The flow of two immiscible fluids between parallel plates

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Vloeistofmechanica
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THE FLOW OF TWO IMMISCIBLE FLUIDS

BETWEEN PARALLEL PLATES

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DELFt, 16 FEBRUARY 1974
I. INTRODUCTION

The flow of two immiscible fluids between parallel plates or thru a capillary tube has attracted the attention of a significant number of investigators. The main reason of this interest was the fact that the solution and understanding of these similar viscous flow problems will be of great help in explanation and solution of other problems in different fields of science. In chemistry, dynamic contact angle problem, capillary viscometers and the influence of meniscus between fluids on the concentration of the solid particles that are contained by the fluids, etc. In civil engineering the above problems are connected with helical-shaw analog, which is used in the investigations of the flows thru porous media.

In the following analysis a modified creeping flow model is used, where the interface between the fluids is assumed to have a finite radius of curvature but a negligible slope. Since the interface preserves its shape, the problem is reduced to a stationary problem with respect to a coordinate system attached to the interface. In solution of the biharmonic equation, which is obtained from the creeping flow model, an infinite series consisted of the product of exponential and trigonometric terms is used.

The modification is used in order to introduce some parameters representing the physical parameters of fluids and of the materials of the plates. This is achieved by replacing the no-slip condition by a condition for finite stresses and a prescribed slip velocity distribution.

The resulting interfaces are drawn with different Taylor numbers and different slip velocity distribution.
NOMENCLATURE

\[ A_{2n-1}, B_n, C_n = \text{coefficients for infinite series for fluid-1 region.} \]

\[ A'_{2n-1}, B'_n, C'_n = \text{coefficients for infinite series for fluid-2 region.} \]

\[ \alpha = \text{half the distance between the plates.} \]

\[ f_1(y), f_2(y) = \text{weighting functions for fluid-1 and fluid-2 respectively.} \]

\[ T = \text{surface tension} \]

\[ U_m = \text{the velocity of parallel plates.} \]

\[ v_1(y), v_2(y) = \text{slip velocity distributions for fluid-1 and fluid-2 regions respectively.} \]

\[ \lambda, \lambda' = \text{parameters which are the coefficients of the approximating functions for the weighting functions } f_1(y), f_2(y) \text{ respectively.} \]

\[ \mu_1, \mu_2 = \text{viscosity of fluid-1 and fluid-2.} \]

\[ \theta = \text{contact angle.} \]
I-4. STATEMENT OF THE PROBLEM

In this analysis the flow of two immiscible viscous fluids between two closely spaced horizontal plates is investigated. As a consequence of immiscibility the thickness of the interface between the fluids is neglected in comparison the macroscopic dimension of the flow region, and it is replaced by a line which is dividing the two different fluid continua. Further the shape of the interface is considered to be invariable. This hypothesis is based on the previous experimental evidences that after a transition period the interface preserves its shape provided that the conditions governing the flow are not varied. The possibility of a Rayleigh–Taylor instability is eliminated by the presence of the strong surface tension effect due to the close spacing of the plates, see Rose and Heins [17].

To reduce the problem to a steady flow the reference point is taken on the interface which yields to a model with a stationary interface and two parallel plates that are moving with a constant velocity which will be equal to the velocity of the interface if the plates were taken fixed as in the original problem.

The thin layer of displaced fluid which is left behind the interface on the plates is also disregarded due to its negligible thickness which is comparable with the interface thickness. Latter in the analysis an attempt is made to represent the effects of the thickness of the interface and the layer deposited on the plates by allowing a slip velocity distribution in the vicinity of the contact points. In order to simplify the geometry of the problem the displacements of the interface with respect to a plane configuration is disregarded. This assumption yields to a plane interface at the origin. In accordance with this assumption the slope of the interface is also neglected. But in order to take the surface tension effects into consideration the radius of curvature of the interface is assumed to be finite, (fig. 2), page 22.
FIG. 1: THE FLOW OF TWO IMMISCIBLE FLUIDS

$\tau_f =$ THICKNESS OF FLUID 2 LAYER LEFT ON THE PLATES

$\tau_i =$ THICKNESS OF THE INTERFACE
II. Previous Studies and Recent Developments

The first remarkable investigation of the flow of two immiscible fluids has been done by West G.D [1]. He studied the motion of a mercury index in an otherwise air-filled capillary tube. Three decades later, Yarnold [2] has made a similar study, both investigators observed a flow from center of the tube towards the walls of the tube near the advancing meniscus and a flow from the walls of tube towards the centerline at the back of index, etc. Receding meniscus see (Fig. 3). In his study Yarnold suggested the existence of non-Poiseuille flow in the vicinity of the ends of the index. This flow pattern has been called "the fountain effect" by Rose [3]. Later Schwartz et al. [4] have observed the flow pattern in a capillary tube by inserting a dye into the index, the effect of the surface tension together with the viscous forces had been investigated by Fairbrother & Stubbs [5] and the ratio of the product of index velocity with viscosity to the surface tension was suggested to be the controlling factor in capillary flows of two immiscible fluids.

One of the first theoretical investigations of the flow of two immiscible fluids has been done by Garabedian [6]. Saffman and Taylor [7] have given a theoretical analysis of an inviscid fluid finger moving in a narrow Hele-Shaw cell, which is filled with a viscous liquid. See (Fig. 4). In their investigation they neglected the effect of surface tension and their conformal mapping technique resulted in non-unique interface shape. The same authors have also given an experimental curve for the finger width versus finger velocity, starting from a geological problem Bhattacharya and Savic [8] gave a solution for the capillary flow of two fluids induced by a solid piston. They applied a Fourier transformation method for the solution of Stokes-Beltrami equation in case of a tube and biharmonic equation in case of parallel plate channel. The flow patterns they have obtained were in good agreement with the observed ones but their solution was too simple to accommodate all the related hydrodynamical conditions of the problem.
FIG. 3: THE FOUNTAIN EFFECT

FIG. 4: SINGLE FINGER IN HELE SHAW CELL
III. DERIVATION OF THE EQUATIONS

\[ y(2) \]

\[ \Rightarrow x(4) \]

**Fig. 1**

1. **BASIC EQUATIONS**

**EQUATION OF MOTION (NAVIER-STOKES EQUATION)**

\[ u_{i,t} + u_j u_{ij} = f_i - \frac{1}{\rho} p_i + \nu \Omega_{ij} \]

**CONTINUITY EQUATION FOR INCOMPRESSIBLE CASE**

\[ u_{i,i} = 0 \]

2. **CONSIDERATIONS OF THE SYSTEM**

THE FLOW IS CONSIDERED TO BE 2-DIMENSIONAL.

THE EFFECTS OF GRAVITATIONAL FORCES ARE NEGLECTED.

\[ f_i = 0 \] (SEE APPENDIX)

THE REFERENCE SYSTEM IS ATTACHED TO THE INTERFACE WHICH IS MOVING WITH A CONSTANT VELOCITY \( u_m \), ACCORDINGLY THE FLOW IS CONSIDERED TO BE STATIONARY,

\[ u_{i,e} = 0 \]

THE MOMENTUM EQUATION IS REDUCED TO

\[ u_j u_{ij} = -\frac{1}{\rho} p_i + \nu \Omega_{ij} \]

OR WITH THE COORDINATE SYSTEM OF FIG.1.

\[ u_i \frac{\partial u_i}{\partial x_i} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \]

\[ u_i \frac{\partial u_2}{\partial x_1} + u_j \frac{\partial u_2}{\partial x_2} = \frac{1}{\rho} \frac{\partial p}{\partial x_2} + \nu \nabla^2 u_2 \]
SUBSTITUTE THE FOLLOWING DIMENSIONLESS MAGNITUDES

\[ \hat{x} = \frac{x}{a} \]
\[ \hat{y} = \frac{y}{a} \]
\[ \hat{u} = \frac{u}{U_m} \]
\[ \hat{v} = \frac{v}{U_m} \]
\[ \hat{p} = \frac{p}{\mu U_m} \]

\[ \frac{U_m^2}{a} \left( \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} \right) = -\frac{\mu U_m \frac{\partial \hat{p}}{\partial \hat{x}}}{\rho a^2} + \frac{\nu U_m}{a^2} \nabla^2 \hat{u} \]

\[ \frac{U_m^2}{a} \left( \hat{v} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right) = -\frac{\mu U_m \frac{\partial \hat{p}}{\partial \hat{y}}}{\rho a^2} + \frac{\nu U_m}{a^2} \nabla^2 \hat{v} \]

