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12 Asymptotic Methods

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12.1 DEFINITIONS

12.1.1 Asymptotic and Order Symbols

Let $f(x)$ and $\phi(x)$ be functions defined on a point set $\mathbf{X}$, and c be a limit point of $\mathbf{X}$. If $f(x)/\phi(x) \to 1$ as $x \to c$, then we say that $f(x)$ is *asymptotic* to $\phi(x)$, and write

$$f(x) \sim \phi(x) \qquad (x \to c \text{ in } \mathbf{X})$$

The point c is called the *distinguished point* of this *asymptotic relation*; c need not belong to $\mathbf{X}$. The set $\mathbf{X}$ may be real or complex; in the latter event it is required that $f(x)/\phi(x)$ approaches its limit uniformly with respect to $\arg x$.

In a similar way, if $f(x)/\phi(x) \to 0$ as $x \to c$ then we write

$$f(x) = o\{\phi(x)\} \qquad (x \to c \text{ in } \mathbf{X})$$

and if $|f(x)/\phi(x)|$ is bounded as $x \to c$, then

$$f(x) = O\{\phi(x)\} \qquad (x \to c \text{ in } \mathbf{X})$$

Lastly, if $|f(x)/\phi(x)|$ is bounded in the *whole* of $\mathbf{X}$, then we write

$$f(x) = O\{\phi(x)\} \qquad (x \in \mathbf{X})$$

Examples:

$$\sinh x \sim x \qquad (x \to 0 \text{ in any point set})$$

$$\sin\left(n\pi + \frac{1}{n}\right) = O\left(\frac{1}{n}\right) \qquad (n \to \infty \text{ through integer values})$$

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$$e^{ix}/(1+x) = O(x^{-1}) \qquad (x \in \text{real line})$$

$$e^{-x} = o(1) \qquad (x \to \infty \text{ in the sector } |\arg x| \leqslant \tfrac{1}{2}\pi - \delta, \text{ where } \delta > 0)$$

The last relation is invalid in the open sector $|\arg x| < \frac{1}{2}\pi$ owing to lack of uniformity with respect to $\arg x$.

The symbol $o\{\phi(x)\}$ or, more briefly, $o(\phi)$ may be used to denote an *unspecified* function with the property stated in the second paragraph. This use is generic, that is, $o(\phi)$ need not denote the same function f at each occurrence. The distinguished point is understood to be the same, however. Similarly for $O(\phi)$. Thus, for example

$$o(\phi) + o(\phi) = o(\phi); \quad o(\phi) = O(\phi); \quad O(\phi)O(\psi) = O(\phi\psi)$$

Relations of this kind are not necessarily reversible. For example, $O(\phi) = o(\phi)$ is false. Another instructive example is supplied by

$$\{1 + o(1)\}\cosh x - \{1 + o(1)\}\sinh x \qquad (x \to +\infty)$$

This expression is $o(e^x)$, not $\{1 + o(1)\}e^{-x}$ because the $o(1)$ terms may represent different functions.

12.1.2 Integration and Differentiation of Asymptotic and Order Relations

Asymptotic and order relations may be integrated subject to obvious restrictions on the convergence of the integrals involved. Suppose, for example, that in the interval (a, ∞) the function $f(x)$ is continuous and $f(x) \sim x^\nu$ as $x \to \infty$, where ν is a real or complex constant. Then as $x \to \infty$*

$$\int_x^\infty f(t)\,dt \sim -\frac{x^{\nu+1}}{\nu+1} \qquad (\operatorname{Re}\nu < -1)$$

and

$$\int_a^x f(t)\,dt \sim \begin{cases} \text{a constant} & (\operatorname{Re}\nu < -1) \\ \ln x & (\nu = -1) \\ x^{\nu+1}/(\nu+1) & (\operatorname{Re}\nu > -1) \end{cases}$$

These results are extendible to complex integrals in a straightforward manner.

Differentiation is permissible only with extra conditions. For example, let $f(z)$ be holomorphic for all sufficiently large $|z|$ in a given sector **S**, and

$$f(z) = O(z^\nu) \qquad (z \to \infty \text{ in } \mathbf{S})$$

*Except where otherwise stated, proofs of all results quoted in the present chapter will be found in Olver 1974a.

where ν is a fixed real number. Then

$$f^{(m)}(z) = O(z^{\nu-m}) \qquad (z \to \infty \text{ in } \mathbf{S}')$$

where $\mathbf{S}'$ is any sector properly interior to $\mathbf{S}$ and having the same vertex. This result also holds if the symbol O is replaced in both places by o.

12.1.3 Asymptotic Solution of Transcendental Equations

Let $f(\xi)$ be a strictly increasing function of the real variable ξ in an interval (a, ∞), and

$$f(\xi) \sim \xi \qquad (\xi \to \infty)$$

Then for $u > f(a)$ the equation $f(\xi) = u$ has a unique root $\xi(u)$ in (a, ∞), and

$$\xi(u) \sim u \qquad (u \to \infty)$$

As an example, consider the equation

$$x^2 - \ln x = u$$

In the notation just given we may take $\xi = x^2$, $f(\xi) = \xi - \frac{1}{2}\ln \xi$, and $a = \frac{1}{2}$. Then $\xi(u) \sim u$ as $u \to \infty$, implying that

$$x = u^{1/2}\{1 + o(1)\} \qquad (u \to \infty)$$

Higher approximations can be found by successive resubstitutions. Thus

$$x^2 = u + \ln x = u + \ln\,[u^{1/2}\{1 + o(1)\}] = u + \tfrac{1}{2}\ln u + o(1)$$

and thence

$$x = u^{1/2}\left\{1 + \frac{\ln u}{4u} + o\left(\frac{1}{u}\right)\right\}$$

and so on.

The same procedure can be used for complex variables, provided that the function $f(\xi)$ is analytic and the parameter u restricted to a ray or sector properly interior to the sector of validity of the relation $f(\xi) \sim \xi$.

12.1.4 Asymptotic Expansions

Let $f(x)$ be a function defined on a real or complex unbounded point set $\mathbf{X}$, and $\sum a_s x^{-s}$ a formal power series (convergent or divergent). If for each positive integer n

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_{n-1}}{x^{n-1}} + O\left(\frac{1}{x^n}\right) \qquad (x \to \infty \text{ in } \mathbf{X})$$

then $\sum a_s x^{-s}$ is said to be the *asymptotic expansion* of $f(x)$ as $x \to \infty$ in $\mathbf{X}$, and we write

$$f(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \qquad (x \to \infty \text{ in } \mathbf{X})$$

It should be noticed that the symbol $\sim$ is being used in a different (but not inconsistent) sense from that of 12.1.1.

A necessary and sufficient condition that $f(x)$ possesses an asymptotic expansion of the given form is that

$$x^n \left\{ f(x) - \sum_{s=0}^{n-1} \frac{a_s}{x^s} \right\} \to a_n$$

as $x \to \infty$ in $\mathbf{X}$, for each $n = 0, 1, 2, \ldots$. In the special case when $\sum a_s x^{-s}$ converges for all sufficiently large $|x|$, the series is automatically the asymptotic expansion of its sum in any point set.

In a similar manner, if c is a finite limit point of a set $\mathbf{X}$ then

$$f(x) \sim b_0 + b_1(x - c) + b_2(x - c)^2 + \cdots \qquad (x \to c \text{ in } \mathbf{X})$$

means that the difference between $f(x)$ and the n^{th} partial sum of the right-hand side is $O\{(x - c)^n\}$ as $x \to c$ in $\mathbf{X}$.

Asymptotic expansions having the same distinguished point can be added, subtracted, multiplied, or divided in the same way as convergent power series. They may also be integrated. Thus if $\mathbf{X}$ is the interval $[a, \infty)$ where $a > 0$, and $f(x)$ is continuous with an asymptotic expansion of the above form as $x \to \infty$, then

$$\int_a^x f(t)\,dt \sim A + a_0 x + a_1 \ln x - \frac{a_2}{x} - \frac{a_3}{2x^2} - \frac{a_4}{3x^3} - \cdots \qquad (x \to \infty)$$

where

$$A = \int_a^\infty \left\{ f(t) - a_0 - \frac{a_1}{t} \right\} dt - a_0 a - a_1 \ln a$$

The last integral necessarily converges because the integrand is $O(t^{-2})$ as $t \to \infty$.

Differentiation of an asymptotic expansion is legitimate when it is known that the derivative $f'(x)$ is continuous and its asymptotic expansion exists. Differentiation is also legitimate when $f(x)$ is an analytic function of the complex variable x, provided that the result is restricted to a sector properly interior to the sector of validity of the asymptotic expansion of $f(x)$.

If the asymptotic expansion of a given function exists, then it is unique. On the other hand, corresponding to any prescribed sequence of coefficients $a_0, a_1, a_2, \ldots,$ there exists an infinity of analytic functions $f(x)$ such that

$$f(x) \sim \sum_{s=0}^{\infty} \frac{a_s}{x^s} \qquad (x \to \infty \text{ in } \mathbf{X})$$

The point set $\mathbf{X}$ can be, for example, the real axis or any sector of finite angle in the complex plane. Lack of uniqueness is demonstrated by the null expansion

$$e^{-x} \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \cdots \qquad (x \to \infty \text{ in } |\arg x| \leqslant \tfrac{1}{2}\pi - \delta)$$

where δ is a positive constant not exceeding $\frac{1}{2}\pi$.

12.1.5 Generalized Asymptotic Expansions

The definition of an asymptotic expansion can be extended in the following way. Let $\{\phi_s(x)\}$, $s = 0, 1, 2, \ldots$, be a sequence of functions defined on a point set $\mathbf{X}$ such that for every s

$$\phi_{s+1}(x) = o\{\phi_s(x)\} \qquad (x \to c \text{ in } \mathbf{X})$$

Then $\{\phi_s(x)\}$ is said to be an *asymptotic sequence* or *scale.* Additionally, suppose that $f(x)$ and $f_s(x)$, $s = 0, 1, 2, \ldots$, are functions such that for each nonnegative integer n

$$f(x) = \sum_{s=0}^{n-1} f_s(x) + O\{\phi_n(x)\} \qquad (x \to c \text{ in } \mathbf{X})$$

Then $\sum f_s(x)$ is said to be a *generalized asymptotic expansion of $f(x)$ with respect to the scale* $\{\phi_s(x)\}$, and we write

$$f(x) \sim \sum_{s=0}^{\infty} f_s(x); \quad \{\phi_s(x)\} \text{ as } x \to c \text{ in } \mathbf{X}$$

Some, but by no means all, properties of ordinary asymptotic expansions carry over to generalized asymptotic expansions.

12.2 INTEGRALS OF A REAL VARIABLE

12.2.1 Integration by Parts

Asymptotic expansions of a definite integral containing a parameter can often be found by repeated integrations by parts. Thus for the Laplace transform

$$Q(x) = \int_0^\infty e^{-xt} q(t)\, dt$$

assume that $q(t)$ is infinitely differentiable in $[0, \infty)$, and for each s

$$q^{(s)}(t) = O(e^{\sigma t}) \qquad (0 \leqslant t < \infty)$$

where σ is an assignable constant. Then for $x > \sigma$

$$Q(x) = \frac{q(0)}{x} + \frac{q'(0)}{x^2} + \cdots + \frac{q^{(n-1)}(0)}{x^n} + \epsilon_n(x)$$

where n is an arbitrary nonnegative integer, and

$$\epsilon_n(x) = \frac{1}{x^n} \int_0^\infty e^{-xt} q^{(n)}(t)\, dt$$

With the assumed conditions

$$|\epsilon_n(x)| \leqslant \frac{A_n}{x^n} \int_0^\infty e^{-xt+\sigma t}\, dt = \frac{A_n}{x^n(x-\sigma)} \qquad (x > \sigma)$$

A_n being assignable. Thus $\epsilon_n(x) = O(x^{-n-1})$, and

$$Q(x) \sim \sum_{s=0}^{\infty} \frac{q^{(s)}(0)}{x^{s+1}} \qquad (x \to \infty)$$

An example is furnished by the incomplete Gamma function:

$$\Gamma(\alpha, x) = e^{-x} x^{\alpha} \int_0^\infty e^{-xt}(1+t)^{\alpha-1}\, dt \sim e^{-x} x^{\alpha-1} \sum_{s=0}^{\infty} \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-s)}{x^s}$$

as $x \to \infty$, α being fixed. The $\sim$ sign is now being used in the sense that

$$\Gamma(\alpha, x)/(e^{-x} x^{\alpha-1})$$

has the sum as its asymptotic expansion, as defined in 12.1.4. In the present case a straightforward extension of the analysis shows that if α is real and $n \geqslant \alpha - 1$, then the n^{th} error term of the asymptotic expansion is bounded in absolute value by the first neglected term and has the same sign.

12.2.2 Watson's Lemma

Let $q(t)$ now denote a real or complex function of the positive real variable t having a finite number of discontinuities and infinities, and the property

$$q(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu} \qquad (t \to 0+)$$

where λ and μ are constants such that $\operatorname{Re} \lambda > 0$ and $\mu > 0$. Assume also that the Laplace transform of $q(t)$ converges throughout its integration range for all sufficiently large x. Then formal term-by-term integration produces an asymptotic expansion, that is

$$\int_0^\infty e^{-xt} q(t)\,dt \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \qquad (x \to \infty)$$

This result is known as *Watson's lemma*, and is one of the most frequently used methods for deriving asymptotic expansions. It should be noticed that by permitting $q(t)$ to be discontinuous the case of a finite integration range $\int_0^b$ is automatically included.

