext}}^{(8)} = -p_{\text{yield}} \iint \left( \overline{G}_1 \times \overline{G}_2 \right) \cdot \delta \tilde{\Phi} \bigg|_{\xi = 1/2} \, d\xi^1 \, d\xi^2
\]

(5.3.8.b)
Chapter 6 - Solution strategy

6.1 - Introduction

As mentioned in Chapter 2 and Chapter 5, equilibrium of the tube under a given loading condition is captured by the principle of virtual work. This principle has been introduced in a general form in section 2.5, see (2.5.1). The virtual work equation states that, given equilibrium, the internal virtual work equals the external virtual work for arbitrary kinematically admissible displacements. The expression for the internal virtual work, embedding the assumed kinematics, is stated by (2.5.11). The general expression for the external virtual work is formulated in (5.1.1), where the definitions of the end loading and the surface loading are given in (5.2.8) and (5.3.7) - (5.3.8). From the virtual work equation, a governing set of equations is obtained. In the present chapter, the strategy for solving this governing set of equations for a certain loading history is discussed.

In the present application, the governing set of equations is strongly non-linear. In this context, Newton-Raphson methods are widely used. In the solution, limit points and bifurcation points may occur. Therefore, it is important to apply stable and efficient methods for solving the governing equations. In cases where limit points and bifurcation points may occur, adequate continuation methods are required in order to pass these points. Such continuation methods are firstly proposed in a mechanics formulation by WEMPNER (1971) and RIKS (1972). Since then, a large variety of methods based on this proposal is published. In CRISFIELD (1991), a comprehensive review on continuation methods is given.

In analyses with a single load parameter, the above mentioned solution strategy works satisfactory. In that case, the governing set of equations is augmented with one equation, which captures the continuation method. However, in the present model, multiple loading parameters may be used. In that case, the governing set of equations should be augmented with a set of equations. One of the equations in this set captures the continuation method. The other equations are either used for the definition of the relations between the loading parameters, or for the definition of certain boundary conditions. In DUFFETT & REDDY (1986), a set of equations with a similar structure is considered. In their article, particular attention is paid to the bifurcation behaviour. In the present chapter, attention is focused solely on tracing of the solution curve which is determined by such an augmented set.
6.2 - Augmented set of equations

In order to explain the solution strategy, it is convenient to work with general expressions of the governing equations. The detailed discussion of the individual equations is postponed to sections 6.4 - 6.6. Thus, in the present section and section 6.3, a mathematical formulation based on general expressions is offered.

The principle of virtual work is phrased into a finite set of equations as follows. Introduce the "column vectors" $\tilde{D}$ and $\tilde{L}$, which contain the degrees-of-freedom of the model and the load parameters, respectively. Herewith, the following general expression for the principle of virtual work is obtained:

$$\delta W = \delta W_{\text{int}} - \delta W_{\text{ext}} \equiv \tilde{W}(\tilde{D}, \tilde{L}) \cdot \delta \tilde{D} = 0 \quad (6.2.1)$$

If, in this expression, $\delta \tilde{D}$ denote kinematically admissible variations of $\tilde{D}$, then it follows from (6.2.1) that

$$\tilde{W}(\tilde{D}, \tilde{L}) = \tilde{0} \quad (6.2.2)$$

Vector $\tilde{D}$ is assumed to be an ND-dimensional vector and $\tilde{L}$ an NF-dimensional vector. ND represents the total number of degrees-of-freedom in the model and NF the total number of load parameters. By definition ND $\geq$ 1 and NF $\geq$ 1. Set (6.2.2) constitute ND equations in (ND+NF) unknown parameters. In order to arrive at a solvable system, (6.2.2) is augmented with NF equations; i.e.

$$\begin{cases}
\tilde{W}(\tilde{D}, \tilde{L}) = \tilde{0} \\
\tilde{g}(\tilde{D}, \tilde{L}) = \tilde{0} \\
h(\tilde{D}, \tilde{L}, \tilde{S}) = 0
\end{cases} \quad (6.2.3)$$

Here, $\tilde{g} = \tilde{0}$ consist(s) of (NF-1) equations and give(s) the definition of a specific loading case. Therefore, $\tilde{g}$ is referred to as the "load-case function". The single value function $h = 0$ determines the continuation method which is used within the load-case. The scalar $\tilde{S}$ is introduced as an additional parameter and will be referred to as the "continuation parameter". The constraint equations which are implemented in the present tube model are based on load control, displacement control and arc-length control.

It is common practice to apply an incremental method in order to solve non-linear equations. This leads to a successive set of equilibrium states of the model. In the present work, a full Newton-Raphson iteration scheme is used. This scheme will be discussed in
section 6.3. In section 6.4, details will be presented about the evaluation of the vector $\tilde{W}$. A general discussion on the load-case function(s) $\tilde{g}$ will be given in section 6.5. Finally, the details of the continuation methods, captured by the function $h$, are presented in section 6.6.

### 6.3 - Full Newton-Raphson iteration process

The set of equations (6.2.3) is strongly non-linear. It is common practice in computational mechanics that for such non-linear equations, a Newton-Raphson iteration process is adopted. This process is based on a linearisation of (6.2.3) about a known solution. Suppose this solution to be $(\tilde{\mathbf{D}}^{(0)}$, $\tilde{\mathbf{L}}^{(0)}$, $\mathbf{s}^{(0)})$. The objective of the iteration process is to find, for $\mathbf{s}^{(0)} + \Delta \mathbf{s}$, a solution $(\tilde{\mathbf{D}}^{(0)} + \Delta \tilde{\mathbf{D}}$, $\tilde{\mathbf{L}}^{(0)} + \Delta \tilde{\mathbf{L}}$). For this purpose, the following Taylor series expansion of (6.2.3) is considered:

\[
\begin{bmatrix}
\tilde{W}^{(0)} \\
\tilde{g}^{(0)} \\
h^{(0)}
\end{bmatrix} +
\begin{bmatrix}
W_D^{(0)} & W_L^{(0)} \\
G_D^{(0)} & G_L^{(0)} \\
h_D^{(0)} & h_L^{(0)}
\end{bmatrix}
\begin{bmatrix}
\Delta \tilde{\mathbf{D}} \\
\Delta \tilde{\mathbf{L}}
\end{bmatrix} + \ldots = 0
\]

where the following definitions are used:

\[
W_D = \frac{\partial \tilde{W}}{\partial \tilde{D}}, \quad W_L = \frac{\partial \tilde{W}}{\partial \tilde{L}}
\]

\[
g_D = \frac{\partial \tilde{g}}{\partial \tilde{D}}, \quad g_L = \frac{\partial \tilde{g}}{\partial \tilde{L}}
\]

\[
h_D = \frac{\partial h}{\partial \tilde{D}}, \quad h_L = \frac{\partial h}{\partial \tilde{L}}
\]

The superscript $(0)$ indicates that the corresponding quantity is evaluated for $(\tilde{\mathbf{D}}^{(0)}$, $\tilde{\mathbf{L}}^{(0)}$, $\mathbf{s}^{(0)} + \Delta \mathbf{s})$. The notation $\Delta \tilde{\mathbf{D}}$ represents a finite difference of the degrees-of-freedom from the known values in $\tilde{\mathbf{D}}^{(0)}$. Likewise, $\Delta \tilde{\mathbf{L}}$ represents a finite difference of the loading parameters from the known values in $\tilde{\mathbf{L}}^{(0)}$.

By truncating the Taylor series to the terms which are shown in (6.3.1.a) and solving the resulting system of equations, a solution for $(\Delta \tilde{\mathbf{D}}$, $\Delta \tilde{\mathbf{L}}$) is obtained. From this, the solution $(\tilde{\mathbf{D}}^{(0)} + \Delta \tilde{\mathbf{D}}$, $\tilde{\mathbf{L}}^{(0)} + \Delta \tilde{\mathbf{L}}$, $\mathbf{s}^{(0)} + \Delta \mathbf{s}$) follows. Since, in general, this solution will not satisfy (6.2.3), an iteration process is required. A Newton-Raphson iteration scheme states that the values of $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{L}}$ at the successive iterations $(\text{iter} = 1$, $(\text{iter} = 2)$ etc. are obtained according to the following scheme:
REPEAT (for $\Delta$ fixed)

$$\begin{bmatrix} W_D^{(\text{itr})} & W_L^{(\text{itr})} \\ g_D^{(\text{itr})} & g_L^{(\text{itr})} \\ h_D^{(\text{itr})} & h_L^{(\text{itr})} \end{bmatrix} \cdot \begin{bmatrix} \Delta^{(\text{itr})} \tilde{D} \\ \Delta^{(\text{itr})} \tilde{L} \end{bmatrix} = \begin{bmatrix} -\tilde{W}^{(\text{itr})} \\ -\tilde{g}^{(\text{itr})} \\ -\tilde{h}^{(\text{itr})} \end{bmatrix}$$ (6.3.2)

with the update

$$\begin{cases} \tilde{D}^{(\text{itr+1})} := \tilde{D}^{(\text{itr})} + \Delta^{(\text{itr})} \tilde{D} \\ \tilde{L}^{(\text{itr+1})} := \tilde{L}^{(\text{itr})} + \Delta^{(\text{itr})} \tilde{L} \end{cases}$$ (6.3.3)

$$\text{itr} := \text{itr} + 1$$

UNTIL CONVERGENCE.

More details about the convergence criterion are discussed in section 6.3.1. The matrix $W_D$, as stated in (6.3.2), is called the "tangent stiffness matrix", whereas the complete matrix at the left-hand side of (6.3.2) will be referred to as the "system matrix".

Several options for the frequency of the evaluation of the tangent stiffness matrix have been proposed in literature, see e.g. the text-book of CRISFIELD (1991). In the present work, this matrix is explicitly evaluated for every iteration within an increment. Other techniques, so-called quasi Newton-Raphson schemes, do not require the explicit evaluation of the tangent stiffness matrix for every iteration. In these schemes, the computational effort per iteration is usually less than in a full Newton-Raphson iteration scheme. However, the convergence characteristics are generally worse.

A partitioning method will be used for solving (6.3.2). Therefore, the upper relation in (6.3.2) is worked out such that an expression for $\Delta^{(\text{itr})} \tilde{D}$ is obtained. Thus,

$$\Delta^{(\text{itr})} \tilde{D} = \{W_D^{(\text{itr})}\}^{-1} \cdot \left\{ -\tilde{W}^{(\text{itr})} - W_L^{(\text{itr})} \cdot \Delta^{(\text{itr})} \tilde{L} \right\}$$ (6.3.4)

This expression substituted in the lower two relations in (6.3.2) yields

$$\begin{bmatrix} g_L^{(\text{itr})} - g_D^{(\text{itr})} \cdot \{W_D^{(\text{itr})}\}^{-1} \cdot W_L^{(\text{itr})} \\ h_L^{(\text{itr})} - h_D^{(\text{itr})} \cdot \{W_D^{(\text{itr})}\}^{-1} \cdot W_L^{(\text{itr})} \end{bmatrix} \cdot \Delta^{(\text{itr})} \tilde{L} = \begin{bmatrix} -\tilde{g}^{(\text{itr})} + g_D^{(\text{itr})} \cdot \{W_D^{(\text{itr})}\}^{-1} \cdot \tilde{W}^{(\text{itr})} \\ -\tilde{h}^{(\text{itr})} + h_D^{(\text{itr})} \cdot \{W_D^{(\text{itr})}\}^{-1} \cdot \tilde{W}^{(\text{itr})} \end{bmatrix}$$ (6.3.5)

The sets of equations (6.3.4) and (6.3.5) are solved in terms of $(\Delta^{(\text{itr})} \tilde{D}, \Delta^{(\text{itr})} \tilde{L})$ for each iteration.

Note that the number of equations in (6.3.5) is NF, which is the total number of load parameters. This is generally a relatively small number as compared to ND. Therefore,
(6.3.5) gives a relatively small set of equations. In general, the matrix at the left-hand side of (6.3.5) is not symmetric. As will be discussed in section 6.4, the tangent stiffness matrix \( W_D^{(ir)} \) is symmetric. Full advantage of this symmetry is taken by solving (6.3.4) and (6.3.5) according to the following sequence:

1. Apply Crout decomposition of the tangent stiffness matrix \( W_D^{(ir)} \). For a discussion on the Crout decomposition, see e.g. HUGHES (1987).

2. From the decomposition, work out \( [W_D^{(ir)}]^{-1} \cdot W_L^{(ir)} \) as well as \( [W_D^{(ir)}]^{-1} \cdot \vec{w}^{(ir)} \) by means of back-substitution.

3. Calculate \( \Delta^{(ir)} \vec{L} \) from solving the relatively small \((NF \times NF)\)-system (6.3.5).

4. Substitute \( \Delta^{(ir)} \vec{L} \) in (6.3.4) and derive \( \Delta^{(ir)} \vec{D} \).

### 6.3.1 - Convergence criterion

In combination with the Newton-Raphson iteration process discussed above, a convergence criterion is applied which is based on the iterative change of both the degrees-of-freedom and the load parameters. The "iterative change" is introduced as the change within the current iteration; i.e. \( \Delta^{(ir)} \vec{D} \) and \( \Delta^{(ir)} \vec{L} \). These iterative changes are scaled with the respective "incremental changes", which are defined as the changes within the current increment; i.e. \( \vec{D}^{(ir)} - \vec{D}^{(0)} \) and \( \vec{L}^{(ir)} - \vec{L}^{(0)} \), respectively. Basically, the incremental change is the sum of the iterative changes within an increment.

The convergence criterion is defined as follows

\[
\frac{\| \Delta^{(ir)} \vec{D} \|}{\| \vec{D}^{(ir)} - \vec{D}^{(0)} \|} + \frac{\| \Delta^{(ir)} \vec{L} \|}{\| \vec{L}^{(ir)} - \vec{L}^{(0)} \|} < \varepsilon
\]

\[(6.3.6)\]

where \( \varepsilon \) is a small parameter. The norm \( \| (\cdot) \| \) in (6.3.6) is defined in (1.3.17) since all the involved vectors are so-called "column vectors". From numerical experience, it is observed that in general \( \varepsilon = 10^{-6} \) yields sufficiently accurate results.

### 6.4 - Basic equations

In the present section, the link is made between the vectors \( \vec{D} \), \( \vec{L} \) and \( \vec{W} \) and the matrices \( W_D \) and \( W_L \), used in the previous sections, on the one hand and the formulation of the tube model on the other hand.
Based on the discretisation, summarised in (4.3.16), the vector with the degrees-of-freedom, $\tilde{D}$, is taken according to

$$
\tilde{D} = \begin{bmatrix}
\zeta^B - 1 & \Theta^B & \frac{U_{10}}{R} & \frac{U_{11}}{R} & \ldots & \frac{VW_{q1}}{R} & \frac{V_{q2}}{R} & \ldots \\
\frac{W_{00}}{R} & \frac{W_{q2}}{R} & \ldots & \frac{\Lambda_{00}}{t} & \frac{\Lambda_{q1}}{t} & \ldots
\end{bmatrix}
$$

(6.4.1.a)

Thus, the first two components of vector $\tilde{D}$ contain information about the "beam deformation" and the other components contain the (normalised) participation factors from the "shell deformation". Vector $\tilde{L}$ contains the load parameters defined in (5.2.8) and (5.3.7) according to the following ordering:

$$
\tilde{L} = \begin{bmatrix}
\lambda^{(1)} & \lambda^{(2)} & \ldots & \lambda^{(N-1)} & \lambda^{(N)}
\end{bmatrix}
$$

(6.4.1.b)

The vector $\tilde{W}$ will be separated into two contributions, which follow from the internal virtual work and the external virtual work, respectively. Thus,

$$
\tilde{W} = \tilde{R}_{\text{int}} - \tilde{R}_{\text{ext}}
$$

(6.4.2.a)

The matrices $W_D$ and $W_L$ follow by differentiation of $\tilde{W}$ with respect to $\tilde{D}$ and $\tilde{L}$, respectively. For this purpose, the following notation is introduced:

$$
W_D = \frac{\partial \tilde{R}_{\text{int}}}{\partial \tilde{D}} - \frac{\partial \tilde{R}_{\text{ext}}}{\partial \tilde{D}} \equiv R_{\text{int,D}} - R_{\text{ext,D}}
$$

(6.4.2.b)

$$
W_L = \frac{\partial \tilde{R}_{\text{int}}}{\partial \tilde{L}} - \frac{\partial \tilde{R}_{\text{ext}}}{\partial \tilde{L}} \equiv R_{\text{int,L}} - R_{\text{ext,L}}
$$

(6.4.2.c)

In sections 6.4.1 and 6.4.2, above expressions will be discussed. For the evaluation of the quantities $\tilde{W}$, $W_D$ and $W_L$, certain geometric information is required. This geometric information will be discussed in section 6.4.3. Some of the contributions to the internal/external virtual work are stated in terms of integral expressions. The evaluation of these integrals will be discussed in section 6.4.4.

