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DOI
10.1109/TAC.2018.2886181

Publication date
2019

Document Version
Accepted author manuscript

Published in
IEEE Transactions on Automatic Control

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2886181, IEEE Transactions on Automatic Control

A classification of nodes for structural controllability

Christian Commault and Jacob van der Woude

***Abstract*— In this paper we consider (large and complex) interconnected networks. We assume that each state/node, not belonging to a set of forbidden nodes of the network, can be selected to act as a steering node, meaning that such a node then is influenced by its own individual control. We aim to achieve structural controllability and we present a classification of the associated steering nodes as being essential (always required to be present), useful (present in certain configurations) and useless (never necessary in whatever configuration). The classification is based on two types of decomposition that naturally show up in the context of the two conditions (connection condition and rank condition) for structural controllability. The underlying methods are related to well-known and efficient network algorithms.**

***Index Terms*— Controllability, structured system theory, input connection condition, rank condition, steering node.**

I. Introduction

The controllability of complex networks has received a lot of attention in the recent years. Especially, the question of interest is where to put so-called driver nodes by which the behaviour of the network can be controlled, see [7], [12], [15], [16].

Here we call a state/node of the network a steering node if it can be influenced from the outside of the network by a control. We assume that not all nodes may be directly controlled, so that the steering nodes must belong to a given set of effective nodes, see [14].

We aim to achieve structural controllability and we present a classification of the associated steering nodes as being essential (always required to be present), useful (present in certain configurations) and useless (never necessary in whatever configuration). In engineering, the classification of sensors in terms of their importance for preserving some property (such as observability) is an active research field, see [3].

It should be noted that, although we use here some concepts and tools of [3], the present paper is much more than just a dualisation of the results of [3]. The main difference is that here the classification does not rely on a given set of inputs and corresponding actuators. Indeed, in this paper the inputs are not yet present and have to be chosen. Moreover, they directly influence only one state.

The steering node classification in this paper resembles a similar classification (in critical, redundant and intermittent driver nodes) in [10]. However, opposite to [10], [12], we suppose that a control can only influence one steering node and that not all nodes can be chosen as steering nodes.

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An extended version of this paper, containing detailed proofs and a lot of examples, is available in [6].

The outline of this paper is as follows. In section 2 we formulate the problem of steering node selection for controllability and introduce the notions of useless, useful and essential nodes for a specific property. We also introduce structured systems together with their digraph representation and recall the two well-known conditions for structural controllability. In section 3 the two conditions are further analysed, using connectivity aspects and the rank condition. The latter is done by means of a DM-decomposition of the bipartite graph associated to the structured systems. In section 4 we present criteria for a node to be useless, useful or essential for each of the two conditions separately, and for the two conditions simultaneously, yielding structural controllability. We conclude by section 5 with a summary of the results of this paper and with some topics for future research.

II. Problem formulation

A. The controllability problem

In this paper we consider a large scale system composed of n states interacting together with linear dynamics. We assume that we can represent the behaviour of the whole system by the simple equation

$$\dot{x}(t) = Ax(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and A is a real constant $n \times n$ matrix.

We will distinguish m states, called the *steering nodes* $\mathcal{S} = \{x_{i_1}, \ldots, x_{i_m}\}$, with $i_j \in \{1, \ldots, n\}$ and $i_1 < i_2 < \cdots < i_m$. To each steering node x_{i_j}, we associate a control input u_j that acts only on this state node. In this way we obtain a system that can be represented as

$$\Sigma : \ \dot{x}(t) = Ax(t) + Bu(t), \tag{2}$$

where matrix B has m columns and its j-th column has all its entries equal to 0 except for $b_{i_j}(\neq 0)$, being the i_j-th component of column j of B. Hence, the node set $\mathcal{S}$ is in 1-1 correspondence with the (structure of) matrix B. In the following we will be looking for a set of steering nodes such that the pair (A, B), as introduced above, is controllable.

As in [14], we assume that there is a set of forbidden nodes, which cannot be used as steering nodes. This situation is frequently met in applications.

Let us denote by $\mathcal{F} \subseteq \mathcal{X}$, where $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, the set of forbidden nodes, and the complementary set by $\mathcal{E} = \mathcal{X} \backslash \mathcal{F}$. The nodes of $\mathcal{E}$ will be called *effective nodes*. These effective nodes $\mathcal{E} = \{x_{k_1}, \ldots, x_{k_p}\}$, with $k_j \in \{1, \ldots, n\}$ and

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$k_1 < k_2 < \cdots < k_p$, can be associated with control inputs in order to define a steering node set $\mathcal{S} \subseteq \mathcal{E}$.