Example: Consider

$$\int_0^\infty e^{-x \cosh \tau}\,d\tau = e^{-x} \int_0^\infty \frac{e^{-xt}}{(2t+t^2)^{1/2}}\,dt$$

Since

$$(2t+t^2)^{-1/2} = \sum_{s=0}^{\infty} (-)^s \frac{1 \cdot 3 \cdot 5 \cdots (2s-1)}{s!\, 2^{2s+(1/2)}}\, t^{s-(1/2)} \qquad (0<t<2)$$

the above result is applied with $\lambda = \frac{1}{2}$ and $\mu = 1$, to give

$$\int_0^\infty e^{-x \cosh \tau}\,d\tau \sim e^{-x} \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} (-)^s \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2s-1)^2}{s!\,(8x)^s} \qquad (x \to \infty)$$

12.2.3 Riemann-Lebesgue Lemma

Let a be finite or $-\infty$, b be finite or $+\infty$, and $q(t)$ continuous in (a, b) save possibly at a finite number of points. Then

$$\int_a^b e^{ixt}\, q(t)\, dt = o(1) \qquad (x \to \infty)$$

provided that this integral converges uniformly at a, b, and the exceptional points, for all sufficiently large x. This is the *Riemann-Lebesgue lemma.*

It should be noticed that if the given integral converges absolutely throughout its range, then it also converges uniformly, since x is real. On the other hand, it may converge uniformly but not absolutely. For example, if $0 < \delta < 1$ then by integration by parts it can be seen that

$$\int_0^\infty \frac{e^{ixt}}{t^\delta}\, dt$$

converges uniformly at both limits for all sufficiently large x, but converges absolutely only at the lower limit.

12.2.4 Fourier Integrals

Let a and b be finite, and $q(t)$ infinitely differentiable in $[a, b]$. Repeated integrations by parts yield

$$\int_a^b e^{ixt} q(t)\, dt = \sum_{s=0}^{n-1} \left(\frac{i}{x}\right)^{s+1} \{e^{iax} q^{(s)}(a) - e^{ibx} q^{(s)}(b)\} + \epsilon_n(x)$$

where

$$\epsilon_n(x) = \left(\frac{i}{x}\right)^n \int_a^b e^{ixt} q^{(n)}(t)\, dt$$

As $x \to \infty$ we have $\epsilon_n(x) = o(x^{-n})$, by the Riemann-Lebesgue lemma. Hence the expansion just obtained is asymptotic in character.

A similar result applies when $b = \infty$. Provided that each of the integrals

$$\int_a^\infty e^{ixt} q^{(s)}(t)\, dt \qquad (s = 0, 1, \ldots)$$

converges uniformly for all sufficiently large x, we have

$$\int_a^\infty e^{ixt} q(t)\, dt \sim \frac{ie^{iax}}{x} \sum_{s=0}^{\infty} q^{(s)}(a) \left(\frac{i}{x}\right)^s \qquad (x \to \infty).$$

Whether or not b is infinite, an error bound is supplied by

$$|\epsilon_n(x)| \leqslant x^{-n}\, \mathcal{V}_{a,b}\{q^{(n-1)}(t)\}$$

where $\mathcal{V}$ is the *variational operator*, defined by

$$\mathcal{V}_{a,b}\{f(t)\} = \int_a^b |f'(t)\, dt|$$

12.2.5 Laplace's Method

Consider the integral

$$I(x) = \int_a^b e^{-xp(t)} q(t)\, dt$$

in which x is a positive parameter. The peak value of the factor $e^{-xp(t)}$ is located at the minimum t_0, say, of $p(t)$. When x is large this peak is very sharp, and the overwhelming contribution to the integral comes from the neighborhood of t_0. It is therefore reasonable to approximate $p(t)$ and $q(t)$ by the leading terms in their ascending power-series expansions at t_0, and evaluate $I(x)$ by extending the integration limits to $-\infty$ and $+\infty$, if necessary. The result is *Laplace's approximation* to $I(x)$.

For example, if t_0 is a simple minimum of $p(t)$ which is interior to (a, b) and $q(t_0) \neq 0$, then

$$I(x) \doteqdot \int_a^b e^{-x\{p(t_0)+(1/2)(t-t_0)^2 p''(t_0)\}} q(t_0)\, dt$$

$$\doteqdot q(t_0) e^{-xp(t_0)} \int_{-\infty}^{\infty} e^{-(1/2)x(t-t_0)^2 p''(t_0)}\, dt = q(t_0) e^{-xp(t_0)} \sqrt{\frac{2\pi}{xp''(t_0)}}$$

In circumstances now to be described, approximations obtained in this way are asymptotic representations of $I(x)$ for large x.

By subdivision of the integration range and change of sign of t (if necessary) it can always be arranged for the minimum of $p(t)$ to be at the left endpoint a. The other endpoint b may be finite or infinite. We further assume that:

a. $p'(t)$ and $q(t)$ are continuous in a neighborhood of a, save possibly at a.
b. As $t \to a$ from the right

$$p(t) - p(a) \sim P(t-a)^{\mu}; \quad q(t) \sim Q(t-a)^{\lambda-1}$$

and the first of these relations is differentiable. Here P, μ, Q, and λ are constants such that $P > 0$, $\mu > 0$, and $\operatorname{Re} \lambda > 0$.

c. $\displaystyle\int_a^b e^{-xp(t)} q(t)\, dt$ converges absolutely throughout its range for all sufficiently large x.

With these conditions

$$\int_a^b e^{-xp(t)} q(t)\, dt \sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{-xp(a)}}{(Px)^{\lambda/\mu}} \qquad (x \to \infty)$$

Example: Consider

$$I(x) = \int_0^\infty e^{x\tau - (\tau-1)\ln \tau}\, d\tau$$

The maximum value of the integrand is located at the root of the equation

$$x - 1 - \ln \tau + (1/\tau) = 0$$

For large x the relevant root is given by

$$\tau \sim e^{x-1} = \zeta$$

say; compare 12.1.3. To apply the Laplace approximation the location of the peak needs to be independent of the parameter x. Therefore we take $t = \tau/\zeta$ as new integration variable, giving

$$I(x) = \zeta^2 \int_0^\infty e^{-\zeta p(t)} q(t)\, dt$$

where

$$p(t) = t(\ln t - 1); \quad q(t) = t$$

The minimum of $p(t)$ is at $t = 1$, and Taylor-series expansions at this point are

$$p(t) = -1 + \tfrac{1}{2}(t-1)^2 - \tfrac{1}{6}(t-1)^3 + \cdots; \quad q(t) = 1 + (t-1)$$

In the notation introduced above we have $p(a) = -1$, $P = \frac{1}{2}$, $\mu = 2$ and $Q = \lambda = 1$. Hence

$$\int_1^\infty e^{-\zeta p(t)}\, q(t)\, dt \sim \left(\frac{\pi}{2\zeta}\right)^{1/2} e^{\zeta}$$

On replacing t by $2 - t$, we find that the same asymptotic approximation holds for the corresponding integral over the range $0 \leqslant t \leqslant 1$. Addition of the two contributions and restoration of the original variable yields the required approximation

$$I(x) \sim (2\pi)^{1/2}\, e^{3(x-1)/2}\, \exp\,(e^{x-1}) \qquad (x \to \infty)$$

12.2.6 Asymptotic Expansions by Laplace's Method

The method of 12.2.5 can be extended to derive asymptotic expansions. Suppose that in addition to the previous conditions

$$p(t) \sim p(a) + \sum_{s=0}^{\infty} p_s(t-a)^{s+\mu}; \quad q(t) \sim \sum_{s=0}^{\infty} q_s(t-a)^{s+\lambda-1}$$

as $t \to a$ from the right, and the first of these expansions is differentiable. Then

$$\int_a^b e^{-xp(t)}\, q(t)\, dt \sim e^{-xp(a)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \qquad (x \to \infty)$$

where the coefficients a_s are defined by the expansion

$$\frac{q(t)}{p'(t)} \sim \sum_{s=0}^{\infty} a_s v^{(s+\lambda-\mu)/\mu} \qquad (v \to 0+)$$

in which $v = p(t) - p(a)$. By reversion and substitution the first three coefficients are found to be

$$a_0 = \frac{q_0}{\mu p_0^{\lambda/\mu}}; \quad a_1 = \left\{\frac{q_1}{\mu} - \frac{(\lambda+1)p_1 q_0}{\mu^2 p_0}\right\} \frac{1}{p_0^{(\lambda+1)/\mu}}$$

$$a_2 = \left[\frac{q_2}{\mu} - \frac{(\lambda+2)p_1 q_1}{\mu^2 p_0} + \{(\lambda+\mu+2)p_1^2 - 2\mu p_0 p_2\} \frac{(\lambda+2)q_0}{2\mu^3 p_0^2}\right] \frac{1}{p_0^{(\lambda+2)/\mu}}$$

In essential ways Watson's lemma (12.2.2) is a special case of the present result.

12.2.7 Method of Stationary Phase

This method applies to integrals of the form

$$I(x) = \int_a^b e^{ixp(t)}\, q(t)\, dt$$

and resembles Laplace's method in some respects. For large x the real and imaginary parts of the integrand oscillate rapidly and cancel themselves over most of the range. Cancellation does not occur, however, at (i) the endpoints (when finite) owing to lack of symmetry; and at (ii) the zeros of $p'(t)$, because $p(t)$ changes relatively slowly near these *stationary points.*

Without loss of generality the range of integration can be subdivided in such a way that the stationary point (if any) in each subrange is located at the left endpoint a. Again the other endpoint b may be finite or infinite. Other assumptions are:

a. In (a, b), the functions $p'(t)$ and $q(t)$ are continuous, $p'(t) > 0$, and $p''(t)$ and $q'(t)$ have at most a finite number of discontinuities and infinities.

b. As $t \to a+$

$$p(t) - p(a) \sim P(t-a)^{\mu}; \quad q(t) \sim Q(t-a)^{\lambda-1}$$

the first of these relations being twice differentiable and the second being differentiable. Here P, μ, and λ are positive constants, and Q is a real or complex constant.

c. $q(t)/p'(t)$ is of bounded variation in the interval (k, b) for each $k \in (a, b)$ if $\lambda < \mu$, or in the interval (a, b) if $\lambda \geqslant \mu$.

d. As $t \to b-$, $q(t)/p'(t)$ tends to a finite limit, and this limit is zero when $p(b) = \infty$.

With these conditions $I(x)$ converges uniformly at each endpoint for all sufficiently large x. Moreover,

$$I(x) \sim e^{\lambda \pi i/(2\mu)} \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{ixp(a)}}{(Px)^{\lambda/\mu}} \qquad (x \to \infty)$$

if $\lambda < \mu$, or

$$I(x) = -\lim_{t \to a+} \left\{\frac{q(t)}{p'(t)}\right\} \frac{e^{ixp(a)}}{ix} + \lim_{t \to b-} \left\{\frac{q(t) e^{ixp(t)}}{p'(t)}\right\} \frac{1}{ix} + o\left(\frac{1}{x}\right) \qquad (x \to \infty)$$

if $\lambda \geqslant \mu$.

Example: The Airy integral of negative argument

$$\mathrm{Ai}(-x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3} v^3 - xv\right) dv \qquad (x > 0)$$

The stationary points of the integrand satisfy $v^2 - x = 0$, and the only root in the range of integration is $\sqrt{x}$. To render the location of the stationary point independent of x, we substitute $v = \sqrt{x}\,(1 + t)$, giving

$$\mathrm{Ai}(-x) = \frac{x^{1/2}}{\pi} \int_{-1}^{\infty} \cos\,\{x^{3/2} p(t)\}\, dt$$

where $p(t) = -\frac{2}{3} + t^2 + \frac{1}{3}t^3$. With $q(t) = 1$, it is seen that as $t \to \infty$ the ratio $q(t)/p'(t)$ vanishes and its variation converges. Accordingly, the given conditions are satisfied. For the range $0 \leqslant t < \infty$ we have $p(0) = -\frac{2}{3}$, $\mu = 2$, and $P = Q = \lambda = 1$. The role of x is played here by $x^{3/2}$, and we derive

$$\int_0^{\infty} \exp\,\{ix^{3/2} p(t)\}\, dt \sim \frac{1}{2}\pi^{1/2} e^{\pi i/4} x^{-3/4} \exp\left(-\frac{2}{3} i x^{3/2}\right)$$

The same approximation is found to hold for $\int_{-1}^{0}$, and on taking real parts we arrive at the desired result:

$$\mathrm{Ai}(-x) = \pi^{-1/2} x^{-1/4} \cos\left(\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}\pi\right) + o(x^{-1/4}) \qquad (x \to \infty)$$

As in the case of Laplace's method, the method of stationary phase can be extended to the derivation of asymptotic expansions; see Erdélyi 1956, section 2.9, and Olver 1974b.

12.3 CONTOUR INTEGRALS

12.3.1 Watson's Lemma

The result of 12.2.2 can be extended to complex values of the large parameter. Again, let $q(t)$ be a function of the positive real variable t having a finite number of discontinuities and infinities, and the property

$$q(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu} \qquad (t \to 0+)$$

with $\mathrm{Re}\,\lambda > 0$ and $\mu > 0$. Assume also that the abscissa of convergence (section 11.2) of the transform

$$Q(z) = \int_0^{\infty} e^{-zt} q(t)\, dt$$

is finite or $-\infty$. Then

$$Q(z) \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{z^{(s+\lambda)/\mu}}$$

as $z \to \infty$ in the sector $|\arg z| \leqslant \frac{1}{2}\pi - \delta$ $(< \frac{1}{2}\pi)$, the power $z^{(s+\lambda)/\mu}$ having its principal value.

When $q(t)$ is an analytic function of t, the region of validity of the last expansion can often be increased by rotation of the integration path about the origin. The general result is as follows. In addition to the foregoing conditions, assume that $q(t)$ is holomorphic in the sector $\alpha_1 < \arg t < \alpha_2$, α_1 being negative and α_2 positive. Assume also that for each $\delta \in (0, \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_1)$ the given asymptotic expansion of $q(t)$ for small $|t|$ applies in the sector $\alpha_1 + \delta \leqslant \arg t \leqslant \alpha_2 - \delta$, and $q(t) = O(e^{\sigma|t|})$ as $t \to \infty$ in the same sector. Here σ is an assignable constant. Then $Q(z)$ can be continued analytically into the sector $-\alpha_2 - \frac{1}{2}\pi < \arg z < -\alpha_1 + \frac{1}{2}\pi$, and the given asymptotic expansion for large z holds when $-\alpha_2 - \frac{1}{2}\pi + \delta \leqslant \arg z \leqslant -\alpha_1 + \frac{1}{2}\pi - \delta$.