### 6.4.1 - Link with the internal virtual work

The vector $\tilde{R}_{\text{int}}$, as stated in (6.4.2.a), follows from the internal virtual work expression (2.5.11); i.e.,

$$
\tilde{R}_{\text{int}} \cdot \delta \tilde{D} = \iint \left\{ n^{\theta} \delta \varepsilon_{step} + m^{\theta} \delta \rho_{step} + n^{\gamma} \delta \chi \right\} \sqrt{a} \, d\xi^1 \, d\xi^2
$$
with \( \delta e_{\alpha \beta} = \frac{\partial e_{\alpha \beta}}{\partial \bar{D}} \cdot \delta \bar{D} \), \( \delta p_{\alpha \beta}^* = \frac{\partial p_{\alpha \beta}^*}{\partial \bar{D}} \cdot \delta \bar{D} \), \( \delta \chi_3 = \frac{\partial \chi_3}{\partial \bar{D}} \cdot \delta \bar{D} \) \hspace{0.5cm} (6.4.3)

\[ \Rightarrow \tilde{\mathbf{R}}_{\text{int}} = \iint \left\{ n^{\alpha \beta} \frac{\partial e_{\alpha \beta}}{\partial \bar{D}} + m^{\alpha \beta} \frac{\partial p_{\alpha \beta}^*}{\partial \bar{D}} + n^{33} \frac{\partial \chi_3}{\partial \bar{D}} \right\} \sqrt{\alpha} \, d\xi^1 \, d\xi^2 \]

The derivatives of the strain measures with respect to the degrees-of-freedom are obtained from expression (2.4.13); i.e.

\[ \frac{\partial e_{\alpha \beta}}{\partial \bar{D}} = \frac{1}{2} \left( \frac{\partial \bar{A}_{\alpha}}{\partial \bar{D}} \cdot \bar{A}_\beta + \bar{A}_\alpha \cdot \frac{\partial \bar{A}_\beta}{\partial \bar{D}} \right) \]

\[ \frac{\partial p_{\alpha \beta}^*}{\partial \bar{D}} = \frac{\partial \bar{A}_\alpha}{\partial \bar{D}} \cdot \bar{N} + \frac{\lambda}{\lambda} \frac{\partial \bar{A}_{\alpha \beta}}{\partial \bar{D}} \cdot \bar{N} + \frac{\lambda}{\lambda} \bar{A}_{\alpha \beta} \cdot \frac{\partial \bar{N}}{\partial \bar{D}} \]

\[ \frac{\partial \chi_3}{\partial \bar{D}} = \Lambda \frac{\partial \bar{A}_3}{\partial \bar{D}} \] \hspace{0.5cm} (6.4.4)

**Derivative of \( \tilde{\mathbf{R}}_{\text{int}} \) with respect to \( \bar{D} \)**

The derivative of \( \tilde{\mathbf{R}}_{\text{int}} \) with respect to \( \bar{D} \) is worked out from (6.4.3). Thus,

\[ R_{\text{int}, \bar{D}} = \iint \left\{ \frac{\partial n^{\alpha \beta}}{\partial \bar{D}} \frac{\partial e_{\alpha \beta}}{\partial \bar{D}} + n^{\alpha \beta} \frac{\partial^2 e_{\alpha \beta}}{\partial \bar{D}^2} + \frac{\partial m^{\alpha \beta}}{\partial \bar{D}} \frac{\partial p_{\alpha \beta}^*}{\partial \bar{D}} + m^{\alpha \beta} \frac{\partial^2 p_{\alpha \beta}^*}{\partial \bar{D}^2} + \right. 
\]

\[ + \left. \frac{\partial n^{33}}{\partial \bar{D}} \frac{\partial \chi_3}{\partial \bar{D}} + n^{33} \frac{\partial^2 \chi_3}{\partial \bar{D}^2} \right\} \sqrt{\alpha} \, d\xi^1 \, d\xi^2 \]

\hspace{0.5cm} (6.4.5)

Based on this expression, the following decomposition is carried out:

\[ R_{\text{int}, \bar{D}} = K_{\text{mat}} + K_{\text{geo}} \] \hspace{0.5cm} (6.4.6)

\[ K_{\text{mat}} = \iint \left\{ \frac{\partial n^{\alpha \beta}}{\partial \bar{D}} \frac{\partial e_{\alpha \beta}}{\partial \bar{D}} + \frac{\partial m^{\alpha \beta}}{\partial \bar{D}} \frac{\partial p_{\alpha \beta}^*}{\partial \bar{D}} + \frac{\partial n^{33}}{\partial \bar{D}} \frac{\partial \chi_3}{\partial \bar{D}} \right\} \sqrt{\alpha} \, d\xi^1 \, d\xi^2 \]

\[ K_{\text{geo}} = \iint \left\{ n^{\alpha \beta} \frac{\partial^2 e_{\alpha \beta}}{\partial \bar{D}^2} + m^{\alpha \beta} \frac{\partial^2 p_{\alpha \beta}^*}{\partial \bar{D}^2} + n^{33} \frac{\partial^2 \chi_3}{\partial \bar{D}^2} \right\} \sqrt{\alpha} \, d\xi^1 \, d\xi^2 \]

Matrix \( K_{\text{mat}} \) is called the "material part of the tangent stiffness matrix", whereas matrix \( K_{\text{geo}} \) is called the "geometric part of the tangent stiffness matrix".
For the evaluation of the matrix $K_{\text{mre}}$, the derivatives of the stress measures $n^{\alpha \beta}$, $m^{\alpha \beta}$ and $n^{33}$ with respect to $\tilde{D}$ are required. These derivatives are written in terms of derivatives of $\varepsilon_{\alpha \beta}$, $\rho_{\alpha \beta}^*$ and $\chi_3$ with respect to $\tilde{D}$ by using the "2D consistent moduli" as given in (3.4.13). Due to the symmetry of the 2D consistent moduli, matrix $K_{\text{mre}}$ is also symmetric.

The second derivatives of $\varepsilon_{\alpha \beta}$, $\rho_{\alpha \beta}^*$ and $\chi_3$ with respect to $\tilde{D}$, required for the evaluation of the matrix $K_{\text{geo}}$, are derived from (6.4.4); i.e.

$$\frac{\partial^2 \varepsilon_{\alpha \beta}}{\partial \tilde{D}^2} = \frac{\partial \tilde{A}_\alpha}{\partial \tilde{D}} \cdot \frac{\partial \tilde{A}_\beta}{\partial \tilde{D}} + \frac{1}{2} \left( \tilde{A}_\alpha \cdot \frac{\partial^2 \tilde{A}_\beta}{\partial \tilde{D}^2} + \frac{\partial^2 \tilde{A}_\alpha}{\partial \tilde{D}^2} \cdot \tilde{A}_\beta \right)$$

$$\frac{\partial^2 \rho_{\alpha \beta}^*}{\partial \tilde{D}^2} = \frac{\partial^2 \Lambda}{\partial \tilde{D}^2} \frac{\lambda}{\lambda} \cdot \tilde{N} + \frac{\lambda}{\lambda} \frac{\partial^2 \tilde{A}_{\alpha \beta}}{\partial \tilde{D}^2} \cdot \tilde{N} + \frac{\lambda}{\lambda} \frac{\partial \tilde{A}_{\alpha \beta}}{\partial \tilde{D}} \cdot \frac{\partial \tilde{N}}{\partial \tilde{D}} + 2 \frac{\partial \tilde{A}_{\alpha \beta}}{\partial \tilde{D}} \cdot \frac{\partial \tilde{N}}{\partial \tilde{D}}$$

$$(6.4.7)$$

$$\frac{\partial^2 \chi_3}{\partial \tilde{D}^2} = \frac{\partial \Lambda}{\partial \tilde{D}} \frac{\partial \Lambda}{\partial \tilde{D}} + \Lambda \frac{\partial^2 \Lambda}{\partial \tilde{D}^2}$$

From the series expansion of the thickness function $\Lambda$ in (4.4.4), which is linear in $\tilde{D}$, it follows that the second derivative of $\Lambda$ with respect to $\tilde{D}$ is zero. As a consequence, the underlined terms in (6.4.7) vanish. The vectors in (6.4.7) are non-linear in $\tilde{D}$. Therefore, the second derivatives of these vectors are non-zero.

The second derivatives of $\varepsilon_{\alpha \beta}$, $\rho_{\alpha \beta}^*$ and $\chi_3$ with respect to $\tilde{D}$, stated in (6.4.7), are symmetric with respect to the components of $\tilde{D}$. Therefore, the matrix $K_{\text{geo}}$ is symmetric.

In the present formulation, the tangent stiffness matrix is not sparse. This is due to the choice of the approximate functions for the "shell deformation", which is such that at every material point all functions contribute to the deformation. Consequently, there is a coupling between all the degrees-of-freedom in the model.

**Derivative of $\tilde{R}_{\text{int}}$ with respect to $\tilde{L}$**

For certain loading conditions, e.g. thermal loading, the internal virtual work depends on loading parameters. However, in the present model, the loading parameters are introduced in the external virtual work expression only. In this case,

$$R_{\text{int},L} = 0$$

$$(6.4.8)$$
6.4.2 - Link with the external virtual work

The vector $\mathbf{\tilde{R}}_{\text{ext}}$, as stated in (6.4.2.a), follows from the external virtual work expressions in (5.2.8) and (5.3.7) - (5.3.8); i.e.

$$\mathbf{\tilde{R}}_{\text{ext}} \cdot \delta \mathbf{D} = \sum_{k=1}^{\text{NF}} \lambda^{(k)} \delta W^{(k)}_{\text{ext}} = \sum_{k=1}^{\text{NF}} \lambda^{(k)} \mathbf{\tilde{R}}^{(k)}_{\text{ext}} \cdot \delta \mathbf{D}$$

$$\Rightarrow \quad \mathbf{\tilde{R}}_{\text{ext}} = \sum_{k=1}^{\text{NF}} \lambda^{(k)} \mathbf{\tilde{R}}^{(k)}_{\text{ext}}$$

(6.4.9.a)

with

$$\mathbf{\tilde{R}}^{(1)}_{\text{ext}} = N_{\text{yield}} \frac{\partial \mathbf{F}^{\mathbf{M}}}{\partial \mathbf{D}} \bigg|_{\xi^1 = +\ell/2} \cdot \mathbf{\hat{t}}_1$$

$$\mathbf{\tilde{R}}^{(2)}_{\text{ext}} = N_{\text{yield}} \frac{\partial \mathbf{F}^{\mathbf{M}}}{\partial \mathbf{D}} \bigg|_{\xi^1 = -\ell/2} \cdot \mathbf{\hat{t}}_1$$

$$\mathbf{\tilde{R}}^{(3)}_{\text{ext}} = N_{\text{yield}} \frac{\partial \mathbf{F}^{\mathbf{M}}}{\partial \mathbf{D}} \bigg|_{\xi^2 = +\ell/2} \cdot \mathbf{\hat{t}}_2$$

$$\mathbf{\tilde{R}}^{(4)}_{\text{ext}} = N_{\text{yield}} \frac{\partial \mathbf{F}^{\mathbf{M}}}{\partial \mathbf{D}} \bigg|_{\xi^2 = -\ell/2} \cdot \mathbf{\hat{t}}_2$$

$$\mathbf{\tilde{R}}^{(5)}_{\text{ext}} = -M_{\text{yield}} \frac{1}{2} \frac{\partial \mathbf{F}^{\mathbf{B}}}{\partial \mathbf{D}}$$

(6.4.9.b)

$$\mathbf{\tilde{R}}^{(6)}_{\text{ext}} = M_{\text{yield}} \frac{1}{2} \frac{\partial \mathbf{F}^{\mathbf{B}}}{\partial \mathbf{D}}$$

$$\mathbf{\tilde{R}}^{(7)}_{\text{ext}} = \int \int \mathbf{\bar{g}}_1 \times \mathbf{\bar{g}}_2 \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{D}} \bigg|_{\xi = -1/2} d\xi^1 d\xi^2$$

$$\mathbf{\tilde{R}}^{(8)}_{\text{ext}} = -\int \int \mathbf{\bar{g}}_1 \times \mathbf{\bar{g}}_2 \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{D}} \bigg|_{\xi = 1/2} d\xi^1 d\xi^2$$

The derivatives of the mean position vector $\mathbf{\bar{F}}^{\mathbf{M}}$ with respect to $\mathbf{\bar{D}}$ at $\xi^1 = \pm \ell/2$ are obtained from (5.2.3). The derivative of the position vector $\mathbf{\bar{F}}$ with respect to $\mathbf{\bar{D}}$ follows from (2.2.8); i.e.

$$\frac{\partial \mathbf{\bar{F}}}{\partial \mathbf{\bar{D}}} = \frac{\partial \mathbf{\bar{F}}}{\partial \mathbf{\bar{D}}}^0 + \xi \left\{ \frac{\partial \Delta \mathbf{\bar{N}}}{\partial \mathbf{\bar{D}}} + \Lambda \frac{\partial \mathbf{\bar{N}}}{\partial \mathbf{\bar{D}}} \right\}$$

(6.4.10)
Derivative of $\tilde{R}_{\text{ext}}$ with respect to $\tilde{D}$

In some cases, such as fluid loading, the loading depends on the actual deflections in the model. This results into a "loading part of the tangent stiffness matrix". This part can be non-symmetric. In the present model, the deformation dependency of the loading is fully accounted for in $\tilde{R}_{\text{ext}}$, while its influence in the tangent stiffness matrix $W_{\text{T}}$ is symmetrised. In this case, from (6.4.9.a), we have

$$K_{\text{load}} = \frac{1}{2} \left( R_{\text{ext},D} + (R_{\text{ext},D})^T \right) \quad \text{where} \quad R_{\text{ext},D} = \sum_{k=1}^{NF} \lambda^{(k)} R_{\text{ext},D}^{(k)} \quad (6.4.11.a)$$

The derivatives of the contributions $\tilde{R}_{\text{ext}}$ with respect to $\tilde{D}$ follow from (6.4.9.b). The results are

$$R_{\text{ext},D}^{(1)} = N_{\text{yield}} \frac{\partial^2 \tilde{G}^{(M)}}{\partial \tilde{D}^2} \left| \xi = +\ell/2 \right. \left. \cdot \tilde{I}_1 \right.$$  

$$R_{\text{ext},D}^{(2)} = N_{\text{yield}} \frac{\partial^2 \tilde{G}^{(M)}}{\partial \tilde{D}^2} \left| \xi = -\ell/2 \right. \left. \cdot \tilde{I}_1 \right.$$  

$$R_{\text{ext},D}^{(3)} = N_{\text{yield}} \frac{\partial^2 \tilde{G}^{(M)}}{\partial \tilde{D}^2} \left| \xi = +\ell/2 \right. \left. \cdot \tilde{I}_2 \right.$$  

$$R_{\text{ext},D}^{(4)} = N_{\text{yield}} \frac{\partial^2 \tilde{G}^{(M)}}{\partial \tilde{D}^2} \left| \xi = -\ell/2 \right. \left. \cdot \tilde{I}_2 \right.$$  

$$R_{\text{ext},D}^{(5)} = 0$$  

$$R_{\text{ext},D}^{(6)} = 0 \quad (6.4.11.b)$$

$$R_{\text{ext},D}^{(7)} = p_{\text{yield}} \int \left\{ \frac{\partial (\tilde{G}_1 \times \tilde{G}_2)}{\partial \tilde{D}} \cdot \frac{\partial \tilde{D}}{\partial \tilde{D}} + (\tilde{G}_1 \times \tilde{G}_2) \cdot \frac{\partial^2 \tilde{D}}{\partial \tilde{D}^2} \right\} d\tilde{\xi}^1 d\tilde{\xi}^2 \left| \xi = -1/2 \right.$$  

$$R_{\text{ext},D}^{(8)} = -p_{\text{yield}} \int \left\{ \frac{\partial (\tilde{G}_1 \times \tilde{G}_2)}{\partial \tilde{D}} \cdot \frac{\partial \tilde{D}}{\partial \tilde{D}} + (\tilde{G}_1 \times \tilde{G}_2) \cdot \frac{\partial^2 \tilde{D}}{\partial \tilde{D}^2} \right\} d\tilde{\xi}^1 d\tilde{\xi}^2 \left| \xi = 1/2 \right.$$  

From the first derivatives of $\tilde{G}^{(M)}$ and $\tilde{D}$ with respect to $\tilde{D}$, mentioned in the previous section, the second derivatives required in (6.4.11.b) are obtained. The derivative of $\tilde{G}_1 \times \tilde{G}_2$ with respect to $\tilde{D}$ follows from
\[
\frac{\partial (\mathbf{G}_1 \times \mathbf{G}_2)}{\partial \mathbf{\tilde{D}}} = \mathbf{G}_1 \times \frac{\partial \mathbf{G}_2}{\partial \mathbf{\tilde{D}}} + \frac{\partial \mathbf{G}_1}{\partial \mathbf{\tilde{D}}} \times \mathbf{G}_2
\]

(6.4.12.a)

where, according to (5.3.2) and the discussion thereafter, we have

\[
\frac{\partial \mathbf{G}_\alpha}{\partial \mathbf{\tilde{D}}} = \frac{\partial \mathbf{A}_\alpha}{\partial \mathbf{\tilde{D}}} + \xi \left( \frac{\partial \Lambda}{\partial \mathbf{\tilde{D}}} \mathbf{\tilde{N}}_\alpha + \Lambda \frac{\partial \mathbf{\tilde{N}}_\alpha}{\partial \mathbf{\tilde{D}}} \right)
\]

(6.4.12.b)

**Derivative of \( \mathbf{\tilde{R}}_{\text{ext}} \) with respect to \( \mathbf{\tilde{L}} \)**

It simply follows from (6.4.9) that

\[
\mathbf{R}_{\text{ext,L}} = \begin{bmatrix} \mathbf{\tilde{R}}_{\text{ext}}^{(1)} & \mathbf{\tilde{R}}_{\text{ext}}^{(2)} & \ldots & \mathbf{\tilde{R}}_{\text{ext}}^{(N-1)} & \mathbf{\tilde{R}}_{\text{ext}}^{(N)} \end{bmatrix}
\]

(6.4.13)

**6.4.3 - Geometric information for the tube model**

By inspection of the results in the previous sections, it is concluded that the following geometric information is required for the evaluation of the matrices \( W_D \) and \( W_L \) and the vector \( \mathbf{\tilde{W}} \):

\[
\begin{pmatrix}
\mathbf{\tilde{A}}_\alpha, \mathbf{\tilde{A}}_{\alpha \beta}, \mathbf{\tilde{N}}, \mathbf{\tilde{N}}_\alpha, \Lambda \\
\frac{\partial \mathbf{\Phi}^M}{\partial \mathbf{\tilde{D}}} , \frac{\partial \mathbf{\Phi}^0}{\partial \mathbf{\tilde{D}}} , \frac{\partial \mathbf{\tilde{A}}_\alpha}{\partial \mathbf{\tilde{D}}} , \frac{\partial \mathbf{\tilde{A}}_{\alpha \beta}}{\partial \mathbf{\tilde{D}}} , \frac{\partial \mathbf{\tilde{N}}}{\partial \mathbf{\tilde{D}}} , \frac{\partial \mathbf{\tilde{N}}_\alpha}{\partial \mathbf{\tilde{D}}} , \frac{\partial \Lambda}{\partial \mathbf{\tilde{D}}} \\
\frac{\partial^2 \mathbf{\Phi}^M}{\partial \mathbf{\tilde{D}}^2} , \frac{\partial^2 \mathbf{\Phi}^0}{\partial \mathbf{\tilde{D}}^2} , \frac{\partial^2 \mathbf{\tilde{A}}_\alpha}{\partial \mathbf{\tilde{D}}^2} , \frac{\partial^2 \mathbf{\tilde{A}}_{\alpha \beta}}{\partial \mathbf{\tilde{D}}^2} , \frac{\partial^2 \mathbf{\tilde{N}}}{\partial \mathbf{\tilde{D}}^2} , \frac{\partial^2 \mathbf{\tilde{N}}_\alpha}{\partial \mathbf{\tilde{D}}^2}
\end{pmatrix}
\]

(6.4.14)

From a computational point of view, it has been experienced that using components of the vectors in (6.4.14) with respect to the \{ \mathbf{\tilde{E}}_k \} -basis is most convenient.

