Giant current fluctuations in an overheated single-electron transistor

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Interplay of cotunneling and single-electron tunneling in a thermally isolated single-electron transistor leads to peculiar overheating effects. In particular, there is an interesting crossover interval where the competition between cotunneling and single-electron tunneling changes to the dominance of the latter. In this interval, the current exhibits anomalous sensitivity to the effective electron temperature of the transistor island and its fluctuations. We present a detailed study of the current and temperature fluctuations at this interesting point. The methods implemented allow for a complete characterization of the distribution of the fluctuating quantities, well beyond the Gaussian approximation. We reveal and explore the parameter range where, for sufficiently small transistor islands, the current fluctuations become gigantic. In this regime, the optimal value of the current, its expectation value, and its standard deviation differ from each other by parametrically large factors. This situation is unique for transport in nanostructures and for electron transport in general. The origin of this spectacular effect is the exponential sensitivity of the current to the fluctuating effective temperature.

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I. INTRODUCTION

By its statistical physics definition, the temperature of an open electron system in equilibrium is fixed, with a value equal to the temperature of the reservoir it is connected to.[1] The energy of the system can still fluctuate because of the constant exchange of energy with its surroundings. Out of equilibrium, the entire concept of a temperature becomes ill defined. If the internal relaxation in the system is strong enough, however, the electron energy distribution still follows the equilibrium Fermi function with some effective temperature that is determined by a balance of the energy currents flowing into and out of the system.[2] In this out-of-equilibrium situation the intrinsic fluctuations of the energy currents translate directly to fluctuations in the effective temperature of the system, provided that the electron-electron relaxation time is significantly shorter than the energy relaxation time to the reservoirs or the phonon bath.[3] If the electron-phonon interaction in the system is weak, the dominant energy flows are to the reservoirs which are used to drive the system out of equilibrium. In this case the system is said to be fully overheated. As the size of the system becomes ever smaller, full overheating becomes easier to achieve, or, harder to avoid.

One example of a system where overheating is important to take into account is the single-electron transistor (SET).[4] When the size of the Coulomb island is decreased ever further in the pursuit for sensitivity, the temperature of the island begins to be affected by the electronic heat flows to the surroundings, e.g., leads.[5–7] In an earlier paper[8] we studied the fully overheated single-electron transistor, and found out that the inclusion of inelastic cotunneling,[9] along with the often dominant sequential tunneling of single electrons, is important and leads to peculiar overheating effects. The electron transport through the overheated SET is divided into three regimes: cotunneling dominated, competition, and single-electron dominated regimes. We also found that the electric current noise in the crossover interval is mostly contributed by the frequency interval corresponding to typical time scale of temperature equilibration. This is because of the anomalously strong temperature sensitivity of the current in this interval, which makes the temperature fluctuations directly visible in the current (noise).

In this paper we explore and quantify further the temperature fluctuations and the associated current fluctuations in an overheated SET. We study the SET under conditions of well-developed Coulomb blockade so that the junction conductances $G_{T,L}, G_{T,R} \ll e^2/\hbar$. We assume that the level spacing on the island is small compared to its temperature and the charging energy, $\delta_I \ll T \ll E_C \equiv e^2/2C$, which is the case in metallic SETs. To have a clear model, we assume a vanishing temperature in the leads.[10] Since a possible asymmetry of the SET does not affect our results qualitatively, we assume $G_{T,L}=G_{T,R}=G_T$ and define a dimensionless conductance $g_T \equiv G_T\hbar/e^2 \ll 1$. Our focus is mainly on the fluctuations around the crossover between the competition and single-electron regimes, $V_b \approx V_C=(\sqrt{2}-1)V_{\mathrm{th}}$, i.e., well below the zero-temperature Coulomb blockade threshold, $eV_{\mathrm{th}}=2(E_C-eV_G)$ (see Fig. 1). Note that in the actual experimental setup eV_G must be replaced with a combination of the source, drain, and gate voltages together with the respective mutual capacitances.[11] Here we lump all this into one gate voltage dependence. The electron-electron relaxation time, $\tau_{e\text{-}e}$, is assumed short compared to the characteristic time scale for the energy relaxation to the leads, τ_E, which in turn is assumed small compared to the electron-phonon relaxation time, $\tau_{e\text{-}ph}$.[12] These conditions ensure that the island is fully overheated and in quasiequilibrium.

Within the Coulomb blockade, the temperature of the island is fixed by a balance between energy flows in and out of the island due to inelastic cotunneling and sequential tunneling (see Fig. 2). Inelastic cotunneling leaves behind an electron-hole excitation on the island, always heating it up. Sequential tunneling can either cool or heat the island, depending on bias voltage. As discussed in Ref. 8, there are three qualitatively different regions of bias voltages, revealed

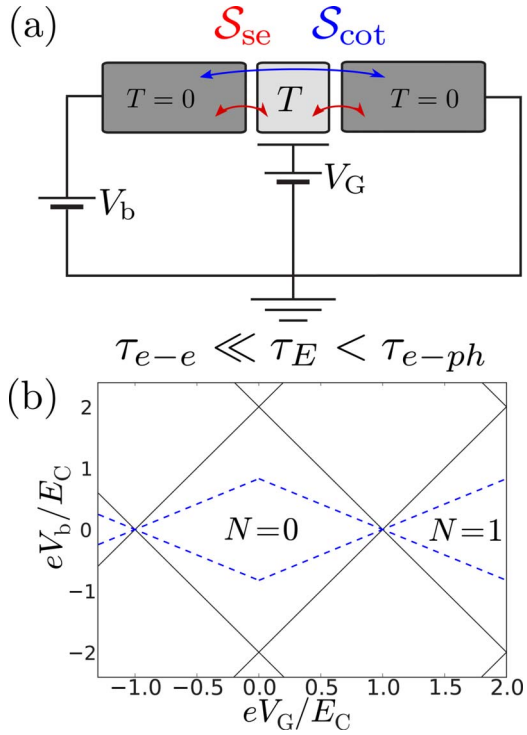

FIG. 1. (Color online) (a) A schematic of a single-electron transistor, biased by voltage V_b. The charge on the island can be tuned with the gate voltage V_G. (b) Coulomb diamonds in a symmetric SET. Dashed line shows the threshold voltage V_C. Changing the gate voltage has the effect of changing V_C.