Example: Consider

$$Q(z) = \int_0^{\infty} e^{-zt} \ln(1 + \sqrt{t})\, dt$$

The singularities of $q(t) \equiv \ln(1 + \sqrt{t})$ are given by $\sqrt{t} = -1$, hence $q(t)$ is holomorphic in the sector $|\arg t| < 2\pi$. Within the unit circle

$$q(t) = \sum_{s=1}^{\infty} (-)^{s-1} \frac{t^{s/2}}{s}$$

With $\lambda = \mu = 2$ we derive

$$Q(z) \sim \frac{1}{2z} \sum_{s=1}^{\infty} (-)^{s-1} \frac{\Gamma(\frac{1}{2}s)}{z^{s/2}} \qquad (z \to \infty,\ |\arg z| \leqslant \tfrac{5}{2}\pi - \delta)$$

δ being any positive constant less than $\frac{5}{2}\pi$.

12.3.2 Laplace's Method

Extensions of the results of 12.2.5 and 12.2.6 to complex variables necessitate great care in the choice of branches of the many-valued functions which are used. Let $\mathcal{P}$ denote the path for the contour integral

$$I(z) = \int_a^b e^{-zp(t)} q(t)\, dt$$

and assume that the endpoint a is finite. The other endpoint b may be finite or infinite. Also, let ω denote the angle of slope of $\mathcal{P}$ at a, that is, the limiting value of $\arg(t-a)$ as $t \to a$ along $\mathcal{P}$. The functions $p(t)$ and $q(t)$ are assumed to be holomorphic in an open domain **T** which contains $\mathcal{P}$, with the possible exception of the endpoints a and b.

Further assumptions are:

a. In the neighborhood of a there are convergent series expansions

$$p(t) = p(a) + \sum_{s=0}^{\infty} p_s(t-a)^{s+\mu}; \quad q(t) = \sum_{s=0}^{\infty} q_s(t-a)^{s+\lambda-1}$$

in which $p_0 \neq 0$, $\operatorname{Re}\lambda > 0$, and $\mu > 0$. When μ or λ is nonintegral (and this can only happen when a is a boundary point of **T**) the branches of $(t-a)^{\mu}$ and $(t-a)^{\lambda}$ are determined by

$$(t-a)^{\mu} \sim |t-a|^{\mu} e^{i\mu\omega}; \quad (t-a)^{\lambda} \sim |t-a|^{\lambda} e^{i\lambda\omega}$$

as $t \to a$ along $\mathcal{P}$, and by continuity elsewhere.

b. The parameter z is confined to a sector or single ray, given by $\theta_1 \leqslant \theta \leqslant \theta_2$, where $\theta = \arg z$ and $\theta_2 - \theta_1 < \pi$.

c. $I(z)$ converges at b absolutely and uniformly for all sufficiently large $|z|$.

d. $\operatorname{Re}\{e^{i\theta} p(t) - e^{i\theta} p(a)\}$ attains its minimum on $\mathcal{P}$ at $t = a$ (and nowhere else).

The last condition is crucial; it demands that the endpoint a is the location of the peak value of the integrand when $|z|$ is large.

With the foregoing conditions

$$I(z) \sim e^{-zp(a)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{z^{(s+\lambda)/\mu}}$$

as $z \to \infty$ in $\theta_1 \leqslant \arg z \leqslant \theta_2$. In this expansion the branch of $z^{(s+\lambda)/\mu}$ is

$$|z|^{(s+\lambda)/\mu} e^{i\theta(s+\lambda)/\mu}$$

and the coefficients a_s are determined by the method and formulas of 12.2.6, with the proviso that in forming the powers of p_0, the branch of $\arg p_0$ is chosen to satisfy

$$|\arg p_0 + \theta + \mu\omega| \leqslant \tfrac{1}{2}\pi$$

This choice is always possible, and it is unique.

12.3.3 Saddle-Points

Consider now the integral $I(z)$ of 12.3.2 in cases when the minimum value of $\operatorname{Re}\{zp(t)\}$ on the path $\mathscr{P}$ occurs not at a but an interior point t_0, say. For simplicity, assume that $\theta(\equiv \arg z)$ is fixed, so that t_0 is independent of z. The path may be subdivided at t_0, giving

$$I(z) = \int_{t_0}^{b} e^{-zp(t)}\, q(t)\, dt - \int_{t_0}^{a} e^{-zp(t)}\, q(t)\, dt$$

For large $|z|$ the asymptotic expansion of each of these integrals can be found by application of the result of 12.3.2, the role of the series in Condition (a) being played by the Taylor-series expansions of $p(t)$ and $q(t)$ at t_0. If $p'(t_0) \neq 0$, then it transpires that the asymptotic expansions of the two integrals are exactly the same, and on subtraction only the error terms are left. On the other hand, if $p'(t_0) = 0$ then the μ of Condition (a) is an integer not less than 2; in consequence, *different* branches of $p_0^{1/\mu}$ are used in constructing the coefficients a_s, and the two asymptotic expansions no longer cancel.

Cases in which $p'(t_0) \neq 0$ can be handled by deformation of $\mathscr{P}$ in such a way that on the new path the minimum of $\operatorname{Re}\{zp(t)\}$ occurs either at one of the endpoints or at a zero of $p'(t)$. As indicated in the preceding paragraph, the asymptotic expansion of $I(z)$ may then be found by means of one or two applications of the result of 12.3.2. Thus the zeros of $p'(t)$ are of great importance; they are called *saddle-points*. The name derives from the fact that if the surface $|e^{p(t)}|$ is plotted against the real and imaginary parts of t, then the tangent plane is horizontal at a zero of $p'(t)$, but in consequence of the maximum-modulus theorem this point is neither a maximum nor a minimum of the surface. Deformation of a path in the t-plane to pass through a zero of $p'(t)$ is equivalent to crossing a mountain ridge *via* a pass.

The task of locating saddle-points for a given integral is generally fairly easy, but the construction of a path on which $\operatorname{Re}\{zp(t)\}$ attains its minimum at an endpoint or saddle-point may be troublesome. An intelligent guess is sometimes successful, especially when the parameter z is confined to the real axis. Failing this, a partial study of the conformal mapping between the planes of t and $p(t)$ may be needed.

In constructing bounds for the error terms in the asymptotic expansion of $I(z)$ it is advantageous to employ integration paths along which $\operatorname{Im}\{zp(t)\}$ is constant. On the surface $|e^{zp(t)}|$ these are the paths of steepest descent from the endpoint or saddle-point. In consequence, the name *method of steepest descents* is often used. For the purpose of deriving asymptotic expansions, however, use of steepest paths is not essential.

Example: Bessel functions of large order—An integral of Schläfli for the Bessel function of the first kind is given by

$$J_\nu(\nu \operatorname{sech} \alpha) = \frac{1}{2\pi i} \int_{\infty - \pi i}^{\infty + \pi i} e^{-\nu p(t)}\, dt$$

where

$$p(t) = t - \operatorname{sech}\alpha \sinh t$$

Let us seek the asymptotic expansion of this integral for fixed positive values of α and large positive values of ν.

The saddle-points are the roots of $\cosh t = \cosh \alpha$, and are therefore given by $t = \pm\alpha$, $\pm\alpha \pm 2\pi i, \ldots$. The most promising is α, and as a possible path we consider that indicated in Figure 12.3-1. On the vertical segment we have $t = \alpha + i\tau$ where $-\pi \leqslant \tau \leqslant \pi$, and therefore

$$\operatorname{Re}\{p(t)\} = \alpha - \tanh\alpha \cos\tau > \alpha - \tanh\alpha \qquad (\tau \neq 0)$$

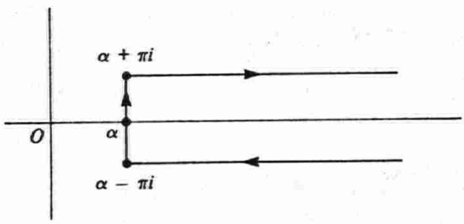

Fig. 12.3-1 *t*-plane.

On the horizontal segments $t = \alpha \pm \pi i + \tau$ where $0 \leqslant \tau < \infty$, and

$$\operatorname{Re}\{p(t)\} = \alpha + \tau + \operatorname{sech}\alpha \sinh(\alpha + \tau) \geqslant \alpha + \tanh\alpha$$

Clearly $\operatorname{Re}\{p(t)\}$ attains its minimum on the path at α, as required.

The Taylor series for $p(t)$ at α is given by

$$p(t) = \alpha - \tanh\alpha - \tfrac{1}{2}(t-\alpha)^2 \tanh\alpha - \tfrac{1}{6}(t-\alpha)^3 - \tfrac{1}{24}(t-\alpha)^4 \tanh\alpha + \cdots$$

In the notation of 12.3.2, we have $\mu = 2$, $p_0 = -\frac{1}{2}\tanh\alpha$, $p_1 = -\frac{1}{6}$, $p_2 = -\frac{1}{24}\tanh\alpha$, and $\lambda = q_0 = 1$. On the upper part of the path $\omega = \frac{1}{2}\pi$, and since $\theta = 0$ the correct choice of branch of $\arg p_0$ is $-\pi$. The formulas of 12.2.6 yield

$$a_0 = (\tfrac{1}{2}\coth\alpha)^{1/2} i; \quad a_1 = \tfrac{1}{3}\coth^2\alpha; \quad a_2 = (\tfrac{1}{2} - \tfrac{5}{6}\coth^2\alpha)(\tfrac{1}{2}\coth\alpha)^{3/2} i$$

Hence from 12.3.2

$$\int_{\alpha}^{\infty + \pi i} e^{-\nu p(t)}\, dt \sim e^{-\nu(\alpha - \tanh\alpha)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{\nu^{(s+1)/2}}$$

The corresponding expansion for $\int_{\alpha}^{\infty - \pi i}$ is obtained by changing the sign of i throughout. Combination of the results yields

$$J_\nu(\nu \operatorname{sech} \alpha) \sim \frac{e^{-\nu(\alpha - \tanh \alpha)}}{(2\pi\nu \tanh \alpha)^{1/2}} \left\{1 + \left(\frac{1}{8} \coth \alpha - \frac{5}{24} \coth^3 \alpha\right) \frac{1}{\nu} + \cdots\right\}$$

This is *Debye's expansion.* No expression for the general term is available; the easiest way of calculating higher terms is *via* differential-equation theory (12.8.2).

Conformal mapping was not required in this example because a suitable path was easily guessed. For the corresponding problem with complex ν and α, however, the mapping is almost unavoidable.

12.4 FURTHER METHODS FOR INTEGRALS

12.4.1 Logarithmic Singularities

Watson's lemma, Laplace's method, and the method of stationary phase can be extended in a straightforward manner to cases in which the integrand has a logarithmic singularity at the saddle-point.

For example, with the conditions of 12.2.2

$$\int_0^\infty e^{-xt} q(t) \ln t \, dt \sim \sum_{s=0}^{\infty} \Gamma'\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} - \ln x \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}}$$

In other words, formal differentiation of the general result of 12.2.2 with respect to the exponent λ (or μ) is legitimate. Such differentiations may be repeated any number of times.

12.4.2 Generalizations of Laplace's Method

The underlying idea of Laplace's method may be applied to integrals in which the parameter x enters in a more general way than in sections 12.2 and 12.3. Consider the integral

$$I(x) = \int_a^b \exp\{-xp(t) + x^\alpha r(t)\}\, q(t)\, dt$$

in which $p(t)$ and $q(t)$ satisfy the conditions of 12.2.5, $r(t)$ is independent of x, and α is a constant. What kind of behavior can be permitted in $r(t)$ at $t = a$ without changing the result already obtained for the case $r(t) = 0$? A sufficient condition is, in fact

$$r(t) \sim R(t-a)^\nu \quad (t \to a+)$$

where R and ν are constants such that $R \neq 0, \nu \geqslant 0$, and $\nu > \mu\alpha$. In the case $R > 0$ we must also have $\alpha < 1$.

When $\nu \leqslant \mu\alpha$ the term $x^{\alpha} r(t)$ may no longer be treated as a negligible perturbation. The case $\nu < \mu\alpha$ can be handled simply by interchanging the roles of $p(t)$ and $r(t)$, and regarding x^{α} instead of x as the large parameter.

The case $\nu = \mu\alpha$ is more interesting, because $I(x)$ can no longer be approximated satisfactorily in terms of elementary functions. The simplest integral having the same character is *Faxén's integral*

$$\mathrm{Fi}(\xi, \eta; y) = \int_0^{\infty} \exp\left(-\tau + y\tau^{\xi}\right)\tau^{\eta-1}\, d\tau \qquad (0 \leqslant \mathrm{Re}\,\xi < 1, \mathrm{Re}\,\eta > 0)$$

This is used as approximant in the following general result.