REARRANGING

\[ \frac{U_m a}{\nu} \left( \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{x}} + \nabla^2 \hat{u} \]

\[ \frac{U_m a}{\nu} \left( \hat{v} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{y}} + \nabla^2 \hat{v} \]

SINCE \[ \frac{U_m a}{\nu} = \frac{Re}{2} \]

\[ \frac{Re}{2} \left[ \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right] = -\frac{\partial \hat{p}}{\partial \hat{x}} + \nabla^2 \hat{u} \quad (NON-DIMENSIONAL) \]

\[ \frac{Re}{2} \left[ \hat{v} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right] = -\frac{\partial \hat{p}}{\partial \hat{y}} + \nabla^2 \hat{v} \]

FOR \[ Re \ll 1 \]; CREEPING FLOW CONDITION
\[- \frac{\partial F}{\partial x} + \nabla^2 U = 0 \]
\[- \frac{\partial F}{\partial y} + \nabla^2 \tilde{U} = 0 \]

(Stokes Equations)

From the above equations, the pressure term is eliminated in the following manner,
\[ \frac{\partial}{\partial y} \left[ -\frac{\partial F}{\partial x} + \nabla^2 U \right] - \frac{\partial}{\partial x} \left[ -\frac{\partial F}{\partial y} + \nabla^2 \tilde{U} \right] = 0 \]

As a result, the equation of motion for creeping flow regime is reduced to
\[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 U}{\partial x^2 \partial y} - \frac{\partial^2 U}{\partial y^2} = 0 \]

in dimensional form.

Introduce a stream function \( \psi \) satisfying the equation of continuity,
\[ \frac{\partial \psi}{\partial x} = -U \]
\[ \frac{\partial \psi}{\partial y} = U \]

Substitution of above relationships in the momentum equation yields to the biharmonic equation,
\[ \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \]

or
\[ \nabla^4 \psi = 0 \]
3. **Boundary Conditions**

In the region far from the interface, where the disturbances due to interface are negligible, Poiseuille type flow prevails,

\[ u_1 = \frac{U_m}{2} \left( y - \frac{y^3}{3x} \right) \quad \text{for fluid 1} \]

\[ u_2 = \frac{U_m}{2} \left( y - \frac{y^3}{3x} \right) \quad \text{for fluid 2} \]

At the both infinities flow is parallel to x-axis,

\[ u_1 = \frac{\partial u_1}{\partial x} = 0 \quad \text{as} \quad x \to -\infty \]

\[ u_2 = -\frac{\partial u_2}{\partial x} = 0 \quad \text{as} \quad x \to +\infty \]

Normal component of the velocity on the plates is zero and the tangential component of the velocity is equal to the velocity of the plates.

\[ -\frac{\partial u_1}{\partial y} = u_1 = 0 \quad \text{at} \quad y = \pm a \]

\[ -\frac{\partial u_2}{\partial y} = u_2 = 0 \quad \text{at} \quad y = \pm a \]

\[ \frac{\partial u_1}{\partial y} = -U_m \quad \text{at} \quad y = \pm a \]

\[ \frac{\partial u_2}{\partial y} = -U_m \quad \text{at} \quad y = \pm a \]
4. **Kinetic Conditions on the Interface**

The interface is assumed to be nearly straight with negligible slope but finite radius of curvature. With respect to the reference system, the velocity of the interface in \( x \)-direction is zero,

\[
\frac{\partial u_1}{\partial y} = u_1 = 0 \quad \text{at } x = 0
\]

\[
\frac{\partial u_2}{\partial y} = u_2 = 0 \quad \text{at } x = 0
\]

\[
\phi = \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \quad \text{at } x = 0
\]

Since there can be no slip at the interface between the fluids, the velocities in \( y \)-direction are equal in magnitude and direction,

\[
\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} \quad \text{at } x = 0
\]

7. **Stress Conditions on Interface** (Landau L.D., Lifshitz)

**Stress Tensor**

\[
\sigma_{ik} = -p \delta_{ik} + \Pi_k
\]

Where, \( \Pi_k \) is the viscous stress tensor;

\[
\Pi_k = \mu \left( \frac{\partial u_i}{\partial y} + \frac{\partial u_k}{\partial x} \right)
\]

\( \Delta \) and \( P \) is hydrostatic pressure. Stress on the boundaries are expressed in the following tensorial form,

\[
-\sigma_{ik} n_k = p n_i - \Pi_k n_k
\]

On a interface between two immiscible liquids, the stresses exerted by fluids must be equal and opposite in the absence of surface tension:

\[
\sigma_{ik}^{(1)} n_k - \sigma_{ik}^{(2)} n_k = 0
\]
TAKING INTO ACCOUNT THE SURFACE TENSION $T$ BETWEEN THE FLUIDS,

$$\sigma_{ik}^{(n)} \theta_{jk}^{(e)} = T \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \eta_i$$

AFTER NECESSARY SUBSTITUTIONS ABOVE EQUATION BECOMES,

$$[P^{(n)} - P^{(2)}] \eta_i = (P_{ik}^{(n)} - P_{ik}^{(2)}) \eta_k - T \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \eta_k$$

OR

$$[P^{(n)} - P^{(2)}] \eta_i = \left[ \mu_1 \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) - \mu_2 \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \right] \eta_k - T \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \eta_i$$

$i = 1, 2$

$k = 1, 2$

FOR $i = 1$, $k = 1, 2$

$$(P^{(n)} - P^{(2)}) \eta_1 = \left[ \mu_1 \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) - \mu_2 \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \right] \eta_1 + \left[ \mu_1 \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) - \mu_2 \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \right] \eta_2 - T \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \eta_1$$

FOR $i = 2$, $k = 1, 2$

$$(P^{(n)} - P^{(2)}) \eta_2 = \left[ \mu_1 \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) - \mu_2 \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \right] \eta_1 + \left[ \mu_1 \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) - \mu_2 \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \right] \eta_2 - T \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \eta_2$$

FOR THE SAKE OF SIMPLICITY THE FOLLOWING SUBSTITUTIONS ARE MADE;

$$P^{(n)} = P_1$$

$$P^{(2)} = P_2$$

$$x_1 = x$$

$$x_2 = y$$

$$U_1^{(n)} = U_1$$

$$U_1^{(2)} = U_2$$

$$U_1 = U_1$$

$$U_2 = U_2$$
For \( n_1 = 1 \), \( n_2 = 0 \)

The above equations reduce to,

\[
P_1 - P_2 = 2\mu_1 \left( \frac{\partial u_1}{\partial x} \right) - 2\mu_2 \left( \frac{\partial u_2}{\partial x} \right) - T \left( \frac{1}{\kappa} \right)
\]

\[
\mu_1 \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) - \mu_2 \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) = 0
\]

And since

\[
u_1 = \frac{\partial u_1}{\partial y} \quad \nu_1 = -\frac{\partial u_1}{\partial x}
\]

\[
u_2 = \frac{\partial u_2}{\partial y} \quad \nu_2 = -\frac{\partial u_2}{\partial x}
\]

The following equations are obtained,

\[
P_1 - P_2 = 2\mu_1 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial y^2} - 2\mu_2 \frac{\partial u_2}{\partial x} \frac{\partial^2 u_2}{\partial y^2} - T \left( \frac{1}{\kappa} \right)
\]

\[
\mu_1 \left( -\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) - \mu_2 \left( -\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) = 0
\]
II. SOLUTION OF THE PROBLEM

EQUATIONS

\[ \nabla^4 \psi_1 = 0 \quad \text{(1)} \]
\[ \nabla^4 \psi_2 = 0 \quad \text{(1') } \]

\[ \frac{\partial p_1}{\partial x} = \mu_1 \nabla^2 \psi_1 \quad \text{(2.a)} \]
\[ \frac{\partial p_1}{\partial y} = \mu_1 \nabla^2 \psi_1 \quad \text{(2.b)} \]

\[ \frac{\partial p_2}{\partial x} = \mu_2 \nabla^2 \psi_2 \quad \text{(2.a')} \]
\[ \frac{\partial p_2}{\partial y} = \mu_2 \nabla^2 \psi_2 \quad \text{(2.b')} \]

BOUNDARY CONDITIONS AT INFINITIES

\[ \psi_1 = \frac{U_0}{2} \left[ y - \frac{y^3}{2 \bar{a}^2} \right] \quad \text{as } x \to -\infty \quad \text{(3)} \]
\[ \psi_2 = \frac{U_0}{2} \left[ y - \frac{y^3}{2 \bar{a}^2} \right] \quad \text{as } x \to +\infty \quad \text{(3')} \]

\[ \frac{\partial \psi_1}{\partial x} = 0 \quad \text{as } x \to -\infty \quad \text{(4)} \]

\[ \frac{\partial \psi_2}{\partial x} = 0 \quad \text{as } x \to +\infty \quad \text{(4')} \]