The differentiation of the \{ \mathbf{\tilde{E}}_k \} -basis with respect to the mid-surface coordinates \( \xi^a \) follows from (4.2.14) - (4.2.16); i.e.

\[
\begin{align*}
\mathbf{\tilde{E}}_{1,1} &= -\xi^B K^B \mathbf{\tilde{E}}_2 \\
\mathbf{\tilde{E}}_{2,1} &= \xi^B K^B \mathbf{\tilde{E}}_1 \\
\mathbf{\tilde{E}}_{1,2} &= \mathbf{\tilde{E}}_1 \\
\mathbf{\tilde{E}}_{2,2} &= \mathbf{\tilde{E}}_2
\end{align*}
\]

(6.4.15)
The mid-surface position vector \( \vec{\Phi}^0 \) expressed in terms of components on the \( \{ \vec{E}_k \} \)-basis is obtained from (4.3.16). From the vector \( \vec{\Phi}^0 \), the mid-surface base vectors \( \vec{A}_\alpha \) are found by virtue of (2.2.11). Using (6.4.15), we thus obtain

\[
\vec{A}_1 = A^k_1 \vec{E}_k
\]

\[
\begin{align*}
A^1_1 &= (1 + U^3_1 S^B) + \zeta^B K^B \left[ V^S \cos(\psi_2) + W^S \sin(\psi_2) \right] \\
A^2_1 &= -U^S S^B \zeta^B K^B + V^S_1 \cos(\psi_2) + W^S_1 \sin(\psi_2) \\
A^3_1 &= -V^S_1 \sin(\psi_2) + W^S_1 \cos(\psi_2)
\end{align*}
\]

\( .(6.4.16.a) \)

\[
\vec{A}_2 = A^k_2 \vec{E}_k
\]

\[
\begin{align*}
A^1_2 &= U^3_2 S^B + U^S S^3_2 \\
A^2_2 &= \left( 1 + V^3_2 \right) \cos(\psi_2) + \left( -\frac{V^S_2}{R} + \frac{W^S_2}{R} \right) \sin(\psi_2) \\
A^3_2 &= \left( -\frac{V^S_2}{R} + \frac{W^S_2}{R} \right) \cos(\psi_2) - \left( 1 + V^3_2 \right) \sin(\psi_2)
\end{align*}
\]

\( .(6.4.16.b) \)

The vectors \( \vec{A}_{\alpha,\beta} \) follow by partial differentiation of \( \vec{A}_\alpha \) with respect to \( \xi^\beta \); i.e.

\[
\begin{align*}
\vec{A}_{1,1} &= (A^1_1 + \xi^B K^B A^3_1) \vec{E}_1 + (A^1_2 - \xi^B K^B A^3_2) \vec{E}_2 + (A^1_3) \vec{E}_3 \\
\vec{A}_{1,2} &= (A^1_1) \vec{E}_1 + (A^2_2) \vec{E}_2 + (A^3_2) \vec{E}_3 \\
\vec{A}_{1,3} &= (A^1_2 + \xi^B K^B A^3_2) \vec{E}_1 + (A^2_3 - \xi^B K^B A^3_1) \vec{E}_2 + (A^3_3) \vec{E}_3 = \vec{A}_{1,2} \\
\vec{A}_{2,2} &= (A^1_2) \vec{E}_1 + (A^2_2) \vec{E}_2 + (A^3_2) \vec{E}_3
\end{align*}
\]

\( .(6.4.17) \)

From the definition of the unit normal vector \( \vec{N} \) in (2.2.12), it is clear that both \( \vec{A}_1 \times \vec{A}_2 \) and its norm needs to be evaluated. From (6.4.16) and the fact that \( \{ \vec{E}_k \} \) is a right-handed orthonormal basis, we obtain

\[
\vec{X}_{1,2} = \vec{A}_1 \times \vec{A}_2 = (A^1_1 A^3_2 - A^3_1 A^1_2) \vec{E}_1 + (A^2_1 A^3_2 - A^3_1 A^2_2) \vec{E}_2 +
\]

\[
+ (A^1_1 A^2_2 - A^2_1 A^1_2) \vec{E}_3
\]

\( .(6.4.18) \)

The norm of this vector is defined according to (1.3.16).

The components of the vectors \( \vec{N}_\beta \) can be obtained by virtue of Weingarten's formula, mentioned in section 2.4; i.e.

\[
\vec{N}_\beta = -B^\gamma_\beta \vec{A}_\gamma = -\left( \vec{A}_{\beta,\alpha} \cdot \vec{N} \right) \vec{A}^{\alpha} = -\left( \vec{A}_{\alpha,\beta} \cdot \vec{N} \right) \vec{A}^{\alpha}
\]

\( .(6.4.19) \)
These expressions can be worked out by using the results from (6.4.16)-(6.4.18).

From the vectors mentioned in (6.4.16)-(6.4.19), the first and second derivatives with respect to the degrees-of-freedom can be obtained. Since, the evaluation of the first and second derivatives of the unit normal vector $\vec{N}$ with respect to the degrees-of-freedom is more complicated than the derivatives of the other geometric quantities, these are next given explicitly.

Due to the allowance of finite membrane strains, both the direction and the length of the vector $\vec{X}_{1:2}$ may change significantly. For this reason, it is essential that the derivative of the unit normal vector $\vec{N}$ involves not only the derivative of the vector $\vec{X}_{1:2}$, but also the derivative of its norm. Therefore, the first and second derivatives of the unit normal vector $\vec{N}$ read

\[
\frac{\partial \vec{N}}{\partial \tilde{D}} = \frac{1}{\| \vec{X}_{1:2} \|} \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} - \left[ \frac{1}{\| \vec{X}_{1:2} \|^2} \left( \vec{X}_{1:2} \cdot \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} \right) \right] \vec{X}_{1:2} = , \tag{6.4.20.a}
\]

\[
= \frac{1}{\| \vec{X}_{1:2} \|} \left( \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} - \left[ \vec{N} \cdot \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} \right] \vec{N} \right)
\]

\[
\frac{\partial^2 \vec{N}}{\partial \tilde{D}^2} = \frac{1}{\| \vec{X}_{1:2} \|} \frac{\partial^2 \vec{X}_{1:2}}{\partial \tilde{D}^2} - \frac{2}{\| \vec{X}_{1:2} \|^2} \left( \vec{N} \cdot \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} \right) \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} +
\]

\[
- \frac{1}{\| \vec{X}_{1:2} \|^2} \left( \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} \cdot \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} \right) \vec{N} - \frac{1}{\| \vec{X}_{1:2} \|} \left( \vec{N} \cdot \frac{\partial^2 \vec{X}_{1:2}}{\partial \tilde{D}^2} \right) \vec{N} +
\]

\[
+ \frac{3}{\| \vec{X}_{1:2} \|^2} \left( \vec{N} \cdot \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} \right) \left( \vec{N} \cdot \frac{\partial \vec{X}_{1:2}}{\partial \tilde{D}} \right) \vec{N}. \tag{6.4.20.b}
\]

The derivatives of $\vec{X}_{1:2}$ with respect to $\tilde{D}$ are derived from (6.4.18).

### 6.4.4 - Mid-surface integration

The integrations which are required for the evaluation of the vector $\vec{R}_{es}$, see (6.4.3), and the matrices $K_{ma}$ and $K_{gc},$, see (6.4.6), are carried out over the total range of the material coordinates $\xi_1$ and $\xi_2$. For the vector $\vec{R}_{es}$ and the matrices $R_{es,D}$ and $R_{es,L}$, given in (6.4.9), (6.4.11) and (6.4.13), respectively, the integration area depends on the load parameters to be employed. In the case of internal and/or external pressure (i.e. $\lambda^{(7)}$, $\lambda^{(8)}$), the integrations are again to be carried out over the total range of the material coordinates.
In the present section, the numerical evaluation of such integrals, based on Gaussian quadrature, is discussed.

The decision on the total number of integration points (required for an accurate evaluation of the integrals) is based on the set of degrees-of-freedom employed in the analysis. The beam-like degrees-of-freedom \( \xi^B \) and \( \Theta^B \) are always considered "active". The degrees-of-freedom governing the "shell deformation" are activated by selection of the appropriate Fourier functions in \( (U^S, V^S, W^S, \Lambda) \), see the Summary (4.3.16) and (4.4.4).

From the "active" functions in \( (U^S, V^S, W^S, \Lambda) \), the smallest sinusoidal wavelengths (corresponding to the highest wave-number) in the axial and the circumferential directions are determined. The total integration interval is divided into sub-intervals, which are determined by the smallest sinusoidal wave pattern. This is done in both the axial and the circumferential direction.

For each sub-interval a separate Gauss quadrature rule is applied. The Gauss integration rule is, for example, discussed in HUGHES (1987). From numerical experience, it is concluded that 4 Gaussian integration points per sinusoidal half-wave suffice for an accurate evaluation of the concerned integrals.

The Gauss rule with 4 integration points reads

\[
\int_A^B f(\xi^a) d\xi^a \equiv \sum_{int=1}^4 f(\xi^a_{int}) W_{int}
\]

(6.4.21)

where the location of the integration points and the corresponding weight factors are given in Table 6.1.

<table>
<thead>
<tr>
<th>int</th>
<th>( \xi^a_{int} )</th>
<th>( W_{int} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2}(A+B) - 0.43057(B-A) )</td>
<td>0.17393(B-A)</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2}(A+B) - 0.16999(B-A) )</td>
<td>0.32607(B-A)</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{2}(A+B) + 0.16999(B-A) )</td>
<td>0.32607(B-A)</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{2}(A+B) + 0.43057(B-A) )</td>
<td>0.17393(B-A)</td>
</tr>
</tbody>
</table>

Table 6.1 - Location of the integration points and the corresponding weight factors according to a Gauss rule with 4 integration points.
In Chapter 4, the assumption of symmetric deformations with respect to $\psi_2 = \pi/2$ is made. For this reason, the integration in the circumferential direction is carried out over the interval $\xi_2/R = \psi_2 \in [-\pi/2, \pi/2]$ and the result is multiplied with the factor 2.

6.5 - Definition of the "load-case function"

The specific information which is required for the definition of the "load-case function" $\tilde{g}$ is as follows:

1. the "active" set of load parameters. This set of parameters defines the external loading on the tube.

2. in the case that the "active" set of load parameters consists of more than 1 item, relations between the "active" load parameters.

3. kinematical boundary conditions. In the present model, this applies for the beam deformation only.

In the model, the "load-case function" $\tilde{g}$ is taken to be linear in both the degrees-of-freedom and the load parameters. Thus,

$$\tilde{g}(\mathbf{D}, \mathbf{L}) = \tilde{g}_0 + g_D \cdot \mathbf{D} + g_L \cdot \mathbf{L}$$  \hspace{1cm} (6.5.1)

Since the relations between the active load parameters are assumed to be linear, specification of the ratios between the parameters suffices. The linear loading paths are specified by the $(\text{NF} - 1) \times \text{NF}$-dimensional matrix $g_L$, which contains the ratios between the active load parameters. In this case, the vector $\tilde{g}_0$ is a $(\text{NF} - 1)$-dimensional zero-vector and the matrix $g_D$ is a $(\text{NF} - 1) \times \text{ND}$-dimensional zero-matrix.

As already mentioned in Chapter 5, a load-case (which is a set of active basic loads) should be defined as an equilibrium system. Such equilibrium equations are captured by setting the appropriate ratios in the matrix $g_L$. In Chapter 7, specific examples of equilibrium equations between load parameters are given.

In the present model, the kinematical boundary conditions are applicable for the beam deformation only. This is handled as follows. In the case of a tube without centroidal stretching (inextensional tube), the value of $\zeta^B$ is fixed in the load-case. If the centroidal stretching $\zeta^B$ is the first degree-of-freedom, $D(1)$, then one of the equations in $\tilde{g}$ reads

$$g(\mathbf{D}, \mathbf{L}) = D(1) - \zeta^B_{\text{fixed}} = 0$$  \hspace{1cm} (6.5.2)
The value $\zeta_{\text{fixed}}$ is constant within the current load-case and is determined from the previous load-case. A load parameter or some combination of load parameters from the vector $\tilde{\mathbf{L}}$ could represent the accompanying reaction loading.

Similarly, in the case that the over-all curvature of the tube is kept constant within the current load-case, the value of $\Theta^B$ is fixed. In this case, one of the equations in $\tilde{g}$ reads

$$g\left(\tilde{\mathbf{D}}, \tilde{\mathbf{L}}\right) = D(2) - \Theta^B_{\text{fixed}} = 0$$

(6.5.3)

where the over-all bend-angle $\Theta^B$ is the second degree-of-freedom, $D(2)$, in the model. The value $\Theta^B_{\text{fixed}}$ is determined from the previous load-case. Again, a load parameter or some combination of load parameters from the vector $\tilde{\mathbf{L}}$ could represent the accompanying reaction loading.

### 6.6 - Constraint equation

The implementation of the model is such that for every load-case, a specific continuation method can be chosen. As mentioned, the continuation method is constituted by the equation $h = 0$. In the present model, the following methods are implemented:

- load control (incrementing a certain load parameter),
- displacement control (incrementing a specific degree-of-freedom) and
- arc-length control (incrementing a combination of load parameters and degrees-of-freedom).

The control methods mentioned above are discussed in the sections 6.6.1 to 6.6.3. Finally (in section 6.6.4), a technique is discussed for optimisation of the step-size.

### 6.6.1 - Load control

In this case, a specific ("active") load parameter is prescribed to a certain value. Suppose the controlled load parameter to be denoted with $L_{\text{master}}$. Further suppose the step-size in the current increment to be given by $\Delta_{\text{INC}}^S$. The increment number is denoted with a subscript. If the converged value from the previous increment of the controlled load parameters is denoted with $L_{\text{INC-1}}(\text{master})$, then the constraint equation in the current increment, $h_{\text{INC}} = 0$, reads

$$h_{\text{INC}}\left(\tilde{\mathbf{D}}, \tilde{\mathbf{L}}, \Delta_{\text{INC}}^S\right) = L(\text{master}) - L_{\text{INC-1}}(\text{master}) - \Delta_{\text{INC}}^S = 0$$

(6.6.4)
The derivatives of \( h \), required in (6.3.2), read

\[
\begin{align*}
    h_D &= 0_{ND} \\
    h_L &= \begin{bmatrix} 0_{\text{master-1}} & 1 & 0_{\text{NF-master}} \end{bmatrix}
\end{align*}
\] . (6.6.5)

The method based on load control is illustrated in figure 6.1, which shows a part of the solution curve for a system with two unknown parameters. The point marked with (INC\(-1\)) is the last converged situation and is the starting point for the Newton-Raphson iteration process in the current increment. The line \( t^{(\text{ITR}=1)} = 0 \) is the tangent to the solution curve at the point (INC\(-1\)). The point marked with (ITR = 1) denotes the first iterative solution. Note that for every iteration within the increment, constraint equation \( h_{(\text{INC})} = 0 \) is identical. As a consequence, all iterative solutions are situated on the constraint \( h_{(\text{INC})} = 0 \). The point marked with (INC) is the final converged situation.

*Figure 6.1 - Load control.*
6.6.2 - Displacement control

This method prescribes a certain degree-of-freedom within the current increment to a certain level. Suppose the concerned degree-of-freedom to be denoted with \( D_{\text{displ}} \). Further assume the increment for \( D_{\text{displ}} \) to be given by \( \Delta_{\text{(INC)}}S \). In this case, constraint equation \( h_{\text{(INC)}} = 0 \) becomes

\[
h_{\text{(INC)}} (\bar{D}, \bar{L}, \Delta_{\text{(INC)}}S) = D_{\text{displ}} - D_{\text{(INC-1)}}(\text{displ}) - \Delta_{\text{(INC)}}S = 0 \quad (6.6.6)
\]

where the derivatives of \( h \) required in (6.3.2) read

\[
\begin{align*}
h_D &= \begin{bmatrix} O_{\text{displ-1}} & 1 & O_{\text{ND-displ}} \end{bmatrix} \\
h_L &= O_{NF}
\end{align*}
\quad (6.6.7)
\]

An illustration of displacement control is offered in figure 6.2, which is identical to figure 6.1 except for the constraint equation \( h_{\text{(INC)}} = 0 \) which has been changed into the type of (6.6.6). Again, observe that for the first iteration, the tangent to the solution curve is denoted with \( t^{(\text{ITR}=1)} = 0 \) and that the first iterative solution is denoted with \( (\text{ITR} = 1) \). Since for every iteration within the increment, constraint equation \( h_{\text{(INC)}} = 0 \) is identical, all the iterative solutions are situated on the constraint \( h_{\text{(INC)}} = 0 \).

---

*Figure 6.2 - Displacement control.*
6.6.3 - Arc-length control

It is well-known that load control breaks down at a limit value for the load parameter involved, whereas displacement control breaks down at a limit value for the displacement involved. A method capable to overcome these problems is by using a more "sophisticated" constraint equation (e.g. the arc-length method). The constraint equation employed in this method is adapted such that, for every increment, it is approximately orthogonal to the solution path.

The mathematical formulation of the arc-length method has been first translated into a computational mechanics formulation by WEMPNER (1971) and RIKS (1972). In the sequence, this work will be referred to as the "general arc-length method". Amongst others, SCHWEIZERHOF & WRIGGERS (1986) and FORDE & STIEMER (1987) have advocated linearised forms of the general arc-length method. In this section, the formulation of a linearised version with a so-called constant constraint equation is worked out.

Application of the "general arc-length method" requires the derivation of the intersection of a (hyper-)sphere with the tangent to the solution curve, which is a (hyper-)line. The intersection between a (hyper-)sphere and a (hyper-)line is not always uniquely determined. From the possible solutions, the solution in the proper direction should be selected. This involves additional computational effort. In order to make the computation more straightforward, linearised versions have been introduced. In these versions, the (hyper-)sphere is approximated with its tangent (hyper-)plane. In this case, the intersection of a (hyper-)plane with a (hyper-)line should be derived. If such an intersection exists, then the solution is unique.

The constraint equation $h=0$ within increment number (INC) is defined as the tangent to a (hyper-)sphere as follows. Consider the (hyper-)sphere in terms of the vector with active degrees-of-freedom $\tilde{D}$ and the vector with load parameters $\tilde{L}$ with the centre at the values for (INC−1) and the radius $\Delta_{(INC)}S$. A solution curve for a system with one load parameter and one degree-of-freedom is shown in figure 6.3.