As was done for the steering node set above, matrix $B_\mathcal{E}$ has p columns and its j-th column has all its entries equal to 0 except for $b_{k_j}(\neq 0)$, being the k_j-th component of column j of $B_\mathcal{E}$

Given the set of effective nodes $\mathcal{E}$, it is clear that there exists a steering node set for controllability of the system (1) if and only if the pair $(A, B_\mathcal{E})$ is controllable.

When the pair $(A, B_\mathcal{E})$ is controllable, the node set $\mathcal{E}$ is said to be *c-effective*, with c for controllable.

B. Node classification

When a steering node set $\mathcal{S} \subseteq \mathcal{E} \subseteq \mathcal{X}$, defining matrix B, and therefore also the pair (A, B), is such that a given property $\mathcal{P}$ is true, we call $\mathcal{S}$ an *admissible steering node set* for property $\mathcal{P}$.

For a given property $\mathcal{P}$, a node $x_i \in \mathcal{E}$ can be classified as follows, see for instance [3], [4].

1) Node x_i is called a *useless* node if for any admissible steering node set $\mathcal{S}$ for $\mathcal{P}$ containing x_i, $\mathcal{S}\backslash\{x_i\}$ is still an admissible steering node set for $\mathcal{P}$, where $\mathcal{S}\backslash\{x_i\}$ is the set $\mathcal{S}$ minus the node x_i.
2) A node which is not useless is called a *useful* node. Hence, node x_i is useful if there is an admissible steering node set $\mathcal{S}$ for $\mathcal{P}$ such that $x_i \in \mathcal{S}$, while $\mathcal{S}\backslash\{x_i\}$ is not admissible for $\mathcal{P}$.
3) Node x_i is called an *essential* node if x_i belongs to any admissible steering node set $\mathcal{S}$ for $\mathcal{P}$. Hence, x_i is an essential node if $\mathcal{S}\backslash\{x_i\}$ is not admissible for any admissible steering node set $\mathcal{S}$ for $\mathcal{P}$. The set of essential nodes is a subset of the set of useful nodes.

In this paper we will focus our attention on the search and classification of steering nodes for the controllability in the context of structured systems.

C. Linear structured systems and structural controllability

In the remainder we assume that system (1) is *structured*, meaning that we assume that only the zero/non-zero pattern of (the entries in) matrix A is known.

A structured system of type (1) can be associated with a directed graph $G(A) = (\mathcal{X}, \mathcal{W})$ as follows:

- the node set is $\mathcal{X}$, being the set of state nodes $\{x_1, x_2, \ldots, x_n\}$,
- the edge set is $\mathcal{W} = \{(x_i, x_j)|a_{ji} \neq 0\}$, where $a_{ji} \neq 0$ means that the (j, i)-th entry of matrix A is a structural non-zero, and (x_i, x_j) stands for an edge from node x_i to node x_j.

In the graph $G(A)$ we define a path from a node x_{i_0} to a node x_{i_q} to be a sequence of edges $(x_{i_0}, x_{i_1}), (x_{i_1}, x_{i_2}), \ldots, (x_{i_{q-1}}, x_{i_q})$, such that $x_{i_t} \in \mathcal{X}$ for $t = 0, 1, \ldots, q$, and $(x_{i_{t-1}}, x_{i_t}) \in \mathcal{W}$ for $t = 1, 2, \ldots, q$.

For a structured system we can study generic properties, *i.e.*, properties which are true for almost any value of the matrix entries. One such property is, for instance, the generic controllability of a structured system. Another such property is the generic rank of a structured matrix. Given a structured matrix Q, the rank of Q for almost all values of the non-zero entries will be denoted by g-rank$\,Q$.

Example 1: Consider the system defined by a structured matrix A given by

$$A = \begin{pmatrix} 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with * representing a non-zero entry. It follows by a simple inspection that g-rank$\,A = 6$. The associated digraph $G(A)$ is depicted in Figure 1. In the figure also the strongly connected components are already indicated. These components will be introduced and used later on.