BOUNDARY CONDITION ON THE AXIS OF SYMMETRY

- \[ \frac{\partial \psi_1}{\partial y} = 0 \quad \text{at } y = 0 \quad \text{(5)} \]

- \[ \frac{\partial \psi_2}{\partial y} = 0 \quad \text{at } y = 0 \quad \text{(5')} \]
1. **Boundary Conditions on the Plates.**

\[ \frac{\partial w_1}{\partial y} = -U_m \quad y = \pm a \quad (6) \]

\[ \frac{\partial w_2}{\partial y} = -U_m \quad y = \pm a \quad (6') \]

\[ -\frac{\partial w_1}{\partial x} = 0 \quad y = \pm a \quad (7) \]

\[ -\frac{\partial w_2}{\partial x} = 0 \quad y = \pm a \quad (7') \]

**Boundary Conditions on the Interface**

\[ \frac{\partial w_1}{\partial y} = 0 \quad \text{or} \quad w_1 = 0 \quad \text{at} \quad x = 0 \quad (8) \]

\[ \frac{\partial w_2}{\partial y} = 0 \quad \text{or} \quad w_2 = 0 \quad \text{at} \quad x = 0 \quad (8') \]

\[ \frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial x} \quad \text{at} \quad x = 0 \quad (9) \]

\[ P_1 - P_2 = 2\mu_1 \left( \frac{\partial^2 w_1}{\partial x \partial y} \right) - 2\mu_2 \left( \frac{\partial^2 w_2}{\partial x \partial y} \right) - T \frac{\partial^2 X(y)}{\partial y^2} \quad \text{at} \quad x = 0 \quad (10) \]

\[ \mu_1 \left( -\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_2}{\partial y^2} \right) - \mu_2 \left( -\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} \right) = 0 \quad \text{at} \quad x = 0 \quad (11) \]
The creeping flow of a viscous fluid and the bending of semi-infinite plates are given by the same partial differential equation, e.g. biharmonic equation.

\[ \nabla^4 W = F \]

In case of semi-infinite plates, \( F \) is \( \frac{q}{D} \) where \( q \) is the intensity of load and \( D \) is the flexural rigidity of the plate, but in case of flow of immiscible fluids between parallel plates, \( F \) is equal to zero, and \( W \) is the stream function which corresponds to deflection in the problem of semi-infinite plates.

Following the A. Nádai's solution let

\[ W_i = X_m \sin \frac{m\pi y}{a} \quad \text{or} \quad X_m \cos \frac{m\pi y}{a} \]

The substitution of above expressions of \( W \) into biharmonic equation yields

\[ \frac{d^4 X_m}{dx^4} - 2\left(\frac{m^2 \pi^2}{a^2}\right) \frac{d^2 X_m}{dx^2} + \frac{m^4 \pi^4}{a^4} X_m = 0 \]

The general solution of this differential equation gives,

\[ X_m = C_1 e^{\frac{m\pi x}{a}} + C_2 e^{\frac{m\pi x}{a}} + C_3 e^{-\frac{m\pi x}{a}} + C_4 x e^{-\frac{m\pi x}{a}} \]

\[ W_i = \left[ e^{\frac{m\pi x}{a}} (C_1 + C_2 x) + e^{-\frac{m\pi x}{a}} (C_3 + C_4 x) \right] \left[ \sin \frac{m\pi y}{a} \text{ or } \cos \frac{m\pi y}{a} \right] \]
IF,
\[ W_2 = e^{\frac{\alpha}{\beta} Y_n} \text{ or } e^{-\frac{\alpha}{\beta} Y_n} \]

AFTER SUBSTITUTION BIHARMONIC EQUATION BECOMES,
\[ \frac{\partial^4 Y_n}{\partial y^4} + 2 \left( \frac{\partial^2 Y_n}{\partial x^2} \right) \frac{\partial^2 Y_n}{\partial y^2} + \frac{\partial^4 Y_n}{\partial x^4} Y_n = 0 \]

AND
\[ Y_n = k_1 \sin \frac{\pi y}{a} + k_2 y \sin \frac{\pi y}{a} + k_3 \cos \frac{\pi y}{a} + k_4 y \cos \frac{\pi y}{a} \]

ACCORDINGLY
\[ W_2 = \left[ \sin \frac{\pi y}{a} (k_1 + k_2 y) + \cos \frac{\pi y}{a} (k_2 - k_4 y) \right] \left[ e^{\frac{\alpha}{\beta} Y_n} - e^{-\frac{\alpha}{\beta} Y_n} \right] \]

SINCE BOTH W_1 AND W_2 ARE SOLUTIONS OF BIHARMONIC EQUATION THEIR SUM MUST ALSO BE A SOLUTION
\[ W_1 + W_2 = (D_1 + D_2 Y) e^{\frac{\alpha}{\beta} Y_n} \sin \frac{\pi y}{a} + (D_3 + D_4 Y) e^{\frac{\alpha}{\beta} Y_n} \cos \frac{\pi y}{a} \]
\[ + D_5 x e^{\frac{\alpha}{\beta} Y_n} \sin \frac{\pi y}{a} + D_6 x e^{\frac{\alpha}{\beta} Y_n} \cos \frac{\pi y}{a} \]
\[ + (D'_1 + D'_2 Y) e^{-\frac{\alpha}{\beta} Y_n} \sin \frac{\pi y}{a} + (D'_3 + D'_4 Y) e^{-\frac{\alpha}{\beta} Y_n} \cos \frac{\pi y}{a} \]
\[ + D'_5 x e^{-\frac{\alpha}{\beta} Y_n} \sin \frac{\pi y}{a} + D'_6 x e^{-\frac{\alpha}{\beta} Y_n} \cos \frac{\pi y}{a} \]

THE STREAM FUNCTION IS TAKEN AS,
\[ (Y'_1 = \frac{U_m}{a} (y - \frac{y^2}{a^2}) + \sum \left( D_1 + D_2 y + D_3 x \right) e^{\frac{\alpha}{\beta} Y_n} + \sum \left( D_3 + D_4 y + D_6 x \right) e^{-\frac{\alpha}{\beta} Y_n} \]
FOR THE REGION \(-\infty < x < 0 , \ -a \leq y \leq a\)

AND
\[ (Y'_2 = \frac{U_m}{a} (y - \frac{y^2}{a^2}) + \sum \left( D'_1 + D'_2 y + D'_3 x \right) e^{-\frac{\alpha}{\beta} Y_n} + \sum \left( D'_3 + D'_4 y + D'_6 x \right) e^{\frac{\alpha}{\beta} Y_n} \]
FOR THE REGION \(0 < x < \infty , \ -a \leq y \leq a\)

THE ABOVE SOLUTIONS ARE EXPECTED TO SATISFY THE FOLLOWING BOUNDARY CONDITIONS;
\[ U(x,0) = -\frac{\partial \psi}{\partial x} = 0 \quad \text{AND} \quad \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = 0 \quad \text{AT} \ y = 0 \ (i) \]

WHICH IMPLIES THAT STREAM FUNCTION MUST BE ODD FUNCTION OF Y IN ORDER TO HAVE SYMMETRY IN VELOCITY FIELD AROUND X-AXIS.
NORMAL COMPONENT OF VELOCITY MUST BE ZERO ON THE PLATES,

\[ u(x, \pm 2) = -\frac{\partial u}{\partial x} = 0 \quad \text{at } y = \pm 2 \quad \text{(i)} \]

AT THE BOTH INFINITIES THE FLOW IS POISEUILLE TYPE,

\[ u(\pm \infty, y) = -\frac{\partial u}{\partial x} = 0 \quad \text{at } x = \pm \infty \quad \text{(iii)} \]

\[ u(\pm \infty, y) = \frac{U_m}{2} \left( y - \frac{y^3}{3a^2} \right) \quad \text{at } x = \pm \infty \quad \text{(iv)} \]

FROM (i)

\[ D_2 = D_3 = D_6 = 0 \]

AND

\[ D_2' = D_3' = D_6' = 0 \]

FROM (iii)

FOR \( \cos m \frac{y}{a} \) TO BE ZERO, \( m = \frac{2n-1}{2} \) FOR TERMS WITH \( \cos \frac{2n-1}{2} \)

CONDITIONS (iii) AND (iv) ARE SATISFIED WITH THE PRESENCE OF EXPONENTIAL TERMS.

