Outer-\((J_1, J_2)\)-lossless factorizations of linear discrete time-varying systems

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Abstract

In this paper the outer-J-lossless factorization for linear discrete time-varying systems is treated. Lossless operators and its corresponding J-lossless chain-scattering operators are studied. Then the factorization is treated by first ‘taking out’ the anticausal part, and then considering the outer-J-lossless factorization of the causal part.

1 Introduction

In reality most physical and economical systems demonstrate a time-varying and/or non-linear behavior. Taking into account the fact that non-linear systems operating around a particular trajectory within their operation envelope can adequately be described by linear, time-varying (LTV) systems, the development of several system theoretical concepts as exist for linear time-invariant systems can be motivated for linear time-varying systems. In this section, we introduce the notation used in representing linear discrete time-varying systems. We use the setting of the linear discrete time-varying system theory as is developed in e.g. [2, 4]. We consider the scattering and chain-scattering operators of a time-varying system, where the chain-scattering operator of a system is a useful tool for the design of a controller for the original system. In the time-invariant case we refer to [1], where the J-lossless factorization is studied for chain-scattering representations for the purpose of \(H_\infty\) control.

In Section 2 we treat the preliminaries and notation, as is reported in e.g. [4]. Then in Section 3, we study lossless scattering operators and its corresponding J-lossless chain-scattering operators in a time-varying setting. In Section 4 we continue with the development of an outer-J-lossless factorization by separating the system in a causal and anticausal part.

2 Preliminaries

In this section, we introduce the notation used in representing linear Time-Varying (LTV) systems. To be consistent with earlier literature in which this notation was defined, e.g. [4], we think of sequences as row vectors, and of operators as acting on the sequences at the left, so that we will write \(uT\) rather than \(Tu\), which is the usual notation for time-invariant systems in the control literature.

A state space realization of the LTV system \(P\) to be controlled, is denoted on a local time scale as:

\[
\begin{align*}
x_{k+1} &= x_k A_k + u_k B_k \\
y_k &= x_k C_k + u_k D_k
\end{align*}
\]

(1)

where \(x_k, u_k\) and \(y_k\) are (finite dimensional) row vectors in respectively \(\mathbb{C}^{N_k}\), \(\mathbb{C}^{M_k}\) and \(\mathbb{C}^{L_k}\) and the matrices \(\{A_k, B_k, C_k, D_k\}\) are bounded matrices of appropriate dimensions. Remark that this notation is compatible with the earlier work on LTV systems as reported in e.g. [4].

Dene the state space representation more compactly, we introduce as done in e.g. [4], the vector sequence space \(\mathcal{B}\) (which contains information on the dimensions),

\[
\mathcal{B} = \cdots \times \mathcal{B}_0 \times \mathcal{B}_1 \times \cdots,
\]

where \(\mathcal{B}_k = \mathbb{C}^{N_k}\) and the square box identifies the space of the 0-th entry. In a similar way, we introduce the dimension space sequence \(\mathcal{M}\) and \(\mathcal{N}\) from the integer sequences \(\{M_k\}\) and \(\{N_k\}\). It is allowed that some integers in these sequences are zero. The space of sequences in \(\mathcal{B}\) with finite 2-norm will be denoted by \(\ell_2^\infty\). Next we stack the sequence of state vectors \(x_k\), input vectors \(u_k\) and output vectors \(y_k\) into \(\infty\)-dimensional row vectors \(x\), \(u\) and \(y\); denoted explicitly for the state vector sequence as,

\[
\begin{bmatrix}
\cdots & x_{-1} & x_0 & x_1 & \cdots
\end{bmatrix}
\]

where the square identifies the position of the 0-th entry. Let \(\mathcal{B}^{(-1)}\) denote the shifted dimension space sequence of \(\mathcal{B}\), i.e.,

\[
\mathcal{B}^{(-1)} = \cdots \times \mathcal{B}_0 \times \mathcal{B}_1 \times \cdots,
\]

and let \(\mathcal{D}(\mathcal{M}, \mathcal{N})\) denote the Hilbert space of bounded diagonal operators \(\ell_2^\infty \to \ell_2^\infty\), then we can stack the system operators \(A_k, B_k, C_k\) and \(D_k\) into the diagonal operators \(A, B, C\) and \(D\), as (denoted only explicitly for \(A\)):

\[
\begin{align*}
A &= \text{diag} \begin{bmatrix} \cdots & A_{-1} & A_0 & A_1 & \cdots \end{bmatrix} \in \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)}), \\
C &\in \mathcal{D}(\mathcal{B}, \mathcal{N}), \quad B \in \mathcal{D}(\mathcal{M}, \mathcal{B}^{(-1)}), \quad D \in \mathcal{D}(\mathcal{M}, \mathcal{N}).
\end{align*}
\]

Let the causal bilateral shift operator on sequences be denoted by \(Z\), such that,

\[
\begin{bmatrix}
\cdots & x_{-1} & x_0 & x_1 & \cdots
\end{bmatrix} \in \mathcal{B}
\]

then a compact notation on a global time scale of the state space representation (1) is:

\[
\begin{align*}
xZ^{-1} &= xA + uB \\
y &= xC + uD
\end{align*}
\]

(2)

With this notation it is possible to represent a LTV system as an operator. If the system is asymptotically stable, then the inverse of the
3 Lossless and J-lossless operators

Lossless and J-lossless operators (functions) play an important role in system and control engineering because of their many useful and elegant properties. We consider lossless and J-lossless operators and their properties in linear discrete time-varying context for the purpose of $H_\infty$ control.