Let

$$I(x) = \int_0^{b} \exp\left\{-xp(t) + x^{\nu/\mu} r(t) + s(x, t)\right\} q(x, t)\, dt$$

in which b is finite, and

a. In the interval $(0, b]$ the functions $p'(t)$ and $r(t)$ are continuous and $p'(t) > 0$.

b. As $t \to 0+$

$$p(t) = p(0) + Pt^{\mu} + O(t^{\mu_1}); \quad p'(t) = \mu P t^{\mu-1} + O(t^{\mu_1 - 1}); \quad r(t) = Rt^{\nu} + O(t^{\nu_1})$$

where $P > 0$, $\mu_1 > \mu > \nu \geqslant 0$, and $\nu_1 > \nu$.

c. For all sufficiently large x the functions $s(x, t)$ and $q(x, t)$ are continuous in $0 < t \leqslant b$, and

$$|s(x, t)| \leqslant Sx^{\gamma} t^{\sigma}; \quad |q(x, t) - Qt^{\lambda-1}| \leqslant Q_1 x^{\beta} t^{\lambda_1 - 1}$$

where $S, \gamma, \sigma, Q, \lambda, Q_1, \beta$, and λ_1 are independent of x and t, and*

$$\sigma \geqslant 0; \quad \lambda > 0; \quad \lambda_1 > 0; \quad \gamma < \min(1, \sigma/\mu); \quad \beta < (\lambda_1 - \lambda)/\mu$$

Then

$$I(x) = \frac{Q}{\mu}\, \mathrm{Fi}\left(\frac{\nu}{\mu}, \frac{\lambda}{\mu}; \frac{R}{P^{\nu/\mu}}\right) \frac{e^{-xp(0)}}{(Px)^{\lambda/\mu}} \left\{1 + O\left(\frac{1}{x^{\tilde{\omega}/\mu}}\right)\right\} \qquad (x \to \infty)$$

where $\tilde{\omega} = \min(\mu_1 - \mu, \sigma - \mu\gamma, \lambda_1 - \lambda - \mu\beta, \nu_1 - \nu)$.

*None of γ, β, or $\lambda_1 - \lambda$ is required to be positive.

12.4.3 Properties of Faxén's Integral

Commonly needed pairs of values of the parameters are $\xi = \eta = \frac{1}{2}$ and $\xi = \eta = \frac{1}{3}$. For these cases

$$\mathrm{Fi}\left(\tfrac{1}{2}, \tfrac{1}{2};\ y\right) = \sqrt{\pi}\, e^{y^2/4}\ \{1 + \mathrm{erf}\left(\tfrac{1}{2} y\right)\}$$

$$\mathrm{Fi}\left(\tfrac{1}{2}, \tfrac{1}{2};\ y\right) + \mathrm{Fi}\left(\tfrac{1}{2}, \tfrac{1}{2};\ -y\right) = 2\sqrt{\pi}\, e^{y^2/4}$$

$$\mathrm{Fi}\left(\tfrac{1}{3}, \tfrac{1}{3};\ y\right) = 3^{2/3}\, \pi \mathrm{Hi}(3^{-1/3}\, y)$$

and

$$e^{-\pi i/6}\, \mathrm{Fi}\left(\tfrac{1}{3}, \tfrac{1}{3};\ y e^{\pi i/3}\right) + e^{\pi i/6}\, \mathrm{Fi}\left(\tfrac{1}{3}, \tfrac{1}{3};\ y e^{-\pi i/3}\right) = 3^{2/3}\, 2\pi\, \mathrm{Ai}(-3^{-1/3}\, y)$$

Here $\mathrm{Hi}(x)$ denotes *Scorer's function*, defined by

$$\mathrm{Hi}(x) = \frac{1}{\pi} \int_0^\infty \exp\left(-\tfrac{1}{3} t^3 + xt\right) dt$$

and Ai is the Airy integral.

Example: Parabolic cylinder functions of large order—An integral representation for the parabolic cylinder function is supplied by*

$$U\left(n + \frac{1}{2}, y\right) = \frac{e^{-y^2/4}}{\Gamma(n+1)} \int_0^\infty e^{-yw - (w^2/2)}\, w^n\, dw \qquad (n > -1)$$

We seek an asymptotic approximation for large positive n and fixed y.

The integrand attains its maximum at $w = -\frac{1}{2} y + \sqrt{\frac{1}{4} y^2 + n}$. Since this is asymptotic to $\sqrt{n}$ for large n we make the substitution $w = \sqrt{n}\,(1 + t)$; compare the example at the end of 12.2.5. Accordingly

$$U\left(n + \frac{1}{2}, y\right) = \exp\left(-y\sqrt{n} - \frac{1}{2} n - \frac{1}{4} y^2\right) \frac{n^{(n+1)/2}}{\Gamma(n+1)} \int_{-1}^\infty e^{-np(t) - yt\sqrt{n}}\, dt$$

where

$$p(t) = t + \tfrac{1}{2} t^2 - \ln(1 + t) = t^2 - \tfrac{1}{3} t^3 + \cdots \qquad (t \to 0)$$

*This notation is due to Miller (1955). In the older notation of Whittaker, $U(n + \frac{1}{2}, y)$ is denoted by $D_{-n-1}(y)$.

The general result of 12.4.2 is applied with $x = n$, $r(t) = -yt$, $s(x, t) = 0$, and $q(x, t) = 1$. Thus $\mu = 2, \mu_1 = 3, \nu = 1, \widetilde{\omega} = 1$, and

$$\int_0^b e^{-np(t)-yt\sqrt{n}}\,dt = \frac{\mathrm{Fi}(\frac{1}{2}, \frac{1}{2}; -y)}{2\sqrt{n}}\left\{1 + O\left(\frac{1}{\sqrt{n}}\right)\right\}$$

for any fixed value of the positive number b. Similarly,

$$\int_{-b}^0 e^{-np(t)-yt\sqrt{n}}\,dt = \frac{\mathrm{Fi}(\frac{1}{2}, \frac{1}{2}; y)}{2\sqrt{n}}\left\{1 + O\left(\frac{1}{\sqrt{n}}\right)\right\}$$

provided that $0 < b < 1$. The contributions from the tails $\int_b^\infty$ and $\int_{-1}^{-b}$ are exponentially small when n is large, hence by addition and use of Stirling's approximation (7.2-11) we derive the required result

$$U\left(n + \frac{1}{2}, y\right) = \frac{\exp\left(-y\sqrt{n} + \frac{1}{2}n\right)}{\sqrt{2}\,n^{(n+1)/2}}\left\{1 + O\left(\frac{1}{\sqrt{n}}\right)\right\} \qquad (n \to \infty)$$

12.4.4 More General Kernels

Watson's lemma (12.2.2 and 12.3.1) may be regarded as an inductive relation between two asymptotic expansions; thus

$$q(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu} \qquad (t \to 0+)$$

implies

$$\int_0^\infty e^{-xt} q(t)\,dt \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right)\frac{a_s}{x^{(s+\lambda)/\mu}} \qquad (x \to +\infty)$$

provided that Re $\lambda > 0$, $\mu > 0$, and the integral converges. Similar induction of series occurs for integrals in which the factor e^{-xt} is replaced by a more general kernel $g(xt)$. Thus

$$\int_0^\infty g(xt) q(t)\,dt \sim \sum_{s=0}^{\infty} G\left(\frac{s+\lambda}{\mu}\right)\frac{a_s}{x^{(s+\lambda)/\mu}} \qquad (x \to +\infty)$$

in which $G(\alpha)$ denotes the Mellin transform of $g(t)$:

$$G(\alpha) = \int_0^\infty g(\tau)\tau^{\alpha-1}\, d\tau$$

Special cases include

$$\int_0^\infty \mathrm{Ai}(xt)q(t)\,dt \sim \sum_{s=0}^\infty 3^{-(s+\lambda+2\mu)/(3\mu)}\Gamma\left(\frac{s+\lambda}{\mu}\right)\left\{\Gamma\left(\frac{s+\lambda+2\mu}{3\mu}\right)\right\}^{-1} \frac{a_s}{x^{(s+\lambda)/\mu}}$$

and

$$\int_0^\infty K_0(xt)q(t)\,dt \sim \frac{1}{4}\sum_{s=0}^\infty \left\{\Gamma\left(\frac{s+\lambda}{2\mu}\right)\right\}^2 a_s\left(\frac{2}{x}\right)^{(s+\lambda)/\mu}$$

where K_0 is the modified Bessel function. It is assumed that for large t, $q(t)$ is $O(e^{\sigma t^{3/2}})$ for the Ai kernel and $O(e^{\sigma t})$ for the K_0 kernel, σ being an assignable constant.

12.4.5 Bleistein's Method

Let

$$I(\alpha, x) = \int_0^k e^{-xp(\alpha,t)}q(\alpha, t)t^{\lambda-1}\, dt$$

where k and λ are positive constants (k possibly being infinite), α is a variable parameter in the interval $[0, k)$, and x is a large positive parameter. Assume that $\partial^2 p(\alpha, t)/\partial t^2$ and $q(\alpha, t)$ are continuous functions of α and t, and also that for given α the minimum value of $p(\alpha, t)$ in $[0, k)$ is attained at $t = \alpha$, at which point $\partial p(\alpha, t)/\partial t$ vanishes but both $\partial^2 p(\alpha, t)/\partial t^2$ and $q(\alpha, t)$ are nonzero. For large x Laplace's method gives

$$I(\alpha, x) \sim e^{-xp(\alpha,\alpha)}q(\alpha, \alpha)\alpha^{\lambda-1}\left\{\frac{x}{2\pi}\left[\frac{\partial^2 p(\alpha, t)}{\partial t^2}\right]_{t=\alpha}\right\}^{-1/2}$$

if $\alpha \neq 0$, or

$$I(\alpha, x) \sim \frac{1}{2}e^{-xp(0,0)}q(0, 0)\Gamma\left(\frac{1}{2}\lambda\right)\left\{\frac{x}{2}\left[\frac{\partial^2 p(0, t)}{\partial t^2}\right]_{t=0}\right\}^{-\lambda/2}$$

if $\alpha = 0$.

Whether or not $\lambda = 1$, the first of these approximations does not reduce to the second as $\alpha \to 0$. This abrupt change means that the first approximation is nonuniform for arbitrarily small values of α.

To obtain a uniform approximation, we introduce a new integration variable w, given by

$$p(\alpha, t) = \tfrac{1}{2} w^2 - aw + b$$

where a and b are functions of α chosen in such a way that the endpoint $t = 0$ corresponds to $w = 0$, and the stationary point $t = \alpha$ corresponds to the stationary point $w = a$. Thus

$$b = p(\alpha, 0); \quad a = \{2p(\alpha, 0) - 2p(\alpha, \alpha)\}^{1/2}$$

and

$$w = \{2p(\alpha, 0) - 2p(\alpha, \alpha)\}^{1/2} \pm \{2p(\alpha, t) - 2p(\alpha, \alpha)\}^{1/2}$$

the upper or lower sign being taken according as $t >$ or $< \alpha$. The relationship between t and w is one-to-one, and because

$$\frac{dw}{dt} = \pm \frac{1}{\{2p(\alpha, t) - 2p(\alpha, \alpha)\}^{1/2}} \frac{\partial p(\alpha, t)}{\partial t}$$

the relationship is free from singularity at $t = \alpha$.

Transformation to w as variable gives

$$I(\alpha, x) = e^{-xp(\alpha,0)} \int_0^{\kappa} \exp\left\{-x\left(\frac{1}{2} w^2 - aw\right)\right\} f(\alpha, w)\, w^{\lambda-1}\, dw$$

where

$$f(\alpha, w) = q(\alpha, t)\left(\frac{t}{w}\right)^{\lambda-1} \frac{dt}{dw}$$

and $\kappa = \kappa(\alpha)$ is the value of w at $t = k$. The factor $f(\alpha, w)$ is expanded in a Taylor series centered at the peak value $w = a$ of the exponential factor. This series has the form

$$f(\alpha, w) = \sum_{s=0}^{\infty} \phi_s(\alpha)(w - a)^s$$

in which the coefficients $\phi_s(\alpha)$ are continuous at $\alpha = 0$. The required uniform expansion is then obtained in a similar manner to Laplace's method: κ is replaced by

∞ and the series integrated term by term. Thus with the notation

$$F_s(y) = \int_0^\infty \exp\left(-\frac{1}{2}\tau^2 + y\tau\right)(\tau - y)^s \tau^{\lambda-1}\, d\tau$$

we derive

$$I(\alpha, x) \sim \frac{e^{-xp(\alpha,0)}}{x^{\lambda/2}} \sum_{s=0}^{\infty} \phi_s(\alpha) \frac{F_s(a\sqrt{x})}{x^{s/2}} \qquad (x \to \infty)$$

in the sense that this series is a generalized asymptotic expansion with respect to an appropriate scale (12.1.5).

Example: Let

$$I(\alpha, x) = \int_0^{\pi/2} e^{x(\cos\theta + \theta \sin\alpha)}\, d\theta$$

where $0 \leqslant \alpha < \frac{1}{2}\pi$, and x is a large positive parameter. In the present notation

$$p(\alpha, \theta) = -\cos\theta - \theta\sin\alpha; \quad \partial p(\alpha, \theta)/\partial\theta = \sin\theta - \sin\alpha$$

The minimum of $p(\alpha, \theta)$ in the range of integration is $\theta = \alpha$. Since this approaches an endpoint as $\alpha \to 0$ we have exactly the situation described above.

