Weak Capture and the Weak Stability Boundary

An investigation into the restricted problem of three bodies

K. Kumar

Delft, 17 June 2008
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To my parents
This thesis is in partial fulfilment of the Master of Science in Aerospace Engineering degree at Delft University of Technology, The Netherlands. The research presented in this report is the culmination of an extensive scientific investigation carried out at: Department of Aerospace Engineering at Delft University of Technology; Department of Aerospace Engineering & Engineering Mechanics, University of Texas at Austin; and Department of Aerospace Engineering at Politecnico di Milano.

I was introduced to the specialist field of astrodynamics during the first year of my Master of Science degree in Aerospace Engineering in Delft. I was fascinated and captivated by space mission analysis and trajectory design, having taken classes covering fundamental theory relating to these disciplines. Hence, I decided to choose a thesis topic within this subject area. The majority of research in astrodynamics performed within the Astrodynamics & Satellite Systems Group focusses on space mission analysis and trajectory optimisation techniques. Specifically, global trajectory optimisation has become a core specialisation within the group. Despite the wealth of knowledge and expertise in this area of astrodynamics within the group, I chose for a slightly alternative path by selecting a thesis topic relating to a more fundamental treatise of the dynamics governing the motion of celestial bodies.

The problem of three bodies, describing the motion of three masses under mutual gravitation, is fundamental to understanding chaotic processes in celestial mechanics. In addition, recent developments have indicated that it is possible to find low-energy trajectories that harness the dynamics of the problem to provide unparalleled mission performance. The Hiten mission, Japan’s first mission to the Moon, was rescued by using such a trajectory. Hence, there is interest currently in developing our understanding of the underlying dynamics of the problem. This is the background to the thesis subject that I selected. In this report, the findings of a study into the dynamics of the Restricted 3-Body Problem is presented.

I would like to thank ir. R. Noomen, my thesis advisor in Delft, for giving me the opportunity to explore my area of interest and take on a subject of this nature. I value the insights he provided me during my thesis work and the support he offered during my numerous adventures around the world. I would also like to thank Dr. F. Topputo for his extensive involvement in my research and invaluable advice. Without his guidance, much of the work presented in this report would not have been possible. I would like to thank Prof. F. Bernelli Zazzera for providing me with the opportunity to work on my research at the Department of Aerospace Engineering at Politecnico di Milano and for sitting on my graduation committee. I would like to extend my gratitude to Prof. C. Ocampo as well, for providing me...
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Finally, I would like to make special mention of the contribution my family has made. Thanks to my brother for always being ready to take my mind off of things by whacking a cricket ball around. Thanks to my parents for being my rock to lean on and for supporting my many foreign escapades. Without them the work presented in this report would not have been achievable.

As another chapter comes to a close, it is time to explore new horizons. New beginnings beckon whilst old friends remains. Onwards and upwards!

Delft, 17 June 2008
The 3-Body Problem (3BP) in astrodynamics has been investigated by numerous mathematicians and scientists, including leading figures such as Euler, Lagrange and Poincaré. Poincaré in particular contributed many findings that shed light on the underlying dynamics of the problem. His investigations have stood the test of time and to a large extent shape our current understanding of some key aspects. Poincaré was able to establish the chaotic nature of motion in the 3-Body Problem (3BP). His research laid the foundations for an entirely new area of mathematics that has seen widespread applications. Despite his extensive investigations and the efforts of others, much remains unknown about the 3BP.

The 3BP describes the motion of any three massive bodies under the influence of their mutual gravitational attraction. Such systems are non-integrable, meaning that closed-form solutions to describe the motion of the masses are not available using current methods. Considering the third body in the system to be of negligible mass leads to formulation of the Restricted 3-Body Problem (R3BP). The two massive bodies in the system are referred to as primaries. The model can be simplified further by imposing the requirement that the primaries must orbit their common gravitational barycentre in circular orbits. Further, the third body can be required to move in the same plane of motion as the primaries. With these assumptions, it is possible to formulate the Planar Circular Restricted 3-Body Problem (PCR3BP). Remarkably, despite these simplifications, the PCR3BP preserves some of the most interesting features of the general problem of three bodies.

To investigate the nature of the problem, numerical methods have been employed extensively in recent years. Such research has confirmed the chaotic nature of this deterministic system and has shed light on a number of interesting phenomena. Weak capture is a particularly interesting and describes the occurrence of natural transit motion between the primaries. Understanding weak capture is essential to improving our knowledge of the underlying dynamics of the problem.

By studying weak capture it is possible that theories in celestial mechanics can be improved upon, such as formation theories of our solar system, formation theories of planetary ring systems, the dynamics of the main asteroid belt, and the dynamics of triple star systems, to mention but a few.

Recently, interest has been gathering for the use of weak capture in space mission design. Weak capture transfers, that describe transfers between primaries requiring zero deterministic propellant mass, have been analysed for space mission applications since the 1960’s. Conley conjectured that low-energy transfers to the Moon exist that would surpass the performance of the traditional optimum Hohmann transfer [Conley, 1968]. This conjecture was only proven in flight when
the Hiten mission successfully brought the failed Muses-A Japanese spacecraft to the Moon. This was the first instance of a low-energy transfer being used for space mission design and subsequently led to extensive research into the dynamics of such trajectories.

Despite these successes, our understanding of the global solution space is restricted. The Weak Stability Boundary (WSB), which describes the set of points in phase space that exhibit weak capture behaviour, has been investigated by many research groups, including [Belbruno et al., 2008], [Garcia and Gomez, 2006], [Koon et al., 2006]. Yet despite this, a rigorous mathematical definition remains elusive. Specifically, a robust analytical definition of the WSB region has not yet been established. Initial attempts at identifying and describing this set have paved the way for further analysis.

Beyond a mathematician’s interest in establishing and improving such a definition, there are real applications that can be envisioned. Global trajectory optimisation is notoriously difficult and slow, especially when the search space is large, as is common to many problems in astrodynamics. To aid in the design of trajectories for future missions that harness weak capture, it is therefore beneficial to find ways to reduce the search space. By establishing an accurate analytical definition of the WSB region, it is possible that global trajectory optimisation for such missions can be streamlined.

With this in mind, the study presented in this report is aimed at investigating the WSB region further, with a view to shedding further light on its internal structure and possibly improving on the latest analytical definition. In the PCR3BP, transit motion is intimately linked with the location of invariant manifold structures associated with periodic orbits about special equilibrium points, known as Lagrange points. These invariant manifolds are separatrices for the motion: trajectories lying within these topological tubes are transitory whilst those lying outside are non-transitory. Hence, the weak capture problem in the case of the PCR3BP can be analysed by considering the location of these invariant manifolds. The link between the WSB region and the location of invariant manifolds has been extensively charted within the last decade. In this study, the PCR3BP is used to visually illustrate this link and establish the correctness of the definition of the WSB region. Specifically, the internal structure of the WSB is tied directly to the location of the invariant manifolds. This is done by numerically analysing the problem by using Poincaré sections or Poincaré maps, which offer insight into the underlying dynamics by taking a slice out of the flow. The problem with such Poincaré sections is that they do not provide insight into the stability of trajectories within the WSB region. Hence, extended-Poincaré sections are introduced.

Simulations were performed during this study for the Earth-Moon PCR3BP. Based on the results of these simulations, a number of general conclusions can be drawn. Firstly, it is apparent that the use of invariant manifold intersections on Poincaré sections to determine the internal structure of the WSB region is limited. A stochastic method was introduced using the concept of extended-Poincaré sections. Such sections were generated for three different Jacobi energy values (C) and provide additional insight into the internal structure of the WSB region. The cases considered are complex, in that the WSB region is difficult to approximate analytically. This is specifically the case because of the complex invariant tori structure on the right side of the Moon i.e the far side. The Jacobi energy levels
considered were selected such that the associated WSB regions would be complex, to highlight the advantage of using extended-Poincaré sections to study its internal structure.

It would seem from the extended-Poincaré sections that the WSB region consists of many points that remain stable around the Moon for relatively few orbits. Histogram data plotted to determine the distribution of these points amongst different trajectory classes hints at the fact that with decreasing values of $C$, trajectories in the WSB region as a whole become less stable around the Moon. This is expected, as with decreasing values of $C$ the system becomes more energetic, hence weak capture trajectories are more likely to escape the gravitational pull of the Moon. This is hinted at by a few other features as well. For instance, the resonant island which surrounds the periodic orbit on the right of the Moon noticeably decreases in size as the value of $C$ decreases.

Angular momentum was introduced as a condition to try to remove invariant tori on the left side of the Moon from the WSB set. In addition, the observation was made that only points lying between the L1 and L2 Lagrange points (equilibrium points in the PCR3BP), would be necessary. This was deemed to be the case since trajectories which transit through the neighbourhood of these points intersect the Poincaré section near the Moon; hence it is superfluous to include points beyond these Lagrange points, as they would belong to trajectories that already yield intersections near the Moon. The conditions imposed to take this into account were approximated by vertical straight line boundaries at the Lagrange points.

The largest Jacobi energy value for which transit is possible, is the energy level associated with L1 ($C_1$), as in this case the physical Jacobi energy surface opens up for the first time to allow trajectories to pass between the Earth and Moon: an interior weak capture transfer. For this reason, from this study the upper bound in $C$, for which the WSB region exists, is determined to be equal to $C_1$. The existing analytical definition, given by [Belbruno et al., 2008], uses the Jacobi energy associated with L2 ($C_2$) as the upper bound. In this report, Jacobi energy values which are larger than $C_2$ clearly illustrate that transit between the Earth and Moon is possible. If the WSB region is generically defined as the set of points in phase space supporting such behaviour, then it is clear that the upper bound for the definition must be $C_1$. This is an improvement upon the existing analytical definition given by [Belbruno et al., 2008].

One of the main problems with the current analytical definition is that it overestimates the WSB region. It includes a number of different regions of points that do not exhibit transit motion: invariant tori around the Earth, and invariant tori on both sides of the Moon. An attempt was made to remove some of these regions from the WSB set. By imposing the L1 and L2 constraints, the invariant tori around the Earth are removed. The angular momentum condition was imposed to try and remove invariant tori on the left side of the Moon i.e. the near-Earth side. For the Jacobi energy values considered, it is demonstrated that this can be achieved quite successfully. It is seen that most but not all of the invariant tori on the left side are removed. This is definitely an improved approximation of the WSB region, and was possible because it was noted that for most of the invariant tori on the left side, there is a sign change in angular momentum compared to points lying in the chaotic sea.
Subsequent analysis of the behaviour of the angular momentum boundary as a function of decreasing Jacobi energy value indicates, however, that the approximation for lower values of $C$ deteriorates. Hence, it can be concluded that the angular momentum condition is not robust: it is only applicable to a subset within the Jacobi energy range for which the WSB region is defined. It can be considered sufficiently accurate for the range $C_1 > C > C^*_H$, where $C^*_H \approx 3.1752$. The aim therefore of improving the analytical definition was not entirely successfully accomplished. Nevertheless, the results obtained during this study might prove to be a good starting point to investigate other possibilities to improve the analytical definition.

The WSB region in the PCR3BP is highly complex. During this study attempts were made to improve our understanding of the WSB set. The use of extended-Poincaré sections was successful in highlighting the internal structure of the WSB set. It was noted that the stability of trajectories in the WSB set is linked to the location of the invariant manifolds; however using these manifolds to analyse stability proved to be complicated by the fact that the topological tubes break up. Extended-Poincaré sections are more robust in investigating the internal structure. An attempt was also made to improve the analytical definition of the WSB region to match the numerically generated set; however this proved to be quite difficult.
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**Acronyms**

- 2BP: 2-Body Problem
- 3BP: 3-Body Problem
- CR3BP: Circular Restricted 3-Body Problem
- ER3BP: Elliptic Restricted 3-Body Problem
- ESOC: European Space Operations Centre
- ESA: European Space Agency
- PCR3BP: Planar Circular Restricted 3-Body Problem
- R3BP: Restricted 3-Body Problem
- MMO: Mercury Magnetospheric Orbiter
- MPO: Mercury Planet Orbiter
- NBP: N-Body Problem
- SEP: Solar Electric Propulsion
- SPM: State Propagation Matrix
<table>
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<tr>
<td>ToF</td>
<td>Time-of-Flight</td>
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<td>WSB</td>
<td>Weak Stability Boundary</td>
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Sir Isaac Newton’s *Law of Universal Gravitation* was formulated for the first time in his work entitled *Philosophiae Naturalis Principia Mathematica*, published in 1687 [Barrow-Green, 1997]. This physical law describes the mutual force of attraction between two point masses that acts along the line that connects them. More specifically, Newton states that this force is proportional to the product of their masses and inversely proportional to the square of their distance of separation. Combined with Newton’s *Laws of Motion*, these physical laws form the backbone of classical mechanics.

Astrodynamics, or orbital mechanics, describes the motion of bodies in space under the influence of the gravitational attraction of one or many bodies. A classical result in astrodynamics considers the motion of a point of negligible mass (zero point mass) solely under the gravitational pull of a massive, central point mass. In such systems, the motion of the orbiting zero point mass describes a conic section: an ellipse, a parabola, or a hyperbola. The case of an ellipse was formalised by Johannes Kepler in his *Laws of Planetary Motion*. This system is called the 2-Body Problem (2BP) and is in essence a dynamical system described by a set of twelve first-order, coupled, differential equations. This system can be solved analytically, as it can be shown that exactly ten algebraic integrals of motion exist and two additional equations can be eliminated through proper choice of the time variable and coordinate reference frame. If the zero point mass is replaced with a massive body, then it can be shown that the two bodies in the system will orbit in conic sections about their common gravitational centre: the barycentre.

A natural extension of the 2BP is the 3-Body Problem (3BP). This problem describes the motion of three gravitational masses under the influence of their mutual gravitational attraction. The simplicity of the statement of the problem belies its true nature. As with the 2BP, it can be shown that a total of ten algebraic integrals of motion exist for this system, and two additional reduction equations based on clever choice of temporal and spatial coordinates. However, unlike the 2BP, the 3BP cannot be solved analytically, as the dynamical system is described by eighteen coupled, first-order differential equations. Hence, it is only possible to reduce this system to six coupled, first-order differential equations. No methods currently exist to solve such a system analytically in closed-form.

Mathematicians and physicists alike have tried their hand at deciphering the nature of this dynamical system. None has been more successful than Henri
Poincaré [Barrow-Green, 1997]. Poincaré made significant contributions to a number of areas of physics and mathematics. His pioneering work on the 3BP represented the first rigorous study of a chaotic deterministic system. In fact, the roots of the field of chaos theory are found in his treatise of such phenomena in the 3BP. Many other mathematicians and physicists such as Euler, Lagrange, Hill, Sundman, to name but a few, have devoted research efforts to studying the 3BP, each making important contributions. Nevertheless, a general solution to the 3BP is yet to be found.

The 3BP is not merely a theoretical artefact. It finds applications in astronomy and astrodynamics. In nature, triple star systems are known to exist and are of interest to astronomers studying the nature and origin of solar systems and galaxies. Further, many of the principles established by studying the 3BP can be extended to the many-body problem, which is important in modelling, for instance, solar systems, galaxies, black holes and ring systems. In astrodynamics, application of our understanding of the 3BP is currently keenly researched. This follows the successful transfer of a failed Japanese lunar mission to the Moon. The transfer was conceived and realised by E. Belbruno in 1991 to bring the Hiten spacecraft to the Moon using the concept of weak capture and the related Weak Stability Boundary (WSB) [Belbruno, 2004]. This transfer is characterised by a much lower fuel mass consumption in getting to the Moon, compared to more traditional methods (Hohmann), and is termed a low-energy transfer. Conley conjectured that such a transfer could exist between the Earth and the Moon [Conley, 1968], and the Hiten spacecraft was the first to successfully demonstrate its existence in mission context.

The nature and dynamics associated with weak capture and, in particular, the region termed the WSB, are not fully understood yet. The aim of the current thesis is to explore the WSB region further, in the hope of being able to uncover more of the underlying dynamics. This is done through a predominantly numerical study, supplemented by analytical methods. The structure of the WSB region in the 3BP is difficult to ascertain because of its sensitivity to system parameters, such as the mass parameter and Jacobi energy level. The WSB region has been visualised in the PCR3BP by using Poincaré sections, which provide an illustration of a slice of the flow of the dynamical system [Belbruno et al., 2008]. These sections are effective in illustrating the chaotic nature of the WSB region. Additionally, using these sections, an analytical definition of the WSB region, which is a description of intersections of a number of different mathematical sets, has been developed [Belbruno et al., 2008]. This analytical definition overestimates the WSB region however. In this study, an attempt is made to improve upon the analytical definition. At present, the internal structure of the WSB region has not been visualised. Specifically, the stability of trajectories around the smaller of the two primaries has not been visualised using Poincaré sections. The aim of this thesis is to establish a robust methodology to investigate the internal structure of the WSB region. The results obtained will be analysed based on knowledge of the underlying dynamics, namely, the existence of periodic solutions around the collinear Lagrange points, and the existence of associated invariant manifold structures.

In this report, the results presented pertain to the Earth-Moon system, to place them in context of previous studies, such as [Belbruno et al., 2008]. The methods used are, however, generally applicable to arbitrary values of the mass parameter.
A number of different Jacobi energy values are considered. Jacobi energy values for which the L1 Lagrange point exists but the L2 Lagrange point does not are considered. These Jacobi energy values are particularly interesting because they permit transit between the Earth and Moon, however they prevent transit to the outer realm. These energy levels have not been investigated in previous studies, such as [Belbruno et al., 2008]. Thus the results presented in this report are complementary to existing results.

In Chapter 2, an overview of some basic ideas relating to dynamical systems theory will be provided. The fundamental laws governing motion in astrodynamics, the N-Body Problem (NBP), and the 2BP will be briefly summarised in Chapter 3. Chapter 4 outlines the Circular Restricted 3-Body Problem (CR3BP) in detail. The characteristics of the planar case (PCR3BP) are described in detail, with the equations of motion outlined using both the Newton and Hamiltonian formulation. Regularisation of the equations of motion is also described. A brief overview of the Elliptic Restricted 3-Body Problem (ER3BP) will be given in Chapter 5. In Chapter 6, the concepts of periodic orbits and associated invariant manifolds will be outlined. Numerical algorithms to calculate them will be summarised and some results will be presented against benchmark data. The central ideas relating to the WSB, including the current state of knowledge, will be outlined in full in Chapter 7. Some of the problems of the current WSB definitions will be highlighted. In Chapter 8, the methods employed to probe the WSB region will be outlined, and subsequently the results obtained from numerous numerical simulations will be presented. Some preliminary analysis will be provided and conclusions based on the results obtained will be discussed. A flavour of the possible applications of the research pursued in this thesis to space mission design will be offered in Chapter 9. In Chapter 10, overall conclusions based on the investigations carried out will be outlined. Finally, in Chapter 11, a number of recommendations and aspects for future work will be mentioned.
Introduction to Dynamical Systems

The theory of dynamical systems attempts to describe the motion of mechanical systems through the use of mathematical models. These models are based on systems of differential equations. The aim of this chapter is to succinctly describe the essential concepts necessary to establish a consistent framework to analyse such systems. The theories and properties of dynamical systems outlined in this chapter form the mathematical basis for the analysis presented in the rest of this report. The material presented in this chapter is highly mathematical in nature, but is necessary to obtain a holistic view of the analyses presented in later chapters.

In Section 2.1, the systems of differential equations that will be addressed in this chapter will be formally defined. The concept of phase space will be introduced in Section 2.2. The notion of phase space is central to the analysis of the Planar Circular Restricted 3-Body Problem (PCR3BP) that will be presented throughout the rest of this report. In Section 2.3, the concepts of critical points and periodic solutions of these systems will be stated. In the PCR3BP, five critical points, or equilibrium points are known to exist (Lagrange points). Their existence and associated properties are elaborated upon in full in Chapter 4. It can be shown that periodic solutions can be found in the neighbourhood of the collinear Lagrange points. The existence of these periodic solutions is established in Chapter 4. Numerical computation of such solutions is extensively treated in Chapter 6. The concepts of Poincaré mapping and integrals of motion will be briefly introduced in Section 2.4. Poincaré maps or Poincaré sections are presented in Chapters 7 8 to analyse the dynamical structure of the PCR3BP. The basics of Hamiltonian systems will then be sketched in Section 2.7. The PCR3BP is a time-independent Hamiltonian system and the properties outlined in Section 2.7 are key to understanding the underlying dynamics. Finally, stability of critical points and periodic solutions of dynamical systems will be outlined in Section 2.8. The application of these properties to non-linear systems will also be explained. Stability properties are essential when considering weak capture in the PCR3BP. The stability concepts presented in this chapter lay the foundations for analyses presented in the rest of this report.

A brief introduction to dynamical systems theory can be found in most standard textbooks such as [Verhulst, 2000], [Wiggins, 1990], [Arnold, 1989], and [Arnold et al., 1993]. The development presented here will, to a large extent, follow [Verhulst, 2000] and [Wiggins, 1990].
2.1 Systems of Differential Equations

Differential equations of the form:

\[ \dot{x} = \frac{dx}{dt} = f(t, x) \]  

(2.1)

where \( t \) is a scalar, \( t \in \mathbb{R} \), commonly identified as time, are known as non-autonomous. The vector function \( f : G \rightarrow \mathbb{R}^n \) is continuous in \( t \) and \( x \); \( G \) is an open subset of \( \mathbb{R}^{n+1} \), so \( x \in \mathbb{R}^n \). The vector function \( x(t) \) is a solution of Equation (2.1) on an interval \( I \subset \mathbb{R} \) if \( x : I \rightarrow \mathbb{R}^n \) is continuously differentiable and if \( x(t) \) satisfies Equation (2.1).

A particular case of Equation (2.1) where the independent variable \( t \) does not occur explicitly is often of interest:

\[ \dot{x} = \frac{dx}{dt} = f(x) \]  

(2.2)

Equations of the form given by Equation (2.2) are called autonomous. In this chapter, characteristics of systems of autonomous differential equations will be discussed.

2.2 Phase Space

The space in which the behaviour of the variables \( x = (x_1, \ldots, x_n) \) is described, parametrised by \( t \), is called phase space. A point in phase space with coordinates \( (x_1(t), \ldots, x_n(t)) \), at a certain point in time \( t \), is denoted a phase point. This phase point will in general, for increasing \( t \), move through phase space.

Generally the solution curves to Equation (2.2) are not known; however it is possible to formulate the behaviour of the orbits in phase space. Equation (2.2) in terms of its components reads:

\[ \dot{x}_i = f_i(x), \quad i = 1, \ldots, n \]  

(2.3)

Using one of these components e.g. \( x_1 \) as an independent variable, and given that \( f(x_1) \neq 0 \), this system can be cast into:

\[ \frac{dx_i}{dx_1} = \frac{f_i(x)}{f_1(x)}, \quad i = 2, \ldots, n \]  

(2.4)

Solutions to this set of equations are denoted orbits in phase space. Based on existence and uniqueness theorems it can be shown that such orbits in phase space never intersect in finite time for autonomous systems. Problems arise in this construction for points \( a = (a_1, \ldots, a_n) \) if:

\[ f_1(a_1) = \ldots = f_n(a_n) = 0 \]  

(2.5)

In that case point \( a \in \mathbb{R}^n \) is a zero of the vector function \( f(x) \) and is denoted a critical point.
### 2.3 Critical Points and Periodic Solutions

**Definition 2.1** Suppose that the vector function \( f(x) \) has a zero at \( x = a \), then this point is called a critical point of the equation \( \dot{x} = f(x) \).

Such a critical point corresponds with an equilibrium solution of the system of differential equations, as \( x(t) = a \) satisfies the equation for all time. Periodic solutions can be defined as follows:

**Definition 2.2** Suppose that \( x = \phi(t) \) is a solution of \( \dot{x} = f(x) \), \( x \in D \subset \mathbb{R}^n \) and suppose there exists a positive number \( T \) such that \( \phi(t + T) = \phi(t) \) for all \( t \in \mathbb{R} \), then \( \phi(t) \) is called a periodic solution of the equation with period \( T \).

For autonomous systems, periodic solutions correspond to closed orbits or cycles in phase-space. The reverse also holds true:

**Lemma 2.1** A periodic solution of an autonomous system \( \dot{x} = f(x) \) corresponds with a closed orbit in phase-space and a closed orbit corresponds with a periodic solution.

Proof of this lemma can be found in [Verhulst, 2000].

### 2.4 Poincaré Mapping

The following autonomous system in \( \mathbb{R}^n \) is considered:

\[
\dot{x} = f(x)
\]  

(2.6)

with periodic solution \( \phi(t) \), which corresponds with a closed orbit in \( n \)-dimensional phase-space. A \((n - 1)\)-dimensional transversal \( V \) (a manifold punctured by the closed orbit and nowhere tangent to it) to the closed orbit is constructed. The concept of a transversal is illustrated in Figure 2.1.

![Figure 2.1 Transversal punctured by closed orbit [Verhulst, 2000].](image-url)

The closed orbit intersects this transversal at point \( a \), as illustrated in Figure 2.1. Consider an orbit denoted \( \gamma(x_0) \) starting at \( x_0 \in V \); this orbit is followed until it returns to \( V \). The choice of \( x_0 \) is such that it is close enough to \( a \) for the return orbit to return to \( V \). This is guaranteed to occur for a particular neighbourhood around \( a \) according to the continuous dependence of the solution on the initial value chosen.
Supposing that the orbit does return to intersect $V$, then the point $x_0$ is said to be mapped in $V$. This is called the return-map or Poincaré-map. The point $P(x_0) \in V$ found after the first mapping can be mapped again. This second mapping is then given by $P^2(x_0)$. Subsequent mappings follow similarly. The point $a \in V$ is a fixed point of the map $P$, as the return orbit maps the point $a$ onto itself.

### 2.5 Linearisation

The nature of the flow of a dynamical system in the vicinity of a critical point can be analysed by linearising the equations of motion in the neighbourhood of such a point. The assumption is made that $f(x)$ has a Taylor series expansion of the first degree about $x = a$ with additional higher order terms. Therefore it follows that $\dot{x} = f(x)$, in the neighbourhood of $x = a$, can be written as:

$$\dot{x} = \frac{\partial f(a)}{\partial x}(x - a) + O(x^2)$$ (2.7)

Since $x = a$ is a critical point, it is noted that $\frac{\partial f}{\partial x}(a) = 0$. In general terms, linear equations with constant coefficients read:

$$\dot{y}' = \frac{\partial f(a)}{\partial y'}(y' - a)$$ (2.8)

If the point $a$ is shifted to the origin this equation reads:

$$\dot{y} = \frac{\partial f(a)}{\partial y} y$$ (2.9)

This can be written in matrix notation as:

$$\dot{y} = Ay$$ (2.10)

where $A$ is an $n \times n$ matrix with constant coefficients.

Linearisation is also possible in the neighbourhood of periodic solutions. Suppose that $\phi(t)$ is a periodic solution of Equation (2.1) and that $x = \phi(t) + y$. Substitution into Equation (2.1) and expanding the vector function about $\phi(t)$ yields:

$$\dot{\phi}(t) + \dot{y} = f(t, \phi(t) + y) = f(t, \phi(t)) + \frac{\partial f}{\partial x}(t, \phi(t))y + \ldots$$ (2.11)

Given that $\dot{\phi}(t) = f(t, \phi(t))$ it follows that:

$$\dot{y} = \frac{\partial f}{\partial x}(t, \phi(t))y + \ldots$$ (2.12)

This equation is in general difficult to solve however. It is possible though to qualitatively assess this linearisation, to examine the behaviour of trajectories in the neighbourhood of the periodic solution.

A Poincaré map $P$ is linearised in the neighbourhood of a fixed point $a$ as follows:

$$\frac{\partial P}{\partial x_0}(a)(x_0 - a)$$ (2.13)
2.6 First Integrals and Integral Manifolds

In certain cases, when time is eliminated from the system of autonomous differential equations, as in Equation (2.4), it is possible to perform closed-form integration yielding relations between the components of the solution vector. Such relations are denoted first integrals in general. More specifically:

**Definition 2.3** Consider the differentiable function $F: \mathbb{R}^n \rightarrow \mathbb{R}$ and the vector function $x: \mathbb{R} \rightarrow \mathbb{R}^n$. The derivative $L_t$ of the function $F$ along the vector function $x$, parametrised by $t$ is:

$$L_tF = \frac{\partial F}{\partial x} \dot{x} = \frac{\partial F}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial F}{\partial x_n} \dot{x}_n$$

(2.14)

where $x = (x_1, \ldots, x_n)$. $L_t$ is denoted the orbital derivative. It is noted that the partial derivative with respect to time is zero since the dynamical system being considered is autonomous.

**Definition 2.4** Consider the equation $\dot{x} = f(x)$, $x \in D \subset \mathbb{R}^n$; the function $F(x)$ is called a first integral of the equation if in $D$ it holds that:

$$L_tF = 0$$

(2.15)

with respect to the vector function $x(t)$.

The equation $F(x) = constant$ defines a family of orbits and is denoted an integral manifold.

2.7 Hamiltonian Systems

The purpose of this section is to outline the fundamental principles of Hamiltonian systems. The principles of Hamiltonian dynamics are key to discovering the properties of many problems in celestial mechanics, including the restricted problem of three bodies. By using the framework of Hamiltonian dynamics, it is possible to recast problems into more convenient variables that reflect remarkable properties. In this section the basic principles of Hamiltonian dynamics will be introduced. The theory presented in this section will largely follow [Szebehely, 1967] and [Verhulst, 2000].

In an $n$-dimensional configuration space the coordinates of a point are given by the $n$ generalised coordinates $q_i$ of the dynamical system. In the corresponding $2n$-dimensional phase space each point is defined by the generalised coordinates $q_i$ and the generalised momenta $p_i$. Points in the configuration space define configurations of the system whilst points in the phase space define the states of the system.

The phase space representation of a system is related to Hamiltonian dynamics, which in turn makes use of the canonical transformation. Depending on the nature of the system being analysed, the choice of the set of variables to represent this system can be of fundamental importance. A formalistic approach to Hamiltonian dynamics will therefore be offered here to highlight the salient aspects.
A configuration of a dynamical system with \( n \) degrees of freedom and \( n \) independent coordinates \( q_1, \ldots, q_n \) is fully determined if the coordinates are known as a function of time. The \( 2n \) initial conditions \( q_i(t_0) = q_i^0, \dot{q}_i(t_0) = \dot{q}_i^0 \) determine the solution \( q_i = q_i(t, q_i^0, \dot{q}_i^0) \). The configuration space is related to Lagrangian dynamics where the equations of motion are given by:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0
\]  

(2.16)

where \( i = 1, \ldots, n \) and \( L = T - V \) is the Lagrangian function, with \( T \) the kinetic energy and \( V \) the potential energy of the system. These equations also result from solution of the variational problem, governed by Hamilton’s principle for conservative systems, which states that the motion of the system from \( t_1 \) to \( t_2 \) is a local extremum for the path taken.

The phase-space description of the system is composed of generalised coordinates \( (q_1, \ldots, q_n) \) and momenta \( (p_1, \ldots, p_n) \), the latter defined by the Legendre transformation:

\[
p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}
\]

(2.17)

The Hamiltonian of this system is then given by:

\[
H(q, p, t) = \sum_{i=1}^{n} \dot{q}_ip_i - L(q, \dot{q}, t)
\]

(2.18)

where the generalised velocities are eliminated by solving Equation (2.17) for \( \dot{q}_i = f_i(q, p, t) \). In general, this is possible since the kinetic energy is of positive definite form. The total differential of \( H \) according to Equation (2.18) is given by:

\[
dH = \sum_{i=1}^{n} \left( \dot{q}_ip_i + p_id\dot{q}_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i \right) - \frac{\partial L}{\partial t} dt
\]

(2.19)

Since \( H = H(q, p, t) \), it also follows that:

\[
dH = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt
\]

(2.20)

By comparing Equations (2.19) and (2.20) it follows that:

\[
\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}
\]

(2.21a)

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}
\]

(2.21b)

\[
\dot{p}_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}
\]

(2.21c)

Equations (2.21b) and (2.21c) constitute a \( 2n \)th-order system called the canonical equations of Hamilton. According to Equation (2.18):

\[
\frac{dH}{dt} = \sum_{i=1}^{n} \left( p_i\ddot{q}_i + \dot{p}_i\dot{q}_i - \frac{\partial L}{\partial q_i}\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) - \frac{\partial L}{\partial t}
\]

(2.22)
Using Equations (2.17) and (2.21c) it follows that:

\[ \frac{dH}{dt} = -\frac{\partial L}{\partial t} \]  

(2.23)

The variables \( q_i, p_i \) form a canonical system.

An important property of time-independent Hamiltonian systems is summarised by the following theorem:

**Theorem 2.1 (Liouville)** The flow generated by a time-independent Hamiltonian system is volume-preserving.

Theorem 2.1 refers to the change of the volume of an element in phase space, where change is caused by the flow of an autonomous differential equation. In a time-independent Hamiltonian system it can be shown that the volume of such an element remains constant. A volume element in two-dimensions is effectively an area element. The PCR3BP is a time-independent Hamiltonian system and Theorem 2.1 is a fundamental property that will be used in Chapter 8.

## 2.8 Stability

Critical points and periodic solutions were defined in Section 2.3. The time evolution of solutions starting in the neighbourhood of such critical points or periodic solutions is often of interest when exploring global dynamics. Analysis and evaluation of such behaviour falls within the scope of stability theory. Mathematical notions of stability have always posed difficulties; nevertheless there are a number of theories applicable to such solutions to map their stability. In this section, a number of these theories will be presented pertaining to both autonomous and non-autonomous dynamics systems (Section 2.1). It is assumed that \( x = 0 \) is a critical point of both of these systems without loss of generality.

### 2.8.1 Critical Points

It is important to sketch the various characterisations possible for critical points before delving into issues related to their stability. The following definitions provide a brief overview of several classical descriptions of critical points.

**Definition 2.5** A critical point \( x = 0 \) of the Equation (2.2) is called a positive attractor if there exists a neighbourhood \( \Omega_0 \subset \mathbb{R}^n \) such that \( x(t_0) \in \Omega_0 \) implies \( \lim_{t \to \infty} x(t) = 0 \).

**Definition 2.6** A critical point \( x = 0 \) of Equation (2.2) is called a negative attractor if there exists a neighbourhood \( \Omega_0 \subset \mathbb{R}^n \) such that \( x(t_0) \in \Omega_0 \) implies \( \lim_{t \to -\infty} x(t) = 0 \).

The following describes another important characterisation of critical points:
Definition 2.7 The point \( x = 0 \) is called a non-degenerate critical point of the function \( F(x) \) if:

\[
\left| \frac{\partial^2 F(0)}{\partial x^2} \right| \neq 0
\]  
(2.24)

This leads to another important definition:

Definition 2.8 If \( x = 0 \) is a non-degenerate critical point of the \( \mathcal{C}^\infty \) function \( F(x) \), then \( F(x) \) is called a Morse-function in a neighbourhood of \( x = 0 \).

It can be shown that the behaviour of the Morse-function in a neighbourhood of a critical point \( x = 0 \) is determined by the quadratic part of the Taylor expansion of the function. Since it is known that for a critical point \( \frac{dF}{dx}(0) = 0 \), it follows that the lowest order of the Taylor expansion is quadratic. For instance, given that \( x = 0 \) is a non-degenerate critical point of the Morse-function \( F(x) \) with Taylor expansion about this critical point given by:

\[
F(x) = F(0) - c_1 x_1^2 - c_2 x_2^2 - \ldots - c_k x_k^2 + c_{k+1} x_{k+1}^2 + \ldots + c_n x_n^2 + O(x^3)
\]  
(2.25)

with positive coefficients \( c_1, \ldots, c_2 ; k \) is called the index of the critical point. There exists a transformation \( x \rightarrow y \) in a neighbourhood of the critical point such that \( F(x) \rightarrow G(y) \) where \( G(y) \) is a Morse-function with critical point \( x = 0 \), with the same index \( k \), and apart from \( G(0) \) only quadratic terms. This can be stated as a lemma:

Lemma 2.2 (Morse) Consider the \( \mathcal{C}^\infty \) function \( F : \mathbb{R}^n \rightarrow \mathbb{R} \) with non-degenerate critical point \( x = 0 \), index \( k \). In a neighbourhood of \( x = 0 \) there exists a diffeomorphism which transforms \( F(x) \) to the form:

\[
G(y) = G(0) - y_1^2 - y_2^2 - \ldots - y_k^2 + y_{k+1}^2 + \ldots + y_n^2
\]  
(2.26)

Proof of this statement can be found in [Verhulst, 2000]. If \( k = 0 \) this implies that level sets in the neighbourhood of the critical point correspond with a positive definite quadratic form. In the case that \( n = 2 \) this implies closed orbits around the critical point.

The Morse lemma can therefore be used to establish the non-linear character of a critical point in a corresponding non-linear system.

If the critical point is assumed to be non-degenerate (Equation (2.24)), this implies that:

\[
\det A \neq 0
\]  
(2.27)

The eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( A \) can be determined by solving the characteristic equation:

\[
\det A - \lambda I = 0
\]  
(2.28)

The characteristic of the critical point in this linear system is determined by the values of these eigenvalues. Solutions are then of the general form \( x(t) = ce^{\lambda t} \), where \( c \) is a constant. For two-dimensional linear systems the different cases possible are listed below:
Node: The eigenvalues are real and have the same sign. In the case that the eigenvalues are unequal the critical point is a node. If \( \lambda_1, \lambda_2 < 0 \) then the critical point is a positive attractor. If \( \lambda_1, \lambda_2 > 0 \) then the critical point is a negative attractor. If the eigenvalues are equal in the phase-plane the orbits are straight lines through the origin.

Saddle Point: The eigenvalues are real and are of different sign. The orbits in phase-space are hyperbolic.

Focus: The eigenvalues are complex conjugates \( (\lambda_{1,2} = \mu \pm i\omega) \). The orbits in phase-space spiral in and out with respect to the origin. In the case that \( Re(\lambda_{1,2}) < 0 \), the origin is a positive attractor. In the case that \( Re(\lambda_{1,2}) > 0 \), the origin is a negative attractor.

Centre: The eigenvalues are purely imaginary. The orbits in phase-space are circles.

In an \( n \)-dimensional system the number of cases quickly increases. These cases however are generally a combination of the cases illustrated above for two-dimensional systems.

It is possible to define the following types of stability for flow in the neighbourhood of critical points.

**Definition 2.9** Consider Equation (2.1) and a neighbourhood \( D \subset \mathbb{R}^n \) of \( x = 0 \); the solution starting at \( t = t_0 \) in \( x = x_0 \in D \) is indicated by \( x(t; t_0, x_0) \). The solution \( x = 0 \) is called Lyapunov-stable if for each \( \epsilon > 0 \) and \( t_0 \) a \( \delta(\epsilon, t_0) > 0 \) can be found such that \( \|x_0\| \leq \delta \) yields \( \|x(t; t_0, x_0)\| \leq \epsilon \) for \( t \geq t_0 \).

**Definition 2.10** If the critical point \( x = 0 \) of Equation (2.1) is not Lyapunov-stable then it is called unstable.

**Definition 2.11** The critical point \( x = 0 \) of Equation (2.1) is called asymptotically stable if \( x = 0 \) is stable and if there exists a \( \delta(t_0) > 0 \) such that:

\[
\|x_0\| \leq \delta_0 \Rightarrow \lim_{t \to \infty} \|x(t; t_0, x_0)\| = 0 \tag{2.29}
\]

**2.8.2 Periodic Orbits**

Considering Equation (2.1) again, it is possible to extend the definitions of Lyapunov and asymptotic stability, established for critical points, to periodic solutions as well.

**Definition 2.12** Consider Equation (2.1) with periodic solution \( \phi(t) \). The periodic solution is Lyapunov-stable if for each \( t_0 \) and \( \epsilon > 0 \) it is possible to find \( \delta(\epsilon, t_0) > 0 \) such that:

\[
\|x_0 - \phi(t_0)\| \leq \delta \Rightarrow \|x(t; t_0, x_0) - \phi(t)\| \leq \epsilon \tag{2.30}
\]

for \( t \geq t_0 \).

Lyapunov stability for periodic solutions implies that orbits starting in the neighbourhood of the periodic solution remain in the neighbourhood, but in the sense
that the phase points which start out being close together remain near each other. This is quite an exceptional case of stability.

Asymptotic stability for periodic solutions is less straightforward to define. To tackle this issue, stability of periodic solutions is redefined as stability of the Poincaré-map (Section 2.4) in the neighbourhood of \( a \).

**Definition 2.13** Given is Equation (2.2) with periodic solution \( \phi(t) \), transversal \( V \) and Poincaré-map \( P \) with fixed point \( a \). The solution \( \phi(t) \) is stable if for each \( \epsilon > 0 \) it is possible to find \( \delta(\epsilon) \) such that:

\[
\| x_0 - a \| \leq \delta, x_0 \in V \Rightarrow \| P^n(x_0) - a \| \leq \epsilon, n = 1, 2, 3, \ldots
\] (2.31)

A periodic solution exhibiting this type of stability is sometimes referred to as being orbitally stable.

**Definition 2.14** Given is Equation (2.2) with periodic solution \( \phi(t) \), transversal \( V \) and Poincaré-map \( P \) with fixed point \( a \). The solution \( \phi(t) \) is asymptotically stable if it is stable and if there exists a \( \delta > 0 \) such that:

\[
\| x_0 - a \| \leq \delta, x_0 \in V \Rightarrow \lim_{n \to \infty} P^n(x_0) = a
\] (2.32)

These two definitions apply specifically to autonomous equations. Non-autonomous equations are treated by simply rewriting the system of equations with an explicit relation for the independent variable e.g. \( t \). This means that an \( n \)-dimensional non-autonomous system is rewritten as an \( (n + 1) \)-dimensional autonomous system. Hence it follows that the transversal \( V \) is \( n \)-dimensional.

It is now possible to characterise the stability of periodic solutions of the original non-autonomous system.

### 2.8.3 Non-Linear Systems

A number of theories can be stated relating to the transfer of properties from linear systems to the corresponding non-linear systems. Assuming that Equation (2.2) can be rewritten as:

\[
\dot{x} = Ax + g(x)
\] (2.33)

with \( A \) a non-singular \( n \times n \) matrix and that:

\[
\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0
\] (2.34)

then it is possible to state the following theorems:

**Theorem 2.2** If \( x = 0 \) is a positive/negative attractor for the linearised equation, then \( x = 0 \) is a positive/negative attractor for the non-linear equation.