**Definition 3.1** $\Sigma \in \mathcal{X}$ is an isometry if $\Sigma^* \Sigma = I$, a coisometry if $\Sigma \Sigma^* = I$ and unitary if both $\Sigma^* \Sigma = I$ and $\Sigma \Sigma^* = I$.

A special case for an isometric operator or a co-isometric operator occurs when the operator is upper.

**Definition 3.2** An isometric operator $\Sigma$ is called lossless iff $\Sigma \in \mathcal{U}$. A coisometric operator $\Sigma$ is called co-lossless iff $\Sigma \in \mathcal{U}$. A unitary operator $\Sigma$ is called inner iff $\Sigma \in \mathcal{U}$. In this case, $\Sigma$ is both lossless and co-lossless.

The next theorem gives a characterization of isometric and co-isometric operators.

**Theorem 3.3** Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_N, B_N, C_N, D_N\}$ and $\|A_N\| < 1$. $\Sigma = D_N + B_N (I - A_N Z)^{-1} C_N$. $\Sigma$ is an isometry iff there exists a Hermitian operator $Q \in \mathcal{D}$ such that

$$
\begin{bmatrix}
A_N & C_N
\end{bmatrix}
\begin{bmatrix}
Q^{-1} & I
\end{bmatrix}
\begin{bmatrix}
A_N & C_N
\end{bmatrix}^* =
\begin{bmatrix}
Q & I
\end{bmatrix}^*
$$

(4)

$\Sigma$ is a coisometry if there exists a Hermitian operator $P \in \mathcal{D}$ such that

$$
\begin{bmatrix}
A_N & C_N
\end{bmatrix}^* 
\begin{bmatrix}
P & I
\end{bmatrix}
\begin{bmatrix}
A_N & C_N
\end{bmatrix} =
\begin{bmatrix}
P^{-1} & I
\end{bmatrix}^*
$$

(5)

$\Sigma$ is unitary iff both (4) and (5) are satisfied. If $\{A_N, B_N, C_N, D_N\}$ is a uniform realization, then $Q \succ 0$ and $P \succ 0$; if $\Sigma$ is also unitary, then $P = Q^{-1}$.

**Proof:** Sufficiency: Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_N, B_N, C_N, D_N\}$ and $\|A_N\| < 1$. Assume that (4) is satisfied. Then $\Sigma \Sigma^* = I$. With the expression of $\Sigma$ we have

$$
\Sigma \Sigma^* = [D_N + B_N (Z - A_N)^{-1} C_N] [D_N^* + C_N^* (Z - A_N^*)^{-1} B_N^*]
$$

By expanding the right hand side of the above expression with the conditions given by (4), we can obtain that $\Sigma \Sigma^* = D_N D_N^* + B_N Q^{-1} B_N^* = I$. $\Sigma \Sigma^* = I$ can be proved in a similar way.

Necessity: Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_N, B_N, C_N, D_N\}$ and $\|A_N\| < 1$. Assume that $\Sigma \Sigma^* = I$. Then, conditions given by (4) are satisfied. In particular we have $P_0(\Sigma \Sigma^*) = \Sigma \Sigma^* = I$. Define $F_0 = (I - A_N Z) C_N$, so that $\Sigma = D_N + B_N Z F_0$.

Hence,

$$
P_0(\Sigma \Sigma^*) = D_N D_N^* + B_N P_0(Z F_0 F_0^* Z^*) B_N^* = I
$$

(6)

Let $Q = P_0(F_0 F_0^*)$, then $\Sigma \Sigma^* = I$ indicates $D_N D_N^* + B_N Q^{-1} B_N^* = I$ and $Q$ satisfies the recursion $Q = C_N C_N^* + A_N Q^{-1} A_N^*$. Consider

$$
P_0(Z \Sigma^* \Sigma^* Z) = P_0(Z \Sigma^* D_N D_N^* Z) + P_0(Z \Sigma^* D_N F_0^* Z B_N^*) + P_0(Z \Sigma^* B_N Z F_0 F_0^* Z B_N^*)
$$

If $n > 0$, the first and second terms on the right hand side of the above equation are equal to zero. The third term yields $P_0(Z \Sigma^* \Sigma^* Z F_0 F_0^* Z B_N^*) = B_N^* A_N^* (n-1) C_N^* D_N^*$ and the fourth term results in $P_0(Z \Sigma^* \Sigma^* Z F_0 F_0^* Z B_N^*) = B_N^* A_N^* (n-1) A_N Q^{-1} B_N^*$. Substituting the results of the third and fourth terms back into $P_0(Z \Sigma^* \Sigma^*)$, we obtain $P_0(Z \Sigma^* \Sigma^*) = B_N^* A_N^* (n-1) C_N^* D_N^* + A_N Q^{-1} B_N^*$. Since $\Sigma \Sigma^*$ is diagonal, $P_0(Z \Sigma^* \Sigma^*) = 0$ for $n \neq 0$. Then $C_N^* D_N^* + A_N Q^{-1} B_N^* = 0$. In a similar way, we can prove that if $\Sigma \Sigma^* = I$, then (5) is satisfied. The rest of the proof follows immediately from the definitions of uniform reachability and uniform observability.

