On the hodograph transformation of irrotational conical flow

by

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Summary.

A discussion is given of the mapping of irrotational conical flow onto a surface \( w(u,v) \) in the hodograph space. A differential geometrical analysis of this surface is given. If the velocity normal to the radius \( U \) is smaller than the velocity of sound \( (U < a) \) the flow is termed conical-subsonic and the points on the \( w(u,v) \) surface are hyperbolic points. If \( U > a \), the flow is called conical-supersonic and these points are elliptic points. It is shown that singularities in the transformation occur when the flow is conical-sonic \( (U = a) \). The Jacobian determinant \( \Delta \) \( (\Delta = w_{uu} w_{vv} - w_{uv}^2) \) is equal to the Gaussian curvature of the surface and becomes zero in parabolic points of this surface. Parabolic points are shown to represent limit cones, being conical limit lines. A continuous flow over a conical-sonic line is possible if the Jacobian changes sign going through infinity. The two other cases where \( \Delta \to \infty \) are Prandtl-Meyer flow, mapped on a sharp edge in the \( w(u,v) \) surface and parallel flow, which is bounded by a Mach cone, and mapped onto a conical point of this surface.

Conical flow with axial symmetry is investigated from the point of view of the hodograph transformation. As an example of conical flow without axial symmetry the flow on the expansion side of a flat plate delta wing with supersonic leading edges is discussed.
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1. Introduction

Conical flow as occurring in the supersonic flow around conical bodies has been studied extensively by means of linear and non-linear theory. In non-linear theory the flow around a specific body is obtained as a numerical solution of the non-linear equations of conical flow. (ref.1,2,3). The co-ordinates in the physical space are usually taken as the independent variables and boundary conditions are specified in the physical space. Solution of the problem in the hodograph space is in general more complicated because of the difficulties in prescribing the boundary conditions. In order to get a better understanding of the general properties of conical flow it may be of interest, however, to consider the transformation of conical flow into the hodograph space. It especially helps to determine the proper boundary conditions in applying a numerical method of solution.

2. Equation for irrotational conical flow in the hodograph space.

Conical flows are spatial flows for which the velocity, density and pressure are constant on rays through one point of the physical space. These physical quantities therefore are dependent on two length co-ordinates instead of three, as in general is the case in spatial flows. As independent variables also two velocity components may be chosen. The conical flow around a given body is then given by a surface in the hodograph space. This surface may be obtained as an integral surface of the differential equation for conical flow in the hodograph space. A derivation of the equation for irrotational conical flow was given by Busemann (ref.4) assuming the existence of a velocity potential. A similar derivation in which this assumption is not made from the beginning is given below. Let to the physical space a co-ordinate system \(x, y, z\) be fixed with the origin in the apex of the conical flow, and let \(u, v\) and \(w\) be the velocities along the axes, respectively. A direction in the physical space then corresponds to the same direction in the hodograph space.
Under the usual assumptions of inviscid, isentropic flow without heat conduction the equation of gasdynamics may be written as:

\[ u_x (1- \frac{u_x^2}{a^2}) + v_y (1- \frac{v_y^2}{a^2}) + w_z (1- \frac{w_z^2}{a^2}) - \frac{uv}{a^2} (u_x + v_y) - \frac{uw}{a^2} (v_y + w_z) - \frac{uw}{a^2} (w_z + u_x) = 0 \]  

(1)

where \( a \) is the velocity of sound related to the velocity components by

\[ a^2 = \frac{\gamma + 1}{2} a^2 = \frac{\frac{\gamma - 1}{2}}{u^2 + v^2 + w^2} \]  

(2)

and \( a \) is the critical velocity of sound.

Eq. (1) may be specialized for conical flow with the aid of the following equations, which can be found using the property of conical flow that velocity components do not change along rays through the origin. They read:

\[ \begin{align*}
\frac{du}{dx} &= u_x x + u_y y + u_z z = 0 \\
\frac{dv}{dy} &= v_x x + v_y y + v_z z = 0 \\
\frac{dw}{dz} &= w_x x + w_y y + w_z z = 0
\end{align*} \]  

(3)

The transformation to the hodograph space is obtained by noting that as velocity components along a radius do not vary, this radius must have the same direction as the normal to the \( w(u,v) \) surface in the hodograph space. Therefore,

\[ \frac{w_u}{z} = - \frac{x}{z} ; \quad \frac{w_v}{z} = - \frac{y}{z} \]  

(4)

Differentiating eq. (4) in a plane \( z = \text{constant} \) yields:

\[ \begin{align*}
\frac{dx}{dz} &= -zw_{uu} = -zw_{uu} \frac{du}{dz} - zw_{uv} \frac{dv}{dz} \\
\frac{dy}{dz} &= -zw_{v} = -zw_{vu} \frac{du}{dz} - zw_{vv} \frac{dv}{dz}
\end{align*} \]  

(5)

Solving eq. (5) for \( du \) and \( dv \) gives:
\[ du = \frac{-w_{yy} \, dx + w_{uv} \, dy}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \]
\[ dv = \frac{w_{vu} \, dx - w_{uu} \, dy}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \] (6)

Since also,
\[ du = u_x \, dx + u_y \, dy \]
\[ dv = v_x \, dx + v_y \, dy \] (7)

