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A DICHOTOMY CONCERNING UNIFORM BOUNDEDNESS OF RIESZ TRANSFORMS ON RIEMANNIAN MANIFOLDS

ALEX AMENTA AND LEONARDO TOLOMEO

ABSTRACT. Given a sequence of complete Riemannian manifolds (M_n) of the same dimension, we construct a complete Riemannian manifold M such that for all $p \in (1, \infty)$ the L^p -norm of the Riesz transform on M dominates the L^p -norm of the Riesz transform on M_n for all n . Thus we establish the following dichotomy: given p and d , either there is a uniform L^p bound on the Riesz transform over all complete d -dimensional Riemannian manifolds, or there exists a complete Riemannian manifold with Riesz transform unbounded on L^p .

1. INTRODUCTION

Given a Riemannian manifold M , one can consider the Riesz transform $R := \nabla(-\Delta)^{\frac{1}{2}}$, where ∇ is the Riemannian gradient and Δ is the (negative) Laplace–Beltrami operator. In the Euclidean case $M = \mathbb{R}^n$, this can be identified with the vector of classical Riesz transforms (R_1, \dots, R_n) , as can be seen by writing R as a Fourier multiplier (see [12, §5.1.4]).

It is easy to show that R is bounded from $L^2(M)$ to $L^2(M; TM)$, and substantially harder to determine whether R extends to a bounded map from $L^p(M)$ to $L^p(M; TM)$ for $p \neq 2$. We let

$$R_p(M) := \sup_{\|f\|_{L^p} \leq 1} \|R(f)\|_{L^p}$$

denote the (possibly infinite) L^p -norm of the Riesz transform on M . Various conditions, often involving the heat kernel on M and its gradient, are known to imply finiteness of $R_p(M)$; see for example [2, 3, 4, 5, 6, 7, 8, 9, 13, 14]. These results usually entail finiteness of $R_p(M)$ for all $p \in (1, 2)$, or for some range of $p > 2$. On the other hand, there exist manifolds M for which $R_p(M)$ is known to be infinite for some (or all) $p > 2$: see [1, 5, 6, 7, 8, 13].

Remark 1.1. When M has finite volume we abuse notation and write $L^p(M)$ to denote the space of p -integrable functions *with mean zero*. This modification ensures that $(-\Delta)^{-1/2}$ is densely defined. When M has infinite volume, $L^p(M)$ denotes the usual Lebesgue space.

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The Euclidean case is now classical: for all $p \in (1, \infty)$ there is a constant $C_p < \infty$ such that $R_p(\mathbb{R}^n) \leq C_p < \infty$ for all $n \in \mathbb{N}$ ([16]). This behaviour is expected to persist for all complete Riemannian manifolds, at least for $p < 2$. More precisely, in [9] it is conjectured that for all $p \in (1, 2)$ there exists a constant $C_p < \infty$ such that $R_p(M) \leq C_p$ for all complete Riemannian manifolds M . Such uniform bounds have been proven for all $p \in (1, \infty)$ under curvature assumptions; rather than provide an overview of the vast literature on this topic we simply point to the recent paper [10] and references therein.

One could weaken the conjecture slightly and guess that $R_p(M)$ is finite for all M , given $p \in (1, 2)$. In this article we show that this can only hold if the bound is uniform among all manifolds of a fixed dimension. This observation follows from the following dichotomy.

Theorem 1.2. *Fix $d \in \mathbb{N}$ and $p \in (1, \infty)$. Then the following dichotomy holds: either*

- *there exists a constant $C_{p,d} < \infty$ such that $R_p(M) \leq C_{p,d}$ for all complete d -dimensional Riemannian manifolds M , or*
- *there exists a complete $(d+1)$ -dimensional Riemannian manifold M such that $R_p(M) = \infty$.*

This follows from the following proposition, which we prove by an explicit construction.