AS A RESULT THE FOLLOWING EXPRESSIONS ARE OBTAINED,

\[ u_1 = \frac{U_m}{2} \left( y - \frac{y^3}{3a^2} \right) + \sum_{n=1}^{\infty} 2y A_n e^{-l_{2n-1} \frac{y}{2a}} \cos (2n-1) \frac{\pi y}{2a} + \sum_{n=1}^{\infty} \beta_n e^{-l_{2n-1} \frac{y}{2a}} \sin \eta \frac{y}{a} \]

\[ + \sum_{n=1}^{\infty} C_n e^{-l_{2n-1} \frac{y}{2a}} \sin \eta \frac{y}{a} \quad \text{FOR } -\infty \leq x \leq 0 \quad \text{(v)} \]

\[ u_2 = \frac{U_m}{2} \left( y - \frac{y^3}{3a^2} \right) + \sum_{n=1}^{\infty} 2y A_n e^{-l_{2n-1} \frac{y}{2a}} \cos (2n-1) \frac{\pi y}{2a} + \sum_{n=1}^{\infty} \beta_n e^{-l_{2n-1} \frac{y}{2a}} \sin \eta \frac{y}{a} \]

\[ + \sum_{n=1}^{\infty} C_n e^{-l_{2n-1} \frac{y}{2a}} \sin \eta \frac{y}{a} \quad \text{FOR } 0 \leq x \leq +\infty \quad \text{(vi)} \]
3. THE EQUATIONS AND BOUNDARY CONDITIONS

THE EQUATIONS FOR THE STREAM FUNCTIONS ARE

\[
\psi_1 = y \sum_{n=1}^{\infty} A_{2n-1} e^{-(2n-1) \pi x / a} \cos((2n-1) \pi y / a) + \frac{1}{2} \sum_{n=1}^{\infty} B_{2n} e^{-(2n) \pi x / a} \sin((2n) \pi y / a) + C_{0} e^{0} \sin(y / a) + \frac{1}{2} \sum_{n=1}^{\infty} C_{2n} e^{-(2n) \pi x / a} \sin((2n) \pi y / a) \]

\[
\psi_2 = y \sum_{n=1}^{\infty} A_{2n-1} e^{-(2n-1) \pi x / a} \cos((2n-1) \pi y / a) + \frac{1}{2} \sum_{n=1}^{\infty} B_{2n} e^{-(2n) \pi x / a} \sin((2n) \pi y / a) + C_{0} e^{0} \sin(y / a) + \frac{1}{2} \sum_{n=1}^{\infty} C_{2n} e^{-(2n) \pi x / a} \sin((2n) \pi y / a) \]

THESE EQUATIONS SATISFY THE FOLLOWING BOUNDARY CONDITIONS:

\[x \to -\infty \rightarrow \psi_1 \to \frac{U_m}{2} \left[ y - \frac{y^3}{a^2} \right] \]

\[x \to +\infty \rightarrow \psi_2 \to \frac{U_m}{2} \left[ y - \frac{y^3}{a^2} \right] \]

\[y = 0 \Rightarrow \frac{\partial \psi_1}{\partial x} = 0 \]

\[y = 0 \Rightarrow \frac{\partial \psi_2}{\partial x} = 0 \]

\[y = \pm a \Rightarrow \frac{\partial \psi_1}{\partial x} = 0 \]

\[y = \pm a \Rightarrow \frac{\partial \psi_2}{\partial x} = 0 \]

TO DETERMINE THE COEFFICIENTS OF THE SERIES EXPRESSIONS, THE FOLLOWING BOUNDARY CONDITIONS ARE GOING TO BE USED:

\[y = \pm a \Rightarrow \frac{\partial \psi_1}{\partial y} = -U_m \quad (6) \]

\[y = \pm a \Rightarrow \frac{\partial \psi_2}{\partial y} = -U_m \quad (6') \]

\[y = \pm a \Rightarrow \left\{ \begin{array}{c} \psi_1 = 0 \quad \text{(from} \frac{\partial \psi_1}{\partial y} = 0, \frac{\partial \psi_2}{\partial y} = 0) \end{array} \right\} \quad (8) \]

\[y = \pm a \Rightarrow \left\{ \begin{array}{c} \frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x} \end{array} \right\} \quad (8') \]

\[y = \pm a \Rightarrow \left\{ \begin{array}{c} \mu_1 \left( \frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} \right) = \mu_0 \left( \frac{\partial^2 \psi_2}{\partial x^2} + \frac{\partial^2 \psi_2}{\partial y^2} \right) \end{array} \right\} \quad (10) \]
4. Determination of the Coefficients $A_{2n-1}B_n, C_n$ and $A_{2n-1}, B_n, C_n$.

The coefficients are to be determined from the six boundary conditions; (6), (6) representing the no-slip condition on the plates of both fluid particles; (8), (8) conditions implying stationary interface at $x=0$; (9) and (11) continuity of tangential velocity and tangential stresses across the interface respectively. But the conditions of no-slip on the moving plates and the stationary interface are not in a good agreement with each other, and they imply infinite velocity gradients at the contact points, e.g. at $y=\pm a$, $x=0$. Besides this contradiction the pressure equations obtained from Stokes's equation for both of the fluids yield to divergent series at the contact points. To overcome these difficulties the boundary conditions are modified in the following manner.

The condition of no-slip on the plates is released for both of the fluids. Instead of this condition a condition for finite pressures is used at contact points and a slip velocity distribution is allowed on the plates.

\[
y = \pm a \quad \frac{\partial \psi_1}{\partial y} = v_1(x) \quad (6.1)
\]

\[
y = \pm a \quad \frac{\partial \psi_2}{\partial y} = v_2(x) \quad (6.2)
\]

Where $v_1(x)$ and $v_2(x)$ are slip velocity distributions.

The condition of stationary interface is maintained:

\[
x = 0 \quad \psi_1 = 0 \quad (8)
\]

\[
x = 0 \quad \psi_2 = 0 \quad (8')
\]

Since $v_1(x)$ and $v_2(x)$ are not known, two of the coefficients for each fluid are going to be determined from (8) and (8') with the help of two weighting functions.
The conditions for the continuity of the tangential velocity and the stresses are also maintained.

\[ \frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial x} \quad \text{AT } x=0 \] (3)

\[ \mu_1 \left(- \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right) = \mu_2 \left(- \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\right) \quad \text{AT } x=0 \] (4)

The condition of finite pressures is introduced as a new boundary condition for the pressure equation in both fluids. This condition shall be used in determination of weighting functions.

From these boundary conditions the coefficients are determined in the following way;

- The coefficients \( A_{2n-1}, C_n, A_{2n-1}', \) and \( C_n' \) are determined from the conditions (3) and (4) with the help of two weighting functions \( f_1(y), f_2(y) \) which are used in (3) and (4) respectively.

From (3)

\[ y^2 A_{2n-1} \cos(2n-1) \frac{\pi y}{2a} + \sum_1^\infty C_n \sin \frac{n \pi y}{a} = \frac{Um}{2} \left( \frac{y^3}{a^2} - y \right) \] (12)

For calculation of \( A_{2n-1} \) and \( C_n \) it is assumed that a certain part of the R.H.S function is expanded in the cosine series and the other part is expanded in sine series, such that the fraction of the R.H.S function is given by the weight function \( f_1(y) \).

As a result the following relations are obtained

\[ f_1(y) \frac{Um}{2} \left[ \frac{y^2}{a^2} - 1 \right] = \sum_1^\infty A_{2n-1} \cos(2n-1) \frac{\pi y}{2a} \] (13)

AND

\[ [1 - f_1(y)] \frac{Um}{2} \left[ \frac{y^3}{a^2} - y \right] = \sum_1^\infty C_n \sin \frac{n \pi y}{a} \] (14)
With the same method from (8') similar relations are obtained for fluid-2:

\[ f_2(y) \frac{y^2}{2} \left( \frac{y^3}{a^2} - 1 \right) = \sum A_{2n+1} \cos((2n-1)\beta y) \frac{y^m}{a^2} \tag{13'} \]

\[ \left[ 1 - f_2(y) \right] \frac{y^2}{2} \left( \frac{y^3}{a^2} - y \right) = \sum C_n \sin n \beta y \frac{y^m}{a^2} \tag{14} \]

The unknown weighting functions are approximated by linear combinations of suitably chosen functions \( \phi_0, \phi_1, \ldots, \phi_m \) and \( \phi', \phi'_1, \ldots, \phi'_m \) of the form

\[ f_i(y) = \sum_{k=0}^{m} \lambda_k \phi_k \]

\[ f_i'(y) = \sum_{k=0}^{m} \lambda'_k \phi'_k \]

For the sake of simplicity the functions are chosen as powers of \( \frac{y}{a} \),

\[ f_1(y) = \lambda_0 + \lambda_1 \frac{y}{a} + \lambda_2 \frac{y^2}{a^2} + \ldots + \lambda_m \frac{y^m}{a^m} \]

\[ f_1'(y) = \lambda'_0 + \lambda'_1 \frac{y}{a} + \lambda'_2 \frac{y^2}{a^2} + \ldots + \lambda'_m \frac{y^m}{a^m} \]

The coefficients \( A_{2n+1}, C_n, A'_{2n+1}, C'_n \) are determined from (13'), (14), (13'), (14') with the help of the above approximations of weighting functions. The conditions (9) and (10) are used in determination of \( B_m \) and \( B'_m \). The resulting coefficients contain the constants of the approximations of the weighting functions, as well as the physical parameters \( \eta, \eta_1, \eta_2, a \). By means of the determined coefficients from Stokes' equation the pressure equations and from (6.2), (6.2) sup velocity distribution are obtained, which contain the \( \lambda \)-constants.