Comparing eq. (6) and eq. (7) yields:

\[ u_x = \frac{-w_{yy}}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \]
\[ u_y = \frac{w_{uv}}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \]
\[ v_x = \frac{w_{vu}}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \]
\[ v_y = \frac{-w_{uu}}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \] (8)

Furthermore,
\[ dw = w_u \, du + w_v \, dv \]
\[ dw = w_x \, dx + w_y \, dy \] (9)

From eq. (6) and eq. (9) then follows:

\[ w_x = \frac{-w_u \, w_{vv} + w_v \, w_{vu}}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \]
\[ w_y = \frac{w_u \, w_{uv} - w_v \, w_{uu}}{z(w_{uu} \, w_{vv} - w_{uv} \, w_{vu})} \] (10)
Thus far the condition that the flow is free of rotation has not been used and the relations given above would be equally valid for rotational flow. The components of vorticity along the x, y and z axis are, respectively,

\[ \psi = w_y - v_z = \frac{w_u (w_{uv} - w_{vu})}{z (w_{uu} w_{vv} - w_{uv} w_{vu})} \]

\[ \eta = u_z - w_x = \frac{w_v (w_{uv} - w_{vu})}{z (w_{uu} w_{vv} - w_{uv} w_{vu})} \]

\[ \zeta = v_x - u_y = \frac{w_v (w_{uv} - w_{vu})}{z (w_{uu} w_{vv} - w_{uv} w_{vu})} \] (11)

In order for the function \( w(u, v) \) to represent a surface in the hodograph space \( w_{uv} \) and \( w_{vu} \) must be continuous and as a consequence be equal. From eq. (11) then follows that the flow is free of rotation. Irrotationality therefore appears to be a necessary condition for the possibility of the transformation of conical flow into an Euclidian hodograph space.

By means of eqs. (1), (3), (4), (8) and (10) the differential equation for irrotational conical flow in the hodograph space may then be written as. \(^x\)

\[ w_{uu} \left[ 1 - \frac{v^2}{a^2} + w_v \left( 1 - \frac{w^2}{a^2} \right) - 2 \frac{u}{a^2} \frac{w_v}{a} \right] + w_{vv} \left[ 1 - \frac{u^2}{a^2} + w_u \left( 1 - \frac{w^2}{a^2} \right) - 2 \frac{u}{a^2} \frac{w_u}{a} \right] + \\
+ 2 \frac{u}{a^2} \frac{w_{uv}}{a} - w_u w_v \left( 1 - \frac{w^2}{a^2} \right) + w_v \frac{uv}{a^2} + w_u \frac{vw}{a^2} = 0 \] (12)

It may be noted that if \( w(u, v) \) is a solution of this equation, \(- w(-u, -v)\) also is a solution. In irrotational conical flow around a body the streamlines may therefore be reversed.

\(^x\) This formula corrects a printing error appearing in ref. 4, eq (10).
Eq. (12) is a non-linear partial differential equation of the second order and may be of the elliptic, parabolic or hyperbolic type. The characteristics of the equation are given by

\[
\left(\frac{dV}{du}\right)^2 \left[ \frac{1-v^2}{a^2} + v^2 \left(\frac{1-w^2}{a^2}\right) - 2w_\nu \frac{w}{a^2} \right] - 2\left(\frac{dV}{du}\right) \left[ \frac{w}{a^2} - \nu \frac{w}{a^2} \right] + w_\nu \frac{w}{a^2} + w_\mu \frac{w}{a^2} + \left[ \frac{1-u^2}{a^2} + u^2 \left(\frac{1-w^2}{a^2}\right) - 2u_\nu \frac{w}{a^2} \right] = 0
\] (13)

As has been noted by Busemann (ref. 4) it is convenient to use a co-ordinate system, wherein velocity components are measured along and perpendicular to the radius under consideration. The \(u,v,w\) system therefore is rotated in such a way that the \(w\) axis has the direction of the radius and that the velocity in the \(v\) direction is zero. The new co-ordinate system will be indicated by \(U,V\) and \(W, U\) being the velocity perpendicular to and \(W\) the velocity along the radius. In this notation eq. (13) then gives, since \(w_\mu = w_\nu = 0\), for the direction of the characteristics:

\[
\left(\frac{dV}{dU}\right)_{\text{char}} = \pm \sqrt{\frac{U^2}{a^2} - 1}
\] (14)

It may be seen from eq. (14) that if \(U > a\) the two characteristic directions are real and different. The differential equation is of the hyperbolic type and the flow will be termed conical-supersonic flow. If \(U = a\) the two characteristic directions are real and equal. The differential equation is of the parabolic type and the flow will be termed conical-sonic flow. If \(U < a\) the characteristic directions are imaginary and the differential equation is of elliptic type. The flow will be called conical-subsonic flow.

3. General discussion of the hodograph transformation for irrotational conical flow.

From eqs. (5) and (6) it may be seen that the transformation is governed by the Jacobian \(\Delta = w_{uu} w_{vv} - w^2_{uv}\). 

If the transformation is locally one-to-one, that is that one radius in the physical space corresponds to one point in the hodograph space and vice versa, $\Lambda$ is finite and different from zero. Singularities in the transformation occur for $\Lambda = 0$ and $\Lambda, w_v$.