The appropriate transformation is given by

$$\cos\theta + \theta\sin\alpha = 1 + aw - \tfrac{1}{2}w^2$$

where

$$a = \sqrt{2}\,(\cos\alpha + \alpha\sin\alpha - 1)^{1/2}$$

Thus

$$w = a \pm \sqrt{2}\,\{\cos\alpha + (\alpha - \theta)\sin\alpha - \cos\theta\}^{1/2} \qquad (\theta \gtrless \alpha)$$

The new integral is

$$I(\alpha, x) = e^x \int_0^{\kappa} \exp\left\{-x\left(\frac{1}{2}w^2 - aw\right)\right\} \frac{d\theta}{dw}\, dw$$

where

$$\kappa = a + \sqrt{2}\,\{\cos\alpha + (\alpha - \tfrac{1}{2}\pi)\sin\alpha\}^{1/2}$$

and

$$\frac{d\theta}{dw} = \frac{w - a}{\sin\theta - \sin\alpha} = \sum_{s=0}^{\infty} \phi_s(\alpha)(w - a)^s$$

In the last expansion the first three coefficients are given by

$$\phi_0(\alpha) = \frac{1}{(\cos\alpha)^{1/2}}; \quad \phi_1(\alpha) = \frac{\sin\alpha}{3\cos^2\alpha}; \quad \phi_2(\alpha) = \frac{5 - 2\cos^2\alpha}{24(\cos\alpha)^{7/2}}$$

Write

$$X_s(\alpha, x) = \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}) + (-)^s\gamma\{\tfrac{1}{2}s + \tfrac{1}{2}, x(\cos\alpha + \alpha\sin\alpha - 1)\} \qquad (s = 0, 1, \ldots)$$

where $\gamma(\alpha, x)$ is the incomplete Gamma function

$$\gamma(\alpha, x) = \int_0^x e^{-t}\, t^{\alpha-1}\, dt \qquad (\operatorname{Re}\alpha > 0)$$

Then we may express the required asymptotic expansion in the form

$$I(\alpha, x) = \frac{e^{x(\cos\alpha + \alpha\sin\alpha)}}{(2x)^{1/2}} \left\{ \sum_{s=0}^{n-1} \phi_s(\alpha) X_s(\alpha, x) \left(\frac{2}{x}\right)^{s/2} + O\left(\frac{1}{x^{n/2}}\right) \right\} \qquad (x \to \infty)$$

where n is an arbitrary nonnegative integer. The O-term is uniform in any interval $0 \leqslant \alpha \leqslant \alpha_0$ for which α_0 is a constant less than $\pi/2$.

For fixed α and large x the incomplete Gamma function can be approximated in terms of elementary functions; compare 12.2.1. Then the uniform asymptotic expansion reduces to either the first or second Laplace approximation given at the beginning of this subsection, depending whether $\alpha > 0$ or $\alpha = 0$. This is, of course, to be expected, both in the present example and in the general case.

12.4.6 Method of Chester, Friedman, and Ursell

Let

$$I(\alpha, x) = \int_{\mathscr{P}} e^{-xp(\alpha, t)}\, q(\alpha, t)\, dt$$

be a contour integral in which x is a large parameter, and $p(\alpha, t)$ and $q(\alpha, t)$ are analytic functions of the complex variable t and continuous functions of the parameter α. Suppose that $\partial p(\alpha, t)/\partial t$ has two zeros which coincide for a certain value $\hat{\alpha}$, say, of α, and at least one of these zeros is in the range of integration. The problem of obtaining an asymptotic approximation for $I(\alpha, x)$ which is uniformly valid for α in a neighborhood of $\hat{\alpha}$ is similar to the problem treated in 12.4.5. In the present case we employ a cubic transformation of variables, given by

$$p(\alpha, t) = \tfrac{1}{3}w^3 + aw^2 + bw + c$$

The stationary points of the right-hand side are the zeros $w_1(\alpha)$ and $w_2(\alpha)$, say, of the quadratic $w^2 + 2aw + b$. The values of $a = a(\alpha)$ and $b = b(\alpha)$ are chosen in such a way that $w_1(\alpha)$ and $w_2(\alpha)$ correspond to the zeros of $\partial p(\alpha, t)/\partial t$. The other coefficient, c, is prescribed in any convenient manner.

The given integral becomes

$$I(\alpha, x) = e^{-xc} \int_{\mathcal{Q}} \exp\left\{-x\left(\frac{1}{3}w^3 + aw^2 + bw\right)\right\} f(\alpha, w)\, dw$$

where $\mathcal{Q}$ is the w-map of the original path $\mathcal{P}$, and

$$f(\alpha, w) = q(\alpha, t)\frac{dt}{dw} = q(\alpha, t)\frac{w^2 + 2aw + b}{\partial p(\alpha, t)/\partial t}$$

With the prescribed choice of a and b, the function $f(\alpha, w)$ is analytic at $w = w_1(\alpha)$ and $w = w_2(\alpha)$ when $\alpha \neq \hat{\alpha}$, and at the confluence of these points when $\alpha = \hat{\alpha}$. For large x, $I(\alpha, x)$ is approximated by the corresponding integral with $f(\alpha, w)$ replaced by a constant, that is, by an Airy or Scorer function, depending on the path $\mathcal{Q}$.

Example: Let us apply the method just described to the integral

$$A(\alpha, x) = \int_0^\infty e^{-x(\operatorname{sech}\alpha \sinh t - t)}\, dt$$

in which $\alpha \geqslant 0$ and x is large and positive. The integrand has saddle-points at $t = \alpha$ and $-\alpha$. The former is always in the range of integration, and it coincides with the latter when $\alpha = 0$. We seek an asymptotic expansion of $A(\alpha, x)$ which is uniform for arbitrarily small values of α.

By symmetry, the appropriate cubic transformation has the form

$$\operatorname{sech}\alpha \sinh t - t = \tfrac{1}{3}w^3 - \zeta w$$

The stationary points of the right-hand side are $w = \pm\zeta^{1/2}$. Since they are to correspond to $t = \pm\alpha$, the value of the coefficient ζ is determined by

$$\tfrac{2}{3}\zeta^{3/2} = \alpha - \tanh\alpha$$

Then

$$A(\alpha, x) = \int_0^\infty \exp\left\{-x\left(\frac{1}{3}w^3 - \zeta w\right)\right\} \frac{dt}{dw}\, dw$$

The peak value of the exponential factor in the new integrand occurs at $w = \zeta^{1/2}$. We expand t in a Taylor series at this point, in the form

$$t = \alpha + \sum_{s=0}^{\infty} \frac{\phi_s(\alpha)}{s+1} (w - \zeta^{1/2})^{s+1}$$

The coefficients $\phi_s(\alpha)$ can be found, for example, by repeatedly differentiating the equation connecting t and w and then setting $t = \alpha$ and $w = \zeta^{1/2}$. In particular

$$\phi_0(\alpha) = \left(\frac{4\zeta}{\tanh^2\alpha}\right)^{1/4}; \quad \phi_1(\alpha) = \frac{2 - \{\phi_0(\alpha)\}^3}{3\phi_0(\alpha)\tanh\alpha}$$

It is easily verified that each of these expressions tends to a finite limit as $\alpha \to 0$.

The desired asymptotic expansion is now obtained by termwise integration. Thus

$$A(\alpha, x) \sim \pi \sum_{s=0}^{\infty} \phi_s(\alpha) \frac{\mathrm{Qi}_s(x^{2/3}\zeta)}{x^{(s+1)/3}} \qquad (x \to \infty)$$

where

$$\mathrm{Qi}_s(y) = \frac{1}{\pi}\int_0^\infty \exp\left(-\frac{1}{3}t^3 + yt\right)(t - y^{1/2})^s\, dt \qquad (s = 0, 1, \ldots)$$

These integrals are related to Scorer's function (12.4.3) by

$$\mathrm{Qi}_0(y) = \mathrm{Hi}(y); \quad \mathrm{Qi}_1(y) = \mathrm{Hi}'(y) - y^{1/2}\,\mathrm{Hi}(y); \quad \mathrm{Qi}_2(y) = \pi^{-1} - 2y^{1/2}\,\mathrm{Qi}_1(y)$$

and

$$\mathrm{Qi}_s(y) = (-)^s \pi^{-1} y^{(s-2)/2} - 2y^{1/2}\,\mathrm{Qi}_{s-1}(y) + (s-2)\,\mathrm{Qi}_{s-3}(y) \qquad (s \geqslant 3)$$

If α is restricted to a finite interval $[0, \alpha_0]$, then the error on truncating the asymptotic expansion of $A(\alpha, x)$ at its n^{th} term is $O\{x^{-(n+1)/3}\,\mathrm{Qi}_n(x^{2/3}\zeta)\}$ uniformly with respect to α, provided that n is even. For α ranging over the infinite interval $[0, \infty)$ this error is uniformly $O\{(1+\zeta)^{(n+1)/4}x^{-(n+1)/3}\,\mathrm{Qi}_n(x^{2/3}\zeta)\}$, in this case provided that n is even and nonzero.

12.5 SUMS AND SEQUENCES

12.5.1 Bernoulli Polynomials

The polynomials $B_0(x), B_1(x), \ldots$, defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{s=0}^{\infty} B_s(x)\frac{t^s}{s!} \qquad (|t| < 2\pi)$$

are called the *Bernoulli polynomials*. Their values at $x = 0$ are the *Bernoulli numbers* $B_s = B_s(0)$.

The first few Bernoulli numbers are given by

$$\begin{array}{lllll} B_0 = 1 & B_1 = -\frac{1}{2} & B_2 = \frac{1}{6} & B_3 = 0 & B_4 = -\frac{1}{30} \\ B_5 = 0 & B_6 = \frac{1}{42} & B_7 = 0 & B_8 = -\frac{1}{30} & B_9 = 0 \end{array}$$

the only nonvanishing B_s of odd subscript being, in fact, B_1. Corresponding polynomials are

$$B_0(x) = 1; \qquad B_1(x) = x - \tfrac{1}{2}$$

$$B_2(x) = x^2 - x + \tfrac{1}{6}; \qquad B_3(x) = x(x - \tfrac{1}{2})(x - 1)$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}; \qquad B_5(x) = x(x - \tfrac{1}{2})(x - 1)(x^2 - x - \tfrac{1}{3})$$

$$B_6(x) = x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{42}$$

Important properties include

$$B_s(x) = \sum_{j=0}^{s} \binom{s}{j} B_{s-j}x^j$$

$$B_s'(x) = sB_{s-1}(x); \quad \int_0^1 B_s(x)\,dx = 0 \quad (s \geqslant 1); \quad B_s(1-x) = (-)^s B_s(x)$$

$$\sum_{j=1}^{n} j^s = \frac{1}{s+1}\{B_{s+1}(n+1) - B_{s+1}\} \qquad (s \geqslant 1)$$

$$\sum_{j=1}^{\infty} \frac{1}{j^{2s}} = (-)^{s-1}\frac{(2\pi)^{2s}B_{2s}}{2(2s)!} \qquad (s \geqslant 1)$$

and

$$\int_0^\infty \frac{x^{2s-1}}{e^{2\pi x}-1}\,dx = (-)^{s-1}\frac{B_{2s}}{4s} \quad (s \geqslant 1)$$

When $s \geqslant 1$, the only zeros of $B_{2s+1}(x)$ in the interval $[0, 1]$ are $0, \frac{1}{2}$, and 1, and the only zeros of $B_{2s}(x) - B_{2s}$ in the same interval are 0 and 1. Also,

$$|B_{2s}(x)| \leqslant |B_{2s}| \quad (0 \leqslant x \leqslant 1)$$

12.5.2 The Euler-Maclaurin Formula

If a, m, and n are integers such that $a < n$ and $m > 0$, and $f^{(2m)}(x)$ is absolutely integrable over the interval (a, n), then

$$\sum_{j=a}^{n} f(j) = \int_a^n f(x)\,dx + \frac{1}{2}f(a) + \frac{1}{2}f(n) + \sum_{s=1}^{m-1}\frac{B_{2s}}{(2s)!}\{f^{(2s-1)}(n) - f^{(2s-1)}(a)\}$$
$$+ \int_a^n \frac{B_{2m} - B_{2m}(x - [x])}{(2m)!} f^{(2m)}(x)\,dx$$

Here $[x]$ denotes the integer in the interval $(x - 1, x]$; in consequence, as a function of x, $B_{2m}(x - [x])$ is periodic and continuous, with period 1.

The formula just given is the *Euler-Maclaurin formula*. Its uses include numerical quadrature, numerical summation of slowly convergent series, and asymptotic approximation of sums of series. Another version of the formula is furnished by

$$\sum_{j=a}^{n} f(j) = \int_a^n f(z)\,dz + \frac{1}{2}f(a) + \frac{1}{2}f(n) - 2\int_0^\infty \frac{\operatorname{Im}\{f(a+iy)\}}{e^{2\pi y}-1}\,dy$$
$$+ \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(n) + 2\frac{(-)^m}{(2m)!}\int_0^\infty \operatorname{Im}\{f^{(2m)}(n + i\vartheta_n y)\}\frac{y^{2m}\,dy}{e^{2\pi y}-1}$$

ϑ_n being some number in the interval $(0, 1)$. This second form is valid with the conditions:

a. $f(z)$ is continuous throughout the strip $a \leqslant \operatorname{Re} z \leqslant n$, and holomorphic in its interior.

b. $f(z)$ is real on the intersection of the strip with the real axis.

c. $f(z) = o(e^{2\pi|\operatorname{Im} z|})$ as $\operatorname{Im} z \to \pm\infty$ in $a \leqslant \operatorname{Re} z \leqslant n$, uniformly with respect to $\operatorname{Re} z$.

d. $\displaystyle\int_0^\infty \frac{\operatorname{Im}\{f(a+iy)\}}{e^{2\pi y}-1}\,dy$ converges

e. $f^{(2m)}(z)$ is continuous on the line $\operatorname{Re} z = n$.

Example: Let us seek the asymptotic expansion of the sum

$$S(n) = \sum_{j=1}^{n} j \ln j$$

for large n. Setting $f(x) = x \ln x$, we have

$$\int f(x)\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2; \quad f'(x) = \ln x + 1; \quad f^{(s)}(x) = (-)^s \frac{(s-2)!}{x^{s-1}} \quad (s \geqslant 2)$$

The first form of the Euler-Maclaurin formula leads to

$$S(n) = \frac{1}{2}n^2 \ln n - \frac{1}{4}n^2 + \frac{1}{2}n \ln n + \frac{1}{12}\ln n + C - \sum_{s=2}^{m-1} \frac{B_{2s}}{2s(2s-1)(2s-2)n^{2s-2}} - R_m(n)$$

where m is an arbitrary integer exceeding unity,

$$C = \frac{1}{4} - \frac{1}{720} - \frac{1}{12}\int_1^\infty \frac{B_4(x-[x])}{x^3}\,dx$$

and

$$R_m(n) = \int_n^\infty \frac{B_{2m} - B_{2m}(x-[x])}{2m(2m-1)x^{2m-1}}\,dx$$

The final result in 12.5.1 shows that $B_{2m} - B_{2m}(x - [x])$ is bounded in absolute value by B_{2m} and has the same sign. Hence

$$|R_m(n)| \leqslant \frac{|B_{2m}|}{2m(2m-1)(2m-2)n^{2m-2}} \quad (m \geqslant 2)$$

Since the last quantity is $O(1/n^{2m-2})$ as $n \to \infty$, the expansion for $S(n)$ is an asymptotic expansion, complete with error bound.