The (hyper-)sphere, which for this particular case is a circle, is denoted with "general". The location of the tangent to the (hyper-)sphere is determined from the previous increment (see figure 6.3, where the tangent to the (hyper-)sphere is denoted with "$h_{(INC)}=0\)"). Iterative solutions are obtained by intersection of the constraint $h_{(INC)}=0$ and the tangent to the solution curve. For a full Newton-Raphson iteration scheme, the tangent to the solution curve changes for every iteration. In figure 6.3, the tangent to the solution curve for the first iteration, $t^{(TR=1)}=0$, is shown. The constraint equation $h_{(INC)}=0$ will be defined such that it is constant for every iteration within the current increment. Therefore, all iterative solutions are located on $h_{(INC)}=0$. 
Figure 6.3 - Linearised arc-length control with a constant constraint.

Mathematically, the method is governed by

\[ h_{(INC)} = \Delta_{(INC-1)} \tilde{D} \cdot \left( \tilde{D} - \tilde{D}_{(INC-1)} \right) + \Delta_{(INC-1)} \tilde{L} \cdot \left( \tilde{L} - \tilde{L}_{(INC-1)} \right) + \left( \| \Delta_{(INC-1)} \tilde{D} \| + \| \Delta_{(INC-1)} \tilde{L} \| \right) \Delta_{(INC)} S = 0 \]  

\hspace{10cm} (6.6.8)

where the increments \( \Delta_{(INC-1)} \tilde{D} \) and \( \Delta_{(INC-1)} \tilde{L} \) are the converged "incremental changes" from the previous increment.

The derivatives of \( h \), which are required in (6.3.2), read

\[
\begin{align*}
  h_D &= \Delta_{(INC-1)} \tilde{D} \\
  h_L &= \Delta_{(INC-1)} \tilde{L}
\end{align*}
\]  

\hspace{10cm} (6.6.9)

In the implementation of the model, dimensionless degrees-of-freedom are used in the vector \( \tilde{D} \), see (6.4.1.a), which in the analyses performed are of the order of unity. The vector \( \tilde{L} \) contains the load parameters \( \lambda^{(k)} \), see (6.4.1.b), which are also dimensionless. In the analyses carried out, the values of \( \lambda^{(k)} \) are also of the order of unity. Therefore, scaling of the different contributions in the constraint (6.6.8) to appropriate dimensions for the
degrees-of-freedom and the load parameters, as discussed in CRISFIELD (1991), is not necessary.

Observe from (6.6.9), that the derivatives of h are fully determined before the iteration process within the increment is started. This leads to relatively straightforward calculations. However, a drawback of the presented method is that the converged solutions generally differ from the solutions based on the "general arc-length method" (although the solutions for both methods are on the exact solution curve). The linearised arc-length method may lead to convergence problems for solution curves that show a rapid change in the slope.

Finally, it is mentioned that in the case of arc-length control, the analysis is started with an increment using load control. After convergence of the solution in this first increment, the corresponding arc-length is determined and applied in the second increment. From this increment onwards, arc-length control is used.

6.6.4 - Adaptation of the step-size

One of the objectives of the continuation method is to derive information about the solution curve in a (computational) cost-effective way. In order to do so, the step-size $\Delta_{(inc)}S$ should be adapted to the actual solution curve. In the case that the convergence to the solution curve is slow, smaller increments should be taken. In the case that the convergence is faster, larger increments could be taken.

An effective technique is due to CRISFIELD (1991). This technique requires the input of the "desired number of iterations" in order to achieve convergence (this number is denoted with $I_{(D)}$). If $I_{(INC-1)}$ denotes the "used number of iterations" in the previous increment, then the step-size for the current increment reads

$$\Delta_{(INC)}S = \sqrt{\frac{I_{(D)}}{I_{(INC-1)}}} \Delta_{(INC-1)}S$$  \hspace{1cm} (6.6.10)

Due to this, the step-size is adapted until $I_{(INC)} = I_{(D)}$ is achieved.

In the case that plastic deformation is traced, the step-size should be taken sufficiently small. This is due to the obtained accuracy for the integration of the rate-equations in terms of the finite difference scheme as discussed in section 3.4.
Chapter 7 - Examples

7.1 - Input specifications

The formulation as discussed in the previous chapters is implemented in a (special-purpose) numerical tool, called TUBEHAVE. This package is dedicated to the analysis of the deformation behaviour of tubes and thus, a minimal amount of input is required in order to run the package. Although the package is relatively easy to use, a certain level of expertise on material data input and on the meaning of the deformation modes is a prerequisite. In the present section, the input data required to run TUBEHAVE is discussed. The input is divided into five main categories.

geometry data

The formulation is based on the shell-like assumptions as discussed in Chapter 2. Thus, in the reference configuration, not only the position vector to the mid-surface of the shell, but also the thickness function should be specified. Following the decomposition as discussed in Chapter 4, the position vector to the beam (\( \mathbf{\varphi}^\text{B.o} \)) and the additional vector \( \mathbf{\varphi}^\text{B.o} \) need to be defined.

The geometry of the beam in the reference configuration is defined with four parameters:

- the length along the line-of-centroids (denoted with \( \ell \)).
- a characteristic radius of the cross-sections (denoted with \( R \)).
- a characteristic wall-thickness (denoted with \( t \)).
- the bend-angle (denoted with \( \Theta^B \)), which is the angle between both end planes of the tube in the reference configuration. Angle \( \Theta^B \) is measured in radians.

The specific set of degrees-of-freedom governing the "shell deformation" should also be specified by the user. In the sequel, this set of degrees-of-freedom is abbreviated with DOFD. For reasons of a quick overview, the following special notation for DOFD is introduced:
DOFD = labels \{ (\psi_1 \text{ range} ); (\psi_2 \text{ range} ) \} , (7.1.1)

where labels denotes the displacement components $U^s, V^s, W^s$ (either separately or in combinations) and/or the thickness function $\Lambda$. The first set $\{ \}$ defines the range of the terms in the Fourier series expansion in the axial direction, whereas the second set gives the range of the Fourier terms in the circumferential direction. For example, $\text{DOFD} = UVWA\{ (0,2); (0,3,6,9,12) \}$ means that the Fourier terms for both $U^s, V^s, W^s$ and $\Lambda$ are given by the $0^{th}$ and the $2^{nd}$ term in the axial coordinate and the $0^{th}, 3^{rd}, 6^{th}, 9^{th}$ and $12^{th}$ term in the circumferential coordinate (these Fourier terms are defined in Chapter 4).

The specific set of degrees-of-freedom used for the definition of the reference configuration is denoted with $\text{DOFD}_R$. For example, $\text{DOFD}_R = VW(\{ 0 \}; \{ 0,2 \})$, which indicates that the Fourier terms for both $V^s$ and $W^s$ are given by the $0^{th}$ term in the axial coordinate and the $0^{th}$ and the $2^{nd}$ term in the circumferential coordinate.

**material data**

The elastic material properties are determined by the following two items:

- Young's modulus, denoted with $E$.
- Poisson's ratio, denoted with $\nu$.

The input required for the plastic part of the constitutive relations is the definition of the uniaxial stress-strain curve of the material. This curve is determined in either of the following two ways:

- a fit by means of a power-law hardening function which, in relation to the yield criterion (3.2.15), is stated by

\[
\kappa(\bar{\varepsilon}^p) = \sigma(\bar{\varepsilon}^p) = \sigma_y \left( \frac{\bar{\varepsilon}^p}{\varepsilon_y} + h \right)^h , (7.1.2)
\]

for $\bar{\varepsilon}^p = \varepsilon - \varepsilon_y \geq 0 \quad (\varepsilon_y = E \sigma_y)$

where $\kappa$ is the uniaxial true Kirchhoff stress, $\sigma$ is the uniaxial true Cauchy stress and $\varepsilon$ is the (total) uniaxial logarithmic strain. The difference between the Kirchhoff and the Cauchy stress is due to elastic compressibility of the material, which is small. More details on the definition of the stress and strain measures in (7.1.2) are offered in section 7.7, dealing with a tube under axial tension.
• a numerical data file which contains a point-wise approximation of the uniaxial stress-strain curve (in terms of the uniaxial true Kirchhoff stress and the (total) uniaxial logarithmic strain). A piece-wise linear fit is assumed between the data points.

In both cases the value for $\sigma_y$ should be specified. In the case that a data file is used, the stress-value for 0.2% strain is recommended. The hardening exponent $m$ is only input when the power-law fit is used.

**loading data**

In this section, input is required concerning the loading history. If required, a sequence of combinations of the basic loads as discussed in Chapter 5 might be specified. Also a choice on the load-stepping algorithm should be made: load control, displacement control or arc-length control. In addition, an option on adaptation of the step-size is implemented, together with the choice of the desired number of iterations until convergence within an increment.

The definition of the kinematic boundary conditions is also considered to be a part of the loading data. The user should specify whether the axial stretch of the beam (determined by $\xi^B$) is fixed within a load-case and/or the overall bend-angle of the beam ($\Theta^B$) is fixed.

**integration data**

This input defines the total number of integration points to be used in the analysis. Information is required on the amount of integration points in the three directions of the material coordinates. In the axial and circumferential direction, the specified number represents the amount of Gauss integration points per smallest sinusoidal half-wave in the Fourier series expansions. From this amount, the total number of integration points in the respective directions is then determined (see the discussion in section 6.4.3). In the direction through-the-thickness, the specified value represents the number of layers to be used, see section 2.6. It is recommended to use 4 Gauss integration points per smallest sinusoidal half-wave and 7 layers through-the-thickness.

In some cases, a certain periodicity in the displacements is known in advance. In such cases it is effective to limit the integration to a certain section of the circumference and/or to a section of the length. Special options are implemented which allow for divisions to be made in the integration area of the modelled tube.
storage data

Storage of numerical results is tailored to the chosen post-processing package. The following options are available:

- PATRAN, a general package. This package is recommended when complicated plots with stress-contours etc. need to be processed.

- DANPLOT, a PC-based package. This package is suitable for plots of the deformed configuration.

- "Spreadsheet"-packages, which are recommended for simple load-displacement curves.

No further details on the input will be given here, as it is of minor interest to the reader in understanding the following examples.
7.2 - External pressure

In the case of marine pipelines, external pressure occurs during installation (when the pipes are empty) and also when for some reason the transportation through the line is shut off for a certain period. Another example of a tube under external pressure is the casing used in a well. Due to the extremely high pressure in the well and the possible contraction of the surrounding rock, the external (over-) pressure on the casing can be significant.

Practical situations of failure of tubes under external pressure show the occurrence of collapse, which is characterised by ovalisation of the cross-sections. Due to inhomogeneities in either the geometry or the loading, the ovalisation takes place at a small part of the tube length. Depending on the specific situation, the ovalisation continues until two opposite points on the tube wall contact each other. From then onwards, the ovalisation spreads to increasing parts of the tube length (the so-called propagating buckle).

In the present example, the class of problems is limited to tubes where both the initial shape of the cross-section and the external pressure are not axially dependent. In this particular case, the deformation of the tube remains independent of the axial coordinate. Propagating buckling is not considered, for this would require an adequate contact algorithm (which is not implemented in TUBEHAVE).

The aim of the present example is to verify the results for some specific cases of tubes under external pressure with results which are obtained by others. To this end, the elastic response as well as the influence of plastic deformation on the response is investigated. From observations, already done by Levy in 1884, the critical (buckling) mode of a perfect tube under homogeneous external pressure takes the shape of an oval cross-section (constant in axial direction). In the present example, the influence of initial imperfections in the shape of this buckling mode on the deformation behaviour will be considered.

The ovalisation of a cross-section under homogeneous external pressure shows two symmetry planes, therefore a quarter of the circumference is representative for the whole cross-section. In the present example, additional symmetry with respect to $\psi_2 = 0$, apart from the symmetry choice in the model (with respect to $\psi_2 = \pi/2$), is assumed. Thus, the numerical integrations are performed for a quarter of the circumference of the tube.

The boundary conditions at both ends of the tube length are the general sets as discussed in section 4.3. As a consequence, both ends of the tube length remain plane. In addition, ovalisation of the cross-sections in the "end planes" is allowed for.
The introduction of initial ovality is done by setting initial values in the components of \( \mathbf{\tilde{e}}^{a,0} \), i.e. \( (u^a, v^a, w^a) \), see (4.3.1). An appropriate measure for ovality is the parameter \( \Delta_0 \) defined as
\[
\Delta_0 = \frac{D_{\text{max}} - D_{\text{min}}}{D_{\text{max}} + D_{\text{min}}}
\]
where \( D_{\text{max}} \) is the maximum outer diameter of the tube and \( D_{\text{min}} \) the minimum outer diameter. Introduction of \( \Delta_0 \) into the definition of the reference configuration is done by setting the participation factors \( w_{a2} = \Delta_0 R \) and \( v_{a2} = -1/2 \Delta_0 R \), where \( R \) is the mean radius of the tube. This is considered to give the most realistic description of initial ovality. Thus, \( \text{DOFD}_R = vw(\langle 0 \rangle ; \langle 2 \rangle) \).

**Elastic response**

A relatively thin-walled tube (with \( R / t = 96.15 \)) and a relatively thick-walled tube (\( R / t = 9.615 \)) are considered. Both tubes are axially free to expand. By setting the initial yield stress \( \sigma_y \) sufficiently large, elastic material behaviour is enforced throughout the total deformation. The elastic material properties are \( E = 200 \text{ GPa} \) and \( \nu = 0.333 \). For both tubes \( \ell / R = 1.0 \) is taken.

In figure 7.2.1, the elastic response for both tubes with a small initial ovality \( \Delta_0 = 0.1\% \) is shown. The value along the horizontal axes in figure 7.2.1 is the maximum radial displacement normalised with the mean (initial) radius. The calculations are stopped when this value passes unity, for then the upper and lower half of the tube cross-section are in contact with each other and the problem changes into a contact problem. On the vertical axes, the external pressure (in MPa) is denoted. In the figures also the critical (buckling) pressure, due to TIMOSHENKO (1936), is noted. This value reads
\[
P_c = \frac{E}{4} \left( \frac{t}{R} \right)^3 \quad \text{(axially free tube)}
\]

The curves in the figure show the results obtained with TUBEHAVE. Clearly, around the pressure \( P_c \), the curves deviate from the initial course. The numbers next to the curves refer to the highest wave-length number in the Fourier series expansion. Due to the additional symmetry in the circumferential direction (with respect to \( \psi_2 = 0 \)), \( \text{DOFD} \) is given by \( \text{UVWA}(\langle 0 \rangle ; \langle 0, 2, 4, \ldots, X \rangle) \).
Figure 7.2.1 - Elastic response for relative thin and thick (axially free) tubes with initial ovality $\Delta_0 = 0.1\%$ for different number of Fourier terms.

In the analyses, the beam-stretch ($\zeta^B$) is essential, for the tube is axially free to expand ($\zeta^B$ leads to a deformation, which is constant in the axial direction). From the analyses, we found that for the thin-walled tube ($R/t = 96.15$), the $\psi_2$-dependency in the thickness function $\Lambda$ is not required. This is due to the fact that $\Lambda$ is essential for membrane deformations only. In the analyses for the thin-walled tube, the membrane deformation almost vanishes with respect to the bending deformation. However, in the analyses for both tubes, the full $\psi_2$-dependency in $\Lambda$ is taken.

The points denoted with $x$ in figure 7.2.1 are results from a (shell) finite element computation. The finite element formulation is described in VAN KEULEN (1993). In this calculation, 40 elements are used for the modelling of a quarter of the circumference (which
yielded sufficiently accurate results). In the formulation by VAN KEULEN, the pressure is defined on the mid-surface, whereas TUBEHAVE takes the exact external surface. Thus, in TUBEHAVE the pressurised surface is larger (in comparison, a factor $(9.615 + 0.5)/9.615 = 1.05$ for the thick tube and likewise, a factor $1.005$ in the thin case). This implies that, for the same pressures, the effective loading in TUBEHAVE is larger. After correction with this scale factor, the results in figure 7.2.1 fully agree.

It is concluded from the comparison in figure 7.2.1, that an accurate description of the ovalisation of a thin (elastic) tube requires more terms in the Fourier expansion then the description of the ovalisation of a thick tube. Clearly from figure 7.2.1, an accurate description of the ovalisation of the thin (elastic) tube under external pressure up to where contact occurs requires a Fourier expansion up to the terms in $(8\psi_2)$. In the case of the thick tube, the terms up to $(6\psi_2)$ suffice. The reason is that the ratio bending stiffness / membrane stiffness (i.e. $\theta(t^2/L^2)$) is much higher for the relatively thick tube. As a consequence, the thick tube deforms more smoothly than the thin tube.

**Elastic-plastic response**

It is well-known that plastic deformation reduces the material stiffness. Therefore, it is expected that the pressure which the tube is able to sustain becomes lower when plastic deformation occurs. The objective of the present analyses is to estimate the influence of plastic deformation. This is done for some values of the initial ovality $\Delta_0$.

To this end, the thick "axially free" tube ($R/t = 9.615$ and $\ell/R = 1.0$) is considered again. The elastic material properties are $E = 200$ GPa and $\nu = 0.333$. The hardening behaviour of the material is taken to be given by a piece-wise linear fit. This fit is shown in figure 7.2.2, where the value $\sigma_y = 300$ MPa. The stress-ratio along the vertical axis denotes the axial stress, normalised with $\sigma_y$.

![Figure 7.2.2 - Piece-wise linear uniaxial stress-strain curve.](image)
A comparison of the results with the solution based on elastic material properties shows the occurrence of a maximum loading point. Thus, the solution changes from a stable into an unstable one (stable in the sense of pressure-controlled applications like water pressure). In the analyses presented in this section, the Fourier expansions up to the terms in $(6\psi_4)$ suffices because of the fact that the displacements stay relatively small up to the maximum loading point. Thus, using the general definition of DOFD, UVWA$\{0\};\{0,2,4,6\}$ is taken.

The maximum loading points are verified against the results from YEH & KYRIAKIDES (1986). They apply equivalent kinematics as well as a uniaxial stress-strain curve, which is a smooth curve through the data points shown in figure 7.2.2. However, they assume the pressure to act on the mid-surface. Another difference (being the assumption of plane stress through-the-thickness by Yeh & Kyriakides versus the constant stress component through-the-thickness approximated by TUBEHAVE) is not important in bending dominated problems. A comparison of the maximum loading points for the various values of the initial ovality is given in the table below. (N.B. the values by Yeh & Kyriakides are estimated from figure 11 in their article.) The quantity $\delta$ is defined as the maximum radial displacement.