**Theorem 2.3** If matrix \( A \) has an eigenvalue with positive real part, then the critical point \( x = 0 \) is not a positive attractor for the nonlinear equation.
An important theorem can also be stated regarding the existence of stable and unstable manifolds. For a linear system, $E(\lambda)$ is the generalised eigenspace of the $n \times n$ matrix $A$ for the corresponding eigenvalue $\lambda$. A stable manifold of this linear system is defined as the linear subspace $E_s$ of $\mathbb{R}^n$, which is equal to the sum over the generalised eigenspaces with eigenvalues $\lambda$ with negative real part. Similarly, for a linear system an unstable manifold is defined as the linear subspace $E_u$ of $\mathbb{R}^n$, which is equal to the sum over the generalised eigenspaces with eigenvalues $\lambda$ with positive real part.

For the non-linear case the following theorem is an important result:

**Theorem 2.4 (Existence of stable and unstable manifolds)** Consider Equation (2.33) where $A$ is a constant $n \times n$ matrix with $n$ eigenvalues with non-zero real part and $g(x)$ is $C^k$ in the neighbourhood of $x = 0$ with:

$$\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0 \quad (2.35)$$

Then there exists a $C^k$ manifold $W_s$, called the stable manifold of $x = 0$, with the following properties:

1. $0 \in W_s$, $W_s$ has the same dimension as $E_s$ and the tangent space of $W_s$ at $x = 0$ is equal to $E_s$.
2. If we have $x(t_0) \in W_s$ for a solution $x(t)$, then $x(t) \in W_s$ for all $t \geq t_0$ and $\lim_{t \to \infty} x(t) = 0$.
3. If $x(t_0) \notin W_s$ for a solution $x(t)$, then $\|x(t)\| \geq \delta$ for some real, positive $\delta$, for a suitable $t_1 \geq t_0$ and $t \geq t_1$.

Similarly there exists a $C^k$ manifold $W_u$, called the unstable manifold of $x = 0$, with the following properties:

1. $0 \in W_u$, $W_u$ has the same dimension as $E_u$ and the tangent space of $W_u$ at $x = 0$ is equal to $E_u$.
2. If we have $x(t_0) \in W_u$ for a solution $x(t)$, then $x(t) \in W_u$ for all $t \leq t_0$ and $\lim_{t \to -\infty} x(t) = 0$.
3. If $x(t_0) \notin W_u$ for a solution $x(t)$, then $\|x(t)\| \geq \delta$ for some real, positive $\delta$, for a suitable $t_1 \leq t_0$ and $t \leq t_1$.

Proof of this theorem is found in [Hartman, 1964] or [Knobloch and Kappel, 1974].

The following theorems sketch the concept of asymptotic stability and instability in the neighbourhood of $x = 0$ for a non-autonomous, non-linear system. Equations of the following form in $\mathbb{R}^n$ are considered:

$$\dot{x} = Ax + B(t)x + f(t,x), x(t_0) = x_0, t_0 \in \mathbb{R} \quad (2.36)$$

where $A$ is a constant $n \times n$ matrix, $B(t)$ is a continuous $n \times n$ matrix with the property that $\lim_{t \to -\infty} \|B(t)\| = 0$. The vector function $f(t,x)$ is continuous in $t$ and $x$ and Lipschitz continuous in $x$ in a neighbourhood of $x = 0$ with $\lim_{\|x\| \to 0} \frac{\|f(t,x)\|}{\|x\|} = 0$ uniformly in $t$. 

Theorem 2.5 (Poincaré-Lyapunov) If Equation (2.36) is considered such that all eigenvalues of $A$ have negative real part, then there exist positive constants $C$, $t_0$, $\delta$, $\mu$ such that $\|x\| \leq \delta$ implying that $\|x(t)\| \leq C \|x_0\| e^{-\mu(t-t_0)}$, $t \geq t_0$. The critical point $x = 0$ is asymptotically stable and the attraction is exponential in a $\delta$-neighbourhood of this critical point.

Theorem 2.6 If Equation (2.36) is considered such that at least one eigenvalue of $A$ has positive real part, then the solution $x$ is unstable.

Proofs for both of these theorems can be found in [Verhulst, 2000]
In this chapter, the essential aspects of classical orbital mechanics will be considered. Firstly, the fundamental law of gravitation that governs the motion of heavenly bodies will be introduced. The equations of motion for the N-Body Problem (NBP) will then be briefly developed. Finally, the specific case of the 2-Body Problem (2BP) will be addressed. As these fundamentals can be found in most references on orbital mechanics, they will only be briefly outlined. The discussions in this chapter are restricted to the classical formulation of celestial mechanics. The following is adapted from [Prussing and Conway, 1993], [Roy, 1965], [Wakker, 2005].

3.1 Fundamental Laws

3.1.1 Newton’s Laws of Motion

The fundamental laws of motion described by Newton in his Principia form the backbone of classical celestial mechanics. Newton formulated his laws of motion as follows:

First Law  Every particle continues in its state of rest or uniform motion in a straight line relative to an inertial reference frame, unless it is compelled to change that state by forces acting upon it.

Second Law  The time rate of change of linear momentum of a particle relative to an inertial reference frame is proportional to the resultant of all forces acting upon that particle and is collinear with and in the direction of the resultant force.

Third Law  If two particles exert forces on each other, these forces are equal in magnitude and opposite in direction.

Newton’s Second Law can mathematically be formulated as follows:

\[ \vec{F} = m \frac{d\vec{v}}{dt} = m \frac{d^2\vec{r}}{dt^2} \]  \hspace{1cm} (3.1)

Equation (3.1) is only valid for bodies of constant mass and when their motion is considered with respect to an inertial reference frame.
3.1.2 Universal Law of Gravitation

Newton’s observation of the motion of heavenly bodies through the sky led to the formulation of his universal law of gravitation. This law states that:

Two particles attract each other with a force directly proportional to their masses and inversely proportional to the square of the distance between them.

In mathematical notation, the universal law of gravitation reads:

\[
\vec{F}_{ij} = G \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij} \tag{3.2}
\]

where \( \vec{F}_{ij} \) is the gravitational force exerted by body \( j \) on body \( i \), \( m_i \) and \( m_j \) are the masses of the two particles respectively, \( \vec{r}_{ij} \) is the position vector from \( i \) to \( j \), and the proportionality constant \( G \) known as the Universal Gravitational Constant has a value of \( 6.668 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2} \). This law gives the force that body \( j \) exerts on body \( i \), with the outward direction from body \( i \) being positive.

The universal law of gravitation can be rewritten by introducing a scalar potential function:

\[
U_i = -G \frac{m_j}{r_{ij}} + U_{i0} \tag{3.3}
\]

where \( U_{i0} \) is an arbitrary constant. \( U_i \) is therefore the potential of the force field generated by \( m_j \) at the location of \( m_i \) per unit mass of \( m_i \). Then it follows that:

\[
\vec{F}_{ij} = -m_i \nabla_i U_i = -\frac{\partial U}{\partial \vec{r}_i} \tag{3.4}
\]

3.2 N-body Problem

3.2.1 Equations of Motion

In this section, the equations of motion of the N-Body Problem (NBP) will be developed. Using Newton’s Second Law and the Universal Law of Gravitation, it is possible to establish the equations governing the motion of a system of particles.

Consider a system of \( n \) particles illustrated in Figure 3.1. Consider bodies \( i \) and \( j \) with masses \( m_i \) and \( m_j \) and coordinates \( x_i, y_i, z_i \) and \( x_j, y_j, z_j \) respectively. Their position vectors with respect to the origin of the inertial reference frame are given by \( \vec{r}_i \) and \( \vec{r}_j \) respectively. The position vector of body \( j \) with respect to body \( i \) can be represented by:

\[
\vec{r}_{ij} = \vec{r}_j - \vec{r}_i \tag{3.5}
\]

The magnitude of this vector given in the position coordinates of the two bodies is:

\[
r_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} \tag{3.6}
\]
Assuming that the only force governing the motion of the system of \(n\) bodies is gravitation, it is possible to state the equations of motion of the \(i^{th}\) body under the influence of all the other bodies in the system. This is accomplished by equating Newton’s Second Law and the Universal Law of Gravitation:

\[
m_i \frac{d^2 \vec{r}_i}{dt^2} = \sum_j^* G \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij} \tag{3.7}
\]

The notation adopted here for the summation indicates that the summation is taken for all \(j = 1, \ldots, n\) but excludes \(j = i\) (indicated by \(*\)). Equation (3.7) can be written as six scalar first-order differential equations.

In general for a system of \(n\) bodies, a system of \(6n\) first-order differential equations can be written. Such a system completely describes the motion of the \(n\) bodies under the influence of mutual gravitation with respect to an inertial reference frame.

### 3.2.2 Integrals of Motion

Ten independent algebraic integrals of motion for the NBP can be shown to exist [Wakker, 2005]. Six of these integrals are derived from considerations of the linear momentum of the system. A further three integrals are derived from considerations of the total angular momentum of the system. The last integral is derived from energy considerations. For purposes of completeness, these integrals are briefly derived here.

**Linear Momentum** Summing Equation (3.7) over all \(i\):

\[
\sum_i m_i \frac{d^2 \vec{r}_i}{dt^2} = \sum_i \sum_j^* G \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij} \tag{3.8}
\]
The right-hand side of Equation (3.8) is equal to zero, as the double summation means all contributions come in anti-symmetric pairs e.g. \( \bar{r}_{23} = -\bar{r}_{32} \). Hence:

\[
\sum_i m_i \frac{d^2 \bar{r}_i}{dt^2} = \frac{d^2}{dt^2} \left( \sum_i m_i \bar{r}_i \right) = 0 \tag{3.9}
\]

It should also be noted that the position vector of the centre of mass of the system is given by:

\[
\bar{r}_{cm} = \frac{\sum_i m_i \bar{r}_i}{\sum_i m_i} \tag{3.10}
\]

Using Equation (3.10) and integrating Equation (3.9):

\[
\frac{d}{dt} (\bar{r}_{cm}) = \bar{a} \tag{3.11a}
\]

\[
\bar{r}_{cm} = \bar{a}t + \bar{b} \tag{3.11b}
\]

Equations (3.11a) and (3.11b) in scalar form give six integrals of motion relating to the linear momentum (or velocity when the equations are considered per unit mass) and initial (at \( t = 0 \)) position of the centre of mass with respect to an inertial reference frame.

**Angular Momentum** Taking the vector product of Equation (3.7) and \( \bar{r}_i \) and summing over all \( i \):

\[
\sum_i \bar{r}_i \times \left( m_i \frac{d^2 \bar{r}_i}{dt^2} \right) = \sum_i \bar{r}_i \times \left( \sum_j^* G m_i m_j \bar{r}_{ij} \right) \tag{3.12}
\]

The right-hand side of Equation (3.12) is equal to zero, as the contributions from the summation all come in anti-symmetric pairs. Hence:

\[
\frac{d}{dt} \left( \sum_i m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt} \right) = 0 \tag{3.13}
\]

Integrating:

\[
\sum_i m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt} = \bar{H} \tag{3.14}
\]

where \( \bar{H} \) denotes the total angular momentum of the N-Body system. This vector quantity is normal to the invariable plane defined by Laplace that contains the centre of mass. In scalar form this introduces three more integrals of motion.

**Energy** Taking the scalar product of (3.7) and \( \frac{d\bar{r}_i}{dt} \) and summing over all \( i \):

\[
\sum_i \frac{d\bar{r}_i}{dt} \cdot \left( m_i \frac{d^2 \bar{r}_i}{dt^2} \right) = \sum_i \frac{d\bar{r}_i}{dt} \cdot \left( \sum_j^* G m_i m_j \frac{1}{\bar{r}_{ij}^3} \bar{r}_{ij} \right) \tag{3.15}
\]

This can be rewritten as:

\[
\frac{d}{dt} \left( \frac{1}{2} \sum_i m_i \frac{d\bar{r}_i}{dt} \cdot \frac{d\bar{r}_i}{dt} \right) = \sum_i \sum_j^* G m_i m_j \frac{1}{\bar{r}_{ij}^3} \frac{d\bar{r}_i}{dt} \cdot (\bar{r}_j - \bar{r}_i) \tag{3.16}
\]
Algebraic manipulation yields the following:

\[
\frac{d}{dt} \left( \frac{1}{2} \sum_i m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt} \right) = \frac{d}{dt} \left( \frac{1}{2} G \sum_i \sum_j^* \frac{m_i m_j}{r_{ij}} \right) \tag{3.17}
\]

After integration:

\[
\sum_i \frac{1}{2} m_i \left( \frac{d\vec{r}_i}{dt} \right)^2 - \frac{1}{2} G \sum_i \sum_j^* \frac{m_i m_j}{r_{ij}} = E_T \tag{3.18}
\]

Equation (3.18) can be rewritten as:

\[
E_k + E_p = E_T \tag{3.19}
\]

where \( E_k \) describes the total kinetic energy and \( E_p \) describes the total potential energy of the N-Body system in an inertial reference frame. Equation (3.19) illustrates the fact that the sum of the total kinetic and potential energy of the system remains constant. This is the tenth and last integral of motion.

### 3.3 2-Body Problem

The 2BP of celestial mechanics is well understood. The fundamental equations of motion read:

\[
\frac{d^2\vec{r}_1}{dt^2} = G \frac{m_2}{r_{12}^3} \vec{r}_{12} \tag{3.20a}
\]

\[
\frac{d^2\vec{r}_2}{dt^2} = G \frac{m_1}{r_{21}^3} \vec{r}_{21} \tag{3.20b}
\]

From the above equations it can be seen that the motion of both Body 1 and 2 is only dependent on the mass of the other body in the system. This system of equations describes individually the motion of Body 1 and Body 2 with respect to an inertial reference system with arbitrary origin. By centring this reference system about Body 1, it is possible to rewrite this system of equations as a single equation. This is achieved by subtracting (3.20b) from (3.20a):

\[
\frac{d^2\vec{r}_1}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = G \frac{m_2}{r_{12}^3} \vec{r}_{12} - G \frac{m_1}{r_{21}^3} \vec{r}_{21} \tag{3.21}
\]

If \( \vec{r} \equiv \vec{r}_1 - \vec{r}_2 \) and \( \vec{r}_{12} = -\vec{r}_{21} \) then:

\[
\frac{d^2\vec{r}}{dt^2} = -G \frac{m_1 + m_2}{r^3} \vec{r} \tag{3.22}
\]

This gives the relative motion of Body 1 with respect to Body 2. Introducing \( \mu \equiv G(m_1 + m_2) \) yields:

\[
\frac{d^2\vec{r}}{dt^2} = -\frac{\mu}{r^3} \vec{r} \tag{3.23}
\]
This second order differential equation has a well-known, closed-form solution:

$$r = \frac{p}{1 + e \cos(\theta)}$$

(3.24)

where \(r\) indicates the distance between Body 1 and Body 2, \(p\) is known as the semi-latus rectum, \(e\) is the eccentricity, and \(\theta\) is the true anomaly. The motion of Body 1 can be shown to be confined to a single plane using the integrals of motion derived for the NBP. If it is assumed that \(m_2 \ll m_1\), then it follows that, to good approximation, the location of \(m_1\) is fixed at one of the foci of the ellipse, whilst \(m_2\) travels along the ellipse. Equation (3.24) is the equation of a conic section and describes motion in an ellipse, providing that \(0 < e < 1\), or a circle if \(e = 0\). Depending on the numerical value of \(e\), the solution given by Equation (3.24) can change slightly to accommodate motion in a parabola \((e = 1)\) or hyperbola \((e > 1)\).

The formulation of the solution for these two cases is essentially the same. Figure 3.2 illustrates the case of elliptical motion.

![Figure 3.2 Parameters of the motion of Body 1 relative to Body 2, described by the equation of a conic section.](image)

It is noted that in the case of the 2BP relative motion of the two bodies can be demonstrated in closed-form by analytically solving the second-order nonlinear differential equation (Equation (3.23)). Two properties of Kepler orbits with \(0 \leq e < 1\) will be outlined here.

### 3.3.1 Kepler Energy

Using Equation (3.23), it is possible to derive a law of conservation of energy for the 2BP. The scalar product of Equation (3.23) and \(\frac{d\vec{r}}{dt}\) yields:

$$\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} + \frac{\mu}{r^3} \frac{d\vec{r}}{dt} \cdot \vec{r} = 0$$

(3.25)
This can be rewritten as follows:

\[
\frac{1}{2} \frac{d}{dt} \left( \frac{d\bar{r}}{dt} \cdot \frac{d\bar{r}}{dt} \right) + \frac{1}{2} \frac{\mu}{r^3} \frac{d}{dt} (\bar{r} \cdot \bar{r}) = \frac{1}{2} \frac{d}{dt} (V^2) - \frac{d}{dt} \left( \frac{\mu}{r} \right) = 0
\]  

(3.26)

Integrating this equation yields an energy conservation law:

\[
\frac{1}{2} V^2 - \frac{\mu}{r} = E_T
\]  

(3.27)

This conservation law describes the fact that for the 2BP the total energy, which consists of contributions of kinetic and potential energy, remains constant.

For elliptical and circular orbits it can be shown that:

\[
E_T = \frac{1}{2} V^2 - \frac{\mu}{r} < 0
\]  

(3.28)

For parabolic orbits it can be shown that:

\[
E_T = \frac{1}{2} V^2 - \frac{\mu}{r} = 0
\]  

(3.29)

For hyperbolic orbits it can be shown that:

\[
E_T = \frac{1}{2} V^2 - \frac{\mu}{r} > 0
\]  

(3.30)

The reader is referred to [Wakker, 2005] for a complete derivation of these conditions.

### 3.3.2 Angular Momentum

A conservation law for angular momentum can be derived for the 2BP by taking the vector product of Equation (3.23) and \( \bar{r} \), yielding:

\[
\bar{r} \times \frac{d^2\bar{r}}{dt^2} + \bar{r} \times \frac{\mu}{r^3} = 0
\]

\[
\frac{d}{dt} \left( \bar{r} \times \frac{d\bar{r}}{dt} \right) = 0
\]

(3.31)

Integrating Equation (3.31) yields:

\[
\bar{r} \times \bar{V} = \bar{H}
\]

(3.32)

Equation (3.32) shows that in the 2BP the angular momentum remains constant. This also follows from the general angular momentum condition derived for the NBP (Equation (3.14)).
Chapter 4

Circular Restricted 3-Body Problem

The 3-Body Problem (3BP) in celestial mechanics has been studied by Euler, Lagrange, Poincaré [Barrow-Green, 1997] and many others. Such studies have given rise to a new understanding of dynamical systems in general. Given the difficulties involved in analysing the general 3BP, various simplified versions have been studied. In this chapter a simplified model, often the starting point for many discussions relating to the 3BP will be recalled. The equations of motion for the 3BP read (derived from Equation (3.7)):

\[
\begin{align*}
\frac{d^2 \vec{r}_1}{dt^2} &= G \frac{m_2}{r_{12}^3} \vec{r}_{12} + G \frac{m_3}{r_{13}^3} \vec{r}_{13} \\
\frac{d^2 \vec{r}_2}{dt^2} &= G \frac{m_1}{r_{21}^3} \vec{r}_{21} + G \frac{m_3}{r_{23}^3} \vec{r}_{23} \\
\frac{d^2 \vec{r}_3}{dt^2} &= G \frac{m_1}{r_{31}^3} \vec{r}_{31} + G \frac{m_2}{r_{32}^3} \vec{r}_{32}
\end{align*}
\] 

(4.1a) (4.1b) (4.1c)

Using these equations, the Planar Circular Restricted 3-Body Problem (PCR3BP) will be studied. The PCR3BP is succinctly described as follows [Szebehely, 1967]:

Two bodies revolve around their centre of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of three bodies is to describe the motion of this third body.

The equations of motion for this model will be derived in two different ways. Using these two different approaches, various characteristics of the system will be explored. The Newton approach, outlined in Section 4.1, can be found in most standard works on celestial mechanics such as [Szebehely, 1967], [Wakker, 2005], and [Roy, 1965]. The development of the equations of motion based on this approach will to a large extent follow [Szebehely, 1967]. The Hamiltonian approach, outlined in Section 4.2, is more abstract. This development of the equations of motion based on this approach will to a large extent follow [Koon et al., 2006]. A local regularisation of the equations of motion of the CR3BP will be presented in Section 4.3. This is necessary to be able to study impact trajectories and nearly-impact trajectories.

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4.1 Newton Approach

The CR3BP problem will be approached here using the classical Newton formulation. The two main bodies in this problem, \( m_1 \) and \( m_2 \) respectively, will be referred to as the primaries. These bodies can both be represented as point masses. Analysis of the 2-Body Problem (2BP) indicates that the motion of these two bodies is restricted to a single plane (conservation of angular momentum). The third body has mass \( m \) such that it does not influence the motion of the primaries.

4.1.1 Equations of Motion

An inertial, rectangular, reference frame is chosen such that motion of the primaries is contained in the \( XY \)-plane, with the origin being located at the gravitational barycentre. The equations of motion of the third body, \( m \), in this inertial reference frame is then given by:

\[
\frac{d^2 X}{dt^2} = -G \left[ \frac{m_1}{R_1^3} (X - X_1) + \frac{m_2}{R_2^3} (X - X_2) \right] \tag{4.2a}
\]

\[
\frac{d^2 Y}{dt^2} = -G \left[ \frac{m_1}{R_1^3} (Y - Y_1) + \frac{m_2}{R_2^3} (Y - Y_2) \right] \tag{4.2b}
\]

\[
\frac{d^2 Z}{dt^2} = -G \left[ \frac{m_1}{R_1^3} X + \frac{m_2}{R_2^3} X \right] \tag{4.2c}
\]

The distances \( R_1 \) and \( R_2 \) describe the relative separation distances between \( m \) and the primaries. These distances are given by:

\[
R_1^2 = (X - X_1)^2 + (Y - Y_1)^2 + Z^2 \tag{4.3a}
\]

\[
R_1^2 = (X - X_2)^2 + (Y - Y_2)^2 + Z^2 \tag{4.3b}
\]

In these equations, the coordinates of the primaries are given by \((X_1, Y_1)\) and \((X_2, Y_2)\), for \( m_1 \) and \( m_2 \) respectively.

The circular motion of \( m_1 \) and \( m_2 \) about their common centre of mass requires balance between the gravitational and centrifugal forces:

\[
G \frac{m_1 m_2}{R_{12}^2} = m_1 R_{10} n^2 = m_2 R_{20} n^2 \tag{4.4}
\]

where \( G \) is the Universal Gravitational constant, \( n \) is the common angular velocity or mean motion of \( m_1 \) and \( m_2 \), \( R_{12} \) is their separation distance, and \( R_{10} \) and \( R_{20} \) are their respective distances from an arbitrary origin. From Equation (4.4) it follows that:

\[
G (m_1 + m_2) = n^2 R_{12}^2 \tag{4.5}
\]

This is Kepler’s Third Law. Based on this, the angular velocity of the circular motion of the primaries, \( n \), is normalised to 1. In addition, the distance of separation between the two primaries, \( R_{12} \), is also normalised to 1. A mass parameter is also introduced to describe this system: \( \mu = \frac{m_2}{m_1 + m_2} \). The mass of \( m_1 \) is given in dimensionless units by \( 1 - \mu \), whilst the mass of \( m_2 \) is given by \( \mu \). By convection, the mass parameter must also satisfy \( 0 \leq \mu \leq \frac{1}{2} \). This is due to the symmetry
of the masses. It then follows from Equation (4.5) that in normalised coordinates $G = 1$.

To simply the description of this system further, a rotating reference frame is introduced. Figure 4.1 illustrates a rotating reference frame $(xyz)$ in addition to the inertial reference frame $(XYZ)$. This reference frame rotates at the same angular velocity ($n$) as the primaries do about their common gravitational barycentre, which is located at the origin of both reference frames. The rotating and inertial reference frames are chosen to coincide at $t = t_0$. In the rotating reference frame, the primaries are located on the $x$-axis: in other words, in the rotating reference frame, the primaries are seen to be fixed. By introducing this reference frame, it is possible to rewrite the equations of motion in a form that reveals an important integral of motion (Jacobi’s integral derived in Section 4.1.2). The $Z$-

![Diagram of inertial and rotating reference frames](image)

Figure 4.1 Definition of inertial (XYZ) and rotating (xyz) reference frames. The rotating reference frame rotates counter-clockwise with respect to the inertial reference frame about a common origin. and $z$-axes extend out of the page. It turns out that the equations of motion for the three-dimensional (spatial) CR3BP case can be easily extended from the two-dimensional (planar) case. Hence, the derivation outlined here will pertain to the planar CR3BP (PCR3BP). The final result for the spatial case will also be provided.

From the conservation of linear momentum for the N-Body Problem (NBP) (Section 3.2.2) it follows that, in the rotating reference frame, $m_1$ is located at $(\mu, 0)$ and $m_2$ is located at $(1 - \mu, 0)$. All distances in the rotating reference frame are scaled based on the distance of separation between the two primaries. Since the angular velocity is normalised to 1, it follows that the angular separation at any given point in time between the two rotating reference frames is given by $\theta = t$, with $\theta_0 = t_0 = 0$.

The coordinate transformation to transform coordinates in the rotating reference frame to coordinates in the inertial reference frame follows from geometric considerations. Using Figure 4.1, it can be seen that the coordinate transformation can be written as:

$$X = x \cos t - y \sin t$$  \hspace{1cm} (4.6a)  

$$Y = x \sin t + y \cos t$$  \hspace{1cm} (4.6b)
This can be rewritten in matrix notation as:
\[
\begin{bmatrix}
X \\
Y
\end{bmatrix} =
\begin{bmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\] (4.7)

The transformation of velocity components is obtained by differentiating Equation (4.7) with respect to time. This yields:
\[
\begin{bmatrix}
\dot{X} \\
\dot{Y}
\end{bmatrix} =
\begin{bmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{bmatrix}
\begin{bmatrix}
\dot{x} - \dot{y} \\
\dot{y} + \dot{x}
\end{bmatrix}
\] (4.8)

Figure 4.2 illustrates the description of the problem in the rotating reference frame. To simplify the manipulation of the equations of motion using the transformations given by Equations (4.7) and (4.8), complex notation is introduced:
\[W = we^{it}\] (4.9)

where \(W = X + iY\) and \(w = x + iy\). Equations (4.2a) and (4.2b) can be written compactly as:
\[
\frac{d^2W}{dt^2} = - \left[ \frac{1 - \mu}{R_1^3}(W - W_1) + \frac{\mu}{R_2^3}(W - W_2) \right]
\] (4.10)

Using Equation (4.9) it is possible to rewrite Equation (4.10) in the rotating system as:
\[
\frac{d^2w}{dt^2} + 2i \frac{dw}{dt} - w = - \left[ \frac{(1 - \mu)}{|w - w_1|^3}(w - w_1) + \frac{\mu}{|w - w_2|^3}(w - w_2) \right]
\] (4.11)

The location of the primaries in the rotating reference frame, \(w_1\) and \(w_2\), are given by \(x_1\) and \(x_2\) respectively (\(y_1 = y_2 = 0\)). The separation distances between the bodies show no explicit dependence on time in the rotating system. The real and imaginary parts give the equations of motion in the \(x\) and \(y\) directions respectively for the rotating reference frame:
\[
\frac{d^2x}{dt^2} - 2 \frac{dy}{dt} - x = - \left[ \frac{(1 - \mu)}{r_1^3}(x - x_1) + \frac{\mu}{r_2^3}(x - x_2) \right]
\] (4.12a)
\[
\frac{d^2y}{dt^2} + 2\frac{dx}{dt} - y = - \left[ \frac{(1 - \mu)}{r_1^3} y + \frac{\mu}{r_2^3} y \right]
\]  
(4.12b)

It is clear that the right-hand side of these equations no longer shows explicit
dependence on time.

The equations of motion for the planar problem can be written compactly as:
\[
\ddot{x} - 2\dot{y} = \Omega_x \\
\ddot{y} + 2\dot{x} = \Omega_y
\]  
(4.13a)

(4.13b)

where the dots denote derivatives with respect to dimensionless time \(t\) and:
\[
\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1 - \mu)
\]  
(4.14)

where \(r_1^2 = (x + \mu)^2 + y^2\) and \(r_2^2 = (x + \mu - 1)^2 + y^2\). The constant term
\(\frac{1}{2}\mu(1 - \mu)\) is arbitrary, however is included as per convention [Belbruno et al.,
2008]. The equations of motion for the three-dimensional case are given by Equations
(4.13a) and (4.13b), and:
\[
\ddot{z} = \Omega_z
\]  
(4.15)

In this case, \(r_1^2 = (x + \mu)^2 + y^2 + z^2\) and \(r_2^2 = (x + \mu - 1)^2 + y^2 + z^2\).

4.1.2 An Invariant Relation

It can be shown that Equations (4.13a) and (4.13b) possess an integral of motion,
if they are multiplied by \(\frac{dx}{dt}\) and \(\frac{dy}{dt}\) respectively, summed, and integrated with
respect to \(t\):
\[
\frac{1}{2} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right] = \int_{t_0}^{t} \left( \frac{\partial\Omega}{\partial x} \frac{dx}{dt} + \frac{\partial\Omega}{\partial y} \frac{dy}{dt} \right)
\]  
(4.16)

Since \(\Omega\) is not explicitly a function of \(t\):
\[
d\Omega = \frac{\partial\Omega}{\partial x} dx + \frac{\partial\Omega}{\partial y} dy
\]  
(4.17)

Hence:
\[
\frac{1}{2} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right] = \Omega - \frac{C}{2}
\]  
(4.18)

This is known as the Jacobi integral and is the only known integral in the PCR3BP.
It can also be written as:
\[
V^2 = 2\Omega - C
\]  
(4.19)

where \(V\) is the total velocity of \(m\) in the rotating reference frame and \(C\) is a con-
stant of integration denoted the Jacobi constant. The value of \(C\) is fully determined
by the position and velocity of \(m\) at \(t = 0\).

The Jacobi integral is a combination of the energy and angular momentum
integrals of the N-Body Problem (NBP). In literature it is sometimes stated as an
energy surface:
\[
E = \frac{1}{2} V^2 - \Omega = -\frac{1}{2} C
\]  
(4.20)

There are no other integrals to constrain the motion of a body in the PCR3BP;
the PCR3BP is an analytically non-integrable problem.
4.1.3 Surfaces of Hill

If $m$ has zero velocity, the Jacobi integral becomes:

$$2\Omega = C \quad (4.21)$$

In other words:

$$\left(x^2 + y^2\right) + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} + \frac{1}{2} \mu(1 - \mu) = C \quad (4.22)$$

This equation describes the Surfaces of Hill, which are surfaces in $\mathbb{R}^2$ in the PCR3BP on which the velocity of $m$ is zero. Accordingly, these surfaces are also called zero-velocity surfaces. Since it is necessary that $V^2 \geq 0$ for any real body, it follows that:

$$\left(x^2 + y^2\right) + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} \geq C \quad (4.23)$$

Equation (4.23) defines, for a given value of $C$, the boundaries of the regions in which $m$ must be found. It is stressed that though Equation (4.23) reflects the regions that are accessible to $m$, it does not yield information about its orbit.

To study the qualitative properties of these surfaces, the projection on the $xy$-plane is considered. Qualitative analysis of these surfaces for different values of $C$ yields an illustration of the nature of the PCR3BP. Here, five cases will be considered. Figure 4.3 illustrates the first four cases. The regions shaded in white are accessible to $m$. In this figure, each case is a function of the value of $E$; this is related to the Jacobi Constant through Equation (4.20).

**Case (a):** If the value of $E$ is less than $E_1$, the surfaces are closed. If $m$ is initially in the vicinity of $m_1$, it will remain within a region around $m_1$. A similar scenario exists if $m$ is initially located in the vicinity of $m_2$. Additionally, if $m$ is located in the exterior realm, it cannot penetrate through to the interior realms in the neighbourhood of $m_1$ and $m_2$.

**Case (b):** For a value of $E$ such that $E_1 < E < E_2$ the interior realms associated with $m_1$ and $m_2$ merge. The point at which these surfaces first touch is denoted the L1 Lagrange point and will be discussed in further detail in the following section. As the interior realms grow, the boundary with the outer realm shrinks. In such a case it is possible for $m$, if located initially in the vicinity of either of the primaries, to travel to a neighbourhood of the other primary. However, it is still not possible in such a scenario to bridge the interior and exterior realms.

**Case (c):** For a value of $E$ such that $E_2 < E < E_3$ the interior and exterior realms are bridged as an opening exists. The point at which the interior and exterior realms meet is denoted the L2 Lagrange point. Now $m$ can move away from a neighbourhood around the primaries to the exterior realm of space.

**Case (d):** For a value of $E$ such that $E_3 < E < E_4 = E_5$ the interior and exterior realms touch at another point, denoted the L3 Lagrange point. The two realms merge further.
4.1 Newton Approach

Figure 4.3  Surfaces of Hill projected on $xy$-plane for increasing values of $E$. The shaded regions indicate the forbidden realms (regions inaccessible to the third body in the system) [Koon et al., 2006].

**Case (e):** For a value of $E$ equal to $E_5$ the forbidden realms shrink and disappear in two points, denoted the L4 and L5 Lagrange points. This case is not illustrated in Figure 4.3, as the forbidden regions disappear, meaning that there are no regions of phase space that are inaccessible to $m$.

The existence and location of the five Lagrange points are outlined in the next section.

4.1.4 Lagrange Libration Points

Classically it is shown that there are five equilibrium points that exist in the PCR3BP. To determine and locate these equilibrium points the following is assumed:

\[ \ddot{x} = \dot{y} = 0 \quad (4.24a) \]
\[ \dot{x} = \dot{y} = 0 \quad (4.24b) \]

From these assumptions it follows from Equation (4.23) that:

\[ \Omega_x = 0 \quad (4.25a) \]
\[ \Omega_y = 0 \quad (4.25b) \]
This implies that:

\[ x - (1 - \mu) \frac{(x + \mu)}{r_1^3} - \mu \frac{(x + \mu - 1)}{r_2^3} = 0 \]  
(4.26a)

\[ y \left( 1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} \right) = 0 \]  
(4.26b)

The first of these equations is quintic and coupled; it cannot be solved in closed-form. Through iterative techniques it is possible to determine that there are five equilibrium points with coordinates determined by the solutions of these equations. There are many such iterative schemes that can be used. The reader is referred to [Szebehely, 1967] for further information on possible solution methods.

The five equilibrium points are denoted the Lagrange (libration) points. A body located at these points will not experience acceleration with respect to the rotating reference frame. These five points are denoted: L1, L2, L3, L4 and L5. L1, L2, and L3 all lie on the x-axis: L2 is located to the right of \( m_2 \), L3 is located to the left of \( m_1 \), and L1 is located between \( m_1 \) and \( m_2 \) and is situated closer to \( m_2 \). L4 and L5 lie symmetrically above and below the x-axis such that an equilateral triangle is formed with the primaries. The location of these Lagrange points is illustrated in Figure 4.4.

![Figure 4.4 Projection of the location of the five Lagrange points on the xy-plane, in the rotating reference frame [Koon et al., 2006].](image)

To compute the location of the collinear Lagrange points \((y = 0)\) requires the solution of:

\[ x - \frac{1 - \mu}{(x + \mu)^2} - \frac{\mu}{(x + \mu - 1)^2} = 0 \]  
(4.27)
4.1 Newton Approach

as \( y = 0 \). The iterative methods suggested in [Szebehely, 1967] provide insight into the location of these points using series expansions.

The computation of the location of \( L_4 \) and \( L_5 \) is more straightforward. It can be shown that for these points \( r_1 = r_2 = 1 \) [Wakker, 2005]. In other words:

\[
\begin{align*}
    r_1^2 &= 1 = (x + \mu)^2 + y^2 \quad (4.28a) \\
    r_2^2 &= 1 = (x + \mu - 1)^2 + y^2 \quad (4.28b)
\end{align*}
\]

Solving for \( x \) and \( y \) yields:

\[
\begin{align*}
    x &= \frac{1}{2} - \mu \quad (4.29a) \\
    y &= \pm \frac{1}{2}\sqrt{3} \quad (4.29b)
\end{align*}
\]

4.1.5 Stability in the Lagrange Libration Points

Stability in these five Lagrange points can be investigated by linearising the equations of motion in their neighbourhood. Linearisation of Equations (4.13a) and (4.13b) leads to the following system of equations:

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}_x \\
\dot{y} \\
\dot{v}_y 
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{\partial^2 \Omega}{\partial x^2} & 0 & \frac{\partial^2 \Omega}{\partial x \partial y} & 2 \\
0 & 0 & 0 & 1 \\
\frac{\partial^2 \Omega}{\partial y \partial x} & -2 & \frac{\partial^2 \Omega}{\partial y^2} & 0 
\end{bmatrix}
\begin{bmatrix}
x \\
v_x \\
y \\
v_y 
\end{bmatrix}
\]

where \( v_x = \dot{x} \) and \( v_y = \dot{y} \). The characteristic equation for this system is obtained by evaluating the following determinant:

\[
\begin{vmatrix}
-\lambda & 1 & 0 & 0 \\
\frac{\partial^2 \Omega}{\partial x^2} & -\lambda & \frac{\partial^2 \Omega}{\partial x \partial y} & 2 \\
0 & 0 & -\lambda & 1 \\
\frac{\partial^2 \Omega}{\partial y \partial x} & -2 & \frac{\partial^2 \Omega}{\partial y^2} & -\lambda 
\end{vmatrix}
\]

This leads to the following characteristic equation:

\[
\lambda^4 + \lambda^2 \left( 4 - \frac{\partial^2 \Omega}{\partial x^2} - \frac{\partial^2 \Omega}{\partial y^2} \right) + \frac{\partial^2 \Omega}{\partial x^2} \frac{\partial^2 \Omega}{\partial y^2} - \left( \frac{\partial^2 \Omega}{\partial x \partial y} \right)^2 = 0 \quad (4.32)
\]

A note is made here that the commutativity of the mixed second derivatives is assumed. Equation (4.32) is a quadratic equation in \( \lambda^2 \). The solution hereof is given by:

\[
\alpha_1^2 = \frac{-(4 - \Omega_{xx} - \Omega_{yy}) + \sqrt{(4 - \Omega_{xx} - \Omega_{yy})^2 - 4(\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2)}}{2} \quad (4.33a)
\]
\[ \alpha_2^2 = \frac{-(4 - \Omega_{xx} - \Omega_{yy}) - \sqrt{(4 - \Omega_{xx} - \Omega_{yy})^2 - 4(\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2)}}{2} \]  
(4.33b)

Hence it follows that the eigenvalues of this problem are:

\[ \lambda_1 = \alpha_1 \]  
(4.34a)

\[ \lambda_2 = -\alpha_1 \]  
(4.34b)

\[ \lambda_3 = \alpha_2 \]  
(4.34c)

\[ \lambda_4 = -\alpha_2 \]  
(4.34d)

It is clear that these eigenvalues come in conjugate pairs. To evaluate these eigenvalues the second derivatives of \( \Omega(x, y) \) with respect to \( x \) and \( y \) must be calculated.

These derivatives are given below:

\[ \Omega_{xx} = 1 - \frac{1 - \mu}{r_1^3} + 3 \frac{1 - \mu}{r_1^5} (x + \mu)^2 - \frac{\mu}{r_2^3} (x + \mu - 1)^2 \]  
(4.35a)

\[ \Omega_{yy} = 1 - \frac{1 - \mu}{r_1^3} + 3 \frac{1 - \mu}{r_1^5} y^2 - \frac{\mu}{r_2^3} + 3 \frac{\mu}{r_2^5} y^2 \]  
(4.35b)

\[ \Omega_{xy} = 3y \left( \frac{1 - \mu}{r_1^5} (x + \mu) + \frac{\mu}{r_2^3} (x + \mu - 1) \right) \]  
(4.35c)

For each of the Lagrange points these derivatives can be evaluated; hence the eigenvalues can be evaluated. Using these eigenvalues, the corresponding eigenvectors \( (k_i) \) can be calculated by solving the following system:

\[
\begin{bmatrix}
-\lambda_i & 1 & 0 & 0 \\
\frac{\partial^2 \Omega}{\partial x^2} & -\lambda_i & \frac{\partial^2 \Omega}{\partial x \partial y} & 2 \\
0 & 0 & -\lambda_i & 1 \\
\frac{\partial^2 \Omega}{\partial y \partial x} & -2 & \frac{\partial^2 \Omega}{\partial y^2} & -\lambda_i
\end{bmatrix}_{(x,y)=L_j(x,y)} \begin{bmatrix} k_{i,1} \\ k_{i,2} \\ k_{i,3} \\ k_{i,4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]  
(4.36)

where \( i = 1, \ldots, 4 \) and \( j = 1, \ldots, 5 \). This system is evaluated at each Lagrange point, for each eigenvalue, yielding a set of 20 eigenvectors. The general solution of the linear system can be formulated using these eigenvalues and corresponding eigenvectors as follows, provided that the four eigenvalues obtained are uniquely different and noting that they come in conjugate pairs:

\[ x(t) = A_1e^{\lambda_1 t} + A_2e^{-\lambda_1 t} + A_3e^{\lambda_3 t} + A_4e^{-\lambda_3 t} \]  
(4.37a)

\[ y(t) = B_1e^{\lambda_1 t} + B_2e^{-\lambda_1 t} + B_3e^{\lambda_3 t} + B_4e^{-\lambda_3 t} \]  
(4.37b)

This provides an approximation of the solution in the neighbourhood of the Lagrange points. Since there is no closed-form analytical expression for the location of the collinear Lagrange points, this system cannot be solved exactly. However by qualitative considerations, the nature of the linear solution in the neighbourhood of these Lagrange points can be explored.
4.1 Newton Approach

Collinear Lagrange Points

For the collinear points the following holds:

\[ y = 0 \] (4.38a)

\[ r_1^2 = (x + \mu)^2 \] (4.38b)

\[ r_2^2 = (x + \mu - 1)^2 \] (4.38c)

Therefore it follows by substitution that:

\[ \Omega_{xx} = 1 + 2K \] (4.39a)

\[ \Omega_{yy} = 1 - K \] (4.39b)

\[ \Omega_{xy} = 0 \] (4.39c)

where:

\[ K = \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} \] (4.40)

Hence it follows that the eigenvalues are given by:

\[ \lambda^2 = \frac{(K - 2) \pm \sqrt{K(9K - 8)}}{2} \] (4.41)

It follows from Equation (4.40) that \( K > 0 \). For stable equilibrium i.e. for the equilibrium point considered to be a positive attractor, as stated in Section 2.8.1, the real part of the four eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) must be negative. However, since these eigenvalues come in conjugate pairs it follows that they must be purely imaginary. This implies that \( \lambda^2 < 0 \). In other words:

\[ Re \left( \frac{(K - 2) \pm \sqrt{K(9K - 8)}}{2} \right) < 0 \] (4.42)

\[ Im \left( \frac{(K - 2) \pm \sqrt{K(9K - 8)}}{2} \right) = 0 \] (4.43)

The imaginary part of \( \lambda^2 = 0 \) if:

\[ K(9K - 8) \geq 0 \] (4.44)

This leads to the condition that \( K \geq \frac{8}{9} \). If \( K \geq \frac{8}{9} \) then it follows that:

\[ (K - 2) \pm \sqrt{K(9K - 8)} < 0 \] (4.45)

The case where the contribution of the root term is positive yields:

\[ (K - 2) + \sqrt{K(9K - 8)} < 0 \] (4.46)

Based on this, it is possible to state a strong condition. If this inequality is algebraically manipulated, it follows that:

\[ (2K + 1)(K - 1) < 0 \] (4.47)
Since $K$ is positive, it follows that this inequality can only be true if $K < 1$. For a negative contribution of the root, Equation (4.45) is satisfied for all $K \geq \frac{8}{9}$. For the equilibrium point to be a positive attractor however, it must hold that $\lambda^2 < 0$ for both solutions. Therefore, the constraints placed on $K$ are summarised as:

$$\frac{8}{9} < K < 1 \quad (4.48)$$

The location of the collinear Lagrange points, given by Equation (4.26a), can be rewritten as:

$$x \left(1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3}\right) - \frac{\mu(1 - \mu)}{r_1^3} + \frac{\mu(1 - \mu)}{r_2^3} = 0 \quad (4.49)$$

This can then be rewritten as:

$$1 - K = \frac{\mu(1 - \mu)}{x} \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \quad (4.50)$$

It is clear from this equation that for L1, L2 and L3 (locations given in Figure 4.4):

$$1 - K < 0 \quad (4.51)$$

Hence $K > 1$, which contradicts the requirement stated in Equation (4.48) necessary for equilibrium point considered to be stable. Hence, by negation it follows that the collinear equilibrium points must all be unstable.