For determination of \( \lambda \)-values the condition of finite pressures gives the first relation of the \( m+1 \) relations needed for each set of constants \( \lambda_k, \lambda'_k \). The other \( n \) relations can be obtained by prescribing the magnitude of velocity distribution at \( n \) points in each of the fluid regions.

Since the shape of interface and the sup velocity distribution are determined by \( \lambda \)-values for every given physical situation they will be called \( \lambda \)-parameters.
Since even functions can be expanded in cosine series and odd functions can be expanded in sine series, the terms of power series with odd powers can be left out because their product with main function is odd function in relations (13), (13') and in case of relations (14), (14') they produce even functions which are not affecting the calculation of the coefficients.

7. The calculation of the $A_{2n-1}, B_n, C_n, A'_{2n-1}, B'_n$ and $C'_n$

In the following calculations first, two terms of the approximation are considered yielding two $\lambda$-parameters for each fluid, and the results are compared with the solution of single $\lambda$-value solution before a sophisticated matrix analysis is attempted.

In accordance with above approximation the equations (13), (13'), (14) and (14') are written as

\[
(\lambda_0 + \lambda_2 \frac{y^2}{a^2}) \frac{U_m}{2} \left[ \frac{y^2}{a^2} - 1 \right] = \sum_{i=1}^{\infty} A_{2n-1} \cos(2n-1) \frac{y_i}{2a} \\
(1 - \lambda_0 - \lambda_2 \frac{y^2}{a^2}) \frac{U_m}{2} \left[ \frac{y^2}{a^2} - 1 \right] = \sum_{i=1}^{\infty} C_n \sin \pi \frac{y_i}{a} \\
(\lambda'_0 + \lambda'_2 \frac{y^2}{a^2}) \frac{U_m}{2} \left[ \frac{y^2}{a^2} - 1 \right] = \sum_{i=1}^{\infty} A'_{2n-1} \cos(2n-1) \frac{y_i}{2a} \\
(1 - \lambda'_0 - \lambda'_2 \frac{y^2}{a^2}) \frac{U_m}{2} \left[ \frac{y^2}{a^2} - 1 \right] = \sum_{i=1}^{\infty} C'_n \sin \pi \frac{y_i}{a}
\]
BY APPLYING THE METHOD FOR THE DETERMINATION OF FOURIER COEFFICIENTS TO EQUATIONS (13.1), (13.2), (14.1) AND (14.1) THE FOLLOWING RESULTS ARE OBTAINED.

\[
A_{2n-1} = \frac{16 U_m (-1)^n}{\pi^2} \left[ \frac{4B \lambda_2}{\pi^2} - (7\lambda_2 + \lambda_0) \right] 
\]

\[
C_n = \frac{U_m a (-1)^n}{\pi^2} \left[ \frac{6 - 6\lambda_0 - 14\lambda_2 + \frac{120\lambda_2}{\pi^2}}{\pi^2} \right] 
\]

\[
A'_{2n-1} = \frac{16 U_m (-1)^n}{\pi^2} \left[ \frac{4B \lambda_2'}{\pi^2} - (7\lambda_2 + \lambda_0) \right] 
\]

\[
C_n' = \frac{U_m a (-1)^n}{\pi^2} \left[ \frac{6 - 6\lambda_0 - 14\lambda_2 + \frac{120\lambda_2}{\pi^2}}{\pi^2} \right] 
\]

THE OTHER COEFFICIENTS \( B_n \) & \( B_n' \) ARE DETERMINED FROM THE EQUATIONS (9) AND (11).

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \alpha \quad x=0 \quad (9)
\]

\[
M_1 \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - M_2 \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right) = 0
\]

AFTER SUBSTITION OF THE EXPRESSIONS FOR \( \psi \) & \( \psi' \) AND REARRANGING FOLLOWING RELATIONS ARE OBTAINED;

FROM EQUATION (9)

\[
\sum_{i=1}^{\infty} \left( B_i' - B_i \right) \sin \pi \frac{y}{a} = y \sum_{i=1}^{\infty} \left( A_{2i-1} + A'_{2i-1} \right) (2n-1) \frac{\pi}{2a} \cos \pi \frac{y}{a} 
\]

\[
+ \sum_{i=1}^{\infty} \left( C_i + C_i' \right) \frac{\pi}{a} \sin \pi \frac{y}{a} \quad (17)
\]

BY THE METHOD FOR DETERMINATION OF FOURIER COEFFICIENTS THE EXPRESSION FOR \( B_n - B_n' \) BECOMES

\[
B_k - B_k' = \frac{16 \left( A_{2n-1} + A'_{2n-1} \right) (2n-1)^2 k}{(4k-1)^2 \pi} + \left( C_k + C_k' \right) \frac{\pi k}{a} \quad (18)
\]
CONTACT ANGLE $\theta$

INTERFACE DISPLACEMENT $s$

INTERFACE

CONTACT POINT

FIG. 2: GEOMETRY
\[
\sum_{n=1}^{\infty} \left( \frac{\mu_2 A_{2n-1} - \mu_1 A_{2n-1}}{2n-1} \cdot \frac{\pi}{2a} \right) \sin \left( \frac{(2n-1)\pi y}{2a} \right) + \sum_{n=1}^{\infty} \left( \frac{\mu_2 A_{2n} - \mu_1 A_{2n}}{2n} \cdot \frac{\pi}{2a} \right) \sin \left( \frac{2n\pi y}{2a} \right) = \sum_{n=1}^{\infty} \left( \frac{\mu_1 B_n + \mu_2 B_n'}{n^2} \right) \frac{\pi}{2a} \sin \left( \frac{n\pi y}{2a} \right)
\]

AND THE METHOD OF FOURIER COEFFICIENTS YIELDS,

\[
M_2 B'_k + M_1 B_k = -\frac{3(M_2 - M_1) U_0 (1)}{k^2 \pi^2} + \frac{(M_2 C'_k - M_1 C_k) \pi}{k \pi} + \frac{(M_2 A_{2k-1} - M_1 A_{2k-1}) \pi}{(4k-1) \pi} \left( \frac{2k-1}{(4k-1)^2} \right) \left[ \frac{(3k-1)^2 - 4k + 1}{(4k-1)^2} \right]
\]

BY SOLVING (18) AND (19) SIMULTANEOUSLY THE EXPRESSIONS FOR \( B_n \) AND \( B'_n \) ARE OBTAINED AS,

\[
B_n = -\frac{3(M_2 - M_1) U_0 (1)}{(M_2 + M_1) \pi^2} + \frac{16 A_{2n-1} \pi}{(4n-1)^2 \pi^2} + \frac{4(M_2 A_{2n-1} - M_1 A_{2n-1}) \pi}{(M_2 + M_1) \pi} \left( \frac{C_1 \pi}{a} \right)
\]

\[
B'_n = \frac{3(M_2 - M_1) U_0 (1)}{(M_2 + M_1) \pi^2} - \frac{16 A_{2n} \pi}{(4n)^2 \pi^2} + \frac{4(M_2 A_{2n} - M_1 A_{2n}) \pi}{(M_2 + M_1) \pi} \left( \frac{C_1 \pi}{a} \right)
\]