<table>
<thead>
<tr>
<th>$\Delta_0$</th>
<th>Max. pressure (MPa)</th>
<th>$\delta / R$ at max. pressure</th>
<th>Max. pressure (MPa)</th>
<th>$\delta / R$ at max. pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>27.5</td>
<td>0.007</td>
<td>25.9</td>
<td>0.010</td>
</tr>
<tr>
<td>0.01</td>
<td>21.5</td>
<td>0.013</td>
<td>20.8</td>
<td>0.017</td>
</tr>
<tr>
<td>0.05</td>
<td>12.0</td>
<td>0.027</td>
<td>11.6</td>
<td>0.027</td>
</tr>
</tbody>
</table>

*Table 7.2.1 - Comparison of the maximum loading point for an elastic/plastic (axially free) tube with initial ovalities $\Delta_0 = 0.1\%$, 1.0\% and 5.0\%.*

The difference between the values for $p_{\text{max}}$ reported by Yeh & Kyriakides and those obtained with TUBEHAVE is ranging from 3.5\% - 6\%. Realising that Yeh & Kyriakides assume the pressure to work on the mid-surface (as mentioned, this yields 5\% reduction), it is clear that the resemblance between the values for $p_{\text{max}}$ is quite good.

A comparison of the values for $\delta / R$ at $p_{\text{max}}$ is less satisfactory. However, it should be remarked that due to the horizontal slope of the curves at $p_{\text{max}}$, assessment of the value of $\delta / R$ at $p_{\text{max}}$ is extremely sensitive. Therefore, a few percent difference in $p_{\text{max}}$, which is observed above, may lead to a significant difference in $\delta / R$. 
7.3 - Pure bending

A typical situation leading to bending of a tube is the installation process of pipelines and risers in offshore applications. The "S"-lay method, as shown in figure 7.3.1, induces high curvatures especially when the pipe is pulled through the "J"-tube. In order to show some of the capabilities of the model for simulation of such operations, the deformation patterns that develop in bending of tubes are investigated.

![Diagram of S-lay method](image)

**Figure 7.3.1 - The "S"-lay method.**

In 1927, Brazier pointed out that bending of relatively thin-walled tubes induces ovalisation of the tube cross-section. Ovalisation causes a reduction of the moment of inertia of the cross-section and thus lowers the bending stiffness of the tube. Eventually, a maximum value for the bending moment is reached. It is known from experimental work of e.g. REDDY (1979) and KYRIAKIDES & JU (1992), that the limit load is often preceded by the development of short wave-length ripples at the compressed side of the tube. Soon after the appearance, the ripples start to localise to smaller regions and finally lead to a sharp kink in the tube. For thicker tubes, the experimental work has shown that the development of the axial ripples and the final kink exhibits to be less abrupt.

The aim of the present example is to demonstrate some of the capabilities of the model. We show how the types of deformation which successively develop in bending of tubes can be modelled. This is most clearly demonstrated for a thin-walled elastic tube. Therefore, a thin elastic tube under bending is considered first. Next, the influence of plastic deformation on the bending response is shown. Both analyses were performed using
"displacement control" on the bend angle \( \Theta^B \), up to the range where localisation of the axial ripples exhibits. In the analyses performed, the beam stretch \( \xi^B \) is taken to be free.

**Thin elastic tube**

An elastic tube with \( R = 100.0 \) mm, \( t = 1.0 \) mm and \( E = 200 \) GPa, \( v = 0.333 \) is considered. For reasons of comparison with the results by KYRIAKIDES & JU (1992.2), \( \ell = 95.8 \) mm is taken. The boundary conditions enforce both ends of the tube to remain plane and allow for ovalisation of the cross-sections in the "end planes", see section 4.3. Furthermore, symmetry with respect to the middle of the tube length is assumed. The loading is determined by two opposite bending moments at both end-sections of the tube; thus \( \lambda^{(5)} = -\lambda^{(6)} \), which load parameters have been defined in (5.2.8).

The result of the analyses for this tube configuration is shown in figure 7.3.2. Along the horizontal axis, the overall curvature of the tube is denoted: \( K^B = \Theta^B / (\xi^B \ell) \), see (4.2.15). The value of the applied bending moment is denoted on the vertical axis. The different curves in figure 7.3.2 show the influence of the types of deformation as mentioned above and will be discussed in more detail next.

![Plot showing influence of different types of deformation on the bending response of a thin-walled (elastic) tube.](image)

*Figure 7.3.2 - Influence of different types of deformation on the bending response of a thin-walled (elastic) tube.*

The curve denoted with "\{0/1\}" shows the response of the tube where the cross-sections are enforced to stay circular during the deformation. Set DOF is given by \( UVW\Lambda(0; \{0,1\}) \). These terms are essential for the description of pure bending. The 1st order terms in \( (\psi_d) \) are necessitated by the application of three-dimensional constitutive
relations. Due to the overall bending, the strain in the axial direction ($\eta_{11}$) varies from compressive to tensile around the circumference, leading to a $\sin(\psi_2)$-dependency. Without the 1st order terms in ($\psi_2$), the constitutive relations would result in an unrealistic three-dimensional stress situation. The introduction of the 1st order terms in ($\psi_2$) in DOFD leads to appropriate strains $\eta_{22}$ and $\eta_{33}$, such that a uniaxial stress situation is recovered, which is in agreement with the "classical result".

The curve denoted with "{0/1}" in figure 7.3.2 is in full agreement with the analytical expression (the "classical result")

$$K^B = \frac{M}{EI} , \text{ where } I = \pi R^4 t$$

which is valid for circular thin-walled cross-sections.

Secondly, ovalisation is introduced and it is assumed that the ovalisation is independent of the axial coordinate. It is experienced that the truncation of the Fourier series expansion in the circumferential coordinate at the 6th order term yields sufficiently accurate results (the ovalisation remains rather small). Thus, the set of degrees-of-freedom DOFD is taken as $UVWA\{\{0\}; \{0,1,2,\ldots,6\}\}$. Following from numerical comparisons, it is concluded that an extension using higher order terms does not alter the results as shown.

The influence of ("axially" constant) ovalisation is shown in figure 7.3.2, where the response is denoted with "{0/6}". Clearly, a limit load type of instability occurs. The curve is in perfect agreement with the result of REISSNER (1959). He derived the following relation (which is translated in our notation):

$$\frac{M}{EI} = \kappa^B \left\{ 1 - \frac{3}{2} \left( 1 - \nu^2 \right) \left( \frac{R}{t} \right)^2 R^2 \left( \kappa^B \right)^2 \right\}$$

For reasons of comparison with the results of KYRIAKIDES & JU (1992), the set of degrees-of-freedom is extended with the $\sin/\cos$ terms in ($4\psi_1$). (It is emphasised that the terms in ($4\psi_1$) are added only; the introduction of other terms in $\psi_1$ follows.) Thus, the set of degrees-of-freedom determining the "shell deformation" reads $UVWA\{\{0,4\}; \{0,1,2,\ldots,6\}\}$. Initial imperfections are introduced in DOFD $R = w\{\{4\}; \{0\}\}$, by setting

$$w = w_{40} \cos(4\psi_1) \text{ with } w_{40} = -10^{-4} R \text{ and also } -5.10^{-4} R$$

The moment-curvature response for both cases are denoted with "{0&4/6}"; the upper curve refers to the smaller value of the initial imperfection. Clearly from figure 7.3.2, for the smaller value of the initial imperfection the response deviates abruptly from the
"0/6"-response before the limit moment is reached. Comparison of this "deviation point" with the "bifurcation point" reported in KYRIAKIDES & JU (1992.2) yields a perfect agreement. The bifurcation point is also shown in figure 7.3.2.

Finally, localisation of the axial ripples is allowed for. Therefore, the axial dependency is taken to be given by the sin/cos terms up to $(6 \psi_1)$, where symmetry with respect to the middle of the tube length is assumed. Thus, DOFD is taken according to

$$UWVA\left\{0,2,4,6; 0,1,2, \ldots, 6 \right\}.$$

In this case, the initial imperfections are captured in $DOFD_R = w(2,4; 0)$ by setting

$$w = w_{20} \cos(2 \psi_1) + w_{40} \cos(4 \psi_1) \quad \text{with} \quad w_{20} = w_{40} = -5 \cdot 10^{-4} R,$$

which is a choice leading to localisation in the middle of the tube length. The result of this calculation is shown in figure 7.3.2 by the curve "(6/6)". It should be remarked that this response is not checked against the result of others. In KYRIAKIDES & JU (1992.2), no results for this case are given.

This example clearly shows the capabilities of the model in the case of pipe bending. The method is successful in tracing the types of deformation that govern the problem including the localisation process. A rigorous description of the localisation up to the development of the final kink would require more Fourier terms in both directions than employed in the "(6/6)"-case.

**Influence of plastic deformation**

A tube with $R = 22.0$ mm, $t = 1.0$ mm, $\epsilon = 32.6$ mm and $E = 67.2$ GPa, $\nu = 0.333$ is considered. In this case, the effect of plastic deformation is included. The uni-axial stress-strain curve of the tube material (a type of aluminum) is the piece-wise linear curve shown in figure 7.3.4 ($\sigma_s = 304$ MPa).

![Figure 7.3.4 - Piece-wise linear uniaxial stress-strain curve.](image-url)
This curve is taken for reasons of comparison with some of the results of KYRIAKIDES & JU (1992.2).

For this specific tube configuration, calculations with the four sets of degrees-of-freedom as mentioned in the previous section are performed. A difference with the elastic analyses is that the value of the participation factors determining the initial imperfections in (7.3.3) and (7.3.4) in the respective analyses are taken to be \(-10^{-3}\) R.

The moment-curvature responses from the analyses are given in figure 7.3.5 and denoted likewise as in the elastic case. Clearly, the curves are more smooth than in the elastic case.

![Figure 7.3.5 - The effect of axial rippling for a relatively thick-walled tube.](image)

Verification of the "\{0/6\}" curve is done with the curve presented in KYRIAKIDES & JU (1992.2). Specific values for the maximum bending moment and the corresponding bend-angle in the "\{0/6\}" case are \(M = 0.56 \times 10^6\) Nmm (0.57 \(\times 10^6\)) at \(\kappa^B = 0.00072\) mm\(^{-1}\) (0.00069). The values between the brackets are estimated from the curves in KYRIAKIDES & JU (1992.2). The correspondence between the values is quite satisfactory. A comparison of the limit loading point in the "\{0&4/6\}" curve yields \(M = 0.55 \times 10^6\) Nmm (0.56 \(\times 10^6\)) at \(\kappa^B = 0.00064\) mm\(^{-1}\) (0.00059), which is also satisfactory. As for the elastic case, a comparison for the "\{6/6\}" curve is not possible.
7.4 - Combined bending and external pressure

In pipeline laying processes (like for example the "S"-lay method, see figure 7.3.1), the pipes are empty payed out into the sea. As a consequence, the pipes are subjected to external hydrostatic pressure. The external (over-) pressure on an empty pipe increases linearly with the water depth. In combination with the external pressure, the pipe is subjected to the curvatures induced by the laying process. Therefore, questions that could arise for a safe installation of offshore pipelines are amongst others: What is the maximum allowable curvature of the tube given a certain external pressure? Or, the other way around: What is the maximum allowable external (over-) pressure given a certain curvature?

As demonstrated in the previous examples (see sections 7.2 and 7.3), the deformation behaviour of tubes under external pressure as well as bending of tubes is characterised by ovalisation of the cross-sections. Therefore, it is expected that the presence of external pressure makes the ovalisation due to bending more severe.

It is well-known that plastic deformation is path-dependent. This means that the equilibrium solution for a certain actual loading on the system is not uniquely determined. Additional information concerning the loading history is required for a unique solution. In terms of the current application, this means that pressurisation up to $P_l$ followed by bending with end bending moments $M_l$ at the one hand and first bending with end moments $M_l$ followed by pressurisation up to $P_l$ at the other hand, will not necessarily lead to identical tube configurations.

The aim of the present example is to show the path dependency of the deformation for a typical tube configuration under combined external pressure and bending. To this end, two types of loading histories are considered: "bending first / bending & pressure next" (abbreviated, the "M-P"-case) as well as "pressure first / pressure & bending next" (the "P-M"-case). Verification of the results is done against experimental data reported by CORONA & KYRIAKIDES (1988). The verification of the analyses will be restricted to the limit values for the second loading condition.

For reasons of verification of the results, a tube with the following geometric and material properties is taken: $R = 17.35 \text{ mm}$, $t = 1.0 \text{ mm}$ and $\ell = 147.0 \text{ mm}$ with an initial ovality of the cross-sections determined by $\Delta_0 = 0.06\%$, which is not depending on the axial coordinate. The elastic material properties read $E = 186.0 \text{ GPa}$ and $\nu = 0.333$. A Ramberg-Osgood fit on the uniaxial stress-strain curve of the material is given by CORONA & KYRIAKIDES (1988). As an approximation of this curve, the piece-wise linear fit as shown in figure 7.4.1 (where $\sigma_y = 224 \text{ MPa}$) has been used.
Figure 7.4.1 - A piece-wise linear approximation of the Ramberg-Osgood fit on the uniaxial stress-strain curve of the material used in the experiments by CORONA & KYRIAKIDES (1988).

It is assumed that the ovalisation of the cross-sections is constant in the "axial" direction of the tube. The tube is axially free to expand, thus the beam-like stretch factor $\xi^B$ is not restrained. The set of degrees-of-freedom governing the "shell deformation" reads $UW\Lambda\{\{0\};\{0,2,4,6,8\}\}$. The introduction of initial ovality $\Delta_0$ in the model is done by setting $w_{02} = \Delta_0 R$ as well as $v_{02} = -1/2 \Delta_0 R$, where $DOF_D.R = v\{\{0\};\{2\}\}$.

In the case of bending, the loading is determined by two opposite bending moments at both end-sections of the tube, thus $\lambda^{(5)} = -\lambda^{(6)}$. These load parameters have been defined in (5.2.8).

**bending first / bending & pressure next**

In this case, the tube is incrementally loaded to a chosen value of the overall bend angle at zero pressure. At the chosen value, the angle is fixed and the external pressure is incremented. The calculation is performed until a limit value of the pressure is obtained. The results for the limit values of the pressure under given fixed values of the overall bend angle are denoted with the drawn line in figure 7.4.2. The other points in the figure are experimental results as plotted by CORONA & KYRIAKIDES (1988). The loading history is also indicated in the figure. The quantity "dimensionless pressure" along the vertical axis denotes the value of load parameter $\lambda^{(8)}$, see (5.3.7). A discussion of the results follows.
Figure 7.4.2 - Comparison of the limit external pressure at fixed overall bend angles (see load history) with experimental data.

pressure first / pressure & bending next

In this case, the tube is first pressurised up to a chosen value and afterwards subjected to bending. At the chosen value, the external pressure is fixed. The calculation is performed until a limit value of the end bending moments is achieved.

Figure 7.4.3 - Comparison of the overall bend angle corresponding to the limit moment at fixed external pressure with experiments.
The drawn line in figure 7.4.3 shows the value of the prescribed external pressure as a function of the overall bend angle, which corresponds to the limit moment. Again, the other points in the figure are experimental results as plotted by CORONA & KYRIAKIDES (1988).

In figure 7.4.3 (the "P-M"-case), the correspondence between the drawn line and the (scattered) experimental data is satisfactory. For the "M-P"-case, see figure 7.4.2, especially at the higher values of the fixed overall bend angle, the difference between the results by TUBEHAVE and the experimental data in CORONA & KYRIAKIDES (1988) becomes larger. A reason for this difference could be the following.

In the numerical analyses presented here, the ovalisation is assumed to be independent of the axial coordinate. Thus, the description of the development of axial ripples under bending loading, as demonstrated in section 7.3, is not included. It is concluded from section 7.3 that the axial ripples develop at larger overall curvatures and lowers the limit value of the applied bending moment. This could explain the difference observed in figure 7.4.2, since in the experiments the axial ripples could have been developed while in the numerical analyses the description of such ripples is not included.

It is doubted whether the Bauschinger effect could be another reason for the observed difference. The Bauschinger effect is observed in solids which unload elastically after being loaded into the plastic range, but in which (upon reloading), renewed plasticity begins before reaching the reversed stress level. In the "M-P"-case, the first loading condition (bending) leads to an axial stress component, which causes yielding in some parts of the material. At the next stage, driven by the external pressure, a negative hoop stress and a negative stress through-the-thickness develops. At those integration points with three negative stress components, the increasing external pressure makes the effective stress slightly lower (thus, elastic unloading occurs). However, the effective stress will never become such low that the reversed stress level is reached.
7.5 - Internal pressure

Transportation of a fluidum through a pipeline is caused by (or, vice versa, leads to) a pressure-drop over the total length of the line. Therefore, a pipeline in the "normal" production status requires an internal pressure which is higher at the inlet than at the outlet. Internal pressures at the inlet may be significant. In the present example, the response of some typical tube configurations under internal pressure is investigated.

In tubes under internal pressure, it is observed that the following deformation modes develop. In first instance, the tube expands uniformly, which (due to contraction and the incompressibility of the material under plastic deformation) leads to thinning of the tube wall. The thinning of the tube wall may lead to the occurrence of a maximum pressure. Localisation of the deformation is caused by small imperfections, which are always present in tube configurations in practice. It is observed in practice that, for the thicker tubes, localisation occurs after the attainment of the pressure maximum.

Corrosion is a major problem in subsea oil/gas pipelines, see the inventory of pipeline failures by MANDKE (1990). This article is focused on the Gulf of Mexico, however the failure behaviour of pipelines in other regions cannot be much different. A severe corrosion pattern which has been observed is caused by carbon dioxide dissolved in water. Water is also present in the produced fluidum. Due to the higher relative weight of water than oil, this type of corrosion takes place at the bottom of the line. In most cases, the corrosion pattern develops at relative long axial length. The capabilities of the model in the description of the deformation behaviour of pipes with this type of corrosion is shown in the second part of the example.

The analyses performed in the present example are subdivided into two idealised types (in practice all sorts of combinations of types will occur):

- **Straight tube with a uniform wall-thickness** (section: "perfect" pipes). In these analyses, circular cross-sections are assumed throughout the deformation. Apart from assessment of the pressure maximum, also the localisation in the axial direction of the (axi-symmetric) expansion is investigated.

- **Straight tube with an initial wall-thickness which varies along the circumference** (section: "corroded" pipe). In this case, the deformation is assumed to be constant in the axial direction. Localisation of the expansion in the circumferential direction is investigated.

The material for both types of tubes is taken to be a steel with the API grade X52. As an approximation of the uniaxial stress-strain curve of such material, we take the power-law
hardening function (7.1.2). The parameters read $E = 210$ GPa, $\sigma_y = 249$ MPa, and $h = 0.105$. Graphically, the curve is shown in figure 7.5.1. For completeness of the elastic constitutive properties, $\nu = 0.333$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig751.png}
\caption{A power-law hardening function fitted on the uniaxial stress-strain curve for steel grade X52.}
\end{figure}

"Perfect" pipes

The geometric properties are taken as follows: $R = 16.0$ mm, $t = 1.0$ mm and $\ell = 48.0$ mm. At both "end planes" of the tube, radial expansion is allowed for. The "shell deformation", at first instance, is taken according to $\text{DOFD} = \text{UVWA}(\{0\}; \{0\})$. In this section, both an axially free and an axially restrained tube are considered.