**Equilateral Lagrange Points**

For the equilateral Lagrange points, $L_4$ and $L_5$, stability analysis is straightforward, as the locations of these equilibrium points is known in closed-form. The conditions for these Lagrange points are summarised as:

$$x = \frac{1}{2} - \mu \quad (4.52a)$$

$$y = \pm \frac{1}{2} \sqrt{3} \quad (4.52b)$$

$$r_1 = r_2 = 1 \quad (4.52c)$$

Using these relations it follows that:

$$\Omega_{xx} = \frac{3}{4} \quad (4.53a)$$

$$\Omega_{yy} = \frac{9}{4} \quad (4.53b)$$

$$\Omega_{xy} = \pm \frac{3}{4} \sqrt{3}(1 - 2\mu) \quad (4.53c)$$

Substituting into Equation (4.33a) yields:

$$\lambda^2 = \frac{-1 \pm \sqrt{1 - 2\sqrt{3}\mu(1 - \mu)}}{2} \quad (4.54)$$

For $\lambda^2 < 0$ it follows that the $Im(\lambda^2) = 0$, which means that:

$$1 - 2\sqrt{3}\mu(1 - \mu) \geq 0 \quad (4.55)$$
For \( \Re(\lambda^2) < 0 \) it follows that:

\[
-1 + \sqrt{1 - 27\mu(1 - \mu)} < 0
\]

(4.56)

Given that \( 0 \leq \mu \leq \frac{1}{2} \), Equation (4.55) yields:

\[
\mu \leq \frac{1}{2} - \frac{\sqrt{621}}{54}
\]

(4.57)

Substituting the bounds \( 0 < \mu \leq \frac{1}{2} - \frac{\sqrt{621}}{54} \approx 0.0385 \) into Equation (4.56) indicates that this inequality is always satisfied. Hence, for these bounds it is clear that the \( L_4 \) and \( L_5 \) Lagrange points are stable equilibrium points. For \( \mu > \frac{1}{2} - \frac{\sqrt{621}}{54} \) these Lagrange points are unstable.

### 4.1.6 Motion about the Lagrange Libration Points

The motion of a small mass \( m \) in the neighbourhood of the Lagrange points can be described by the linearised system given by Equation (4.30). By analysing the eigenvalues of this system further, it is possible to sketch the nature of the motion of this body in the vicinity of the Lagrange points. This will be done in turn for the collinear and the equilateral equilibrium points.

#### Collinear Lagrange Points

From the results from the previous section, it follows that for the collinear Lagrange points \( \Omega_{xx} = 1 + 2K > 3, \Omega_{yy} = 1 - K < 0, \Omega_{xy} = 0 \). The characteristic equation of the linearised system is given by Equation (4.32). This can be rewritten as:

\[
\lambda^4 + (2 - K)\lambda^2 - (2K + 1)(K - 1) = 0
\]

(4.58)

The roots of this equation are given by:

\[
\lambda^2 = \frac{K - 2}{2} \pm \sqrt{\frac{(K - 2)^2 + 4(2K + 1)(K - 1)}{2}}
\]

(4.59)

Introducing \( \alpha = 1 - \frac{1}{2}K \) and \( \beta^2 = (1 + 2K)(1 - K) \), this equation can be rewritten as:

\[
\lambda^2 = -\alpha \pm \sqrt{\alpha^2 + \beta^2}
\]

(4.60)

From this it follows that the two roots of Equation (4.59) are real and opposite in sign. Hence, the eigenvalues associated with the linearised system are purely real and opposite in sign and purely imaginary and opposite in sign respectively. The real roots \( (\lambda_1, \lambda_2) \) therefore yield exponential increase and decrease of the general solution, whereas the imaginary roots \( (\lambda_3, \lambda_4) \) yield periodic motions. The structure of the general solution is given by Equations (4.37a) and (4.37b).

These equations demonstrate a linearised approximation of the motion of the third body in the neighbourhood of the collinear Lagrange points. The coefficients \( A_i \) and \( B_i \) in these equations are not independent. By substituting the general solution form \( x(t) = Ae^{\lambda t} \) and \( y(t) = Be^{\lambda t} \) into Equation (4.30), it follows that:

\[
B_i = \frac{\lambda^2 - 2K - 1}{2\lambda_i}A_i = \gamma_iA_i
\]

(4.61)
where \( i = 1, \ldots, 4 \). Hence with four initial conditions all eight coefficients can be fully determined. Therefore, Equations (4.37a) and (4.37b) can be rewritten as:

\[
x(t) = A_1 e^{\lambda_1 t} + A_2 e^{-\lambda_1 t} + A_3 e^{\lambda_3 t} + A_4 e^{-\lambda_4 t} \\
y(t) = \gamma_1 A_1 e^{\lambda_1 t} + \gamma_2 A_2 e^{-\lambda_1 t} + \gamma_3 A_3 e^{\lambda_3 t} + \gamma_4 A_4 e^{-\lambda_4 t}
\]  

(4.62a)  

(4.62b)

The first two terms in these equations describe the exponential increase and/or decrease of the solution, whilst the last two terms describe periodic behaviour. By introducing the four initial conditions \( x(t_0), \dot{x}(t_0), y(t_0), \dot{y}(t_0) \), it is possible to fully determine the linear approximation of the trajectory of the third mass in the neighbourhood of each of the three collinear Lagrange points. An extensive treatment of the nature of this general linearised solution is given by [Wakker, 2005].

**Equilateral Lagrange Points**

A similar analysis as performed for the collinear Lagrange points can be performed to investigate the motion of the third body in the vicinity of the equilateral Lagrange points. The general solution follows the structure given by Equations (4.62a) and (4.62b). The difference in the analysis stems from the nature of the eigenvalues. For a detailed look at the various cases that occur for the L4 and L5 Lagrange points the reader is referred to [Wakker, 2005].

### 4.2 Hamiltonian Formulation

The equations of motion for the CR3BP can also be derived through the use of a Hamiltonian function. A short derivation will be presented here. Subsequently, the characteristics of the CR3BP derived previously will be re-derived using the Hamiltonian formulation. The treatment here follows [Koon et al., 2006].

#### 4.2.1 Equations of Motion

The Euler-Lagrange equations are given by:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0
\]

(4.63)

where \( q_i \) are generalised coordinates, \( i = 1, \ldots, n \). The Lagrangian function, taken as kinetic minus potential energy, is given in the inertial \( XY \)-plane by:

\[
\tilde{L}(X, Y, \dot{X}, \dot{Y}, t) = \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) - \tilde{U}(X, Y, t)
\]

(4.64)

In the rotating \( xy \)-plane the Lagrangian function in dimensionless coordinates is given by:

\[
L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} ((\dot{x} - y)^2 + (\dot{y} + x)^2) - U(x, y)
\]

(4.65)

where the kinetic energy in the inertial reference frame is written in terms of coordinates of the rotating reference frame by using the transformation given by Equation (4.8). It is noted that the Lagrangian function in the rotating reference
frame is no longer explicitly a function of time. The gravitational potential function is given by:

\[ U(x, y) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} = -\frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \] (4.66)

since in the rotating reference frame both \( r_1 \) and \( r_2 \) are not explicitly a function of time. These distances are given as \( r_1^2 = (x + \mu)^2 + y^2 \) and \( r_2^2 = (x + \mu - 1)^2 + y^2 \).

Using Equation (4.63) it follows that the equations of motion in the rotating reference frame are given by:

\[ \ddot{x} - 2\dot{y} = -\bar{U}_x \] (4.67a)

\[ \ddot{y} + 2\dot{x} = -\bar{U}_y \] (4.67b)

where \( \bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) + U(x, y) \) is denoted the effective potential.

Given the full definition of the Lagrangian system, it is now possible, through use of the Legendre transformation, to transform this system to a Hamiltonian form. The Legendre transformation, given by Equation (2.17) in Section 2.7, is in this case given by:

\[ p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y \] (4.68a)

\[ p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x \] (4.68b)

The Hamiltonian in this case, using Equation (2.18) in Section 2.7, is therefore:

\[ H(x, y, p_x, p_y) = p_x\dot{x} + p_y\dot{y} - L \]
\[ = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) + \bar{U}(x, y) \] (4.69)

The equations in Hamiltonian form are given by Equations (2.21b) and (2.21c):

\[ \dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \] (4.70a)

\[ \dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \] (4.70b)

\[ \dot{p}_x = -\frac{\partial H}{\partial x} = p_y - x - \bar{U}_x \] (4.70c)

\[ \dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - y - \bar{U}_y \] (4.70d)

By using Equations (4.70a) and (4.70b), it is possible to rewrite Equations (4.70c) and (4.70d) as follows:

\[ \ddot{x} - 2\dot{y} = -\bar{U}_x(x, y) \] (4.71a)

\[ \ddot{y} + 2\dot{x} = -\bar{U}_y(x, y) \] (4.71b)

These equations of motion are identical to Equation (4.13a) and (4.13b).
4.2.2 Linearisation

The process of linearisation treated previously will be re-evaluated using the Hamiltonian formulation of the equations of motion of the CR3BP. The analysis here will be restricted to the collinear Lagrange points. The geometry of trajectories in the neck region associated with the L1 and L2 Lagrange points will be considered in detail. This neck region exists for a particular range of values of the Jacobi Constant, illustrated Figure 4.3. In fact the case of most interest lies between case (a) and (b) where the neck region appears for the first time near L1.

To find the linearised equations of motion around the L1 and L2 Lagrange points, the quadratic terms of the Hamiltonian function are necessary. The location of these points in phase space is denoted by \((x_L, y_L, \dot{x}_L, \dot{y}_L) = (x_L, 0, 0, 0)\), based on the analysis presented in Section 4.1.4. The Hamiltonian function must therefore be expanded about \((x_L, y_L, p_{xL}, p_{yL}) = (x_L, 0, 0, x_L)\) (using Equations (4.68a) and (4.68b)). Expanding this Hamiltonian function after a coordinate change (the origin being located at \((x_L, 0, 0, x_L)\)) yields:

\[
H_L = \frac{1}{2} ((p_x + y)^2 + (p_y - x)^2 - ax^2 + by^2) \quad (4.72)
\]

where \(a = 2\bar{\mu} + 1\), \(b = \bar{\mu} - 1\) and:

\[
\bar{\mu} = \frac{1 - \mu}{|x_L + \mu|^3} + \frac{\mu}{|x_L + \mu - 1|^3} \quad (4.73)
\]

Both \(a\) and \(b\) are positive constants. The linearised equations in Hamiltonian form, using Equations (2.21b) and (2.21c), are:

\[
\dot{x} = \frac{\partial H_L}{\partial p_x} = p_x + y \quad (4.74a)
\]

\[
\dot{y} = \frac{\partial H_L}{\partial p_y} = p_y - x \quad (4.74b)
\]

\[
\dot{p}_x = -\frac{\partial H_L}{\partial x} = p_y - x + ax \quad (4.74c)
\]

\[
\dot{p}_y = -\frac{\partial H_L}{\partial y} = -p_x - y - by \quad (4.74d)
\]

The inverse of the Legendre transformation follows from Equations (4.68a) and (4.68b) and is given by:

\[
v_x = p_x + y \quad (4.75a)
\]

\[
v_y = p_y - x \quad (4.75b)
\]

Here \(v_x\) and \(v_y\) are velocities in the rotating reference frame. Using these equations it is also possible to rewrite the linearised equations, given previously in Hamiltonian form, in Lagrangian form. This results in equations that are expressed purely in terms of time-derivatives of positions and velocities as opposed to positions and momenta.

\[
v_x = \dot{x} \quad (4.76a)
\]

\[
v_y = \dot{y} \quad (4.76b)
\]
These equations compare favourably with the linearised equations for the collinear Lagrange points, obtained with the Newton approach (Subsection 4.1.5). The Hamiltonian function becomes:

\[ H_L = \frac{1}{2} (v_x^2 + v_y^2 - ax^2 + by^2) \] (4.77)

which is another form of Jacobi’s Integral as expected.

### 4.2.3 Geometry of Solutions

In this section the geometry of the solutions in the neighbourhood of the collinear Lagrange points will be considered. Having established for the collinear Lagrange points that the four eigenvalues come in conjugate pairs with two being purely real and two being purely imaginary (Subsection 4.1.5), it is possible to investigate further the geometric nature of the solutions by considering the associated flow in the phase plane in the neighbourhood of these equilibrium points. The eigenvalues \( \pm \lambda_1 \) will be denoted \( \pm \lambda \) and the eigenvalues \( \pm \lambda_3 \) will be denoted \( \pm i\nu \).

Equations (4.76a), (4.76b), (4.76c), (4.76d) can be cast into matrix form as follows:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{v}_x \\
\dot{v}_y
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
a & 0 & 0 & 2 \\
0 & -b & -2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
v_x \\
v_y
\end{bmatrix}
\] (4.78)

The characteristic polynomial obtained from this system corresponds directly with the polynomial obtained using the Newton approach. The eigenvalues are therefore indeed the same.

#### Eigenvectors

An eigenvector of this linearised system is denoted \( v = (k_1, k_2, k_3, k_4) \). If \( \alpha \) is an eigenvalue of this system, then it follows that \( Av = \alpha v \), where \( A \) is the \( 4 \times 4 \) matrix in Equation (4.78). Hence it follows that:

\[
\begin{bmatrix}
-\alpha & 0 & 1 & 0 \\
0 & -\alpha & 0 & 1 \\
a & 0 & -\alpha & 2 \\
0 & -b & -2 & -\alpha
\end{bmatrix}
\begin{bmatrix}
k_1 \\
k_2 \\
k_3 \\
k_4
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (4.79)

This leads to the following relations:

\[ k_3 = \alpha k_1 \] (4.80a)
\[ k_4 = \alpha k_2 \quad (4.80b) \]
\[ ak_1 - \alpha k_3 + 2k_4 = 0 \quad (4.80c) \]
\[ bk_2 + 2k_3 + \alpha k_4 = 0 \quad (4.80d) \]

It is evident from these equations that \( k_1 \neq 0 \) as otherwise the eigenvector will be trivial. For convenience, \( k_1 = 1 \) is chosen. This means that \( k_3 = \alpha \). The other degree of freedom is \( k_2 \). Using these equations, it follows that the form of the eigenvector is \( v = (1, k_2, \alpha, \alpha k_2) \). From Equations (4.80c) and (4.80d) the following relations hold:
\[ \alpha^2 = a + 2\alpha k_2 \quad (4.81a) \]
\[ \alpha^2 k_2 = -bk_2 - 2\alpha \quad (4.81b) \]

Substituting the first two eigenvalues of the linearised system identified previously \((\pm \lambda)\) for \( \alpha \), it follows that the associated eigenvectors are:
\[ u_1 = (1, k_{2,1}, \lambda, \lambda k_{2,1}) \quad (4.82a) \]
\[ u_2 = (1, k_{2,2}, -\lambda, -\lambda k_{2,2}) \quad (4.82b) \]

From Equations (4.81a) and (4.81b) it follows that:
\[ \lambda^2 = a + 2\lambda k_{2,1} \quad (4.83a) \]
\[ \lambda^2 k_{2,1} = -bk_{2,1} - 2\lambda \quad (4.83b) \]
\[ \lambda^2 = a - 2\lambda k_{2,2} \quad (4.83c) \]
\[ \lambda^2 k_{2,2} = -bk_{2,2} + 2\lambda \quad (4.83d) \]

From Equations (4.83a) and (4.83c) it follows that:
\[ k_{2,1} = -k_{2,2} \quad (4.84) \]

If \( k_{2,2} = \sigma \) it follows from Equation (4.83d) that:
\[ \sigma = \frac{2\lambda}{\lambda^2 + b} > 0 \quad (4.85) \]

This is confirmed by Equation (4.83b). Hence the eigenvectors for these two eigenvalues are:
\[ u_1 = (1, -\sigma, \lambda, -\lambda \sigma) \quad (4.86a) \]
\[ u_2 = (1, \sigma, -\lambda, -\lambda \sigma) \quad (4.86b) \]

Similarly, considering the third and fourth eigenvalues of the linearised system \((\pm iv)\) it follows that the eigenvectors are:
\[ w_1 = (1, -i\tau, iv, \nu \tau) \quad (4.87a) \]
\[ w_2 = (1, i\tau, -iv, \nu \tau) \quad (4.87b) \]

with:
\[ \tau = -\frac{(\nu^2 + a)}{2\nu} < 0 \quad (4.88) \]

To analyse the properties of phase space, a linear change is introduced such that the coordinate axes are changed from \( x, y, \dot{x}, \dot{y} \) to the eigenvectors \( u_1, u_2, w_1, w_2 \). The new coordinates are denoted \((\xi, \eta, \zeta_1, \zeta_2)\).
In these coordinates, the linearised system of differential equations becomes:

\[
\dot{\xi} = \lambda \xi \tag{4.89a}
\]

\[
\dot{\eta} = -\lambda \eta \tag{4.89b}
\]

\[
\dot{\varsigma}_1 = \nu \varsigma_2 \tag{4.89c}
\]

\[
\dot{\varsigma}_2 = -\nu \varsigma_1 \tag{4.89d}
\]

Hence, the energy integral given by Equation (4.77) becomes:

\[
E_L = \lambda \xi \eta + \frac{\nu}{2} (\varsigma_1^2 + \varsigma_2^2) \tag{4.90}
\]

The solution of Equations (4.89a), (4.89b), (4.89c), and (4.89d) can be written as:

\[
\xi(t) = \xi_0 e^{\lambda t} \tag{4.91a}
\]

\[
\eta(t) = \eta_0 e^{-\lambda t} \tag{4.91b}
\]

\[
\varsigma(t) = \varsigma_1(t) + i\varsigma_2(t) = \varsigma_0 e^{-i\nu t} \tag{4.91c}
\]

The constants \(\xi_0, \eta_0, \varsigma_0\) are determined by the initial values chosen.

These linearised equations admit integrals in addition to the standard Jacobi Integral. These integrals are local and are a result of a theorem formulated by [Moser, 1958]:

**Theorem 4.1 (Moser’s Theorem)** Suppose that \(x = y = 0\) is a critical point of a Hamiltonian system with the following, time-independent, Hamiltonian function:

\[
H(x, y) = \lambda x_1 y_1 + \frac{\nu}{2} (x_2^2 + y_2^2) + O_3(x_1, x_2, y_1, y_2) \tag{4.92}
\]

where \(\pm \lambda\) and \(\pm i\nu\) are eigenvalues of a non-degenerate critical point, and \(O_3(x, y)\) are terms of order 3 or greater. The equations of motion obtained from this Hamiltonian function are:

\[
\dot{x}_1 = \lambda x_1 + O_2(x_1, x_2, y_1, y_2) \tag{4.93a}
\]

\[
\dot{y}_1 = -\lambda y_1 + O_2(x_1, x_2, y_1, y_2) \tag{4.93b}
\]

\[
\dot{x}_2 = \nu y_2 + O_2(x_1, x_2, y_1, y_2) \tag{4.93c}
\]

\[
\dot{y}_2 = \nu x_2 + O_2(x_1, x_2, y_1, y_2) \tag{4.93d}
\]

The linearised equations follow by neglecting the terms of order 2 or greater in the above equations.

In general it can be stated that there exists a (real) analytic transformation given by:

\[
x_1 = \xi + O_2(\xi, \eta, \varsigma, \bar{\varsigma}) \tag{4.94a}
\]

\[
y_1 = \eta + O_2(\xi, \eta, \varsigma, \bar{\varsigma}) \tag{4.94b}
\]

\[
z = x_2 + iy_2 = \varsigma + O_2(\xi, \eta, \varsigma, \bar{\varsigma}) \tag{4.94c}
\]
where $\zeta, \bar{\zeta}$ are complex conjugates, as well as a power series $\alpha$ and $\beta$ in the variables $\chi = \xi \eta$ and $|\zeta|^2$ of the form:

$$\alpha = \lambda + O_{1}(\chi, |\zeta|^2)$$  \hspace{1cm} (4.95a)

$$\beta = -i\nu + O_{1}(\chi, |\zeta|^2)$$ \hspace{1cm} (4.95b)

such that the solutions of the transformed equations are given by:

$$\xi(t) = \xi_0 e^{t \alpha}$$ \hspace{1cm} (4.96a)

$$\eta(t) = \eta_0 e^{-t \alpha}$$ \hspace{1cm} (4.96b)

$$\zeta(t) = \zeta_0 e^{t \beta}$$ \hspace{1cm} (4.96c)

$$\bar{\zeta}(t) = \bar{\zeta}_0 e^{-t \beta}$$ \hspace{1cm} (4.96d)

where $\xi_0, \eta_0, \zeta_0$ are constants determined by initial conditions. The coefficients of $\alpha$ and $\beta$ are real and complex respectively. Hence it follows that $\xi \eta = x_1 y_1 + O_3(x_1, x_2, y_1, y_2)$ and $|\zeta|^2 = x_2^2 + y_2^2 + O_3(x_1, x_2, y_1, y_2)$ are local integrals, as are $\alpha$ and $\beta$. The transformation also results in the following form of the Hamiltonian function:

$$H(x_1, x_2, y_1, y_2) = K(\xi, \eta, \zeta, \bar{\zeta}) = \lambda \xi \eta + \frac{1}{2} |\zeta|^2 + O_2(\chi, |\zeta|^2)$$ \hspace{1cm} (4.97)

which depends only on $(\chi, |\zeta|^2)$.

Hence it follows that the linearised system given by Equation (4.89) admits additional local integrals in addition to the energy integral, namely the functions $\xi \eta$ and $|\zeta|^2 = \zeta_1^2 + \zeta_2^2$ are both constant along solutions.

**Equilibrium Region**

A neighbourhood of the $L_1$ and $L_2$ Lagrange points in the energy surface, where the position space projections are in the neck regions described previously, is denoted $R_1$ and $R_2$ respectively. For brevity, the Lagrange points will generically be denoted $L$ and the neighbourhood in the neck region will be denoted $R$. For a positive $E_0$ and $c$ the region $R$ is determined by $E_L = E_0$ and $|\eta - \xi| \leq c$. This region is homeomorphic (topologically equivalent) to the product of a two-sphere and an interval ($S^2 \times I$). For each fixed value of $\eta - \xi$ on the interval $I = [-c, c]$, the equation $E_L = E_0$ determines the following two-sphere:

$$\frac{\lambda}{4} (\eta + \xi)^2 + \frac{\nu}{2} (\zeta_1^2 + \zeta_2^2) = E_0 + \frac{\lambda}{4} (\eta + \xi)^2$$ \hspace{1cm} (4.98)

**Remark** Two objects are homeomorphic if they can be deformed into each other by a continuous, invertible mapping. A common example is that of a homeomorphism between a coffee cup and a doughnut. It is possible to visualise the fact that a coffee cup can be stretched and bent continuously to mould it into the shape of a doughnut. This example illustrates the fact that these two objects intrinsically possess the same topological form. Hence the coffee cup is homeomorphic to the doughnut [Wolfram, 1999].
4.2 Hamiltonian Formulation

The bounding sphere of \( R \) for which \( \eta - \xi = -c \) is denoted \( n_1 \) and the bounding sphere for which \( \eta - \xi = c \) is denoted \( n_2 \). The set of points on each sphere for which \( \eta + \xi = 0 \) is denoted the equator; the set of points for which \( \eta + \xi > 0 \) or \( \eta + \xi < 0 \) are denoted the north and south hemispheres respectively. Figure 4.5 illustrates the projection of the flow on the \( \eta \xi \)-plane.

![Figure 4.5](image)

**Figure 4.5** The projection of orbits on the \( \eta \xi \)-plane near the equilibrium point \( L \). In (a) the region \( R \) is bounded by dotted vertical line segments on the left and the right and thick lines at the top and bottom that represent hyperbolic segments. The origin represents the periodic orbit centred at the equilibrium point. The \( \xi \) and \( \eta \) axes represent trajectories that are asymptotically winding toward the periodic orbit. In (b) four additional trajectories are shown, illustrating two transit and two non-transit orbits. [Koon et al., 2006].

In Figure 4.5, the coordinate axes have been tilted by 45° in order to correspond with the direction of the flow illustrated in a later figure. In Figure 4.5(a) the region \( R \) is bounded at the top and the bottom (thick lines) by the hyperbola \( \eta \xi = \frac{E_0}{\lambda} \) which correspond with \( |s|^2 = 0 \) (Equation (4.90)). The region \( R \) is bounded on the left and the right by \( \eta - \xi = \pm c \) (dotted vertical lines). These vertical lines correspond with the bounding spheres \( n_1 \) and \( n_2 \) as illustrated in Figure 4.5(b). As \( \eta \xi \) was identified as being an integral of the linearised system, it follows that the solutions in the \( \eta \xi \)-plane follow trajectories that correspond with \( \eta \xi = k \) where \( k \) is a constant, for \( \eta \xi \neq 0 \). If \( \eta \xi > 0 \) the hyperbolic branches connect the vertical bounding line segments. If \( \eta \xi < 0 \) these branches have start and end points on the same vertical bounding line segment.

Figure 4.5(b) can be interpreted as follows: each point in the projection corresponds to a circle in \( R \) with radius given by \( \rho = |s|^2 = constant \). From Equation (4.90) it follows that \( |s|^2 = \frac{2}{\rho}(E_0 - \lambda \xi \eta) \). It follows from this that, for the hyperbolic bounding segments where \( \eta \xi = \frac{E_0}{\lambda}, \rho = 0 \) which means that the circle collapses to a point in \( R \). Therefore the vertical line segments \( \eta - \xi = \pm c \) in the \( \eta \xi \)-plane correspond to two spheres that bound \( R \). This is because each line segment corresponds with a circle, of radius determined by the value of \( \rho \), where the two end circles are pinched to a point.
Based on this projection, nine classes of orbits can be categorised into four groups. These four groups are:

1. The point at the origin of Figures 4.5(a) and 4.5(b) \((\eta = \xi = 0)\) corresponds with a periodic orbit in \(R\). This periodic orbit is denoted the Lyapunov orbit.

2. The four half-open segments on the axes \((\eta \xi = 0, \text{green})\) correspond to four cylinders of orbits asymptotic to the periodic solution at the origin in forward time \((\xi = 0)\) or backward time \((\eta = 0)\). These are asymptotic orbits.

3. The hyperbolic segments for which \(\eta \xi = k < 0\) (red) correspond to two cylinders crossing \(R\) from one bounding sphere to the other \((n_1\text{ to } n_2\text{ or vice-versa})\). These cylinders start and end in the same hemisphere; the cylinders are located in the north hemisphere if they go from \(\eta - \xi = c\) to \(\eta - \xi = -c\) and vice-versa for the south hemisphere. These orbits are called transit orbits. In Figure 4.5(b), both \(T_{12}\) and \(T_{21}\) are examples of transit orbits.

4. The hyperbolic segments for which \(\eta \xi = k > 0\) (blue) correspond to two cylinders of orbits in \(R\) that pass from one hemisphere to the other whilst remaining on the same bounding sphere. If \(\xi > 0\) the bounding sphere is \(n_1\) \((\eta - \xi = -c)\) and the orbits pass from the south to the north \((\eta + \xi < 0\text{ to } \eta + \xi > 0)\). The situation is reversed if the bounding sphere is \(n_2\) \((\xi < 0)\). As the trajectories are constrained to a single sphere, they are denoted non-transit orbits.

The region \(R\) can be visualised better by making use of the McGehee representation [McGehee, 1969]. McGehee proposed that the region \(R\) can be represented by a spherical annulus. The details of the theory behind this representation will not be discussed here, however the interpretation thereof is interesting for the present discussion. Further details about the topological theory required to arrive at the McGehee representation can be found in, for instance, [Waalkens and Wiggins, 2004]. The McGehee representation is illustrated by Figure 4.6.

Figure 4.6(a) presents a cross-section of \(R\). Qualitatively this is the same as Figure 4.5(b). Figure 4.6(b) is obtained by rotating the cross-section about the \(\omega\)-axis. In this representation, the following four groups of orbits correspond with the four groups identified previously:

1. The Lyapunov orbit in the region \(R\) is denoted \(l\) and is unstable, as trajectories can depart from the region \(R\) along the unstable eigenvector.

2. The bounding spheres of \(R\) are denoted \(n\) \((n_1\text{ and } n_2)\) and can be divided into two hemispheres. The hemisphere where the flow enters \(R\) is denoted \(n^+\) and the hemisphere where the flow leaves \(R\) is denoted \(n^-\). The intersections with \(n\) of the cylinders of the orbits asymptotic to the unstable periodic orbit \(l\) are denoted \(a^+\) and \(a^-\) \((|\varsigma|^2 = \rho^*)\). In Figure 4.6(b), \(a^+\) appears as a circle in \(n^+\) and \(a^-\) appears as a circle in \(n^-\).

3. The spherical cap \((|\varsigma|^2 < \rho^*)\) in \(n^+\), bounded by \(a^+\) is denoted \(d^+\). Transit orbits entering \(R\) on \(d^+\) of one bounding sphere exit on \(d^-\) of the other bounding sphere. Similarly, the spherical cap \((|\varsigma|^2 < \rho^*)\) in \(n^-\), bounded by \(a^-\) is denoted \(d^-\). Transit orbits leaving \(R\) on \(d^-\) of one bounding sphere have come from \(d^+\) of the other bounding sphere.
4. The intersection $b$ of $n^+$ and $n^-$ is a circle of tangency points. Orbits tangent at this circle do not enter $R$, instead ‘bouncing off’. The spherical zone bounded by $a^+$ and $b$ is denoted $r^+$ ($|\varsigma|^2 > \rho^*$) and non-transit orbits entering $R$ on $r^+$ (where exit on the same bounding sphere through $r^-$ ($|\varsigma|^2 > \rho^*$) which is bounded by $a^-$ and $b$.

The key observation to made from this study of the equilibrium region $R$ is that orbits asymptotic to the Lyapunov orbit can be viewed as part of stable and unstable cylindrical manifolds associated with the unstable periodic orbit. These manifold structures effectively separate orbits into two distinct classes: transit and non-transit orbits. The transit orbits are found inside the cylindrical manifolds, whilst the non-transit orbits are found outside.

**Trajectories in the Neck Region**

The analysis of the equilibrium region $R$ has been illustrated in the $\eta \xi$-plane. This however is rather abstract and it is more interesting to analyse the consequences of the observations made for trajectories in position space i.e. in the $xy$-plane. The $\eta$ and $\xi$ axes represent the eigenvectors $u_1$ and $u_2$ derived previously. Their projections on the $xy$-plane, $\bar{u}_1$ and $\bar{u}_2$, are important when studying the behaviour of orbits in position space. Four categories of orbits were described previously and the study of the projection of these orbits on the $xy$-plane is necessary to construct a complete, qualitative picture of the flow characteristics in the neck region. The observations presented here summarise the results obtained by [Conley, 1968].

Given that the eigenvalues of the linearised system are $\pm \lambda$ and $\pm \nu$, and the associated eigenvectors are $u_1, u_2, w_1, w_2$, the general solution can be constructed...
as follows:

\[
\bar{x}(t) = (x(t), y(t), \dot{x}(t), \dot{y}(t)) = c_1 u_1 e^{\lambda t} + c_2 u_2 e^{-\lambda t} + 2Re(c_3 w_1 e^{i\nu t})
\] (4.99)

where \(c_1, c_2\) are real constants and \(c_3 = c_{3,1} + ic_{3,2}\) is complex. It is noted that Equation (4.99) is essentially of the same form as Equations (4.37a) and (4.37b). The \(x\)-coordinate equation of Equation (4.99) is:

\[
x(t) = c_1 e^{\lambda t} + c_2 e^{-\lambda t} + 2(c_{3,1} \cos \nu t - c_{3,2} \sin \nu t)
\] (4.100)

It follows from this equation that as \(t \to \infty\) the solution is dominated by the first term \((c_1 e^{\lambda t})\). Hence depending on the value of \(c_1\) the solution either tends to minus infinity (stays on the left of the equilibrium point), is bounded (remains around the equilibrium point), or tends to plus infinity (stays on the right of the equilibrium point). Similarly, as \(t \to -\infty\), the value of \(c_2\) determines the nature of the solution in the neck region. Different combinations of values of \(c_1, c_2\) provide nine classes of orbits which can be placed in four categories that correspond well with the categories previously determined. The four categories in this case have the following distinctive properties:

1. If \(c_1 = c_2 = 0\) the solution obtained is purely periodic; the Lyapunov orbit. It is shown in [Conley, 1968] that the projection of this periodic solution on the \(xy\)-plane corresponds with an ellipse with major axis of length \(2\tau \sqrt{\frac{E_0}{\kappa}}\) in the direction of the \(y\)-axis, and minor axis of length \(2\sqrt{\frac{E_0}{\kappa}}\) in the direction of the \(x\)-axis (\(\kappa\) is a constant). The orientation of the orbit is clockwise, which is opposite to the direction of rotation of the primaries. The size of the ellipse tends to zero as the energy \(E_0\) tends to zero.

2. Orbits for which \(c_1 c_2 = 0\) are asymptotic to the Lyapunov orbit. It is shown in [Conley, 1968] that for \(c_1 = 0\) the projection of the asymptotic orbits in the \(xy\)-plane corresponds with a strip \(S_1\) centred around \(\bar{u}_1\) (the projection of the eigenvector \(u_1\) in the \(xy\)-plane) and bounded by the lines \(y = \sigma x \pm 2\sqrt{E_0 (\sigma^2 + \tau^2)}\). The asymptotic orbits for which \(c_2 = 0\) correspond with strip \(S_2\) when projected in the \(xy\)-plane, centred around \(\bar{u}_2\) (the projection of the eigenvector \(u_2\) in the \(xy\)-plane), bounded by the lines \(y = -\sigma x \pm 2\sqrt{E_0 (\sigma^2 + \tau^2)}\). As with the size of the Lyapunov orbit, the width of the strip goes to zero as the energy \(E_0\) tends to zero.

3. Orbits for which \(c_1 c_2 < 0\) correspond with transit orbits as they traverse the neck region from \(-\infty\) to \(\infty\) (left-hand side to the side-hand side) or vice-versa.

4. Orbits for which \(c_1 c_2 > 0\) correspond with non-transit orbits as they remain on the side of the equilibrium region where they start i.e. if the orbit starts on the left-hand side, it remains on the left-hand side as \(t \to \infty\), and if it starts on the right-hand side, it remains on the right-hand side.

The main characteristics of the flow in the neck region have now been described. A full qualitative picture of the types of motion possible in the equilibrium region \(R\) has been provided. Figure 4.7 illustrates the four categories of motion described.
4.3 Regularisation

A property of the Newtonian gravitational force field, due to the inverse square dependence on the distance of separation, is the fact that forces acting between particles approach infinity as the distance between them approaches zero. In other words, at impact a singularity exists. In reality, actual spacecraft never reach this point as the primaries involved are three-dimensional i.e. have spatial extent; hence the spacecraft would strike the surface of the body before reaching the point of singularity. The singularity is however of utmost importance in numerical computations, where the accuracy of the integration scheme being used deteriorates in its neighbourhood.

This problem can be tackled by choosing an appropriate set of local state variables to eliminate the singularity. The process of choosing this set of local variables and re-deriving the differential equations governing the motion of the spacecraft is termed regularisation. In this section, a brief overview of a local regularisation of the PCR3BP based on the famous Levi-Civita transformation will be presented. The singularity at the smaller of the two primaries, $m_2$ will be eliminated through this process. The treatment here follows the methodology described in [Szebehely, 1967], where local regularisation about the larger primary, $m_1$, is described.

The Levi-Civita relation transforms the $xy$-plane to an $uv$-plane. In traditional
$xy$-coordinates, in the PCR3BP the location of the two primaries and the third body are given as illustrated in Figure 4.8. The location of any point on this plane can be represented as a complex variable as follows: $z = x + iy$. Using the Levi-Civita transformation, the $xy$-plane is transformed to the $uv$-plane, illustrated in Figure 4.9. As with the $xy$-plane, it is possible to represent any point in the $uv$-plane as a complex variable: $w = u + iv$. The Levi-Civita transformation is then given by Equation (4.101).

$$ z = f(w) = w^2 + 1 - \mu $$

(4.101)

This is not the transformation given in most texts, as typically the origin of the $uv$-plane is chosen to coincide with $m_1$. In this case, the origin is chosen to be $m_2$, as this is the primary about which the local regularisation is being performed. This accounts for the $1 - \mu$ term in Equation (4.101). The method used to determine the regularised equations of motion however is exactly the same as for the case that the local regularisation is being performed about $m_1$. Using this transformation, and
substituting the relevant variables, the Levi-Civita transformation can be rewritten as given by Equation (4.102).

\[ x + iy = u^2 + 2uvi - v^2 + 1 - \mu \]  

(4.102)

From this it follows that:

\[ x - 1 + \mu = u^2 + v^2 \]  

(4.103a)

\[ y = 2uv \]  

(4.103b)

Equations (4.103a) and (4.103b) give the complete transformation of the spatial variables. Using this transformation it is apparent that the location of the primaries is mapped as follows from the \( xy \)-plane to the \( uv \)-plane:

\[
\begin{align*}
& m_1: \begin{cases} 
x = -\mu \\
y = 0 
\end{cases} \Rightarrow \begin{cases} 
u = 0 \\
v = \pm 1 
\end{cases} \\

& m_2: \begin{cases} 
x = 1 - \mu \\
y = 0 
\end{cases} \Rightarrow \begin{cases} 
u = 0 \\
v = 0 
\end{cases}
\end{align*}
\]

For \( y = 0 \) it follows that \( u = 0 \) for the location of the primaries, since \( v \) is the coefficient of the imaginary term in Equation (4.103a). It should be noted that the Levi-Civita transformation is not one-to-one. In fact, the \( uv \)-plane is symmetric about the \( u \)-axis as positive and negative coordinates of \( v \) map onto the same spatial position on the \( xy \)-plane. Given that the transformation of the spatial coordinates has been established, it is now necessary to investigate the necessary transformation of the time variable. Derivatives with respect to \( \tau \) will be indicated as primed variables. The time variable is transformed to fictitious time \( \tau \) using the relation given by Equation (4.104).

\[ \frac{dt}{d\tau} = g(w) = |f'|^2 \]  

(4.104)

The basis for using this relation to transform the time variable is treated extensively by [Szebehely, 1967]. From Equation (4.101) it follows that:

\[ \frac{df}{d\tau} = 2w \]  

(4.105)

The function \( g(w) \) can therefore be evaluated as:

\[ g(w) = 4|w|^2 \]  

(4.106)

Therefore it follows that:

\[ \frac{dt}{d\tau} = 4|w|^2 \]  

(4.107)

The transformation of spatial coordinates represents a conformal mapping; it contains the geometric information and it controls the accuracy of the orbit parameters. The transformation of the time coordinate is the essential one, as it controls the kinematic aspects and it performs the actual regularisation.