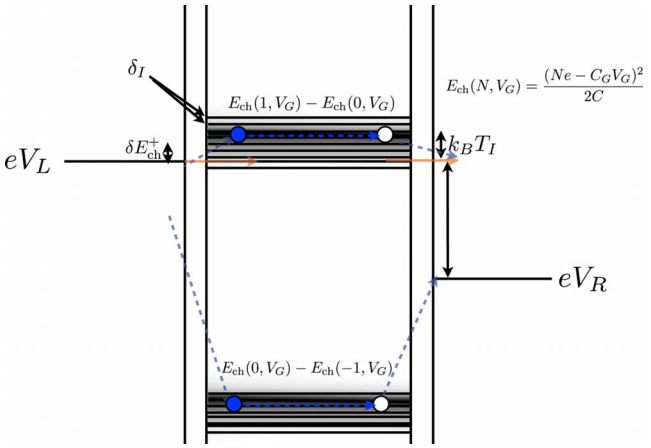

FIG. 2. (Color online) Schematic energy balance of the single-electron transistor within the Coulomb blockade regime, where the allowed charge state energies are outside the bias window. In this case the temperature of the island is fixed by the balance between sequential tunneling (solid red arrows) and inelastic cotunneling (dashed blue arrows), where the strength of the previous depends on the exponentially small $[\propto \exp(-\delta E^+/k_BT)]$ amount of empty (filled) single-electron states below (above) the Coulomb blockade threshold corresponding to the addition (removal) of an electron to (from) the island. This leads to the extreme temperature sensitivity of the current, which is the main topic of this paper.

by the behavior of the average temperature: at the lowest voltages, sequential tunneling is completely suppressed and the average temperature is fixed by the inelastic cotunneling. Close to the Coulomb blockade threshold (and above), inelastic cotunneling effects can be disregarded and sequential tunneling sets the temperature. The most interesting region is the intermediate regime where the competition between these two processes gives rise to both anomalously sensitive temperature dependence of the current and to strongly non-Gaussian temperature fluctuations. In this regime, we find (see Sec. II and Appendix A) for the average temperature $\langle T\rangle \approx T_C$ where T_C satisfies

$$\alpha g_T \left(\frac{V_C}{\sqrt{2}T_C}\right)^3 \exp\left[\frac{1}{\sqrt{2}}\left(\frac{V_C}{T_C}-1\right)\right] = 1, \tag{1}$$

α being a dimensionless coefficient, $\alpha \approx 0.1$. For example, for $g_T = 10^{-3}$, $T_C/V_C \approx 0.1$ and for smaller values of g_T, $T_C/V_C \approx 1/[\sqrt{2}\ln(1/g_T)]$. As we show below, the small value of T_C/V_C allows us to rigorously describe the anomalously large and strongly non-Gaussian temperature and current fluctuations in an overheated SET.

We find that the probability distributions of the temperature and current fluctuations are strongly non-Gaussian, and that the size of the transistor island has a large influence on these distributions. In particular, we find that for small islands with relatively large single-electron level spacing,

$$\delta_I \gtrsim \left(\frac{T_C}{V_C}\right)^5 V_C,$$

the mode, expectation value, and standard deviation of the temperature and current differ from each other by parametrically large factors [see Figs. 6 and 8, and Eqs. (32), (34), and (35)]. This we show by first casting the Keldysh action technique used in Refs. 3 and 8 into the form of a Fokker-Planck (FP) equation, allowing us to describe the full normalized probability distribution, instead of the non-normalized large deviation function $\ln P$. For some realistic values, $T_C/V_C \approx 0.1$ and $E_C \approx 1$ meV, $\delta_I \gtrsim 10^{-8}$ eV, corresponding to roughly 10^9 atoms.[13]

This paper is structured as follows: Sec. II introduces the theoretical methods utilized in this paper. In Sec. III we concentrate on the regime of small fluctuations and in Sec. IV on fluctuations in systems with relatively large single-electron level spacing. We conclude and discuss our findings in Sec. V. In most of the text, we use units such that $\hbar = k_B = e = 1$ but restore them in the important results.

II. APPROACH

We use the extended Keldysh action technique[14] as the basis of our theoretical calculations. Within this framework it is straightforward to calculate the full statistics of, for example, electric and energy current.[15,16] The effective action of the system is augmented with counting fields $\chi_{L(R)}(t)$ and $\xi_{L(R)}(t)$ for charge and energy transfer to the left (right) reservoir, respectively. The probability to obtain a certain realization of fluctuating energy currents is

$$P[\dot{H}_L(t)]P[\dot{H}_R(t)] = \int \mathcal{D}\xi_L(t)\mathcal{D}\xi_R(t)\exp\left\{-\int dt[i\xi_L\dot{H}_L + i\xi_R\dot{H}_R - \mathcal{S}(\xi_L,\xi_R,E)]\right\}, \quad (2)$$

where $\mathcal{S}$ is the effective action describing the transport processes.[17] Analogous expression gives the probabilities for charge currents with the substitution $\xi \mapsto \chi$. To ensure the conservation of energy, E, on the island, i.e., $\dot{H}_L(t)+\dot{H}_R(t) = \dot{E}(t)$, we use a delta functional,[18]

$$\delta[\dot{H}_L(t) + \dot{H}_R(t) - \dot{E}(t)] = \int \mathcal{D}\xi(t) \times \exp\left\{i\int dt\xi(\dot{H}_L + \dot{H}_R - \dot{E})\right\}, \quad (3)$$

to generate the conditional probability

$$P[\dot{H}_L(t),\dot{H}_R(t)] = \int \mathcal{D}\xi_L(t)\mathcal{D}\xi_R(t)\mathcal{D}\xi(t)\mathcal{D}E(t) \times \exp\left\{-\int dt[i(\xi_L-\xi)\dot{H}_L + i(\xi_R-\xi)\dot{H}_R + i\xi\dot{E} - \mathcal{S}(\xi_L,\xi_R,E)]\right\}. \quad (4)$$

In the case of the single-electron transistor adding a delta functional for the conservation of charge is not necessary since the effective action we use already conserves charge. Integrating over $\dot{H}_L$ and $\dot{H}_R$ yields the energy-conserving Keldysh partition function as

$$\mathcal{Z} = \int \mathcal{D}\xi(t)\mathcal{D}E(t)\exp\left\{-\int dt[\xi(t)\dot{E}(t) - \mathcal{S}(\xi,E)]\right\}, \quad (5)$$

where the imaginary unit has been absorbed into ξ.

To calculate the probability of the island having energy between $E^*\cdots E^*+\Delta E$ during time $\tau\cdots\tau+\Delta\tau$ the functional integration over $E(t)$ in Eq. (5) is performed over paths which satisfy this requirement. In general, the functional integration cannot be performed analytically. The variation of the exponent with respect to E and ξ yields the semiclassical equations of motion,

$$\dot{E} = \partial_\xi \mathcal{S}, \quad \dot{\xi} = -\partial_E \mathcal{S}, \quad (6)$$

which, when solved with the desired boundary conditions $E(\tau)=E^*$ and $\xi(-\infty)\to 0$, give the saddle-point trajectories of E and ξ. The probability is then obtained as the saddle-point value of the functional integral.[3] The saddle-point approximation is valid when the fluctuations around the semiclassical trajectories are small. In our case this translates to the condition $\delta_I \ll (T_C/V_C)^3 V_C$.