Referring to Figure 1 (a), let $\Sigma$ be a known operator, mapping input $[a_{12} b_{12}]$ to output $[a_{21} b_{21}]$, i.e.

$$
\begin{bmatrix}
a_{12} & b_{12}
\end{bmatrix} =
\begin{bmatrix}
a_{21} & b_{21}
\end{bmatrix}
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12}
\end{bmatrix}
\begin{bmatrix}
\Sigma_{21}
\end{bmatrix}
$$

(7)

In the figure, the variable with a dot stands for an input of the mapping and without a dot stands for an output. If $\Sigma_{22}$ is invertible, we can derive the mapping from $[a_{11} b_{11}]$ to $[a_{22} b_{22}]$, denoted by $\Theta$ in Figure 1(b) from $\Sigma$ as

$$
\Theta =
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13}
\end{bmatrix}
\begin{bmatrix}
\Sigma_{21} & \Sigma_{22} & \Sigma_{23}
\end{bmatrix}
$$

(8)

**Figure 1:** Scattering operator $\Sigma$, chain scattering operator $\Theta$.

$\Sigma$ is called a scattering operator and $\Theta$ is called the corresponding chain scattering operator.

If we introduce a feedback relation $b_1 = a_1 S$ between $b_1$ and $a_1$, then...
then the closed loop mapping from $b_2$ to $a_2$, denoted by $\Phi$, is given by

$$
\Phi = \Sigma_{21} + \Sigma_{22}(I - \Sigma_{12})^{-1}\Sigma_{11}
$$

$$(9)$$

and

$$
\text{HM}(\Theta; S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21})
$$

where $\text{HM}$ stands for Holographic transformation.

In Figure 1, we use a dot to indicate the variables of the input port. The variables with arrows into the block are input variables and with arrows out of the block are output variables.

Let $J_1 \in \mathcal{D}$ be the input port signature and $J_2 \in \mathcal{D}$ the output port signature matrices which are defined as $J_i = \begin{bmatrix} \cdots & j_{i-1} & j_{i0} & j_{i1} & \cdots \end{bmatrix}$ for $i = 1, 2$, where the entry $j_{ik} = \begin{bmatrix} I_{(p+)} & -I_{(p-)} \end{bmatrix}$ ($k = -\infty, \ldots, +\infty$) is determined by the dimensions of the input and output of the ports at time instant $k$. For a chain scattering operator, the dimension of input variables on the input port is $p+$ and the dimension of output variables on the input port is $p−$. It is reversed on the output port.

**Definition 3.4** Let $J_1$ and $J_2$ be the input and output signature operators respectively of a known operator $\Theta \in \mathcal{X}$. $\Theta$ is a $(J_1, J_2)$-isometry (sometimes shortly called a J-isometry) if $\Theta J_1 \Theta^* = J_1$. $\Theta$ is a $(J_1, J_2)$-coisometry (shortly a J-coisometry) if $\Theta J_1 \Theta^* = J_2$ and $\Theta$ is $J$-unitary if both $\Theta J_2 \Theta^* = J_1$ and $\Theta^* J_1 \Theta = J_2$.

**Theorem 3.5** Let $J_1$ and $J_2$ be the input and output signature operators respectively of a known operator $\Theta \in \mathcal{X}$. Let the operator $\Sigma$ be isometric, coisometric or unitary. If the corresponding chain scattering operator $\Theta$ exists, then it is $J$-isometric, $J$-coisometric or $J$-unitary, respectively. If the corresponding dual chain scattering operator exists, then it is $J$-isometric, $J$-coisometric or $J$-unitary, respectively.

**Proof:** For the proof of the first statement we refer to [4]. The second statement is proved in a similar way. $\square$

If $\mathcal{H}$ is a locally finite $\mathcal{D}$-invariant subspace, then it has some strong basis representation $\mathcal{F}$ such that $\mathcal{H} = \mathcal{D}_2 \mathcal{F}$. Similar to the definition of a Gramian operator by $\Lambda_F = \mathcal{P}_0(\mathcal{F}/\mathcal{F}^*)$, we define the J-Gramian operator of this basis as the diagonal operator: $\Lambda_F^T = \mathcal{P}_0(\mathcal{F}/\mathcal{F}^*) \in \mathcal{D}(\mathcal{F}, \mathcal{B})$. The operator $\mathcal{F}$ is J-orthonormal if $\Lambda_F^T = J_F$, where $J_F$ is some signature operator on $\mathcal{B}$. We call $\mathcal{H}$ regular if the J-Gramian operator of any strong basis of $\mathcal{H}$ is boundedly invertible. Note that $\Lambda_F^T$ boundedly invertible implies the same for $\Lambda_F$, i.e., $\Lambda_F \succ 0$.