The equations of motion of the PCR3BP in \( xy \) coordinates were given in Section 4.1.1. For purposes of completeness they are given here again.

\[ \ddot{x} - 2\dot{y} = \Omega_x \]  

(4.108a)
\[ y' + 2x = \Omega_y \]  

(4.108b)

Derivatives with respect to real time \( t \) are indicated by dot variables. To transform these equations to the new \( uv \) spatial and fictitious time coordinates, the time derivatives have to transformed. The time derivatives are transformed using the chain rule. The first time derivative of the complex variable \( z \) is given by Equation (4.109).

\[
\frac{dz}{dt} = \frac{dz}{dw} \frac{dw}{d\tau} \frac{d\tau}{dt}
\]

(4.109)

This can be rewritten using the aforementioned relations:

\[
\frac{dz}{dt} = f'w' g = \frac{(2w)(u' + iv')}{4|w|^2} = \frac{2(u + iv)(u' + iv')}{4(u^2 + v^2)}
\]

(4.110)

This is further simplified by expanding brackets to yield the following:

\[
x' + iy' = \frac{uu' + uv'i + vu'i - vv'}{2(u^2 + v^2)}
\]

(4.111)

Hence it follows that:

\[
x' = \frac{uu' - vv'}{2(u^2 + v^2)}
\]

(4.112a)

\[
y' = \frac{uv' + uu'}{2(u^2 + v^2)}
\]

(4.112b)

This derivative is necessary to transform the state vector which is composed of positions and velocities. In complex form, the equations of motion in \( xy \) coordinates can be rewritten as:

\[
n + 2iz = \nabla_z U
\]

(4.113)

It is shown by [Szebehely, 1967] that the equations of motion under the Levi-Civita transformation are given by:

\[
w'' + 2i|f'|^2 w' = \nabla_w(|f'|^2 U)
\]

(4.114)

The equations of motion are given here in complex form where:

\[
U = \Omega - \frac{C}{2}
\]

(4.115)

It is known that \( w' = u' + iv' \), \( w'' = u'' + iv'' \), \( f' = 2w \), and \( |f'|^2 = g = 4|w|^2 = 4(u^2 + v^2) \). Hence the left-hand side of Equation (4.114) can be fully evaluated. To evaluate the right-hand side it is necessary to determine \( \nabla_w(|f'|^2 U) \). To achieve this, \( U \) must be evaluated in terms of \( uv \) coordinates. In \( xy \) coordinates, \( U \) is given by Equation (4.116).

\[
U = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1 - \mu) - \frac{C}{2}
\]

(4.116)

Using previously defined relations it follows that:

\[
x = u^2 + v^2 + 1 - \mu
\]

(4.117a)
4.3 Regularisation

\[ y = 2uv \quad (4.117b) \]

The position vectors \( r_1 \) and \( r_2 \), defined in Section 4.1.1 can be evaluated by using Equations (4.117a) and (4.117b).

\[
\begin{align*}
\mathbf{r}_1^2 &= (x + \mu)^2 + y^2 \\
&= (u^2 - v^2 + 1 - \mu + \mu)^2 + 4u^2v^2 \\
&= (u^2 - v^2 + 1)^2 + 4u^2v^2 \\
&= |1 + w^2|^2 \\
\mathbf{r}_1 &= |1 + w^2| \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{r}_2^2 &= (x + \mu - 1)^2 + y^2 \\
&= (u^2 - v^2 + 1 - \mu + \mu - 1)^2 + 4u^2v^2 \\
&= u^4 + 2u^2v^2 + v^4 \\
&= (u^2 + v^2)^2 \\
\mathbf{r}_2 &= u^2 + v^2 \\
&= |w|^2 \\
\end{align*}
\]

The remaining terms in \( U \) are evaluated as follows:

\[
\begin{align*}
\frac{1}{2}(x^2 + y^2) + \frac{1}{2}\mu(1 - \mu) &= \frac{1}{2}[(u^2 - v^2 + 1 - \mu)^2 + 4u^2v^2] + \frac{1}{2}\mu(1 - \mu) \\
&= \frac{1}{2} \left[ u^4 - 2u^2v^2 + v^4 + 1 - 2\mu + \mu^2 + 2u^2 - 2v^2 - 2\mu v^2 + 2\mu^2v^2 + 4u^2v^2 \right] \\
&= \frac{1}{2} \left[ u^4 + 2u^2v^2 + 2u^2 - 2v^2 + v^4 + 1 + \mu(2u^2 + 2v^2 + 1) \right] \\
&= \frac{1}{2}(\mu|w|^4 + (1 - \mu)|1 + w^2|^2) \\
\end{align*}
\]

Hence it follows that \( U \) can be written as:

\[
U = \frac{1}{2}(\mu|w|^4 + (1 - \mu)|1 + w^2|^2) + \frac{1}{2}\mu(1 - \mu) + \frac{\mu}{|1 + w^2|} - \frac{C}{2} \quad (4.121)
\]

Equation (4.114) can be rewritten as two second-order differential equations in \( uv \) coordinates as follows:

\[
\begin{align*}
u'' - 8(u^2 + v^2)v' &= \frac{\partial}{\partial u}(|f'|^2 U) \quad (4.122a) \\
v'' + 8(u^2 + v^2)u' &= \frac{\partial}{\partial v}(|f'|^2 U) \quad (4.122b) \\
\end{align*}
\]

This is done by evaluating the right-hand side using previously identified relations. The right-hand side can be evaluated now since \( U \) is known in \( uv \) coordinates. The
first step is given by Equation (4.123).

\[
|f'|^2 U = 4 |w|^2 U \\
= 4 |w|^2 \left[ \frac{1}{2} \mu |w|^4 + (1 - \mu) |1 + w^2|^2 \right] + \frac{1 - \mu}{|1 + w^2|} + \frac{\mu}{|w|^2} - C \\
= 2 \mu |w|^6 + 2(1 - \mu) |w|^2 |1 + w^2|^2 \\
+ 4 \mu \frac{1 - \mu}{|1 + w^2|} |w|^2 + 4 \mu - 2C |w|^2 \tag{4.123}
\]

To evaluate $|f'|^2 U$ further, the following relations are required:

\[
|w|^2 = u^2 + v^2 \tag{4.124a}
\]

\[
|1 + w^2| = |1 + u^2 + 2uv - v^2| \\
= \sqrt{(u^2 - v^2 + 1)^2 + 4uv^2} \\
= \sqrt{u^4 + 2uv^2 + v^4 + 1 + 2u^2 - 2v^2} \\
= \sqrt{(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)} \tag{4.124b}
\]

Hence it follows that:

\[
|f'|^2 U = 2 \left[ \mu(u^2 + v^2)^3 + (1 - \mu)(u^2 + v^2)[(u^2 + v^2)^2 + 2(u^2 - v^2) + 1] \\
+ \frac{2(1 - \mu)(u^2 + v^2)}{[(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)]} + 2\mu - C(u^2 + v^2) \right]
\]

\[
= 2 \left[ \mu(u^2 + v^2)^3 + (1 - \mu)[(u^2 + v^2)^3 + 2(u^4 - v^4) + u^2 + v^2] \\
+ \frac{2(1 - \mu)(u^2 + v^2)}{[(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)]} + 2\mu - C(u^2 + v^2) \right]
\]

\[
= 2 \left[ (u^2 + v^2)^3 + 2(1 - \mu)(u^4 - v^4) + (1 - \mu)(u^2 + v^2)^2 \\
+ 2\mu + \frac{2(1 - \mu)(u^2 + v^2)}{[(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)]} \right] \tag{4.125}
\]

To fully evaluate the right-hand side of Equations (4.122a) and (4.122b), the derivative of Equation (4.125) with respect to $u$ and $v$ must be determined. It can be shown that these derivatives are given by Equations (4.126a) and (4.126b).

\[
\frac{\partial}{\partial u}(|f'|^2 U) = 4 \left[ \frac{3u(u^2 + v^2)^2 + 4u^3(1 - \mu) + u(1 - \mu - C)}{[(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)]} \\
+ \frac{2u(1 - \mu)(u^2 + v^2)}{[(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)]} \right] \tag{4.126a}
\]

\[
\frac{\partial}{\partial v}(|f'|^2 U) = 4 \left[ \frac{3v(u^2 + v^2)^2 - 4v^3(1 - \mu) + v(1 - \mu - C)}{[(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)]} \\
+ \frac{2v(1 - \mu)(u^2 + v^2)}{[(u^2 + v^2)^2 + 1 + 2(u^2 - v^2)]} \right] \tag{4.126b}
\]

Hence Equations (4.122a) and (4.122b) fully define the equations of motion in $uv$ coordinates, given the derivatives in Equations (4.126a) and (4.126b). The initial
state vector for these equations of motion is given by:

\[
\mathbf{\pi} = \begin{bmatrix} u \\ v \\ V_u \\ V_v \end{bmatrix}
\]  
(4.127)

Collating results shown previously, it follows that given the state vector in \(uv\) coordinates, the state vector in \(xy\) coordinates can be found using the following set of relations.

\[
x = u^2 - v^2 + 1 - \mu \\
y = 2uv \\
\dot{x} = \frac{uu' - vv'}{2(u^2 + v^2)} \\
\dot{y} = \frac{uv' + u'v}{2(u^2 + v^2)}
\]  
(4.128a-d)

The reverse transformation can easily be shown to be:

\[
u = \sqrt{r^2 - (x + \mu - 1))} \\
v = \frac{y}{2u} \\
u' = 2(u\dot{x} + v\dot{y}) \\
v' = 2(u\dot{y} - v\dot{x})
\]  
(4.129a-d)

This transformation is singular for the case that \(y = 0\). This is solved using the following equations instead of Equations (4.129a) and (4.129b):

\[
u = 0 \\
v = \sqrt{x - 1 + \mu}
\]  
(4.130a-b)

This follows simply from the fact that \(u\) can be chosen freely. This is apparent from Equations (4.128a) and (4.128b). Hence it is now possible to map states defined in \(xy\) coordinates to \(uv\) coordinates and vice-versa. The last point to be noted is the fact that when the equations of motion are integrated in \(uv\) coordinates, the time variable is fictitious (\(\tau\)). Equations (4.107) gives a differential equation which relates \(\tau\) with \(t\). To determine \(t\) therefore, the integration of the equations of motion has to be augmented at each time step using the following relation:

\[
t_1 = 4 \int_{\tau_0}^{\tau_1} (u^2 + v^2) d\tau + t_0
\]  
(4.131)

It is now possible to fully determine the state of an impact trajectory at \(m_2\). It is important to note that the Levi-Civita transformation used here is local. In other words, the singularity at \(m_2\) has been removed, however the singularity at \(m_1\) remains. To remove both of these singularities at once a global regularisation
scheme is required such as the Birkhoff transformation. This is explained extensively in [Szebehely, 1967] and goes beyond the scope of applications required for the current investigation.

For numerical investigations the Levi-Civita transformation presented here can be used in the following manner. A disc of specified radius is defined around $m_2$ e.g. $10^{-3}$. The radius of this disc is arbitrary: it should be large enough to ensure that the region of phase space contained within is associated with large gradients of the vector field. Trajectories outside of this disc are integrated using the equations of motion given in $xy$ coordinates. If a trajectory enters this disc, the state vector is transformed to $uv$ coordinates at that point and the integration proceeds using the equations of motion in $uv$ coordinates using the transformed state vector as initial condition. If the trajectory exists the disc, the state vector is transformed back to $xy$ coordinates at that point and the integration proceeds once again using the equations of motion in $xy$ coordinates. This procedure is summarised in Figure 4.10. The reason why the complete integration is not performed using the equations of motion in $uv$ coordinates is two-fold: firstly trajectories that approach $m_1$ are still singular, hence there is not added advantage in this sense, and secondly, and perhaps more importantly, integration using $uv$ coordinates is generally slower. Hence it makes sense to only use the equations of motion in $uv$ coordinates when necessary.

It should be noted that the local regularisation of the equations of motion presented in this section pertains to the two-dimensional case only. The Levi-Civita transformation used here must be adapted to regularise the equations of motion for the three-dimensional case. This is not a trivial task, and a survey of existing literature indicates that derivation of regularised equations for the spatial case would require use of the general theory of regularisation, which is beyond the scope of the research presented in this report.
Chapter 5

Elliptic Restricted 3-Body Problem

The formulation of Elliptic Restricted 3-Body Problem (ER3BP) is quite similar to the Circular Restricted 3-Body Problem (CR3BP) presented in Chapter 4. The main difference is that the assumption that the primaries are restricted to motion about their common centre mass in circular orbits is replaced by the assumption that their motion is restricted to elliptical orbits. The simplicity of this statement belies the true differences in the motion of the third body in both these cases. In this chapter, a brief summary of the problem formulation for the planar case, as given in [Szebehely, 1967], will be offered.

The derivation of the equations of motion and properties of the ER3BP outlined in this chapter are given as background information. The original intent of the research presented in this thesis was to establish the nature of the Weak Stability Boundary (WSB) region in the ER3BP. However, due to the fact that the ER3BP is non-autonomous, and regularisation of the equations of motion is a non-trivial task that requires additional research, this was not carried out. Hence, this chapter serves to summarise the fundamental equations and properties of the ER3BP, to be used as a starting point for future work in this area.

The ER3BP is an important dynamical system to consider for space mission design since it can be used as a prototype for time-dependent systems in astrodynamics. For the Earth-Moon case presented in Chapter 8, the ER3BP is not significant since the motion of the Moon about the Earth is to good approximation circular. In Chapter 9.4, an interesting mission concept to Mercury is presented. For this mission (BepiColombo), the dynamics of the ER3BP is significant since the eccentricity of Mercury’s orbit around the Sun is not negligible. To analyse the underlying dynamics relating to weak capture for missions to Mercury, the ER3BP is therefore a suitable model.

In Section 5.1, the equations of motion for the ER3BP are derived. Subsequently, an invariant relation for this dynamical system is presented in Section 5.2. The Surfaces of Hill associated with the ER3BP are analysed in Section 5.3. Finally, the existence and properties of Lagrange points in this system are outlined in Section 5.3.
5.1 Equations of Motion

As with the CR3BP a rotating reference is introduced, however in the case of the ER3BP this reference frame is also required to pulsate to ensure that this reference frame rotates with the primaries. This pulsation stems from the fact that the separation distance between the primaries varies periodically, as they orbit their common centre of mass. Such a pulsating reference frame is introduced by using the variable distance between the primaries as the basic unit of length in the system. Dimensionless variables are introduced by considering the distance between the primaries as:

\[
r = \frac{a(1 - e^2)}{1 + e \cos \theta}
\]  

(5.1)

where \(a\) is the semi-major axis and \(e\) is the eccentricity of the orbit of either primary around the other, and \(\theta\) denotes the true anomaly. Distances are scaled based on Equation (5.1). To ensure that the rotating reference system rotates with the primaries, a variable angular velocity \(\dot{\theta}\) is introduced:

\[
\frac{d\theta}{dt} = \frac{G(m_1 + m_2)^{1/2}}{a(1 - e^2)^{3/2}} \left(1 + e \cos \theta\right)^2
\]  

(5.2)

Equation (5.2) is derived by considering the principle of conservation of angular momentum.

Following the treatment in Section 6.4.1, Equation (4.10) can be restated to describe motion in an inertial reference frame for the ER3BP as well:

\[
\frac{d^2S}{dt^2} = - \left[ \frac{\mu_1}{r_1^3}(S - S_1) + \frac{\mu_2}{r_2^3}(S - S_2) \right]
\]  

(5.3)

with \(m_1\) and \(m_2\) normalised with respect to the total system mass to give \(\mu_1\) and \(\mu_2\) respectively. Complex notation is introduced here:

\[
S = se^{i\theta}
\]  

(5.4)

where \(S = X + iY\) and \(s = x' + iy'\). In this case, the pulsating, rotating reference frame is \(x'y'z'\). Using Equation (5.4) it is possible to state Equation (5.3) in the pulsating, rotating reference frame:

\[
\frac{d^2s}{dt^2} + 2i\frac{d\theta}{dt} \frac{ds}{dt} = - \left[ \frac{\mu_1}{r_1^3}(s - s_1) + \frac{\mu_2}{r_2^3}(s - s_2) \right] + s \left( \frac{d\theta}{dt} \right)^2 - is \frac{d^2\theta}{dt^2}
\]  

(5.5)

where:

\[
s_1 = \frac{p_1}{1 + e \cos \theta}
\]  

(5.6a)

\[
s_2 = \frac{-p_2}{1 + e \cos \theta}
\]  

(5.6b)

locates the primaries and:

\[
\frac{p_1}{p_2} = \frac{m_1}{m_2} = \frac{\mu_2}{\mu_1}
\]  

(5.7)
where \(a_1\) and \(a_2\) are the semi-major axes of the elliptic orbits of \(m_1\) and \(m_2\) respectively about their common centre of mass. The distance between these primaries and the third body is given by:

\[
\begin{align*}
    r_1 &= |s - s_1| \\ 
    r_2 &= |s - s_2| 
\end{align*}
\] (5.8a)

For the primaries therefore, the following holds:

\[
\begin{align*}
    x'_i &= (-1)^{i+1} \frac{p_i}{a(1 - e^2)} \\ 
    y'_i &= 0 
\end{align*}
\] (5.9a)

where \(i = 1, 2\). By substituting \(\theta = t\), \(e = 0\), \(p_1 = x'_1 = x_1\), and \(p_2 = x'_2 = x_2\) Equation (5.5), it reduces to the case of the CR3BP.

The true anomaly is introduced as an independent variable as follows:

\[
\frac{d}{dt} \theta = \frac{d}{d\theta} \frac{d\theta}{dt} 
\] (5.10)

Hence Equation (5.5) can now be rewritten:

\[
\begin{align*}
    r \left( \frac{d\theta}{dt} \right)^2 &+ \frac{d^2 b}{d\theta^2} + 2i \frac{db}{d\theta} + \frac{d\theta}{dt} \left( r^2 \frac{d^2 \theta}{dt^2} - \frac{d\theta}{dt} \frac{d^2 r}{dt^2} \right) \\
    &= -\mu_2 \frac{b - b_1}{r_1^2} - \mu_2 \frac{b - b_2}{r_2^2} 
\end{align*}
\] (5.11)

A number of relations can be derived involving \(r\), \(\theta\) and their derivatives. Firstly, the integral of the angular momentum reads:

\[
\left( r \frac{d\theta}{dt} \right)^2 = a(1 - e^2)(\mu_1 + \mu_2) 
\] (5.12)

Its derivative with respect to time reads:

\[
r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} = 0 
\] (5.13)

The equation of motion of \(m\) reads:

\[
\frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 = -\frac{\mu_1 + \mu_2}{r^2} 
\] (5.14)

By using Equation (5.12) this can be rewritten as:

\[
\frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 = -\frac{r^2}{a(1 - e^2)} \left( \frac{d\theta}{dt} \right)^2 
\] (5.15)

Substituting these relations into Equation (5.11) yields:

\[
\frac{d^2 s}{d\theta^2} + 2i \frac{ds}{d\theta} = \frac{r}{a(1 - e^2)} \left[ s - \mu_1 \frac{s - s_1}{r_1^2} - \mu_2 \frac{s - s_2}{r_2^2} \right] 
\] (5.16)
This can be written more compactly as:

\[
\frac{d^2 s}{d\theta^2} + 2i \frac{ds}{d\theta} = \nabla_s \omega(s) \tag{5.17}
\]

where:

\[
\omega(s) = \frac{1}{1 + e \cos \theta} \Omega(s) \tag{5.18}
\]

given that:

\[
\Omega(s) = \frac{1}{2} (x'^2 + y'^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \tag{5.19}
\]

The equations of motion in the rotating, pulsating coordinate system for the ER3BP are now fully defined. The real and imaginary parts of Equation (5.17) give the equations of motion. To summarise, these equations of motion read:

\[
\begin{align*}
\frac{d^2 x'}{d\theta^2} - 2 \frac{dy'}{d\theta} &= \frac{\partial \omega}{\partial x'} \\
\frac{d^2 y'}{d\theta^2} + 2 \frac{dx'}{d\theta} &= \frac{\partial \omega}{\partial y'}
\end{align*} \tag{5.20a,b}
\]

where \(\omega(s)\) is given by Equation 5.18. Under conditions mentioned previously, this problem reduces to the CR3BP as expected. The structure of these equations of motion mimics the structure of the CR3BP equations of motion. This suggests that the ER3BP might share similar characteristics.

### 5.2 An Invariant Relation

It was seen that the CR3BP possesses an integral known as Jacobi’s Integral. Following a similar development it is possible to derive a similar property of the ER3BP. Multiplying Equation (5.20a) by \(\frac{dx'}{d\theta}\), Equation (5.20b) by \(\frac{dy'}{d\theta}\), summing and integrating yields:

\[
\left(\frac{dx'}{d\theta}\right)^2 + \left(\frac{dy'}{d\theta}\right)^2 = 2 \int_{\theta_0}^{\theta} (\omega_{x'} dx' + \omega_{y'} dy') \tag{5.21}
\]

The function \(\omega = \omega(x', y', \theta)\) depends on the independent variable \(\theta\). Hence the integrand in this case is not a total differential as:

\[
d\omega = \frac{\partial \omega}{\partial x'} dx' + \frac{\partial \omega}{\partial y'} dy' + \frac{\partial \omega}{\partial \theta} d\theta = \omega_{x'} dx' + \omega_{y'} dy' + \omega_{\theta} d\theta \tag{5.22}
\]

Hence Equation (5.21) can be rewritten as:

\[
\left(\frac{dx'}{d\theta}\right)^2 + \left(\frac{dy'}{d\theta}\right)^2 = 2 \int_{\theta_0}^{\theta} (d\omega - \frac{\partial \omega}{\partial \theta} d\theta) = 2\omega - 2 \int_{\theta_0}^{\theta} \frac{\partial \omega}{\partial \theta} d\theta - C \tag{5.23}
\]

where:

\[
\frac{\partial \omega}{\partial \theta} = \frac{e \sin \theta}{(1 + e \cos \theta)^2} \Omega \tag{5.24}
\]

and \(C\) is an integration constant. Equation (5.21) can therefore be written as:

\[
\left(\frac{dx'}{d\theta}\right)^2 + \left(\frac{dy'}{d\theta}\right)^2 = 2\omega - 2e \int_{\theta_0}^{\theta} \frac{\Omega \sin \theta}{(1 + e \cos \theta)^2} d\theta - C \tag{5.25}
\]

It is clear that when \(e = 0\) this equation reduces to Jacobi’s Integral for the CR3BP.
5.3 Surfaces of Hill

If \( e \neq 0 \) and an orbit is considered for a short interval i.e. if the motion is selected to start at \( \theta = 0 \) then the part of the orbit considered lies between \( \theta = 0 \) and \( \theta = +\epsilon \) where \( \epsilon \) is a sufficiently small positive quantity. Since \( \theta \) reflects the independent variable this amounts to a short time interval during which both primaries describe sufficiently small arcs of motion. The second term on the right-hand side of Equation (5.25) is a product of \( e \) and \( \epsilon \), which is smaller than the first term \( 2\omega \).

Therefore the zero velocity curves can be approximated by:

\[
2\Omega - C(1 + e \cos \theta) = 0 \tag{5.26}
\]

since on these curves the velocity components in the pulsating, rotating reference frame must be zero i.e. \( \frac{dx'}{\theta} = \frac{dy'}{\theta} = 0 \). At every instant (each \( \theta \)) therefore a different set of zero velocity curves describe the forbidden regions for the third body. The variation of these curves along the trajectory of the primaries is governed by the term \( C(1 + e \cos \theta) \). This variation shows periodicity, hence they are termed pulsating surfaces. This approximation only holds if the integral term is is neglected as in reality the integral term introduces a variation which is first order in the eccentricity.

Since the integral is dependent on the independent variable \( \theta \) it follows that, unlike in the case of Jacobi’s Integral, Equation (5.21) is not only dependent on the beginning and end points of the integration but also on the path taken. This means that unless the solutions to the equations of motion are known \((x'(\theta), y'(\theta))\), the integral in this equation cannot be evaluated as \( \Omega = \Omega(x'(\theta), y'(\theta)) \) cannot be evaluated. However, if the time period considered is short it is possible to reason that locally a set of zero velocity curves are defined by the approximation given by Equation (5.26).

5.4 Lagrange Libration Points

In the CR3BP five equilibrium points, denoted the L1, L2, L3, L4 and L5 Lagrange Libration points, were established. These five equilibrium points persist in the ER3BP. By taking \( \frac{dx'}{\theta} = \frac{dy'}{\theta} = \frac{d^2x'}{\theta^2} = \frac{d^2y'}{\theta^2} = 0 \) it follows from Equations (5.18) through to (5.20b) that:

\[
\frac{d\omega}{dx'} = \frac{d\omega}{dy'} = 0 \tag{5.27}
\]

Hence using Equations (5.18) and (5.19) it follows that:

\[
\frac{d\Omega}{dx'} = \frac{d\Omega}{dy'} = 0 \tag{5.28}
\]
Hence it follows that:

\[ \begin{align*}
    x' - \mu_1 \frac{(x' - x'_1)}{r_1^3} - \mu_2 \frac{(x' - x'_2)}{r_2^3} &= 0 \\
y' \left( 1 - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3} \right) &= 0
\end{align*} \]  

(5.29a)

(5.29b)

These correspond directly with the equations obtained for the CR3BP. Hence it is concluded that the location of the five Lagrange points in the ER3BP is the same as in the corresponding CR3BP. It is important to note that the location of these five Lagrange points is invariant only in the \( x' y' z' \) reference frame and not in the \( xyz \) reference frame. In the \( xyz \) these points are seen to pulsate. This characteristic motion is governed by the \( \frac{1}{1 + e \cos \theta} \) term given by Equation (5.18).

As with the CR3BP, the equations of motion for the ER3BP can be linearised in the neighbourhood of the Lagrange points. This linearisation is given as:

\[ \begin{bmatrix}
    \dot{x}'_1 \\
    \dot{x}'_2 \\
    \dot{y}'_1 \\
    \dot{y}'_2
\end{bmatrix}
= \frac{1}{1 + e \cos \theta}
\begin{bmatrix}
    0 & 1 & 0 & 0 \\
    \frac{\partial^2 \Omega}{\partial x^2} & 0 & \frac{\partial^2 \Omega}{\partial x \partial y} & 2 \\
    0 & 0 & 0 & 1 \\
    \frac{\partial^2 \Omega}{\partial y \partial x} & -2 & \frac{\partial^2 \Omega}{\partial y^2} & 0
\end{bmatrix}
\begin{bmatrix}
    x'_1 \\
    x'_2 \\
    y'_1 \\
    y'_2
\end{bmatrix} \]  

(5.30)

where:

\[ \begin{align*}
    x'_1 &= x' \\
    x'_2 &= \dot{x}' \\
    y'_1 &= y' \\
    y'_2 &= \dot{y}'
\end{align*} \]  

(5.31a)

(5.31b)

(5.31c)

(5.31d)

Analysis of the eigenvalues of this system is more complicated as the right-hand side contains periodic terms as opposed to constant coefficients. Nevertheless it is seen that for the collinear points the instability found to exist in the CR3BP for all values of the mass parameters persists in the ER3BP.
Periodic orbits in the Restricted 3-Body Problem (R3BP) have been discovered and analysed by many people over the last century. The advent of high-speed computing has drastically improved the search techniques employed and has benefited the classification of these periodic orbits. Various different classes of periodic orbits have been shown to exist in the R3BP. Poincaré’s famous conjecture emphasises the importance of studying periodic orbits to understand the underlying dynamics of the problem of three bodies. Poincaré’s conjecture states that if a solution of the restricted problem is given, it is always possible to find a periodic solution such that the difference between the original solution and the periodic solution is as small as desired for any given length of time. In other words, small modifications of any set of initial conditions will result in periodic orbits.

Given the fact that the R3BP is a non-integrable dynamics system, in general it can be said that obtaining complete information about the evolution of any solution is impossible. The exception to this however is for asymptotic, periodic, or nearly-periodic trajectories. The long-term behaviour of such trajectories can be discerned based on analytical or semi-analytical techniques. Therefore, studying such solutions is of particular interest, as they provide a window to understand the long-term global dynamics of the problem.

Beyond these reasons, from a scientific and engineering perspective periodic orbits are of interest. Many phenomena in celestial mechanics can be explained by studying such orbits in the restricted problem. In addition, from an engineering perspective, such orbits can be of significance in space mission applications.

In Section 2.3, a general definition of periodic orbits from the perspective of dynamical systems was offered. In Section 4.2, an extensive analysis of the equations of motion of the Circular Restricted 3-Body Problem (CR3BP) was provided. Dynamics associated with the Lagrange points was explored and the existence of periodic orbits was demonstrated. In the planar case (PCR3BP), these periodic orbits are called Lyapunov orbits. An analysis of the dynamics in the neighbourhood of such orbits associated with the collinear Lagrange points was sketched, and the existence of stable and unstable manifolds associated with these periodic orbits was outlined. These manifold tubes represent the set of trajectories that are asymptotic to the periodic orbits in forward and/or backward time.
Transport phenomena in the restricted problem of three bodies is intimately tied to the existence of these invariant manifold structures. This was established in Section 4.2, where the transit and non-transit character of trajectories in relation to the manifold tubes in the PCR3BP was introduced. There are numerous space mission applications that can be explored by analysing these structures. Whilst periodic orbits in the R3BP have been studied extensively for over a hundred years, invariant manifolds have only been investigated within the last few decades. These manifold structures are highly non-linear, and chaotic processes within the R3BP mean that, generally, they are inaccessible to analytical techniques. With the advent of high-speed computing in the 1960’s, it became possible to explore the stable and unstable manifolds associated with periodic orbits numerically.

In this chapter, an introduction is provided to the computation and analysis of periodic orbits and invariant manifolds associated with L1 and L2, the collinear Lagrange points, in the CR3BP. The existence of such periodic orbits, stable and unstable invariant manifolds, and their properties was discussed in Section 4.2. The aim of this chapter is to present numerical algorithms to search for periodic orbits and their associated invariant manifolds. The numerical algorithms outlined in Section 6.1 can be found in various sources including [Howell, 1983], [Ocampo, 1996] and [Koon et al., 2006]. In Section 6.2, a sample set of results will be presented to illustrate such periodic orbits and their associated stable and unstable invariant manifolds.

6.1 Numerical Algorithms

A standard numerical algorithm can be used to locate periodic orbits in the CR3BP. This numerical algorithm uses a differential corrector technique to converge on a periodic solution. The algorithm will be briefly outlined in this section. Once periodic orbits have been found, it is possible to determine the associated invariant manifolds in the CR3BP. The algorithm used to determine both branches of the unstable and stable manifold will be outlined. The algorithms presented to determine periodic orbits and associated invariant manifolds can be found in a number of standard texts, including [Howell, 1983].

6.1.1 State Transition Matrix and Variational Equations

The state transition matrix (STM) and variational equations are key to the numerical algorithms used to determine periodic orbits and their associated invariant manifolds. The STM relates changes in initial state with changes in final state of a trajectory: it is a linear approximation of the sensitivity of the final state to changes in the initial state. The variational equations describe a dynamical system that reflects how a perturbed trajectory evolves in time with respect to a reference trajectory. These concepts are used to determine the location of periodic orbits using differential correction.

To derive the STM and variational equations, an autonomous dynamical system of the following form is considered:

\[ \dot{x} = f(x) \quad (6.1) \]
In this equation, \( x \) and \( f \) are both vector functions as described in Section 2.1. The initial condition for Equation (6.1) is given by \( x(t_0) = x_0 \). An initial reference trajectory of the equations of motion with this initial state can be denoted as \( \phi(t; x_0) \). A trajectory that starts from a perturbed initial condition \( x_0 + \delta x_0 \) with respect to this reference trajectory can then be denoted as \( \phi(t; x_0 + \delta x_0) \). This perturbed trajectory is initially separated from the reference trajectory by a distance \( \delta x_0 \). The separation distance progresses in time based on the following relation:

\[
\delta x(t) = \phi(t; x_0 + \delta x_0) - \phi(t; x_0) \tag{6.2}
\]

At time \( t_1 \) the distance of separation is:

\[
\delta x(t_1) = \phi(t_1; x_0 + \delta x_0) - \phi(t_1; x_0) \tag{6.3}
\]

Figure 6.1 illustrates the progression of the reference and perturbed trajectories. Expanding Equation (6.2) as a Taylor series about the initial state yields:

\[
\delta x = \frac{\partial \phi(t; x_0)}{\partial x_0} \delta x_0 + O((\delta x_0)^2) \tag{6.4}
\]

It is to be noted that the partial derivative with respect to the time variable is zero as the final time \( t \) is fixed. Therefore, to first-order it follows that:

\[
\delta x = \frac{\partial \phi(t; x_0)}{\partial x_0} \delta x_0 \tag{6.5}
\]

The term \( \frac{\partial \phi(t; x_0)}{\partial x_0} \) is known as the STM, and is abbreviated as \( \Phi(t; t_0) \). The STM plays an important role in the differential correction process.

The variational equations are a linearised set of differential equations that describe the evolution in time of the perturbation. These equations can be derived by considering the differential equation that describes the perturbed trajectory:

\[
\dot{x} + \delta x = f(x + \delta x) \tag{6.6}
\]
Expanding this about the unperturbed state yields:

\[ \dot{x} + \delta \dot{x} = f(x) + \frac{\partial f}{\partial x} \delta x + O((\delta x)^2) \]  

(6.7)

Neglecting higher-order terms and using Equation (6.1), to first-order this gives:

\[ \dot{\delta x} = \frac{\partial f}{\partial x} \delta x \]  

(6.8)

Equation (6.8) describes the variational equations and \( \frac{\partial f}{\partial x} \) is known as the State Propagation Matrix (SPM). The variational equations can be related to the STM by differentiating Equation (6.5) to yield:

\[ \dot{\delta x} = \Phi(t; t_0) \delta x(t_0) + \Phi(t; t_0) \dot{\delta x}(t_0) \]  

(6.9)

This follows from the fact that perturbation of the initial state \( (\delta x(t_0)) \) is kept constant; hence \( \dot{\delta x}(t_0) = 0 \). Substituting this relation into Equation (6.8) and using Equation (6.5) yields:

\[ \Phi(t_1; t_0) = \frac{\partial f}{\partial x} \Phi(t; t_0) \]  

(6.10)

Equation (6.10) therefore provides a set of differential equations that can be used to determine the time evolution of the STM. This provides a means, to first-order, to determine the transition between a perturbed initial state and final state without having to recalculate the trajectory. An important property of the STM is the fact that \( \Phi(t; t) = I \), where \( I \) is a 6 \( \times \) 6 identity matrix.

The SPM is time-dependent, which means that Equation (6.10) cannot be solved analytically if an analytical solution for the reference trajectory is not known. Hence, in such a case the SPM must be evaluated at each point along the trajectory and the STM must be determined through numerical integration.

The SPM for the CR3BP is a 6 \( \times \) 6 matrix consisting of four 3 \( \times \) 3 submatrices. The structure of the SPM, determined by using the equations of motion for the CR3BP given in Section 4.1.1, is:

\[ \tilde{F} = \begin{bmatrix} \tilde{0} & \tilde{I} \\ \tilde{0}_{xx} & 2\tilde{A} \end{bmatrix} \]  

(6.11)

In Equation (6.11), \( \tilde{0} \) is a 3 \( \times \) 3 zero matrix, \( \tilde{I} \) is a 3 \( \times \) 3 identity matrix, and \( \tilde{A} \) is given by:

\[ \tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  

(6.12)

The submatrix \( \tilde{0}_{xx} \) is the symmetric matrix of second partial derivatives of \( \Omega \) with respect to \( x, y, z \) evaluated along the trajectory, where \( \Omega \), given in Section 4.1, is:

\[ \Omega = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2} \mu(1 - \mu) \]  

(6.13)
In other words, \( \ddot{\Omega}_{xx} \) is given by:

\[
\ddot{\Omega}_{xx} = \begin{bmatrix}
\frac{\partial^2 \Omega}{\partial x^2} & \frac{\partial^2 \Omega}{\partial x \partial y} & \frac{\partial^2 \Omega}{\partial x \partial z} \\
\frac{\partial^2 \Omega}{\partial y \partial x} & \frac{\partial^2 \Omega}{\partial y^2} & \frac{\partial^2 \Omega}{\partial y \partial z} \\
\frac{\partial^2 \Omega}{\partial z \partial x} & \frac{\partial^2 \Omega}{\partial z \partial y} & \frac{\partial^2 \Omega}{\partial z^2}
\end{bmatrix}
\tag{6.14}
\]

In this equation, \( \frac{\partial^2 \Omega}{\partial x^2} = \frac{\partial^2 \Omega}{\partial y \partial x} = \frac{\partial^2 \Omega}{\partial z \partial x} \), \( \frac{\partial^2 \Omega}{\partial y \partial y} = \frac{\partial^2 \Omega}{\partial y \partial z} \), \( \frac{\partial^2 \Omega}{\partial z \partial y} = \frac{\partial^2 \Omega}{\partial z^2} \), and the partial derivatives are given by:

\[
\frac{\partial^2 \Omega}{\partial x^2} = 1 - \frac{(1 - \mu)}{r_1^3} - \frac{\mu}{r_2^3} + 3 \left[ (x + \mu) \frac{(1 - \mu)}{r_1^3} + (x + \mu - 1) \frac{\mu}{r_2^3} \right]
\tag{6.15a}
\]

\[
\frac{\partial^2 \Omega}{\partial y^2} = 1 - \frac{\mu}{r_1^3} - \frac{\mu}{r_2^3} + 3y^2 \left[ \frac{1 - \mu}{r_1^5} + \frac{\mu}{r_2^5} \right]
\tag{6.15b}
\]

\[
\frac{\partial^2 \Omega}{\partial z^2} = -\frac{1}{r_1^3} - \frac{\mu}{r_2^3} + 3z^2 \left[ \frac{1 - \mu}{r_1^5} + \frac{\mu}{r_2^5} \right]
\tag{6.15c}
\]

\[
\frac{\partial^2 \Omega}{\partial x \partial y} = 3y \left[ (x + \mu) \frac{1 - \mu}{r_1^3} + (x + \mu - 1) \frac{\mu}{r_2^3} \right]
\tag{6.15d}
\]

\[
\frac{\partial^2 \Omega}{\partial x \partial z} = 3z \left[ (x + \mu) \frac{1 - \mu}{r_1^5} + (x + \mu - 1) \frac{\mu}{r_2^5} \right]
\tag{6.15e}
\]

\[
\frac{\partial^2 \Omega}{\partial y \partial z} = 3yz \left[ \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} \right]
\tag{6.15f}
\]

With the SPM determined, the STM matrix at each integration point along the trajectory is found by integrating Equation (6.10). These differential equations for the STM can be written as a set of 36 first-order differential equations. In combination with the 6 differential equations for the state vector, this yields a full set of 42 first-order differential equations, to be integrated simultaneously at each time step. For purposes of completeness, the set of differential equations to be integrated is summarised here:

\[
\dot{x} = f(x)
\tag{6.16a}
\]

\[
\Phi(t; t_0) = \tilde{F} \Phi(t; t_0)
\tag{6.16b}
\]

The initial conditions provided for this set of differential equations are:

\[
x(t_0) = x_0
\tag{6.17a}
\]

\[
\Phi(t_0; t_0) = \bar{I}_{6 \times 6}
\tag{6.17b}
\]

### 6.1.2 Differential Correction

A differential corrector method is used to locate periodic orbits. Provided an initial guess is supplied in the neighbourhood of a periodic orbit, the differential correction process iterates to correct the initial state such that it corresponds with a point on the target periodic orbit. This is a targeting process through which a reference trajectory (initial guess) is modified such that a new trajectory is found,
with the end condition corresponding with that of a periodic orbit. In order for this targeting process to work, sensitivities of changes in the final state to small changes in the initial state must be known. To determine these sensitivities, the STM is used.

Given is a trajectory that is slightly off and has to be corrected at time $t$. This can be expressed as:

$$x(t) = \phi(t, t_0; x_0) = x_d - \delta x$$  \hspace{1cm} (6.18)

The desired final state at $t$ is denoted $x_d$. A perturbed trajectory is given by $\phi(t, t_0; x_0 + \delta x_0)$. Expanding this in a Taylor series about the reference trajectory gives:

$$\phi(t, t_0; x_0 + \delta x_0) = \phi(t, t_0; x_0) + \frac{\partial \phi(t, t_0; x_0)}{\partial x_0} \delta x_0 + O((\delta x_0)^2)$$  \hspace{1cm} (6.19)

Using Equations (6.5) and (6.18) this can be rewritten as:

$$\phi(t, t_0; x_0 + \delta x_0) = x + \delta x + O((\delta x_0)^2)$$

$$= x_d + O((\delta x_0)^2)$$  \hspace{1cm} (6.20)

Hence, to first-order, it is clear that by changing the initial state $x_0$ by $\delta x_0$, the desired state $x_d$ can be achieved. This is the principle idea of differential correction: making a small change at one end to target a desired state at the other end. Since this only provides a first-order approximation of the desired state, the process is iterated until the desired state is achieved to within a specified tolerance. The process of iteration produces convergence to the desired trajectory within the given tolerance: $\phi(t, t_0; x_0 + \Delta x_0) = x_d$, where $\Delta x_0$ is the accumulation of corrections ($\delta x_0$) to the initial state. This is in essence the same process as Newton’s root-finding method for the flow $F(x_0) = \phi(t, t_0; x_0) - x_d$.

### 6.1.3 Computation of Periodic Orbits

Periodic orbits can be located in the CR3BP by using certain properties of such solutions. These orbits must be symmetric with respect to the $xz$-plane ($y = 0$), which is the plane perpendicular to the plane containing the primaries $m_1$ and $m_2$, following from simple geometric considerations. Further, these orbits intersect the $xz$-plane perpendicularly ($\dot{x} = \dot{z} = 0$). Hence, the state vector, on crossing the $y$-axis, is of the form:

$$\vec{x}_0 = (x_0 \ 0 \ z_0 \ 0 \ \dot{y}_0 \ 0)^T$$  \hspace{1cm} (6.21)

This can be used as an initial state vector to locate periodic orbits. To obtain a first guess of this initial state vector, analytical approximations of the periodic orbits, or reference data given in literature such as [Howell, 1983] can be employed. Given this first guess, the trajectory is integrated until the next crossing of the $xz$-plane. For a periodic orbit it must hold that the final state at this crossing must be perpendicular i.e. of the form:

$$\vec{x}_1 = (x_1 \ 0 \ z_1 \ 0 \ \dot{y}_1 \ 0)^T$$  \hspace{1cm} (6.22)
In this equation, the sign of $\dot{y}_1$ must be opposite to that of $\dot{y}_0$. The first crossing occurs at a time equal to one half the period of the orbit. It is likely that at the first crossing $\dot{x}_1$ and $\dot{z}_1$ will be non-zero; hence the trajectory is not a periodic orbit. The goal is then to manipulate the initial state vector in such a manner that the final state at the first crossing of the $xz$-plane is of the form given previously, yielding a periodic orbit. This is achieved through a differential correction process.