For another approach, we expand $\mathcal{S}$ to second order in ξ. Then the Gaussian integral over ξ in Eq. (5) can be performed and the remaining functional integral transformed into a Fokker-Planck equation for the time evolution of the probability distribution of E,[14]

$$\frac{\partial \mathcal{P}(E,t)}{\partial t} = \hat{H}\mathcal{P}(E,t) \equiv -\frac{\partial}{\partial E}\left[D_1(E)\mathcal{P}(E,t) - D_2(E)\frac{\partial \mathcal{P}(E,t)}{\partial E}\right]. \quad (7)$$

Here, the operator $\hat{H}$ is obtained from $\mathcal{S}(\xi,E)$ with the substitution $\xi \mapsto -\partial/\partial E$. Before applying this "quantization rule," the operator $\hat{H}$ must be normally ordered, i.e., ξ must be to the left of all E (see Appendix B). The Fokker-Planck equation also takes into account the Gaussian fluctuations around the saddle-point trajectory, and is therefore more accurate than the saddle-point approximation.

The stationary probability distribution satisfies $\hat{H}\mathcal{P}_{\rm st}(E)=0$ and has the form

$$\mathcal{P}_{\rm st}(E) = \text{const} \times \exp\left[\int dE \frac{D_1(E)}{D_2(E)}\right], \quad (8)$$

provided that $\int dE[D_1(E)/D_2(E)] \to -\infty$ when $E\to\pm\infty$. The normalization condition $\int dE \mathcal{P}_{\rm st}(E)=1$ fixes the value of the constant prefactor.

The time dependence of E can also be described in terms of a Langevin equation. The Fokker-Planck equation is equivalent to a Langevin equation[19]

$$\frac{\partial E(t)}{\partial t} = D_1(E(t)) + \frac{\partial D_2(E(t))}{\partial E(t)} + \eta(t), \quad (9)$$

where $\eta(t)$ is a random "force" with mean $\langle\eta(t)\rangle=0$ and variance $\langle\eta(t)\eta(t')\rangle = 2D_2(E(t))\delta(t-t')$.

To obtain a probability distribution for the effective electron temperature, we must apply a model which describes the relation between E and T. We first start by pointing out that the energy fluctuations show up as fluctuations of the electron distribution function $f(\epsilon)$, where ϵ is the energy of a given excitation. In a free-electron model, the total thermal energy on the island is related to this via

$$E(t) = \frac{1}{\delta_I}\int d\epsilon\epsilon[f(\epsilon,t) - f_0(\epsilon)], \quad (10)$$

where δ_I is the single-particle level spacing and $f_0(\epsilon)$ is the zero-temperature electron distribution function. In the quasiequilibrium limit, $f(\epsilon)$ tends to a Fermi distribution function, $f(\epsilon,t)=(\exp\{\epsilon/[T(t)]\}+1)^{-1}$, with a fluctuating electron temperature. In this case we can integrate Eq. (10) and obtain the usual expression for the free-electron heat capacity $C(T)$,

$$E = \frac{C(T)T}{2} = \frac{\pi^2 T^2}{6\delta_I}. \quad (11)$$

Note that in principle the distribution of the fluctuating electron temperature may depend on the charge state n. However, under Coulomb blockade the relaxation rate of the excited

charge states is larger than the temperature equilibration rate by a large factor, on the order of E_C/δ_I, and this effect can be disregarded in the following.

Single-electron transistor

For the overheated single-electron transistor the single-electron (SE) part of the action in the limit $T \ll V_b, V_C$ reads[8]

$$\mathcal{S}_{\rm se} = -g_T T e^{-W(T^{-1}+\xi)}\{[2 - e^{(W+V_b)\xi} - e^{W\xi}](\xi^{-1} - T)V_{\rm th}^{-1} + e^{W\xi}\}, \tag{12}$$

where $W=(V_{\rm th}-V_b)/2$. In this paper we concentrate on the cases where along the semiclassical trajectory, described by Eq. (6), $\xi \ll 1/V_C, 1/T_C$. In this case, a straightforward expansion to second order in ξ yields

$$\mathcal{S}_{\rm se} = -g_T T V_{\rm th}^{-1} e^{-W/T}\left\{\left[W^2 + 2TW + TV_b - \frac{V_b^2}{2}\right]\xi - \left[\frac{W^3}{3} + TW^2 - \frac{TV_b^2}{2} + \frac{V_b^3}{6}\right]\xi^2\right\}. \tag{13}$$

Near the critical bias voltage, $V_b \approx V_C = (\sqrt{2}-1)V_{\rm th}$, it is convenient to work with dimensionless variables

$$t = \sqrt{2}T/V_C, \quad v = (V_b - V_C)/V_C, \quad x = \xi V_C,$$

so that the action can be expanded to

$$\mathcal{S}_{\rm se} = -g_T t e^{-(1-v/\sqrt{2})/t}\left[(t-v)x - \frac{x^2}{6}\right]\frac{V_C}{2}, \tag{14}$$

valid when $t \ll 1$, $|v| \ll 1$, $|x| \ll 1$. From this form we can immediately see that the relevant $x \approx 6(t-v) \ll 1$, justifying our expansion above.

The cotunneling part of the action can be written as[8]

$$\mathcal{S}_{\rm cot} = \dot{H}_{\rm cot}\xi + \frac{1}{2}S_{\dot{H},\rm cot}\xi^2,$$

$$\approx \alpha\frac{g_T^2 V_C^2 \xi}{2} + \beta\frac{g_T^2 V_C^3 \xi^2}{12}, \tag{15}$$

where $\dot{H}_{\rm cot}$ and $S_{\dot{H},\rm cot}$ are the energy current and its fluctuations due to cotunneling, and α and β numerical factors on the order of 0.1. Here we disregard the weak voltage and temperature dependence of these terms. It turns out that the energy current noise due to cotunneling, i.e., the term proportional to β, can be neglected: the single-electron term is proportional to $g_T t \exp[-(1-v/\sqrt{2})/t]$, whereas the cotunneling term is proportional to g_T^2. Near the crossover the exponential term is on the order of $g_T t^{-3}$ [see Eq. (1)] and the single-electron contribution is therefore $g_T^2 t^{-2} \gg g_T^2$.