Let $T \in \mathcal{U}$ have a uniformly minimal realization $\{A, B, C, D\}$ with $\varepsilon_A < 1$ and $J_1$ and $J_2$ be the input and output signature operators. Then $F^* = BZ(I - AZ)^{-1}$ and $F_0 = (I - AZ)^{-1}C$ are strong bases of $\mathcal{H}(T)$ and $\mathcal{H}_0(\mathcal{T})$, respectively. If $\mathcal{P}_0(FJ_1 F^*)$ and $\mathcal{P}_0(FJ_2 F^*_0)$ are invertible, we say the realization $\{A, B, C, D\}$ is regular. Regular realizations of bounded lower operators or mixed operators are defined in a similar way.

The chain scattering operator of a lossless scattering system is not lossless itself, but has some special features.

**Definition 3.6** If an operator $\Sigma$ is lossless, then we call the corresponding chain scattering operator $\Theta$ is $J$-lossless.

It is easy to obtain a similar characterization of a chain-scattering operator $\Theta \in \mathcal{U}$ being a $J$ isometry as is obtained for of $\Sigma$ being an isometry in Theorem 3.3. But, contrary to the scattering representation, where lossless operators are always upper by definition, the corresponding chain scattering representation can be lower or mixed. Thus we also need the extension of Theorem 3.3 to lower and mixed operators. It is well known that the cascade connection of J-lossless operators results in a J-lossless operator. In particular, the cascade connection of an upper J-lossless operator and a lower J-lossless operator results in a J-lossless operator which is in general not upper or lower anymore.

**Theorem 3.7** Let $\Theta \in \mathcal{X}$ be a locally finite operator and $\{A_1, B_1, C_1, A_2, B_2, C_2, D_0\}$ be a regular realization with $\varepsilon_A < 1$ and $\varepsilon_A < 1$ such that $D_0 = B_1 Z(I - A_1 Z)^{-1} C_1 + B_2 Z^* (I - A_2 Z^*)^{-1} C_2$. $\Theta$ is $(J_1, J_2)$-isometric if there exists a Hermitian operator $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \in \mathcal{D}$ such that:

$$
\begin{bmatrix}
A_1 & C_1 \\
B_1 & D_0 \\
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
\end{bmatrix}
= \begin{bmatrix}
I & A_2 \\
B_2 & I \\
\end{bmatrix}
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
\end{bmatrix}
= \begin{bmatrix}
I & A_1 \\
B_1 & I \\
\end{bmatrix}
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
\end{bmatrix}
$$

(11)

If $Q \succ 0$, then $\Theta$ is $J$-unitary.

**Proof:** The proof follows straightforwardly from writing out the expressions for $\Theta J_2 \Theta^*$, using (11), and reorganizing the expressions. $\square$

The next theorem reveals an important property of J-lossless operators, since it is a very useful if we want to design an $H_\infty$ controller via the chain-scattering representation.

**Theorem 3.8** Let an operator $\Theta \in \mathcal{X}$ be $(J_1, J_2)$-lossless and have a partitioning as $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ and let an operator $S \in \mathcal{U}$ be strictly contractive ($\|S\|_\infty < 1$). Let

$$
\Phi = \text{HM}(\Theta; S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21})
$$

Then $\Phi$ is upper and $\|\Phi\|_\infty < 1$.

**Proof:** First, we show the invertibility of $(S\Theta_{12} + \Theta_{22})$. Since $\Theta$ is $(J_1, J_2)$-lossless, the corresponding $\Sigma \in \mathcal{U}$ is lossless and has a partitioning

$$
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22} \\
\end{bmatrix}
$$

with $\Sigma_{22}$ invertible. Under these conditions, $\|\Sigma_{12}\|_\infty < 1$ and $(I - \Sigma_{12})$ is invertible. With the relation $\Sigma_{12} = -\Theta_{12}\Theta_{22}^{-1}$, we have $(I + S\Theta_{12}\Theta_{22}^{-1})$ invertible and then $(\Theta_{22} + S\Sigma_{12})$ invertible. $\Phi$ can be expressed with $\Sigma$ and $S$ as, $\Phi = \Sigma_{21} + \Sigma_{22}(I - \Sigma_{12})^{-1}\Sigma_{11}$. The
Using the expansion of \((I - S\Sigma_1^2)^{-1}\) yields \(\Phi = \Sigma_{21} + \Sigma_{22}S\Sigma_{11} + \Sigma_{22}S\Sigma_{12}\Sigma\Sigma_{11} + \cdots\). Under the given conditions, the Neumann series converges to an upper matrix, i.e., \(\Phi\) is upper. Now, rewrite equation (12) as

\[
(S\Theta_{12} + \Theta_{22}) \begin{bmatrix} \Phi & I \end{bmatrix} = \begin{bmatrix} S & I \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}
\] (13)

Denote \(\varphi = (S\Theta_{12} + \Theta_{22})\). Multiplying equation (13) on the right side first with the \(J\) operator and then multiplying each side with the conjugate transpose of themselves, we obtain, \(\varphi(\Phi^* - I)\varphi^* = SS^* - I\). From the condition \(|S|_\infty < 1\), we then have that \(|\Phi|_\infty < 1\).

Finally, in this section we introduce some notions on spaces that are of interest for the rest of the paper.

**Definition 3.9** Let \(T \in U\). Then we define the input null space as \(K(T) = \{U \in L_2Z^{-1} : P(UT) = 0\}\), the input state space as \(H(T) = P_{L_2Z^{-1}}(U(T))\), the output state space as \(H_o(T) = \Phi(\Sigma_2Z^{-1})\), and the output null space as \(K_o(T) = \{Y \in U_2 : P_{L_2Z^{-1}}(Y(T)) = 0\}\).