The differential corrector makes use of the STM at the first crossing. The STM, as described previously, relates changes in the initial state to changes in the final state. Given that $\dot{x}_1$ and $\dot{z}_1$ are non-zero, corrections have to be made to the initial state vector. These corrections are given by:

$$\delta \dot{x}_0 = (\delta x_0 \ 0 \ \delta z_0 \ 0 \ \delta \dot{y}_0 \ 0)^T$$

Since at the first crossing it is enforced that $y = 0$, the only desired changes in the final state are $\delta \dot{x} = -\dot{x}$ and $\delta \dot{z} = -\dot{z}$. The corrections to the initial state can be calculated by considering the following Taylor series expansion:

$$\delta x = \frac{\partial \phi(t; x_0)}{\partial x_0} \delta x_0 + \frac{\partial \phi(t; x_0)}{\partial \dot{t}} \delta \left( \frac{T}{2} \right) + O((\delta x_0)^2) + O \left( \left( \delta \left( \frac{T}{2} \right) \right)^2 \right)$$

This is similar to the expansion given in Equation (6.4), however the partial derivative with respect to time is not zero, as the method used to locate the periodic orbit is in fact a two-point boundary value problem where time is allowed to vary. This is reflected by the fact that changes in the initial state vector lead to different trajectories with different orbital periods ($\delta \left( \frac{T}{2} \right)$). To first-order therefore:

$$\delta x = \Phi \left( \frac{T}{2}, 0 \right) \delta x_0 + \frac{\partial \phi(t; x_0)}{\partial \dot{t}} \delta \left( \frac{T}{2} \right)$$

Equation (6.25) represents a set of six scalar equations. It is known that the boundary condition $y \left( \frac{T}{2} \right) = 0$ must hold; hence $\delta y = 0$. Therefore, the second equation yields the following:

$$\delta y = 0 = \Phi_{21} \delta x_0 + \Phi_{23} \delta z_0 + \Phi_{25} \delta \dot{y}_0 + \dot{y} \delta \left( \frac{T}{2} \right)$$

By rewriting this equation, it follows that:

$$\delta \left( \frac{T}{2} \right) = -\frac{1}{\dot{y}} \left( \Phi_{21} \delta x_0 + \Phi_{23} \delta z_0 + \Phi_{25} \delta \dot{y}_0 \right)$$

Since it is desired that only two variables are changed in the final state ($\dot{x}, \dot{z}$), and there are three initial conditions that can be varied ($x_0, z_0, \dot{y}_0$), it follows that one of the initial conditions can be chosen to be fixed. If, for instance, it is chosen that $x_0$ remains fixed i.e. $\delta x_0 = 0$, then the fourth and sixth scalar equations given in Equation (6.25), combined with Equation 6.27, yields the following:

$$\begin{bmatrix} \delta \dot{x} \\ \delta \dot{z} \end{bmatrix} = \begin{bmatrix} \Phi_{43} & \Phi_{45} \\ \Phi_{63} & \Phi_{65} \end{bmatrix} - \frac{1}{\dot{y}} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \Phi_{23} & \Phi_{25} \end{bmatrix} \begin{bmatrix} \delta z_0 \\ \delta \dot{y}_0 \end{bmatrix}$$

In other words, the necessary corrections to the initial state vector are given by:

$$\begin{bmatrix} \delta z_0 \\ \delta \dot{y}_0 \end{bmatrix} = \left( \begin{bmatrix} \Phi_{43} & \Phi_{45} \\ \Phi_{63} & \Phi_{65} \end{bmatrix} - \frac{1}{\dot{y}} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} \begin{bmatrix} \Phi_{23} & \Phi_{25} \end{bmatrix} \right)^{-1} \begin{bmatrix} \delta \dot{x} \\ \delta \dot{z} \end{bmatrix}$$
Similarly, if \( z_0 \) is kept fixed i.e. \( \delta z_0 = 0 \) it follows that:

\[
\begin{bmatrix}
\delta x_0 \\
\delta \dot{y}_0
\end{bmatrix} = \left( \begin{bmatrix}
\Phi_{41} & \Phi_{45} \\
\Phi_{61} & \Phi_{65}
\end{bmatrix} - \frac{1}{\dot{y}} \begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} \begin{bmatrix}
\Phi_{21} & \Phi_{25} \\
\Phi_{61} & \Phi_{65}
\end{bmatrix} \right)^{-1} \begin{bmatrix}
\delta \dot{x} \\
\delta \dot{z}
\end{bmatrix}
\] (6.30)

Finally, if \( \dot{y}_0 \) is kept fixed i.e. \( \delta \dot{y}_0 = 0 \) it follows that:

\[
\begin{bmatrix}
\delta x_0 \\
\delta z_0
\end{bmatrix} = \left( \begin{bmatrix}
\Phi_{41} & \Phi_{43} \\
\Phi_{61} & \Phi_{63}
\end{bmatrix} - \frac{1}{\dot{y}} \begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} \begin{bmatrix}
\Phi_{21} & \Phi_{23} \\
\Phi_{61} & \Phi_{63}
\end{bmatrix} \right)^{-1} \begin{bmatrix}
\delta \dot{x} \\
\delta \dot{z}
\end{bmatrix}
\] (6.31)

Any of these three methods will provide means to assess necessary corrections to the initial state vector to converge upon a periodic orbit. Convergence with this method is generally rapid. Since the orbit is symmetric about the \( xz \)-plane, integration is not necessary to determine the second half of the periodic orbit.

In summary, the numerical recipe for the differential correction process to locate periodic orbits is as follows:

1. Find a suitable initial state vector of the form \( \bar{x}_0 = (x_0 \ 0 \ z_0 \ 0 \ \dot{y}_0 \ 0)^T \).
2. Integrate initial state vector until first crossing of \( xz \)-plane.
3. Check values of \( \dot{x}, \dot{z} \). If larger in magnitude than specified tolerance, proceed with algorithm. If magnitude is less than specified tolerance, a periodic orbit has been located and algorithm can be stopped. Check that final state vector is of the form \( \bar{x}_1 = (x_1 \ 0 \ z_1 \ 0 \ \dot{y}_1 \ 0)^T \).
4. Determine corrections to initial state vector by fixing one of the three non-zero initial state values and using differential correction scheme.
5. Apply corrections to initial state vector and return to Step 2.

### 6.1.4 Computation of Invariant Manifolds

The unstable and stable manifolds associated with a periodic orbit can be computed numerically. For this, the monodromy matrix \( \bar{M} \) is required, which is the STM associated with the periodic orbit after an integration of one cycle i.e. \( \Phi(T, t_0) \), where \( T \) is the period of the periodic orbit. For any point \( x \) along the periodic orbit, the monodromy matrix serves as a linearisation. In fact, a periodic orbit is a fixed point of a Poincaré-map, as the return orbit maps the point onto itself (Section 2.4) and the monodromy matrix serves as a linearisation of the Poincaré-map in the neighbourhood of the fixed point. The characteristics of the local geometry of the phase space can therefore be determined from the eigenvalues and eigenvectors of the monodromy matrix. In particular, the eigenvectors can be used to approximate local invariant manifolds near \( x \).

The eigenvalues of the monodromy matrix turn out to be of the form: \( \lambda_1 > 1, \lambda_2 = \frac{1}{\lambda_1}, \lambda_3 = \lambda_4 = 1, \lambda_5 = \lambda_6, |\lambda_5| = 1 \). The eigenvalues \( \lambda_5 \) and \( \lambda_6 \) are complex conjugates, and \( \lambda_1 \) and \( \lambda_2 \) are real. Each of these eigenvalues contributes to the local dynamics. A more in-depth analysis of these eigenvalues is given by [Koon et al., 2006]. The dynamical interpretation of these eigenvalues is summarised as follows:
\( \lambda_1, \lambda_2 \): These eigenvalues are associated with the stable and unstable characteristics of the periodic orbit. They reflect the hyperbolic nature of the dynamics in the neighbourhood of the periodic orbit. The associated eigenvectors give the direction of the hyperbolic character. Using this, approximations to the full stable and unstable manifold can be generated.

\( \lambda_3, \lambda_4 \): These are neutral eigenvalues that are associated with motion tangent to the orbit and energy variations associated with transitioning across a family of periodic orbits.

\( \lambda_5, \lambda_6 \): These eigenvalues are associated with a rotation and is related to the existence of quasi-periodic orbits in the neighbourhood of the periodic orbit.

To generate approximations to the full stable and unstable manifolds therefore, the eigenvalues \( \lambda_1, \lambda_2 \) are used. The eigenvalue \( \lambda_1 \) and its associated eigenvector \( v_1 \) are used to generate the unstable manifold, and the eigenvalue \( \lambda_2 \) and its associated eigenvector \( v_2 \) are used to generate the stable manifold. These manifolds are generated by using the following initial guesses:

\[
\begin{align*}
    x^s(x_0) &= x_0 \pm \epsilon v^s(x_0) \quad (6.32a) \\
    x^u(x_0) &= x_0 \pm \epsilon v^u(x_0) \quad (6.32b)
\end{align*}
\]

The parameter \( \epsilon \) yields a small displacement from the point \( x_0 \) on the periodic orbit. The magnitude of \( \epsilon \) should be small enough such that linearisation in the neighbourhood of the periodic orbit is valid, but not too small such that the Time-of-Flight (ToF) is too large due to the asymptotic nature of the stable and unstable manifolds. By integrating the estimate for the unstable manifold forwards in time and the estimate for the stable manifold backwards in time, it is possible to generate approximations that shadow the stable and unstable manifolds. The \('\pm'\) sign allows for both branches of these manifolds to be generated. Carrying this out at a number of different points along the periodic orbit produces a tube of trajectories that reflects the behaviour of both the stable and unstable manifolds.

## 6.2 Numerical Results

Results obtained from running the algorithms described in the previous section, to locate periodic orbits and their associated unstable and stable invariant manifolds in the PCR3BP, are presented in this section. To benchmark the accuracy and fidelity of the implementation of these algorithms, reference data from [Howell, 1983] is used. In [Howell, 1983] families of three-dimensional periodic orbits (Halo orbits) in the neighbourhood of the collinear Lagrange points (L1, L2 and L3) are explored. Table 6.1 summarises reference data for a family of Halo orbits about L1 in the CR3BP. The data given in this table is used to generate initial guesses for the initial state vector. The data given in this table provide the precise initial state vector needed to generate Halo orbits. These initial conditions are perturbed slightly, since the aim is then to observe whether the implementation of the differential correction process is able to reproduce the exact reference values. Table 6.2 contains the results obtained after applying this differential correction process to perturbed initial conditions of the L1 family of Halo orbits. The data presented in this case uses the differential correction matrix given by Equation (6.31) i.e. \( y_0 \) is kept constant. Inspecting these results indicates that the implementation of
Table 6.1  Initial conditions for family of Halo orbits at L1 ($\mu = 0.04$) [Howell, 1983].

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$z_0$</th>
<th>$\dot{y}_0$</th>
<th>$\frac{t}{\tau}$</th>
</tr>
</thead>
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<td>0.198019</td>
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<td>0.342029</td>
<td>1.349887</td>
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<td>1.348532</td>
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<td>0.313788</td>
<td>0.271306</td>
<td>0.978635</td>
</tr>
</tbody>
</table>

Table 6.2  Verification of family of Halo orbits at L1 ($\mu = 0.04$).

The numerical algorithms seems to be working, as the initials conditions obtained match the reference data very well. Figure 6.2 illustrates the family of Halo orbits being considered i.e. the twelve Halo orbits generated using the initial conditions given in Table 6.2. Figure 6.3 illustrates the two branches of the stable and unstable manifolds associated with the first L1 Halo orbit given in the reference data. These manifolds were generated using the numerical algorithm outlined in the previous section. The tube-like structure of the manifolds is apparent from this figure. It is important to note that this figure merely represents the structure of the manifolds in configuration space. These structure however occupy phase space. The periodic orbit and associated manifolds have the property of being associated to a single energy level given by the Jacobi energy integral (Section 4.1). Hence these structures are five-dimensional in phase space, which consists of six dimensions.
6.2 Numerical Results

Figure 6.2  Family of Halo orbits at L1 in CR3BP (xy-plane).

Figure 6.3  Stable (red) and unstable (blue) manifolds of Halo orbit at L1. The Halo orbit used is the first one given in Table 6.2.
Chapter 7

The Weak Stability Boundary: Theory

The Weak Stability Boundary (WSB) is a concept that was developed mainly in light of research carried out by E. Belbruno, in particular relating to the rescue of the Hiten spacecraft in 1990 [Belbruno, 2004]. The concept of the WSB is intimately tied to the notion of weak capture. In this chapter, a number of different topics are covered pertaining to capture. Fundamental definitions relating to capture are given in Section 7.1.

Weak capture, or ballistic capture, in the restricted problem of three bodies is of interest for a number of reasons. Firstly, weak capture in the restricted problem of three bodies is linked to transit behaviour of trajectories, which in turn stems from the chaotic nature of the dynamical system. This idea was initially explored in-depth by [Conley, 1968] in the 1960’s. Conley described the existence of transit orbits in the Planar Circular Restricted 3-Body Problem (PCR3BP). By understanding weak capture better, it might be possible to shed further light on the underlying dynamics of the problem. Secondly, weak capture is interesting from a space mission application perspective. A case for this will be made in Chapter 9, where a number of possible mission scenarios and applications will be presented. In Section 7.1.3, a general description of the capture problem is given, which describes the problem of determining a ballistic capture transfer from the vicinity of one of the primaries to the vicinity of the other primary in the restricted problem of three bodies.

In Section 7.2, definitions are given for the WSB, which is a visualisation of the region in phase space that supports weak capture. An algorithmic definition is given, which describes a set of points in phase space that belong to the WSB. Subsequently, a definition is given that is aimed at approximating this set using analytical sets.

7.1 Introduction to Capture

The definitions in this section have been adapted from [Belbruno, 2004]. The development of the concept of capture will be given here in the context of the 3-Body Problem (3BP) of celestial mechanics. Chapter 3 provided an introduction to the basics of celestial mechanics. Chapters 4 and 5 outlined the basic equations of
motion and some properties of the Circular Restricted 3-Body Problem (CR3BP) and Elliptic Restricted 3-Body Problem (ER3BP) respectively. Capture in astrodynamics refers to bounded motion of one of the bodies in the system with respect to the rest of the bodies.

### 7.1.1 Geometric

Here a geometric interpretation of capture will be presented. The position vector of a mass \( m_j \) in a system containing \( n \) bodies, with respect to the barycentre of the system, is denoted by \( r_j = (x_j, y_j, z_j) \) \((j = 1, 2, \ldots, n)\). The position vector of \( m_j \) with respect to another mass \( m_i \) is denoted by \( r_{ij} = (x_{ij}, y_{ij}, z_{ij}) \) where \( i = 1, 2, \ldots, n \) and \( j \neq i \).

**Definition 7.1** Permanent capture for the three-dimensional N-Body Problem occurs in forward time if, as \( t \to \infty \), all \( r_{ij}, j \neq i \), are bounded, and as \( t \to -\infty \), at least one of the distances \( r_{ij} \) tends to infinity. An analogous definition for permanent capture in backward time follows.

Specifically, for the 3-Body Problem (3BP) i.e. \( n=3 \), it has been shown by Chazy, Hopf and others that permanent capture is unlikely to exist, if it exists at all [Belbruno, 2004]. For the ER3BP, the concept of permanent capture can be stated as follows:

**Definition 7.2** Mass \( m \) is permanently captured into the \( m_1, m_2 \)-system in forward time if, as \( t \to \infty \), \(|r(t)|\) is bounded, and, as \( t \to -\infty \), \(|r| \to \infty \). Once again, an analogous definition for permanent capture in backward time follows.

In this definition, mass \( m \) is the third mass in the system and its position vector is given by \( r \). The definitions of capture given in this section will all be in reference to the ER3BP as the CR3BP, described in Chapter 4, can be viewed as a special case where the eccentricity of the orbits of the primaries is equal to zero. Hence the definitions given here apply equally to the CR3BP.

Permanent capture orbits lie at the boundary between hyperbolic and bounded motion, with hyperbolic orbits leading to the third mass escaping from the system and bounded orbits resulting in bounded motion with respect to the other masses in the system for all time. Sitnikov demonstrated, by analysing a special form of the ER3BP (known as the Sitnikov Problem), that permanent capture exists [Belbruno, 2004]. Alekseev furthered this results by pursuing other methods and managed to show that this motion is chaotic [Belbruno, 2004]. Intuitively, it is not difficult to imagine that permanent capture exists, as between bounded and hyperbolic orbits a small velocity increase of mass \( m \) could cause it to become unbounded in backward time with respect to the rest of the system, and a small velocity decrease of mass \( m \) could cause it to become bounded in forward time. Proving this phenomenon, however, turns out to be very difficult.

The boundary between hyperbolic and bounded motion is defined by parabolic orbits. Such orbits were mentioned in the context of the 2-Body Problem (2BP) in Section 3.3. These orbits can be considered critical escape orbits. They divide the orbit space into trajectories which escape to infinity, and those which remain
bounded for all time. The definition of such parabolic orbits in the ER3BP can be given as:

**Definition 7.3** A parabolic orbit for the ER3BP occurs when
\[ \lim_{t \to \pm \infty} |r(t)| = \infty, \lim_{t \to \pm \infty} |\dot{r}(t)| = 0. \]
The case \( t \to \infty \) is denoted as \( \omega \)-parabolic orbits, and the case \( t \to -\infty \) is denoted as \( \alpha \)-parabolic orbits.

For completeness, the definition of hyperbolic orbits is given as:

**Definition 7.4** A hyperbolic orbit for the ER3BP occurs when
\[ \lim_{t \to \pm \infty} |r(t)| = \infty, \lim_{t \to \pm \infty} |\dot{r}(t)| > 0. \]

**Definition 7.5** A mass \( m \) in the ER3BP is said to have undergone unbounded escape from the \( m_1, m_2 \)-system if in forward time \( \lim_{t \to \infty} |r(t)| = \infty \).

An orbit in permanent capture in forward time therefore leads to escape in backward time, according to the definition of permanent capture.

Besides permanent capture, another type of motion can exist between hyperbolic and bounded motion. This is known as oscillatory motion and is defined by the fact that as \( t \to \pm \infty \), \( \lim |r| \) does not exist. In this case, \( m \) oscillates, with the amplitude of the oscillation increasing as \( t \to \infty \). Oscillatory motion is formally defined as:

**Definition 7.6** Unbounded oscillatory motion for the general NBP occurs when
\[ \lim_{t \to \pm \infty} \sup_{j<k} r_{jk} = \infty, \lim_{t \to \pm \infty} \inf_{j<k} r_{jk} < \infty \] for at least one pair of masses, where \( n \geq 3, j<k \).

The \( \sup \) operator stands for *supremum*: a value is a supremum with respect to a set if it is at least as large as any element of that set. The \( \inf \) operator stands for *infimum*: a value is an infimum with respect to a set if all the elements of the set are at least as large as that value. Since a body undergoing unbounded oscillatory motion will bounce back and forth between local minima and maxima, it is clear that the conditions given in Definition 7.1.1 hold.

A body that is not permanently captured can be temporarily captured:

**Definition 7.7** In the ER3BP, mass \( m \) has temporary capture at \( t = t_0, |t| < \infty \), if \( |r(t)| < \infty \) and \( \lim_{t \to \pm \infty} |r(t)| = \infty \).

In other words, mass \( m \) is in temporary capture if during some finite time it is found in the vicinity of one of the primaries, and in infinite time (either backward or forward) escapes. In practice, it might be more practical to define finite temporary capture. A mass \( m \) is in finite temporary capture if \( |r(t)| \leq d > 0 \) at time \( T \). The duration of capture is then defined as \( |T - t_0| \), where \( t_0 \) is a reference value.

The concept of permanent capture is intimately linked to the notion of stability. It is clear from the definition of permanent capture that the motion of \( m \) remains bounded with respect to the rest of the system in forward time. It is possible to consider such motion stable, as it is known that this body will remain within the bounds of the system for all of time. Similarly, hyperbolic trajectories in forward
time can be viewed as unstable, as the body considered is guaranteed not to remain in the vicinity of the rest of the system for all of time. Characterisation of possible capture and escape orbits can therefore be closely linked to their stability. General stability definitions for dynamical systems were outlined in Section 2.8. The definitions presented here can be viewed as an application of those given in Section 2.8, in the context of astrodynamics.

Moser demonstrated the existence of complex dynamics between hyperbolic and bounded motions in the Sitnikov Problem. Llibre and Simó proved analogous results for the Planar Circular Restricted 3-Body Problem (PCR3BP) but only for the case where the Jacobi Constant is very large. Xia proved similar results for smaller values of the Jacobi Constant, which are interesting particularly for low thrust applications. Thus, it is clear that there has been plenty of research in this area [Belbruno, 2004]. However, the problem of chaotic motions still remains complex.

7.1.2 Analytical

Theoretically, the concept of capture can be approached from another perspective. The notions of capture presented thus far are all geometric i.e. they define capture with respect to the geometry of the bodies in the system, both in backward and forward time. Though this is intrinsically a fairly straightforward manner in which to define capture, it does not reflect the dynamics of the motion of the body considered. This can be important for research into the nature of chaotic motion and capture possibilities.

The capture that will be considered here is often called ballistic capture. Ballistic capture will be introduced in the context of the ER3BP. This can be generalised to the NBP quite easily. Ballistic capture in the ER3BP monitors the sign of the two-body Kepler energy function with respect to the smaller of the two primary masses in the system, \( m_2 \). An inertial coordinate system \((X, Y, Z)\), centred at \( m_2 \), is considered. The Kepler energy function of \( m \) with respect to \( m_2 \) in this coordinate system is defined as:

\[
E_{2}(R, \dot{R}) = \frac{1}{2} \left| \dot{R} \right|^2 - \frac{\mu}{R}
\]  

(7.1)

where \( R = R_{23} = (X_{23}, Y_{23}, Z_{23})^T \), and \( \mu \) is the mass parameter with \( 0 \leq \mu < \frac{1}{2} \).

This definition was also given in the context of the 2BP in Section 3.3.1. Ballistic capture is then defined as follows:

**Definition 7.9** \( m \) is ballistically captured at \( m_2 \) at time \( t = t_1 \) if \( E_2(\phi(t)) \leq 0 \), for a solution \( \phi(t) = (R(t), \dot{R}(t)) \) of the ER3BP relative to \( m_2 \), with \( \dot{R}(\phi(t)) > 0 \).

For the ER3BP, \( \phi(t) \) is considered to be a smooth solution for \( t_1 \leq t \leq t_2 \), \( t_1 < t_2 \), \( t_2 \) is finite.

**Definition 7.10** If \( E_2(\phi(t)) \leq 0 \) with \( t_1 < t < t_2 \), then \( \phi(t) \) is called a ballistic capture transfer from \( t = t_1 \) to \( t = t_2 \), relative to \( m_2 \).
The energy function $E_2$ does not need to be a monotonously decreasing function; it can even take on negative or zero values for $t \leq t_2$.

**Definition 7.11** If $E_2(\phi(t_1)) \leq 0$ and $E_2(\phi(t)) > 0$, then $\phi(t)$ is called a ballistic ejection transfer from $t = t_1$ to $t = t_2$, which defines ballistic ejection (or escape) from $m_2$.

Given that $\phi(t)$ is a ballistic capture transfer from $t_1$ to $t_2$, two types can be identified.

**Definition 7.12** A ballistic capture transfer is called unbounded if $t_1 = -\infty$ and $\lim_{t \to -\infty} |\phi(t)| = \infty$, where $\lim_{t \to -\infty} E_2(\phi(t)) > 0$, $E_2(\phi(t_2)) \leq 0$, $t_2$ finite. If $|t_1| < \infty$, then $\phi(t)$ is a bounded ballistic capture transfer.

Thus, for an unbounded capture transfer, $m$ starts infinitely far from $m_2$ and moves to ballistic capture at $m_2$ at $t = t_2$. This ballistic capture is permanent if $|\phi(t)| < \infty$ for all $t \geq t_2$. Temporary ballistic capture is defined as follows:

**Definition 7.13** $\phi(t)$ is a ballistic capture transfer from $t = t_1$ to $t = t_2$. If $E_2(\phi(t)) \leq 0$ for $t_2 \leq t \leq t_3$, $t_2 < t_3 < \infty$, and $E_2(\phi(t)) > 0$ for $t > t_3$, then $\phi(t)$ has temporary ballistic capture for $t_2 \leq t \leq t_3$. If $t_3 = \infty$, then $\phi(t)$ has permanent ballistic capture for $t_2 \leq t \leq \infty$.

It is important to note that permanent ballistic capture does not necessarily imply permanent capture, as if $\phi(t)$ is permanently ballistically captured for $t \geq t_2$, it is not necessarily unbounded as $t \to -\infty$, and not necessarily bounded as $t \to \infty$. If $\phi(t)$ is permanently ballistically captured for $t \geq t_2$, $|\phi(t)| < \infty$ for $t \geq t_2$, and $\lim_{t \to -\infty} |\phi(t)| = \infty$, then $\phi(t)$ is also permanently captured. Figure 7.1 illustrates the notion of temporary ballistic capture, as given by Definition 7.1.2. The definitions presented in this section provide preliminary classification of capture orbits. This classification is by no means absolute; however it does provide a starting point for further studies into capture and escape mechanisms.
The capture problem is defined for space mission applications and will be considered in the context of the CR3BP. The capture problem describes the minimization of the $\Delta V$ (velocity change that has to be brought about by e.g. rocket engines on a spacecraft) required to perform a transfer from a capture orbit around one primary to a capture orbit around the other primary. These capture orbits can be either permanent or temporary.

Traditionally, such optimal transfers have been considered by using the Hohmann transfer method (described in all standard literature on astrodynamics such as [Battin, 1987]), also known as the direct transfer method. The Hohmann transfer method uses a simplified model to determine a transfer trajectory from one primary to the other. The transfer trajectory is given by a conic section, as described in the context of the 2BP in Section 3.3. The Hohmann transfer is a two-impulse transfer: a $\Delta V$ burn is required to depart from one primary and another $\Delta V$ burn is required on arrival at the other primary. These transfers are characterised by the fact that the total $\Delta V$ required is large, relatively speaking. This stems from the fact that on arrival at the second primary, the spacecraft possesses excess velocity (known as hyperbolic excess velocity) that has to be compensated for. Hence, a braking manoeuvre is required to attain capture at the second primary. The Hohmann transfer is the most optimal two-impulse transfer possible using a conic section transfer trajectory. Such trajectories are termed high-energy here because of the relatively large $\Delta V$ required on arrival to compensate for the hyperbolic excess velocity. For space missions, typically non-optimal two-impulse transfers are used, as the flight time required for Hohmann transfers makes them unattractive. Such non-optimal transfers require less flight time, however at the cost of additional $\Delta V$, compared to the Hohmann case. Hence, these transfers also belong to the class of high-energy trajectories.

The reason that ballistic capture transfers are interesting from the perspective of space mission design is because they offer an opportunity to surpass the performance of a Hohmann transfer by minimizing, if not eliminating, the hyperbolic excess velocity on arrival at the second primary. In fact, temporarily, ballistic capture transfers can lead to bounded motion around the second primary, i.e. $E_2 < 0$. The existence of such transfer trajectories was contemplated in the 1960’s by Conley and others [Conley, 1968], however it was thought that numerical construction of practical ballistic capture transfers was impossible. An attempt was made to demonstrate that such trajectories could in fact be constructed numerically. In 1986 such a trajectory was successfully constructed numerically, and the methodology used was applied in a real mission scenario in 1991 to rescue the Japanese mission known as Hiten [Belbruno, 2004]. Some details of this mission are presented in Chapter 9. Ballistic capture transfers of this nature are termed low-energy since they minimize total $\Delta V$, using the dynamics of the 3BP, beyond the optimal performance that can be obtained using the dynamics of the 2BP i.e. the Hohmann transfer. Such transfers make use of the concept of weak capture, which can be demonstrated in the restricted problem of three bodies.

The capture problem in broad terms can therefore be summarised as being a problem of developing a methodology to analyse and construct transfer trajectories that utilise ballistic capture. Investigations reveal that ballistic capture is
intimately tied to the underlying dynamics. The capture problem will be considered in the context of the restricted problem of three bodies.

### 7.2 Definition of the WSB

In the PCR3BP, a special set can be defined within which ballistic capture occurs. This set defines the WSB and is central to studying the dynamics of the PCR3BP, which permits transitory motion between the primaries. The WSB has been tackled in an attempt to solve the capture problem and has been investigated extensively over the last decade. A number of different definitions can be found in literature to describe this set; however, a precise mathematical definition is still not agreed upon, as there are a number of outstanding issues to be resolved. The WSB can be considered "a region in phase supporting a special type of chaotic motion for special choices of elliptic initial conditions with respect to \( m_2 \)," as stated by Belbruno and Miller [1990].

Despite the research effort that has been put into determining and analysing the WSB, much still remains unknown. Two different approaches have been used to define the WSB. The original definition was given by Belbruno and describes a numerical algorithm. This numerical algorithm provides the most precise description available; however, it is cumbersome to use because it necessitates numerical integration. Since numerical integration is not essential to determining the WSB, an attempt has been made to find an analytical definition, which is both accurate and efficient to use. In this section both approaches will be introduced and current knowledge about the WSB will be detailed.

#### 7.2.1 Numerical Algorithmic Definition

The numerical algorithmic definition of the WSB was established to determine the set in phase space that supports weak ballistic capture. The definition given here can be found in Belbruno [2004] and Garcia and Gomez [2006].

Consider a radial segment, \( l(\theta) \) departing from the small primary \( m_2 \) and making an angle \( \theta \) with the \( x \)-axis (Figure 7.2). Take trajectories for the infinitesimal body, \( m \), starting on \( l(\theta) \), which satisfy the following.

1. The particle \( m \) starts its motion at the periapsis of an osculating ellipse around \( m_2 \) at a certain point on \( l(\theta) \). If \( r_2 \) denotes the distance between \( m \) and \( m_2 \) at this point, then \( r_2 = a(1 - e) \), where \( a \) and \( e \) are the semi-major axis and eccentricity of the ellipse (see Section 3.3).

2. The initial velocity vector of the trajectory is perpendicular to the radial segment \( l(\theta) \).

3. The initial 2-Body Kepler energy, \( E_2 \), of \( m \) with respect to \( m_2 \) is negative. The value of \( E_2 \), computed at periapsis, is given by \( E_2 = \mu_m \frac{e - 1}{2r_2} \). Hence if \( e \in [0,1) \), then \( E_2 < 0 \)

4. The eccentricity of the initial 2-Body Kepler motion is held fixed along \( l(\theta) \).

For motions with the above initial conditions, the following definition of stability and instability is given.
**Definition 7.14** The motion of a particle is said to be stable about $m_2$ if after leaving $l(\theta)$ it makes a full cycle about $m_2$ without going around $m_1$ and returns to $l(\theta)$ at a point with negative Kepler energy with respect to $m_2$. The motion will be unstable if this condition is not fulfilled.

In [Belbruno, 2004] it is stated that as the initial conditions vary along $l(\theta)$, there is a finite distance $r^*(\theta, e)$, depending on the polar angle $\theta$ and the eccentricity $e$ of the initial osculating ellipse, such that:

- If $r_2 < r^*$, the motion is stable;
- If $r_2 > r^*$, the motion is unstable.

Furthermore, $r^*(\theta, e)$ is a smooth function of $\theta$ and $e$, which defines the WSB region as:

$$ W = \{ r^*(\theta, e) | \theta \in [0, 2\pi], e \in [0, 1) \} $$

(7.2)

In other words, the WSB is defined here as the set of all points that lie exactly between stable and unstable motion with respect to $m_2$. Figure 7.2 illustrates the geometry of this numerical algorithmic definition. This definition contains some ambiguities however. Firstly, the requirements on the initial conditions fix the magnitude of the velocity and its direction, but do not indicate sense. In other words, for a fixed position on the line segment $l(\theta)$, there are two different initial velocities that meet all of the aforementioned requirements. These two different initial conditions lead to two different orbits with different stability properties. They are related to two different components of the WSB: prograde and retrograde motion about $m_2$. Prograde and retrograde motion refer to the rotation direction of $m$ around $m_2$, with respect to the rotation of $m_2$ around $m_1$ - prograde motion around $m_2$ follows the same direction as the motion of $m_2$ around $m_1$, as depicted in Figure 7.2. Hence it is necessary to explore both of these initial conditions to obtain a complete picture of the WSB region. Secondly, [Belbruno, 2004] does not offer proof of the fact that there is a finite distance $r^*(\theta, e)$ at which transition occurs between stable and unstable motion. Moreover, it is not proven that there
is only a solitary transition point along $l(\theta)$. In fact, [Garcia and Gomez, 2006] show that there are several transition points for a fixed value of $\theta$. There points need not be connected. The WSB region can therefore consist of a set of isolated transition points along $l(\theta)$.

Besides the ambiguities in the algorithmic definition, this recipe is quite cumbersome to use to analyse the WSB region. The biggest numerical difficulty is determining when a trajectory is unstable i.e. determining when a trajectory departing from the line segment $l(\theta)$ leaves the vicinity of $m_2$. Typically, an upper bound is set on the time between intersections of $l(\theta)$, however this is not a robust method. In addition, it is important to note that despite the fact that this algorithm requires integration of trajectories to determine the points that lie in the WSB, essentially the WSB exists independent of the need to integrate. Hence a more elegant and robust solution would be to find an analytical description of the set $W$. The algorithmic definition is useful to locate the WSB in configuration space, but it does not give sufficient insight into the capture dynamics at the heart of the problem.

[Garcia and Gomez, 2006] provides a numerical study of the WSB region in configuration space using the algorithmic definition. Figure 7.3 illustrates the results obtained for initial conditions with positive velocity and eccentricities of $e = 0, 0.3, 0.6, 0.9$. The grey areas given in this figure refer to stable motions. The white

![Figure 7.3: Visualisation of stable regions (grey area) in configuration space using the algorithmic definition for $e = 0, 0.3, 0.6, 0.9$. The origin of the reference system used is set to $m_2$ [Garcia and Gomez, 2006].](image)
areas refer to unstable motions. Hence, the WSB region is given by the curve that lies precisely between the two areas. It is clear from this figure that if a line segment is drawn from the origin at \( m_2 \) in an arbitrary radial direction, there are numerous cases for which there are multiple points lying at the boundary between the stable and unstable regions. This confirms the fact that the algorithmic definition yields, in general, multiple points along \( l(\theta) \) that belong to the WSB.

These figures do provide insight into the WSB; however they lack information about the underlying dynamics as they only illustrate the structure in configuration space. To obtain a better picture of the WSB region, visualisation in phase space is required.

### 7.2.2 Analytical Definition

The numerical algorithmic definition of the WSB is the most precise currently; however as pointed out, there are some difficulties with it. Hence, there has been interest in determining an analytical definition to approximate this set, to help better visualise the WSB and analyse its properties. The first analytical definition of the WSB was given by [Belbruno, 2004]. Recently, an in-depth study was performed to improve upon this definition and provide numerical visualisation of the WSB region [Belbruno et al., 2008]. The methodology used, termed the Energy Method, and the results obtained will be outlined here.

The basic tool used to study the WSB using the Energy Method is the Poincaré section or Poincaré map. Poincaré sections were introduced in terms of basic dynamical systems theory in Section 2.4. The essential idea behind these Poincaré sections is that they provide insight into underlying dynamics through a phase portrait, which characterises phase space.

The Poincaré sections used to study the WSB are two-dimensional. The sections used are defined by the following set.

**Definition 7.15** \( S(C) = \{(x, y, \dot{x}, \dot{y}) \in J(C) | y = 0, \dot{y} > 0\} \)

These sections are therefore two-dimensional sets on three-dimensional energy surfaces for a fixed value of the Jacobi energy integral. By fixing the value of the Jacobi energy, one state variable can be eliminated. This simplifies analysis of the WSB region, as successive Jacobi energy cases can be studied independently. The Jacobi energy integral, given by Equation (4.19), is:

\[
J = 2\Omega - V^2
\]  
and describes a three-dimensional energy surface, \( J(C) \). The set of points belonging to this surface is given by:

\[
J(C) = \{(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 | J(x, y, \dot{x}, \dot{y}) = C\}
\]

In this equation, \( V \) is the total velocity of \( m \), and:

\[
\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1 - \mu)
\]

For the PCR3BP, there are a total of four state variables: \( x, y, \dot{x}, \dot{y} \). The Jacobi energy integral provides an equation that can be used to reduce this to three
degrees of freedom by fixing an energy level. In this case, ˙y is fixed using the chosen energy level. Next, the section is chosen to coincide with the x˙x-plane, i.e. y = 0. This means that there are only two variables left to be determined: x, ˙x. Initial conditions are selected by choosing points belonging to S(C) i.e. a coordinate (x, ˙x). The value of y is known (y = 0), and the magnitude of ˙y is calculated using the Jacobi energy integral given by Equation (7.3). The only remaining degree of freedom is the sense of ˙y. This is fixed by choosing ˙y > 0 for the section. Hence, for any given combination of (x, ˙x), the full initial state vector is defined. With a set of initial conditions chosen that are contained in the section, the trajectories are integrated until they intersect the section again. On intersecting the section, it is necessary that the trajectory is transversal i.e. ˙y ≠ 0. If it shown that the intersection is transversal, then the point of intersection is recorded. This point is then used as initial condition for the next integration. In this manner the whole section can be filled.

This procedure was carried out by [Belbruno et al., 2008] using a Runge-Kutta 7th/8th-order, variable step size integrator, with absolute and relative tolerances of 10$^{-12}$. A set of 100 initial conditions lying on the x-axis was selected for a single Jacobi energy level. Each of these initial conditions was integrated for 1000 intersections i.e. revolutions, and each intersection was recorded. In this manner, the x, ˙x-plane was filled for a single Jacobi energy value, using only 100 initial conditions. The meshgrid used on the x-axis was equidistant. This procedure was carried out for a number of different energy levels. A brief summary of the results obtained will be offered here.

A set of seven different sections were generated, for seven different energy levels. Each of these sections provides insight into the underlying dynamics and the structure of the WSB. The value of the mass parameter used for these sections was $\mu = 0.00121506683$, which is the mass parameter for the Earth-Moon system. In other words, for these sections $m_1$ and $m_2$ represent the Earth and the Moon respectively. Each of the seven cases is also associated with a different choice of the initial eccentricity of the osculating ellipse and will be briefly outlined here.

1. C = 3.1817683176 (e = 0) Figure 7.4(a) illustrates the Poincaré section in the x˙x-plane. The large empty areas are due to the projection of $\mathcal{J}$ on this region, delimiting the areas of phase space where m cannot be located. Within the region where m can be found, there are some very interesting features. The flow in this region is characterised by a chaotic sea of points, where no discernible patterns are visible, interspersed by invariant tori close to the Earth ($x = -\mu$) and the Moon ($x = 1 - \mu$). Invariant tori can be seen associated with a number of different resonance orbits: 4:1, 3:1, 8:3, 7:3, and 2:1. Resonance orbits are defined as follows: an $m:n$ resonant orbit around the Earth, in resonance with the Moon, is an orbit whose period $T$ is related to the Moon’s period $T_2$ by $mT = nT_2$, where m, n are positive integers. The invariant tori located around the Moon are associated with stable orbits (quasi-periodic). These invariant tori exist specifically in the restricted problem of three bodies due to the dynamics associated with the perturbing force of the second primary. This is formally stated by the Kolmogorov-Arnold-Moser theorem (KAM theorem), which describes the persistence of certain invariant tori that exist in the 2BP under the influence of perturbations, due to resonance [Arndt, 1989]. KAM theorem is central to understanding the underlying dynamics of this problem;
however it will not be covered here, as the topic of interest lies in the chaotic sea.

The invariant tori support stable trajectories; hence they do not support transitory motion from one primary to the other, which is necessary for the WSB region. The reader is referred to standard texts on dynamical systems theory such as [Arnold, 1989] for an in-depth discussion of these invariant tori. An exactly resonant, stable periodic orbits exists at the centre of each of these tori. The collection of invariant tori around such a point will be referred to as a resonance island. It can therefore be established that the WSB region must exclude periodic orbits and resonant islands.

Figure 7.4(b) provides a close-up view of the 2:1 resonant island. This illustrates the KAM tori well, with the exactly resonant periodic orbit at the centre. It is important to note that the structures in this figure are ordered, in contrast to the chaotic sea.

Figure 7.5(a) illustrates the same section in the osculating elements \((a, \theta)\). This provides another view of the underlying dynamics of the system. In these coordinates, standard Kepler 2-Body orbits around the Earth would generate vertical straight lines. The global resonant structure is clearly visible, with the 2:1, 3:1, 4:1, 7:3 and 8:3 resonant orbits surrounded by invariant tori. It is important to note that the chaotic sea connects all of the resonances (except 4:1). This allows for resonance hops to occur (explored in detail by [Belbruno et al., 2008]). The gravitational pull of the Moon acts to break the invariant structure \((\theta = \pm \pi)\). This means that \(m\) gets captured by the Moon when it is approximately at apoapsis with respect to the Earth, which is consistent with expectation. Figure 7.5(b) provides a close-up view of the 2:1 island.

It is also important to note from Figure 7.5(a) that there is a barrier separating the 4:1 islands from the chaotic sea, meaning that for this energy level the Earth is enclosed by invariant stable orbits. Hence transit is not permitted between such orbits and translunar trajectories. For space mission applications this means that a \(\Delta V\) manoeuvre would be required from low-altitude orbits around the Earth to break through the barrier and get to the Moon (this problem was solved by Belbruno by using the Sun as a perturbing force to eliminate the barrier) [Belbruno, 2004].

2. \(C=3.1744515840\) \((e = 0.1)\) The value of the Jacobi energy integral for this section is slightly lower than that of the previous case. Figure 7.6 illustrates the Poincaré section for this energy level. There are no qualitative differences of note between this case and Case 1. The tori are slightly different in shape but they persist.

3. \(C=3.1510904302\) \((e = 0.2)\) For this Jacobi energy level there are some distinct differences to be noted. Figure 7.7 illustrates the Poincaré section associated with this energy level. From Figure 7.7(a) it is clear that the chaotic sea of points has become more pronounced. The shape of the 2:1 resonant island has changed: the right-hand side of this island (side closest to Moon) is more complex with appearance of two sharp fins. The 7:3 island has become much less dense. Figure 7.7(b) illustrates the biggest change that has occurred: the right-hand side of the 2:1 island near the Moon has opened up completely. This means that the chaotic sea can now access larger values of \(a\).