A numerical estimate for the constant C can be found by summing directly the first few terms of the series $\sum j \ln j$, and using the bound for $|R_m(n)|$. For example, if $m = 4$ and $n = 5$, then we have

$$|R_4(5)| \leqslant \frac{(1/30)}{8 \cdot 7 \cdot 6} \frac{1}{5^6} \doteqdot 0.6 \times 10^{-8}$$

Direct summation yields $S(5) = 18.27449823$, and subtracting the values of the known terms in the expansion of $S(5)$, we find that $C = 0.24875449$, correct to eight decimal places.

An analytical expression for C can be derived from the second form of the Euler-Maclaurin formula given above. The result is expressible as

$$C = \frac{\gamma + \ln(2\pi)}{12} - \frac{\zeta'(2)}{2\pi^2}$$

where γ denotes Euler's constant, $\zeta(z)$ is the Riemann Zeta function

$$\zeta(z) = \sum_{s=1}^{\infty} \frac{1}{s^z} \qquad (\operatorname{Re} z > 1)$$

and $\zeta'(2)$ the derivative of $\zeta(z)$ at $z = 2$.

12.5.3 Asymptotic Expansions of Entire Functions

The asymptotic behavior, for large $|z|$, of entire functions defined by their Maclaurin series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

can sometimes be found by expressing the sum as a contour integral and applying the methods of sections 12.2 to 12.4.

Example: Consider the function

$$f(\rho, x) = \sum_{j=0}^{\infty} \left(\frac{x^j}{j!}\right)^{\rho}$$

for large positive values of x, where ρ is a constant in the interval $(0, 4]$. From the residue theorem it follows that

$$\sum_{j=0}^{n-1} \left(\frac{x^j}{j!}\right)^{\rho} = \frac{1}{2i} \int_{\mathcal{C}} \left\{\frac{x^t}{\Gamma(t+1)}\right\}^{\rho} \cot(\pi t)\, dt$$

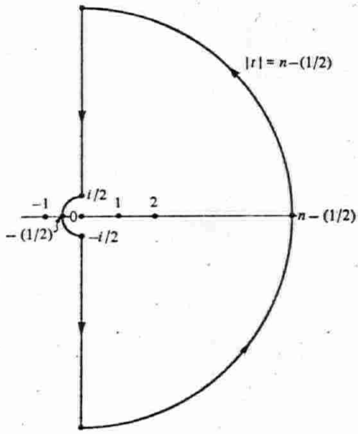

Fig. 12.5-1 t-plane: contour $\mathcal{C}$.

where $\mathcal{C}$ is the closed contour depicted in Figure 12.5-1. Now

$$\frac{\cot(\pi t)}{2i} = -\frac{1}{2} - \frac{1}{e^{-2\pi i t} - 1} = \frac{1}{2} + \frac{1}{e^{2\pi i t} - 1}$$

Hence

$$\sum_{j=0}^{n-1} \left(\frac{x^j}{j!}\right)^\rho = \int_{-1/2}^{n-(1/2)} \left\{\frac{x^t}{\Gamma(t+1)}\right\}^\rho dt - \int_{\mathcal{C}_1} \left\{\frac{x^t}{\Gamma(t+1)}\right\}^\rho \frac{dt}{e^{-2\pi i t} - 1}$$
$$+ \int_{\mathcal{C}_2} \left\{\frac{x^t}{\Gamma(t+1)}\right\}^\rho \frac{dt}{e^{2\pi i t} - 1}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are respectively the upper and lower halves of $\mathcal{C}$.

By means of Stirling's approximation (7.2-11) it is verifiable that the integrals around the large circular arcs vanish as $n \to \infty$, provided that $\rho \leqslant 4$ (which we have assumed to be the case). Also, $|x^{t\rho}| \leqslant 1$ when $x \geqslant 1$ and Re $t \leqslant 0$. Hence

$$f(\rho, x) = \int_0^\infty \left\{\frac{x^t}{\Gamma(t+1)}\right\}^\rho dt + O(1) \quad (x \geqslant 1)$$

The asymptotic behavior of the last integral can be found by use of Stirling's approximation and Laplace's method in the manner of the example treated in 12.2.5. The final result is given by

$$f(\rho, x) \sim \frac{e^{\rho x}}{\rho^{1/2}(2\pi x)^{(\rho-1)/2}} \qquad (x \to \infty)$$

12.5.4 Coefficients in a Maclaurin or Laurent Expansion

Let $f(t)$ be a given analytic function and

$$f(t) = \sum_{n=-\infty}^{\infty} a_n t^n \qquad (0 < |t| < r)$$

a Laurent expansion. What is the asymptotic behavior of a_n as n approaches ∞ or $-\infty$? More specially, what is the asymptotic behavior of the sequence of coefficients in a Maclaurin series?

Problems of this kind can be brought within the scope of sections 12.3 and 12.4 by use of the Cauchy formula

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(t)}{t^{n+1}}\, dt$$

in which $\mathcal{C}$ is a simple closed contour encircling $t = 0$. However, in cases when $f(t)$ has finite singularities other than $t = 0$, the method of the next subsection often yields the required approximation in an easier way.

12.5.5 Method of Darboux

In the complex t-plane let r be the distance of the nearest singularity of $f(t)$ from the origin, and suppose that a 'comparison' function $g(t)$ can be found with the properties:

a. $g(t)$ is holomorphic in $0 < |t| < r$.
b. $f(t) - g(t)$ is continuous in $0 < |t| \leqslant r$.
c. The coefficients in the Laurent expansion

$$g(t) = \sum_{n=-\infty}^{\infty} b_n t^n \qquad (0 < |t| < r)$$

have known asymptotic behavior.

Then by allowing the contour in Cauchy's formula to expand, we deduce that

$$a_n - b_n = \frac{1}{2\pi i} \int_{|t|=r} \frac{f(t) - g(t)}{t^{n+1}}\, dt = \frac{1}{2\pi r^n} \int_0^{2\pi} \{f(re^{i\theta}) - g(re^{i\theta})\}\, e^{-ni\theta}\, d\theta$$

Application of the Riemann-Lebesgue lemma (12.2.3) to the last integral yields

$$a_n = b_n + o(r^{-n}) \qquad (n \to \infty)$$

This is a first approximation to a_n. Often this result is refinable in two ways. First, if $f^{(m)}(t) - g^{(m)}(t)$ is continuous on $|t| = r$ then the integral for $a_n - b_n$ may be integrated m times by parts to yield the stronger result

$$a_n = b_n + o(r^{-n}n^{-m}) \qquad (n \to \infty)$$

Secondly, it is unnecessary for $f(t) - g(t)$—or $f^{(m)}(t) - g^{(m)}(t)$—to be continuous on $|t| = r$; it suffices that the integrals involved converge uniformly with respect to n.

Example: Legendre polynomials of large order—The standard generating function for the Legendre polynomials is given by

$$\frac{1}{(1 - 2t\cos\alpha + t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(\cos\alpha)t^n \qquad (|t| < 1)$$

Let the left-hand side be denoted by $f(t)$. The only singularities of this function are branch-points at $t = e^{\pm i\alpha}$. To insure that these points do not coincide we restrict $0 < \alpha < \pi$ in what follows.

Let $(e^{i\alpha} - t)^{-1/2}$ be the branch of this square root which is continuous in the t-plane cut along the outward-drawn ray through $t = e^{i\alpha}$ and takes the value $e^{-i\alpha/2}$ at $t = 0$. Similarly, let $(e^{-i\alpha} - t)^{-1/2}$ denote the conjugate function. Then $f(t)$ can be factorized as

$$f(t) = (e^{i\alpha} - t)^{-1/2}(e^{-i\alpha} - t)^{-1/2} \qquad (|t| < 1)$$

If $t \to e^{\mp i\alpha}$ from within the unit circle, then

$$(e^{i\alpha} - t)^{-1/2} \to e^{-\pi i/4}(2\sin\alpha)^{-1/2} \qquad (t \to e^{-i\alpha})$$

$$(e^{-i\alpha} - t)^{-1/2} \to e^{\pi i/4}(2\sin\alpha)^{-1/2} \qquad (t \to e^{i\alpha})$$

Accordingly, in the notation used above we set

$$g(t) = e^{-\pi i/4}(2\sin\alpha)^{-1/2}(e^{-i\alpha} - t)^{-1/2} + e^{\pi i/4}(2\sin\alpha)^{-1/2}(e^{i\alpha} - t)^{-1/2}$$

The coefficient of t^n in the Maclaurin expansion of $g(t)$ is

$$b_n = \left(\frac{2}{\sin\alpha}\right)^{1/2}\binom{-\frac{1}{2}}{n}\cos\left\{\left(n + \frac{1}{2}\right)\alpha + \left(n - \frac{1}{4}\right)\pi\right\}$$

By Darboux's method a first approximation to $P_n(\cos\alpha)$ is given by

$$P_n(\cos\alpha) = b_n + o(1) \qquad (n \to \infty)$$

Since, however, by Stirling's approximation

$$b_n = \{2/(\pi n \sin\alpha)\}^{1/2} \cos(n\alpha + \tfrac{1}{2}\alpha - \tfrac{1}{4}\pi) + O(n^{-3/2})$$

this estimate for $P_n(\cos\alpha)$ reduces effectively to $o(1)$.

An improved result is obtainable by observing that the integral of $f'(t) - g'(t)$ around the unit circle converges uniformly with respect to n. Accordingly, we may integrate once by parts and apply the Riemann-Lebesgue lemma to obtain

$$P_n(\cos\alpha) = b_n + o(n^{-1})$$

By synthesizing a function $g(t)$ which matches the behavior of $f(t)$ *and* its first m derivatives at $t = e^{\mp i\alpha}$, we can extend this result into

$$P_n(\cos\alpha) = \left(\frac{2}{\sin\alpha}\right)^{1/2} \sum_{s=0}^{m-1} \binom{-\frac{1}{2}}{s}\binom{s-\frac{1}{2}}{n} \frac{\cos\alpha_{n,s}}{(2\sin\alpha)^s} + O\left(\frac{1}{n^{m+(1/2)}}\right) \qquad (n \to \infty)$$

where $\alpha_{n,s} = (n - s + \frac{1}{2})\alpha + (n - \frac{1}{2}s - \frac{1}{4})\pi$, and m is an arbitrary positive integer.

12.5.6 Haar's Method

Let $f(t)$ be given by an inverse Laplace transform (section 11.2)

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp$$

and $g(t)$ a comparison function having a known transform

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt} G(p)\, dp$$

and known asymptotic behavior for large positive t. By subtraction

$$f(t) - g(t) = \frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} e^{itv} \{F(c+iv) - G(c+iv)\}\, dv$$

If the last integral converges uniformly at each limit for all sufficiently large t, then the Riemann-Lebesgue lemma (12.2.3) shows that

$$f(t) = g(t) + o(e^{ct}) \qquad (t \to \infty)$$

If, in addition, the corresponding integrals with F and G replaced by their derivatives $F^{(j)}$ and $G^{(j)}$, $j = 1, 2, \ldots, m$, converge uniformly, then by repeated integrations by parts and use again of the Riemann-Lebesgue lemma, we derive

$$f(t) = g(t) + o(t^{-m} e^{ct}) \qquad (t \to \infty)$$

This method for approximating a given function $f(t)$ is due to Haar, and is analogous to Darboux's method for sequences. The best results are obtained by translating the integration contour to the left to make the value of c as small as possible.

Example: Bessel functions of large argument—For $t > 0$ and $\nu > -\frac{1}{2}$, the Bessel function $J_\nu(t)$ is representable by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{tp}\,dp}{(p^2+1)^{\nu+(1/2)}} \qquad (c > 0)$$

where $f(t) = \pi^{1/2}(\frac{1}{2}t)^\nu J_\nu(t)/\Gamma(\nu + \frac{1}{2})$, and $(p^2+1)^{\nu+(1/2)}$ has its principal value. To approximate $f(t)$ for large t we deform the path into two loop integrals*

$$\int_{-\infty+i}^{(i+)} + \int_{-\infty-i}^{(-i+)}$$

In the first of these the factor $(p^2+1)^{-\nu-(1/2)}$ is replaced by its expansion in ascending powers of $p - i$. Then using Hankel's integral for the reciprocal of the Gamma function (7.2-9), we derive

$$\frac{1}{2\pi i}\int_{-\infty+i}^{(i+)} \frac{e^{tp}\,dp}{(p^2+1)^{\nu+(1/2)}}$$

$$= \frac{1}{2^{\nu+(1/2)}e^{(2\nu+1)\pi i/4}} \sum_{s=0}^{n-1} \binom{-\nu-\frac{1}{2}}{s} \frac{1}{(2i)^s} \frac{e^{it}}{\Gamma(\nu+\frac{1}{2}-s)\,t^{s-\nu+(1/2)}} + \epsilon_n(t)$$

Here n is an arbitrary integer, and

$$\epsilon_n(t) = \frac{1}{2\pi i}\int_{-\infty+i}^{(i+)} e^{tp}\, O\{(p-i)^{n-\nu-(1/2)}\}\,dp$$

the O-term being uniform on the loop path.

*The notation $\int_a^{(b+)}$ means that the integration path begins at a, encircles the singularity at b once in the positive sense, and returns to its starting point without encircling any other singularity of the integrand.