From differential geometrical considerations $\Lambda$ may be recognized as the Gaussian curvature of the surface $w(u, v)$ in the hodograph space (ref. 5). If the Gaussian curvature is finite an analysis of the geometry of the surface may be based on Dupin's indicatrix. This will be considered in this article. In addition to and connection with it a discussion of the general properties of the transformation will also be given. In the following articles singularities in the transformation will be considered in more detail.

A curve related to Dupin's indicatrix is obtained as the intersection with the $w(u, v)$ surface of a plane parallel to the tangent plane in the point considered at such a distance $C$ that higher order derivatives may be neglected with respect to the second order derivatives, when the shape of the curve of intersection is determined.

Using the $U, V, W$ co-ordinate system attached to a radius with velocities $U_1$ and $W_1$ this curve is given by

$$W = W_1 - C$$

$$W_{uu} U^2 + 2 W_{uv} UV + W_{vv} V^2 - 2U_1 W_{uu} U - 2V_1 W_{uv} V + U_1^2 W_{uu} - 2C = 0$$

(15)

This is the equation of a conic, being an ellipse when $W_{uu} W_{vv} - W_{uv}^2 = \Lambda > 0$, a parabola when $\Lambda = 0$, and a hyperbola when $\Lambda < 0$. Points on the surface are respectively called elliptic, parabolic or hyperbolic points. For an elliptic point the surface is in any direction curved in the same sense; for a parabolic point there is one direction wherein the curvature of the surface disappears; whereas for a hyperbolic point curvatures of the surface of opposite sign occur. For a hyperbolic point there are two directions for which the curvature of the surface becomes zero.
Dupin's indicatrix formed by setting out the square root of the absolute value of the radius of curvature $\sqrt{R}$ in the corresponding direction is an ellipse, two parallel lines, and a hyperbola, respectively.

As has been noted by Busemann (ref.4) from eq.(12) a curvature relation for the $w(u,v)$ surface may be obtained. In the $U,V,W$ co-ordinate system eq.(12) namely simplifies in the point considered to:

$$\bar{W}_{UU} + \bar{W}_{VV} \left(1 - \frac{U^2}{a^2}\right) = 0$$

(16)

Since $\bar{W}_U = \bar{W}_V = 0$ the curvature relation may therefore be written as

$$R_U \left(1 - \frac{U^2}{a^2}\right) + R_V = 0$$

(17)

where $R_U$ and $R_V$ are the radii of curvature in the U and V direction, respectively.

Elliptic points thus correspond to conical-supersonic flow, parabolic points to conical-sonic\(x\), and hyperbolic points to conical-subsonic flow.

The axes of the conic are in the principal directions, and the corresponding curvatures are called the principal radii of curvature $\rho_1$ and $\rho_2$.

The principal directions are given by $\alpha$ ($\alpha$ measured positive in counterclockwise direction) where $\alpha$ may be deduced from

$$\tan 2\alpha = \frac{2\bar{W}_{UV}}{\bar{W}_{UU} - \bar{W}_{VV}}$$

(18a)

The Gaussian curvature is defined as the product of the curvatures in the principal directions:

$$K_G = \frac{1}{\rho_1 \rho_2} = \bar{W}_{UU} \bar{W}_{VV} - \bar{W}_{UV}^2 = \Delta$$

(18b)

and the mean curvature as the sum of these curvatures:

$$K_M = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \bar{W}_{UU} + \bar{W}_{VV}$$

(19)

\(x\) As will be shown later the direction for which $R \rightarrow \infty$ indeed is the $U$ direction.
The Gaussian curvature thus is positive for an elliptic point, zero for a parabolic point and negative for a hyperbolic point.

The asymptotic directions are obtained by equating the higher order part of \((15)\) to zero. Then follows:

\[
\left( \frac{dV}{du} \right)_{\text{asympt.}} = \frac{-W_{uu} \pm \sqrt{-(W_{uu} W_{vv} - W_{uv}^2)}}{W_{vv}} \tag{20}
\]

Thus for an elliptic point \((\Delta > 0)\) the asymptotic directions are imaginary, for a hyperbolic point \((\Delta < 0)\) there are two real asymptotic directions and for a parabolic point \((\Delta = 0)\) the two real asymptotic directions fall together along the axis of the parabola.

The direction of the streamline in the hodograph space can be found from the direction of the streamline in the physical space.

Along a streamline in the physical space there is:

\[
\frac{dx}{dz} = \frac{u}{w} \tag{21}
\]

\[
\frac{dy}{dz} = \frac{v}{w}
\]

Using the relation which regulates the transformation [eq.\((4)\)] then follows:

\[
(dx)_s = -w_u (dz)_s - z (dw_u)_s \tag{22}
\]

\[
(dy)_s = -w_v (dz)_s - z (dw_v)_s
\]

or with eq.\((21)\)

\[
(dx)_s = \frac{-z \left[ w_{uu} (du)_s + w_{uv} (dv)_s \right]}{1 + w_u \frac{w}{u}}
\]

\[
(dy)_s = \frac{-z \left[ w_{uv} (du)_s + w_{vv} (dv)_s \right]}{1 + w_v \frac{w}{v}} \tag{23}
\]

The index \(s\) refers to conditions along a streamline. Again using eq.\((21)\) leads to

\[
\left( \frac{dV}{du} \right)_s = \frac{(v + w w_v) w_{uu} - (u + w w_u) w_{uv}}{(u + w w_u) w_{vv} - (v + w w_v) w_{uv}} \tag{24}
\]
or
\[
\frac{\partial V}{\partial U} S = - \frac{W_{UV}}{w_{VV}}
\]

(25)

The streamline is directed along a principal direction if
\[ W_{UV} = 0, \]

as in axially symmetric flow. From eqs. (20) and (25) follows, that for a parabolic point the streamline is in the direction of the two asymptotic directions, which fall together along the axis of the parabola; so, in the streamline direction \( R \rightarrow \infty \).