The total action, including single-electron and cotunneling contributions with the aforementioned approximations, is then

$$\mathcal{S} = -g_T t e^{-(1-v/\sqrt{2})/t}\left[(t-v)x - \frac{x^2}{6}\right]\frac{V_C}{2} + \alpha\frac{g_T^2 x V_C}{2}. \tag{16}$$

Using Eq. (11) we substitute

$$\xi \mapsto -\frac{\partial}{\partial E} = \frac{6\delta_I}{\pi^2 t V_C^2}\frac{\partial}{\partial t} \tag{17}$$

and obtain for the distribution of temperature,

$$\mathcal{P}(t,\tau) = \frac{\pi^2 V_C^2 t}{6\delta_I}\mathcal{P}\left(E = \frac{\pi^2 V_C^2 t^2}{12\delta_I}, \tau\right), \tag{18}$$

the Fokker-Planck equation

$$\frac{\pi^2}{3\delta_I g_T}\frac{\partial}{\partial\tau}\mathcal{P}(t,\tau) = -\frac{\partial}{\partial t}\left\{D_1(t)\mathcal{P}(t,\tau) - D_2(t)\frac{\partial}{\partial t}\mathcal{P}(t,\tau)\right\}, \tag{19}$$

where

$$D_1(t) = -(t-v)e^{-(1-v/\sqrt{2})/t} + \frac{\alpha g_T}{t} - \frac{\delta_I}{\pi^2 V_C}\frac{1-v/\sqrt{2}}{t^3}e^{-(1-v/\sqrt{2})/t},$$

$$D_2(t) = \frac{\delta_I}{\pi^2 V_C}\frac{1}{t}e^{-(1-v/\sqrt{2})/t},$$

which, using Eq. (8), yield the stationary distribution

$$\mathcal{P}_{\rm st}(t) \propto \exp\left\{-\frac{\pi^2 V_C}{\delta_I}\left[\frac{1}{3}t^3 - \frac{1}{2}t^2 v - \alpha g_T t \exp\left(\frac{1-v/\sqrt{2}}{t}\right) + \alpha g_T(1-v/\sqrt{2})\mathrm{Ei}\left(\frac{1-v/\sqrt{2}}{t}\right)\right] + \frac{1-v/\sqrt{2}}{t}\right\}. \tag{20}$$

Here, $\mathrm{Ei}(z)$ is the exponential integral function.[20]

To focus on the crossover regime, we introduce a new set of dimensionless parameters by changing the variables to

$$t = \sqrt{2}T_C/V_C + 2\theta(T_C/V_C)^2,$$

$$v = \sqrt{2}T_C/V_C + 2\nu(T_C/V_C)^2$$

with the temperature at the crossover, T_C, defined through Eq. (1), see also Appendix A. We also introduce a dimensionless level spacing,

$$\mathcal{D} = \frac{1}{4\sqrt{2}\pi^2}\frac{\delta_I}{eV_C}\left(\frac{eV_C}{k_B T_C}\right)^5,$$

characterizing the relative strength of the fluctuations, and a typical relaxation time for these fluctuations in the crossover interval,

$$\tau_r = \frac{2\sqrt{2}\pi^2}{3\alpha g_T^2}\frac{\hbar}{\delta_I}\left(\frac{k_B T_C}{eV_C}\right)^3.$$

For a typical $g_T = 10^{-3}$, $\tau_r \approx 1.7\times 10^{-10}/\delta_I$ eV s. When $\nu, \theta \ll V_C/T_C$, and $T_C/V_C \ll 1$, we can expand in the small parameter, T_C/V_C, and simplify the Fokker-Planck equation for the overheated SET considerably

$$\tau_r \frac{\partial}{\partial \tau}\mathcal{P}(\theta,\tau) = -\frac{\partial}{\partial \theta}\left\{[-(\theta-\nu)e^{\theta}+1-\mathcal{D}e^{\theta}]\mathcal{P}(\theta,\tau) - \mathcal{D}e^{\theta}\frac{\partial}{\partial \theta}\mathcal{P}(\theta,\tau)\right\}. \quad (21)$$

The stationary distribution becomes

$$\mathcal{P}_{\rm st}(\theta) \propto \exp\left\{-\frac{1}{\mathcal{D}}\left(\frac{1}{2}\theta^2 - \nu\theta + e^{-\theta}\right) - \theta\right\}, \quad (22)$$

which is the main result of this paper. The distribution consists of two parts: the first, proportional to $1/\mathcal{D}$, which is also present in the semiclassical (saddle-point) limit, and the second, independent of $\mathcal{D}$, describing the Gaussian fluctuations around the saddle-point trajectories, which become significant as δ_I (and therefore $\mathcal{D}$) grows. Low-temperature fluctuations are exponentially suppressed since the cooling due to singe-electron tunneling is also exponentially small. High-temperature fluctuations on the other hand do not have such suppressing mechanisms, hence they follow the usual Gaussian form.

III. REGIME OF SMALL FLUCTUATIONS

In the limit of a metallic island, $\mathcal{D} \ll 1$, the last terms in the exponents in Eqs. (20) and (22) can be disregarded. In this limit the typical fluctuations are Gaussian and small: the variance is proportional to δ_I/V_C. It should be stressed that our approach still goes beyond the Gaussian approximation—we can also look at *atypical*, large fluctuations, which are non-Gaussian in character.

On the single-electron side of the crossover but yet not far from the crossover point, $1 \gg v \gg T_C/V_C$, we can in addition neglect the exponential terms, proportional to g_T, in Eq. (20),

$$\mathcal{P}_{\rm st}(t) \propto \exp\left\{-\frac{\pi^2 V_C}{\delta_I}\left[\frac{1}{3}t^3 - \frac{1}{2}t^2 v\right]\right\}. \quad (23)$$

The distribution has a maximum at $t=v$. The cubic term favors low-temperature fluctuations, similar to that obtained in Ref. 3 for the noninteracting voltage biased island.[21] This is because far above the crossover SE processes serve as an efficient cooling mechanism. Similar to the results in Ref. 3, deviations from Gaussian statistics appear for temperature fluctuations on the order of average temperature but their probability is greatly enhanced since $\ln \mathcal{P} \simeq T^3/\delta_I V_C^2 \ll T/\delta_I$.

In the competition regime, $v<0$, we can approximate the exponential integral with its asymptotic form, $\mathrm{Ei}(z) \approx e^z(1/z+1/z^2)$, for large z. The resulting distribution is

$$\mathcal{P}_{\rm st}(t) \propto \exp\left\{-\frac{\pi^2 V_C}{\delta_I}\left[\frac{1}{3}t^3 - \frac{1}{2}t^2 v + \alpha g_T \frac{t^2}{1-v/\sqrt{2}}\exp\left(\frac{1-v/\sqrt{2}}{t}\right)\right]\right\}. \quad (24)$$

In contrast to the previous case, low-temperature fluctuations are exponentially suppressed for reasons explained at the end of Sec. II.