Proposition 1.3. *Fix $d \geq 1$, and let $(M_n)_{n \in \mathbb{N}}$ be a sequence of complete d -dimensional Riemannian manifolds. Then there exists a complete Riemannian manifold M of dimension $d+1$ such that for all $p \in (1, \infty)$,*

$$R_p(M) \geq \sup_{n \in \mathbb{N}} R_p(M_n).$$

The main implication of Theorem 1.2 is as follows: to construct a manifold M for which $R_p(M) = \infty$ for some $p \in (1, 2)$, it suffices to construct a sequence $(M_n)_{n \in \mathbb{N}}$ of manifolds of equal dimension such that $R_p(M_n) \rightarrow \infty$ as $n \rightarrow \infty$. Thus one is led to consider lower bounds for L^p -norms of Riesz transforms. These seem not to have been considered in the literature, excluding of course the well-known computation of the L^p -norm of the Hilbert transform (the Riesz transform on \mathbb{R}) [15]. We hope that our contribution will provoke further interest in such lower bounds.

2. PRELIMINARY LEMMAS

We begin with some basic lemmas. The first says that the range of the Laplace-Beltrami operator is dense in L^p , and the second relates the Riesz transform on a manifold M with that on the M -cylinder $M \times \mathbb{R}$. These cylinders play a key role in the proof of our main theorem.

Lemma 2.1. *Let M be a complete Riemannian manifold. Then the set $S := \Delta(C_c^\infty(M))$ is dense in $L^p(M)$ for all $p \in (1, \infty)$ (recalling that we write $L^p(M)$ for the space of p -integrable mean zero functions when M has finite volume).*

Proof. Let $H \in L^{p'}(M)$ be such that $\langle H, F \rangle = 0$ for every $F \in S$. Then $\langle H, \Delta G \rangle = 0$ for every test function G , so H is harmonic. By [17, Theorem 3], it follows that H is constant, and the result follows. \square

Lemma 2.2. *Let M be a complete Riemannian manifold. Then*

$$R_p(M \times \mathbb{R}) \geq R_p(M).$$

Proof. Consider the following modification of the Riesz transform on $M \times \mathbb{R}$:

$$\tilde{R} := \nabla_M (-\Delta_{M \times \mathbb{R}})^{-\frac{1}{2}} = \nabla_M (-\Delta_M - \partial_t^2)^{-\frac{1}{2}}.$$

This is just the projection of R onto the first summand of the tangent bundle $T(M \times \mathbb{R}) = TM \oplus T\mathbb{R}$, so we have that

$$(1) \quad \|\tilde{R}F\|_{L^p} \leq \|RF\|_{L^p}.$$

Let $F \in C_c^\infty(M \times \mathbb{R})$, and for all $\lambda > 0$ consider the function

$$F_\lambda(x, t) := \lambda^{\frac{1}{p}} F(x, \lambda t),$$

which satisfies $\|F_\lambda\|_{L^p(M \times \mathbb{R})} = \|F\|_{L^p(M \times \mathbb{R})}$. Rescaling the operator \tilde{R} in the variable t , we define

$$\tilde{R}_\lambda := \nabla_M (-\Delta_M - \lambda^2 \partial_t^2)^{-\frac{1}{2}},$$

so that

$$(2) \quad \|\tilde{R}F_\lambda\|_{L^p} = \|\tilde{R}_\lambda F\|_{L^p}.$$

Now take $f \in C_c^\infty(M) \cap D((-\Delta_M)^{-\frac{1}{2}})$ and $\rho \in C_c^\infty(\mathbb{R})$ such that $\|\rho\|_{L^p(\mathbb{R})} = 1$, and consider the function $F(x, t) = f(x)\rho(t)$. Since Δ_M and ∂_t^2 commute, and the function

$$G_\lambda(x, y) = \left(\frac{x}{x + \lambda^2 y} \right)^{\frac{1}{2}}$$

is bounded by 1 for $(x, y) > 0$, and $G_\lambda \rightarrow 1$ pointwise as $\lambda \rightarrow 0$, we have

$$\lim_{\lambda \rightarrow 0} (-\Delta_M - \lambda^2 \partial_t^2)^{-\frac{1}{2}} F = \lim_{\lambda \rightarrow 0} G_\lambda(-\Delta_M, -\partial_t^2) (-\Delta_M)^{-\frac{1}{2}} f \otimes \rho = (-\Delta_M)^{-\frac{1}{2}} f \otimes \rho$$

in L^2 , and thus also as distributions. Therefore $\tilde{R}_\lambda F \rightarrow Rf \otimes \rho$ as distributions, and so

$$\liminf_{\lambda \rightarrow 0} \|\tilde{R}_\lambda F\|_{L^p(M \times \mathbb{R})} \geq \|Rf \otimes \rho\|_{L^p(M \times \mathbb{R})} = \|Rf\|_{L^p(M)}.$$