A generalization of a theorem on \(J\)-unitary operators (see [4]) to \((J_2, J_1)\)-isometries is given as follows:

**Theorem 3.10** Let \(\Theta \in U\) be a \((J_2, J_1)\)-isometry, i.e., \(\Theta J_2 \Theta^* = J_1\), then the output null space is given by \(K_o(\Theta) = U_2 \odot J_2 \Theta K(\Theta^* | U_2)\).

**4 J-lossless Factorization**

Let us consider the factorization \(G = T_0\Theta \in X\) with \(T_0\) invertible and outer, and \(\Theta (J_2, J_1)\)-lossless in the discrete time-varying context. This kind of factorization is called an outer–\(J, J'\)-lossless factorization [1]. Here, we consider the case where the dimension sequence of the output of \(G\) is pointwise greater than or equal to the dimension sequence of the input. With \(T_0\) invertible this means that \(\Theta\) should be of the same size as \(G\).

Assume that an operator \(G \in X\) is specified by the representation,

\[
G = D + B_cZ(I - A_cZ)^{-1}C_c + B_cZ^*(I - A_cZ)^{-1}C_a
\] (14)

with \(\ell_k < 1\) and \(\ell_k < 1\) and the dimension of the output of \(G\) is pointwise greater than or equal to the dimension of the input. Suppose that \(G\) admits a factorization:

\[
G = G_1\Theta_a
\] (15)

where the operator \(\Theta_a \in L\) is anticausal and J-lossless (the subscript ‘\(a\)’ stands for anticausal), and \(G_1\) is causal. Furthermore, suppose that \(G_1\) admits a factorization as

\[
G_1 = T_0\Theta_c
\] (16)

where \(\Theta_c \in U\) (the subscript ‘\(c\)’ stands for causal) is J-lossless and \(T_0 \in U\) is outer. Define

\[
\Theta = \Theta_b\Theta_a
\] (17)

then, \(G\) has an outer–\(J, J'\)-lossless factorization \(G = T_0\Theta\).

With this strategy, we consider the outer–\(J, J'\)-lossless factorization of \(G\) in two steps, first we take out the anticausal J-lossless part and then the causal \(J, J'\)-lossless part.

**4.1 Anticausal J-lossless factorization**

Let \(G \in X\) be a given driver with outer signature matrices \((J_1, J_2)\), specified by (14) with \(\ell_{A_c} < 1\) and \(\ell_{A_c} < 1\) and \((A_a, C_a)\) uniformly observable. Let us consider the factorization in equation (15).

**Proposition 4.1** Let \(G \in X\) be a given driver with outer signature matrices \((J_1, J_2)\), specified by (14) with \(\ell_{A_c} < 1\) and \(\ell_{A_c} < 1\) and \((A_a, C_a)\) uniformly observable. Let \(F_0 = (I - A_cZ^*)^{-1}C_a\). Define a \(J\)-unitary operator \(\Theta_a \in L\) with its anticausal output space \(H_o(\Theta_a) = D_2F_0\). Assume that there is a Hermitian invertible operator \(Q \in D\) such that

\[
A_aQA_a^* - C_aJ_2C_a^* = Q^{-1}
\] (18)

is satisfied. Under this condition, we embed \([A_a, C_a]\) with a pair \([B_{\Theta_0}, D_{\Theta_0}]\) such that:

\[
\begin{bmatrix}
A_a & C_a \\
B_{\Theta_0} & D_{\Theta_0}
\end{bmatrix}
\begin{bmatrix}
Q & -J_2 \\
-J_2 & P
\end{bmatrix}
\begin{bmatrix}
A_a & C_a \\
B_{\Theta_0} & D_{\Theta_0}
\end{bmatrix}^*
= Q^{-1} - J_2
\] (19)

and

\[
\begin{bmatrix}
A_a & C_a \\
B_{\Theta_0} & D_{\Theta_0}
\end{bmatrix}^*
\begin{bmatrix}
P^{-1} & -J_2 \\
-J_2 & P
\end{bmatrix}
\begin{bmatrix}
A_a & C_a \\
B_{\Theta_0} & D_{\Theta_0}
\end{bmatrix}
= P^{-1} - J_2
\] (20)

are satisfied. Define a \(J\)-unitary operator \(\Theta_a = D_{\Theta_0} + B_{\Theta_0}Z(I - A_aZ)^{-1}C_c \in L\) and let \(G_1 = G_2J_2\Theta_a\). Then, \(G_1\) is upper and has a realization

\[
G_1 = D_2 + B_cZ(I - A_cZ)^{-1}C_c
\] (21)

where \(A_c, B_c, C_c, D_c\) are equal to,

\[
A_c = \begin{bmatrix}
A_c & C_cJ_2C_a^* \\
A_c & A_c^*B_{\Theta_0}J_2
\end{bmatrix}
\] (22)

\[
B_c = \begin{bmatrix}
B_c & D_2J_2C_c^* - B_cQ\Phi_a \\
D_2J_2 & D_2J_2
\end{bmatrix}
\] (23)

\[
D_2 = DJ_2D_1J_2 - B_cQ\Phi_a\Phi_a^*J_2
\] (24)

If \(Q \neq 0\), then \(\Theta_a\) is \(J\)-lossless and \(G\) has a factorization \(G = G_1\Theta_a\) with \(G_1\) upper and \(\Theta_a\) lower and \(J_2\)-lossless.