If we restrict $n > \nu - \frac{1}{2}$, then the path in the last integral may be collapsed onto the two sides of the cut through $p = i$ parallel to the negative real axis. Thence it follows that $\epsilon_n(t)$ is $O(1/t^{n-\nu+(1/2)})$ as $t \to \infty$. Similar analysis applies to the other loop integral, and combination of the results gives the required expansion

$$J_\nu(t) = \left(\frac{2}{\pi t}\right)^{1/2} \sum_{s=0}^{n-1} \binom{-\nu - \frac{1}{2}}{s} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2} - s)} \frac{\cos\{t - (\frac{1}{2}s + \frac{1}{2}\nu + \frac{1}{4})\pi\}}{(2t)^s} + O\left(\frac{1}{t^{n+(1/2)}}\right)$$

It can be verified that this agrees with the expansion of 7.2.9.

From the standpoint of Haar, the role of $F(p)$ is played here by $(p^2 + 1)^{-\nu-(1/2)}$ and that of $G(p)$ by

$$\sum_{s=0}^{n-1} \frac{1}{2^{\nu+(1/2)}} \binom{-\nu - \frac{1}{2}}{s} \left\{ \frac{(p - i)^{s-\nu-(1/2)}}{e^{(2\nu+1)\pi i/4}(2i)^s} + \frac{(p + i)^{s-\nu-(1/2)}}{e^{-(2\nu+1)\pi i/4}(-2i)^s} \right\}$$

12.6 THE LIOUVILLE-GREEN (OR WKB) APPROXIMATION

12.6.1 The Liouville Transformation

Let

$$\frac{d^2 w}{dx^2} = f(x) w$$

be a given differential equation, and $\xi(x)$ any thrice-differentiable function. On transforming to ξ as independent variable and setting

$$W = \left(\frac{d\xi}{dx}\right)^{1/2} w$$

we find that

$$\frac{d^2 W}{d\xi^2} = \left[\dot{x}^2 f(x) - \frac{1}{2}\{x, \xi\}\right] W$$

Here the dot signifies differentiation with respect to ξ, and $\{x, \xi\}$ is the *Schwarzian derivative*

$$\{x, \xi\} = -2\dot{x}^{1/2} \frac{d^2}{d\xi^2}(\dot{x}^{-1/2}) = \frac{\dddot{x}}{\dot{x}} - \frac{3}{2}\left(\frac{\ddot{x}}{\dot{x}}\right)^2$$

properties of which include

$$\{x, \xi\} = -\left(\frac{dx}{d\xi}\right)^2 \{\xi, x\}; \quad \{x, \zeta\} = \left(\frac{d\xi}{d\zeta}\right)^2 \{x, \xi\} + \{\xi, \zeta\}$$

The foregoing change of variables is called the *Liouville transformation*. If we now prescribe

$$\xi = \int f^{1/2}(x)\, dx$$

then $\dot{x}^2 f(x) = 1$

$$\frac{d^2 W}{d\xi^2} = \left[1 - \frac{1}{2}\{x, \xi\}\right] W$$

and

$$\{x, \xi\} = \frac{5f'^2(x) - 4f(x)f''(x)}{8f^3(x)} = \frac{2}{f^{3/4}} \frac{d^2}{dx^2}\left(\frac{1}{f^{1/4}}\right)$$

Neglect of the Schwarzian enables the equation in W and ξ to be solved exactly, and this leads to the following general solution of the original differential equation:

$$Af^{-1/4}(x) \exp\left\{\int f^{1/2}(x)\, dx\right\} + Bf^{-1/4}(x) \exp\left\{-\int f^{1/2}(x)\, dx\right\}$$

where A and B are arbitrary constants. This is the *Liouville-Green* or *LG approximation*, also known as the *WKB approximation*. The expressions

$$f^{-1/4} \exp\left(\pm \int f^{1/2}\, dx\right)$$

are called the *LG functions*.

In a wide range of circumstances, described in following subsections, neglect of the Schwarzian is justified and the LG approximation accurate. An important case of failure is immediately noticeable, however. At a zero of $f(x)$ the Schwarzian is infinite, rendering the LG approximation meaningless. Zeros of $f(x)$ are called *transition points* or *turning points* of the differential equation. The reason for the names is that on passing through a zero of $f(x)$ on the real axis, the character of each solution changes from oscillatory to monotonic (or vice-versa). Satisfactory approximations cannot be constructed in terms of elementary functions in the neighborhood of a transition point; see section 12.8 below.

12.6.2 Error Bounds: Real Variables

In stating error bounds for the LG approximation, it is convenient to take the differential equation in the form

$$\frac{d^2w}{dx^2} = \{f(x) + g(x)\}\, w$$

It is assumed that in a given finite or infinite interval (a_1, a_2), $f(x)$ is a positive, real, twice-continuously differentiable function, and $g(x)$ is a continuous, real or complex function. Then the equation has twice-continuously differentiable solutions

$$w_1(x) = f^{-1/4}(x) \exp\left\{\int f^{1/2}(x)\,dx\right\} \{1 + \epsilon_1(x)\}$$

$$w_2(x) = f^{-1/4}(x) \exp\left\{-\int f^{1/2}(x)\,dx\right\} \{1 + \epsilon_2(x)\}$$

with the error terms bounded by

$$|\epsilon_j(x)|,\ \tfrac{1}{2} f^{-1/2}(x)\, |\epsilon_j'(x)| \leq \exp\{\tfrac{1}{2}\mathcal{V}_{a_j,x}(F)\} - 1 \qquad (j = 1, 2)$$

Here $\mathcal{V}$ denotes the variational operator defined in 12.2.4, and $F(x)$ is the *error-control function*

$$F(x) = \int \left\{\frac{1}{f^{1/4}} \frac{d^2}{dx^2}\left(\frac{1}{f^{1/4}}\right) - \frac{g}{f^{1/2}}\right\} dx$$

The foregoing result applies whenever the $\mathcal{V}_{a_j,x}(F)$ are finite.

A similar result is available for differential equations with solutions of oscillatory type. With exactly the same conditions, the equation

$$\frac{d^2w}{dx^2} = \{-f(x) + g(x)\}\, w$$

has twice-continuously differentiable solutions

$$w_1(x) = f^{-1/4}(x) \exp\left\{i\int f^{1/2}(x)\,dx\right\} \{1 + \epsilon_1(x)\}$$

$$w_2(x) = f^{-1/4}(x) \exp\left\{-i \int f^{1/2}(x)\,dx\right\} \{1 + \epsilon_2(x)\}$$

such that

$$|\epsilon_j(x)|,\ f^{-1/2}(x)\,|\epsilon_j'(x)| \leqslant \exp\{\mathcal{V}_{a,x}(F)\} - 1 \qquad (j = 1, 2)$$

Here a is an arbitrary point in the closure of (a_1, a_2)—possibly at infinity—and the solutions $w_1(x)$ and $w_2(x)$ depend on a. When $g(x)$ is real, $w_1(x)$ and $w_2(x)$ are complex conjugates.

12.6.3 Asymptotic Properties with Respect to the Independent Variable

We return to the equation

$$\frac{d^2w}{dx^2} = \{f(x) + g(x)\}\, w$$

The error bounds of 12.6.2 immediately show that

$$w_1(x) \sim f^{-1/4} \exp\left(\int f^{1/2}\,dx\right) \qquad (x \to a_1+)$$

$$w_2(x) \sim f^{-1/4} \exp\left(-\int f^{1/2}\,dx\right) \qquad (x \to a_2-)$$

These results are valid whether or not a_1 and a_2 are finite, and also whether or not f and $|g|$ are bounded at a_1 and a_2. All that is required is that the error-control function $F(x)$ be of bounded variation in (a_1, a_2).

A somewhat deeper result, not immediately deducible from the results of 12.6.2, is that when $\left|\int f^{1/2}\,dx\right| \to \infty$ as $x \to a_1$ or a_2, there exist solutions $w_3(x)$ and $w_4(x)$ with the complementary properties

$$w_3(x) \sim f^{-1/4} \exp\left(\int f^{1/2}\,dx\right) \qquad (x \to a_2-)$$

$$w_4(x) \sim f^{-1/4} \exp\left(-\int f^{1/2}\,dx\right) \qquad (x \to a_1+)$$

The solutions $w_1(x)$ and $w_2(x)$ are unique, but not $w_3(x)$ and $w_4(x)$. At a_1, $w_1(x)$ is said to be *recessive* (or *subdominant*), whereas $w_4(x)$ is *dominant*. Similarly for $w_2(x)$ and $w_3(x)$ at a_2.

Example: Consider the equation

$$\frac{d^2w}{dx^2} = (x + \ln x)w$$

for large positive values of x. We cannot take $f = x$ and $g = \ln x$ because $\int gf^{-1/2}\,dx$ would diverge at infinity. Instead, set $f = x + \ln x$ and $g = 0$. Then for large x, $f^{-1/4}(f^{-1/4})''$ is $O(x^{-5/2})$, consequently $\mathcal{V}(F)$ converges. Accordingly, there is a unique solution $w_2(x)$ such that

$$w_2(x) \sim (x + \ln x)^{-1/4} \exp\left\{-\int (x + \ln x)^{1/2}\,dx\right\} \qquad (x \to \infty)$$

and a nonunique solution $w_3(x)$ such that

$$w_3(x) \sim (x + \ln x)^{-1/4} \exp\left\{\int (x + \ln x)^{1/2}\,dx\right\} \qquad (x \to \infty)$$

These asymptotic forms are simplifiable by expansion and integration; thus

$$w_2(x) \sim x^{-(1/4)-\sqrt{x}} \exp\left(2x^{1/2} - \tfrac{2}{3}x^{3/2}\right); \quad w_3(x) \sim x^{-(1/4)+\sqrt{x}} \exp\left(\tfrac{2}{3}x^{3/2} - 2x^{1/2}\right)$$

12.6.4 Convergence of $\mathcal{V}(F)$ at a Singularity

Sufficient conditions for the variation of the error-control function to be bounded at a finite point a_2 are given by

$$f(x) \sim \frac{c}{(a_2 - x)^{2\alpha+2}}; \quad g(x) = O\left\{\frac{1}{(a_2 - x)^{\alpha-\beta+2}}\right\} \quad (x \to a_2-)$$

provided that c, α, and β are positive constants and the first relation is twice differentiable.

Similarly, when $a_2 = \infty$ sufficient conditions for $\mathcal{V}(F)$ to be bounded are

$$f(x) \sim cx^{2\alpha-2}; \quad g(x) = O(x^{\alpha-\beta-2}) \quad (x \to \infty)$$

again provided that c, α, and β are positive and the first relation is twice differentiable. When $\alpha = \frac{3}{2}$ we interpret the last condition as $f'(x) \to c$ and $f''(x) = O(x^{-1})$; when $\alpha = 1$ we require $f'(x) = O(x^{-1})$ and $f''(x) = O(x^{-2})$.

12.6.5 Asymptotic Properties with Respect to Parameters

Consider the equation

$$\frac{d^2w}{dx^2} = \{u^2 f(x) + g(x)\} w$$

in which u is a large positive parameter. If we again suppose that in a given interval (a_1, a_2) the function $f(x)$ is positive and $f''(x)$ and $g(x)$ are continuous, then the result of 12.6.2 may be applied with $u^2 f(x)$ playing the role of the previous $f(x)$. On discarding an irrelevant factor $u^{-1/2}$ it is seen that the new differential equation has solutions

$$w_j(u, x) = f^{-1/4}(x) \exp\left\{(-)^{j-1} u \int f^{1/2}(x)\, dx\right\} \{1 + \epsilon_j(u, x)\} \qquad (j = 1, 2)$$

where

$$|\epsilon_j(u, x)|, \ \frac{1}{2uf^{1/2}(x)} \left|\frac{\partial \epsilon_j(u, x)}{\partial x}\right| \leqslant \exp\left\{\frac{\mathcal{V}_{a_j, x}(F)}{2u}\right\} - 1$$

the function $F(x)$ being defined exactly as before. Since $F(x)$ is independent of u, the error bound is $O(u^{-1})$ for large u and fixed x. Moreover, if $F(x)$ is of bounded variation in (a_1, a_2), then the error bound is $O(u^{-1})$ uniformly with respect to x in (a_1, a_2). The differential equation may have a singularity at either endpoint without invalidating this conclusion as long as $\mathcal{V}(F)$ is bounded at a_1 and a_2.

Thus the LG functions represent asymptotic solutions in the neighborhood of a singularity (as in 12.6.3), and uniform asymptotic solutions for large values of a parameter. This *double* asymptotic property makes the LG approximation a remarkably powerful tool for approximating solutions of linear second-order differential equations.