Combining this with (2) and (1), and the fact that $C_c^\infty(M) \cap D((-\Delta_M)^{-\frac{1}{2}})$ is dense in $L^p(M)$,¹ yields $R_p(M \times \mathbb{R}) \geq R_p(M)$. \square

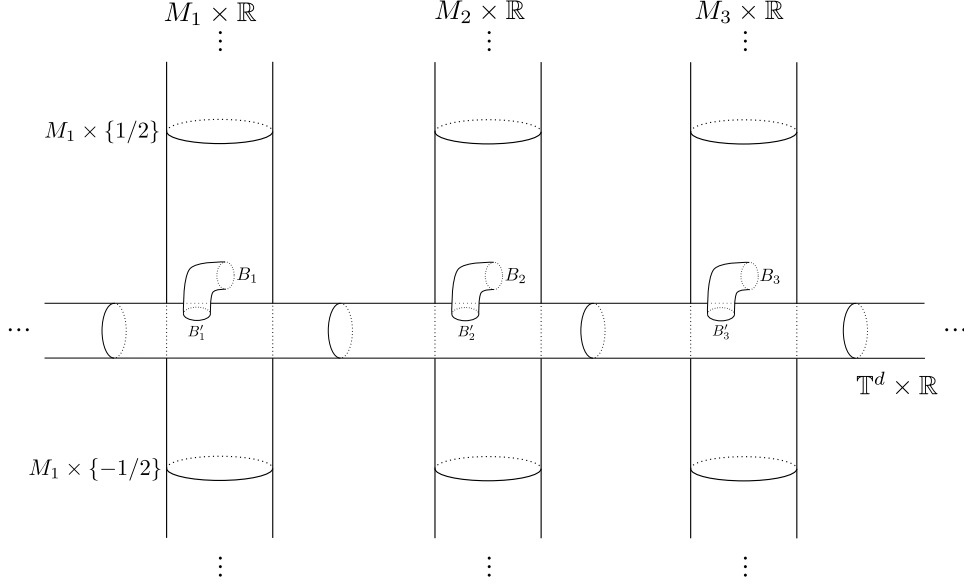
3. PROOF OF THE MAIN THEOREM

In this section we carry out the construction that proves Proposition 1.3, which implies Theorem 1.2.

Consider a sequence $(M_n)_{n \in \mathbb{N}}$ of complete d -dimensional Riemannian manifolds. We will connect the M_n -cylinders $(M_n \times \mathbb{R})_{n \in \mathbb{N}}$ along a \mathbb{T}^d -cylinder $\mathbb{T}^d \times \mathbb{R}$ as follows.² For each $n \in \mathbb{N}$ fix a coordinate chart $U_n \subset M_n \times (-1/2, 1/2)$ and a small ball $B_n \subset U_n$. Similarly, for each $n \in \mathbb{N}$ choose a small coordinate chart $U'_n \subset \mathbb{T}^n \times \mathbb{R}$ such that the charts $(U'_n)_{n \in \mathbb{N}}$ are pairwise disjoint, and a small ball $B'_n \subset U'_n$. For each $n \in \mathbb{N}$, glue the manifold $(M_n \times \mathbb{R}) \setminus B_n$ to $(\mathbb{T}^n \times \mathbb{R}) \setminus B'_n$ along the boundaries

¹This follows from the inclusion $D((-\Delta_M)^{-\frac{1}{2}}) \supseteq D((-\Delta_M)^{-1}) \supseteq \Delta_M(C_c^\infty(M))$, which is dense by Lemma 2.1. See also [11, Lemma 2.2]. Again, recall that $L^p(M)$ denotes the corresponding space of mean zero functions when M has finite volume.

²Of course, one could connect the M_n -cylinders to each other directly, without needing the \mathbb{T}^d -cylinder. This would work just as well.