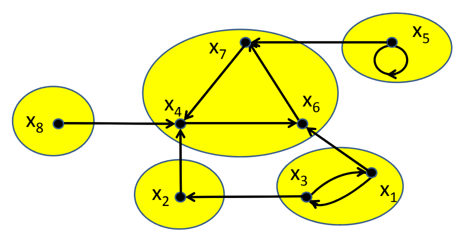

Fig. 1. Digraph of Example 1

As in Subsection II-A, we can select a set of m steering nodes in $\mathcal{X}$, to which we associate m control inputs. This induces an $n \times m$ matrix B with only m non-zero entries, one in each column and at most one in each row.

The graph $G(\Sigma)$ can be obtained from $G(A)$ by adding the m input nodes through the set $\mathcal{U} = \{u_1, \ldots, u_m\}$, and by adding m edges, one from each input node to the corresponding steering node. Hence, $G(\Sigma)$ has node set given by $\mathcal{X} \cup \mathcal{U}$, and the edge set is updated as $\mathcal{W} := \mathcal{W} \cup \{(u_j, x_{i_j})|j = 1, 2, \ldots, m\}$.

In $G(\Sigma)$, a path $(v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{q-1}}, v_{i_q})$, where $v_{i_0} \in \mathcal{U}$ and $v_{i_q} \in \mathcal{X}$, is called an input-state path. The system Σ is said to be *input-connected* if for any state node $x_i, i = 1, \ldots, n$, there exists an input-state path with end node x_i.

The notion of structural controllability was introduced and studied by Lin, who proved a necessary and sufficient condition for structural controllability in terms of graph theoretic objects called cacti, see [11]. The following result can be proved to be equivalent to Lin's result (references are given in [6]).

Theorem 1: Let Σ be the linear structured system defined by (2) with associated graph $G(\Sigma)$. The system is structurally controllable if and only if

1) the system Σ is input-connected,
2) g-rank $[A, B]$=n.

In the following, the conditions 1 and 2 of Theorem 1 will be referred to as the input connection condition and the rank condition, respectively.

Given a structured system of type (1) with associated graph $G(A)$, the steering node selection problem then amounts to extending $G(A)$ with input nodes and edges (input-steering node) in such a way that the conditions of Theorem 1 are fulfilled in the extended graph $G(\Sigma)$.

III. Structural controllability via steering node selection

We will first revisit the two conditions for structural controllability individually and refine Lin's theorem in terms of possible nodes to be impacted by inputs.

A. Input connection condition

Consider the linear structured system defined by (1) with its associated graph $G(A)$. A strongly connected component $\mathcal{C}$ is defined to be a maximum set of nodes of $G(A)$ such that there exists a path, possibly of length zero, between any two nodes of $\mathcal{C}$. The graph can be partitioned into a set of strongly connected components and this set can be endowed with a partial order. A strongly connected component of $G(A)$ with no incoming edge from another strongly connected component is called a *Critical Connection Component* (CCC). Now we can deduce easily, see also [1], [5], that a steering node set $\mathcal{S}$ is admissible for the input connection condition if and only if there exists a node of $\mathcal{S}$ in any Critical Connection Component of $G(A)$.

Example 1 (cont.): The previous notions and results can be illustrated on Example 1. The graph possesses five strongly connected components, namely $\{x_1, x_3\}$, $\{x_2\}$, $\{x_5\}$, $\{x_8\}$ and $\{x_4, x_6, x_7\}$, where $\{x_1, x_3\}$, $\{x_5\}$ and $\{x_8\}$ are the Critical Connection Components. It is clear from the graph, that input connection is verified if and only if x_5, x_8, and either x_1 or x_3 are steering nodes.

B. Rank condition

In order to check the rank condition we introduce a second type of graph by which our structured system can be represented.

1) Generic rank and maximum matching: To a given structured $\mu \times \nu$ matrix L one can associate a bipartite graph $H(L) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$, where the sets $\mathcal{V}^+$ and $\mathcal{V}^-$ are two disjoint node sets, and $\mathcal{W}'$ is the edge set.

- the node set $\mathcal{V}^+$ is described by $\{v_1^+, \ldots, v_\nu^+\}$ and the node set $\mathcal{V}^-$ is given by $\{v_1^-, \ldots, v_\mu^-\}$.
- the edge set $\mathcal{W}'$ is given by $\mathcal{W}' = \{(v_i^+, v_j^-) | L_{ji} \neq 0\}$. In the latter, (v_i^+, v_j^-) denotes the edge between nodes v_i^+ and v_j^-, and, as before, $L_{ji} \neq 0$ means that the (j, i)-th entry of the matrix L is a structural non-zero.