**Proof:** Rewrite equation (18) as

\[
\begin{bmatrix}
A_a & C_a \\
B_{\Theta_0} & D_{\Theta_0}
\end{bmatrix}
\begin{bmatrix}
Q & -J_2 \\
-J_2 & P
\end{bmatrix}
\begin{bmatrix}
A_a & C_a \\
B_{\Theta_0} & D_{\Theta_0}
\end{bmatrix}^*
= Q^{-1}
\]

For \(Q\) invertible, we embed \([A_a, C_a]\) with \([B_{\Theta_0}, D_{\Theta_0}]\) such that (19) and (20) are satisfied. In this case, \(P = Q^{-1}\) and the realization \([A_a, B_{\Theta_0}, C_a, D_{\Theta_0}]\) is regular. We construct \(\Theta_a = D_{\Theta_0} + B_{\Theta_0}Z(I - A_aZ)^{-1}C_c\). With the lower, and \(J\)-unitary version of Theorem 3.3 we know that \(\Theta_a\) is \(J_2\)-unitary. Let \(G_1 = G_2J_2\Theta_a\), then

\[
G_1 = G_2J_2\Theta_aJ_2 + [D + B_cZ(I - A_cZ)^{-1}C_c]J_2C_c^*(I - ZA_c^*)^{-1}ZB_c\Theta_aJ_2
\]

\[
+ B_cZ(I - A_cZ)^{-1}C_cJ_2D_2\Theta_aJ_2 + B_cZ(I - A_cZ)^{-1}C_cJ_2D_2\Theta_aJ_2
\]

\[
+ B_cZ^*(I - A_cZ)^{-1}C_cJ_2D_2\Theta_aJ_2
\]

\[
+ B_cZ^*(I - A_cZ)^{-1}C_cJ_2D_2\Theta_aJ_2
\]

\[
+ B_cZ^*(I - A_cZ)^{-1}C_cJ_2D_2\Theta_aJ_2
\]

\[
+ B_cZ^*(I - A_cZ)^{-1}C_cJ_2D_2\Theta_aJ_2
\]
Let a chain scattering operator 

\[ B_2 Z^*(I-A_2 Z)^{-1} C_{2,2} D_{2,2}^* J_2 \]

\[ + B_2 Z^*(I-A_2 Z)^{-1} C_{2,2} D_{2,2}^* (I-Z A_2^{-1}) Z B_{2,2} J_2 = -B_2 Q B_{2,2} J_2 - B_2 Q A_2 Z (I-Z A_2^{-1}) Z B_{2,2} J_2 \]

Now we see that this part is also upper. Then \( G_1 \) is upper. By combining the first three terms with the last two terms of \( G_1 \), we derive that \( G_1 \) has the realization \( \{ A_2, B_2, C_2, D_2 \} \) of (22), (23), and (24). Since \( \Theta_1^* J_2 \Theta_2 = J_2 \), \( G \) admits a factorization \( G = G_1 \Theta_\alpha \). If \( Q > 0 \), then \( \Theta_\alpha \) is anticausal and \( J_2 \)-lossless as (15) requires.

4.2 Causal J-lossless factorization

In this subsection we continue with the second step, the outer-\( J^* \)-lossless factorization of \( G_1 \). We start with a result on the outer part.

Theorem 4.2 Let \( T \in \mathcal{U} \) with port signature matrices \( (J_1, J_2) \).

Suppose that there exists a \( \Theta \in \mathcal{U} \) which is \( (J_1, J_2) \)-isometric with its realization regular, such that \( \mathcal{T}_{T, T} = \mathcal{U}_T \). Then \( T \) has a factorization \( T = T_0 \Theta \) with \( T_0 \in \mathcal{U} \) outer.

Proof: Define \( T = T J_2 \Theta^* \). Then

\[ \mathcal{T}_{T_0 T_0} = \mathcal{T}_{T_0 T_0} \Theta T_0^* \Theta = \mathcal{T}_{T_0 T_0} \Theta T_0^* \Theta = \mathcal{T}_{T_0 T_0} \Theta T_0^* \Theta = T_2 \Theta \]

so that \( T_0 \) is outer. If \( \Theta \) is J-unitary, \( \Theta^{-1} = J_2 \Theta^* J_1 \), then it is always true that if \( T_0 = T J_2 \Theta^* J_1 \), then \( T = T_0 \Theta \). In the case that \( \Theta \) is only \( (J_1, J_2) \)-isometric but with its realization regular, there always exists an \( \Omega \in \mathcal{U} \) which is the J-complement of \( \Theta \) such that:

\[
\begin{bmatrix}
\Theta & J_1 \\
J_2^* & \Theta^* \end{bmatrix} = \begin{bmatrix}
J_2 & J_1 \\
\Theta & \Theta^* \end{bmatrix}
\]

(25)

where \( J_2 \) is called the complement port signature matrix of \( J_1 \). Then we have:

\[ \Omega J_1 \Omega + \Theta J_1 \Theta = J_2 \text{ or } J_2 \Omega J_1 \Omega = I - J_2 \Theta^* J_1 \Theta \text{ and } \Theta J_2 \Omega = 0. \]

On the other hand, because \( \mathcal{T}_{T_0 T_0} \subseteq \mathcal{T}_{T_0 T_0} \Theta T_0^* \Theta = \mathcal{T}_{T_0 T_0} \Theta T_0^* \Theta = T_2 \Theta \), hence \( T = T_0 \Theta \) if \( \Theta \) has an \( \Omega \) such that the equations of (25) are satisfied. In case of a regular realization of \( \Theta \) there always exists such an \( \Omega \). This proves the theorem.