FIGURE 1. Construction of M from $(M_n)_{n \in \mathbb{N}}$.

of B_n and B'_n ; this is possible since both these balls are ‘Euclidean’ balls sitting inside coordinate charts. This results in a C^0 -Riemannian manifold (M, g') , which is C^∞ away from the set $\Sigma = \cup_n \partial B_n$ on which we glued the manifolds together. Mollify the metric to get a C^∞ -Riemannian manifold (M, g) such that $g = g'$ away from the ε -neighbourhood of Σ for some very small ε . An artist’s impression of this construction, with $M_n = S^1$ for each n , is shown in Figure 1.

For each $n \in \mathbb{N}$ we have an inclusion map

$$i_n: M_n \times (1, \infty) \rightarrow M$$

which is an isometry. From here on we fix n and just write $i = i_n$. Functions on M can be pulled back to $M_n \times (1, \infty)$; the pullback map is denoted i^* , so that for $f: M \rightarrow \mathbb{R}$ the function $i^*f: M_n \times (1, \infty) \rightarrow \mathbb{R}$ is defined by

$$i^*f(x, t) = f(i(x, t)).$$

On the other hand, for $g: M_n \times (1, \infty) \rightarrow \mathbb{R}$ we can define a pushforward $i_*g: M \rightarrow \mathbb{R}$ by setting $i_*g(i(x, t)) := g(x, t)$ on $i(M_n \times (1, \infty))$ and extending by zero to the rest of M . For a function $g: M_n \times \mathbb{R} \rightarrow \mathbb{R}$ and for $s \in \mathbb{R}$ we let $\tau_s g: M_n \times \mathbb{R} \rightarrow \mathbb{R}$ be the translated function $\tau_s g(x, t) := g(x, t - s)$. Similarly if $g: M_n \times (1, \infty) \rightarrow \mathbb{R}$ we can define $\tau_s g: M_n \times (1 + s, \infty) \rightarrow \mathbb{R}$. These concepts apply equally well to vector fields in place of functions.

We will need the following lemma, which relates the heat flow on $M_n \times \mathbb{R}$ to the one on M .

Lemma 3.1. *Let $F: M_n \times \mathbb{R} \rightarrow \mathbb{R}$ be smooth and compactly supported, and fix $\sigma > 0$. Then for every $(x, t) \in M_n \times \mathbb{R}$,*

$$\lim_{s \rightarrow +\infty} (e^{\sigma \Delta_M} i_* \tau_s F)(i(x, t + s)) = (e^{\sigma \Delta_{M_n \times \mathbb{R}}} F)(x, t).$$

Proof. Let $W_{x,t}(\sigma)$ be a Brownian motion on $M_n \times \mathbb{R}$ at time σ starting from the point (x, t) . Since the generator $\frac{1}{2}\Delta_{M_n \times \mathbb{R}}$ satisfies $\frac{1}{2}i_* \circ \Delta_{M \times \mathbb{R}}|_{i(M_n \times (1, +\infty))} = \frac{1}{2}\Delta_M|_{i(M_n \times (1, +\infty))}$, defining the stopping time

$$T(x, t) := \inf \{s : W_{x,t}(s) \in M_n \times (-\infty, 1)\},$$

we have that $i(W_{x,t}(\sigma))$ is a Brownian motion on M for $\sigma < T(x, t)$. Therefore there exists a Brownian motion $\tilde{W}_{i(x,t)}(\sigma)$ on M such that $\tilde{W}(\sigma) = i(W(\sigma))$ for $\sigma < T$; if \bar{W} is a Brownian motion on M , we can take for example

$$\tilde{W}_{i(x,t)}(\sigma) = \begin{cases} i(W_{x,t}(\sigma)) & \text{if } \sigma < T, \\ \bar{W}_{i(W_{x,t}(T))}(\sigma - T) & \text{if } \sigma \geq T. \end{cases}$$

We have that

$$\begin{aligned} & (e^{\sigma\Delta_M} i_* \tau_s F)(i(x, t+s)) \\ &= \mathbb{E}[(i_* \tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))] \\ &= \mathbb{E}[(i_* \tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma < T}] + \mathbb{E}[(i_* \tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma \geq T}] \\ &= \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))\mathbb{1}_{2\sigma < T}] + \mathbb{E}[(i_* \tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma \geq T}] \\ &= \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))] \\ &\quad - \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))\mathbb{1}_{2\sigma \geq T}] + \mathbb{E}[(i_* \tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma \geq T}] \\ &= (e^{\sigma\Delta_{M_n \times \mathbb{R}}} \tau_s F)(x, t+s) \\ &\quad - \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))\mathbb{1}_{2\sigma \geq T}] + \mathbb{E}[(i_* \tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma \geq T}]. \end{aligned}$$