A *matching* in the bipartite graph $H(L) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$ is a set of edges $\mathcal{M} \subseteq \mathcal{W}'$ such that the edges in $\mathcal{M}$ have no common node. A node is *covered* by a matching if there exists an edge in the matching that is *incident* to the node. A matching $\mathcal{M}$ is called *maximum* if its cardinality is maximum. The *maximum matching problem* consists of finding a matching of maximum cardinality. It can be solved by using efficient combinatorial algorithms, see for example [9].

It is a well-known result of combinatorics that the generic rank of a structured matrix L is the cardinality of a maximum matching in the corresponding bipartite graph $H(L)$, see [13].

The previous analysis can be applied to the A matrix, with bipartite graph $H(A)$, node sets $\mathcal{V}^+ = X^+ = \{x_1^+, \ldots, x_n^+\}$, $\mathcal{V}^- = X^- = \{x_1^-, \ldots, x_n^-\}$, and edges corresponding to the non-zero entries in A. The same can be done for $[A, B]$, with bipartite graph $H([A, B])$, nodes sets $\mathcal{V}^+ = X^+ \cup U$, $\mathcal{V}^- = X^-$ and $U = \{u_1, \ldots, u_m\}$, and edges corresponding to the non-zero entries in $[A, B]$. The latter can be applied to check the rank condition of Theorem 1, by looking for a maximum matching in the bipartite graph $H([A, B])$, see [5].

On the other hand, starting with a structured system as in (1), with the generic rank of A being the size of a maximum matching in $H(A)$, the rank defect, defined as $d_r(A) := n-$ g-rank A, is the minimal number of steering nodes needed to make the rank condition of Theorem 1 become true, see [12].

2) Dulmage-Mendelsohn decomposition: We present now the Dulmage-Mendelsohn decomposition, see [8], abbreviated as DM-decomposition. The DM-decomposition is a useful tool to parameterize all maximum matchings in a bipartite graph. The DM-decomposition of the bipartite graph $H(L) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$ is the uniquely defined family of bipartite subgraphs $H_i = (\mathcal{V}_i^+, \mathcal{V}_i^-; \mathcal{W}_i')$, called the DM-components, where the collection of subsets $\{\mathcal{V}_0^+, \mathcal{V}_1^+, \cdots, \mathcal{V}_r^+, \mathcal{V}_\infty^+\}$ is a partition of $\mathcal{V}^+$, and likewise for $\mathcal{V}^-$. The set $\mathcal{W}_i'$ is the set of edges in $\mathcal{W}'$ incident with nodes from $\mathcal{V}_i^+$ as well $\mathcal{V}_i^-$, for $i = 0, 1, \ldots, r, \infty$. In the decomposition, the bipartite subgraph H_0 is called *minimal inconsistent part*, the bipartite subgraph H_∞ is called *maximal inconsistent part*, and the other subgraphs $H_i, i = 1, \ldots, r$, are called *consistent parts*.

The DM-decomposition and the above components have the following properties, for details see [13] and in particular Proposition 3.1.

Proposition 1: Let $H(L) = (\mathcal{V}^+, \mathcal{V}^-; \mathcal{W}')$ be a bipartite graph with its DM-decomposition, and with $H_i = (\mathcal{V}_i^+, \mathcal{V}_i^-; \mathcal{W}_i')$, $i = 0, 1, \ldots, r, \infty$, as its DM-components. Then we have the following properties, where $|\mathcal{V}_i^+|$ denotes the cardinality of $\mathcal{V}_i^+$, and likewise for $|\mathcal{V}_j^-|$.

1) A maximum matching on $H(L)$ is a union of maximum matchings on the DM-components H_i, $i = 0, 1, \ldots, r, \infty$.
2) Every node of $\mathcal{V}_0^-$ (or $\mathcal{V}_\infty^+$, or $\mathcal{V}_i^+$, $\mathcal{V}_i^-$, $i = 1, \ldots, r$) is covered by any maximum matching on $H(L)$.
3) A node $v^+ \in \mathcal{V}^+$ belongs to the minimal inconsistent part H_0, *i.e.*, $v^+ \in \mathcal{V}_0^+$, if and only if there exists a maximum matching on $H(L)$ that does not cover node v^+, implying that $|\mathcal{V}_0^+| > |\mathcal{V}_0^-|$.
4) A node $v^- \in \mathcal{V}^-$ belongs to the maximal inconsistent part H_∞, *i.e.*, $v^- \in \mathcal{V}_\infty^-$, if and only if there exists a maximum matching on $H(L)$ that does not cover node v^-, implying that $|\mathcal{V}_\infty^-| > |\mathcal{V}_\infty^+|$.