The responses for both tubes, in terms of the dimensionless expansion $W/R$ versus the internal pressure, are shown in figure 7.5.2. In view of DOFD, the expansion $W = W_0$, thus $W$ is uniform over the entire tube model. The results for the current calculations are denoted with "ideal". The distinction between the "axially free" and the "axially restrained" case is also shown in the figure.

The maximum loading points in both curves (denoted with x) reasonably compare with the results obtained with formulae given by KLEVER (1992). With the current choice of parameters, his formulae yield for the axially free tube: $p_{\text{max}} = 27.3$ MPa (27.6 MPa) at $W/R = 0.070$ (0.073), where the values between the brackets are obtained with TUBEHAVE. For the axially restrained tube, the results are: $p_{\text{max}} = 31.0$ MPa (31.0 MPa) at $W/R = 0.053$ (0.056). The values for the pressure obtained with the formulae by KLEVER (1992) are scaled to pressure loading on the internal surface.
Figure 7.5.2 - Internal pressure - radial expansion curves for an axially free and axially restrained tube. The localisation of the expansion for both cases is also shown.

Secondly, the localisation of the radial expansion is investigated. To this end, axially dependent deformation is allowed for, where the requirement of circular cross-sections is still retained. The analyses are performed for both the axially free and the axially restrained tube. The initial imperfection is defined by a small variation in the wall-thickness as follows:

\[ \lambda = \lambda_{20} \cos(2\psi_1) \quad \text{with} \quad \lambda_{20} = -0.1\% \ t \]

which leads to a tube which has a slightly smaller value of the wall thickness in the middle than at both the "end planes" (thus, DOFD_R = \( \lambda(\{2\};\{0\}) \)). In the calculations, symmetry with respect to the middle of the tube length is assumed, where the "shell deformation" is captured by DOFD = UVWA(\(\{0,2,4,6\};\{0\})\). In figure 7.5.2, the pressure-expansion response for the cross-sections at the middle ("middle"-curves) as well as at the "end planes" ("right side"-curves) are shown.

The effect of the wall thickness variation given in (7.5.1) is that localisation of the deformation is initiated at the middle of the analysed tube. This can be seen from the "axially free"-curves in figure 7.5.2, where the dimensionless expansion (given a certain value for the internal pressure) is larger at the middle of the tube than at the end planes. This bulging effect is much less pronounced for the axially restrained tube. This is due to the relatively large axial displacements associated with the radial displacements required for the bulging. In the axially free case, the axial displacements do not lead to an axial loading on both end-sections. However, for the axially restrained case, the axial displacements give rise to relatively large axial stresses which in turn limit the bulging.
"Corroded" pipe

The axially restrained tube as analysed in the previous section is considered again. However in this case, a corrosion pattern as shown in figure 7.5.3 is assumed. This pattern is constant in the axial direction.

\[ \lambda(\psi_2) = \begin{cases} 
\text{thin} & , \psi_2 \in \left[\frac{-\pi}{2}, \frac{-\pi}{3}\right] \\
\text{thin} + \frac{\text{thick-thin}}{\pi/6} \left( \psi_2 + \frac{\pi}{3} \right) & , \psi_2 \in \left(\frac{-\pi}{3}, \frac{-\pi}{6}\right) \\
\text{thick} & , \psi_2 \in \left[\frac{-\pi}{6}, \frac{\pi}{2}\right] 
\end{cases} \]  

(7.5.2)

Figure 7.5.3 - Idealised corrosion pattern, which is assumed to be constant in the axial direction.

Note that the thickness over a 60°-part of the circumference is "thin" and over a 240°-part thick. In between these parts, the thickness varies linearly from "thin" to "thick". It is granted that the geometry as shown above is a simplification of the reality, where corrosion takes place at the inner surface. However, it is expected that the deformation approximates the reality.

For the present application, the deformation is taken to be constant in the axial direction. The wall-thickness as shown in figure 7.5.3 is approximated by setting appropriate values to the participation factors in the thickness function in the reference configuration. These values are obtained by means of a Fourier transform. To this end, \( \lambda \) is defined according to

\[ \lambda(\psi_2) = \begin{cases} 
\text{thin} & , \psi_2 \in \left[\frac{-\pi}{2}, \frac{-\pi}{3}\right] \\
\text{thin} + \frac{\text{thick-thin}}{\pi/6} \left( \psi_2 + \frac{\pi}{3} \right) & , \psi_2 \in \left(\frac{-\pi}{3}, \frac{-\pi}{6}\right) \\
\text{thick} & , \psi_2 \in \left[\frac{-\pi}{6}, \frac{\pi}{2}\right] 
\end{cases} \]  

(7.5.2)
The Fourier transform reads:

\[
\begin{align*}
\lambda_{0,0} &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \lambda(\psi_2) \, d\psi_2 \\
\lambda_{0,2n} &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \lambda(\psi_2) \cos(2n\psi_2) \, d\psi_2 \\
\lambda_{0,(2n-1)} &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \lambda(\psi_2) \sin((2n-1)\psi_2) \, d\psi_2
\end{align*}
\] (7.5.3)

In the present application, the convergence characteristics with respect to the amount of terms in DOFD for one value of thin/thick are investigated. A tube with a 33% corrosion depth is taken, thus thin/thick=0.667. In the analyses, DOFD = UVWA\(\langle 0 \rangle; \langle 0,1,2, \ldots \rangle; X \rangle)\) is taken, where the value X is varied. In table 7.5.1, we sequentially list the value of X, the value of the participation factor that is added in the Fourier series of the thickness function \(\lambda\), the maximum value for the internal pressure, and the associated values of the hoop strain (in the thin part \(\psi_2=\pi/2\), at the middle \(\psi_2=0\) and in the thick part \(\psi_2=\pi/2\), respectively).

<table>
<thead>
<tr>
<th>X</th>
<th>(\lambda_{0,X})</th>
<th>(p_{\text{max}}) [MPa]</th>
<th>(\varepsilon_{\text{hoop}}^{\text{thin}})</th>
<th>(\varepsilon_{\text{hoop}}^{\text{mid}})</th>
<th>(\varepsilon_{\text{hoop}}^{\text{thick}})</th>
</tr>
</thead>
<tbody>
<tr>
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<td>28.4</td>
<td>0.056</td>
<td>0.056</td>
<td>0.056</td>
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<tr>
<td>1</td>
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<td>0.031</td>
</tr>
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<td>2</td>
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<td>0.099</td>
<td>0.058</td>
<td>0.008</td>
</tr>
<tr>
<td>3</td>
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<td>25.4</td>
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<td>0.011</td>
</tr>
<tr>
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<td>0.028</td>
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</tr>
<tr>
<td>5</td>
<td>-0.022</td>
<td>22.7</td>
<td>0.14</td>
<td>0.026</td>
<td>0.003</td>
</tr>
<tr>
<td>6</td>
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<td>22.1</td>
<td>0.13</td>
<td>0.022</td>
<td>0.003</td>
</tr>
<tr>
<td>7</td>
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<td>0.12</td>
<td>0.019</td>
<td>0.002</td>
</tr>
<tr>
<td>8</td>
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<td>0.11</td>
<td>0.017</td>
<td>0.002</td>
</tr>
<tr>
<td>9</td>
<td>0.005</td>
<td>21.9</td>
<td>0.11</td>
<td>0.017</td>
<td>0.002</td>
</tr>
<tr>
<td>10</td>
<td>0.004</td>
<td>21.9</td>
<td>0.11</td>
<td>0.017</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Table 7.5.1 - Convergence characteristics for the pressure maximum and the associated hoop strain at different locations with respect to amount of Fourier terms.

A graphical interpretation of table 7.5.1 is given in figure 7.5.4, which shows the obtained value for the pressure maximum as a function of the amount of Fourier terms used.
in DOFD. These results are denoted with "TUBEHAVE (approx. thickness)". It is emphasised that the Fourier approximation of the thickness function in the reference configuration in this case is truncated after the Xth term. In the figure also the results are shown of analyses where the function λ is determined by the first 10 Fourier terms and where the value X refers to the amount of Fourier terms for the displacements only. Thus, in these analyses, DOFD equals UVW(0; 0,1,2, .......,X) & λ(0; 0,1,2, .......,10). The results of the latter analyses are denoted in the figure with "TUBEHAVE (complete thickness)".

Figure 7.5.4 - Maximum pressure as a function of the value X in DOFD, defined as UVW(0; 0,1,2, .......,X): "approx. thickness", as well as UVW(0; 0,1,2, .......,X) & λ(0; 0,1,2, .......,10): "complete thickness".

Apart from the TUBEHAVE results for the corroded pipe, figure 7.5.4 shows three other values for the maximum pressure. First, the upper and lower bound values are mentioned, which are obtained with TUBEHAVE by assuming a constant wall-thickness "thin" and a constant wall-thickness "thick". Secondly, the value obtained with the formula presented by KLEVER (1992) is mentioned. This formula is based on the assumption that the pipe, when deforming, remains circular in cross-section. He found that the maximum pressure of a corroded pipe can best be expressed in terms of the hoop stress in the thin section. A result obtained with his formula (by setting the defect arc-length ratio 1/6) and the TUBEHAVE results show extremely good correspondence.

This particular burst analysis of a corroded pipe has also been simulated using the general finite element package MARC (1993). The response curves (in terms of the strains versus the internal pressure) up to the maximum loading point show good correspondence
with the TUBEHAVE results (with \( \text{DOFD} = \text{UVWA}((0); (0,1,2, \ldots, 8)) \)), see figure 7.5.5. In the MARC-analysis, finite strain shell elements have been used.

![Graph showing comparison of hoop strains](image)

**Figure 7.5.5 - Comparison of the hoop strains at the thin \((\psi_2 = -\pi/2)\) and at the thick part \((\psi_2 = \pi/2)\), between the results obtained with TUBEHAVE (DOFD = UVWA((0); (0,1,2, \ldots, 8))) and MARC.**

Clearly from table 7.5.1 and figure 7.5.5, the deformation accumulates within the thin section. This is confirmed with figure 7.5.6, which shows the deformed mid-surface of the tube at the maximum internal pressure. The dots shown in the figure indicate the positions of the used integration points. The arrows in the figure indicate some typical displacement vectors. It is clear that the thin section is stretched and expanded more than the thick part.

![Graph showing reference and deformed mid-surface](image)

**Figure 7.5.6 - Reference and deformed mid-surface at the maximum internal pressure; the arrows indicate the displacement vectors for some typical integration points.**
7.6 - Axial compression

A certain amount of observed pipe buckling failures is due to axial compression of the line. These failures are caused by, for example, ground movements (whether or not caused by earthquakes) which put the line into compression. Another example leading to axial compressive stresses is the thermal expansion of an axially restrained tube.

Documented pipeline buckling failures due to axial compression fall into two main categories. The smaller diameter lines, usually buried in relatively shallow trenches, tend to buckle like beams. This so-called "upheaval buckling" makes the line tear out of the ground and bend up. In this case, the deformation is often such that the pipe remains intact. The buckling pattern of the somewhat larger diameter pipelines buried in deeper trenches is more locally. Under compression, these buckle over a small section into a mode with a certain number of axial and circumferential waves.

In the present example, the deformation behaviour of tubes (both stubby and slender) under axial compression is investigated. The first part deals with the elastic response of a tube, where a wrinkling mode (short wave-length deformation in both the axial and the circumferential direction) develops. In the second part, the post-buckling behaviour of a stubby (plastic) tube is analysed. Finally, axial compression of a slender (elastic-plastic) tube is considered.

(elastic) Stubby tube

A tube with $\ell=20.0\,\text{cm}$, $R=36.0\,\text{cm}$ and $t=0.125\,\text{cm}$ is considered. The elastic material properties are $E=200\,\text{GPa}$ and $\nu=0.30$. The boundary conditions require both ends of the tube model to remain plane, although deformation of the cross-section in the plane itself is allowed for. The loading is determined by the following combination of basic loads: $\lambda^{(1)} = -\lambda^{(2)}$.

For this tube configuration, the buckling mode is "diamond" shaped. The amount of sinusoidal waves of the radial displacement in both the axial and the circumferential direction is determined upon the geometry and the boundary conditions. The critical axial stresses corresponding to "diamond" buckling modes for tubes without end-constraints is given by TIMOSHENKO (1936). Translated in our notation, the simplified expression for the critical axial stress corresponding to a "diamond" mode with $m$ sinusoidal half-waves in the axial direction and $2n$ sinusoidal half-waves in the circumferential direction reads
\[
\frac{\sigma_c}{E} = \left( \frac{1}{R} \frac{Y}{Y^2} \right)^2 + \frac{1}{Y^2}, \quad Y = \frac{n^2 + \lambda^2}{\lambda}, \quad \lambda = \frac{m \pi R}{\ell} \quad \text{(7.6.1)}
\]

The lowest value for the critical stress is obtained for \( m = 1 \) & \( n = 12 \), which yields \( \sigma_c / E = 0.0021 \).

The post-buckling behaviour for this tube configuration is investigated via the introduction of (small) initial imperfections in the shape of the first critical buckling mode as mentioned above. It is observed that all (i.e. 7) strain measures defined in Chapter 2 play a relevant role in this type of deformation. Therefore, the post-buckling behaviour with respect to the mentioned "diamond" shape is considered to be a reliable benchmark.

KOITER (1945) was the first to formulate a general theory which for axially compressed cylinders reveals the extreme imperfection sensitivity. He also presented asymptotic formulae for the determination of the initial post-buckling behaviour. A more general study on the post-buckling behaviour of axially compressed tubes is given in DONNELL & WAN (1950). A numerical investigation for one typical configuration is presented in the thesis of STANLEY (1985) and also in SIMO et al. (1990.0). For reasons of validation, the same tube configuration is analysed in the present section.

In the circumferential direction, the numerical integrations are performed over a 15°-part only. Defining \( X \) half-waves over the 15°-part of the circumference and \( X \) half-waves over the length of the tube means that the definition of the degrees-of-freedom reads UVWA\{0, 1, 2, \ldots, X; 0, 12, 24, \ldots, 12.X\}. The numbers next to the curves in figure 7.6.1 refer to the chosen value of \( X \). The curves in figure 7.6.1 belong to a tube with an initial imperfection in the shape of the mentioned buckling mode; the amplitude is taken \( w_{112} = 10\% \tau \).

In order to compare the results to those reported by STANLEY (1985), the "axial deflection" \( \Delta U \) is defined as being the deflection at the middle of the tube relative to the right-hand end-plane, thus

\[
\Delta U(\psi_2) = \left( \xi^B - 1 \right) \frac{\ell}{2} + U^B(\psi_1 = 0, \psi_2) - U^B(\psi_1 = \frac{\pi}{2}, \psi_2) \quad \text{(7.6.2)}
\]

At a certain value of \( \psi_2 \), \( \Delta U \) reaches a maximum value. This value is plotted in figure 7.6.1 for different sets of degrees-of-freedom. Along the vertical axis, the mean axial compressive stress normalised with Young's modulus is denoted. The points marked with \( \hat{\circ} \) in the figure are results picked from the graph presented by STANLEY (1985).
Figure 7.6.1 - Normalised axial load, versus maximum axial deflection at the middle of the tube. The convergence with respect to the amount of degrees-of-freedom is shown. The points marked with ◊ are due to STANLEY (1985).

The curve denoted with "lin-elast" shows the result of a computation where geometric non-linearities are not accounted for. On this curve, the critical state (according to Timoshenko's formula (7.6.1)) is marked. The response, the so-called primary path, is dominantly governed by membrane deformations. After the "snapping", bending and membrane deformations both play an important role.

The equilibrium paths are traced by means of displacement control on $W_{t12}$, which works satisfactory (no problems are encountered around the snapping point). It is remarked that, although the axial deflection "snaps", participation factor $W_{t12}$ is monotonically increasing. By application of the general method using arc-length control, the "snapping" part of the curve is also traced. However, around the load-maximum (the transition from a membrane state into a bending state of stress) a number of restarts with a smaller arc-length were required.

Clearly, the convergence of the curves to the "exact" results of STANLEY (1985) is rather slow. It is concluded that "snapping" is accurately traced by setting $DOFD$ equal to $UVWA\{(0,1,2,3,4,5);(0,12,24,36,48,60)\}$. However, in practice, the set of degrees-of-freedom with $X=3$ suffices for it predicts both the maximum and the minimum axial load quite adequately. It is striking that for the case $X=1$ no "snapping" behaviour is traced, which means that the higher order terms in both the axial and circumferential direction are required for a proper description of the buckling behaviour.
The results in figure 7.6.1 are presented for reason of verification of the results. However, the physical meaning of the axial deflection as defined in (7.6.1) is at least vague. In practical circumstances, one experiences the shortening of the pipe. Due to the definition of the "shell deformation", which vanishes at both the end-sections, the shortening simply reads \((\zeta^B - 1)\ell\). In figure 7.6.2, a comparison is made between the maximum axial deflection as defined before and half the shortening of the pipe. The curves show the response for different amplitudes of the initial imperfections; i.e. \(w_{5/12} = 1\%t, 10\%t, 50\%t\) and \(100\%t\). In all computations, the set of degrees of freedom with \(X=5\) is applied. The same qualitative relation between the maximum axial deflection and the half-shortening of the pipe is also shown in the curves as presented in SIMO et al. (1990.7).

\[
\begin{array}{c}
\text{Figure 7.6.2 - Maximum axial deflection and half-shortening versus the axial loading for different initial imperfection amplitudes.}
\end{array}
\]

The above figure clearly shows the extreme imperfection sensitivity. The knock-down factor (which is the ratio between the load maximum for the imperfect versus the perfect tube) for a tube with \(10\%t\)-imperfection is already 0.75. Further, the curves show that the "snapping" behaviour for the half-shortening is sharper than for the maximum axial deflection. However, the tangent to the curves after the "snapping" has occurred is higher for the half-shortening than for the maximum axial deflection.

\(\text{(plastic) Stubby tube}\)

The wrinkling mode as discussed in the previous section is not bound to occur in relatively thick-walled tubes. This is due to the fact that the ratio bending stiffness / membrane
stiffness for thick-walled tubes is much higher than for the more thin-walled tubes. As mentioned by YUN & KYRIAKIDES (1988), thicker tubes tend to bifurcate into an axisymmetric mode instead of a "diamond" mode. In the current example, we demonstrate the capabilities of the model in the description of an axisymmetric deformation mode. One typical tube configuration is analysed.