From differentiation of eq. (12) it may be deduced, that in a parabolic point \( \frac{\partial V}{\partial U} S = \frac{\gamma + 1}{\gamma} \neq 0 \). The Gaussian curvature changes sign going through zero.

Thus, a line of parabolic points separates a region with elliptic points from a region with hyperbolic points, that is a region with conical-supersonic flow from a region with conical-subsonic flow. The flow in the parabolic point is conical-sonic.

From eq. (17) then follows if \( R_V \neq 0 \), that \( R_U \rightarrow \infty \) as \( U \rightarrow a \). The streamline is therefore in the \( U \) direction and \( W_{UV} = 0 \). As \( R_V \) does not change sign in passing the parabolic point it follows from the curvature relation that \( R_U \) changes sign going through infinity. The streamline thus exhibits an inflection point in the parabolic point.

The acceleration is in the direction of the streamline in the hodograph space and can also be expressed in geometrical properties of the \( w(u, v) \) surface.

From eqs. (21), (23) and (24) follows:
\[
(\frac{dx}{du})_S = - zu \frac{w_{uu} w_{VV} - w_{UV}^2}{(u + w w_u w_v - (v + w w_u) w_{uv})} (\frac{du}{u})_S
\]

\( \star \)

For one direction in between the positive and negative \( U \) axis \( R \) becomes zero. The indicatrix of Dupin, thus consists of two infinite straight lines falling together along the \( U \) axis. It does not change Dupin's indicatrix and the character of the point if \( R \) becomes zero in the \( V \) direction. Thus if \( R_U = 0 \) and \( R_V \rightarrow \infty \) the point considered is also a parabolic point.
\[ (dy)_s = - zv \frac{w_{uu}w_{vv} - w_{uv}^2}{(u+ww)_u w_{vv} - (v+ww)_v w_{uv}} (du)_s \]  

\[ (dz)_s = - zw \frac{w_{uu}w_{vv} - w_{uv}^2}{(u+ww)_u w_{vv} - (v+ww)_v w_{uv}} (du)_s \]  

The acceleration of the \( u \) component then follows with eq. (26) to be:

\[ \frac{du}{ds}_s = - \frac{1}{qz} \frac{(u+ww)_u w_{vv} - (v+ww)_v w_{uv}}{w_{uu}w_{vv} - w_{uv}^2} \]  

(27)

For the \( v \) component is obtained with eqs. (24) and (27):

\[ \frac{dv}{ds}_s = - \frac{1}{qz} \frac{(v+ww)_v w_{uu} - (u+ww)_u w_{uv}}{w_{uu}w_{vv} - w_{uv}^2} \]  

(28)

Further:

\[ \frac{dw}{ds}_s = (\frac{du}{ds}_s)_u + (\frac{dv}{ds}_s)_v \]  

(29)

so that with eqs. (27) and (28) may be found that:

\[ \frac{dw}{ds} = - \frac{1}{qz} \frac{(u+ww)_u (w_{uu}w_{vv} - w_{uv}^2) - (v+ww)_v (w_{uu}w_{vv} - w_{uv}^2)}{w_{uu}w_{vv} - w_{uv}^2} \]  

(30)

With \( q \frac{dq}{ds}_s = u(\frac{du}{ds}_s)_s + v(\frac{dv}{ds}_s)_s + w(\frac{dw}{ds}_s)_s \)  

(31)

and eqs. (28), (29) and (30), then the acceleration is given by

\[ \frac{dq}{dt}_s = q(\frac{dq}{ds}_s)_s = - \frac{1}{qz} \frac{(v+ww)_v (w_{uu}w_{vv} - 2(u+ww)_u (v+ww)_v w_{uv} + (u+ww)_u^2 w_{uv} w_{vv})}{w_{uu}w_{vv} - w_{uv}^2} \]  

(32)
or with eq. (12):

\[
\left( \frac{dq}{dt} \right)_s = -\frac{a^2}{qz} \frac{w_{uu} (1+w_v^2) + w_{vv}(1+w_u^2) - 2w_u w_v w_{uv}}{w_{uu} w_{vv} - w_{uv}^2}
\]  

(33)

In the U, V, W system eq. (33) reads in the point under consideration:

\[
\left( \frac{dq}{dt} \right)_s = -\frac{a^2}{qr} \frac{w_{UU} + w_{VV}}{w_{UU} w_{VV} - w_{UUV}^2}
\]  

(34)

where \( r \) is the distance measured along the radius.

When eqs. (18) and (19) are used the acceleration becomes:

\[
\left( \frac{dq}{dt} \right)_s = -\frac{a^2}{qr} \frac{K_M}{K_G} = -\frac{a^2}{qr} (\rho_1 + \rho_2)
\]  

(35)

The acceleration is therefore seen to be simple related to the principal radii of curvature of the \( w(u,v) \) surface.