The DM-decomposition is useful in reordering matrices to get a more insightful form. Indeed, it can be shown, see [13], that after a permutation of the rows and columns based on the partitions of $\mathcal{V}^+$ and $\mathcal{V}^-$, matrix L is brought into the following upper block triangular form

$$\begin{pmatrix} L_{00} & L_{01} & \dots & L_{0r} & L_{0\infty} \\ 0 & L_{11} & \dots & L_{1r} & L_{1\infty} \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & & L_{rr} & L_{r\infty} \\ 0 & 0 & \dots & 0 & L_{\infty\infty} \end{pmatrix},$$

where matrix L_{ij} has dimensions $|\mathcal{V}_i^-| \times |\mathcal{V}_j^+|$, for $i,j = 0,1,2,\dots,r,\infty$, with $i \leq j$.

It can be shown from the properties of Proposition 1 that g-rank $L_{ii} = |\mathcal{V}_i^-| = |\mathcal{V}_i^+|$, for $i = 1,2,\dots,r$, g-rank $L_{00} = |\mathcal{V}_0^-| < |\mathcal{V}_0^+|$, and g-rank $L_{\infty\infty} = |\mathcal{V}_\infty^+| < |\mathcal{V}_\infty^-|$. Moreover, it follows from property 3 that leaving out a column of L_{00} does not decrease its rank. Similarly, from property 4 it follows that leaving out a row from $L_{\infty\infty}$ also does not give a rank drop. Hence, leaving out a column of L with index in $\mathcal{V}_0^+$ does not give a rank drop, and likewise for leaving out a row of L with index in $\mathcal{V}_\infty^-$. Note that the latter also implies that adding a unit column e_i, a column with the only nonzero entry at the i-th place, with $x_i^- \in \mathcal{V}_\infty^-$ does give an increase in rank. The latter observations are insightful/useful in the proofs given later in this paper.

The rank condition for controllability can be expressed using only the maximal inconsistent part of the DM-decomposition of $H(A)$, see [2], which is indeed the maximal dilation in the sense of Lin [11].

Proposition 2: Consider the linear structured system defined by (1) with associated bipartite graph $H(A)$ and the corresponding DM-decomposition.

A steering node set $\mathcal{S}$ is admissible for the rank condition if and only if there exists a maximum matching in the bipartite subgraph H_∞ such that for every node x_i^- in $\mathcal{V}_\infty^-$ that is not covered by the matching, there holds $x_i \in \mathcal{S}$.

Example 1 (cont.): The DM-decomposition corresponding to the bipartite graph associated with Example 1 is given in Figure 2. The maximum size of a matching in $H(A)$ is 6.

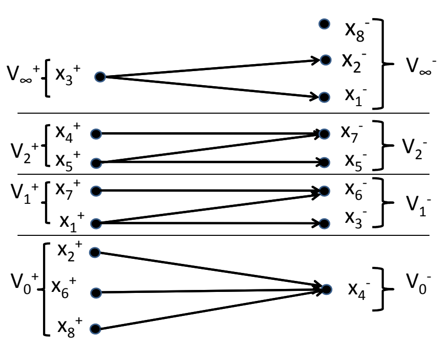

Fig. 2. DM-decomposition of Example 1

Hence, the generic rank of A is equal to 6 and $d_r(A) = 2$. From Proposition 2 it follows that a maximum matching of H_∞ can be (x_3^+, x_1^-), which implies that a possible admissible steering node set for the rank condition is $\{x_2, x_8\}$. However, also $\{x_1, x_8\}$ can act as an admissible steering node set when (x_3^+, x_2^-) is chosen as maximum matching of H_∞.

IV. Node classification for structural controllability

We start with a structured system of type (1), hence only defined by the matrix A. As before, we denote the associated directed graph by $G(A)$ and the associated bipartite graph by $H(A)$.

Using refinements of the two conditions of Theorem 1, obtained in Section III, the controllability condition for the pair $(A, B_\mathcal{E})$ can be reformulated as follows.

Lemma 1: Given the set of effective nodes $\mathcal{E}$, the corresponding structured system $(A, B_\mathcal{E})$ is structurally controllable, and therefore there exists an admissible steering node set for controllability, if and only if

1) for any Critical Connection Component $\mathcal{C}_j$ of $G(A)$ we have $\mathcal{C}_j \cap \mathcal{E} \neq \emptyset$,
2) the size of a maximum matching in $H([A, B_\mathcal{E}])$ is n.