Therefore

$$|(e^{\sigma\Delta_M} i_* \tau_s F)(i(x, t+s)) - (e^{\sigma\Delta_{M_n \times \mathbb{R}}} \tau_s F)(x, t+s)| \leq 2\|F\|_{L^\infty} \mathbb{P}(T(x, t+s) \leq 2\sigma).$$

Since $\Delta_{M_n \times \mathbb{R}}$ is translation invariant in the \mathbb{R} coordinate, we have that

$$\begin{aligned} \mathbb{P}(T(x, t+s) \leq 2\sigma) &\leq \mathbb{P}(\{W_{x,t+s}(\sigma') \in M_n \times (-\infty, 1) \text{ for some } \sigma' \leq 2\sigma + 1\}) \\ &= \mathbb{P}(\{W_{x,t}(\sigma') \in M_n \times (-\infty, 1-s) \text{ for some } \sigma' \leq 2\sigma + 1\}) \end{aligned}$$

and by continuity of $W_{x,t}(\cdot)$, this tends to 0 as $s \rightarrow \infty$. Thus we find that

$$\lim_{s \rightarrow +\infty} \left((e^{\sigma\Delta_M} i_* \tau_s F)(i(x, t+s)) - (e^{\sigma\Delta_{M_n \times \mathbb{R}}} \tau_s F)(x, t+s) \right) = 0.$$

The conclusion follows from translation invariance of $\Delta_{M_n \times \mathbb{R}}$ in \mathbb{R} . \square

We return to the proof of Proposition 1.3. Fix $\varepsilon > 0$, and choose $F = \Delta_{M_n \times \mathbb{R}} H$ for some $H \in C_c^\infty(M_n \times \mathbb{R})$ with $\|F\|_{L^p} = 1$ such that

$$\|R_{M_n \times \mathbb{R}} F\|_{L^p} \geq (R_p(M_n) - \varepsilon) \wedge \varepsilon^{-1}.$$

Such a function exists by Lemmas 2.1 and 2.2. We claim that

$$(3) \quad \lim_{s \rightarrow +\infty} \tau_{-s} i^* R_M(i_* \tau_s F) = R_{M_n \times \mathbb{R}} F$$

as distributions. Assuming (3) for the moment, we have

$$\begin{aligned} \limsup_{s \rightarrow \infty} \|R_M(i_* \tau_s F)\|_{L^p(M)} &\geq \limsup_{s \rightarrow \infty} \|i^* R_M(i_* \tau_s F)\|_{L^p(M_n \times \mathbb{R})} \\ &= \limsup_{s \rightarrow \infty} \|\tau_{-s} i^* R_M(i_* \tau_s F)\|_{L^p(M_n \times \mathbb{R})} \\ &\geq \|R_{M_n \times \mathbb{R}} F\|_{L^p(M_n \times \mathbb{R})} \geq R_p(M_n) - \varepsilon, \end{aligned}$$

while for all $s \in \mathbb{R}$

$$\|i_*\tau_s F\|_{L^p(M)} \leq \|\tau_s F\|_{L^p(M_n \times \mathbb{R})} = \|F\|_{L^p(M_n \times \mathbb{R})} \leq 1.$$

The result follows, so it remains to prove (3).