4. \(C=3.1109096645\) \((e = 0.3)\) For this case again there are significant changes in the Poincaré section, with the 7:3 and 8:3 islands having shrunk considerably.
From Figure 7.8(b), it is apparent that the invariant tori and the barrier still persist, however they have shrunk in size.

5. $C=3.0524780288$ ($e=0.4$) Figure 7.9 illustrates the Poincaré section associated with this energy level. It is apparent from both Figure 7.7(a) and Figure 7.9(b) that the 7:3 and 8:3 islands have both completely disappeared. In addition, small tori in the middle of Figure 7.9(a) has appeared which corresponds to a 3:2 island. The barrier to the left of Figure 7.9(b) is also less pronounced.

6. $C=2.9734250513$ ($e=0.5$) For this energy level, the Hill’s region has opened up considerably and the only invariant tori defined are 4:1, 3:1, 2:1 and 3:2. It is important to note that the invariant tori around the Moon have disappeared, meaning that there are no stable orbits around the Moon any longer. Figure 7.10 illustrates the associated Poincaré section.

7. $C=2.8698501942$ ($e=0.6$) This is the last energy level for which Poincaré sections were generated. Much of the ordered structure has disappeared, as can be seen from Figure 7.11. The only invariant tori that persist are 4:1, 2:1 and 3:1, with the latter being quite difficult to spot. Hence phase space is clearly almost completely filled by the chaotic sea.

The series of Poincaré sections generated serve to illustrate how sensitive the underlying dynamics of the PCR3BP is to the chosen energy level. These sections also clearly illustrate that there are ordered structures and a chaotic sea. A general conclusion can be drawn that as the energy level increases i.e. the Jacobi energy value decreases, phase space is filled by the chaotic sea to a greater extent and the ordered structures are destroyed. This happens in a smooth and continuous manner, as a function of the Jacobi energy value.

As stated previously, trajectories starting on any of the invariant tori are restricted to motion on these tori. Hence the WSB, which requires transitory motion, cannot include points on these tori. In other words, the WSB region must reside in the chaotic sea. This is an important observation in making a step towards finding an analytical description of the WSB.

An analytical definition was first explored by [Belbruno, 2004]. In this definition, the WSB region depends on the value of $C$.

**Definition 7.16** Let the two sets $\sigma$ and $\Sigma$ be given by:

$$\sigma = \{ x, \dot{x}, y, \dot{y} \, | \, \dot{r}_{23} = 0 \}, \quad (7.6)$$

$$\Sigma = \{ x, \dot{x}, y, \dot{y} \, | \, E_2 \leq 0 \}, \quad (7.7)$$

then for $C \in [C^*, C_2)$, the WSB, $\mathcal{W}(C)$, is defined as:

$$\mathcal{W}(C) = \mathcal{J}(C) \cap \Sigma \cap \sigma, \quad (7.8)$$

where $\mathcal{J}(C)$ is the three-dimensional manifold given by Equation (7.4) and $C^*$ is an estimate for the existence of $\mathcal{W}(C)$.

In this definition, $\mathcal{J}(C)$ describes the surface given by the Jacobi energy integral, $\Sigma$ describes the set of points in phase space with the property that the 2-Body Kepler energy with respect to $m_2$ is negative, and $\sigma$ imposes the condition that
the points in phase space belonging to the WSB must be at periapsis or apoapsis, where \( \dot{r}_{23} = 0 \). Additionally, the existence of the WSB is restricted to \( C \in [C^*, C_2] \), where \( C_2 \) is the energy level associated with L2 Lagrange point and \( C^* = 2.2 \) is a numerical estimate obtained by [Belbruno, 2004]. Therefore, in this definition, the WSB does not exist for Jacobi values for which L2 does not exist i.e. the WSB only exists when the inner and outer Hills regions are connected. This definition is known as the classic WSB.

The condition requiring \( \dot{r}_{23} = 0 \) effectively means that the WSB is one-dimensional in the Poincaré section: a subset of the \( x \)-axis, as \( \dot{r}_{23} = 0 \) means that \( \dot{x} = 0 \) since \( \dot{y} = 0 \). The definition of the WSB can be extended however, if ballistic capture is allowed to occur at any point along an osculating ellipse with respect to \( m_2 \) i.e. no restriction on \( \dot{r}_{23} \). This relaxes the conditions on the WSB set. A new set \( B \) is defined by [Belbruno et al., 2008] and is referred to as the extended WSB.

**Definition 7.17** Let \( \mathcal{J}(C) \) and \( \Sigma \) be defined by Equations (7.4) (7.7) respectively, then the extended WSB, \( B \), is defined as:

\[
B = \mathcal{J}(C) \cap \Sigma,
\]

where \( C \in [C^*, C_2] \) and \( C^* \) is estimated such that \( B \) is non-empty.

In other words, the set \( B \) is simply the set \( \mathcal{W}(C) \) without the restriction that \( \dot{r}_{23} = 0 \). A revised numerical estimate of \( C^* \) is provided by [Belbruno et al., 2008]; \( C^* \approx 2.8868 \).

For the first Poincaré section investigated (Case 1: \( C = 3.1817683176 \)), the projection of the sets \( B \) and \( \mathcal{W}(C) \) are illustrated by Figure 7.12. In Figure 7.12, the projections of \( \mathcal{J}(C) \) and \( \Sigma \) are given. All of the points contained in the Poincaré section are contained within the boundary defined by the projection of \( \mathcal{J}(C) \), as expected and required. The \( \Sigma \) set defines the boundary where \( E_2 = 0 \). The set \( B \) is shown as the collection of bold points (green) that lie inside (where the Moon is located) the boundaries defined by \( \mathcal{J}(C) \) and \( \Sigma \). It can be seen here that using this definition, part of the 2:1 island around the Earth is included and the invariant tori around the Moon are also included in the WSB. These points however do not lead to transitory motion. The classic WSB \( \mathcal{W}(C) \) is a subset of \( B \) and is also indicated in this figure; \( \mathcal{W}(C) \) is given by the bold line segments of the \( x \)-axis. It is noted that the set \( B \) offers many more opportunities to be temporarily captured by the Moon than the set \( \mathcal{W}(C) \). The set \( B \) largely overlaps with the chaotic sea on the Poincaré section. In this figure, the set of invariant curves around the Moon (white region with closed curves left of the Moon) are excluded from the set \( B \) by using a lower limit on the Kepler energy value (solid red line). The value for the lower limit of the Kepler energy was determined by trial and error. An analytical method to determine this lower bound has not yet been found.

Currently, the extended WSB set \( B \) is the best analytical approximation available that describes the set of points in phase space that support weak ballistic capture. It is observed that the classic WSB definition is too restricted, and that there are many more points that support such behaviour if \( m \) is allowed to be captured by \( m_2 \) at any point along the osculating ellipse. Figure 7.13 illustrates
how set \( B \) behaves as the Jacobi energy value varies. Figure 7.13(a) illustrates a case that was not studied numerically by [Belbruno et al., 2008]. For this energy level, the WSB set is disconnected. For clarification, it is noted that in this figure \( C_2 \) refers to the energy level associated with the interior Lagrange point, \( L_1 \). Figure 7.13(i) indicates that as the value of \( C \) approaches \( C^* \), the sets \( W \) and \( B \) reduce to a point. This is why the WSB region is not defined for \( C < C^* \). Figure 7.13(b) is the case presented in Figure reffig:WSBSection1wWSB, for which the \( L_1 \) and \( L_2 \) Lagrange points both exist.

To illustrate the set \( \Sigma \) on these Poincaré sections, the 2-Body Kepler energy with respect to the Moon must be derived in terms of rotating coordinates. The 2-Body Kepler energy and angular momentum equations were outlined in Section 3.3 in inertial coordinates. Since the equations of motion of the PCR3BP are given in rotating coordinates (Section 4.1.1), it is necessary to express the Kepler energy and angular momentum with respect to this reference frame. The 2-Body Kepler energy, given by Equation (3.27), and angular momentum, given by Equation (3.32), are restated here with respect to the Moon.

\[
E_M = \frac{1}{2} V_M^2 - \frac{\mu}{R_M} \tag{7.10a}
\]
\[
\vec{H}_M = \vec{R}_M \times \vec{V}_M \tag{7.10b}
\]

The origin of the inertial and rotating reference frames used for the PCR3BP is the gravitation barycentre of the primaries. The position and velocity vectors in the inertial reference frame, with respect to the barycentre, are given by:

\[
\vec{R} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{7.11a}
\]
\[
\vec{V} = \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} \tag{7.11b}
\]

The position and velocity vectors, expressed in terms of the rotating reference frame using the coordinate transformation given by Equations (4.7) and (4.8), are restated here.

\[
\vec{R} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.12a}
\]
\[
\vec{V} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} \dot{x} - y \\ \dot{y} + x \end{bmatrix} \tag{7.12b}
\]

In order to derive \( E_M \) and \( H_M \) in the rotating reference frame with respect to the Moon, the origin has to be shifted. This is achieved by shifting the position vector as follows: \( x \to x - (1 - \mu) \):

\[
\vec{R}_M = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x + \mu - 1 \\ y \end{bmatrix} \tag{7.13a}
\]
\[ V_M = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} \dot{x} - y \\ \dot{y} + x + \mu - 1 \end{bmatrix} \] (7.13b)

For the Poincaré sections that will be studied in Chapter 8, the projection of these vectors on the \( x\dot{x}\)-plane is required i.e. \( y = 0 \). Therefore it follows that:

\[ \bar{R}_M = \begin{bmatrix} (x + \mu - 1) \cos t \\ (x + \mu - 1) \sin t \end{bmatrix} \] (7.14a)

\[ \bar{V}_M = \begin{bmatrix} \dot{x} \cos t - (\dot{y} + x + \mu - 1) \sin t \\ \dot{x} \sin t + (\dot{y} + x + \mu - 1) \cos t \end{bmatrix} \] (7.14b)

From this it follows that:

\[ R_M = |\bar{R}_M| = \sqrt{[(x + \mu - 1) \cos t]^2 + [(x + \mu - 1) \sin t]^2} = |x + \mu - 1| \] (7.15a)

\[ V_M = |\bar{V}_M| = \sqrt{[\dot{x} \cos t - (\dot{y} + x + \mu - 1) \sin t]^2 + [\dot{x} \sin t + (\dot{y} + x + \mu - 1) \cos t]^2} \]

\[ = \sqrt{\dot{x}^2 \cos^2 t - 2\dot{x}(\dot{y} + x + \mu - 1) \cos t \sin t + (\dot{y} + x + \mu - 1)^2 \sin^2 t} \]

\[ + \dot{x}^2 \sin^2 t + 2\dot{x}(\dot{y} + x + \mu - 1) \cos t \sin t + (\dot{y} + x + \mu - 1)^2 \cos^2 t} \]

\[ = \sqrt{\dot{x}^2 + (\dot{y} + x + \mu - 1)^2} \] (7.15b)

By using these equations, it follows that the projection of the Kepler energy with respect to the Moon, on the \( x\dot{x}\)-plane can be written as:

\[ E_M = \frac{1}{2} V_M^2 - \frac{\mu}{R_M} \]

\[ = \frac{1}{2}(\dot{x}^2 + (\dot{y} + x + \mu - 1)^2) - \frac{\mu}{|x + \mu - 1|} \] (7.16)

Similarly, it follows that the projection of the angular momentum with respect to
7.2 Definition of the WSB

the Moon, on the $x\dot{x}$-plane can be written as:

$$\vec{H}_M = \vec{R}_M \times \vec{V}_M$$

$$= \begin{bmatrix}
(x + \mu - 1) \cos t & (x + \mu - 1) \sin t & 0 \\
\dot{x} \cos t - (\dot{y} + x + \mu - 1) \sin t & \dot{x} \sin t + (\dot{y} + x + \mu - 1) \cos t & 0 \\
-(x + \mu - 1) \sin t(\dot{x} \cos t - (\dot{y} + x + \mu - 1) \sin t) & (x + \mu - 1) \dot{x} \cos t \sin t + (x + \mu - 1)(\dot{y} + x + \mu - 1) \cos^2 t & 0 \\
-(x + \mu - 1) \dot{x} \cos t \sin t + (x + \mu - 1)(\dot{y} + x + \mu - 1) \sin^2 t & [(x + \mu - 1)(\dot{y} + x + \mu - 1)] \k & 0 
\end{bmatrix}$$

(7.17)

Using a similar derivation, it can be shown that the Kepler energy and angular momentum with respect to the Earth, on the $x\dot{x}$-plane can be written as:

$$E_E = \frac{1}{2} V_E^2 - \frac{\mu}{R_E}$$

$$= \frac{1}{2}(\dot{x}^2 + (\dot{y} + x + \mu)^2) - \frac{\mu}{|x + \mu|}$$

(7.18)

$$\vec{H}_E = \vec{R}_E \times \vec{V}_E$$

$$= [(x + \mu)(\dot{y} + x + \mu)] \k$$

(7.19)

These equations will be used extensively in Chapter 8 to visualise a possible improvement to current analytical definition of the WSB, given by [Belbruno et al., 2008]. The improvement introduced uses the angular momentum with respect to the Moon to try and eliminate the invariant tori on the left-side of the Moon. A full analysis is presented in Chapter 8.

In addition to providing an extension of the classic analytical definition of the WSB, [Belbruno et al., 2008] provide insight into the role of invariant manifolds associated with Lyapunov orbits in determining the WSB region. It can be shown that the location in phase space of the WSB is intimately linked to the location of the stable and unstable manifolds associated with Lyapunov orbits (see Chapter 6 for a description of periodic orbits in the CR3BP and associated stable and unstable manifolds). Figure 7.14 illustrates the interior branch of the stable and unstable manifolds associated with a Lyapunov orbit at $L_1$ for $C = 3.1817683176$ (Case 1). Figure 7.14(a) illustrates these manifold structures in configuration space. Figure 7.14(b) illustrates the intersection of these manifold structures on the $x\dot{x}$-plane. The first three intersections of the interior branches of the stable and unstable manifolds are shown. The successive intersections shown in Figure 7.14(b) are topological circles until the first transverse intersection occurs; these circles are destroyed by the time the third intersection occurs. It can be seen from this figure that the intersections of the stable and unstable manifolds overlap. This leads to interesting dynamics. The exact points that lie on both the stable and unstable manifolds are known as transverse homoclinic points. Integration of
these points in forward and backward time leads to a trajectory that is asymptotic to the Lyapunov orbit at L1 along both manifolds. These points are important in determining the dynamics of the PCR3BP. It can be shown that the existence of such points implies the existence of a hyperbolic network, which in turn implies the existence of chaotic motion. The proof of this statement mathematically challenging and is covered in a number of different texts. The reader is referred to [Belbruno, 2004] for an extensive treatment of the implication of the existence of a hyperbolic network on the WSB region. Figure 7.15 illustrates the projection of the intersections of the interior branch of the stable and unstable manifolds associated with the Lyapunov orbit at L1 on the Poincaré section obtained for $C = 3.1817683176$ (Case 1). It can be seen from Figure 7.15 that the projections of the stable and unstable manifolds lie in the chaotic sea, between the invariant tori. As was described in Section 4.2.3, trajectories that lie inside these manifolds transit the neck region i.e. these points belong to the WSB. It is logical therefore that the invariant tori are not included inside the projections of the manifolds, as these orbits are stable about the primaries and cannot lead to transitory motion, as stated previously. Visualisation of the extended WSB $B$ therefore provides insight into the underlying dynamics and the link between the invariant manifold structures and weak ballistic capture.
7.2 Definition of the WSB

(a) The \( x\dot{x} \)-plane

(b) 2:1 resonant island

Figure 7.4 Poincaré section in \( (x, \dot{x}) \) coordinates for the flow of the PCR3BP for \( C = 3.1817683176 \) [Belbruno et al., 2008].
Figure 7.5  Poincaré section in $(a, \theta)$ coordinates for the flow of the PCR3BP for $C = 3.1817683176$ [Belbruno et al., 2008].
7.2 Definition of the WSB

(a) The $x\dot{x}$-plane

(b) The $a\dot{\theta}$-plane

Figure 7.6 Poincaré section for the flow of the PCR3BP for $C = 3.1744515840$ [Belbruno et al., 2008].
Figure 7.7  Poincaré section for the flow of the PCR3BP for $C = 3.1510904302$ [Belbruno et al., 2008].
7.2 Definition of the WSB

(a) The $x\dot{x}$-plane

(b) The $a\theta$-plane

Figure 7.8 Poincaré section for the flow of the PCR3BP for $C = 3.1109096645$ [Belbruno et al., 2008].
Figure 7.9 Poincaré section for the flow of the PCR3BP for $C = 3.0524780288$ [Belbruno et al., 2008].
7.2 Definition of the WSB

(a) The $x\dot{x}$-plane

(b) The $a\theta$-plane

Figure 7.10 Poincaré section for the flow of the PCR3BP for $C = 2.9734250513$ [Belbruno et al., 2008].
Figure 7.11  Poincaré section for the flow of the PCR3BP for $C = 2.8698501942$  [Belbruno et al., 2008].
Figure 7.12  Poincaré section for the flow of the PCR3BP for \( C = 3.1817683176 \) containing the classic and extended WSB, \( W \) and \( B \), respectively. The boundaries of the sets \( \mathcal{J}(C) \) (blue line) and \( \Sigma \) (dashed red line) are also shown [Belbruno et al., 2008].

Figure 7.13  Visualisation of the behaviour of the set \( B \) (grey area) on the Poincaré section as a function of Jacobi energy. The set \( W \) (bold line) and the location of the Moon (black dot) is visualised as well [Belbruno et al., 2008].
Figure 7.14  Interior branch of stable (blue) and unstable (red) manifolds associated with Lyapunov orbit for $C = 3.1817683176$ in (a) configuration space and (b) the $x\ddot{x}$-plane [Belbruno et al., 2008].

Figure 7.15  Projection of successive cuts of the interior branch of the stable (blue) and unstable (red) manifolds on the Poincaré section for $C = 3.1817683176$. Projection of the set $\mathcal{J}(C)$ (blue line) and $\Sigma$ (dashed red line) is provided [Belbruno et al., 2008].
Chapter 8

The Weak Stability Boundary: Results

The Weak Stability Boundary (WSB) is a region in phase space that supports weak ballistic capture. In Chapter 7, an extensive treatment of the WSB was provided. A numerical algorithmic and an analytical definition of the WSB region were outlined. The theory presented in Chapter 7 represents the current state of knowledge. For both the algorithmic \cite{Garcia and Gomez, 2006} and analytical \cite{Belbruno et al., 2008} definitions, visualisations of the WSB were provided for the Planar Circular Restricted 3-Body Problem (PCR3BP).

A number of difficulties with current definitions of the WSB region were mentioned. The algorithmic definition is the most precise mathematical description of the set of points in phase space that lie in the WSB region; however this definition is cumbersome to use, as it requires integration of many trajectories, and it does not provide sufficient insight into the underlying dynamics. The analytical definition given is an attempt to circumvent the need for integration to establish the WSB region in phase space. This method, also termed the Energy Method, is an efficient way of representing the WSB region. Poincaré sections in the \((x, \dot{x})\)-plane are used to analyse the robustness of the analytical definition. These Poincaré sections are also useful in examining the underlying dynamics. An extended WSB definition is introduced, which is able to capture a larger set of points that are capable of supporting weak ballistic capture. This definition however fails to exclude certain invariant tori around the Earth, and invariant tori around the Moon. Hence, the analytical definition is a good approximation of the WSB, but it is still inaccurate.

In this chapter, an attempt will be made to improve upon the analytical definition by providing a better approximation of the WSB set. In addition, the Poincaré sections analysed by \cite{Belbruno et al., 2008} do not shed light on the internal structure of the WSB region. A set of Poincaré sections will be analysed in a similar manner as done by \cite{Belbruno et al., 2008}, with the addition of visualisations of the internal structure of the WSB.

In Section 8.1, the methodology used to analyse the WSB region will be summarised. The use of Poincaré sections will be outlined briefly, and the additional visualisations will be explained. Subsequently, in Section 8.2, the results obtained
will be given and preliminary analysis will be provided. Finally, in Section 8.3, a
number of conclusions based on the results obtained will be outlined.

8.1 Methodology

The Energy Method presented in Section 7.2.2 underpins the analysis of the WSB
region presented in this chapter. Essentially, this method makes use of Poincaré
sections in the \((x, \dot{x})\)-plane to study the dynamics of the PCR3BP and investigate
the WSB region. The Poincaré sections were defined as:

\[
S(C) = \{(x, y, \dot{x}, \dot{y}) \in \mathcal{J}(C)| y = 0, \dot{y} > 0\}
\] (8.1)

The set \(\mathcal{J}(C)\) defines a 3-dimensional energy surface given by considering the
Jacobi energy integral \(J = 2\Omega - V^2\). In Section 7.2.2, a number of such Poincaré
sections were presented and analysed for different energy levels. The extended
WSB region was projected on these sections to illustrate the set of points belonging
to \(B\), given as:

\[
B = \mathcal{J}(C) \cap \sigma
\] (8.2)

The set \(\Sigma\) is given by: \(\Sigma = \{x, \dot{x}, y, \dot{y}|E_2 \leq 0\}\), with \(E_2 = \frac{1}{2}R_{23}^2 - \frac{\mu}{R_{23}}\) denoting
the 2-Body Kepler energy with respect to \(m_2\). The intersection of the stable
and unstable invariant manifold structures with the Poincaré sections was also
illustrated.

For the purposes of analysing the WSB region more closely, Poincaré sections
will be generated to detail the dynamics in the neighbourhood of \(m_2\). Figure 8.1

![Figure 8.1](image)

**Figure 8.1** Setup of Poincaré section in configuration space, illustrating the Earth (blue circle), Moon
(black circle), L1 and L2 Lagrange points (green circles) and Poincaré section (red bold line).

illustrates the Poincaré section chosen in configuration space (red bold line). For
the remainder of this chapter, \(m_1\) and \(m_2\) will be referred to as the Earth and
the Moon respectively, as the sections generated will be for the Earth-Moon mass
parameter \((\mu = 0.00121506683)\). The Poincaré section is centred about the Moon
and extends beyond the L1 and L2 Lagrange points. By restricting the \(x\)-domain
of the section, an in-depth study can be done of the dynamics in the vicinity of
the Moon. Initial conditions are chosen on this section and are integrated until
the first intersection of the section. An example of this is illustrated in the figure.
As explained in Section 7.2.2, since the PCR3BP has four state variables, by selecting the section to lie on the $x$-axis, as shown in the figure, it is possible to fix one variable i.e. $y = 0$. It should be noted that this can be considered a line segment $l(0)$, as given in the algorithmic definition, with $\theta = 0$. The Jacobi energy integral is used to fix one more variable: by choosing a value for the Jacobi energy, the value of $\dot{y}$ is fixed. The sense of $\dot{y}$ is given by the definition of the section (Equation (8.1)). Hence this leaves the state variables $(x, \dot{x})$. The Poincaré section is schematically illustrated in Figure 8.2 in the $(x, \dot{x})$-plane. This section is centred at the Moon. Five initial conditions (red circles) lying on the $x$-axis in the $x\dot{x}$-plane are also illustrated. These initial conditions describe initial state vectors when complemented with the variables $y$ and $\dot{y}$, as described previously. Given these initial state vectors, integration is carried out until the trajectories intersect the section again. On intersecting the section, the trajectories yield new $(x, \dot{x})$-coordinates, which are noted. Figure 8.3 schematically illustrates these new points, having integrated the initial conditions. The intersection points are then used as initial conditions for subsequent integration. In this manner, it is possible to fill the Poincaré section. The sections described in Section 7.2.2 were produced in this manner by using 100 initial points on the $x$-axis and integrating each for 1000 intersections i.e. orbits. It turns out that it is sufficient to choose points lying on the $x$-axis, i.e. $\dot{x} = 0$. By integrating each initial condition for 1000 intersections, the $x\dot{x}$-plane can be filled.

The same method will be employed to generate sections here. To allow for impact trajectories to be analysed as well, the regularised equations of motion are used, as described in Section 4.3. The Poincaré sections generated are two-dimensional, and though they provide insight into the underlying dynamics by reproducing the dynamical structures that exist, there is no indication of the stability of the trajectories that lie in the WSB region. Hence, to complement these sections, a third dimension will be added, indicating the number of orbits that the points in the WSB region remain in the vicinity of the Moon. Theoretically, these results should match the location of the intersections of the stable and unstable manifolds (see Chapter 6 for a description of invariant manifolds). In Section 7.2.2,
an example of a projection of the branch of the stable and unstable manifolds towards the Earth on a Poincaré section is given. Such intersections will also be computed here. Figure 8.4 schematically illustrates the new sections that will be generated. Each of the points lying in the Poincaré section, including the initial conditions lying on the meshgrid (x-axis), are integrated forwards and backwards in time to determine their stability with respect to the Moon. This provides the necessary information to extended the Poincaré sections to a third dimension. Accordingly, these sections are denoted extended-Poincaré sections. A number of different sections will be generated for the PCR3BP, for different values of the Jacobi energy. By analysing the data using the tools previously mentioned, the
aim is to enhance understanding of the underlying dynamics and possibly improve the analytical definition of the WSB region.

8.2 Results

In this section, the results obtained from various simulations will be presented. The results will be commented upon in light of identifying and analysing the WSB region. The following proposition is put forward to improve the analytical definition of the WSB region:

**Definition 8.1** Let \( \mathcal{J}(C) \), \( \Sigma \), \( \Gamma \), and \( \Psi \) be defined by:

\[
\mathcal{J}(C) = \{(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 | J(x, y, \dot{x}, \dot{y}) = C\} \tag{8.3a}
\]

\[
\Sigma = \{x, \dot{x}, y, \dot{y} | E_2 \leq 0\}, \tag{8.3b}
\]

\[
\Gamma = \{x, \dot{x}, y, \dot{y} | H_2 > 0\}, \tag{8.3c}
\]

\[
\Psi = \{x, \dot{x}, y, \dot{y} | x_{L1} < x < x_{L2}\}, \tag{8.3d}
\]

Then the 2nd-extended WSB, \( B_2 \), is defined as:

\[
B_2 = \mathcal{J}(C) \cap \Sigma \cap \Gamma \cap \Psi , \tag{8.4}
\]

where \( C \in [C_1^*, C_2^*] \), and \( C_1^*, C_2^* \) are estimated such that \( B_2 \) is non-empty: \( C_1^* = C_1 \) where \( C_1 \) is the Jacobi energy associated with the L1 Lagrange point, \( C_2^* = 2.886 \) is estimated numerically [Belbruno et al., 2008].

In this definition, \( H_2 \) refers to the relative 2-Body angular momentum (see Section 3.3.2) of \( m \) with respect to the Moon (\( m_2 \)). The set \( \Gamma \) therefore refers to all points in phase space that belong to trajectories that exhibit prograde motion around the Moon. As described in Section 7.2.2, \( \mathcal{J}(C) \) describes the surface given by the Jacobi energy integral, and \( \Sigma \) describes the set of points in phase space with the property that the 2-Body Kepler energy with respect to the Moon is negative i.e. these points represent bounded motion with respect to the Moon at the time of evaluation along the trajectories that they lie on. The set \( \Psi \) contains all of the points in phase space that have \( x \)-coordinates that lie between the location of the L1 and L2 Lagrange points.

To study the robustness of Definition 8.2, Poincaré sections were generated for a number of different Jacobi energy values. The values used are summarised in Table 8.1. All the Jacobi energy values selected are greater than \( C_2 \), i.e. L1 exists but L2 does not. The reason for this is because previous studies, such as [Belbruno et al., 2008] have concentrated on values less than \( C_2 \) previously. Hence, it is most fruitful to consider values greater than \( C_2 \). The methodology and analysis presented in this chapter can easily be applied to values greater than \( C_2 \) in future work. To generate Poincaré sections, a variable-order, variable step size integrator is used. A set of 100 equidistant initial points are selected on the \( x \)-axis in the \( xx \)-plane. These initial 100 points are each integrated for 1000 intersections of the \( xx \)-plane. Each of the cases presented in Table 8.1 will be analysed separately. The figures mentioned in the text are given at the end this section.
8.2.1 Jacobi Energy Cases

Case 1: \( C = 3.1980000000 \)

For this Jacobi energy value, the L1 Lagrange point exists, but the L2 Lagrange point does not i.e. \( C_1 > C > C_2 \). Hence the inner and outer Hill's regions are not connected; however there is a connection between the two inner regions, permitting motion between the Earth and the Moon. Figure 8.5 illustrates the Poincaré section obtained.

The location of the Moon (red) and L1 (brown) are also illustrated in this figure. A number of distinct features can be noted in this section. By analysing the points contained within this section, it is possible to identify the WSB region. For the current analysis, the WSB region is essentially characterised by the minimal set of points in phase space, which lead to transit motion between the Earth and Moon in finite time. The transit condition is geometric in nature and quite difficult to implement rigorously in simulation software. This geometric condition is used by the algorithmic WSB definition (Section 7.2.1). The geometric condition for transit relates to motion around one primary switching to motion around the other.

Typically, simulations to identify the algorithmic WSB region, as carried out by [Garcia and Gomez, 2006], make use of the fact that subsequent to transit, the orbital period of the trajectory varies significantly compared to pre-transit motion. In this case, it is identified that for the Jacobi energy value being considered, transit trajectories must pass through the neck region in the vicinity of L1. Hence, as a first approximation, it would seem that a condition that determines whether a trajectory crosses this neck region could be used to identify transit trajectories; hence the WSB region. The condition used is to check whether, during successive intersections of the Poincaré section, the \( x \)-coordinates of the intersection points lie on different sides of L1. For intersections where this condition is true, the implication is that \( m \) is transiting between the Earth and the Moon, for the case being studied here.

Applying this condition involves searching through the set of points contained in the Poincaré section and determining whether in their time history transit occurs. Carrying this procedure out enables the internal structure of the WSB region to be investigated too. The results of this analysis are used to construct three-dimensional plots, as illustrated schematically by Figure 8.4: extended-Poincaré sections. For \( C = 3.1980000000 \), this yields Figure 8.6.

From this figure, it is clear that there is a large gap between a set of points which are stable for at least 1000 orbits, and a set of points which are stable for less than approximately 250 orbits. The set of points which are stable for 0 orbits around the Moon refer to stable trajectories around the Earth that do not transit
to the Moon in finite time, and unstable trajectories around the Earth that transit to the Moon, which are already included in the WSB region due to intersections with the Poincaré section to the right of L1. The unstable trajectories to the left of L1 do lie on weak capture trajectories, however, since they are accounted for by intersections with the Poincaré section to the right of L1, there is no value in including them in the WSB set. The same argument holds for points to the right of L2, for cases where L2 exists. This is the reason for including the set $\Psi$ in $B_2$.

The set of points which are stable for at least 1000 orbits are interesting to consider. As mentioned previously, the initial set of 100 points on the $x$-axis are each integrated for 1000 orbits. The choice of integrating for 1000 orbits is arbitrary and is taken from the study performed by [Belbruno et al., 2008]. The idea behind choosing this number is that it should be sufficient to be able to distinguish unstable motion i.e. motion for which transit occurs in finite time, from stable motion i.e. motion for which transit does not occur at all.

The study performed by [Belbruno et al., 2008] suggests that the choice of 1000 orbits is good in estimating the underlying dynamics of the PCR3BP, and specifically identifying the WSB region, which excludes stable motion. It would seem that for the Jacobi energy value being considered here this is also true i.e. the unstable and stable character of trajectories within respect to the Moon is distinguishable in the Poincaré section. The set of trajectories that remain stable for 1000 orbits are in fact stable for infinite time i.e. they do not support weak capture. The reason this conclusion can be drawn will be reflected upon once intersections of the stable and unstable invariant manifolds associated with the Lyapunov orbit around L1 are computed.

Figure 8.7 provides an illustration of the set of points in the Poincaré section that are stable for at least 1000 orbits, and the set of points that are stable for 0 orbits around the Moon. This figure illustrates a number of interesting features. The predominant features on the right-hand side of this figure (blue) are the closed curves. As described in Section 7.2.2 previously, these closed curves are associated with invariant tori according to the Kolmogorov-Arnold-Moser theorem (KAM theorem), which is a fundamental result for dynamical systems such as the PCR3BP. These invariant tori describe quasi-periodic motion: points located on such curves, when integrated forwards or backwards, are only permitted to intersect the Poincaré section at points belonging to the initial curve. Hence, such motion is stable in the sense that transit motion between the primaries is not permitted. Precisely periodic orbits are found at the centre of the invariant tori lying on both the left and right sides of the Moon. The features on the right side of the Moon are typical of such Poincaré sections. The invariant tori that lie to the extreme left of the figure (red) are in fact tori surrounding the Earth. These points belong to the set that describes stable motion around the Earth.

The features on the right side of the Moon are particularly interesting. On the right side, a number of invariant curves can be identified. Specifically, a ‘fish’-shape can be identified, within which the points are seen to be more chaotic in Figure 8.5. This ‘fish’-shape was thoroughly investigated by [Gidea and Masdemont, 2007] for the case that $\mu = 0.5$, i.e. the masses of the primaries are equal. In [Gidea and Masdemont, 2007], it is demonstrated that the ‘fish’-shape is generated by successive intersections of the stable and unstable manifolds with the Poincaré section. This is important to note, as transit trajectories are required to lie within
such intersections. This was described in Section 4.2.3, where it was noted that in
the planar problem the stable and unstable invariant manifolds function as sepa-
ratrices for the motion: the stable and unstable manifolds differentiate transit and
non-transit trajectories. It can be shown that the ‘fish’-shape noted in Figure 8.7
can also be generated by considering intersections of these manifolds. This will be
demonstrated shortly.

The invariant curve given by the ‘fish’-shape represents a set of points that,
like the closed curves on the left side, lead to stable trajectories around the Moon.
Hence these points do not belong to the WSB region either. Within the ‘fish’-
shape there are a number of points that are also stable for 1000 orbits. A set of
closed curves can be identified, which represent a resonant island about a prograde
periodic orbit. The points that belong to this island also represent stable motion
around the Moon; thus they too must be excluded from the WSB region. It is
noted that other than points belonging to the ‘fish’-shape and the resonant island
contained within it, there are no points which remain stable for 1000 orbits or
more. This suggests that the WSB region does not contain points which are stable
for at least 1000 orbits. Further analysis of the points contained within the ‘fish’-
shape provides an indication of the distribution of stability of weak capture orbits
at the Moon and will be outlined too.

Figure 8.7 also illustrates the set of points that are stable around the Moon
for 0 orbits. As previously mentioned, this set includes both stable and unstable
trajectories. Trajectories that are stable around the Earth and do not transit to
the Moon in finite time are included (red). Further, trajectories that are unstable
around the Earth and do transit to the Moon are also included, since these points
are already accounted for on the other side of L1 (green) It is clear that the tra-
jectories that do not transit to the Moon belong to invariant KAM tori associated
with the Earth. Hence such trajectories cannot depart from the vicinity of the
Earth. The points that lie on weak capture trajectories are found in a chaotic sea,
which is characteristic of the behaviour of transit trajectories in the PC3BP.

Figure 8.8(a) illustrates an extended-Poincaré section for unstable orbits that
remain in the vicinity of the Moon for less than 1000 orbits before transiting to
the Earth. This section excludes the points that are to the left of L1. From
Figure 8.8(a) it is apparent that there is a distribution of points that stay stable
around the Moon, starting at 6 orbits up to and including 217 orbits. From the
figure it would seem that there is a greater density of points that are stable for
less than 100 orbits. It is also apparent that for higher numbers of orbits there are
gaps in the distribution. The location of these points can be analysed further by
considering the projection of these points on the $x\dot{x}$-plane. Figure 8.8(b) provides
such an illustration, where the number of orbits is colour-coded.

From Figure 8.8(b) the ‘fish’-shape can be clearly seen. The outer contour
of the ‘fish’-shape was described previously as being an invariant torus. It is
apparent from this figure that all orbits on the right side of the Moon in the $x\dot{x}$-
plane that lead to transit motion are contained within this contour. The prograde
periodic orbit that was identified, along with the associated invariant tori can
also be spotted within this shape as the white circular gap. It is clear that the
WSB region, if defined as being the set of points in phase space that supports
transit motion, and therefore possibly weak capture, must exclude these invariant
tori. The extended WSB defined by the set $B$ fails to neglect such stable motion completely.

It is difficult to ascertain patterns within this ‘fish’-shape, however through further analysis it is noted that orbits closer to the $x$-axis remain stable for a greater number of orbits. It is also clear that there are few points that remain stable for greater than 100 orbits, in contrast to the number that remain stable for less than 100 orbits. Further, it is apparent that there are a number of transit orbits on the left side of the Moon. This set of points is small and is located in the vicinity of the L1 Lagrange point.

Figure 8.9 illustrates the distribution of points for a sample set of cases: 6, 90, 145 and 217 orbits. It is interesting to note from this figure that the case of 6 orbits seems to be quite ordered, whilst the cases of 90 and 145 orbits are distributed quite chaotically. The ordered structure for 6 orbits will be analysed in detail shortly. The case of 217 orbits quite clearly traces out the ‘fish’-shape. This provides an illustration of how the internal structure of the WSB region within the ‘fish’-shape is built up.

To highlight the distribution of points within this figure, a histogram can be plotted indicating the frequency of points that are stable for different number of orbits around the Moon. Such a histogram is presented in Figure 8.10(a). Cumulative histogram data is presented in Figure 8.10(b). From these histograms it is apparent that transit motion is dominated by trajectories that remain stable around the Moon for less than 100 orbits. The cumulative histogram distribution given by Figure 8.10(b) is the best representation of this fact. From this plot it is clear that successive contributions, in terms of frequency of points for higher numbers of orbits, drops off. This plot indicates that over 80% of the points lie on trajectories that are stable for less than 100 orbits around the Moon.

To analyse the robustness of these histogram plots, another Poincaré section was generated with each initial condition on the $x$-axis integrated for 2000 orbits rather than 1000. This means that the $x\dot{x}$-plane is filled more densely. Figure 8.11 illustrates a comparison between the cumulative histogram distributions generated for both Poincaré sections. From this figure it is evident that the discrepancies between the two data sets are small. Therefore, it is possible to conclude that comparatively there are more trajectories that stay stable for fewer orbits.

This internal structure of the WSB region, which in this particular case largely describes the internal structure of the ‘fish’-shape, can be understood be analysing successive intersections of the stable and unstable invariant manifolds of the Lyapunov orbit at L1 with the Poincaré section. The periodic orbit and associated invariant manifolds are generated using the method described in Section 6.1. Figure 8.12 illustrates the first six intersections of the stable and unstable manifolds with the Poincaré section. The ‘fish’-shape described previously is clearly evident from Figure 8.12 as well. The implication of this is that, as suggested previously, points that lie within this contour are subject to transit to the Earth, as they lie within invariant manifolds, which are separatrices for transit and non-transit motion. The intersection of the first six unstable manifolds is given by the blue areas. The successive intersections are labelled from $W_1^u$ to $W_6^u$. The stable manifolds are given by the red areas and are similarly labelled. Each intersection of the stable and unstable manifold is symmetric about the $x$-axis. It is clear that
as the manifolds intersect more, the closed curves are stretched out and twisted more. With successive integrations, the manifolds tend closer to the $x$-axis. The stretching and twisting of the manifolds is due to the inherent chaotic nature of the PCR3BP, and the mathematics describing this behaviour goes beyond the scope of the current analysis. It suffices to state this as being a property of the PCR3BP to understand the structure and origin of ‘fish’-shape contour.

It is apparent from Figure 8.12 that the manifold areas overlap: in particular the unstable and stable manifold intersections overlap. This is denoted by the yellow and green areas. These areas have particular significance. As stated earlier, it was established in Section 4.2.3 that trajectories that lie within these manifold areas are transitory: lying within an unstable manifold means that the point lies on a trajectory that transits the neck near L1 in backwards time, whilst lying within a stable manifold means that the point lies on a trajectory that transits the neck near L1 in forwards time. The significance of the overlap areas is that points within this area will transit both in forwards and backwards time. Such a point therefore belongs to the WSB region, as it belongs to a trajectory which starts in the neighbourhood of the Earth, transits to the Moon for a certain period of time, before transiting back to the Earth. There is no requirement that transit must occur in both directions for a trajectory to belong to the WSB region. A trajectory that, for instance, starts out near the Earth and transits to the Moon but never returns to Earth also belongs to the WSB region. In essence, a body $m$ following such a trajectory achieves permanent capture with respect to the Moon. It has been shown mathematically, however, that the set containing permanent capture trajectories is of measure zero \cite{Belbruno2004}. To all intents and purposes, such trajectories do not exist in the PCR3BP. Hence, effectively the WSB region contains trajectories that transit to and fro between the Earth and the Moon.

Figure 8.13 provides a schematic illustration of such an overlap between an unstable and a stable manifold. This figure illustrates the fact that points within the intersection of the unstable manifold, $W_u^r$, remain in the vicinity of the Moon for $t > 0$ and transit to the Earth in $t < 0$ after $i$ intersections. Points within the intersection of the stable manifold, $W_s^l$, remain in the vicinity of the Moon for $t < 0$ and transit to the Earth in $t > 0$ after $j$ intersections. Points that lie within the overlap therefore for $t < 0$ transit after $i$ intersections and for $t > 0$ transit after $j$ intersections. The number of orbits that these trajectories remain in the vicinity of the Moon is then given by: $n = i + j - 1$. It is necessary to subtract one from the sum of $i$ and $j$ because of the geometric location of the stable and unstable manifolds. This is better understood when considering an example. Figure 8.12 illustrates an example of a trajectory that lies within the overlap area. From this figure, a total of six loops around the Moon can be identified. The point in the Poincaré section lies in the overlap of the first unstable manifold intersection and sixth stable manifold intersection. This means that using the previously mentioned formula, the trajectory should remain in the vicinity of the Moon for six orbits, which concurs with the figure. In forwards and backwards time, it is clear that the trajectory transits the region near L1 and heads towards the Earth.