In this section we will give a classification of nodes according to the definitions of Section II. We will provide this classification first for each condition (input connection condition and rank condition) and then for controllability.

A. Classification of nodes for input connection

Proposition 3: Consider a linear structured system of type (1) with associated graph $G(A)$ and c-effective node set $\mathcal{E}$. The node x_i in $\mathcal{E}$ is

1) useless for input connection if and only if it does not belong to a Critical Connection Component,
2) useful for input connection if and only if it belongs to a Critical Connection Component,
3) essential for input connection if and only if it is the unique effective node in a Critical Connection Component.

Proof

The proof directly follows from the definition of useless, useful and essential nodes, and from the fact that each CCC must contain an effective steering node for insuring input connection, see Subsection III-A.

B. Classification of nodes for the rank condition

For the rank condition we have the following proposition.

Proposition 4: Consider a linear structured system of type (1) with associated graph $G(A)$, associated bipartite graph $H(A)$ and the corresponding DM-decomposition. Consider a c-effective node set $\mathcal{E}$, with $B_\mathcal{E}$ the corresponding input matrix and $H([A, B_\mathcal{E}])$ the associated bipartite graph with its DM-decomposition. The node x_i in $\mathcal{E}$ is

1) useless for the rank condition if and only if x_i^- does not belong to the set $\mathcal{V}_\infty^-$ of the DM-decomposition of $H(A)$,
2) useful for the rank condition if and only if x_i^- belongs to the set $\mathcal{V}_\infty^-$ of the DM-decomposition of $H(A)$,
3) essential for the rank condition if and only if the corresponding input node u_{k_i} does not belong to the set $\mathcal{V}_0^+$ in $H([A, B_\mathcal{E}])$.

Proof

1) $\Rightarrow$ From [1] Proposition 9, it follows that for a unit vector e_i, g-rank$[A, e_i]$=g-rank$(A)+1$ if and only if for the corresponding state vertex x_i, we have $x_i^- \in \mathcal{V}_\infty^-$, otherwise g-rank$[A, e_i]$= g-rank$(A)$. The proof can then be based on linear algebra arguments as follows. The effective set $\mathcal{E} = \{x_{k_1}, \ldots, x_{k_p}\}$, being such that $(A, B_\mathcal{E})$ is structurally controllable, implies that g-rank$[A, e_{k_1}, \ldots, e_{k_p}]$=$n$ for the corresponding unit vectors. If $x_i \in \mathcal{E}$ is such that $x_i^- \in \mathcal{V}_\infty^-$, we have g-rank$[A, e_i]$=g-rank$(A)$+1. Then, from the incomplete basis theorem, there exists a steering node set $\mathcal{D} = \{x_{l_1}, \ldots, x_{l_{p_1}}\} \subset \mathcal{E}$ containing x_i, of cardinality $n-$g-rank(A), such that g-rank$[A, e_{l_1}, \ldots, e_{l_{p_1}}] = n$. With $\mathcal{D}$ being of minimal cardinality, taking off e_i, and then x_i from $\mathcal{D}$, would violate the rank condition, therefore x_i is not useless.
   $\Leftarrow$ Take $\mathcal{D} \subset \mathcal{E}$ admissible, $x_i \in \mathcal{D}$, and assume that $x_i^- \notin \mathcal{V}_\infty^-$. Then $x_i^- \in \mathcal{V}^- \backslash \mathcal{V}_\infty^-$. Since $\mathcal{D}$ is admissible, by Proposition 2, there exists a maximum matching on the bipartite subgraph H_∞ such that for every $x_j^- \in \mathcal{V}_\infty^-$ not covered by this matching, there holds that $x_j \in \mathcal{D}$. Now fix this matching. Recall that $x_i \in \mathcal{D}$ and leave x_i out of $\mathcal{D}$. Then still for every $x_j^- \in \mathcal{V}_\infty^-$ not covered by the matching there holds that $x_j \in \mathcal{D}\backslash\{x_i\}$. Hence, by Proposition 2, it follows that $\mathcal{D}\backslash\{x_i\}$ is also admissible and therefore x_i is useless.
2) Obvious from point 1.
3) $\Rightarrow$ Notice first that, since the size of a maximum matching in $H([A, B_\mathcal{E}])$ is n, from Proposition 1, the H_∞ part of $H([A, B_\mathcal{E}])$ is empty. Suppose now that u_{k_i} belongs to the set $\mathcal{V}_0^+$ in $H([A, B_\mathcal{E}])$. Then there exists a maximum matching in $H([A, B_\mathcal{E}])$ which does not cover u_{k_i}. Therefore one can build an admissible steering node set which does not contain x_i, so x_i is not essential.
   $\Leftarrow$ Assume that u_{k_i} does not belong to the set $\mathcal{V}_0^+$ in $H([A, B_\mathcal{E}])$. Then, any maximum matching in $H([A, B_\mathcal{E}])$ contains u_{k_i} and discarding u_{k_i} (which is equivalent to discard the corresponding effective node x_i from the admissible set $\mathcal{S}$), would decrease the size of a maximum matching. Then $\mathcal{S}\backslash\{x_i\}$ is not admissible, and x_i is essential.