OR

\[
B_n = -\frac{3(M_2 - M_1) U_0 (1)}{(M_2 + M_1) \pi^2} + \frac{64 U_0 (1)}{(4n-1)^2 \pi^2} \left[ \frac{4 \lambda_2 x}{(4n-1)^2 \pi^2} - \left( \frac{5 \lambda_2 \lambda_0}{(4n-1)^2 \pi^2} \right) \right] \left( \frac{(6-6\lambda_0 - 4\lambda_2 + \frac{120 \lambda_2}{n^2 \pi^2})}{\pi^2} \right)
\]

\[
B'_n = \frac{3(M_2 - M_1) U_0 (1)}{(M_2 + M_1) \pi^2} + \frac{64 U_0 (1)}{(4n)^2 \pi^2} \left[ \frac{4 \lambda_2 x}{(4n)^2 \pi^2} - \left( \frac{5 \lambda_2 \lambda_0}{(4n)^2 \pi^2} \right) \right] \left( \frac{(6-6\lambda_0 - 4\lambda_2 + \frac{120 \lambda_2}{n^2 \pi^2})}{\pi^2} \right)
\]
$\text{V. PRESSURE EQUATION}$

For fluid 1 the equations 2.3 and 2.6 are considered.

$$\frac{\partial P}{\partial x} = \mu_1 \nabla^2 u_1 \quad (2.3)$$

$$\frac{\partial P}{\partial y} = \mu_1 \nabla^2 u_1 \quad (2.6)$$

Substitute \( u_1 = \frac{\partial u_1}{\partial y}, \quad u_1 = -\frac{\partial u_1}{\partial x} \) in 2.3 and 2.6

$$\frac{\partial P}{\partial x} = \mu_1 \nabla^2 \left( \frac{\partial u_1}{\partial y} \right) \quad 22.3$$

$$\frac{\partial P}{\partial y} = \mu_1 \nabla^2 \left( -\frac{\partial u_1}{\partial x} \right) \quad 22.6$$

After substitution of the expression for \( u_1 \) and carrying out the necessary differentiations and integrations the following results are obtained:

$$P = -2\mu_1 \sum_{1}^{(2n-1)} \frac{\pi x}{2a} \Delta_{2n-1} \cos(2n-1) \frac{\pi x}{2a} + 2\mu_1 \sum_{1}^{\infty} \frac{\pi x}{2a} B_n e^{-\cos \frac{\pi x}{2a}}$$

$$- \frac{3\mu_1 M_{yX}}{a^2} + P_0 \quad 23.3$$

$$P = -2\mu_1 \sum_{1}^{(2n-1)} \frac{\pi x}{2a} \Delta_{2n-1} \cos(2n-1) \frac{\pi x}{2a} + 2\mu_1 \sum_{1}^{\infty} \frac{\pi x}{2a} B_n e^{-\cos \frac{\pi x}{2a}}$$

$$- \frac{3\mu_1 M_{yX}}{a^2} + P_0 \quad 23.6$$

Since 23.3 must be equal to 23.6

$$P_0 = P_0 = P_0$$

This implies that \( P_0 \) is not a function of \( x \) and \( y \).
\[ P_i = -2M \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{2a} \frac{\alpha_n^2}{2\pi a} \cos((2n-1)\pi x/a) + 2M \sum_{n=1}^{\infty} \frac{2n \pi}{a} B_n e^{\cos(n\pi a/2)}\]
\[ - \frac{3/M U_m X}{a^2} + P_0 \]  \hspace{1cm} (24)

And with the same procedure from (2.5') and (2.6') the expression for \( P_2 \) is obtained as,
\[ P_2 = 2M_2 \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{2a} \frac{\alpha_n^2}{2\pi a} \cos((2n-1)\pi x/a) + 2M_2 \sum_{n=1}^{\infty} \frac{2n \pi}{a} B_n e^{\cos(n\pi a/2)}\]
\[ - \frac{3/M_2 U_m X}{a^2} + P_0 \]  \hspace{1cm} (25)

These pressure equations will be used in obtaining the first two relations for determining \( \lambda's \) and in calculation of the interface from equation (10).
VI. RELATION OF THE PARAMETERS

In some of the previous treatments of this problem infinite pressures were obtained in the vicinity of the interface and the plates, in order to avoid this the following procedure is utilized.

The convergence of the series expressions of pressure field are investigated term by term.

\[ \Delta T \quad y = \lambda z \quad \text{AND} \quad x = 0 \]

\[ P = 2 \mu \sum_{n=1}^{\infty} \left( \frac{n \pi}{a} \right)^2 B_n (-1)^n + P_0 \]

\[ P_2 = 2 \mu \sum_{n=1}^{\infty} \left( \frac{n \pi}{a} \right)^2 B_n' (-1)^n + P_0' \]

Since \( B_n \) and \( B_n' \) are of order \( \frac{1}{n^2} \) when they are multiplied by \( \frac{n \pi}{a} \) they become divergent harmonic series \( \left( \frac{1}{n} \right) \).

\[ (-1)^n \frac{n \pi}{a} B_n = - \frac{3 (M_2 - M_1) U_m}{(M_2 + M_1) \pi a} + \frac{G U_m \pi}{8 (4n-1)^2 (2n-1)^2} \left[ \frac{4 \beta \lambda_z}{\pi^2 (2n-1)^2} - \frac{1}{(2n-1)^2} \right] \left( -\frac{\beta \lambda_z}{\pi^2 (2n-1)^2} \right) + \frac{U_m}{\pi a} \left[ 6 - 6 \lambda_0 - 14 \lambda_z + \frac{12 \lambda_z}{n^2} \right] \tag{26} \]

\[ (-1)^n \frac{n \pi}{a} B_n' = - \frac{3 (M_2 - M_1) U_m}{(M_2 + M_1) \pi a} + \frac{G U_m \pi}{8 (4n-1)^2 (2n-1)^2} \left[ \frac{4 \beta \lambda_z}{\pi^2 (2n-1)^2} - \frac{1}{(2n-1)^2} \right] \left( -\frac{\beta \lambda_z}{\pi^2 (2n-1)^2} \right) + \frac{U_m}{\pi a} \left[ 6 - 6 \lambda_0 - 14 \lambda_z + \frac{12 \lambda_z}{n^2} \right] \tag{26'} \]

To have finite pressures in both fluids the limit of above terms as \( n \to \infty \) are equated to zero.

\[ \lim_{n \to \infty} (-1)^n \frac{n \pi}{a} B_n = 0 \tag{27} \]

\[ \lim_{n \to \infty} (-1)^n \frac{n \pi}{a} B_n' = 0 \tag{27'} \]

These two conditions are used to obtain a relation between \( \lambda \)-parameters, in the single \( \lambda \)-parameter solution, the values of \( \lambda \) and \( \lambda' \) are determined from these conditions.
FROM (27)
\[
0 = -\frac{3(M_2 - M_1)}{M_2 + M_1} - \frac{8(g\lambda_2 + \lambda_0)}{\pi^2} - [6 - 6\lambda_0 - 14\lambda_2] \tag{28}
\]
AND FROM (27')
\[
0 = -\frac{3(M_2 - M_1)}{M_2 + M_1} + \frac{8(g\lambda_2' + \lambda_0')}{\pi^2} + [6 - 6\lambda_0' - 14\lambda_2'] \tag{28'}
\]
OR
\[
\lambda_0 = 1.17627 + 0.77814 \frac{(M_2 - M_1)}{M_2 + M_1} - 1.91663 \lambda_2 \tag{28.1}
\]
\[
\lambda_0' = 1.17627 - 0.77814 \frac{(M_2 - M_1)}{M_2 + M_1} - 1.91663 \lambda_2' \tag{28.1'}
\]
(28.1) AND (28.1') ARE THE FIRST RELATIONS NEEDED FOR \( \lambda \)-VALUES. THE OTHER TWO RELATIONS ARE GOING TO BE OBTAINED FROM THE SLIP VELOCITY DISTRIBUTIONS ON THE PLATES.