We have defined the input and output signature matrices \( J_1 \) and \( J_2 \) for a chain scattering operator. In general, their entries are time-varying and the relation between \( J_1 \) and \( J_2 \) cannot be given by a simple expression. But in some special cases, \( J_1 \) and \( J_2 \) are explicitly related. Let us consider the relation of \( J_1 \) and \( J_2 \) in a special case which is related to the problem we deal with.

Let a chain scattering operator \( T \in \mathcal{U} \). The factorization we are interested in is \( T = T_0 \Theta \) with \( T_0 \) outer, \( \Theta \in \{ J_1, J_2 \} \)-lossless and upper. Let \( \Theta \) be partitioned as:

\[
\begin{bmatrix}
\Theta_1 & \Theta_2 \\
\Theta_2 & \Theta_3 \\
\Theta_4 & \Theta_5 \\
\Theta_5 & \Theta_6 \\
\end{bmatrix}
\]

where \( \Theta_3 \) is upper and \( \Theta_3 \) is upper. On the other hand, the corresponding scattering operator, \( \Sigma \), is lossless. Thus \( \Theta_3 \) must be upper as well. Let \( \{ D_0, B_0, C_0, D_0 \} \) be a realization of \( \Theta \).

Proposition 4.3 Let \( T \in \mathcal{U} \) be an operator with port signature matrices \( (J_1, J_2) \). Let \( \Theta \) be a \( (J_1, J_2) \)-isometric operator such that \( \mathcal{K}_1^\alpha(\Theta) = \mathcal{T}_{T, T} \). Then, \( \mathcal{H}(\Theta)(T J_2^* X) \in \mathcal{H}(T) \). The proof of Theorem 4.2 is then a consequence of (24).

Proposition 4.4 Let \( T \in \mathcal{U} \) be a locally finite transfer operator with port signature matrices \( (J_1, J_2) \) such that \( J_2 = \begin{bmatrix} I & J_1 \end{bmatrix} \) and a uniformly reachable realization \( \{ A, B, C, D \} \) such that \( \lambda_\alpha < 1 \) and \( (T_0) J_2^{-1} \) exists. \( T \) has a factorization \( T = T_0 \Theta \), where \( T_0 \) is invertible and upper, and \( \Theta \in \mathcal{U} \in \{ J_1, J_2 \} \)-lossless if there is a pair
{A_\Theta, C_\Theta} which corresponds to a J-orthonormal basis representation of \( T_\Omega(\Theta) \), the output state space of \( \Theta \), with \( L_{A_\Theta} < 1 \), and a diagonal operator X such that the following conditions are satisfied,

(i) \( A_\Theta X(-1)^* + C_\Theta J_2 C^* = X \)

(ii) \( A_\Theta X(-1)^* B^* + C_\Theta J_2 D^* = 0 \)

(iii) \( A_\Theta A_\Theta^* + C_\Theta J_2 C^* = I \)

(iv) \( \ker(X) = 0 \)

If such an X exists, it is unique up to a left diagonal unitary factor, i.e., X is unique.

**Proof:** Let \( F = (I - Z A^*)^{-1} Z B^* \) and \( F_o = (I - AZ)^{-1} C \). Suppose that a pair \{A_\Theta, C_\Theta\} and a diagonal operator X fulfilling (i)–(iii) exist and let \( E_o J_2 = (I - A_\Theta Z^{-1})C_\Theta J_2 \), we have the following equations:

\[
E_o J_2 = C_\Theta J_2 + A_\Theta Z E_o J_2 \\
ZF = B^* + A^* F \\
T^* = D^* + C^* F
\]

(28) (29) (30)

As analyzed before, T has a factorization \( T = T_o \Theta \) with \( T_o \) outer and \( \Theta \) \((J_2, J_1)\)-lossless, iff the pairs \( E_o J_2 T^* = XF \) with \( \ker(X) = 0 \) and \( \Theta(J_2, J_1)\)-lossless are satisfied. Uniform reachability implies that \( T(T) = D_2 F \). According to Proposition 4.3, we need to find a \((J_2, J_1)\)-lossless operator \( \Theta \) such that \( \bar{T}_\Omega(\Theta)J_2 T^* \subset \bar{T}(T) \). That is \( E_o J_2 T^* = XF \) for some bounded \( X \in D \). Because \( F \in L_2Z^{-1}, P_{L_2 Z^{-1}}(E_o J_2 T^*) = XF \). With \( P_{L_2 Z^{-1}}(E_o J_2 T^*) = XF \) and equation (29), \( P_{L_2 Z^{-1}}(ZF) = X^{(-1)} P_{L_2 Z^{-1}}(ZF) \).