C. Classification of nodes for controllability

Next we combine the previous results to obtain a classification of steering nodes for structural controllability.

Theorem 2: Consider a linear structured system of type (1) with associated graph $G(A)$, associated bipartite graph $H(A)$ and the corresponding DM-decomposition. Consider a of c-effective node set $\mathcal{E}$, with $B_\mathcal{E}$ the corresponding input matrix and $H([A, B_\mathcal{E}])$ the associated bipartite graph with its DM-decomposition. The node x_i in $\mathcal{E}$ is

1) essential for structural controllability if and only if x_i is the unique effective node in a Critical Connection Component or the corresponding input node u_{k_i} does not belong to the set $\mathcal{V}_0^+$ in $H([A, B_\mathcal{E}])$.
2) useless for structural controllability if x_i belongs to no Critical Connection Component and x_i^- does not belong to the set $\mathcal{V}_\infty^-$ of the DM-decomposition of $H(A)$.

Proof

1) For controllability, being equivalent to input connection and the rank condition, if a node is essential for one of the properties, it is also essential for controllability. Conversely, if a node is essential for controllability, this means that when it is taken away from an admissible steering node set, the controllability is lost. Then at least one of the properties is lost. Therefore, this node is essential for at least one of the properties. The result then follows by combining the characterizations of essential nodes in Propositions 3 and 4.
2) If a node is useless for both properties, this means that when it is taken off from any admissible steering node set, the two properties remain verified. Therefore, controllability remains verified, and this node is also useless for controllability.

Remark 1: In Theorem 2, we characterize only a subset of the useless steering nodes for controllability (namely those which are useless for both sub-properties). Indeed, as can be seen on examples [6], some steering nodes may be useless for controllability, while being useful for one of the sub-properties.

Example 1 (cont.): The previous results can be illustrated on the eight node example whose graph is depicted in Figure 1. We assume that the set of forbidden nodes is $\mathcal{F} = \{x_1, x_4, x_7\}$ and that then $\mathcal{E} = \{x_2, x_3, x_5, x_6, x_8\}$. As seen in Subsection III-A, the Critical Connection Components are $\{x_1, x_3\}$, $\{x_5\}$ and $\{x_8\}$. From Proposition 3, it follows that being the unique effective node in their CCC, the nodes x_3, x_5 and x_8 are essential for input connection, while other nodes are useless for this property.

Concerning the rank condition, nodes x_3, x_5 and x_6 in $\mathcal{E}$ are useless since x_3^-, x_5^- and x_6^- do not belong to the set $\mathcal{V}_\infty^-$ of the DM-decomposition of $H(A)$, see Figure 2. The DM-decomposition of the graph $H([A, B_\mathcal{E}])$, see Figure 3, shows that the input nodes u_1 and u_5 do not belong to the set $\mathcal{V}_0^+$ in $H([A, B_\mathcal{E}])$, so x_2 and x_8 are essential nodes for the rank condition.

In summary, for controllability, nodes x_2, x_3, x_5 and x_8 of $\mathcal{E}$ are essential, and node x_6 of $\mathcal{E}$ is useless.

D. Classification of nodes when there are no forbidden nodes

It is of interest to examine how Theorem 2 simplifies when there are no forbidden nodes, *i.e.* when $\mathcal{F} = \emptyset$.

Theorem 3: Consider a linear structured system of type (1) with associated graph $G(A)$, associated bipartite graph $H(A)$ and the corresponding DM-decomposition. The node x_i is

1) essential for structural controllability if and only if x_i is the unique node in a Critical Connection Component or there exists no edge (x_j, x_i) in $G(A)$.
2) useless for structural controllability if x_i belongs to no Critical Connection Component and x_i^- does not belong to the set $\mathcal{V}_\infty^-$ of the DM-decomposition.