The pressure gradient may from this result found to be

\[
\left( \frac{dp}{ds} \right)_s = \frac{\rho a^2}{qr} \frac{K_M}{K_G} = \frac{\rho a^2}{qr} (\rho_1 + \rho_2)
\]  

(36)

4. Limit cones.

It can be shown that lines of parabolic points represent limit cones, being conical limit surfaces.

Limit lines or surfaces probably were first discovered in some solutions of the hodograph equation for two-dimensional compressible flow. A more systematic investigation of the properties of limit lines has then been given for two- and three-dimensional flows. For an extensive discussion on limit lines giving many references to the literature may be referred to von Mises's book (ref. 6).

If limit lines appear in a solution, regions in the flow are found for which the velocity is many valued. The transformation of the physical space into the hodograph space therefore becomes singular. Regions with a many valued solution for the velocity are bounded by limit lines, so called because the direction of the
streamlines is reversed at these lines and the flow thus finds a limiting boundary which cannot be crossed. Accompanied with the reverse of the streamline the acceleration and the pressure gradient go to infinity. In the hodograph space the streamline is not curved and becomes tangent to the characteristic in points of the limit line.

This physically unacceptable behaviour of the flow indicates that no isentropic solution satisfying the given boundary conditions physically exists. If nevertheless an isentropic solution is sought for, it appears that infinitesimal disturbances travelling along the characteristic surfaces pile up at an envelope of these characteristic surfaces trying to form finite disturbances. If the envelope is formed by downstream characteristics, in many cases solutions with shock waves may be found.

The transition across a conical-sonic line may take place from conical-subsonic to conical-supersonic flow and vice versa, as follows from the reversibility of the streamlines.

As has been shown the streamline in the hodograph space exhibits an inflection point in a parabolic point. The direction of the normal downstream and upstream of the inflection point thus covers the same region in the physical space and a many valued region appears, bounded by the directions of the normals in the parabolic points. In order to proof that the streamlines in the physical space return at the radii corresponding to parabolic points the symmetry of the solution with respect to the origin is noted. If namely, a parabolic point would be passed along the same \( w(u,v) \) surface, the streamline would, according to the region of normals to be covered, return whereas the direction of the velocity would not change, thus leading to a contradiction. A continuation of the streamline downstream of the parabolic point along the surface \( -w(-u,-v) \) covers the same region in the physical space as upstream of that point and also changes the sign of the velocity. The streamline in the physical space thus exhibits a cusp in a parabolic point. From eqs. (34) and (36) follows, that the acceleration and pressure gradient go to infinity in a parabolic point since \( K' = \Delta \to 0 \) for \( K'' \neq 0 \) (or one of the principal radii \( \rho \to \infty \)). Since in a parabolic point the streamline is in the
U direction and \( U = a \) it follows from eq. (14) that in a parabolic point the streamline in the hodograph space is tangent to the characteristics.

If the characteristics in the physical space form an envelope, the envelope will be conical and the characteristic surfaces must be tangent to the radius for radii of the conical envelope. The velocity normal to the radius is therefore sonic and the envelope is transformed in the hodograph space as a conical-sonic line. However it may not be stated in reverse that a conical-sonic line in general is the image of an envelope of characteristic surfaces in the physical space. The characteristic tangent to a radius should therefore not have an inflection point, if an envelope is to be formed. Since the physical streamlines return for a parabolic point and the Mach angle changes continuously in this point the physical characteristics do not show an inflection point and an envelope of the characteristic surfaces is formed.

It is thus shown, that the flow for \( \Delta = 0 \) shows the typical behaviour of that at limit lines and that points for which \( \Delta = 0 \) (parabolic points) are the images of limit cones.

5. Continuous flow over a conical-sonic line.

Thus far points on the \( w \) surface for which the Gaussian curvature \( K_g \) and therefore the Jacobian \( \Delta \) \( (K_g = \Delta) \) remained finite, were considered. When \( \Delta \to \infty \) other singularities are encountered with \( . \) From the curvature relation \( [\text{eq. (17)}] \) it can be seen for example, that if \( R_u \) remains finite a continuous flow over a conical-sonic line is possible if \( R_v \) is allowed to go through zero at this line. When \( R \) becomes zero, where \( \frac{1}{R} = \frac{dg}{ds} \), this not necessarily means that a discontinuity in \( a \) occurs. The \( w(u,v) \) surface thus does not in general show a sharp edge when \( R \to 0 \). From eq. (25) follows again that the streamline is directed along the \( U \) axis, if \( \tilde{\omega}_{UU} \) remains finite, since \( \tilde{\omega}_{VV} \to \infty \). The Gaussian curvature changes sign going through infinity, thus indicating the transition from one type of flow to the other. The curve given by eq. (15) is a conic degenerated into two straight lines falling together along the \( U \) axis. Dupin's indicatrix consists
of two line segments falling together along the U axis.

The acceleration and pressure gradient remain finite as can be concluded from eqs. (35) and (36), and since the streamline is in a principal direction are equal to:

\[
\begin{align*}
\left(\frac{dq}{dt}\right)_{s, \frac{a}{U} = 1} &= -\frac{a^2}{QR} P_U \\
\left(\frac{dp}{ds}\right)_{s, \frac{a}{U} = 1} &= \frac{2a^2}{QR} R_U
\end{align*}
\]

respectively.