On the other hand,

\[
A_\Theta P_{L_2 Z^{-1}}(ZE_o J_2 T^*) = P_{L_2 Z^{-1}}(A_\Theta Z E_o J_2 T^*) - P_{L_2 Z^{-1}}(C_\Theta J_2 T^*) = XF - C_\Theta J_2 C^* F
\]

Since \( P_{L_2 Z^{-1}}(Z E_o J_2 T^*) = P_{L_2 Z^{-1}}(Z F) \), we have \( A_\Theta X(-1)^* A^* F = XF - C_\Theta J_2 C^* F \). The uniform reachability yields \( A_\Theta X(-1)^* A^* + C_\Theta J_2 C^* = X \), i.e., condition (i).

Condition (ii) is derived from the condition that \( E_o J_2 T^* = XF \in L_2Z^{-1} \) as follows

\[
P_0(E_o J_2 T^*) = C_\Theta J_2 D^* + A_\Theta P_0(Z E_o J_2 T^*) = C_\Theta J_2 D^* + A_\Theta X(-1)^* B^* = 0
\]

Condition (iii) is given by the fact that \( E_o J_2 \) is a J-orthonormal basis representation of the outer state space of a J-Lossless operator and condition (iv) has been derived before.

Conversely, if the conditions (i)–(iv) are satisfied, then the conditions for the existence of the outer \((J_2, J_1)\)-lossless factorization \( T = T_o \Theta \) are satisfied. Substitution of the conditions (i)–(ii) into \( E_o J_2 T^* \) yields that \( E_o J_2 T^* = XF \) and that the conditions (iii)–(iv) are the same in both directions.

With the same strategy given by Theorem 3.28 in [4] we can prove that \( H_\Omega^* = P_{\Omega(\Theta)} F_o \). Hence \( P_{L_2 Z^{-1}}(E_o J_2 T^*) = P_{\Omega(\Theta)}(F_o J_2 F_o) \).

Since \( T \) is uniformly reachable, \( X = P_\Omega(E_o J_2 F_o) \). \( X^* X \) is obtained as

\[
X^* X = P_\Omega(F_o J_2 F_o) P_\Omega(E_o J_2 F_o) = P_\Omega(P_\Omega(F_o J_2 E_o) E_o J_2 F_o) = P_\Omega(P_{L_2 Z^{-1}}(F_o J_2 E_o) E_o J_2 F_o) \]

This implies that \( X^* X \) is unique.

In order to obtain \( X \) in a unique manner, we can choose \( X \) at every step to be in an upper triangular form with all its diagonal entries positive. If we have found \( X \) such that the conditions (i)–(iv) are satisfied, then we have the pair \{A_\Theta, C_\Theta\} which corresponds to a realization of a \((J_2, J_1)\)-lossless operator \( \Theta \). Embedding \{A_\Theta, C_\Theta\} with \{B_\Theta, D_\Theta\} such that,

\[
\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} 1 & I \\ J_2 & B_\Theta & C_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} I \\ 1 \end{bmatrix}
\]

then, \( \Theta = D_\Theta + B_\Theta Z(I - A_\Theta Z^{-1})C_\Theta \) and \( \Theta(J_2, J_1) \)-lossless. With \( T = \Theta o \Theta \), the outer operator \( T_o \) is derived as follows

\[
T_o = T_o \Theta o \Theta = D_2 J_2 D_2 o \Theta + B_2 Z(I - A_2 Z^{-1}) C_2 J_2 D_2 o \Theta o \Theta + T_2 C_2 o \Theta (I - Z o A_2 Z^{-1})^* B_2 o \Theta
\]

After rewriting the third term of the above equation we get

\[
T_2 C_2 o \Theta (I - Z o A_2 Z^{-1})^* B_2 o \Theta = B X^{(-1)} B_2 o \Theta + B_2 Z(I - A_2 Z^{-1})^* A X^{(-1)} B_2 o \Theta
\]

By substituting into (31), we obtain the realization of \( T_o \) given by

\[
T_o = \begin{bmatrix} A_2 C_2 D_2 o \Theta + A X^{(-1)} B_2 o \Theta \\ B_2 D_2 o \Theta \end{bmatrix}
\]

(32)

The invertibility of \( T_o \) follows from condition of the invertibility of \( T o \).

If we rewrite the above results together with a special case of Lemma 5.16 in [4] in an algorithm, a problem that remains is the initialization of \( X \). For a finite operator the dimension of the states after time instant 0 is zero, i.e., \( X_0 = [\] \). For a system which is time-varying until time-instant 0, and time invariant after time instant 0, the initial condition is determined by the solution of the time-invariant system. For a periodic system, the initial condition is determined by the solution of the equivalent time-invariant system within one period. The time invariant system solution can be obtained from an analysis of the eigen space of a corresponding Riccati equation. See e.g. [5].

5 Concluding remarks

In this paper we have treated the outer-J-lossless factorization of a linear discrete time-varying system mostly in an operator setting, i.e., the characterization is given in terms of Lyapunov-type of equations. The proposed strategy can be used as a tool for the development of a solution to the \( H_\infty \) control problem. This has been done in [10].

**References**