**Slip Velocity Distributions on the Plates**

A certain slip velocity distribution has been allowed on the plates in order to overcome the infinite velocity gradients at the contact points, and by means of the \( \lambda \)-parameters. The slip region can be confined in to a small region near the contact points but the extent of this region is difficult to decide. In previous experiments such a macroscopic slip region has not been observed. But on the other hand in this creeping flow model the effect of the thin layer of fluid that may be left behind the interface and the influence of the long range intermolecular forces are not considered and the slip velocity distribution is acting also as a compensation for these factors. The thickness of the interface is also a significant factor at contact points, which must be taken into consideration when the possibility of an slip region is questioned.
1. THE SLIP VELOCITY DISTRIBUTION ON PLATES

THE TANGENTIAL VELOCITY ON THE PLATES IS GIVEN BY:

FOR FLUID-1
\[ \frac{\partial w}{\partial x} = V_1(x) \text{ AT } y = \pm 2 \text{ AND } -\infty < x < 0 \]

\[ V_1(x) = -U_m + 2 \sum_{n=1}^{\infty} A_{2n-1} \sin \frac{\pi x}{2a} - n = 0 \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{2a} \]

FOR FLUID-2
\[ \frac{\partial \psi}{\partial x} = V_2(x) \text{ AT } y = \pm 3 \text{ AND } +\infty > x > 0 \]

\[ V_2(x) = -U_m + 2 \sum_{n=1}^{\infty} A_{2n-1} \sin \frac{\pi x}{2a} + n = 0 \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{2a} \]

SINCE \( V_1(x) \) AND \( V_2(x) \) ARE FUNCTIONS OF \( x \), THE VIScosITIES OF THE FLUIDS AND THE \( \lambda \)-VALUES, BY PRESCRIBING CERTAIN MAGNITUDES TO SLIP VELOCITY DISTRIBUTIONS AT CERTAIN \( x \)-COORDINATES, RELATIONS BETWEEN \( \lambda \)-VALUES ARE OBTAINED.

FOR THE TWO PARAMETER CASE (TWO-\( \lambda \) CASE) THE MAGNITUDE OF SLIP IS EQUATED TO ZERO AT DIFFERENT DISTANCES FROM THE INTERFACE.
VII SHAPE OF THE INTERFACE

\[ P_1 - P_2 = 2\mu \left( \frac{\partial^2 \psi}{\partial x \partial y} \right) - 2\mu_2 \left( \frac{\partial^2 \psi_2}{\partial x \partial y} \right) - T \frac{d^2 X(y)}{dy^2} \] \quad \Delta T \quad x = 0

The substitution of the expressions for \( P_1, P_2, \psi, \) and \( \psi_2 \) from equations (24), (25), (v), (vi) respectively, yields:

\[ T \frac{d^2 X(y)}{dy^2} = 4\mu \sum_{n=1}^{\infty} \left( \frac{\pi}{2a} \right) A_{2n-1} \cos \left( \pi \left( \frac{y}{2a} \right) \right) - 2\mu_2 \sum_{n=1}^{\infty} \left( \frac{\pi}{2a} \right) A_{2n-1} \sin \left( \pi \left( \frac{y}{2a} \right) \right) + \frac{2\mu}{a^2} \sum_{n=1}^{\infty} A_{2n-1} \sin \left( \frac{\pi y}{2a} \right) + \frac{4\mu_2}{a^2} \sum_{n=1}^{\infty} C_n \sin \left( \frac{\pi y}{a} \right) + P_1 - P_2 \] \quad (29)

After integrating (29) with respect to \( y \),

\[ T \frac{dX(y)}{dy} = TX(y) = (P_1 - P_2)y + 2\mu \sum_{n=1}^{\infty} \frac{A_{2n-1}}{\pi} \sin \left( \frac{\pi y}{2a} \right) + y \sum_{n=1}^{\infty} \frac{A_{2n-1}'}{\pi} \cos \left( \frac{\pi y}{2a} \right) + \frac{2\mu}{a^2} \sum_{n=1}^{\infty} \frac{C_n}{\pi} \sin \left( \frac{\pi y}{a} \right) + P_1 - P_2 \] \quad (30)

Since at \( y = 0 \) \( X(0) = 0 \), then \( K_1 = 0 \)

By integrating above expression,

\[ X(y) = (P_1 - P_2)y \left[ \frac{y^2}{2} \right] + 2\mu \sum_{n=1}^{\infty} \frac{A_{2n-1}}{\pi} \sin \left( \frac{\pi y}{2a} \right) - \sum_{n=1}^{\infty} \frac{C_n}{\pi} \cos \left( \frac{\pi y}{a} \right) \] \quad (31)

Due to the fact that the reference system is attached to the interface, \( X(0) = 0 \) at \( y = 0 \)

Therefore,

\[ K_2 = 2\mu \sum_{n=1}^{\infty} \frac{C_n}{\pi} + 2\mu_2 \sum_{n=1}^{\infty} \frac{C_n'}{\pi} \] \quad (32)
FROM (30)
\[ \Delta T \frac{\partial y}{\partial y} = \cos \theta \]

THE SUBSTITUTION OF ABOVE VALUES IN (30) YIELDS

\[ (P_1 - P_0) a = T \cos \theta + 2 \mu_1 \sum_{i=1}^{\infty} \Delta_{2n-1} (-1)^{n-1} + 2 \mu_2 \sum_{i=1}^{\infty} \Delta'_{2n-1} (-1)^{n-1} \quad (33) \]

AFTER SUBSTITUTION OF (32) AND (33) IN (31), THE EQUATION OF INTERFACE BECOMES

\[ X(y) = \frac{y^2}{2a} \left[ \cos \theta + 2 \frac{\mu_1}{T} \sum_{i=1}^{\infty} \Delta_{2n-1} (-1)^{n-1} \right] + 2 \mu_1 \left[ Y \frac{a}{2a} \right] \left[ \sum_{i=1}^{\infty} \frac{\Delta_{2n-1} \sin (2n-1) \pi y}{2a} - \sum_{i=1}^{\infty} \frac{C_n \cos n \pi y}{3} \right] \]

\[ + 2 \mu_2 \left[ Y \frac{a}{2a} \right] \left[ \sum_{i=1}^{\infty} \frac{\Delta'_{2n-1} \sin (2n-1) \pi y}{2a} - \sum_{i=1}^{\infty} \frac{C_n \cos n \pi y}{3} \right] + 2 \mu_1 \sum_{i=1}^{\infty} \frac{C_n}{T} + 2 \mu_2 \sum_{i=1}^{\infty} \frac{C_n}{T} \quad (34) \]

OR WHEN THE EXPRESSIONS FOR THE COEFFICIENTS ARE REPLACED AND THE EQUATION IS REARRANGED,

\[ X(y) = \frac{y^2}{2a} \cos \theta + \frac{172a}{\pi^2} \left[ M_{\mu} \mu_{\lambda} + \frac{H_2 \mu_{\lambda}}{T} \right] y \frac{2}{1} \sum_{i=1}^{\infty} \frac{(\mu (2n+1) \cos (2n+1) \pi y)}{2a} - \frac{23}{\pi^3} \left[ \frac{H_2 \mu_{\lambda} \mu_{\lambda}}{T} \right] + \frac{23}{\pi^3} \left[ \frac{H_2 \mu_{\lambda} \mu_{\lambda}}{T} \right] y \frac{2}{1} \sum_{i=1}^{\infty} \frac{(\mu (2n+1) \cos (2n+1) \pi y)}{2a} \]

\[ + \frac{2}{\pi^3} \left[ \frac{H_2 \mu_{\lambda} \mu_{\lambda}}{T} \right] y \frac{2}{1} \sum_{i=1}^{\infty} \frac{(\mu (2n+1) \cos (2n+1) \pi y)}{2a} - \frac{24}{\pi^3} \left[ \frac{H_2 \mu_{\lambda} \mu_{\lambda}}{T} \right] + \frac{24}{\pi^3} \left[ \frac{H_2 \mu_{\lambda} \mu_{\lambda}}{T} \right] y \frac{2}{1} \sum_{i=1}^{\infty} \frac{(\mu (2n+1) \cos (2n+1) \pi y)}{2a} \]

THE SHAPES OF INTERFACES FOR DIFFERENT \( \frac{H_2 \mu_{\lambda}}{T} \) RATIO ARE GIVEN IN FIGS. 7, 6 AND 7.
IX. COMPARISONS

1. COMPARISON OF SINGLE AND DOUBLE PARAMETER SOLUTION

The creeping flow problem which is studied here has been solved for both one and two parameter cases and the results have been analyzed in order to get information on the effect of using more parameter on the results.

In the single parameter solution the value of \( \lambda \) parameter is determined from the finite stress condition in each of the fluids at the three-phase contact point. This solution yields a certain slip velocity distribution on the plates and an interface shape which is a linear function of Taylor number. Due to the fact that this slip velocity distribution does not lose its significance even in the regions far from the three-phase contact point, the single parameter solution is considered to be not accurate enough to describe the flow phenomenon. Besides this unrealistic slip region the shape of the interface shows a reversed curvature resulting in a concave meniscus instead of a convex one which is expected in the absence of the effect of contact angle. The slip velocity distribution and the interface shapes are shown in Fig. 7.