For reasons of verification, a tube with the following geometric properties \( \ell = 3.6 \text{cm}, \ R = 32.0 \text{cm} \) and \( t = 10 \text{cm} \) is considered. Axisymmetric imperfections are assumed to consist of one sinusoidal half-wave over the length \( \ell \), where different amplitudes will be taken. The elastic material properties are \( E = 200 \text{GPa} \) and \( \nu = 0.30 \). The plasticity effect is defined by the power-law fit on the uni-axial stress-strain curve, where \( \sigma_s = 276 \text{ MPa} \) and \( h = 0.10 \). The over-all bend-angle is fixed (\( \Theta^B = 0 \)). Again, the loading is determined by taking \( \lambda^{(1)} = -\lambda^{(2)} \).

The aim of the present analyses is to verify the results for the maximum load levels with the results reported in YUN & KYRIAKIDES (1988). These latter results are obtained with a shell theory based on Kirchhoff-Love assumptions only. The constitutive description applied in this reference (i.e. J2-flow theory with isotropic hardening) is equivalent to the description in TUBEHAVE.

In our analyses, DOFD equals \( UVWX(\{0\};\{0,1,2\}) \), which suffices for the description of the deformation up to the maximum load level. It is experienced that introduction of more terms in DOFD does not alter the results. All analyses revealed a simple limit load type of instability. The numerical data concerning the limit loads for various imperfection amplitudes are given in table 7.6.1 (the data in the columns Yun & Kyriakides are picked from the graph in the article mentioned before).

Clearly, the correspondence for the load maximum is quite good, although better for the loading parameter than for the associated mean axial strain in the tube, which reads \((-)\mu^B\).

<table>
<thead>
<tr>
<th>(w_{01}/t)</th>
<th>Max. axial load ((-N/N_y))</th>
<th>(-\mu^B) at max. axial load</th>
<th>Max. axial load ((-N/N_y))</th>
<th>(-\mu^B) at max. axial load</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 %</td>
<td>1.48</td>
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<td>1.50</td>
<td>0.012</td>
</tr>
<tr>
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<td>1.30</td>
<td>0.0082</td>
<td>1.32</td>
<td>0.0094</td>
</tr>
<tr>
<td>50 %</td>
<td>0.88</td>
<td>0.0082</td>
<td>0.91</td>
<td>0.0092</td>
</tr>
</tbody>
</table>

Table 7.6.1 - Comparison of the maximum load level for an axially compressed tube with different initial imperfections.
**Elastic-plastic) Slender tube**

In the case of axial compression of slender tubes, the buckling mode is of the so-called "Euler type". This means that the associated displacements are such that the cross-sections are shifted within the plane of consideration and that the shape of the cross-sections (at least, in the first instance) remains circular. In this section, the "Euler type" of buckling for a tube is considered. Two types of analyses are performed. In the first type, the material is assumed to behave elastically, whereas in the second type, plastic deformation is also accounted for. The parameters are chosen such that the buckling (in both types of analyses) occurs in the elastic regime.

The problem is defined as follows. Geometrically, the tube is defined by $\ell = 5000.0 \text{mm}$, $R = 25.0 \text{mm}$ and $t = 1.0 \text{mm}$. Further, $\Theta^b = 0$ is taken. Ovalisation in the end-planes is allowed for. Initial imperfections for the line-of-centroids ($\bar{\phi}^m_0$) in the shape of an overall cosine wave are introduced by setting $v_{21} = w_{21} = \delta$, which defines the amplitude of the cosine wave. The fact that this indeed defines an overall cosine wave becomes clear by substitution of $v_{21} = w_{21} = \delta$ into the expression for $\bar{\phi}^{s,0}$, which is defined in Chapter 4. It yields

$$
\bar{\phi}^{s,0} = \left\{ \delta \cos(\psi_2) \cos(2\psi_1) \right\} \bar{b}_2 + \left\{ \delta \sin(\psi_2) \cos(2\psi_1) \right\} \bar{b}_3 = \\
\left\{ \delta \cos(\psi_2) \cos(2\psi_1) \right\} \left\{ \cos(\psi_2) \bar{I}_2 - \sin(\psi_2) \bar{I}_3 \right\} + \\
+ \left\{ \delta \sin(\psi_2) \cos(2\psi_1) \right\} \left\{ \sin(\psi_2) \bar{I}_2 + \cos(\psi_2) \bar{I}_3 \right\} = \\
= \delta \cos(2\psi_1) \bar{I}_2
$$

(7.6.3)

In the results that are shown, $\delta = 0.1R$ and $\delta = R$ is taken.

In the first instance, elastic material behaviour is enforced by setting the (first) yield value $\sigma_y$ sufficiently large. Elastic material properties $E = 200 \text{GPa}$ and $v = 0.30$ are taken. The results are obtained by setting DOFD equal to UVWA$\{\{0, 2, 4, 6, 8, 10\}; \{0, 1, 2\}\}$ in the analyses. The results will be presented in terms of the line of mean positions $\bar{\Phi}^m(\xi_1)$, defined in (4.3.4). In figure 7.6.3, the lines $\bar{\Phi}^m(\xi_1)$ for four different load levels are shown.
Verification of the results is carried out by comparison of the axial shortening and the mid-span deflection with the so-called "elastica" solution given by TIMOSHENKO (1936). The method discussed in the section "Large Deflections of Buckled Slender Bars" is applied to the current case. In terms of the displacement components $x$ and $y$ as depicted in figure 7.6.3, the "elastica" solution reads

$$
\begin{align*}
\text{(axial shortening)} & \quad x = \frac{8}{k} \int_{0}^{\xi} \frac{p^{2} \sin^{2} \phi}{\sqrt{1 - p^{2} \sin^{2} \phi}} d\phi \\
\text{(half mid-span deflection)} & \quad y = \frac{2p}{k}
\end{align*}
$$

(7.6.4.a)

where $p$ is a load parameter which determines the axial force $N$ as follows

$$
k = \frac{4}{L} \int_{0}^{\xi} \frac{d\phi}{\sqrt{1 - p^{2} \sin^{2} \phi}} , \quad k^{2} = \frac{N}{EI}
$$

(7.6.4.b)

This recipe is used as follows. Via incrementing the value of $p$ (in our calculation from 0.0 to 1.0 in 100 increments), relations (7.6.4.b) yield certain values for $k$ and thus also for the axial force $N$. Corresponding to these values, (7.4.6.a) yield successive sets ($x, y$). In our calculation, the elliptic integrals in (7.6.4) are numerically evaluated. The results obtained are shown in figure 7.6.4 and denoted with "elastica". It is clear from the results in the figure that the TUBEHAVE results for both imperfection amplitudes converge to the "elastica"-solution.
Figure 7.6.4 - Load versus shortening and half mid-span deflection; comparison of results obtained with TUBEHAVE against the "elastica"-solution due to TIMOSHENKO (1936) and stated in (7.6.4).

Finally, the influence of plastic deformation is shown. To this end, we re-consider the \( \delta = 1.0 \) case shown above and assume a different material behaviour; i.e. elastic/perfectly plastic. This behaviour is achieved by setting a small value for the hardening coefficient \( h \) in the power-law hardening function (7.1.2). In the analysis carried out, we have put \( h = 0.005 \) and \( \sigma_y = 2.0 \) GPa \( (\varepsilon_y = 1.0\%) \), where the elastic material properties are similar to those above. The response curves are also shown in figure 7.6.4. It is remarked that these curves are added only for illustrative reasons and that no verification of these results is carried out. The curves show that the axial load reaches a maximum value...
and thus the solution after this maximum is unstable, in contrast to the elastic case, where the solution remains stable.

As a final remark, it is emphasised that in reality short wave-length rimpling starts to develop at certain regions of the tube. These regions are allocated where the compressive stresses due to the axial compressive loading is combined with compression due to bending of the tube. Clearly from figure 7.6.3, those regions are situated at both ends and at the middle of the tube model. Further, it is concluded from the same figure that as the axial loading increases, the induced bending deformation localises and the regions where short wave-length rimpling develops becomes smaller.

In principle, TUBEHAVE is capable to describe the phenomenon of short wave-length rimpling due to bending, see section 7.3. However, due to the length scale in the current problem which is much larger than in section 7.3, the description would require an enormous amount of Fourier terms. This shows a limitation of the model.
7.7 - Axial tension

In this section, the analyses of tubes under axial tension are subdivided into two main categories. Firstly, "perfect tubes" (with a constant wall-thickness) are considered. Secondly, a small variation in the thickness in the reference configuration is assumed such that the necking phenomenon is initiated (section: "necking of tubes").

The elastic material behaviour is defined by $E=210\text{ GPa}$ and $\nu=0.333$. For the definition of the plasticity effect, a power-law fit on the uni-axial stress-strain curve is taken with $\sigma_y = 210 \text{ MPa}$, see (7.1.2). In the analyses, the sensitivity of the results with respect to various hardening coefficients $h$ is considered. The values $h = 0.100, 0.050, 0.025, 0.010$ as well as $h = 0.005$ are taken. The uniaxial stress-strain curves for these values of the hardening parameter are plotted in figure 7.7.1.

![Figure 7.7.1 - Uniaxial stress-strain curves for different hardening coefficients.](image)

**Perfect tubes**

In this section, the response in terms of the axial load, versus the axial extension of a perfect tube is considered. Attention will be focused to the assessment of the maximum loading point and the possibility of bifurcation of the solution. In order to have a clear view on the results, it is important to distinguish between the different stress and strain measures which may be used.

In a tensile specimen, the engineering axial stress is defined as the axial load over the *original* area, while the true axial stress equals the axial load over the *deformed* area. In the
uniaxial case, the true stress equals the Cauchy stress. The difference between the Cauchy stress and the Kirchhoff stress is due to elastic compressibility, which is negligible at larger plastic strains. Hence, in the sequence, no distinction will be made between the Cauchy stress and the Kirchhoff stress. The uniaxial stress in the power-law hardening curve, as proposed in (7.1.2), is observed as the Kirchhoff stress. In figure 7.7.2, the difference between the engineering stress and the Kirchhoff stress, both normalised with the (first) yield stress $\sigma_y$, is shown for the case $h = 0.050$. Note that the engineering stress attains a maximum value.

![Graph](image)

**Figure 7.7.2 - Engineering and Kirchhoff stress, versus logarithmic strain for $h = 0.050$.**

Not only for the stress components, but also for the strains, different measures are frequently used. In the model, the uniaxial strain in the power-law hardening curve (7.1.2) is viewed upon as the so-called *logarithmic* strain, which is defined as follows:

$$\varepsilon = \ln(1 + e) \quad \text{where} \quad e = \frac{\Delta L}{L_{\text{ref}}} = \zeta^B - 1$$

(7.7.1)

The latter strain measure ($e$) is called the *engineering* strain which, due to the definitions in the model, turns out to be the beam-like stretch factor $\zeta^B - 1$.

In KLEVER (1992), an approximation for the maximum load level of a tube under axial tension is given. According to this approximation, the Kirchhoff stress at the maximum load level (i.e. the load where the engineering stress reaches its maximum), equals the tangent modulus to the uniaxial stress-strain curve, in terms of the Kirchhoff stress versus the logarithmic strain. Thus,

$$\tau_{(\text{max})} = E_t = \frac{dk}{de} \quad \text{when} \quad N = N_{\text{max}}$$

(7.7.2)
The tangent $E_t$ is derived from (7.1.2). The maximum load $N_{\text{max}}$, together with the corresponding logarithmic strain then become

$$N_{\text{max}} = N_y \left( \varepsilon_y \right)^{-h} \exp\left( -h - (1-h) \varepsilon_y \right), \quad N_y = 2 \pi R t \sigma_y$$

at $\varepsilon_{\text{max}} = h + (1-h) \varepsilon_y$ . (7.7.3)

In the table below, the results obtained with TUBEHAVE (geometry data: $R=15.0 \text{cm}$, $\ell=60.0 \text{cm}$ and $t=1.0 \text{cm}$ and $\text{DOFD} = \text{UVWA}(0;0)$) are verified against the predictions due to equations (7.7.3). Note that the axial load as mentioned in the table could be observed as either the engineering stress multiplied with the original area or the true stress multiplied with the deformed area. It is clear that the correspondence between the TUBEHAVE results and the predictions due to (7.7.3) is perfect.

<table>
<thead>
<tr>
<th>hardening coefficient</th>
<th>max. axial load [MN]</th>
<th>$\zeta^B - 1$ at max. axial load</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>2.03 (2.04)</td>
<td>0.0059 (0.0060)</td>
</tr>
<tr>
<td>0.010</td>
<td>2.09 (2.09)</td>
<td>0.011 (0.011)</td>
</tr>
<tr>
<td>0.025</td>
<td>2.28 (2.29)</td>
<td>0.026 (0.026)</td>
</tr>
<tr>
<td>0.050</td>
<td>2.65 (2.65)</td>
<td>0.052 (0.052)</td>
</tr>
<tr>
<td>0.100</td>
<td>3.56 (3.56)</td>
<td>0.11 (0.11)</td>
</tr>
</tbody>
</table>

*Table 7.7.1 - Maximum loading points for different hardening coefficients. The values between brackets are due to (7.7.3).*

**Bifurcation analyses**

Since HILL (1958) published a general theory of uniqueness and stability in elastic-plastic solids, a vast amount of work has been done on simulations of necking phenomena. It is mentioned by HILL & HUTCHINSON (1975) that the state of uniaxial tension is unique prior to the attainment of the maximum load. In other words, if necking bifurcation occurs, it will necessarily take place at a (true) axial stress above that of the maximum load. HUTCHINSON & NEALE (1983) stipulate that the neck remains localised with nearly all the subsequent elongation occurring in the neck, provided that the uniaxial stress-strain curve peaks and then falls monotonically.

In the present section, tubes with the same geometric and material properties as in the previous section are analysed. Thus, the load versus deformation responses will be identical to those in the previous section. However, in this case the possibility of a bifurcation of the solution is checked. This is done by carrying out an eigenvalue analysis on the "tangent
stiffness matrix" at the start of each increment: $W_0^{(\text{int})}$, see (6.3.2). Non-positive

eigenvalues are encountered in the case of passing a maximum loading point or a bifurcation

point.

In the examples that will be shown, bifurcation into a necking mode with an
axisymmetric full cosine wave over the tube length is considered. Therefore, DOFD is taken
equal to UVWA$\{ (0,2); \{0\}\}$. Note that in this case, no imperfections should be defined
in the model. The analyses showed a first negative eigenvalue at the maximum loading
point. A second negative eigenvalue occurred at the axial loading and stretching as shown
in table 7.7.2.

<table>
<thead>
<tr>
<th>hardening coefficient</th>
<th>bifurcation load [MN]</th>
<th>$\xi_b - 1$ at bifurcation load</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>2.03 (2.03)</td>
<td>0.0070 (0.0069)</td>
</tr>
<tr>
<td>0.010</td>
<td>2.09 (2.09)</td>
<td>0.0131 (0.0128)</td>
</tr>
<tr>
<td>0.025</td>
<td>2.28 (2.28)</td>
<td>0.0316 (0.0311)</td>
</tr>
<tr>
<td>0.050</td>
<td>2.65 (2.65)</td>
<td>0.0622 (0.0619)</td>
</tr>
<tr>
<td>0.100</td>
<td>3.55 (3.54)</td>
<td>0.1240 (0.127)</td>
</tr>
</tbody>
</table>

Table 7.7.2 - Bifurcation points for different hardening coefficients.
The values between brackets follow from (7.7.5).

The terms between the brackets in table 7.4 follow from expressions given by
TVERGAARD (1990). In this work, homogeneous (pre-buckling) stress states are assumed
and the eventual occurrence of bifurcations is checked based on HILL’s uniqueness
criterium (1958). The expression governing the bifurcation behaviour in the case of an
homogeneous axial stress only, reads

$$(2\beta \alpha^2 - \alpha^2) \tau^2 + (-3\beta L_{11} \alpha^4 + L_{11} \alpha^2 - L_{22}) \tau +
+ (\beta L_{11}^2 \alpha^4 + L_{11} L_{22} - L_{12}^2) = 0 \quad \text{(7.7.4.a)}$$

where in the case of $\nu = 1/2$,

$$L_{11} = E_t + \frac{1}{3}E \quad , \quad L_{22} = \frac{4}{3}E \quad , \quad L_{12} = \frac{2}{3}E \quad \text{(7.7.4.b)}$$

The geometric parameters $\alpha$ and $\beta$ are defined by

$$\alpha = \frac{\pi R}{L} k \quad , \quad \beta = \frac{1}{12} \left(\frac{k}{R}\right)^2 \quad \text{(7.7.4.c)}$$
where \( k \) is the number of axisymmetric cosine waves. The first critical bifurcation stress is determined by the lowest value of \( \tau \) and occurs for \( k = 1 \). By using the uniaxial stress-strain curve, the associated critical value for the axial stretch is obtained. The Kirchhoff stress should be multiplied with the deformed area in order to obtain the axial loading at bifurcation. For the present geometry, the axial load together with the associated axial stretch read

\[
N_{\text{crit}} = \frac{N_y}{2\pi R t} \left( \frac{\varepsilon_y}{1-h} \right)^{1-h} \exp(-\mu_{B,\text{crit}}^{\varepsilon_y}) , \quad N_y = 2\pi R t \sigma_y
\]

at \( \mu_{B,\text{crit}}^{\varepsilon_y} = 1.183 h + (1-h)\varepsilon_y \)  

(7.7.5)

Values of \( N_{\text{crit}} \) and \( \mu_{B,\text{crit}}^{\varepsilon_y} \) for the different values of \( h \) are given in table 7.7.2. Clearly from table 7.7.2, the correspondence between these values and the predictions obtained with TUBEHAVE is perfect.