The streamline in the hodograph space is tangent to the two characteristics of which the directions fall together in the direction of the U axis, as follows from eq. (14). The characteristic surfaces in the physical space are tangent to the radius. Since the physical streamlines continue across the conical-sonic line and the Mach angle changes continuously in this point the physical characteristics show an inflection point on the conical-sonic line. Therefore the characteristic surfaces do not show an envelope.

In general the streamlines in the hodograph space form an angle with the conical-sonic line on the \(w(u,v)\) surface.

The acceleration therefore makes an angle with the conical-sonic line.


A special type of continuous transition across a conical-sonic line occurs when the angle between the streamline in the hodograph space and the conical-sonic line becomes zero. Different streamlines on the hodograph surface flow together along the conical-sonic line, which in this case is a sharp edge in the surface. The normal in any point of the sharp edge may be said to have all values in between and including the normals to the surface in that point. The velocity and pressure in a plane within the radii given by the direction of these normals is constant. The velocity normal to this plane, being the U component for one of the
outer normals is sonic. The flow is thus recognized to be that of plane acoustic waves travelling through still air or of Prandl-Meyer flow. Since the velocity normal to the plane of parallel flow is sonic, the flow is conical-supersonic or conical-sonic.

If the flow is that of centered waves the acceleration is perpendicular to the centering edge, and the sharp edge in the hodograph surface is lying in a flat plane. In figure 2 the four types of centered simple wave flow from and into parallel flow are sketched.

It is seen that expanding flow along downstream Mach waves and compressing flow along upstream Mach waves are possible in a continuous way. For expanding flow along upstream Mach waves and compressing flow along downstream Mach waves limit cones again appear in the solution. Analysis of the flow pattern may be performed using the conventional sweepback theory.

As in two-dimensional flow the waves do not have to be centered. Since the properties of Prandl-Meyer flow are well known, they will not be discussed further in this report.

7. Regions of parallel flow.

If the flow within a cone with a spatial cone angle different from zero is mapped into one point in the hodograph space the flow obviously is parallel and $|\Delta| \rightarrow \infty$. Such a point on the $w(u,v)$ surface is a conical point, unless the point is isolated.

The normals to the $w(u,v)$ surface in this point form a cone congruent with and having the same position in the hodograph space as the cone in the physical space bounding the region of parallel flow. If the conical point is approached in the $U$ direction corresponding to the normal under consideration $R_U$ vanishes with respect to $R_v$ and it follows from the curvature relation [eq. (17)] that $U \rightarrow a$. Since this is true independent of the direction of

*) This case corresponds to parallel flow bounded by a shock wave or parallel flow throughout the physical space.
approach it is seen that the cone formed by the normals is congruent with and has the same position in the hodograph space as the Mach cone in the physical space emanating from the centre of the conical field and corresponding to the velocity in the parallel flow. In a conical flow the velocity in a region of parallel flow therefore must be supersonic. The parallel flow may either be inside or outside of this Mach cone, which may be pointing upstream or downstream.

Parallel flow inside the Mach cone is conical-subsonic and outside the Mach cone conical-supersonic. The flow does not necessarily change type across a conical-sonic line bounded on one side by a region of parallel flow. In order to recognize this and to investigate the behaviour of the flow near the Mach cone in more detail the \( u, v, w \) axes are rotated in such a way that the \( w \) axis is directed along the velocity vector \( q \) of the parallel flow. Then, it is assumed that the \( w(u, v) \) surface in the neighbourhood the conical point may be expanded in a series of which the first three terms are given by

\[
\begin{align*}
  w &= q + A (\psi) p + B (\psi) p^2 \\
  \text{where } u &= p \cos \psi \text{ and } v = p \sin \psi.
\end{align*}
\]

Substitution of eq. (39) in eq. (12) and collecting terms of the same order yields two solutions for \( A (\psi) \).

The first solution is

\[
A (\psi) = C_1 \sin (\psi - \psi_0)
\]

and corresponds to a point, where the surface has a finite Gaussian curvature or a point on a sharp edge.

The second solution is

\[
A (\psi) = \pm \frac{1}{\sqrt{1 - \frac{\psi}{\psi_0}}}
\]

which corresponds to a conical point \( (M = \frac{q}{a}) \).

For the function \( B'(\psi) \) connected with eq. (41), so for a conical point, may be found:

\[
B (\psi) = B = \frac{(\psi + 1)M^3}{2a(M^2 - 1)^2}
\]
Thus to a second approximation the surface is axially symmetric in the neighbourhood of the conical point. Since in the parallel flow the acceleration is zero the acceleration, pressure gradient and curvature of the physical streamline exhibit a finite discontinuity at the Mach cone. From eqs. (41) and (42) the radius of curvature of the hodograph surface may be obtained and filled out in eq. (35).