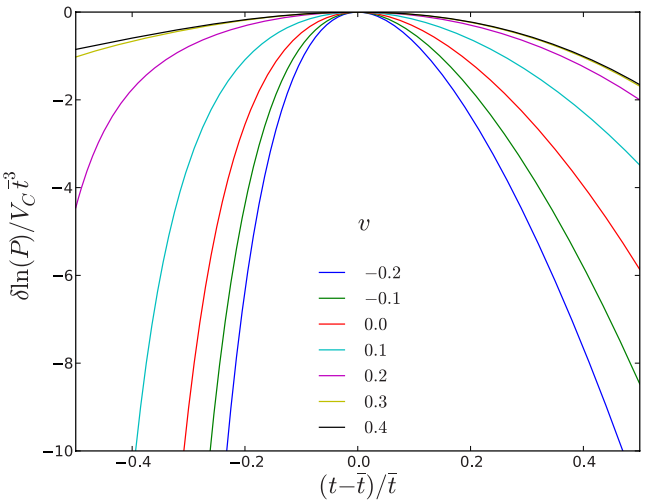

FIG. 3. (Color online) Logarithm of the temperature fluctuation probability for some values of bias voltage $v=(V_b-V_C)/V_C$. In this plot $\alpha=0.1$ and $g_T=10^{-3}$.

Around the crossover, $v \lesssim T_C/V_C$, we have from Eq. (22)

$$\mathcal{P}_{\rm st}(\theta) \propto \exp\left\{-\frac{1}{\mathcal{D}}\left(\frac{1}{2}\theta^2 - \nu\theta + e^{-\theta}\right)\right\}. \quad (25)$$

This distribution is "half-Gaussian," i.e., Gaussian for high-temperature fluctuations but exponentially suppressed for low-temperature fluctuations. In this case the non-Gaussian character appears already for deviations of temperature on the order of $\delta T \simeq (T_C/V_C)T_C$ and their probability is even more enhanced, $\ln \mathcal{P} \simeq (T_C/V_C)^4(T_C/\delta_I)$. The logarithm of these probability distributions is plotted in Figs. 3 and 4, which clearly show the half-Gaussian characteristics for voltages close to and below V_C.

IV. GIANT FLUCTUATIONS

In the previous section, we assumed that the fluctuations are small, $\mathcal{D} \ll 1$. If $\mathcal{D} \simeq 1$, the typical fluctuations become

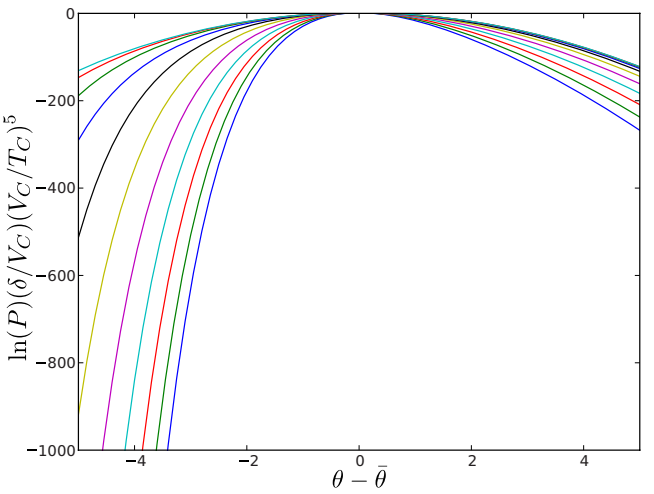

FIG. 4. (Color online) Logarithm of the temperature fluctuation probability for bias voltages $\nu=-5\cdots5$, bottom to top.

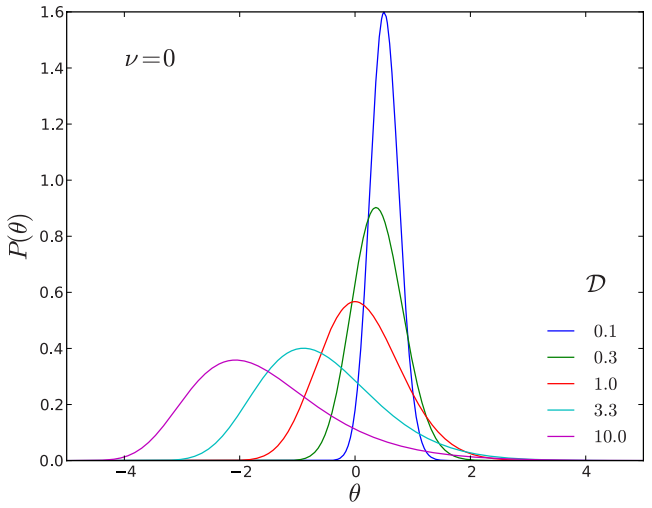

FIG. 5. (Color online) Probability distribution for the temperature of the island. Note how the maximum of the distribution is shifted and the non-Gaussian tail becomes more prominent as $\mathcal{D}$ is increased.

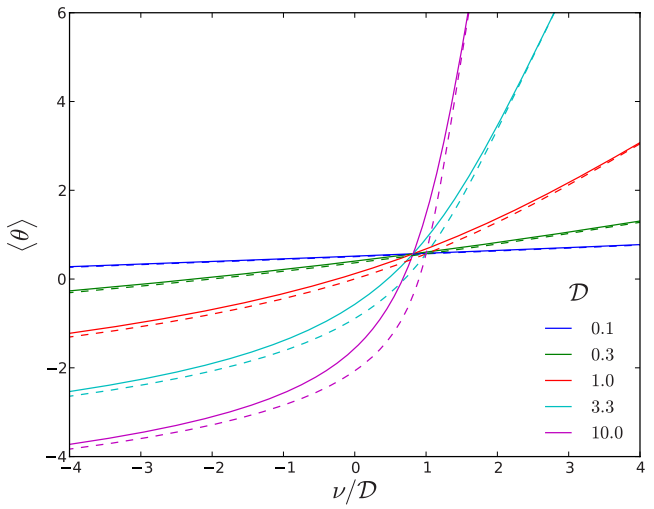

FIG. 6. (Color online) Expectation value of the temperature as a function of bias voltage. Dashed lines show the mode of θ, i.e., the maximum of the distribution. The two tend to each other for $|\nu| \gtrsim \mathcal{D}$.

non-Gaussian. For temperature, the fluctuations are still small, occurring at the scale of $\delta T \simeq T_C(T_C/V_C)$. Since current is anomalously sensitive to temperature, these fluctuations manifest in much stronger fluctuations of the current. Throughout this section we work with the scaled variables, ν and θ, and the results presented are valid for $\nu, \theta \ll V_C/T_C$.

A. Temperature fluctuations

For a finite but small level spacing δ_l, the distribution of temperature is given by Eq. (22) and plotted in Fig. 5. The maximum of the distribution is located at

$$\theta_{\max} = \begin{cases} \nu - \mathcal{D} & \nu > 0 \\ -\ln(-\nu + \mathcal{D}) & \nu < 0 \end{cases}, \quad |\nu| \gg \mathcal{D}. \tag{26}$$

The maximum is shifted to lower temperatures with increasing $\mathcal{D}$ and it differs from the expectation value of the temperature when $|\nu|/\mathcal{D}$ is small as shown in Fig. 6. The variance of θ is plotted in Fig. 7.