The yellow regions in Figure 8.12 illustrate all of the overlapping areas in the figure that contain trajectories that are stable for six orbits around the Moon. Similarly, the green areas indicated the overlapping areas containing trajectories that are stable for seven orbits around the Moon. One interesting feature of this
figure is its area-preserving nature. In Section 2.7, it was stated that a property of time-independent, Hamiltonian systems is their volume-preserving nature (Liouville’s Theorem). The PCR3BP is such a system. It is seen that the total area of each successive unstable manifold intersection remains constant. Similarly, the total area of each successive stable manifold intersection remains constant. Further, the successive overlapping areas i.e. the yellow areas and green areas, are also all constant. This verifies the expected property of the PC3BP. The internal structure of the WSB region can be mapped in this manner.

It is interesting to estimate the percentage of trajectories in the WSB region that are stable for a given number of orbits. For six orbits, this means that all possible combinations of overlapping areas between stable and unstable manifolds that lead to stability for six orbits around the Moon have to be considered. For six orbits, using the previous formula, the possible combinations of intersections are: \( W_s^1 \) and \( W_u^6 \), \( W_u^1 \) and \( W_s^6 \), \( W_s^2 \) and \( W_u^5 \), \( W_u^2 \) and \( W_s^5 \), \( W_s^3 \) and \( W_u^4 \), and \( W_u^3 \) and \( W_s^4 \). The yellow areas in Figure 8.12 illustrate precisely these combinations.

The area of a single yellow area is found to be \( 9.427 \times 10^{-4} \). Since the yellow areas are all equal and there are six in total, this results in a total yellow area of \( 5.656 \times 10^{-3} \). It can be computed that for this Jacobi energy value that the total area of the ‘fish’ shape is approximately \( 6.240 \times 10^{-2} \). Hence it follows that the percentage of trajectories that are stable within the ‘fish’ for six orbits is approximately \( 9.064\% \). Based on the histogram data, the percentage for the Poincaré section generated with 1000 intersections per initial condition is approximately \( 5.599\% \). To first order this stochastic estimate is good. For the Poincaré section generated with 2000 intersection per initial condition, the percentage is approximately \( 5.784\% \). It is understandable that the stochastic method underestimates the area, as the Poincaré section is not completely filled with points: rather the Poincaré section consists of a sample set of points within the \( x\dot{x} \)-plane that are unevenly distributed. Nevertheless, to first-order, the stochastic method provides a good indication of the relative frequency of trajectories that stay in the vicinity of the Moon for differing numbers of orbits.

The manifold method is naturally the most accurate, as the estimated areas include the whole set of points for each trajectory class. It turns out however that the manifold-area method is not a robust way of determining the internal structure of the WSB region. This stems from the chaotic nature of the PCR3BP. In Figure 8.12, all of the manifold intersections are closed curves. This means that estimating the area contained within is fairly straightforward. The problem however is that the chaotic nature of the PC3BP tends to break up these closed curves, or topological circles. This can be illustrated by considering for instance the seventh unstable manifold intersection in addition to the first six. Figure 8.15 illustrates these manifold intersections.

From this figure, it is clear that the seventh unstable manifold intersection is stretched and twisted. The closed curve apparent from the sixth intersection is broken up into several parts. Part of the seventh manifold intersection is contained within the fish shape, part is located on the left side of the Moon, and part transits through the L1 neck region and is found near the Earth. This is possible, as even though the unstable manifold is asymptotic to the Lyapunov orbit in backwards time, there is no condition preventing it from transiting through the neck region in forward time. Trajectories which are asymptotic in forwards and backwards time
to the Lyapunov orbit are special and are denoted homoclinic. Since the manifold intersection breaks up in this manner, it is no longer possible to determine an area contained within and to identify areas of overlap.

The problem becomes worse for successive intersections, with the distribution of the points becoming ever more chaotic. Hence, determining the total area containing points which are on trajectories that are stable around the Moon for seven orbits is not possible in the manner described previously for six orbits. This is because although the green areas seen in Figure 8.15 are closed, it is not possible to find a closed curve to determine the intersection between the first intersection of the stable manifold and the seventh of the unstable manifold. From the figure, it is apparent that they do intersect, but it is clear that isolating a single closed curve is not possible.

The stochastic methodology proposed in this chapter, using extended-Poincaré sections, is a possible solution, to probe the internal structure of the WSB region robustly. It is interesting to consider whether the stochastic regions given in Figure 8.8 match with the first six manifold intersections. Figure 8.16 illustrates the superposition of these manifold intersections on the stochastic data set for the WSB region. For clarity, only points that belong to trajectories that orbit the Moon for less than ten orbits on the right side are shown.

From this figure, it is clear that the extended-Poincaré section corresponds well with the location of the manifold intersections. It should be noted however that because the invariant manifolds tend to break up, there are some results that are not apparent when purely considering their intersections with the Poincaré section. For instance, from this figure it is apparent that there is a point, denoted P, which lies within the overlapping area where one would expect trajectories to remain stable around the Moon for six orbits, that actually remains stable for eight orbits. This anomalous result can be explained by the fact that after the sixth manifold intersection, the closed curve breaks up, and the points lying on the manifold are distributed chaotically within the $x\dot{x}$-plane. Hence, although the yellow areas in Figure 8.15 include all points that are stable for six orbits, they also include points that are stable for more orbits. In other words, the union of the set given by all the yellow areas is larger than the set of points that remains stable for exactly six orbits. This provides another illustration of the fact that purely considering the manifold intersections is insufficient when probing the internal structure of the WSB region.

In fact, the manifold method is quite restricted when it comes to charting the set of points in phase space that support weak capture. Specifically, identifying this set in the full spatial 3BP, in a time-dependent variant such as the Elliptic Restricted 3-Body Problem (ER3BP) or a differentially-corrected ephemeris model, or in a problem with more than three bodies such as the Restricted 4-Body Problem (R4BP), turns out to be very difficult if not impossible when using only the manifold-method. Herein lies the strength of the WSB concept. The WSB concept is much more generic, in the sense that the algorithmic definition or the Poincaré section method is valid irrespective of the model being considered. This is however the case in theory. In practice generating data in, for instance, the CR3BP or ER3BP turns out to lead to added complications. These complications include the fact that extra dimensions are added to the problem, which means that the initial state vector cannot be described in the same manner as done for the
PCR3BP. Additional assumptions are required, and a more robust methodology has to be investigated. In a R4BP the situation is further complicated by the fact that there is no integral of motion that can be used to monitor the accuracy of the trajectory integration. Isolating a set of points belonging to a single energy level is also not straightforward. Possible solutions to these problems will be proposed in Chapter 11.

It is useful to study the WSB region in the PCR3BP, as it can provide clues as to how this set can be identified in more complicated models. The analytical WSB definition presented in Section 7.2.2 was shown to be an attempt to identify this set without resorting to trajectory integration. The extended WSB set $B$, is defined by the intersection of a number of different sets. The projection of Jacobi energy surface ($\mathcal{J}(C)$) on the Poincaré section indicates the boundary between regions where $m$ can and cannot be located. The set $\Sigma$ provides an indication of the boundary between points which have positive and negative Kepler energy with respect to the Moon.

Since the aim of this study is to understand the underlying dynamics better and possibly improve the analytical definition of the WSB, it is useful to visualise some key aspects. The 2-Body Kepler energy plays a central role in defining the WSB region in the analytical definition, for both the classic ($W$) and extended ($B$) cases. It is prudent to visualise the structure of the Kepler energy set within the Hill region. The Kepler energy set can be determined in rotating coordinates using Equation (7.16).

Figure 8.17 illustrates the Kepler energy set within the inner Hill region as a two-dimensional plot for the Jacobi energy level being considered. From this figure, it is apparent that the Kepler energy with respect to the Moon is more positive on the right side compared to the left side of the Moon. This is indicative of the fact that trajectories on the left side are more stable. Trajectories on the left side are predominantly retrograde orbits. Further, from this figure, it seems that there is an inflexion point in the neighbourhood of L1. This is related to the fact that the invariant tori on the left side of the Moon terminate in the vicinity of the neck region. The shape of the invariant tori on the left side can be seen clearly (dark blue area). This provides insight into the structure of the set $B$.

The set $B_2$, given by Equation (8.4), which describes the 2nd-extended WSB, also includes an intersection of the angular momentum boundary with the set $B$, for the Jacobi energy level considered in this case. The projection of the angular momentum with respect to the Moon on the Poincaré section can be visualised in the same manner as the projection of the Kepler energy. Figure 8.18 illustrates a two-dimensional representation.

From this figure, it is apparent that there is a local maximum at L1. Much in the same manner as the Kepler energy set, this invariant tori shape is reflected on the left side of the Moon. The main difference between the Kepler energy set and the angular momentum set is that in defining this shape, the angular momentum seems to differentiate the tori from the chaotic sea by a sign change. In other words, the boundary where the angular momentum set is zero with respect to the Moon seems to trace the shape of the invariant tori. This is a very useful property as it means that it is perhaps possible to exclude the invariant tori on the left side of the Moon from the WSB definition, as desired. It is imperative to note
that the angular momentum set being considered is a projection on the Poincaré sections generated. Hence an angular momentum of zero does not imply that the velocity components must be zero i.e. \( \dot{x}, \dot{y} \). Rather, an angular momentum of zero relates to a specific type of trajectory: one that does not exhibit prograde or retrograde motion around the Moon. In essence, such trajectories are precisely impacting trajectories at the Moon. This follows by considering Equation (7.17). From this equation, it is clear that one of the solutions that results in zero angular momentum is \( x = 1 - \mu \), which is the precise location of the Moon.

To demonstrate whether this is true, the set \( B_2 \) must be projected on the Poincaré section. Firstly, the projection of the set \( B \) is provided in Figure 8.19. In Figure 8.19, the WSB region is defined as the intersection of the Jacobi energy surface projection and the Kepler energy projection. The points given in red all belong to the WSB region according to the extended-WSB analytical definition. It is clear that this set \( B \) includes points that should not be in the WSB region. The invariant tori on both the left and right sides of the Moon should not be included as these points do not lie on transit trajectories. Further, with this definition, there are points to the left of \( L_1 \) that are included too. These points are superfluous, as the trajectories that they lie on are already included in the WSB set since their intersections when orbiting the Moon are included in the set of points to the right of \( L_1 \).

If the WSB region is defined as the minimal set of points in phase space that support weak capture, then effectively the only points that should be included are those illustrated in red in Figure 8.20. The aim therefore is to improve the analytical definition such that the mathematical set approximates the set of points in red in this figure.

Including the conditions that the angular momentum with respect to the Moon must be greater than zero, and the points must lie to the right of \( L_1 \) yields the approximation of the WSB region given by Figure 8.21. It is apparent that the approximation given by the set of red points in this figure is a much better fit to Figure 8.20 than Figure 8.19. Almost all of the invariant tori on the left side of the Moon have been excluded from the WSB set.

Nevertheless, there are still a few problems with this approximation. Firstly, there are a few invariant tori beyond the angular momentum boundary on the left side. These are quite interesting as they depart the Poincaré section such that \( \dot{y} > 0 \), however their angular momentum is positive i.e. the same sense as the trajectories on the right side of the Moon. Additionally, it is clear that on the right side of the Moon, the invariant tori have not been excluded. It turns out that it is quite difficult to approximate the ‘fish’-shape analytically. Hence to resolve this problem requires extensive analytical work. An interesting note though can be made based on Figure 8.22. Figure 8.22 illustrates the set of points in the Poincaré section that belong to impact trajectories (red). Impact trajectories at the Moon are defined as trajectories that cross into a disc of radius \( 10^{-3} \) around the Moon. The interesting thing to note is that these impact trajectories are located on invariant tori on both sides of the Moon. On the right side it is apparent that these impact trajectories shadow the ‘fish’-shape. Hence, although it might not be possible to generate the ‘fish’-shape analytically, it can be observed that a trajectory entering the numerically defined disc around the Moon approximates

\[ x = 1 - \mu, \]
the shape for this Jacobi energy level: a nearly-impacting trajectory. Finding an analytical expression for this set however requires further research.

Figure 8.23 illustrates a number of different regions in the Poincaré section being considered. To be able to visualise the nature of trajectories in each of these regions, a sample set is described in Appendix A. Trajectories in the sample set are integrated in forwards and backwards time. A full description of the trajectories is given in Appendix A.

**Case 2**: $C = 3.4000000000$

For this Jacobi energy value, the L1 and L2 Lagrange points do not exist, i.e. $C > C_1$. Hence the inner Hill’s regions are not connected; motion between the Earth and the Moon is not permitted. Figure 8.24 illustrates the extended-Poincaré section obtained for $C = 3.4000000000$. From Figure 8.24, it is clear that as expected, there are no transit trajectories contained in this Poincaré section. Hence it can be concluded that the WSB region only exists when the Hill’s regions are connected. This means that the lower bound $C_1^*$ is defined such that the L1 Lagrange point exists i.e. $C_1^* = C_1$.

**Case 3**: $C = 3.1990000000$

For this Jacobi energy value, only the L1 Lagrange point exists, as in Case 1. Hence motion is once again only permitted between the inner Hill’s regions, connecting the Earth and the Moon. Figure 8.25 illustrates the extended-Poincaré section obtained for $C = 3.1990000000$. In Figure 8.25 it is apparent that the ‘fish’-shape noted for Case 1 persists. A number of differences can be noted from Figure 8.25 when compared to Figure 8.8. Figure 8.25(a) shows clearly that the maximum number of orbits that trajectories within the WSB remain stable around the Moon is significantly higher. The number of orbits that trajectories in the neighbourhood of the prograde resonant island remain stable around the Moon is also higher. The resonant island itself is larger (white gap is larger). To fully understand the distribution of points in this figure, histograms are generated, as for Case 1. Figure 8.26 illustrates the histogram and cumulative histogram data for this Jacobi energy level. From Figure 8.26, it is apparent that there is a big gap between the maximum number of orbits and the rest. Percentage-wise only approximately 60% of the trajectories remain stable for less than 100 orbits around the Moon. Hence, on the whole for this energy level it seems that the orbits are more stable. This can be understood by the fact that for this Jacobi energy level the neck region near L1 is smaller than compared to Case 1. Hence the Lyapunov orbit is smaller and the associated invariant manifold intersections are also smaller. This means effectively that it is more difficult for trajectories near the Moon to escape through the neck region to the Earth.

Figure 8.27 illustrates the Poincaré section associated with the Jacobi energy level, with the conditions for the set $B_2$ included. From this figure it can be seen that $B_2$ can be used fairly successfully in approximating the WSB region (red). The problem noted in Case 1 that a number of invariant tori on the left side of the Moon are included in $B_2$ persists. It can also been seen from this figure that the ‘fish’-shape and resonant island within persist on the right side of the Moon.
Case 4: $C = 3.1970000000$

For this Jacobi energy value, only the L1 Lagrange point exists, as in Case 1 and Case 3. Hence motion is for this case also only permitted between the inner Hill’s regions, connecting the Earth and the Moon. Figure 8.28 illustrates the extended-Poincaré section obtained for $C = 3.1970000000$. In Figure 8.28 it is apparent that the ‘fish’-shape noted for Case 1 and Case 3 persists. From Figure 8.28(b), it can be seen that there are many trajectories that remain stable for more than 100 orbits along the contour of the ‘fish’-shape. It is also apparent that the resonant island within the ‘fish’-shape is smaller. The set of points belonging to the WSB region on the left side of the Moon is noticeably larger. To understand the distribution of points, histograms are generated as for Case 1 and Case 3. These are given in Figure 8.29. From Figure 8.29, it is apparent that the results are quite similar to Case 1. The main different is the fact that there seems to be a spike in Figure 8.29(b) for the frequency of trajectories remaining stable for more than 200 orbits. This is also apparent from Figure 8.28.

Figure 8.30 illustrates the Poincaré section associated with the Jacobi energy level, with the conditions for the set $B_2$ included. From this figure it is apparent that the set $B_2$ roughly approximates the WSB region (red), however the problem with invariant tori on the left side of the Moon persists. The fact that the resonant island within the ‘fish’-shape has almost disappeared can also clearly be seen from this figure.

8.2.2 Overall Results

Having analysed the WSB region for a few different Jacobi energy values, a number of general results can be commented upon. In this chapter, a new stochastic method of probing the internal structure of the WSB region was presented. For the cases analysed, histogram data was generated that indicated the distribution of stability within the WSB region. To obtain a better comparison of this data, Figure 8.31 illustrates the cumulative histogram data for the three cases analysed, for which transit between the Earth and the Moon occurred (Cases 1, 3, and 4).

From Figure 8.31 it seems that decreasing the Jacobi energy value causes an increase in the percentage of trajectories in the WSB region that remain stable for few orbits. It must be noted that this is very much a preliminary result. To be able to state this as a clear conclusion, a more robust analysis is required with a greater number of cases being studied. Nevertheless, this figure provides food for thought.

In addition to introducing the idea of extended-Poincaré sections, a proposition was put forward at the beginning of this section to improve the analytical definition of the WSB region. One of the main problems with the current analytical definition (Section 7.2.2) is that it includes stable trajectories around the Earth and the Moon, which are non-transitory. The behaviour of the set $B$ as a function of the Jacobi energy level was presented in Figure 7.13. In a similar manner, the behaviour of the set $B_2$ can be studied. Table 8.2 summarises the Jacobi values used to analyse the behaviour of $B_2$. Figure 8.32 illustrates the set $B_2$ for each of these Jacobi energy values. It is clear from this figure that the angular momentum condition (green line) is not robust. For $C = C_3$, the angular momentum boundary
<table>
<thead>
<tr>
<th>C</th>
<th>Jacobi energy</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
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<tr>
<td>9</td>
<td>2.8880000000</td>
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Table 8.2 Jacobi energy values used to analyse behaviour of $B_2$.

on the left side of the Moon passes through the neck region and is connected with the boundary to the left of L1 (L1 and L2 boundaries are given by the blue lines). For decreasing values of $C$, the angular momentum boundary on the left side no longer shadows the invariant tori. In fact, the Kepler energy boundary (red line) becomes a larger subset of the angular momentum condition for decreasing $C$. For $C = C_8$, the Kepler energy set is a full subset of the angular momentum with respect to the Moon. The Kepler energy is a subset of $H_M$, which means that using the $B_2$, the WSB region has disappeared from the left side of the Moon. The angular momentum condition is therefore clearly not accurate over the entire range of Jacobi values. The two boundaries touch for $C_{H}^* \approx 3.1752$. Hence, it can be stated that the angular momentum condition works well for the range $C_1 > C > C_{H}^*$.

Nevertheless, for large values of $C$ such as $C_1$, the angular momentum condition does seem to provide the means to remove most of the invariant tori from the left side of the Moon. Further research is necessary to ascertain whether this angular momentum condition can be incorporated in the analytical WSB definition more effectively.
Figure 8.5  Poincaré section in the $x\dot{x}$-plane for $C = 3.1980000000$. Also illustrated are the Moon (red) and L1 (brown).
Figure 8.6  Extended-Poincaré section in the $x\dot{x}$-plane for $C = 3.1980000000$, illustrating number of orbits around the Moon that $m$ remains stable for.

Figure 8.7  Set of points for $C = 3.1980000000$ that are stable for at least 1000 orbits (blue) around the Moon and 0 orbits. The set of points that belong to the 0-orbits set is comprised of stable (red) and unstable (green) trajectories.
Figure 8.8  Extended-Poincaré section in the $x\dot{x}$-plane for $C = 3.1980000000$ illustrating points leading to unstable motion around the Moon. These points are stable for less than 1000 orbits and more than 0 orbits around the Moon and lie to the right of L1. (a) 3D-view of extended-Poincaré section, (b) 2D-view of extended-Poincaré section.
Figure 8.9 Sample set of cases within extended-Poincaré section in the $x\dot{x}$-plane for $C = 3.1980000000$ illustrating points leading to unstable motion around the Moon; (a) 6 orbits, (b) 90 orbits, (c) 145 orbits, and (d) 217 orbits.

Figure 8.10 Distribution of transit orbits for $C = 3.1980000000$; (a) histogram frequency distribution, (b) cumulative histogram percentage frequency distribution.
Figure 8.11  Comparative cumulative histogram of distribution of transit orbits for \( C = 3.1980000000 \). The cumulative distribution is given for a Poincaré section generated with 1000 (red) and 2000 (blue) orbits per initial condition.

Figure 8.12  First six intersections of the stable (red) and unstable (blue) invariant manifolds with Poincaré section for \( C = 3.1980000000 \). Points that belong to trajectories that are stable for 6 (yellow) and 7 (green) orbits around the Moon are indicated.
Figure 8.13  Schematic illustration of overlapping area between the \( i \)-th intersection of the unstable (red) and \( j \)-intersection of the stable (blue) manifold intersection in a Poincaré section [Gidea and Masdemont, 2007].

Figure 8.14  Example of trajectory that lies within overlap of intersections of unstable and stable manifolds in the Poincaré section. The point lying in the Poincaré section is integrated forwards (red) and backwards (blue) in time. The Moon is also illustrated (black).
Figure 8.15  First six intersections of the stable (red) and unstable (blue) invariant manifolds, and seventh intersection of unstable (black) invariant manifold with Poincaré section for $C' = 3.1980000000$. Points that belong to trajectories that are stable for 6 (yellow) and 7 (green) orbits around the Moon are indicated.
Figure 8.16  First six intersections of the stable and unstable invariant manifolds and stochastic data in extended-Poincaré section for $C = 3.1980000000$. Points in the WSB region on the right side of the Moon that remain stable for less than 10 orbits around the Moon are illustrated.

Figure 8.17  Two-dimensional projection of Kepler energy with respect to Moon on Poincaré section within the inner Hill region for $C = 3.1980000000$. 
Figure 8.18 Two-dimensional projection of angular momentum with respect to Moon on Poincaré section within the inner Hill region for $C = 3.1980000000$.

Figure 8.19 Poincaré section for $C = 3.1980000000$ illustrating the extended-WSB set $B$. The red points belong to the set $B$, which approximates the WSB region. The projections of the Jacobi energy surface (black line) and the Kepler energy boundary (red line) are also illustrated.
Figure 8.20  Poincaré section for $C = 3.1980000000$ illustrating the minimum WSB set (red). The points in blue are not included in the WSB region.

Figure 8.21  Poincaré section for $C = 3.1980000000$ illustrating the 2nd-extended WSB set $B_2$. The red points belong to the set $B_2$, which approximates the WSB region. The projections of the Jacobi energy surface (black line), the Kepler energy boundary (red line), and the angular momentum boundary (green line) are also illustrated.
Figure 8.22  Poincaré section for $C = 3.1980000000$ illustrating impact trajectories with the Moon (red).

Figure 8.23  Poincaré section for $C = 3.1980000000$ illustrating different regions: invariant tori around the Earth (blue), transit points to the left of L1 (green), invariant tori with positive angular momentum and transit points on the left side of the Moon to the right of L1 (cyan), invariant tori on the left side of the Moon with negative angular momentum and invariant tori on the right side of the Moon with positive angular momentum (black), and transit points on the right side of the Moon (red).
Figure 8.24  Extended-Poincaré section in the $xx$-plane for $C = 3.4000000000$; (a) 3D-view of extended-Poincaré section, (b) 2D-view of extended-Poincaré section.
Figure 8.25  Extended-Poincaré section in the $xx$-plane for $C' = 3.1990000000$; (a) 3D-view of extended-Poincaré section, (b) 2D-view of extended-Poincaré section.
Figure 8.26 Distribution of transit orbits for $C = 3.1990000000$; (a) histogram frequency distribution, (b) cumulative histogram percentage frequency distribution.

Figure 8.27 Poincaré section for $C = 3.1990000000$ illustrating the WSB region (red) and the conditions for the 2nd-extended WSB definition. Projections of the Jacobi energy surface (black line), Kepler energy boundary (red line), L1 boundary (blue line), and angular momentum boundary (green line) are illustrated.
Figure 8.28 Extended-Poincaré section in the \( xx \)-plane for \( C = 3.1970000000 \); (a) 3D-view of extended-Poincaré section, (b) 2D-view of extended-Poincaré section.
Figure 8.29 Distribution of transit orbits for $C = 3.1970000000$; (a) histogram frequency distribution, (b) cumulative histogram percentage frequency distribution.

Figure 8.30 Poincaré section for $C = 3.1970000000$ illustrating the WSB region (red) and the conditions for the 2nd-extended WSB definition. Projections of the Jacobi energy surface (black line), Kepler energy boundary (red line), L1 boundary (blue line), and angular momentum boundary (green line) are illustrated.
Figure 8.31 Comparative cumulative histogram data illustrating Case 1 (red), Case 3 (green), and Case 4 (blue); (a) 0-250 Orbits, (b) 0-1000 Orbits
Figure 8.32  Visualisation of 2nd-extended Weak Stability Boundary set $B_2$ (grey) on Poincaré section for decreasing values of $C$. Projections of the Jacobi energy surface (black line), Kepler energy with respect to the Moon (red line), angular momentum with respect to the Moon (green line) are illustrated. Vertical conditions at L1 and L2 are also illustrated (blue lines).
8.3 Conclusions

Based on the simulations performed and the analysis outlined in Section 8.2, a number of general conclusions can be drawn. Firstly, it is apparent that the use of invariant manifold intersections on Poincaré sections to determine the internal structure of the WSB region is limited. A stochastic method was introduced using the concept of extended-Poincaré sections. Such sections were generated for three different Jacobi energy values and provide additional insight into the internal structure of the WSB region. The cases considered are quite complicated, in that the WSB region is quite difficult to approximate analytically. This is specifically the case because of the complex invariant tori structure on the right side of the Moon. The Jacobi energy levels considered were selected such that the associated WSB region would be as complex as possible, to highlight the advantage of using extended-Poincaré sections to study its internal structure. For Jacobi energy values greater than that associated with L2, i.e. $C_2$, other studies indicate that the complex invariant tori structure on the right side of the Moon disappears [Belbruno et al., 2008].

It would seem from the extended-Poincaré sections that the WSB region consists of many points that remain stable around the Moon for relatively few orbits. Histogram data plotted to determine the distribution of these points amongst different trajectory classes hints at the fact that with decreasing values of $C$ (increasing energy), trajectories in the WSB region as a whole become less stable around the Moon. Additionally, it is seen that for the largest value of $C$ investigated, for which transit motion occurs ($C = 3.1990000000$), there is a comparatively large set of trajectories that remain stable for over 800 orbits. For the other two Jacobi energy values considered ($C = 3.1980000000$ and $C = 3.1970000000$), the maximum number of orbits of stability around the Moon is much less. This is understandable, as with decreasing values of $C$ the system becomes more energetic, hence weak capture trajectories are more likely to escape the gravitational pull of the Moon. This is hinted at by a few other features as well. For instance, the resonant island which surrounds the periodic orbit on the right side of the Moon noticeably decreases in size as the value of $C$ decreases.

The remarkable result is that this change, as with the other changes, is clearly visible between the three cases, despite the fact that the value of $C$ does not vary greatly. This highlights how sensitive the PCR3BP is to changes in Jacobi energy and re-emphasises why error control of its value is crucial in generating the sections. One of the problems faced during integration was maintaining the error in the Jacobi energy within acceptable tolerances. The absolute tolerance for Jacobi energy, used for integrating a single initial condition on the $x$-axis for 1000 orbits, was $10^{-6}$. This tolerance level is not to be confused with the absolute error tolerance of $10^{-12}$ used for error accumulation due to the integration scheme adopted. When a trajectory violated this error tolerance level the integration was stopped. Initially, this resulted in a smaller data set for the Poincaré sections and extended-Poincaré sections than expected. The problem was fixed for the Jacobi energy cases considered by tuning the absolute and relative tolerances of the integrator itself. The problem however is that this is not robust, in the sense that the integrator would have to be re-tuned in all likelihood for other Jacobi energy values.
In addition to generating extended-Poincaré sections, an attempt was made to improve upon the existing analytical WSB definition given by set $B$. Angular momentum was introduced as a condition to try to remove invariant tori from the left side of the Moon from the WSB set. In addition, the observation was made that only points lying between L1 and L2 would be necessary. This was deemed to be the case since trajectories which transit through the neighbourhood of these points intersect the Poincaré section near the Moon; hence it is superfluous to include points beyond these Lagrange points as they would belong to trajectories that already yield intersections near the Moon. The conditions imposed to take this into account were approximated by vertical straight line boundaries at the Lagrange points.

The first observation to be made stems from analysing the extended-Poincaré section generated for a Jacobi energy value for which transit motion should not occur: $C = 3.400000000$. As expected, in this case there are no points that transit between the Earth and the Moon; hence the WSB region is empty. The largest Jacobi energy value for which transit is possible is the energy level associated with L1, as in this case the physical Jacobi energy surface opens up for the first time to allow trajectories to pass between the Earth and Moon. For this reason, the lower bound in $C$ value for which the WSB region is defined i.e. $C_1^*$ is found to be equal to $C_1$. This is different from the definition of the set $B$, for which the lower bound is set at $C_2$, which is the Jacobi energy value associated with L2. The Jacobi energy values studied, for which transit is permitted, are all larger than $C_2$. Clearly transit between the Earth and Moon is possible in these cases. If the WSB region is generally defined as the set of points in phase space supporting such behaviour, then it is clear that the lower bound for the definition must be $C_1$.

One of the main problems noted for the set $B$ is that it overestimates the WSB region. It includes a number of different regions of points that do not exhibit transit motion: invariant tori around the Earth, and invariant tori on the prograde and retrograde sides of the Moon. An attempt was made to remove some of these regions from the WSB set. By imposing the L1 and L2 constraints, the invariant tori around the Earth were removed. The angular momentum condition was imposed to try and remove invariant tori on the retrograde side of the Moon. For the Jacobi energy values considered, it is demonstrated that this can be achieved quite successfully. It is seen that most but not all of the retrograde invariant tori are removed. This is definitely an improved approximation of the WSB region, and was possible because it was noted that for most of the retrograde invariant tori, there is a sign change in angular momentum compared to the WSB region.

Subsequent analysis of the behaviour of the angular momentum boundary as a function of decreasing Jacobi energy value indicates however that the approximation for lower values of $C$ deteriorates. Hence, it can be concluded that the angular momentum condition is not robust: it is only applicable to a subset within the Jacobi energy range for which the WSB region is defined: $C_1 > C > C_2^*$, for which $C_2^* \approx 3.1752$ is numerically estimated. The aim therefore of improving the analytical definition was not entirely successfully accomplished, as the set $B_2$ does not approximate the WSB region well for the full range of Jacobi energy values. Nevertheless, this might prove to be a good starting point to investigate other possibilities to improve the analytical definition.

An interesting result was obtained that might prove to be useful as the start-
ing point to try and find an analytical condition to remove the invariant tori on the prograde side of the Moon. These invariant tori are difficult to analytically approximate due to their highly complex contours. The contour of the ‘fish’-shape identified, which contains the WSB region on this side of the Moon, turns out to be shadowed by nearly-impact trajectories at the Moon. This is an aspect for further study.

Approximating the part of the WSB region on the retrograde side of the Moon is difficult too, as it is related to the location of the stable and unstable invariant manifolds. The numerical algorithm used to approximate the location of the invariant manifold intersections is not robust in close proximity to the Lyapunov orbit at L1; hence these intersections are not shown on the Poincaré sections generated. Nevertheless, by analysing sample trajectories (Appendix A), it is apparent that the behaviour in this neighbourhood is closely linked to the manifolds.

Further studies are necessary to be able to investigate the WSB more thoroughly. The analysis presented in this chapter outlined a possible framework to achieve this.
Chapter 9

Space Mission Design Applications

The investigation reported presently is largely theoretical in nature. Despite this, a number of tangible applications of the methods employed and the results obtained, in space mission design, can be identified. In this chapter, a brief overview of such applications will be provided, with a number of observations regarding applications to mission analysis work.

A number of space missions have made use of the weak capture concept in recent times. The Hiten mission was the first to successfully demonstrate a low-energy transfer to the Moon using the weak capture concept. An overview of this mission is provided in Section 9.1. In Section 9.2, an overview is provided of a mission that was successfully executed to collect solar wind particles and return them to the Earth for analysis using a weak capture trajectory that deterministically only required one manoeuvre. The concept of a multi-moon orbiter, with application to the Jovian system, is presented in Section 9.3. Such a tour of the moons of Jupiter, for instance, is facilitated by patching a number of 3-Body Problems (3BP) together. ESA’s BepiColombo mission, designed to study features of Mercury, uses weak capture to mitigate the risks of single-point failures that can occur for traditional chemical orbital insertion manoeuvres. The trajectory designed for BepiColombo uses weak capture to ensure a total of five recovery loops, during which the mission can be rescued in the advent of propulsion system failures. A brief overview of the mission analysis for BepiColombo is provided in Section 9.4. Finally, in Section 9.5, the concept of a cycler is used, with applications in the Earth-Moon system described. Such a trajectory could be used in the future to transport materials necessary to established a lunar base. Concluding remarks are provided in Section 9.6.

9.1 Hiten Mission

The first mission to successfully demonstrate the viability of low-energy transfers was Hiten [Belbruno, 2004]. The Japanese ISAS institute launched two coupled spacecraft into an elliptic Earth orbit in January 1990. The smaller spacecraft, Muses-B, detached from the larger one, Muses-A, and was intended to travel to the Moon using a traditional, high-energy, direct transfer. The communications system on Muses-B failed however and its mission was hence unfulfilled. Muses-B
was intended as a communications relay satellite for Muses-A and was equipped with scientific instrumentation to perform experiments whilst in Earth orbit.

In an attempt to salvage the mission, ISAS looked to the possibility of using the limited propellant mass available on Muses-A to transfer it to the Moon and obtain capture at a lunar periapsis altitude of 100 km. The propellant mass available on-board was however insufficient to capture the spacecraft into such a lunar orbit using the traditional direct transfer method. Nevertheless, there was a strong desire to explore the possibility of alternative methods to reach the Moon using the on-board propellant.

Muses-A was given new life with a radically new mission concept and was promptly renamed Hiten. A solution was found by Belbruno and Miller [Belbruno and Miller, 1990] at the Jet Propulsion Laboratory in June 1990 that enabled Hiten to reach the Moon on a ballistic capture transfer. With the limited propellant on-board, Belbruno and Miller successfully demonstrated that it was possible to reach the Moon by using the weak capture concept.

Belbruno and Miller made use of a 4-Body Problem, in which the Earth, Moon and Sun were modelled using the DE403 planetary ephemeris. The Sun was added to the problem to enable the transfer trajectory to bridge the starting point at the Earth with the ballistic capture transfer. Without the gravitational pull of the Sun, an energy barrier exists (similar to the barrier described in Section 7.2.2), that prevents the spacecraft from ballistically approaching the Earth. The gravitational pull of the Sun helps to break this energy barrier down, thus providing the means of determining a fully ballistic capture transfer from the Earth to the Moon. To optimise the trajectory fully, the full spatial problem was considered. Belbruno and Miller termed this the 3-Dimensional Pseudocircular Restricted 4-Body Problem (PR4BP-3D). The problem is referred to as being ‘pseudocircular’ since the orbits of the primary bodies, modelled by the ephemeris, are not purely circular. Nevertheless, Belbruno and Miller successfully demonstrated in this case that a numerical approximation of the Weak Stability Boundary (WSB) for the Circular Restricted 3-Body Problem is a good starting point to determine a full trajectory from the Earth to the Moon.

Given the target periapsis altitude at the Moon, Belbruno and Miller integrated backwards after having established an approximation of the WSB region using the Algorithmic definition (Section 7.2.1). This trajectory arc, labelled II, is illustrated in Figure 9.1 and went out to approximately four times the Earth-Moon distance at its apoapsis with respect to the Earth. Thus this arc, starting at the patch point denoted $Q$, represents a ballistic capture transfer to the Moon. The time-of-flight (TOF) for arc II is approximately 45 days. The choice of the location of the patch point is chosen such that arc II can be patched with an arc from the Earth. This forward arc, arc I, was calculated by forward integrating from the starting point at the Earth to the patch point $Q$ and requires a TOF of approximately 100 days. The forward and backward trajectories match in position at $Q$, however a mismatch exists in velocity. This is bridged by providing a spacecraft manoeuvre using the on-board propulsion system. In terms of $\Delta V$, the sum of the manoeuvre at the Earth necessary for the spacecraft to follow arc I and the patching manoeuvre at $Q$ was minimised to approximately 44 m/s. The $\Delta V$ to attain temporary capture at the Moon was deterministically zero. Such a trajectory represents a marked improvement compared to a Hohmann transfer, which would require
9.2 Genesis Discovery Mission

The Genesis Discovery mission was launched in August 2001 \[Lo et al., 1998\], \[Wilson and Williams, 2003\], \[Koon et al., 2006\], and was the fifth mission selected as part of NASA’s Discovery program. It was designed to collect solar dust using collector arrays and return to the Earth safely to allow for the samples to be analysed. These samples were analysed to improve our understanding of the origin of the Solar System. The sample return capsule brought the solar dust back to the Earth successfully in September 2004. This marked a historic moment in space research with the first ever samples successfully brought back from beyond the Moon’s orbit.

Genesis was made possible by the unique trajectory design adopted, which enabled the payload capacity to be maximised and the propellant mass to be minimised. Figure 9.2 illustrates the trajectory that was flown. As illustrated in this figure, the outgoing leg of the mission from the Earth sent the spacecraft towards a Halo (3-dimensional periodic orbit) around the L1 Lagrange point in the Sun-Earth system, as this would provide uninterrupted access to the solar wind beyond the Earth’s magnetosphere. The L1 point in the Sun-Earth system is located approximately 1.5 million km from the Earth towards the Sun. Figure 9.3 illustrates three orthogonal projections of the trajectory presented in Figure 9.2. The incoming leg of this trajectory is also very interesting, as it was designed specifically to use the dynamics of the 3-Body Problem (3BP) related to the WSB, by passing in the vicinity of the L2 Lagrange point after departing from the Halo orbit. Both legs used low-energy transfers to bring the spacecraft to its destination. The main feature of the trajectory as a whole is that the three-year mission, from launch to Earth return, is that a marginal amount of $\Delta V$ is required to inject the spacecraft into the target Halo orbit. On-board fuel is also required for station-keeping and guidance and navigation purposes.
Figure 9.2 Genesis Discovery mission trajectory in rotating reference frame [Koon et al., 2006].

The outgoing leg took approximately three months, followed by a 22-month science acquisition phase during which the spacecraft orbits around L1. The spacecraft remained in orbit around L1 for approximately four orbits, after which it followed a natural path towards L2 before heading back to the Earth. Computation of the Halo orbit is well understood, however transfer to and from such an orbit using low-energy transfers remains somewhat of a novelty. The trajectory for Genesis was established by patching various invariant manifolds associated with the Halo orbit. For the outgoing leg, the trajectory was designed such that the spacecraft would follow the stable manifold, taking it naturally towards the Halo orbit without the need for manoeuvres. The incoming leg, a free return trajectory, was constructed by making use of the unstable manifold. The trajectory requires no manoeuvres to depart from the Halo orbit and follow the unstable manifold. This makes the trajectory fuel-efficient. The incoming trajectory departs from the vicinity of L1 and heads towards L2 before returning to the Earth. The trajectory was designed in this manner to fulfil the requirement of a day-side return at the arrival point on the Earth’s surface.

The use of invariant manifolds to design the Genesis trajectory ties in with the notion of weak capture. Hence, these solutions also exist in the WSB region associated with the Sun-Earth system. It is noted that finding a near-optimal trajectory such as used for Genesis using traditional algorithms is difficult. The Genesis trajectory was computed using a patching method which involves tracing the relative positions of the stable and unstable invariant manifolds associated with the target Halo orbit and determining necessary deterministic manoeuvres to bridge any ‘energy gaps’. Whilst this method is shown to be successful, it becomes more difficult to use in time-dependent, spatial problems, where the underlying dynamics is much more complex. To find the global optimum for such a low-energy transfer, it is prudent to identify the set of points in phase space that support weak capture, i.e. the WSB region, and use this as the search space for global optimisation. Hence, improving our understanding of the WSB region can
9.3 Multi-Moon Orbiter

One of the interesting mission scenarios that uses weak capture being considered is a multi-moon orbiter. The concept of such a mission is to launch a spacecraft to a planet with several interesting moons, and to design a trajectory such that multiple moons can be visited during the duration of the mission. Such a trajectory would be engineered to use low-energy transfers to minimise fuel consumption, therefore maximising the available payload mass for scientific instrumentation.

Recently, there has been growing scientific interest in planning a mission to study a number of the icy moons of Jupiter [Koon et al., 2006]: a Jovian Grand Tour. Europa is considered to be one of the best candidates to host life because of the vast liquid oceans that exist under its icy crust. Ganymede and Callisto are also thought to have liquid water under their surfaces. A proposed mission to tour these moons would attempt to map the regions of liquid water.

The multi-moon orbiter concept is radically new in that it allows for long term studies to be performed of many moons using a single spacecraft. Low-energy transfers and weak capture are central to enabling such missions to be planned. The use of low-energy transfers and resonant gravity assists allow for the total fuel consumption to be minimised. Resonant gravity assists with the moons can be understood as follows: performing small manoeuvres to achieve certain spacecraft-moon geometries allows for the spacecraft to jump between mean
motion resonance states with respect to the moon. This ‘jumping’ effectively provides the spacecraft with an energy kick, which in turn reduces the deterministic $\Delta V$ required for the entire trajectory. Such resonance kicks can be identified by studying trajectories that hop from the vicinity of one resonance state to another. This can be done efficiently by studying Poincaré sections, such as those presented in Chapters 7 and 8. Figure 9.4 provides a schematic illustration of a multi-moon orbiter mission concept for a tour of Jupiter’s icy moons. Such a trajectory can be successfully designed by using a Patched 3-Body Problem (P3BP). The method of Patched Conics has been used extensively to design interplanetary trajectories, and considers each leg of the trajectory as a 2-Body Problem (2BP). Although this has proven to be an efficient and robust methodology, it does not allow for dynamics of the 3-Body Problem (3BP), such as weak capture, to be harnessed. The P3BP, which is described in full by [Koon et al., 2006], includes the dynamics of the 3BP and can be used to determine fuel-efficient multi-moon orbiter missions.