For the acceleration then follows: \( x \)

\[
\left( \frac{dq}{ds} \right)_s = \pm \frac{a^2}{r} \frac{1}{\sqrt{M^2 - 1}} \frac{1}{2 + (\gamma - 1)M^2} \tag{43}
\]

and the pressure gradient is, with eq. (36), given by:

\[
\left( \frac{dP}{ds} \right)_s = \pm \frac{\rho a^2}{r} \frac{1}{\sqrt{M^2 - 1}} \frac{1}{2 + (\gamma - 1)M^2} \tag{44}
\]

From the given directions and curvatures in a conical point the conical flow patterns which are possible when parallel flow is involved can be determined. They are illustrated in figure 3.\( x \)

It may be seen that expanding flow from or into a parallel region is bounded by a Mach cone extending downstream and that compressing flow from or into a parallel region is bounded by a Mach cone extending upstream. The flow changes type across the conical-sonic line. Solutions for expanding flow bounded by a upstream Mach cone and compressing flow bounded by a downstream Mach cone may be obtained by reversing the parallel flow in the flows given in figure 3. These solutions are sketched in figure 4, and although permitted mathematically are not physically possible.

\( x \) The sign of the acceleration is determined by the definition of the positive direction of the radius of curvature. Furthermore if the flow direction is reversed, the sign of \( r \) changes with respect to the sign of the velocity along the radius. From the direction in which the streamlines in the hodograph space pass, however, the sign of the acceleration is easily obtained.

\( xx \) The direction in which the normal should be taken positive follows from the requirement that passing along a hodograph streamline corresponds to passing through the physical space, as determined by the normals to the surface, in the same direction as the velocity.
They have a significance similar to the limit cones discussed before and may be regarded as limit cones involving parallel flow on one side. As a difference with limit cones discussed in article 4, it may be noted that across such limit cones the flow changes type, whereas in limit cones involving parallel flow, this is not the case.

From figures 3 and 4 it can be noted that all curved flows bounding a region of a parallel flow are concave toward the axis of the Mach cone. Also it may be seen, that by reversing the streamlines of figures a figs. d may be obtained and by reversing the streamlines of figs. b the streamlines of figures c are given.

8. Conical flow with axial symmetry.

Conical flow with axial symmetry may be considered from the point of view of the hodograph transformation, as discussed in the preceding articles and it may be investigated what types of conical flow with axial symmetry are possible.

In the conical flow around a circular cone at zero angle of attack Busemann (ref. 7) in 1929 studied for the first time an example of conical flow with axial symmetry. This type of flow was then given considerable attention and became known as the Taylor-Maccoll solution (ref. 8). An investigation of all possible types of conical flow with axial symmetry emerging from an initially parallel flow was given by Busemann (ref. 4). Using a different set of equations for the flow this analysis, leaving the assumption of initially parallel flow was also given by Birkhoff and Walsh (ref. 8).

From the axial symmetry it follows that the hodograph surface is a surface of revolution. Singularities as discussed in articles 5 and 6 giving a continuous flow over a conical-sonic line or Prandtl-Meyer flow do therefore not occur. Limit cones and regions of parallel flow are possible.

The equation for irrotational conical flow may now be written as (ref. 4):

\[ vv_{uu} = 1 + v_u^2 - \frac{(u + vv_u)^2}{a^2} \]  

(45)
where \( v \) now is the velocity perpendicular to the axis of symmetry, the latter to be taken along the \( x \) axis.

Since the hodograph surface is a surface of revolution a conical point is only possible if this point falls on the axis of revolution. In reverse it can be shown that if there is a point on the axis of symmetry it is a conical point, so that the axis of symmetry is bounded by a region of parallel flow. This follows from the fact that in the series expansion given in eq. (39) the solution for \( A \) depending on \( \Psi \) does not hold because of the axial symmetry and the series reduces to that used by Busemann (eq (14a) in ref. 4). It has subsequently been shown in a different manner by Birkhoff and Walsh (ref. 3). The parallel region is bounded by the upstream or downstream extending Mach cone, originating in the centre of the conical field and corresponding to the Mach number in the parallel flow. This follows from the curvature relation when approaching the conical point, as was shown before and may also be concluded from the series expansion. It was proofed along in a different way in ref. 8. The conical point thus falls on the supersonic part of the \( u \) axis, whereas points on this part of the axis are always conical point. Unless the flow is parallel throughout the physical space and the point on the \( u \) axis is isolated the curve, obtained by cutting the hodograph surface with a meridian plane an leave and arrive at the \( u \) axis in two directions.

If no flow around a body is involved therefore four types of flow are possible. They consist of all combinations of types of transition from or into parallel flow, as indicating in figs. 3 and 4 and are sketched in fig. 5.

It is seen that fig. 5 b and d may be obtained by reversing the flow direction in fig. 5a and c, respectively.

Elliptic and hyperbolic points may be distinguished on the surface. Since a meridian curve of the hodograph surface arrives at the \( u \) axis with an opposite curvature as when it starts an inflection point in this curve, being a parabolic point of the surface, occurs. In all these solutions there are thus limit cones. In order to turn the flow again in the direction of the initial flow another limit cone with parallel flow on one side has to be introduced.
All the flow patterns thus have two limit cones. The physical streamlines show an inflection point, where the acceleration is in the direction of the streamline. Since in a conical flow the acceleration must be perpendicular to the radius, the streamline has an inflection point, where he is perpendicular to the radius. In the hodograph this point is found be drawing the tangent to the hodograph curve, going through the origin.

With regard to the physical possibility of these flows it can be remarked that the type given in fig. 5a is the only one where a shock wave may be placed in between the two limit cones to match solutions in front of the upstream limit cone and behind the downstream limit cone. This is the well-known Busemann diffusor flow. In the other type of compressing flow, given in fig. 5c, a shock wave in between the limit cones would deviate the flow towards the axis, that is opposite to the direction required by the solution behind the downstream limit cone.