**Necking of tubes**

In this section, the influence of the hardening coefficient on the necking behaviour is investigated. The neck is assumed to be axisymmetric. Also, a small variation of the thickness in the reference configuration is assumed; i.e. 1.0% thinner at the middle of the analysed tube length than at both ends of the model. The symmetric part (symmetry with respect to \( \psi_1 = 0 \)) of the thickness function reads

\[
\lambda(\psi_1) = \begin{cases} 
0.99 \text{ cm} & , \quad \psi_1 \in \left[0, \frac{\pi}{10}\right] \\
0.99 + \frac{0.01}{\pi/10} (\psi_1 - \frac{\pi}{10}) \text{ cm} & , \quad \psi_1 \in \left(\frac{\pi}{10}, \frac{\pi}{5}\right) \\
1.00 \text{ cm} & , \quad \psi_1 \in \left[\frac{\pi}{5}, \frac{\pi}{2}\right]
\end{cases}
\]

(7.7.6)

Note that over 60% of the length the thickness equals 1.00 cm, over 20% of the length 0.99 cm and in between the thickness varies linearly. The Fourier transform of this (symmetric) function is given by

\[
\begin{align*}
\lambda_{0,0} &= \frac{2}{\pi} \int_0^{\pi/2} \lambda(\psi_1) \, d\psi_1 \\
\lambda_{2n,0} &= \frac{4}{\pi} \int_0^{\pi/2} \lambda(\psi_1) \cos(2n\psi_1) \, d\psi_1
\end{align*}
\]

(7.7.7)
In the analyses, the following terms are accounted for: \( \lambda_{0,0} = 0.997 \), \( \lambda_{2,0} = -0.0051 \), \( \lambda_{4,0} = -0.0028 \) and \( \lambda_{4,0} = -0.0006 \). It is experienced that \( \lambda \) as defined in (7.7.6) leads to initiation of necking. At both ends of the tube model, the usual symmetry conditions are assumed. Moreover, the modelled length is \( \ell = 60.0 \text{cm} \) and the reference radius \( R = 15.0 \text{cm} \). In the analyses carried out, DOFD is taken to be UVW:\langle (0, 2, 4, 6); (0) \rangle. The loading is determined by setting \( \lambda^{(1)} = -\lambda^{(2)} \).

The materials of the analysed pipes are identical to those used in the previous examples, see figure 7.7.1. In the first instance, the results will be presented in terms of the axial strain versus the axial loading. In figure 7.7.3, the results for \( h = 0.005 \), \( h = 0.025 \) and \( h = 0.050 \) are shown, where the curves for the perfect tube (discussed in the previous section) are also shown. The curves "(thin)" denote the axial strains at \( \psi_1 = 0 \), the curves "(thick)" at \( \psi_1 = \pi/2 \), whereas the curves "(mean)" give the overall axial strains determined by \( \zeta^B \). It is clear from this picture that the neck starts to develop after the attainment of the load maximum, according to a statement in HILL & HUTCHEONSON (1975).

![Figure 7.7.3 - Necking response for some hardening coefficients.](image)

In order to quantify the localisation of the axial strain, figure 7.7.4 is added. In this figure a comparison is made between the axial strain in the thin (\( \psi_1 = 0 \)) and the thick section (\( \psi_1 = \pi/2 \)). It is clear from the lower graph in this figure that a certain "transition point" might be allocated where the ratio between the axial strains changes drastically. These "points" give an estimation where the necking phenomenon is initiated. At these "points", the axial strains in the thin sections are roughly the same as in the bifurcation analyses, see table 7.7.2.
Figure 7.7.4 - Localisation of the axial strain.

It concluded from this section, that the initiation of the necking phenomenon is traced, since the bifurcation point is estimated correctly. Also, the development of the neck seems to be included. However, the development of a neck is associated with shear deformation, which is not included in the formulation. Therefore, the validity of the results in figures 7.7.3 and 7.7.4 is questionable.
Discussion & Recommendations

The model presented in this thesis is mainly focused on the prediction of the ultimate load behaviour of moderately thick-walled tubes. The formulation is based on shell kinematic assumptions and allows for relatively large membrane strains (the bending strains are required to remain small). The deformation of the tube is decomposed into "beam deformation" and "shell deformation". The "beam deformation" is determined by a constant axial stretch and a constant curvature in one plane. The discretisation of the "shell deformation" is based on Fourier series expansions for the unknown fields.

The correspondence of the computational results presented in this work with analytical / numerical / experimental work presented in the literature is generally quite good. The ultimate load levels are properly described. However, the strain levels at the ultimate loads often show some deviations with respect to experimentally obtained values. Despite the model is intended for moderately thick-walled tubes, it also performs properly in the thin shell limit.

In the present chapter, some final remarks on the major underlying assumptions of the model are made.

Shell kinematic assumptions

The model includes the description of relatively large membrane strains. The bending strains are assumed to remain small. The proper performance of the model in the large membrane strain regime is shown in the case of internal pressure and in the case of axial tension. Not only for these large membrane strain cases, but also for the cases where the membrane strains and the bending strains remain relatively small (e.g. the buckling sensitive cases), an adequate description is obtained.

In the kinematic description of the model, shear deformation is not included. For the examples being considered, comparison of the results with experimental results and a finite element simulation shows that the inclusion of shear deformation really could be omitted. However, for the description of the deformation of even thicker shells, shear deformations could become important (see the recommendation on page 147).

Decomposition of the deformation

The decomposition of the deformation of the tube into "beam deformation" and "shell deformation" works satisfactory. The concept of decomposition is appropriate for the
formulation of a more sophisticated (finite) pipe element. However, the "beam deformation"-modes being selected for the present computer program are not yet sufficient for this purpose (see the recommendation on page 147).

The end cross-sections of the tube model are defined by the orientation (determined by the beam deformation only), the shape and the position (determined by the shell deformation only). By definition, the end cross-sections are plane. The kinematical boundary conditions which the user may define are: a constant axial stretch and/or a constant angle between the end planes. Extension of the "beam deformation"-modes accompanied with a proper extension of possible kinematic boundary conditions will be necessary to make the model applicable to more general 3D-pipe deformation problems and the above mentioned formulation of a more sophisticated (finite) pipe element.

Discretisation of the unknown fields

The degrees-of-freedom in the model are defined as follows. The "beam deformation" is captured by two degrees-of-freedom; i.e. one for the definition of the axial stretch and another for the definition of the curvature of the beam. The "shell deformation" is discretised by means of Fourier series approximations for the displacement components and the thickness function. For every specific analysis, the set of degrees-of-freedom may be chosen. This gives the opportunity to take an active set of degrees-of-freedom which is tailored to the expected deformation mode. Some of the results presented are obtained with less than 10 active degrees-of-freedom. Most of the results are obtained with 10 - 50 degrees-of-freedom. In cases where the deformation localises, the required number of degrees-of-freedom for an accurate description increases sharply. For example, the analyses in section 7.3 on localisation of the axial deformation (the "{6/6}"-cases) are carried out with 102 active degrees-of-freedom.

In the present model, the main computational effort is required for the set-up of the governing set of equations. In view of the number of degrees-of-freedom, the effort for solving the system is much less. The experience with the model is that the required CPU-time is linear in the total number of mid-surface integration points used in the model. Furthermore, the required CPU-time increases quadratically with the number of active degrees-of-freedom. The implementation of the model on an IBM/RS6000 workstation resulted in the following estimate:

\[
\text{EFFORT (in CPU-sec)} \equiv 2 \times 10^{-5} \times (\text{number of mid-surface intg. pts.}) \times ND^2 ,
\]

which is the effort required for one iteration in the full Newton-Raphson scheme. The number ND represents the total number of degrees-of-freedom in the model.
A comparison with the required effort for the MARC-analysis presented in section 7.5 shows the following result. On the same workstation, one full Newton-Raphson iteration for a model with finite strain shell elements using 5 integration points through-the-thickness required approximately 0.512 CPU-seconds. This particular TUBEHAVE-analysis has been performed by using 32 integration points on the mid-surface and 26 active degrees-of-freedom. Thus, the effort for the TUBEHAVE-analysis = \(2 \times 10^{-5} \times (32) \times (26)^2 = 0.433\) CPU-seconds. Hence, it follows from this example that the required effort for a TUBEHAVE-analysis and a comparable MARC-analysis are of the same order of magnitude. It is remarked that the TUBEHAVE code is not yet optimised with respect to the CPU-effort (see the recommendation(s) below).

**Constitutive model**

The employed 3D stress-update procedure yields satisfactory results as long as cyclic plastic loading is excluded. The link of this model to the stress and strain measures from the shell formulation is such that any other 3D constitutive model may be used.

**Recommendations for future work**

The presented formulation is viewed upon as a first step towards a dedicated model for the description of local effects in general piping systems under various loading circumstances. To meet this objective, the following items for future work are recommended:

- Optimisation with respect to CPU-effort.
- Constitutive model, directly formulated in stress-resultants. This could also save CPU-time.
- Other loading conditions, such as temperature loading and contact forces. This depends on requirements from practice.
- Inclusion of shear deformation.
- An extension of the "beam deformation"-modes together with possible kinematic boundary conditions. For the formulation of a more sophisticated (finite) pipe element, a relation between parameters for the position, orientation and the shape of the end-cross-sections with corresponding quantities in the pipe-elements should be specified.
Appendix A - "Corrected moduli" versus "3D continuum moduli"

In this appendix, the influence of the term $\tilde{\eta}^{(1\star)}_B$ on the constitutive equations is demonstrated under the following simplifying assumptions:

(A.I) Thin shell with $\sqrt{a} = 1$ and $\sqrt{g} = \lambda$. (These geometric quantities are defined in (2.2.6) and (2.2.7).)

(A.II) Linear elastic, isotropic material behaviour.

(A.III) The material convected base vectors $\{\tilde{G}_i\}$ constitute an orthonormal basis.

The matrix $M^{(e/p)} = M^{(e)}$, relating the stress rate $\dot{\sigma}$ to the strain rate $\dot{\varepsilon}$, under these assumptions, reads

$$M^{(e/p)}_{KL} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 \\
\nu & \nu & \nu & 1-2\nu & 0 \\
\nu & \nu & \nu & \nu & \frac{1-2\nu}{2}
\end{bmatrix}. \tag{A1}
$$

These moduli are obtained from (3.2.30) together with (3.2.13).

In the first instance, the shell kinematic assumptions from Chapter 2, i.e. without the term $\tilde{\eta}^{(1\star)}_B$, are introduced in the strain rate. This yields

$$\dot{\varepsilon} = \tilde{\eta}^{(0)} - \xi \lambda \tilde{\eta}^{(1)} \tag{A2}
$$

The vectors $\tilde{\eta}^{(0)}$ and $\tilde{\eta}^{(1)}$ are defined in (3.3.1.b). Using (A2), the constitutive relations become

$$\dot{\sigma}_K = M^{(e/p)}_{KL} \left[ \tilde{\eta}^{(0)}_{KL} - \xi \lambda \tilde{\eta}^{(1)}_{KL} \right]. \tag{A3}$$
Integration of (A3) is carried out by means of the layered model. Application of (3.3.16.b) and (3.3.19) yields

\[
\begin{align*}
\delta^{(0)}_K &= Q^{(0)}_{KL} \tilde{\eta}^{(0)}_L + Q^{(1)}_{KL} \tilde{\eta}^{(1)}_L \\
\delta^{(1)}_K &= Q^{(1)}_{KL} \tilde{\eta}^{(0)}_L + Q^{(1)}_{KL} \tilde{\eta}^{(1)}_L
\end{align*}
\]

(A4)

The \(Q\)-matrices are readily obtained after realising assumption (A.1) and also that the moduli \(M_{KL}^{(\varepsilon/p)}\), given in (A1), are no functions of the through-the-thickness coordinate \(\xi\). Then

\[
\begin{align*}
Q^{(0)}_{KL} &= \lambda M_{KL}^{(\varepsilon/p)} \\
Q^{(1)}_{KL} &= 0 \quad \text{for } K, L = 1 \ldots 6 \\
\frac{\lambda^3}{12} M_{KL}^{(\varepsilon/p)}
\end{align*}
\]

(A5)

which are the exact values of the through-the-thickness integrals (in the layered model \(NL \rightarrow \infty\)). These moduli are referred to as the "3D continuum moduli".

Next, the term \(\tilde{\eta}^{(1)*}\) is introduced. In this case, the following strain rates

\[
\dot{\tilde{\epsilon}} = \ddot{\tilde{\eta}}^{(0)} - \xi \lambda \ddot{\tilde{\eta}}^{(1)*}
\]

(A6)

serve as input for the constitutive relations. The theory based on this definition is worked out in Chapter 3. It is discussed in section 3.3 that the "corrected moduli" are obtained according to (3.3.17). Working out of these expressions with the results for the \(Q\)-matrices given above, yields the "corrected moduli" as

\[
\begin{align*}
H^{(0)}_{KL} &= \lambda M_{KL}^{(\varepsilon/p)} \\
H^{(1)}_{KL} &= H^{(0)}_{KL} = 0 \quad \text{for } K, L = 1 \ldots 6
\end{align*}
\]

(A7)

(A8)

\[
H^{(1)}_{KL} = \frac{E\lambda^3}{12(1-v^2)} \begin{bmatrix}
1 & v & 0 & 0 & 0 & 0 \\
. & 1 & 0 & 0 & 0 & 0 \\
. & . & 0 & 0 & 0 & 0 \\
. & . & . & \frac{1-v}{2} & 0 & 0 \\
. & . & . & . & 0 & 0 \\
. & . & . & . & . & 0
\end{bmatrix}
\]

(A9)
It is clear for the example discussed above that, due the introduction of the term \( \hat{\eta}_{ij}^{(0)} \),

- the membrane stiffness is not changed, thus \( H_{KL}^{(0/0)} = Q_{KL}^{(0)} \).
- the bending stiffness is lower, \( H_{KL}^{(0/0)} < Q_{KL}^{(0)} \), since \( \nu > 0 \).

Finally, it is remarked that the model including the term \( \hat{\eta}_{ij}^{(0)} \), yields bending moduli \( H_{KL}^{(0/0)} \) which are identical to those used in the "classical theory of thin shells", see e.g. KOITER (1966). The membrane moduli \( H_{KL}^{(0/0)} \) are not identical to those used in the "classical theory". This is due to the fact that in the present model, no plane stress assumption is made.

If indeed plane stress (\( \sigma_3 = \hat{\sigma}_3 = 0 \)) is assumed in the model, then we obtain from \( \hat{\delta}_3^{(0)} = 0 \) that

\[
H_{3L}^{(0/0)} \hat{\eta}_L^{(0)} = 0
\]

Under this plane stress assumption, the thickness stretch \( \eta_{ij}^{(0)} \) becomes a dependent variable, which in rate form follows from (A10). The result is

\[
\hat{\eta}_{ij}^{(0)} = \frac{-\nu}{1-\nu} \hat{\eta}_L^{(0)} + \frac{-\nu}{1-\nu} \hat{\eta}_2^{(0)}
\]

Using this result yields the following membrane constitutive relations

\[
\begin{bmatrix}
\hat{\delta}_1^{(0)} \\
\hat{\delta}_2^{(0)} \\
\hat{\delta}_3^{(0)}
\end{bmatrix} = \frac{E\lambda}{(1-\nu^2) \text{ symm}} \begin{bmatrix}
1 & \nu & 0 \\
. & 1 & 0 \\
. & 1-\nu & 2
\end{bmatrix} \begin{bmatrix}
\hat{\eta}_L^{(0)} \\
\hat{\eta}_2^{(0)} \\
\hat{\eta}_4^{(0)}
\end{bmatrix}
\]

These membrane moduli are identical to those used in the "classical theory".
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Samenvatting

Proefschrift: Een (grote vervormingen) schalenmodel voor het analyseren van middelmatig dikwandige buizen

In dit proefschrift wordt een theorie behandeld die erop gericht is het belasting/vervormingsgedrag van rechte of gekromde buizen zoals dat in de praktijk optreedt, te simuleren. Dunwandige zowel als redelijk dikwandige buizen met een willekeurige vorm van de dwarsdoorsneden en wanddikte-varianties kunnen worden behandeld.

De formulering van het probleem gaat uit van de Kirchhoff-Love aannamen. Een afwijking van deze aannamen is noodzakelijk wanneer grote membraanvervormingen worden toegelaten (aangenomen is dat de buigvervormingen klein blijven). Met het oog op deze behandeling is een onafhankelijke functie geïntroduceerd die de eindige dikte-rek bepaalt. Afschuwvervormingen zijn buiten beschouwing gelaten.

In het model wordt onderscheid gemaakt tussen twee vervormingstypes: zogenaamde "balkvervorming" en "schaalvervorming". De "balkvervorming" bepaalt de deformatie van de buis volgens de balk-theorie van Bernoulli. Additionele verplaatsingen, zoals ovalisatie en insnoering, worden door de "schaalvervorming" beschreven.

De "balkvervorming" is volledig bepaald door een uniforme axiale rek (de eerste vrijheidsgraad) en een uniforme kromming (de tweede vrijheidsgraad). De velden die de "schaalvervorming" weergeven zijn gedискreteerd met behulp van Fourier reeksen. De amplitude-factoren behorend bij de afzonderlijke termen worden als vrijheidsgraden gezien.

De gebruikte constitutieve relaties zijn gebaseerd op een 3D (grote rekken) elastisch-plastisch model, wat gebruik maakt van het Von Mises vloeicriterium, een associatieve vloeiregel en isotrope versteviging van het materiaal. De toepassing van dit model op de spannings- en vervormingsgrootheden zoals gebruikt in de formulering van het probleem is zodanig dat elk ander 3D constitutief model relatief eenvoudig kan worden ingepast.

De belasting op de buis kan bestaan uit een reeks van combinaties van basisbelastingen. Als basisbelastingen zijn gedefinieerd: invendige en uitwendige druk en ook krachten en momenten aan de beide uiteinden van de buis.

Het belasting/vervormingsgedrag wordt bepaald door een set van incrementele oplossingen. Met het oog hierop zijn de volgende methodes ingebouwd: het incrementeel voorschrijven van een belastingsparameter of van een specifieke vrijheidsgraad en ook een gelineariseerde vorm van de booglengte methode.
Gebaseerd op de gepresenteerde theorie, is er een computer-programma ontwikkeld (genaamd TUBEHAVE). De resultaten die met dit programma zijn behaald komen goed overeen met het analytisch / numeriek en/of experimenteel werk van anderen. Verder laten de resultaten zien dat het gebruik van Fourier reeksen voor de discretisatie van de onbekende velden geschikt is voor de beschrijving van het vervormingsgedrag. Tenslotte kan er uit de gepresenteerde analyses geconcludeerd worden dat de vervormingsgrootheden zoals in het model gehanteerd, voldoende zijn voor een nauwkeurige beschrijving van de vervorming van zowel dunwandige als redelijk dikwandige buizen (in ieder geval tot aan de maximale draagkracht).

Auteur: G. van den Berg
Curriculum Vitae


Na zijn afstuderen ging hij werken aan een promotie-opdracht binnen het Koninklijke/Shell Exploratie en Productie Laboratorium (KSEPL) te Rijswijk. Hij heeft hier onder supervisie van Dr. Ir. F.J. Klever gewerkt aan het onderzoek dat tot dit proefschrift heeft geleid.

Per 1 februari 1995 is hij in dienst van VECTRA Technologies Ltd. te Leiden.