Analytical approximations for the expectation value and variance can be obtained in the limit $|\nu| \gg \mathcal{D}$, $\nu > 0$ and $\nu < 0$. In the first case the weight of the distribution is shifted to large θ, and we can neglect the term $e^{-\theta}$ in Eq. (22). In the latter case the weight is at small θ, and we can neglect the quadratic term in Eq. (22). We obtain

$$\langle\theta\rangle = \begin{cases} \nu - \mathcal{D}, & \nu > 0 \\ -\ln(-\nu), & \nu < 0 \end{cases}, \quad |\nu| \gg \mathcal{D}, \tag{27}$$

$$\mathrm{Var}(\theta) = \begin{cases} \mathcal{D}, & \nu > 0 \\ -\mathcal{D}/\nu, & \nu < 0 \end{cases}, \quad |\nu| \gg \mathcal{D}. \tag{28}$$

For large positive or negative ν, $\langle\theta\rangle \approx \theta_{\max}$. Variance is proportional to $\mathcal{D}$ as in the regime of small fluctuations.

B. Induced current fluctuations

In the overheated SET the electric current near the critical bias is given by[8]

$$I(\theta) = \frac{e^2}{\hbar} \frac{\alpha g_T^2 e^2 V_C^3}{4 k_B^2 T_C^2} e^{\theta}. \tag{29}$$

A fluctuating temperature leads directly to a fluctuating electric current. Since the current depends exponentially on the temperature, small fluctuations of temperature lead to large fluctuations of current. The Fokker-Planck equation for temperature is easily converted to a corresponding equation for the dimensionless current, $j \equiv e^{\theta}$, with the prescription

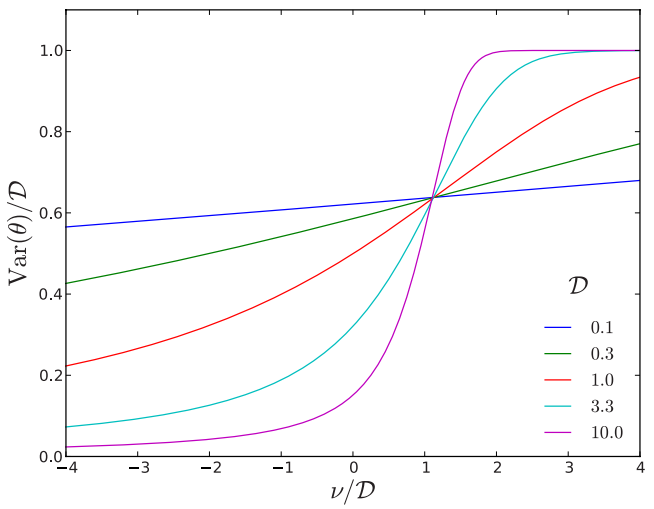

FIG. 7. (Color online) Variance of the temperature as a function of bias voltage. For $\nu/\mathcal{D} \gg 1$ the variance approaches $\mathcal{D}$.

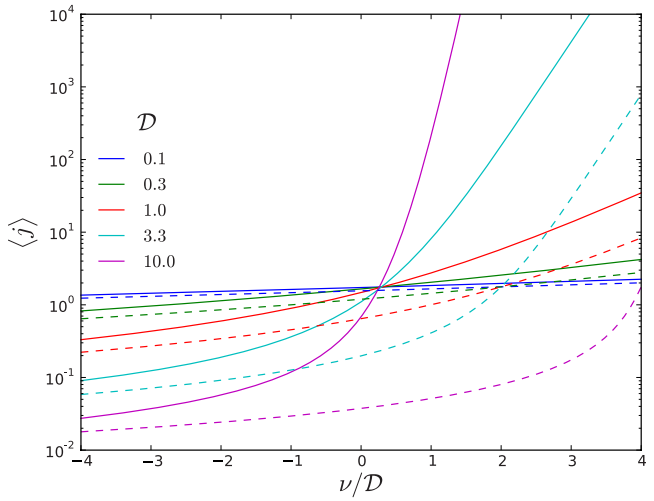

FIG. 8. (Color online) Expectation value of the current as a function of bias voltage. Dashed lines show the mode of j, i.e., the maximum of the distribution. The ratio of these tend to $\exp(3\mathcal{D}/2)$ for $\nu/\mathcal{D} \gg 1$, $\mathcal{D} \gg 1$.

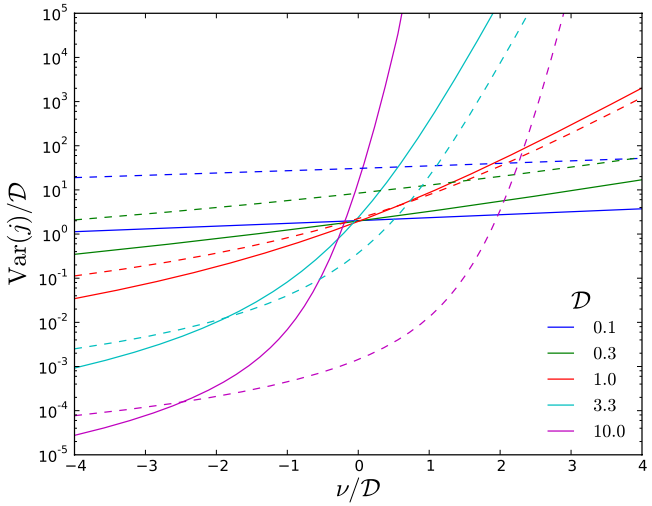

FIG. 9. (Color online) Variance of the current as a function of bias voltage. Dashed lines show the square of the average current, $\langle j \rangle^2/\mathcal{D}$, for comparison. The ratio of these tend to $\exp(\mathcal{D})$ for $\nu/\mathcal{D} \gg 1$, $\mathcal{D} \gg 1$.

$$\frac{\partial}{\partial \theta} = j\frac{\partial}{\partial j}, \quad \mathcal{P}(j,t) = \frac{1}{j}\mathcal{P}(\theta = \ln j, t),$$

resulting in

$$\tau_r \frac{\partial}{\partial t}\mathcal{P}(j,t) = -\frac{\partial}{\partial j}\left\{ j[j(\nu - \ln j - 2\mathcal{D}) + 1]\mathcal{P}(j,t) - \mathcal{D}j^3 \frac{\partial}{\partial j}\mathcal{P}(j,t) \right\}. \tag{30}$$

Note that this equation only includes the dominating contribution to the current fluctuations due to the temperature fluctuations, and the intrinsic fluctuations (thermal and shot noise) are disregarded. The stationary distribution of current is then

$$\mathcal{P}_{\mathrm{st}}(j) \propto \exp\left\{ -\frac{1}{\mathcal{D}}\left[\frac{1}{2}(\ln j)^2 - \nu \ln j + \frac{1}{j} \right] - 2 \ln j \right\}. \tag{31}$$

Similarly to the case of temperature, the maximum of the distribution is shifted to lower values of j when $\mathcal{D}$ is increased. The maximum of the distribution is located at

$$j_{\max} = \begin{cases} \exp(\nu - 2\mathcal{D}) & \nu > 0 \\ (-\nu + 2\mathcal{D})^{-1} & \nu < 0 \end{cases}, \quad |\nu| \gg 2\mathcal{D}. \tag{32}$$

Due to the log normal character of the current distribution, the expectation value and the most probable value of the current deviate as ν is increased. This is shown in Fig. 8. For a small $\mathcal{D}$, this deviation is proportional to $\mathcal{D}$. The variance of j is plotted as a function of the (reduced) bias voltage in Fig. 9. Note that $\mathrm{Var}(j)$ is the variance of the instantaneous electric current, which is different from the zero-frequency spectral noise power, more often encountered in the literature.