If the flow is on one side tangent to a conical boundary again four types of flow, corresponding to the four ways for the hodograph meridian curve to arrive at the u axis, are possible. They are sketched in fig. 6. Again it is seen that fig. 6b and d may be obtained by reversing the flow direction in fig. 6a and 6c, respectively. Since on the cone surface the last term in eq.(45) disappears the curvature of the hodograph meridian curve is determined by the sign of v at the surface. The curvature is then such that an inflection point in the hodograph meridian curve occurs if this curve has to turn in to the u axis with the proper curvature.

Again a limit cone appears in the solution and another limit cone with parallel flow on one side has to be introduced to return the flow in the original direction. In fig. 6a the conical flow around a circular cone placed in a parallel stream is given. As follows from calculations given in refs. 4 and 9 it is possible for a semi cone angle less than 57.6° to find a range of Mach numbers of the parallel flow for which a shock wave may match this flow with the converging flow near the cone surface.
Extensive numerical data on Taylor-Maccoll flow are given in the M.I.T. tables (ref.10). It is not possible to fit a shock wave in the conical flow around a circular cone extending upstream as sketched in fig.5b, because the flow in front of the upstream limit is conical-subsonic. Such a flow therefore does not exist physically, although mathematically there is a solution as in the case of the conical flow around a cone extending downstream. As was pointed out in ref.11, a consequence of this is that in the compressible flow around a pointed afterbody of revolution at zero angle of attack for all Mach numbers a stagnation point exists in the rear point. If in figs. 6a and 6b the semi cone angle is increased above 90° the flows of figs. 6c and d are obtained. They might be considered to be conical flows inside a circular cone, although a streamline passes twice through the cone surface and they do not have physical reality.

As has been noted in ref.4 and 8 flow in front of the upstream limit cone in fig. 5a may with a shock wave be connected with the flow behind the downstream limit cone in fig. 6a. In this way, the most general type of axisymmetric conical flow is obtained, the Taylor-Maccoll solution and the Busemann diffusor flow being the limiting cases thereof.

9. Conical flow without axial symmetry.

Usually a conical flow around a body contains a shock wave, which in non-axially symmetric conical flow generates vorticity in the flow field. In the preceding discussion it is assumed however that the flow is isentropic. Although the foregoing results are therefore strictly valid for isentropic flow and a formal investigation of the transformation of rotational conical flow into a non-Euklidean or curved hodograph space would be necessary, one might expect that at least for conical flow with weak entropy gradients an indication of the behaviour of the flow would be given by the foregoing analysis.

* ) In ref. 10 parabolic points on the hodograph surface are called second-order discontinuities of the equations of flow.
One of the few examples of isentropic non-axially symmetric conical flow would be the flow on the expansion side of a flat plate delta wing with supersonic leading edges. This flow has been investigated by Maslen (ref. 1) and Fowell (ref. 12). A centered Prandtl-Meyer expansion fan is generated by the leading edge, behind which the flow is parallel and directed towards the centreline of the wing. In order to turn the flow in the direction of the centreline of the wing again, compressing flow must begin at the downstream Mach cone of this parallel flow. As has been shown before such a type of transition in isentropic flow occurs only at a limit cone (fig. 4c), thus indicating that in the real flow for all angles of attack a shock wave appears, behind which the flow is no longer isentropic. This inboard shock wave was also experimentally shown by Fowell (ref. 12). It was expected in ref. 12 that the isentropic solution would breakdown at a critical angle of attack, where the Prandtl-Meyer expansion fans from the leading edges would meet in the plane of symmetry, thus prescribing contradictory conditions in this plane. In the experiments however no critical angle of attack was found, what supports the explanation given by the theory of limit cones.
10. References.

2. Mixed type conical flow without axial symmetry; Summary of recent work performed at FIBAL; A. Ferri, R. Vaglio-Laurin, and Nathan Ness, FIBAL 264, December, 1954.
Fig. 1 Prandtl-Meyer flow in the hodograph space.

Expanding flow along downstream Mach waves.  

b) Expanding flow along upstream Mach waves.

Compressing flow along downstream Mach waves.  

d) Compressing along upstream Mach waves.

Fig. 2 The four types of centered Prandtl-Meyer flow bounded by parallel flow.
a) Expanding from a parallel flow.  

b) Expanding into a parallel flow.

Expanding flow (downstream extending Mach cone).

c) Compressing from a parallel flow.  
d) Compressing into a parallel flow.

Compressing flow (upstream extending Mach cone).

Fig. 3. Physically possible flow patterns in conical flow involving regions of parallel flow.
Excess flow (upstream extending Mach cone).

Compressing flow (downstream extending Mach cone).

Fig. 4. Flow patterns in conical flow involving limit cones with a region of parallel flow on one side.
Compressing flow along an upstream Mach cone.

Expanding flow along an upstream Mach cone.

Compressing flow along a downstream Mach cone.

Expanding flow along a downstream Mach cone.

Fig. 5: Solutions for axially symmetric conical flow, where no flow around a body is involved.
a. Compressing flow around a circular cone.

b. Expanding flow around a circular cone.

c. Compressing flow "inside" a circular cone.

d. Expanding flow "inside" a circular cone.

Fig. 6. Solutions for axially symmetric flow "inside" and outside