Proof

For input connection, it is clear that points 1 and 2 of Proposition 3 are unchanged, while in point 3 the word effective has just to be removed.

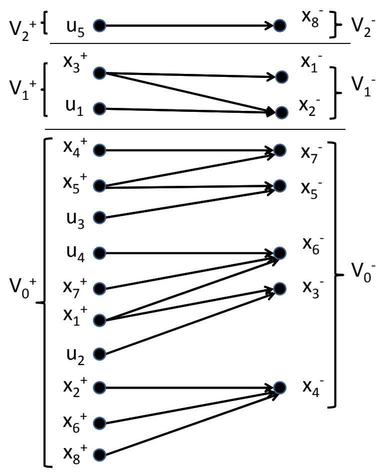

Fig. 3. DM-decomposition of $H([A, B_\mathcal{E}])$

For the rank condition, again points 1 and 2 of Proposition 4 are unchanged, while point 3 needs a particular attention. In this case, all the nodes are effective and in the bipartite graph $H([A, B_\mathcal{E}])$ any vertex x_i^- is connected with the corresponding input node u_i. Assume that there is no edge (x_j^+, x_i^-) in $H([A, B_\mathcal{E}])$, or equivalently that there is no edge (x_j, x_i) in $G(A)$. In this case, any size n matching in $H([A, B_\mathcal{E}])$ contains the edge (u_i, x_i^-), therefore the node x_i is essential for the rank condition. Conversely, if there exists an edge (x_j^+, x_i^-) in $H([A, B_\mathcal{E}])$, the matching composed of all edges (u_k, x_k^-) for $k \neq i$ and of edge (x_j^+, x_i^-) is a size n matching in $H([A, B_\mathcal{E}])$ not containing u_i. Then u_i belongs to the set $\mathcal{V}_0^+$ in $H([A, B_\mathcal{E}])$, therefore x_i is not essential for the rank condition from point 3 of Proposition 4. In summary, a node x_i is essential for the rank condition if and only if there is no edge (x_j, x_i) in $G(A)$.

By gathering the results concerning input connection and the rank condition, we obtain a complete proof of the Theorem.

Example 1 (cont.): We illustrate Theorem 3 on Example 1. Its graph $G(A)$ is depicted in Figure 1, and its bipartite graph $H(A)$ with DM-decomposition are shown in Figure 2.

From Proposition 3, with $\mathcal{F} = \emptyset$, it follows that nodes x_2, x_4, x_6, x_7 are useless for input connection, while nodes x_1, x_3, x_5, x_8 are useful, with nodes x_5, x_8 being essential, because they correspond to Critical Connection Components of cardinality one. Note that all nodes are effective.

From Proposition 4, it follows that nodes x_3, x_4, x_5, x_6, x_7 are useless for the rank condition, nodes x_1, x_2, x_8 are useful, with node x_8 being essential, because it has no incoming edge.

From Theorem 3, it follows that nodes x_5, x_8 are essential steering nodes for controllability, while nodes x_4, x_6, x_7 are useless for controllability. It can be checked by inspection that besides nodes x_5, x_8, also the nodes x_1, x_2, x_3 are useful for controllability. For example, node x_2 cannot be discarded from the admissible steering node set $\{x_2, x_3, x_5, x_8\}$ because of the rank condition.

Note the role of forbidden/effective nodes by comparing the latter results, where $\mathcal{F} = \emptyset$, to the node characterisations obtained previously in case $\mathcal{F} \neq \emptyset$.

V. Conclusions and outlook

In this paper we studied steering nodes in order, for a large complex system, to become structurally controllable when the nodes of the system can be divided into forbidden and effective (=non-forbidden) nodes. We then provided a classification of steering nodes into useless, useful or essential ones. For the individual conditions for structural controllability, being the input connection condition and the rank condition, this classification could be given completely. However, for their combination, culminating in a classification of nodes for structural controllability, this still is not settled completely as far as useless steering nodes are concerned. This will remain a topic for further research. The methods underlying the obtained classifications are based on well-understood algorithms coming from the theory of flows in networks.

Of course, by duality, the results of this paper provide with a classification of nodes for structural observability.

Acknowledgment

The authors wish to thank Jean-Michel Dion for his invaluable collaboration.

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