The introduction of a second parameter improves the results a great deal, as the result of the two parameter solution different slip velocity distributions and more realistic shapes of interface are obtained which are corresponding to smaller slip regions if the region of slip is reduced to a magnitude smaller than half the space between the plates, significant reversed slip distributions are obtained out side of the positive slip region which are not desirable [Fig. 8]. Therefore by using two parameters the slip region in each fluid can be reduced to a magnitude which is approximately equal to the spacing between the plates [Fig. 7]. Since the magnitude of the slip region is still large the use of more parameters are needed for further reduction of the size of the slip region. This can be done by considering some higher order terms of the approximation.
2. COMPARISONS WITH THE RESULTS OF EXPERIMENTAL AND THEORETICAL STUDIES

ROSE AND HEINS [17]

ROSE AND HEINS HAVE DONE EXPERIMENTS IN CAPILLARY TUBES, AND IN THEIR PAPER PHOTOGRAPHS OF THE INTERFACES FOR THE CASE NUJOL DISPLACING AIR HAVE BEEN GIVEN. A COMPARISON OF THE CURVATURES CALCULATED BY DOUBLE PARAMETER METHOD WITH THE CURVATURES OBTAINED BY ROSE AND HEINS AT THE CENTERLINE OF THE TUBE SHOWS THAT THE ALLOWABLE SUP REGION MUST BE TAKEN MUCH SMALLER THAN THE RADIUS OF THE TUBE. SINCE THE RADIUS OF CURVATURES AND CORRESPONDING INTERFACIAL VELOCITIES ARE CALCULATED FROM THE PHOTOGRAPHS ONLY A ROUGH ESTIMATE OF THE VALUES OF $\lambda_0$ AND $\lambda_2$ CAN BE OBTAINED AS A RESULT OF THIS COMPARISON.

ELLIOT AND RIDDIFORD [18]

THESE AUTHORS HAVE MADE SOME EXPERIMENTS USING PARALLEL PLATES WITH LIQUID-LIQUID AND LIQUID-GAS SYSTEMS. SINCE THEIR RESULT CONTAIN ONLY GRAPHS SHOWING THE RELATIONS BETWEEN DYNAMIC CONTACT ANGLES AND "THE VELOCITY OF THE INTERFACE" BY THE SLOPE OF THEIR GRAPH IN THE LINEAR REGION USING THE EQUATION (33) THE VALUES OF PARAMETERS ARE OBTAINED. THE VALUES OF $\lambda_0$ AND $\lambda_2$ FOR WATER-AIR SYSTEM WERE FOUND TO BE LARGE AND THIS IMPLIES AS IN THE PREVIOUS CASE A VERY SMALL SUP REGION WHICH CAN BE ASSUMED TO BE NON-EXISTANT. BUT SINCE LARGE VALUES OF $\lambda_0$ AND $\lambda_2$ YIELDS TO VERY LARGE OSCILLATIONS OF THE SUP VELOCITY DISTRIBUTION CURVE, AN ACCURATE ESTIMATE OF THEIR VALUE FROM THIS COMPARISON WAS IMPOSSIBLE.
The results of this study show that the shape of interface is greatly dependent on the values of $\lambda_1, \lambda_2, \lambda_3, \text{and } \lambda_4$. Since these parameters correspond to certain slip distributions on the plates, the most realistic interface can be obtained with the parameters corresponding to negligible slip velocity regions. But this condition is not possible to achieve with two parameters therefore for better results more parameters must be introduced by considering higher order terms of the approximation of weighting functions.

The calculations showed that in case of two different viscosity fluids the magnitudes of slip are smaller in the less viscous fluid.

In order to improve this method either more parameters must be introduced such that the slip region is reduced or a function of this parameter must be introduced, which will reduce the slip region.

For eliminating the assumption of plane interface some corrections due to the interface shape can be used.

Analysis of this problem with the help of marker and cell method can be very effective since a great deal of information will be obtained in connection with the interface shape and the flow conditions near the interface.
REFERENCES

APPENDIX

A. THE EFFECT OF GRAVITY [14]

In order to neglect the effect of gravity on the interface the following inequality must be satisfied:

\[
\frac{(\rho' - \rho'') g \Delta z}{\frac{T}{R}} \ll 1
\]

where

\( \rho' \) = Density of Fluid 1
\( \rho'' \) = Density of Fluid 2
\( g \) = Gravitational Acceleration
\( \Delta z \) = Vertical extent of the interface
\( T \) = Surface Tension
\( R \) = Radius of curvature

In case of small \( \Delta z \) as in the capillary flows or equal density fluids the effect of gravity can be neglected.
B. RELATION BETWEEN CAPILLARY TUBES AND PARALLEL PLATES

Most of the studies concerning the flow of two immiscible fluids have for the case of capillary tubes, in order to use their results for making comparisons a relation between the flow thru a capillary tube and the flow between parallel plates is needed.

In the flows with very low Reynolds number (Stokes regime) the flow is induced by the pressure gradient if the gravity is neglected. Since most of the creeping flows are belonging to this group, a relation in terms of pressure gradients can be obtained between tube and parallel plate flows.

For the case of tube flows the ratio of pressure gradient due to viscous effect to the capillary pressure is expressed in the following way:

\[ \frac{\Delta P}{P_c} = -\frac{8\mu U_m}{\alpha^2} \]

\[ P_c = \frac{2\eta}{R_t} \]

\[ \frac{\Delta P}{P_c} = -\frac{4R_t}{\alpha^2} \left(\frac{\mu U_m}{\eta}\right) = -\frac{4R_t}{\alpha^2} \left(\text{Tay No}\right) \]

In case of the parallel plates

Pressure gradient \( \Delta P = -\frac{3\mu U_m}{\alpha^2} \)

Capillary pressure \( P_c = \frac{\delta}{R_p} \)

\[ \frac{\Delta P}{P_c} = -\frac{3R_p}{\alpha^2} \left(\frac{\mu U_m}{\eta}\right) = -\frac{3R_p}{\alpha^2} \left(\text{Tay No}\right) \]

Where

\( \alpha = \text{radius of tube or half the distance between the plates} \)

\( \eta = \text{surface tension} \)

\( \mu = \text{abs. viscosity} \)
\[ R_t = \text{radius of curvature of interface in tube} \]
\[ R_p = \text{radius of curvature of interface in plate} \]

If the comparison is going to be made from the point of view of the shape and curvature of interfaces, the Taylor number, which determines the shape of interface, must be same for both cases.

\[ \frac{1}{4R_t} \left( \frac{\Delta P}{\rho} \right)_{\text{tube}} = \frac{1}{3R_p} \left( \frac{\Delta P}{\rho} \right)_{\text{plates}} \]

The ratios of pressure gradients are the same when;

\[ R_p = \frac{4}{3} R_t \]

This relation between the radius of curvatures can be used in comparing the results from point of view of the shape of the interface.
c. Numerical Methods that are Applicable to this Problem.

The flow of two immiscible fluids between parallel plates can be studied by means of numerical methods in a rather effective way. One of the most important methods is the marker and cell method. This is a modified finite difference technique which uses marker particles for indication of the configuration of the fluids, position of interface, indication of fluid particles, and flow trajectories. The study of Rayleigh-Taylor instability with the help of this method is a good example for its usefulness in solving the interface problems [6].

Another effective method is the finite element method. A variational problem of creeping flows can be expressed as the minimisation of the integral of viscous dissipation over the flow region. This is based on the principle of Korteweg, Rayleigh, Lamb, H. That "the viscous dissipation for the flows in which the inertial effects can be neglected, is finite, unique and a minimum for a given set of boundary conditions. A solution of creeping flow between parallel plates has been done by Atkinson, Card and Irons."
SLIP DISTRIBUTION

\[ \lambda_1' = 1.1563 \quad \lambda_2' = 1.1563 \]

\[ \lambda_1 = 0 \quad \lambda_2 = 0 \]

\[ M_1 = 125 \quad M_2 = 125 \]

FIG. 5: SINGLE PARAMETER SOLUTION
$M_1 = M_2$

$\lambda_0 = -2.4521$
$\lambda = 1.8827$
$\lambda'_0 = -2.4521$
$\lambda'_2 = 1.8827$

**SLIP DISTRIBUTION**

**FIG. 7: DOUBLE PARAMETER SOLUTION**
\[ \mathcal{M}_1 = \mathcal{M}_2 \]

\[ \text{TAY N} = .1 \]

\[ \lambda_0 = -13.796 \]
\[ \lambda_2 = 7.801 \]
\[ \lambda'_0 = -13.796 \]
\[ \lambda'_2 = 7.801 \]

**SLIP DISTRIBUTION**

**FIG. 6: DOUBLE PARAMETER SOLUTION**