To comprehend the unusual properties of the distribution, let us first turn to the limit $\mathcal{D} \gg 1$, $|\nu|/\mathcal{D} \simeq 1$. In this case, the distribution reduces to a power-law function

$$\mathcal{P}_{\mathrm{st}}(j) \simeq \frac{1}{j^{2-\nu/\mathcal{D}}}. \tag{33}$$

This form cannot be valid at all values of the current: it is cut off at small currents, $j_s \simeq 1/2\mathcal{D}$, and crosses over to the log-normal distribution at large currents, $j_b \simeq \exp(\mathcal{D})$, $j_s \ll 1 \ll j_b$. Analytical approximations for the expectation value and variance can be obtained in the limit $|\nu| \gg \mathcal{D}$, $\nu > 0$ and $\nu < 0$. Similarly to the case of the temperature distribution, in the first case the weight of the distribution is shifted to large j, and we can neglect the term $1/j$ in Eq. (31). In the latter case we can neglect the $(\ln j)^2$ term. We obtain

$$\langle j \rangle = \begin{cases} \exp(\nu - \mathcal{D}/2) & \nu > 0 \\ -1/\nu & \nu < 0 \end{cases}, \quad |\nu| \gg \mathcal{D}, \tag{34}$$

$$\mathrm{Var}(j) = \begin{cases} \exp(2\nu - \mathcal{D})(\exp(\mathcal{D}) - 1) & \nu > 0 \\ \mathcal{D}[\nu^2(-\nu - \mathcal{D})]^{-1} & \nu < 0 \end{cases}, \quad |\nu| \gg \mathcal{D}. \tag{35}$$

We notice that for $\nu > 0$, $\langle j \rangle$ parametrically exceeds $j_{\max}$, $\langle j \rangle / j_{\max} = \exp(3\mathcal{D}/2)$, and $\sqrt{\mathrm{Var}(j)}$ parametrically exceeds $\langle j \rangle$, $\sqrt{\mathrm{Var}(j)}/\langle j \rangle = \exp(\mathcal{D}/2)$. This signals a highly unusual distribution not satisfying, for instance, the central limit theorem conditions.

We stress that the exponent of the power-law distribution can be readily tuned with bias voltage, $dV_b = 2V_C(T_C/V_C)d\nu$. We are not aware of any other physical systems where a quantity exhibits a power-law distribution with a tunable exponent.

Using Eq. (9) allows us to write down a Langevin equation for the time dependence of the current

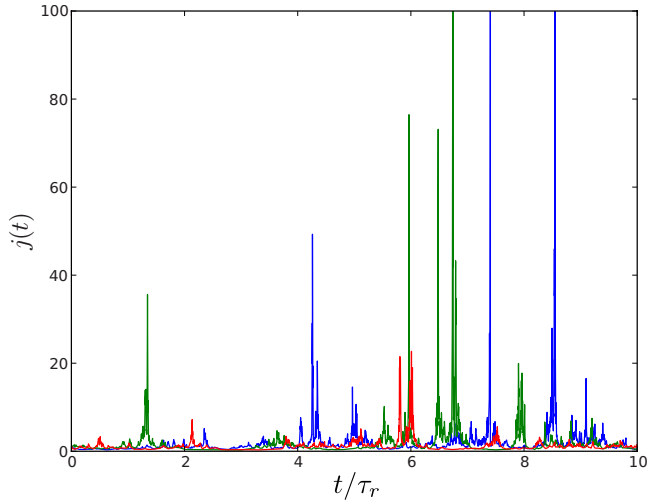

FIG. 10. (Color online) Three simulated realizations of current in an overheated SET at $\nu=0$ and $\mathcal{D}=3.3$. The average current (for $\mathcal{D}\to 0$) is $j\approx 1.7$.

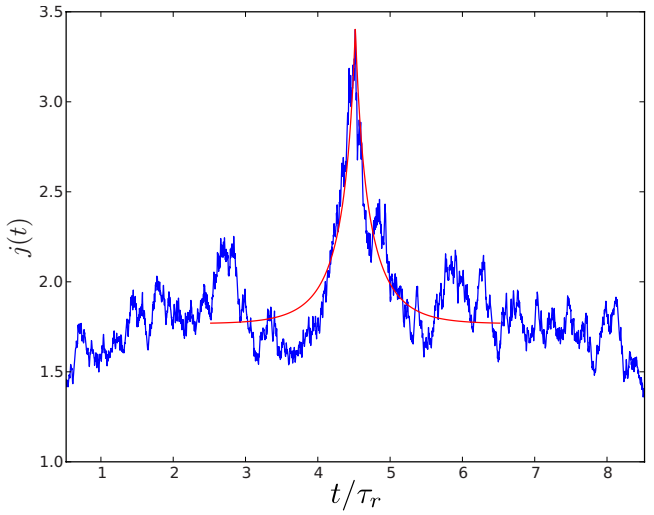

FIG. 11. (Color online) Comparison of a saddle-point fluctuation to the Langevin result at $\nu=0$ and $\mathcal{D}=0.02$.

$$\tau_r \frac{\partial}{\partial t} j(t) + \gamma(j(t))j(t) = \eta(t), \tag{36}$$

where $\gamma(j)=-j(\nu-\ln j+\mathcal{D})-1$ and $\eta(t)$ satisfies $\langle \eta(t)\rangle=0$ and $\langle \eta(t)\,\eta(t')\rangle=2\mathcal{D}j(t)^3\delta(t-t')$. The current exhibits huge peaks at a low frequency that are orders of magnitude larger than the average current, as can be seen from Fig. 10. The time scale for these fluctuations, τ_r, is on the order of milliseconds for $g_T=10^{-3}$ and $\delta_l=0.01$ K.