In the case of a Jovian Grand Tour, the three-body system at a particular instance in time would be comprised of Jupiter, one of the moons, and the spacecraft. Essentially, transfer to and from a particular moon is then computed based on weak capture dynamics. This ties in closely with the computation of invariant manifolds. By patching a number of such three-body systems together, it is therefore possible to construct a trajectory in theory which enters into the Jovian system and visits a number of different moons to perform science, at minimal $\Delta V$ cost. An extensive treatment of such a methodology is presented by [Koon et al., 2006]. By analysing and locating the WSB region, it would therefore be possible to improve the optimisation process for such a multi-moon orbiter mission.

### 9.4 BepiColombo Mission

The European Space Agency’s (ESA) BepiColombo mission is scheduled to be launched in August 2013 and will spend six years en route to the planet Mercury. BepiColombo will deliver two satellites (the Mercury Planet Orbiter (MPO) and
the Mercury Magnetospheric Orbiter (MMO)) in orbits around Mercury. These satellites will study various phenomena on Mercury, of interest to the scientific community. BepiColombo will be the first space mission undertaken by ESA to use solar electric propulsion (SEP) to reach another planet. This propulsion method offers engineers the possibility to incorporate a fail-safe measure in the trajectory of the spacecraft, in addition to boosting payload capacity due to the higher specific impulse of the SEP compared to conventional chemical rocket propulsion. In particular, given the relatively low approach velocity of the spacecraft at Mercury, the engineers have decided to harness the gravitational pull of the Sun to ensure temporary capture around Mercury. This capture is shown to require deterministically zero $\Delta V$ i.e. other than for correction manoeuvres, the spacecraft will not have to expend fuel to achieve temporary capture around Mercury [Campagnola et al., 2006]. This is attractive as it offers mission analysts additional flexibility in achieving permanent capture around Mercury.

An analysis of the low-thrust approach to Mercury and subsequent weak capture is presented by [Campagnola et al., 2006]. The interplanetary trajectory selected for BepiColombo is complex and involves flybys at the Moon, Earth, twice at Venus, and twice at Mercury. ESA mission analysts at the European Space Operations Centre (ESOC), who engineered the trajectory to Mercury, considered traditional chemical orbit insertion a possible single-point failure; if the chemical engine fails to provide the necessary thrust to brake the hyperbolic excess velocity, the spacecraft is likely to be kicked back into interplanetary space and the mission will most likely be unrecoverable. The use of SEP is an option as the braking is smeared out over an arc, however the thrust level is too low to attain direct stable capture around Mercury. Hence, the mission analysts chose to make use of the gravitational pull of the Sun to slowdown the spacecraft and obtain weak capture around Mercury, and subsequently use chemical orbit insertion. Such an approach trajectory would provide recovery options in the event of single or even multiple orbit insertion failures.

The target orbit around Mercury is polar with a periherm altitude of 400 km and an apoherm altitude of 12000 km above the surface. The trajectory is determined by forward and backward propagation from the periherm of the orbit where chemical orbit insertion is planned to occur ($400 \times 180,000$ km). The backward propagation is used to find trajectories where the spacecraft leaves the attraction of Mercury due to the gravitational pull of the Sun. The objective is to find backward trajectories that exceed a distance of 300,000 km from Mercury, such that they are accessible from interplanetary space. The integration time chosen for these backward trajectories is 50 days. Trajectories that are not accessible from interplanetary space through backward propagation are denoted ‘incoming closed’, and those that escape directly from Mercury are denoted ‘incoming open’. All the trajectories are also propagated forward in time for 50 days. Trajectories that remain within 300,000 km of Mercury are denoted ‘outgoing: closed’ and those that leave the system within 50 days are denoted ‘outgoing: open’. Figure 9.5 illustrates the options for chemical orbit insertion for an entire orbit of Mercury around the Sun. Clearly, trajectories in the areas in black and white are not plausible options as they are ‘incoming: closed’, meaning that the are not accessible to spacecraft on arrival at Mercury from interplanetary space. The red area (dark grey) illustrates trajectories that can be accessed from space but are not bound to Mercury for 50 days. The green area (light grey) presents the best possible
Figure 9.5 Orbit insertion possibilities of BepiColombo during Mercury’s orbit around the Sun [Campagnola et al., 2006].

trajectories as the spacecraft can enter the Mercury system from interplanetary space and will remain weakly captured around Mercury for at least 50 days. This is desirable from ESA’s perspective in the event of nominal orbit insertion failure. After an analysis of the requirements and the possible trajectories, the mission analysts at ESOC chose an arrival date of 5th of January, 2017 (true anomaly of Mercury is 66.4°). The mission analysts at ESOC simulated a contingency plan in the case of nominal orbit insertion failure. The trajectory passes near the L1 Lagrange point of the Sun-Mercury system on its approach to Mercury. If orbit insertion fails on the 5th of January, 2017, the engineers are guaranteed five revolutions around Mercury before the spacecraft escapes into interplanetary space again. It is during these five recovery loops that the engineers hope to be able
to recover the spacecraft if nominal orbit insertion fails. The approach trajectory and the five revolutions around Mercury are illustrated in Figures 9.6 and 9.7. In Figure 9.7, $\delta V_{rec}$ indicates the possible recovery manoeuvres in case of orbit insertion failure. The values of $\delta V_{rec}$ are quite marginal (less than 10 m/s) and lead to stable capture orbits around Mercury with quite similar orbital parameters to the nominal case. The mission analysts used numerical methods primarily to characterise possible capture trajectories before making a final selection. This method, though effective in producing the desired end result i.e. a description of a ballistic capture orbit at Mercury, does not offer much insight into the underlying dynamics. Understanding and sketching the underlying dynamics is essential to mission analysts, as it can help to streamline the numerical methods. Additionally, any understanding of the fundamental principles governing the motion of the spacecraft in the Mercury system will provide insight into a vast array of problems in our solar system. For instance, mission analysts would be better placed to judge whether such a trajectory is also viable for spacecraft heading to Mars.

An analysis of the effect of eccentricity of Mercury’s orbit about the Sun on the capture trajectory possibilities is presented by [Naessens, 2006]. The Elliptic Restricted 3-Body Problem (ER3BP) is presented as a plausible model to reflect the primary dynamical effects in the Sun-Mercury system. A necessary condition for the capture orbit on arrival at Mercury is that the Hill Surfaces (zero velocity curves; explained in Chapters 4 and 5) are open. This means that the spacecraft must arrive at Mercury with a minimum energy requirement. Similarly, it is a necessary condition that the Hill Surfaces are open when the spacecraft escapes from the system. However, the Hill Surfaces only provide a coarse demarcation of regions accessible to the spacecraft; they do not reflect the true dynamics involved in the motion of the spacecraft. This is amply illustrated by a number of results obtained for different arrival conditions at Mercury. Once again, these results have been obtained through forward and backward propagation for a period of 50 days with respect to nominal orbit insertion at periherm. Figure 9.8 presents a forward and backward propagation for the case when the true anomaly of Mercury’s
orbit around the Sun is $\theta_{\text{Mercury}} = 85^\circ$. In Figure 9.8, the backward trajectory remains within the system for 50 days despite the fact that the Hill Surfaces are open. Hence, clearly outlining the nature of the Hill Surfaces is not sufficient to characterise the motion of the spacecraft in the Mercury system. To gain insight into the capture mechanisms involved and the likelihood that the spacecraft will escape from the system at a given point in time, the governing dynamics of the system have to be studied.

Mapping the WSB region is key to gaining further insight. By identifying the set of points that can support weak capture, it would be possible to understand why certain geometries of Sun-Mercury-spacecraft permit weak capture and others do not. Identifying the WSB region in the ER3BP is more difficult than in the CR3BP since it is a time-dependent system. This means therefore that the WSB region changes as a function of $\theta_{\text{Mercury}}$. A robust method is required to chart how the WSB region changes. This will provide more insight into the opportunities of designing such a trajectory. Further, studying the internal structure of the WSB region is crucial. The design goal of determining a trajectory which can provide a number of recovery loops is linked to determining the stability of trajectories within the WSB region.

### 9.5 The Cycler

The cycler mission concept is very interesting, particularly within the scope of NASA’s Vision for Space Exploration (VSE), which embodies a strong ambition for man to return to the Moon, and land on Mars. The cycler uses a phenomenon that is particular to the 3-Body Problem (3BP). As described extensively in Chapter 7, phase space is comprised of transit and non-transit trajectories. Transit trajectories are those that pass between the two primaries. It turns out that a trajectory that is transitory can be considered to exhibit such behaviour for all time.
Hence a trajectory that transits from the Earth to the Moon in the Earth-Moon system, can in general be expected to transit back to the Earth in finite time. In the PCR3BP, it is possible to find solutions that are resonant with the motion of the Moon around the Earth. In other words, it is possible to find trajectories that bounce back and forth between the Earth and the Moon regularly. This in effect sets up a chain, which can have very interesting applications. This type of trajectory is denoted a cycler, as it cycles between the two primaries.

Such a cycler might be considered a conveyor belt to transport payload to and from the Moon: and Earth-Moon cycler. The central idea for such a mission scenario would be to put a spacecraft, the transporter, on a cycler such that it spends a certain time around the Earth before transiting to the Moon for a given period of time, and then transiting back to the Earth to repeat the cycle. It is possible to imagine in this scenario that, as the spacecraft approaches the Earth, another spacecraft carrying payload to be transported to the Moon performs a rendezvous and docking manoeuvre to attach itself to the transporter. The transporter then carries the spacecraft to the Moon. The payload spacecraft then detaches from the transporter in the vicinity of the Moon and it can enter into a capture orbit by providing a small manoeuvre. The transporter returns to the vicinity of the Earth, where the cycle is repeated. This could provide a very useful mechanism to transport payload to the Moon to construct a lunar habitat.

Such an Earth-Moon cycler is illustrated by Figure 9.9. This trajectory, given in rotating coordinates, is not perfectly resonant, however it captures the general idea of the back and forth motion between the Earth and Moon that creates the conveyor belt. Another application of such a cycler could be for a spacecraft that performs science at the Moon. The rate at which data can be downloaded from a spacecraft is a function of its proximity to the Earth. Hence, a mission scenario is conceivable in which the spacecraft spends a few months around the Moon collecting data using a scientific payload and then transits to the Earth, where the data can be downloaded at high bit rate. Commands can also be uploaded to the spacecraft before it transits back to the Moon to collect the batch of data.
Such a cycler transits to and fro between the Earth and Moon for all of time in the PCR3BP. Hence, in essence the $\Delta V$ required to maintain the trajectory is deterministically zero. In reality however, such infinitely stable trajectories do not exist as perturbations, such as solar wind and the gravitational pull of other planets, cause them to be destabilised. It is however conceivable that with minimal $\Delta V$, it would be possible to maintain such cyclic behaviour for an extended period of time e.g. a few years. Further research is required to establish the existence of these long-term unstable cyclers in the full ephemeris model.

By analysing and studying the WSB region it would be possible to characterise and identify cyclers with specific stay times at the Earth and Moon. This could therefore be used as a starting point for further optimisation. Hence there is value in charting both the behaviour of the WSB region to changes in various system parameters such as energy and mass, and mapping its internal structure to reflect stability of the weak capture trajectory segments at both primaries.

9.6 Conclusions

In this chapter, a flavour of the types of applications of weak capture trajectories was presented. These mission scenarios make use of the dynamics of the 3BP to lower the $\Delta V$ that would be required to attain such performance using traditional techniques. Quite clearly, harnessing the dynamics of the 3BP opens up a plethora of interesting applications that would otherwise be impractical. Understanding the dynamics governing such trajectories is key to be able to apply them in future missions.

The Hiten mission proved that weak capture is in fact a concept that is applicable to real space missions. Low-energy transfers provide the means to maximise on-board scientific payload mass, to provide recovery options in case of propulsion system failures, and maximum scientific return by frequenting different bodies of interest.

The WSB region encapsulates the set of points in phase space that support weak capture behaviour. Therefore locating the WSB region is a fruitful endeavour. Identifying the WSB region and determining its internal structure can aid in the optimisation of trajectories for spacecraft missions of the future such as a multi-moon orbiter, BepiColombo or an Earth-Moon cycler. Despite the fact that the WSB region in the PCR3BP does not account for the full dynamics of motion for such missions, it does provide a starting point to determine suitable candidate trajectories. Hence, it is useful to study the WSB region in the PCR3BP, with a view to expanding the definition to include its behaviour in time-dependent, fully spatial models, such as the ER3BP or a differentially-corrected ephemeris model.
The 3-Body Problem (3BP) in astrodynamics has been investigated by numerous mathematicians and scientists, including leading figures such as Euler, Lagrange and Poincaré. The 3BP describes the motion of three massive bodies under the influence of their mutual gravitational attraction. Poincaré was the first to establish the chaotic nature of this deterministic system, and in doing laid the foundations for a new field of mathematics. This has led to advancements in fundamental mathematical methods and in our understanding of the underlying dynamics governing phenomena in our natural world. In particular, our understanding of the origin of our solar system, the governing principles of triple star systems, planetary ring formation processes, and mass transfer processes in celestial mechanics has benefited greatly from advances in research in this field. Recently, the 3BP has seen applications in space mission design, as the underlying dynamics is used to enable more fuel-efficient missions to be planned. The Japanese Hiten mission was the first such mission to successfully demonstrate the possible gain in mission performance achievable by using the 3BP [Belbruno, 2004].

Specifically, the notion of weak capture, which embodies the concept of low-energy transfers, is not sufficiently understood. Weak capture or ballistic capture refers to the process through which a body under consideration, typically a spacecraft, can transit between two large masses, termed primaries in the 3BP, without the requirement for deterministic fuel consumption i.e. without the need for rocket engine burns. This transitory behaviour is characteristic of the 3BP and understanding these characteristics is key to improving our understanding of the general problem of three bodies.

In this study, weak capture was investigated in the Planar Circular Restricted 3-Body Problem (PCR3BP). The PCR3BP is a simplified model based on a number of assumptions, i.e. the third body is considered to have negligible mass compared to the other two main masses (the primaries), thus not influencing their motion, the primaries are assumed to follow circular orbits about their common gravitational barycentre, and the motion of the third body is constrained to the plane of motion of the primaries. With these assumptions, it becomes possible to probe the underlying dynamics using semi-analytical and numerical techniques. Remarkably, the PCR3BP preserves some of the most essential dynamics of the general problem of three bodies; hence the results can be extended to more complicated models.
The Weak Stability Boundary (WSB) is a region in phase space that supports weak capture and can be described as a mathematical set of points that belong to transit trajectories. Efforts have been made to study this set and its behaviour in response to the variation of system parameters such as the masses of the primaries or the energy level of the trajectories (Jacobi energy in the case of the PCR3BP) \cite{Belbruno2008, Garcia2006}. A numerical algorithmic definition exists, which locates the WSB region through integration of trajectories and imposition of geometric criteria on the motion of the third body: if the third body rotates about one primary continuously the motion is said to be stable; if the third body rotates about one primary and transfers to rotating about the other the motion is said to be unstable. The WSB region in this definition is interpreted as being the complete set of points that lie at the transition point between stable and unstable motion. This definition, although robust, is difficult to use in practice and requires the integration of many trajectories, which requires significant computing power. Further, the algorithmic definition does not lend itself well to understanding the underlying dynamics of the problem and is very restrictive in terms of space mission applications.

An analytical definition exists, which attempts to mathematically capture the set of points describing the WSB region in phase space without the need for trajectory integration \cite{Belbruno2008}. This definition is based on the Energy method, and uses the intersection of some key mathematical sets such as the Jacobi energy surface, 2-Body Kepler energy with respect to one of the primaries, and a periapsis requirement with respect to the chosen primary. The WSB region can be efficiently generated using the Energy method, however it lacks accuracy. An extended definition has been recently introduced which expands the WSB region to include many more points which exhibit transitory behaviour between the primaries \cite{Belbruno2008}. This allows for the WSB to be used in a more practical sense in trajectory design and mission analysis. The extended definition is therefore more accurate in terms of capturing the essence of weak capture. Despite these improvements, the latest extended definition is also inaccurate.

The ultimate aim of studying the WSB region is to find an analytical definition which is able to precisely reproduce the set of points in phase space that support weak capture in the PCR3BP. The goal of this study was to attempt to find new criteria to improve the existing definition, whilst at the same time providing insight into the internal structure of the WSB region. In the PCR3BP, transit motion is intimately linked with the location of invariant manifold structures associated with periodic orbits about special equilibrium points, known as Lagrange points. These invariant manifolds are separatrices for the motion: trajectories lying within these topological tubes are transitory whilst those lying outside are non-transitory. Hence, the weak capture problem in the case of the PCR3BP can be analysed by considering the location of these invariant manifolds. The link between the WSB region and the location of invariant manifolds has been extensively charted within the last decade. In this study, the PCR3BP is used to visually illustrate this link and establish the correctness of the definition of the WSB region. Specifically, the internal structure of the WSB is tied directly to the location of the invariant manifolds. This is done by numerically analysing the problem by using Poincaré sections or Poincaré maps, which offer insight into the underlying dynamics by taking a slice out of the flow. The limitation of such Poincaré sections is that they
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do not provide insight into the stability of trajectories within the WSB region. Hence, extended-Poincaré sections are introduced in this study.

As part of the study, simulations were carried out for the Earth-Moon PCR3BP. Based on these simulation, a number of general conclusions can be drawn. Firstly, it is apparent that the use of invariant manifold intersections on Poincaré sections to determine the internal structure of the WSB region is limited. A stochastic method was introduced using the concept of extended-Poincaré sections. Such sections were generated for three different Jacobi energy values and provide additional insight into the internal structure of the WSB region. The cases considered are quite complicated, in that the WSB region is quite difficult to approximate analytically. This is specifically the case because of the complex invariant tori structure on the left side of the Moon. The Jacobi energy levels considered were selected such that the associated WSB region would be as complex as possible, to highlight the advantage of using extended-Poincaré sections to study its internal structure.

From extended-Poincaré sections, it would appear that the WSB region consists of many points that remain stable around the Moon for relatively few orbits. Histogram data plotted to determine the distribution of these points amongst different trajectory classes hints at the fact that with decreasing values of $C$, trajectories in the WSB region as a whole become less stable around the Moon. Additionally, it is seen that for the largest value of $C$ investigated, for which transit motion occurs ($C = 3.1990000000$), there is a comparatively large set of trajectories that remain stable for over 800 orbits. For the other two Jacobi energy values considered ($C = 3.1980000000$ and $C = 3.1970000000$), the maximum number of orbits of stability around the Moon is much less. This is logical, as with decreasing values of $C$ the system becomes more energetic, hence weak capture trajectories are more likely to escape the gravitational pull of the Moon. This is also hinted at by the fact that the resonant island, which surrounds the periodic orbit on the right side of the Moon, noticeably decreases in size as the value of $C$ decreases.

Remarkably, this change, as with the other changes, is clearly visible between the three cases, despite the fact that the value of $C$ does not vary greatly. This highlights how sensitive the PCR3BP is to changes in Jacobi energy and re-emphasises why error control of its value is crucial in generating the sections. One of the problems faced during integration was maintaining the error in the Jacobi energy within acceptable tolerances. When a trajectory violated the error tolerance level permitted, the integration was stopped. Initially, this resulted in a smaller data set for the Poincaré sections and extended-Poincaré sections than expected. The problem was fixed for the Jacobi energy cases considered by tuning the absolute and relative tolerances of the integrator itself. The problem however is that this is not robust, in the sense that the integrator would have to be re-tuned in all likelihood for other Jacobi energy values. Hence, there is a need to establish a more rigorous integration scheme to enhance the datasets obtained.

In addition to generating extended-Poincaré sections, an attempt was made to improve upon the current analytical WSB definition. Angular momentum was introduced as a condition to try to remove invariant tori on the left side of the Moon from the WSB set. It was reasoned that only points lying between the L1 and L2 Lagrange points (equilibrium points in the PCR3BP), would be necessary. Trajectories which transit through the neighbourhood of these points intersect the
Poincaré section near the Moon, hence it is superfluous to include points beyond these Lagrange points, as they would belong to trajectories that already yield intersections near the Moon. The conditions imposed to take this into account were approximated by vertical straight line boundaries at the Lagrange points.

The first observation to be made stems from analysing the extended-Poincaré section generated for the Jacobi energy value investigated for which transit motion should not occur: \( C = 3.4000000000 \). Transit motion in this case is not permitted as the Jacobi energy surface is closed at L1, meaning that the regions around the Earth and the Moon are not connected. The results indicate, as expected, that in this case there are no points that transit between the Earth and the Moon; hence the WSB region is empty. The largest Jacobi energy value for which transit is possible is the energy level associated with L1 (\( C_1 \)), as in this case the physical Jacobi energy surface opens up for the first time to allow trajectories to pass between the Earth and the Moon. For this reason, the lower bound in \( C \) value for which the WSB region exists is found to be equal to \( C_1 \). For the existing analytical definition, the lower bound is set at the Jacobi energy associated with L2 (\( C_2 \)). The Jacobi energy values studied, for which transit is permitted, are all larger than \( C_2 \). Clearly transit between the Earth and the Moon is possible in these cases. If the WSB region is generally defined as the set of points in phase space supporting such behaviour, then it is clear that the lower bound for the definition must be \( C_1 \).

One of the main problems with the current analytical definition is that it overestimates the WSB region. It includes a number of different regions of points that do not exhibit transit motion: invariant tori around the Earth, and invariant tori on both sides of the Moon. An attempt was made to remove some of these regions from the WSB set. By imposing the L1 and L2 constraints, the invariant tori around the Earth are removed. The angular momentum condition was imposed to try and remove invariant tori on the left side of the Moon. For the Jacobi energy values considered, it is demonstrated that this can be achieved quite successfully. It is seen that most but not all of the invariant tori on the left side are removed. This is definitely an improved approximation of the WSB region, and was possible because it was noted that for most of the invariant tori on the left side, there is a sign change in angular momentum compared to the actual WSB region.

Subsequent analysis of the behaviour of the angular momentum boundary as a function of decreasing Jacobi energy value indicates however that the approximation for lower values of \( C \) deteriorates. Hence, it can be concluded that the angular momentum condition is not robust: it is only applicable to a subset within the Jacobi energy range for which the WSB region is defined. It can be considered sufficiently accurate for the range \( C_1 > C > C_H^* \), where \( C_H^* \approx 3.1752 \). The aim therefore of improving the analytical definition was not entirely successfully accomplished. Nevertheless, the results obtained during this study might prove to be a good starting point to investigate other possibilities to improve the analytical definition.

An interesting result was discovered that might prove to be useful as the starting point to try and find an analytical condition to remove the invariant tori on the right side of the Moon. These invariant tori are difficult to analytically approximate for the Jacobi energy values studied, due to their highly complex contours. The contour of the ‘fish’-shape identified, which contains the WSB region on this
side of the Moon, turns out to be shadowed by nearly-impact trajectories at the Moon. This is an aspect for further study.

Approximating the part of the WSB region on the left side of the Moon is difficult too, as it is related to the location of the stable and unstable invariant manifolds. The numerical algorithm used to approximate the location of the invariant manifold intersections is not robust in close proximity to the Lyapunov orbit at L1; hence these intersections are not shown on the Poincaré sections generated.

The WSB region in the PCR3BP is highly complex. In this study, attempts were made to improve our understanding of the WSB set. The use of extended-Poincaré sections was successful in highlighting the internal structure of the WSB set. The stability of trajectories in the WSB set is linked to the location of the invariant manifolds; however using these manifolds to analyse stability proved to be complicated by the fact that the topological tubes break up. Extended-Poincaré sections are more robust in investigating the internal structure. An attempt was also made to improve the analytical definition of the WSB region to match the numerically generated set; however this proved to be difficult.

A number of different mission scenarios were described that have used or plan to use the weak capture concept in flight. These mission scenarios make use of the dynamics of the 3BP to lower the $\Delta V$ that would be required to attain performance levels beyond those possible through the use of traditional techniques. Quite clearly, harnessing the dynamics of the 3BP opens up a plethora of interesting applications that would otherwise be impractical. Understanding the dynamics governing such trajectories is key to be able to apply them in future missions.

The Hiten [Belbruno, 2004] and Genesis Discovery [Koon et al., 2006] missions proved that weak capture is in fact a concept that is applicable to real space missions. Low-energy transfers provide the means to maximise on-board scientific payload mass, to provide recovery options in case of propulsion system failures, and maximum scientific return by frequenting different bodies of interest. Since the WSB region encapsulates the set of points in phase space that support weak capture behaviour, locating it is fruitful to the design of future missions. Identifying the WSB region and determining its internal structure can aid in the optimisation of trajectories for spacecraft missions of the future such as a multi-moon orbiter, BepiColombo or an Earth-Moon cycler. Despite the fact that the WSB region in the PCR3BP does not account for the full dynamics of motion for such missions, it does provide a starting point to determine suitable candidate trajectories. With the vast range of possible applications and the drive to improve our understanding of the underlying dynamics, there is a case to be made for further research in this field.
A number of recommendations can be made based on the results obtained during the investigations outlined in this report. The study detailed in this report also provides the impetus for future research in a number of different areas. In this chapter, recommendations and possible future work will be described in full.

One of the problems noted during the analysis performed was the fact that the integrator scheme was sensitive to the Jacobi energy value. In particular, the error tolerance in the Jacobi energy value could not be met in a robust manner when an initial condition of the $x$-axis was integrated for 1000 orbits. Hence it is recommended that a more accurate integration scheme is adopted such that the error tolerance is always met. It might be fruitful to consider the use of a conservative integrator. The integrator scheme used in this study controls the Jacobi energy value by imposing an external constraint. A conservative integrator scheme employs a more natural way of controlling the error in the Jacobi energy. The Jacobi energy is inherently included in the scheme. A study is necessary to assess whether there are any suitable conservative integrators available and whether the computation time when using such an integrator remains tractable.

Only results a limited number of Jacobi energy values ($C$) were presented in this study. The framework was provided to study the WSB region using extended-Poincaré sections. To be able to make robust conclusions about the nature of the WSB region and its dependence on system parameters, a more extensive investigation is necessary that charts its behaviour for a larger range of $C$. It is recommended that values of $C$ presented in other studies, such as Belbruno et al., 2008, are used to verify and extend existing results. Additionally, the mass parameter ($\mu$) was not varied during this study. It would be interesting to determine the influence of $\mu$ on the shape and internal structure of the WSB region. This too can be achieved by generating additional extended-Poincaré sections. By generating histogram data associated with the WSB region for a range of values of $C$ and $\mu$, it might be possible to determine a three-dimensional surface that describes the global behaviour of the WSB region. This could prove essential in improving the analytical WSB definition.

Further research is also needed to establish criteria to remove Kolmogorov-Arnold-Moser (KAM) invariant tori around the Earth and Moon from the current analytical definition of the WSB region. In this study, angular momentum with respect to the Moon was investigated as a possible criterion to achieve this goal. It
was noted that for low values of $C$, this criterion does not work well. The fundamental dynamics associated with KAM tori and Hamiltonian systems needs to be investigated more thoroughly to determine whether there are basic properties that distinguish them from the chaotic sea, where the WSB region resides. Identifying such properties would provide the means to analytically remove the WSB set.

It was noted during this study that the WSB region exists in other models as well. It would be interesting to determine the WSB region in the full spatial Circular Restricted 3-Body Problem (CR3BP). It turns out that extending the current study to the CR3BP is not trivial. This is due to the fact that in the three-dimensional case, two additional dimensions are included ($z, \dot{z}$). This means that the initial state vector is no longer fully determined by selection of $C$ and a two-dimensional Poincaré surface. Additional assumptions are required. A robust methodology is necessary that either makes use of higher-dimensional Poincaré surfaces, or manages to reconstruct phase space through clever selection of and combination of two-dimensional Poincaré surfaces.

Investigating the WSB region in a time-dependent system would also be interesting for future work. For instance, the Elliptic Restricted 3-Body Problem (ER3BP), which was described in full in this report, is often used as a prototype model for time-dependent systems in astrodynamics. The advantage of using this model is that an integral of motion is known to exist. This integral of motion is time-dependent however, hence a new method would be required to deal with the additional dimension associated with the autonomous dynamical system describing the ER3BP. Performing such a study could be key to extending analysis of the WSB region to other time-dependent systems, such as the Restricted 4-Body Problem (R4BP).

The ultimate goal of introducing these other models to space mission design, might be to identify a set of points that support weak capture for real space missions. Therefore the aim would be to generate a method to identify the WSB in a differentially-corrected N-Body ephemeris model. Investigating the WSB region in the CR3BP and ER3BP would provide the stepping stones necessary to develop a methodology for an ephemeris model. Even if an ephemeris model is not used, by introducing time-dependence and the additional spatial dimensions, it might be possible to analyse the WSB region in a model that has close correspondence to the primary dynamics utilised by the optimum trajectory in an ephemeris model. In the Earth-Moon system for instance, the R4BP is seen to be a good in reproducing the essential dynamics of weak capture necessary for space mission applications [Belbruno, 2004].

By analysing the WSB region in a time-dependent, spatial problem, one of the aims for future work could be to develop a toolbox for global optimisation of low-energy trajectories that harness ballistic capture transfers. This toolbox would analyse the WSB region in phase space and identify its extent. Global trajectory optimisation is notoriously difficult and slow, especially when the search space is large, as is common to many problems in astrodynamics. To aid in the design of trajectories for future missions that harness weak capture, it is therefore beneficial to find ways to reduce the search space. By establishing an accurate analytical definition of the WSB region, it is possible that global trajectory optimisation for such missions can be streamlined.
An additional problem that has to be dealt with when analysing the WSB region in for instance the CR3BP or ER3BP, is that the equations of motion have to be regularised to deal with impact trajectories. Either local or global regularisation could be introduced. Further work is necessary to be able to establish the necessary transformations of the equations of motion for both of these models.

It is safe to state that there is plenty of scope for further research. A number of the aspects mentioned here require extensive research, however, before tangible results are likely to be achievable.


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Appendix A

Sample Trajectories for $C = 3.1980000000$

In Chapter 8, a number of different Poincaré sections were investigated. For a Jacobi energy value of $C = 3.1980000000$ an extensive analysis was performed. A number of different classes of orbits were identified. Figure A.1 illustrates a number of different regions. Here, a sample set of trajectories are given that highlight some of these classes of orbits. For each case, the trajectory is given in $xy$-plane in the rotating frame, in the $XY$-plane in an Earth-centred inertial reference frame (ECI), and in the $XY$-plane in a Moon-centred inertial reference frame (MCI).

Figure A.1  Poincaré section for $C = 3.1980000000$ illustrating different regions: invariant tori around the Earth (blue), transit points to the left of L1 (green), invariant tori with positive angular momentum and transit points on the left side of the Moon to the right of L1 (cyan), invariant tori on the left side of the Moon with negative angular momentum and invariant tori on the right side of the Moon with positive angular momentum (black), and transit points on the right side of the Moon (red).
A.1 Invariant tori around Earth: \( x < x_{L1} \)

Figure A.2 illustrates forward and backwards integration for a point belonging to the invariant tori around the Earth in Figure A.1 (blue region). The trajectory illustrated in Figure A.2(a) is quasi-periodic around the Earth, as expected. As can be seen, this trajectory does not transit to the Moon. Figure A.2(b) illustrates the fact that this trajectory is approximately a perturbed ellipse.

![Figure A.2](image)

Figure A.2 Visualisation of trajectory belonging to invariant tori around the Earth in forwards (red line) and backwards time (blue line): (a) trajectory in the \( xy \)-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), L1 (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the \( XY \)-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the \( XY \)-plane in a Moon-centred inertial reference frame, with the Earth’s orbit (green circle) and Moon (black dot) illustrated.
Figure A.3 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.2. The variation of all of these parameters is oscillatory. From Figure A.3(a), it is apparent that the Kepler energy with respect to the Moon is negative locally at certain moments in time. This can be understood by the fact that if the apoapsis of the trajectory around the Earth is close to the Earth-Moon line, the Moon acts as a strong perturbation. Through this interaction, the Kepler energy can be locally negative. This is also true for the angular momentum given in Figure A.3(b). The Kepler energy and angular momentum with respect to the Earth remains positive.

![Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to invariant tori around the Earth; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.](image)
A.2 Transit region: $x < x_{L1}$

Figure A.4 illustrates forward and backwards integration for a point belonging to the transit region to the left of L1 in Figure A.1 (green region). The trajectory illustrated in Figure A.4 is captured around the Earth and captured around the Moon at different periods of time. The motion around the Moon is prograde. It is clear from Figure A.4(a) that transit to and from the Moon occurs in the vicinity of L1.

Figure A.4 Visualisation of trajectory belonging to transit region left of L1 in forwards (red line) and backwards time (blue line); (a) trajectory in the $xy$-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), L1 (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the $XY$-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the $XY$-plane in a Moon-centred inertial reference frame, with the Moon (black dot) illustrated.
A.2 Transit region: $x < x_{L1}$

Figure A.5 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.4. The variation of all of these parameters is oscillatory. In Figures A.5(a) and A.5(b) the period that $m$ orbits the Moon can be clearly identified by the U-shaped minima in the Kepler energy and angular momentum with respect to the Moon. It can be seen that the transit itself occurs rapidly, hence transit to the U-shape is almost a step function. The Kepler energy and angular momentum with respect to the Earth also show variation during the lunar-orbiting phase: the oscillatory behaviour is of a higher frequency and amplitude.

Figure A.5 Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to transit region left of L1; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.
Figure A.6 illustrates a close-up view of the evolution of the Kepler energy and angular momentum with respect to the Earth and the Moon, given in Figure A.5. The Kepler energy with respect to the Moon is negative during the lunar-orbiting phase, whilst the angular momentum is positive. The angular momentum is expected to be positive since the motion around the Moon is prograde. The Kepler energy and angular momentum with respect to the Earth are both positive during this phase.
A.3 Transit region: $x > x_{L1}, x < x_{Moon}$

Figure A.7 illustrates forward and backwards integration for a point belonging to the transit region to the right of L1 and left of the Moon in Figure A.1 (green region). The trajectory illustrated in Figure A.7 is captured around the Earth and captured around the Moon at different periods of time. The motion around the Moon is prograde. It is clear from Figure A.7 that the trajectory transits from the Moon to perturbed elliptical orbits around the Earth.

Figure A.7 Visualisation of trajectory belonging to transit region right of L1 and left of the Moon in forwards (red line) and backwards time (blue line); (a) trajectory in the $xy$-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), L1 (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the $XY$-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the $XY$-plane in a Moon-centred inertial reference frame, with the Moon (black dot) illustrated.
Figure A.8 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.7. The variation of all of these parameters is oscillatory. In Figures A.8(a) and A.8(b) the period that \( m \) orbits the Moon can be clearly identified by the U-shaped minima in the Kepler energy and angular momentum with respect to the Moon. It can be seen that the transit itself occurs rapidly, hence transit to the U-shape is almost a step function. The Kepler energy and angular momentum with respect to the Earth also show variation during the lunar-orbiting phase: the oscillatory behaviour is of a markedly higher frequency and amplitude.

Figure A.8  Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to transit region right of L1 and left of the Moon; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.
A.3 Transit region: $x > x_{L1}, x < x_{Moon}$

Figure A.9 illustrates a close-up view of the evolution of the Kepler energy and angular momentum with respect to the Earth and the Moon, given in Figure A.8. The Kepler energy with respect to the Moon is negative during the lunar-orbiting phase, whilst the angular momentum is positive. The angular momentum is expected to be positive since the motion around the Moon is prograde. The Kepler energy and angular momentum with respect to the Earth is predominantly positive during this phase. They oscillate such that they are locally negative during certain periods of time.

![Graphs showing evolution of Kepler energy and angular momentum with respect to Moon and Earth](image)

**Figure A.9** Close-up of evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth, whilst orbiting the Moon, for trajectory belonging to transit region right of $L1$ and left of the Moon: (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.
A.4 Invariant tori around Moon: $x > x_{L1}, x < x_{Moon}, H_{Moon} > 0$

Figure A.10 illustrates forward and backwards integration for a point belonging to the invariant tori region to the right of $L1$ and left of the Moon with positive angular momentum, in Figure A.1 (yellow region). As can be seen, this trajectory does not transit to the Earth.

![Visualization of trajectory belonging to the invariant tori region](image)

Figure A.10 Visualisation of trajectory belonging to the invariant tori region to the right of $L1$ and left of the Moon with positive angular momentum, in forwards (red line) and backwards time (blue line); (a) trajectory in the $xy$-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), $L1$ (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the $XY$-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the $XY$-plane in a Moon-centred inertial reference frame, with the Moon (black dot) illustrated.
Figure A.11 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.10. The variation of all of these parameters is oscillatory. It can be seen that the Kepler energy with respect to the Moon remains strongly negative, whilst the angular momentum with respect to the Moon remains positive. Both the Kepler energy and angular momentum with respect to the Earth are predominantly positive, with local minima that are negative.

Figure A.11  Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to the invariant tori region to the right of L1 and left of the Moon with positive angular momentum; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.
To be able to interpret the motion of \( m \) around the Moon for this case better, the simulation time is reduced and the figures are regenerated. Figure A.12 illustrates forward and backwards integration of the trajectory for the short simulation time. It can be seen that the motion around the Moon is prograde. The trajectory around the Moon follows a perturbed ellipse.

Figure A.12  Visualisation of trajectory belonging to the invariant tori region to the right of L1 and left of the Moon with positive angular momentum, for shorter simulation time, in forwards (red line) and backwards time (blue line); (a) trajectory in the \( xy \)-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), L1 (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the \( XY \)-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the \( XY \)-plane in a Moon-centred inertial reference frame, with the Moon (black dot) illustrated.
Figure A.13 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.12. The variation of all of these parameters is once again oscillatory. It is again apparent that the Kepler energy with respect to the Moon remains strongly negative, whilst the angular momentum with respect to the Moon remains positive. The sign of the angular momentum is expected, given the fact that the motion around the Moon is prograde. The Kepler energy and angular momentum with respect to the Earth are predominantly positive, with local minima that are negative.

Figure A.13  Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to the invariant tori region to the right of L1 and left of the Moon with positive angular momentum, for shorter simulation time; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.
A.5 Invariant tori around Moon: $x > x_{L1}, x < x_{Moon}$, $H_{Moon} < 0$

Figure A.14 illustrates forward and backwards integration for a point belonging to the invariant tori region to the right of L1 and left of the Moon with negative angular momentum, in Figure A.1 (yellow region). As can be seen, this trajectory does not transit to the Earth. The motion around the Moon is retrograde and follows a perturbed ellipse.

Figure A.14 Visualisation of trajectory belonging to the invariant tori region to the right of L1 and left of the Moon with negative angular momentum, in forwards (red line) and backwards time (blue line); (a) trajectory in the $xy$-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), L1 (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the $XY$-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the $XY$-plane in a Moon-centred inertial reference frame, with the Moon (black dot) illustrated.
Figure A.15 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.14. The variation of all of these parameters is oscillatory and regular. It can be seen that both the Kepler energy and angular momentum with respect to the Moon remain negative. Since the motion is retrograde, it is expected that the angular momentum with respect to the Moon is negative. The Kepler energy with respect to the Earth is predominantly positive, with local minima that are negative. The angular momentum with respect to the Earth.

Figure A.15: Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to the invariant tori region to the right of L1 and left of the Moon with negative angular momentum; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.
A.6 Invariant tori around Moon: $x > x_{Moon}, \ x < x_{L2}$

Figure A.16 illustrates forward and backwards integration for a point belonging to the invariant tori region to the right of the Moon, in Figure A.1 (yellow region). As can be seen, this trajectory does not transit to the Earth. The motion around the Moon is prograde and follows a perturbed ellipse.

![Figure A.16](image)

Figure A.16 Visualisation of trajectory belonging to the invariant tori region to the right of the Moon, in forwards (red line) and backwards time (blue line): (a) trajectory in the $xy$-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), L1 (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the $XY$-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the $XY$-plane in a Moon-centred inertial reference frame, with the Moon (black dot) illustrated.
Figure A.17 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.16. The variation of all of these parameters is oscillatory. It can be seen that the Kepler energy with respect to the Moon remains negative, whilst the angular momentum with respect to the Moon remains positive. Since the motion is prograde, it is expected that the angular momentum with respect to the Moon is positive. The Kepler energy and angular momentum with respect to the Earth are predominantly positive, with local minima that are negative.

Figure A.17 Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to the invariant tori region to the right of the Moon; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.
A.7 Transit region: $x > x_{Moon}, x < x_{L2}$

Figure A.18 illustrates forward and backwards integration for a point belonging to the transit region to the right of the Moon, in Figure A.1 (green region). The trajectory illustrated in Figure A.18 is captured around the Earth and captured around the Moon at different periods of time. The motion around the Moon is prograde. It is clear from Figure A.18 that the trajectory transits from the Earth to perturbed elliptical orbits around the Moon.

Figure A.18 Visualisation of trajectory belonging to the transit region to the right of the Moon in forwards (red line) and backwards time (blue line); (a) trajectory in the $xy$-plane in the rotating reference frame, with the Earth (green dot), Moon (black dot), L1 (cyan star), and Surface of Hill (yellow line) illustrated, (b) trajectory in the $XY$-plane in an Earth-centred inertial reference frame, with the Earth (green dot) and Moon’s orbit (black circle) illustrated, (c) trajectory in the $XY$-plane in a Moon-centred inertial reference frame, with the Moon (black dot) illustrated.
Figure A.19 illustrates the evolution in time of the Kepler energy and angular momentum with respect to the Moon and Earth, in forwards and backwards time, for the trajectory given in Figure A.18. The variation of all of these parameters is oscillatory. In Figures A.19(a) and A.19(b) the period that \( m \) orbits the Moon can be clearly identified by the U-shaped minima in the Kepler energy and angular momentum with respect to the Moon. It can be seen that the transit itself occurs rapidly, hence transit to the U-shape is almost a step function. The Kepler energy with respect to the Moon remains negative during lunar-orbiting, whilst the angular momentum with respect to the Moon remains positive. The Kepler energy and angular momentum with respect to the Earth also show variation during the lunar-orbiting phase. The Kepler energy with respect to the Earth remains positive, whilst the angular momentum with respect to the Earth reaches a negative minimum value during the lunar-orbiting phase.

Figure A.19  Evolution in forwards (blue) and backwards (red) time of Kepler energy and angular momentum with respect to the Moon and Earth for trajectory belonging to transit region left of L2 and right of the Moon; (a) Evolution of Kepler energy with respect to Moon, (b) Evolution of angular momentum with respect to Moon, (c) Evolution of Kepler energy with respect to Earth, (d) Evolution of angular momentum with respect to Earth.