For s sufficiently large, we have that

$$i_*\tau_s F = i_*\tau_s(\Delta_{M_n \times \mathbb{R}} H) = i_*(\Delta_{M_n \times \mathbb{R}} \tau_s H) = \Delta_M i_*\tau_s H,$$

therefore $i_*\tau_s F \in D(\Delta_M^{-1}) \subseteq D((-\Delta_M)^{-\frac{1}{2}})$, and hence

$$R(i_*\tau_s F) = \nabla \left((-\Delta_M)^{-\frac{1}{2}} i_*\tau_s F \right)$$

as a distribution. To test the distributional convergence, let X be a smooth compactly supported vector field in $M_n \times \mathbb{R}$. For large s we have that

$$\begin{aligned} \langle \tau_{-s} i^* R_M(i_*\tau_s F), X \rangle &= \langle R_M(i_*\tau_s F), i_*\tau_s X \rangle \\ &= \left\langle (-\Delta_M)^{-\frac{1}{2}} i_*\tau_s F, \operatorname{div}(i_*\tau_s X) \right\rangle \\ &= \left\langle (-\Delta_M)^{-\frac{1}{2}} i_*\tau_s F, i_*\tau_s \operatorname{div}(X) \right\rangle. \end{aligned}$$

Therefore it is enough to show that for every $G \in C_c^\infty(M_n \times \mathbb{R})$,

$$(4) \quad \lim_{s \rightarrow \infty} \left\langle (-\Delta_M)^{-\frac{1}{2}} i_*\tau_s F, i_*\tau_s G \right\rangle = \left\langle (-\Delta_{M_n \times \mathbb{R}})^{-\frac{1}{2}} F, G \right\rangle.$$

By the well-known formula

$$(-\Delta)^{-\frac{1}{2}} = \pi^{-\frac{1}{2}} \int_0^{+\infty} \sigma^{-\frac{1}{2}} e^{\sigma \Delta} d\sigma,$$

(4) is equivalent to showing that

$$(5) \quad \lim_{s \rightarrow \infty} \int_0^{+\infty} \sigma^{-\frac{1}{2}} \langle e^{\sigma \Delta_M} i_*\tau_s F, i_*\tau_s G \rangle d\sigma = \int_0^{+\infty} \sigma^{-\frac{1}{2}} \langle e^{\sigma \Delta_{M_n \times \mathbb{R}}} F, G \rangle d\sigma.$$

Note that

$$\left| \sigma^{-\frac{1}{2}} \langle e^{\sigma \Delta_M} i_*\tau_s F, i_*\tau_s G \rangle \right| \leq \sigma^{-\frac{1}{2}} \|i_*\tau_s F\|_{L^2} \|i_*\tau_s G\|_{L^2} \leq \sigma^{-\frac{1}{2}} \|F\|_{L^2} \|G\|_{L^2}$$

and

$$\left| \sigma^{-\frac{1}{2}} \langle e^{\sigma \Delta_M} i_*\tau_s F, i_*\tau_s G \rangle \right| = \left| \sigma^{-\frac{3}{2}} \langle e^{\sigma \Delta_M} \sigma \Delta_M i_*\tau_s H, i_*\tau_s G \rangle \right| \lesssim \sigma^{-\frac{3}{2}} \|H\|_{L^2} \|G\|_{L^2}.$$

Since the function $\min(\sigma^{-\frac{1}{2}}, \sigma^{-\frac{3}{2}})$ is integrable, by dominated convergence (5) will be proved if we show

$$(6) \quad \lim_{s \rightarrow \infty} \langle e^{\sigma \Delta_M} i_*\tau_s F, i_*\tau_s G \rangle = \langle e^{\sigma \Delta_{M_n \times \mathbb{R}}} F, G \rangle$$

for every $\sigma > 0$. We show (6) by writing

$$\begin{aligned} \lim_{s \rightarrow \infty} \langle e^{\sigma \Delta_M} i_*\tau_s F, i_*\tau_s G \rangle &= \lim_{s \rightarrow \infty} \langle \tau_{-s} i^* e^{\sigma \Delta_M} i_*\tau_s F, G \rangle \\ &= \lim_{s \rightarrow \infty} \int_{1-s}^{+\infty} \int_{M_n} (e^{\sigma \Delta_M} i_*\tau_s F)(i(x, t+s)) G(x, t) dx dt \\ &= \int_{\mathbb{R}} \int_{M_n} (e^{\sigma \Delta_{M_n \times \mathbb{R}}} F)(x, t) G(x, t) dx dt \\ &= \langle e^{\sigma \Delta_{M_n \times \mathbb{R}}} F, G \rangle, \end{aligned}$$

using Lemma 3.1 and dominated convergence (by $\|F\|_{L^\infty} |G(x, t)|$). This completes the proof of Proposition 1.3, and hence establishes Theorem 1.2.

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