Another way to access the time dependence of the current fluctuations is to study the saddle-point equations for small fluctuations [Eq. (6)], i.e., vanishing δ_l,

$$\tau_r \dot{x} = -\frac{\partial}{\partial \theta} S,$$

$$\tau_r \dot{\theta} = \frac{\partial}{\partial x} S, \tag{37}$$

where the action,

$$S = -[e^{\theta}(\theta-\nu)-1]x + e^{\theta}\frac{x^2 V_C^2}{12T_C^2}, \tag{38}$$

has been written in terms of the dimensionless, scaled, variables, ν, θ, and x. As explained in Ref. 3, the saddle-point solutions follow the trajectory given by $S(\theta(t),x_S(\theta(t)))=0$ with $x_S(\theta)\neq 0$ until a "measurement" at time τ forces them to a trajectory with $x_S\equiv 0$. As shown in Fig. 11, the shape of a fluctuation obtained from the saddle-point equations agrees with the Langevin result. This is expected, since for small $\mathcal{D}$ the fluctuations around the saddle-point trajectory are negligibly small.

V. CONCLUSIONS

We have studied the temperature fluctuations and the associated current fluctuations in an overheated SET. Focusing on the crossover region between the competition of single electron and cotunneling, and pure sequential tunneling, we have found several interesting features, not seen in other commonly considered nanostructures. In overheated SETs with small islands the expectation value and the most probable value of the temperature differ from each other due to a tail in the probability distribution of temperature, extending to significantly high temperatures. Due to exponential temperature sensitivity of the electric current, huge peaks, occurring at frequencies on the order of a few kilohertz, should be visible in an instantaneous measurement of the electric current.

Experimentally, the challenge is to manufacture a transistor island where, despite high resistance tunnel contacts, the electron-phonon heat current is negligible compared to the heat current to the leads. In earlier experiments on the measurement of noise in SETs (see, for example, Ref. 22), this has not been the case. According to our earlier evaluation[8] a transistor island with a volume of $\mathcal{V}=10^{-4}$ μm^3 connected with $g_T=10^{-3}$ tunnel contacts should allow one to detect these large current surges experimentally.

ACKNOWLEDGMENTS

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APPENDIX A: DETAILS ON THE RESCALING

The total energy current in the SET can be found from the action, Eq. (16), via

$$\dot{H} = \left.\frac{\partial \mathcal{S}}{\partial x}\right|_{x=0} = -g_T t e^{-(1-v/\sqrt{2})/t}(t-v)\frac{V_C}{2} + \alpha\frac{g_T^2 V_C}{2}. \tag{A1}$$

The heat balance follows from the requirement for the total energy current to vanish, leading to

$$-te^{-(1-v/\sqrt{2})/t}(t-v)+\alpha g_T=0,$$

$$\Leftrightarrow v=-\alpha g_T t^{-1}e^{(1-v/\sqrt{2})/t}+t. \quad \text{(A2)}$$

Next we make a change of variables to $t=t_C+\theta t_C^2$ and $v=t_C+\nu t_C^2$, where t_C is for now an arbitrary number. If $\theta,\nu\ll 1/t_C$, we can expand in these small parameters to obtain

$$\nu=-\alpha g_T t_C^{-3}e^{1/t_C-1/\sqrt{2}-\theta}+\theta. \quad \text{(A3)}$$

Selecting

$$\alpha g_T t_C^{-3}\exp\left(\frac{1}{t_C}-\frac{1}{\sqrt{2}}\right)=1\Rightarrow t_C\approx\frac{1}{\ln(1/g_T)}\ll 1, \quad \text{(A4)}$$

simplifies the heat balance to

$$\nu=-e^{-\theta}+\theta \quad \text{(A5)}$$

and also implies $t_C\ll 1$, thus justifying the expansion above for a large range of ν and θ. We define the crossover point from the competition between single-electron and cotunneling to pure single-electron tunneling to be at $\theta=0$, i.e., $t=t_C=\sqrt{2}T_C/V_C$, defining the temperature at the crossover. This happens when $\nu=-1$, i.e., at a voltage of $v=t_C-t_C^2$. This rescaling simplifies also the Fokker-Planck equation, as seen in the main text, since it allows us to approximate

$$e^{-(1-v/\sqrt{2})/t}\approx\frac{\alpha g_T V_C^3}{2\sqrt{2}T_C^3}e^{\theta}, \quad \text{(A6)}$$

in many of the formulas.

APPENDIX B: RELATION BETWEEN THE FOKKER-PLANCK EQUATION AND THE FUNCTIONAL INTEGRAL

Let us write the FP equation in the form

$$\frac{\partial}{\partial t}f(q,t)=-\frac{\partial}{\partial q}[D_1(q,t)f(q,t)]+\frac{\partial^2}{\partial q^2}[D_2(q,t)f(q,t)],$$

$$\frac{\partial}{\partial t}f(q',t)=\int dqf(q,t)\left[-D_1(q,t)\frac{\partial}{\partial q}\delta(q'-q)+D_2(q,t)\frac{\partial^2}{\partial q^2}\delta(q'-q)\right] \quad \text{(B1)}$$

and integrate over time from t to $t+\Delta t$,

$$f(q',t+\Delta t)=\int dqf(q,t)\left[\delta(q'-q)-\Delta tD_1(q,t)\frac{\partial}{\partial q}\delta(q'-q)+\Delta tD_2(q,t)\frac{\partial^2}{\partial q^2}\delta(q'-q)+\mathcal{O}(\Delta t^2)\right]. \quad \text{(B2)}$$

Now we use the identity $\delta(q'-q)=\int dp/2\pi\exp[ip(q'-q)]$ to write

$$f(q',t+\Delta t)=\int dq\frac{dp}{2\pi}f(q,t)e^{ip(q'-q)}\times[1+ip\Delta tD_1(q,t)-p^2\Delta tD_2(q,t)+\mathcal{O}(\Delta t^2)]. \quad \text{(B3)}$$

Iterating N times so that $(t_N-t_0)/\Delta t=N$ gives

$$f(q_N,t_N)=\prod_{i=0}^{N-1}\int dq_i\frac{dp_i}{2\pi}f(q_0,t_0)\exp\left\{\sum_{i=0}^{N-1}\Delta t\left[ip_i\frac{q_{i+1}-q_i}{\Delta t}+ip_iD_1(q_i,t_i)-p_i^2D_2(q_i,t_i)\right]\right\}, \quad \text{(B4)}$$

which admits a functional integral representation

$$f(q,t)=\int\mathcal{D}q\mathcal{D}pf(q_0,t_0)\exp\left\{\int dt[ip\dot{q}+ipD_1(q,t)-p^2D_2(q,t)]\right\}. \quad \text{(B5)}$$

This form of the functional integral is not unique [unlike Eq. (B4)] but depends on the discretization procedure.[23] With the substitution $ip\mapsto-\xi$, $q\mapsto E$, we get a functional integral in the form of Eq. (5). This proves that the FP equation [Eq. (7)] corresponds to the partition function of Eq. (5).

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