Example: Parabolic cylinder functions of large order—The parabolic cylinder functions satisfy the equation

$$\frac{d^2w}{dx^2} = \left(\frac{1}{4}x^2 + a\right) w$$

a being a parameter. In the notation of 12.6.2, we take $f(x) = \frac{1}{4}x^2 + a$ and $g(x) = 0$. Referring to 12.6.4, we see that $\mathcal{V}(F)$ is finite at $x = +\infty$. Hence there exist solutions which are asymptotic to $f^{-1/4}e^{\pm\xi}$ for large x, where

$$\xi = \int \left(\frac{1}{4}x^2 + a\right)^{1/2} dx$$

On expansion and integration, we find that

$$\xi = \tfrac{1}{4}x^2 + a \ln x + \text{constant} + O(x^{-2}) \qquad (x \to \infty)$$

Hence the asymptotic forms of the solutions reduce to constant multiples of $x^{a-(1/2)}e^{x^2/4}$ and $x^{-a-(1/2)}e^{-x^2/4}$. The principal solution $U(a, x)$ is specified (uniquely) by the condition

$$U(a, x) \sim x^{-a-(1/2)}e^{-x^2/4} \qquad (x \to \infty)$$

How does $U(a, x)$ behave as $a \to +\infty$? Making the transformations $a = \frac{1}{2}u$ and $x = (2u)^{1/2}t$, we obtain

$$\frac{d^2w}{dt^2} = u^2(t^2 + 1)w$$

A solution of this equation which is recessive at infinity is given by

$$w(u, t) = (t^2 + 1)^{-1/4}e^{-u\hat{\xi}(t)}\{1 + \epsilon(u, t)\}$$

where

$$\hat{\xi}(t) = \int (t^2 + 1)^{1/2}\, dt = \frac{1}{2}t(t^2 + 1)^{1/2} + \frac{1}{2}\ln\{t + (t^2 + 1)^{1/2}\}$$

The error term is bounded by

$$|\epsilon(u, t)| \leqslant \exp\{\mathcal{V}_{t,\infty}(F)/(2u)\} - 1$$

with

$$F(t) = \int (t^2 + 1)^{-1/4}\{(t^2 + 1)^{-1/4}\}''\, dt = -\frac{t^3 + 6t}{12(t^2 + 1)^{3/2}}$$

The solutions $w(u, t)$ and $U(\frac{1}{2}u, \sqrt{2u}\,t)$ must be in constant ratio as t varies, since both are recessive at infinity. The value of the ratio may be found by comparing the asymptotic forms at $t = +\infty$. Thus we arrive at the required approximation:

$$U(\tfrac{1}{2}u, \sqrt{2u}\,t) = 2^{(u-1)/4}e^{u/4}u^{-(u+1)/4}(t^2 + 1)^{-1/4}e^{-u\hat{\xi}(t)}\{1 + \epsilon(u, t)\}$$

This result holds for positive u and all real values of t, or, on returning to the original variables, positive a and all real x. For fixed u (not necessarily large) and large positive t, we have $\epsilon(u, t) = O(t^{-2})$. On the other hand, since $\mathcal{V}_{-\infty,\infty}(F) < \infty$ we have $\epsilon(u, t) = O(u^{-1})$ for large u, uniformly with respect to $t \in (-\infty, \infty)$. These estimates illustrate the doubly asymptotic nature of the LG approximation.

Incidentally, the result of the example in 12.4.3 is obtainable from the present more general result by setting $u = 2n + 1$, $t = y/\sqrt{4n + 2}$ and expanding $\hat{\xi}(t)$ for small t.

12.6.6 Error Bounds: Complex Variables

Let $f(z)$ and $g(z)$ be holomorphic in a complex domain **D** in which $f(z)$ is non-vanishing. Then the differential equation

$$\frac{d^2 w}{dz^2} = \{f(z) + g(z)\} w$$

has solutions which are holomorphic in **D**, depend on arbitrary (possibly infinite) reference points a_1 and a_2, and are given by

$$w_j(z) = f^{-1/4}(z) \exp \{(-)^{j-1} \xi(z)\} \{1 + \epsilon_j(z)\} \qquad (j = 1, 2)$$

where

$$\xi(z) = \int f^{1/2}(z)\, dz$$

and

$$|\epsilon_j(z)|, \quad |f^{-1/2}(z)\epsilon_j'(z)| \leqslant \exp \{\mathcal{V}_{a_j,z}(F)\} - 1$$

Here $F(z)$ is defined as in 12.6.2, with $x = z$.

In contrast to the case of real variables, the present error bounds apply only to subregions $\mathbf{H}_j(a_j)$ of **D**. These subregions comprise the points z for which there exists a path $\mathcal{P}_j$ in **D** linking a_j with z, and along which $\operatorname{Re}\{\xi(z)\}$ is nondecreasing ($j = 1$) or nonincreasing ($j = 2$). Such a path is called ξ-*progressive*. In the bound $\exp\{\mathcal{V}_{a_j,z}(F)\} - 1$ the variation of $F(z)$ has to be evaluated along a ξ-progressive path. Parts of **D** excluded from $\mathbf{H}_j(a_j)$ are called *shadow zones*. The solutions $w_j(z)$ exist and are holomorphic in the shadow zones, but the error bounds do not apply there.

Asymptotic properties of the approximation with respect to z in the neighborhood of a singularity, or with respect to large values of a real or complex parameter, carry over straightforwardly from the case of real variables.

12.7 DIFFERENTIAL EQUATIONS WITH IRREGULAR SINGULARITIES

12.7.1 Classification of Singularities

Consider the differential equation

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)\,w = 0$$

in which the functions $f(z)$ and $g(z)$ are holomorphic in a region which includes the punctured disc $0 < |z - z_0| < a$, z_0 and a being given finite numbers.

If both $f(z)$ and $g(z)$ are analytic at z_0, then z_0 is said to be an *ordinary point* of the differential equation. In this event all solutions are holomorphic in the disc $|z - z_0| < a$.

If z_0 is not an ordinary point, but both $(z - z_0)f(z)$ and $(z - z_0)^2 g(z)$ are analytic at z_0, then this point is said to be a *regular singularity* or *singularity of the first kind.* In this case independent solutions can be constructed in series involving fractional powers of $z - z_0$ and also, possibly, $\ln(z - z_0)$. The series converge when $|z - z_0| < a$; compare 6.4.6.

Lastly, if z_0 is neither an ordinary point nor a regular singularity, then it is said to be an *irregular singularity*, or *singularity of the second kind.* If an integer r exists such that $(z - z_0)^{r+1} f(z)$ and $(z - z_0)^{2r+2} g(z)$ are both analytic at z_0, then the least value of r is said to be the *rank* of the singularity. By analogy, a regular singularity is sometimes said to have zero rank.

In the neighborhood of an irregular singularity it is usually impossible to find convergent series expansions for the solutions in terms of elementary functions. Instead, asymptotic expansions are employed. From section 12.6, especially 12.6.4,* it can be seen that the LG functions furnish asymptotic approximations at an irregular singularity. The purpose of the present section is to extend these approximations into asymptotic expansions for singularities of finite rank. We begin with the simplest and commonest case in applications.

12.7.2 Singularities of Unit Rank

Without loss of generality the singularity is assumed to be at infinity: a finite singularity z_0 can always be projected to infinity by taking $(z - z_0)^{-1}$ as new independent variable. Thus we consider the differential equation of 12.7.1 with

$$f(z) = \sum_{s=0}^{\infty} \frac{f_s}{z^s}; \quad g(z) = \sum_{s=0}^{\infty} \frac{g_s}{z^s}$$

these series converging for sufficiently large $|z|$. Not all of the coefficients f_0, g_0, and g_1 vanish, otherwise the singularity would be regular.

*The symbols f and g are now being used differently.

Formal series solutions in descending powers of z can be constructed in the form

$$w = e^{\lambda z} z^{\mu} \sum_{s=0}^{\infty} \frac{a_s}{z^s}$$

Substituting in the differential equation and equating coefficients, we obtain in turn

$$\lambda^2 + f_0 \lambda + g_0 = 0$$

$$(f_0 + 2\lambda)\mu = -(f_1 \lambda + g_1)$$

and

$$(f_0 + 2\lambda) s a_s = (s - \mu)(s - 1 - \mu) a_{s-1} + \{\lambda f_2 + g_2 - (s - 1 - \mu) f_1\} a_{s-1}$$
$$+ \{\lambda f_3 + g_3 - (s - 2 - \mu) f_2\} a_{s-2} + \cdots + \{\lambda f_{s+1} + g_{s+1} + \mu f_s\} a_0$$

The first of these equations yields two possible values

$$\lambda_1, \lambda_2 = -\tfrac{1}{2} f_0 \pm (\tfrac{1}{4} f_0^2 - g_0)^{1/2}$$

for λ, called the *characteristic values.* The next equation determines the corresponding values μ_1 and μ_2, of μ. Then the values of a_0, say $a_{0,1}$ and $a_{0,2}$ in the two cases, may be assigned arbitrarily and higher coefficients $a_{s,1}$ and $a_{s,2}$, $s = 1, 2, \ldots$, determined recursively. The process fails if, and only if, $\lambda_1 = \lambda_2$, that is, $f_0^2 = 4g_0$. This case is treated below.

In general the formal series diverge. Corresponding to each, however, there is a unique solution $w_j(z), j = 1, 2$, of the differential equation with the property

$$w_j(z) \sim e^{\lambda_j z} z^{\mu_j} \sum_{s=0}^{\infty} \frac{a_{s,j}}{z^s}$$

as $z \to \infty$ in the sector

$$|\arg\{(\lambda_2 - \lambda_1)z\}| \leqslant \tfrac{3}{2}\pi - \delta \quad (j = 1); \quad |\arg\{(\lambda_1 - \lambda_2)z\}| \leqslant \tfrac{3}{2}\pi - \delta \quad (j = 2)$$

δ being an arbitrary positive constant.

In the sector $|\arg\{(\lambda_2 - \lambda_1)z\}| < \pi/2$, the solution $w_1(z)$ is recessive, and the two branches of $w_2(z)$ are dominant. These roles are interchanged in $|\arg\{(\lambda_1 - \lambda_2)z\}| < \pi/2$. Although both $w_1(z)$ and $w_2(z)$ can be continued analytically to any range of $\arg z$, the given sectors of validity of the asymptotic expansions are maximal (unless the expansions happen to converge).

The case in which the characteristic values λ_1 and λ_2 are equal can be handled by the preliminary transformation

$$w = e^{-f_0 z/2} W; \quad t = \sqrt{z}$$

In the new equation either the singularity at $t = \infty$ is regular, or it is irregular with unequal characteristic values (and therefore amenable to the foregoing analysis).

Example: Bessel's equation—Bessel functions of real or complex order ν satisfy

$$\frac{d^2 w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) w = 0$$

In the present notation $f_1 = g_0 = 1$, $g_2 = -\nu^2$, and all other coefficients vanish. The equations for λ and μ yield $\lambda_1 = -\lambda_2 = i$, and $\mu_1 = \mu_2 = -\frac{1}{2}$. With $a_{0,1} = a_{0,2} = 1$ it is found that

$$a_{s,1} = i^s A_s(\nu); \quad a_{s,2} = (-i)^s A_s(\nu)$$

where

$$A_s(\nu) = (4\nu^2 - 1^2)(4\nu^2 - 3^2) \cdots \{4\nu^2 - (2s - 1)^2\}/(s!\, 8^s)$$

Renormalizing $w_1(z)$ and $w_2(z)$ by the factors $(2/\pi)^{1/2} \exp\{\mp(\frac{1}{2}\nu + \frac{1}{4})\pi i\}$, we obtain solutions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ with the properties

$$H_\nu^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i\zeta} \sum_{s=0}^{\infty} i^s \frac{A_s(\nu)}{z^s} \qquad (z \to \infty \text{ in } -\pi + \delta \leqslant \arg z \leqslant 2\pi - \delta)$$

$$H_\nu^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-i\zeta} \sum_{s=0}^{\infty} (-i)^s \frac{A_s(\nu)}{z^s} \qquad (z \to \infty \text{ in } -2\pi + \delta \leqslant \arg z \leqslant \pi - \delta)$$

where δ is an arbitrary positive constant, and $\zeta = z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi$. These are the Hankel functions of order ν; $H_\nu^{(1)}(z)$ is recessive at infinity in the upper half-plane, or, more precisely when $0 < \arg z < \pi$; $H_\nu^{(2)}(z)$ is recessive when $-\pi < \arg z < 0$.

12.7.3 Stokes' Phenomenon

The functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ introduced in the last example can be continued analytically to any range of $\arg z$. Appropriate asymptotic expansions can be constructed by means of the continuation formulas

$$H_\nu^{(1)}(ze^{m\pi i}) = -\operatorname{cosec}(\nu\pi)\,[\sin\{(m-1)\nu\pi\}\, H_\nu^{(1)}(z) + e^{-\nu\pi i}\sin(m\nu\pi)\, H_\nu^{(2)}(z)]$$

$$H_\nu^{(2)}(ze^{m\pi i}) = \operatorname{cosec}(\nu\pi)\,[e^{\nu\pi i}\sin(m\nu\pi)\, H_\nu^{(1)}(z) + \sin\{(m+1)\nu\pi\}\, H_\nu^{(2)}(z)]$$

in which m is an integer. For example, if we take $m = 2$ in the first formula, substitute in the right-hand side by means of the expansions of 12.7.2, and then replace z by $ze^{-2\pi i}$, we arrive at

$$H_\nu^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \left\{ e^{i\zeta} \sum_{s=0}^{\infty} i^s \frac{A_s(\nu)}{z^s} + (1 + e^{-2\nu\pi i})\, e^{-i\zeta} \sum_{s=0}^{\infty} (-i)^s \frac{A_s(\nu)}{z^s} \right\}$$

valid when $z \to \infty$ in $\pi + \delta \leqslant \arg z \leqslant 3\pi - \delta$.

It will be observed that two different asymptotic expansions are available for $H_\nu^{(1)}(z)$ in the sector $\pi + \delta \leqslant \arg z \leqslant 2\pi - \delta$, namely, the expansion just given and the original expansion of 12.7.2. In this sector, however, $e^{-i\zeta}$ is exponentially small at infinity compared with $e^{i\zeta}$, hence the *whole* contribution of the series it multiplies is absorbable, in Poincaré's sense, in any of the error terms associated with the first series.* Accordingly, there is no inconsistency.

Extensions to other phase ranges may be found in the same manner by taking other values of m. In each case an expansion of the form

$$\left(\frac{2}{\pi z}\right)^{1/2} \left\{ \alpha_m(\nu)\, e^{i\zeta} \sum_{s=0}^{\infty} i^s \frac{A_s(\nu)}{z^s} + \beta_m(\nu)\, e^{-i\zeta} \sum_{s=0}^{\infty} (-i)^s \frac{A_s(\nu)}{z^s} \right\}$$

is obtained, where $(m - 1)\pi + \delta \leqslant \arg z \leqslant (m + 1)\pi - \delta$, and $\alpha_m(\nu)$ and $\beta_m(\nu)$ are independent of z. The need for *discontinuous* changes in these coefficients as $\arg z$ is continuously increased (or decreased) is called the *Stokes phenomenon.* It is not confined to solutions of Bessel's differential equation.

12.7.4** Singularities of Higher Rank

When the differential equation of 12.7.1 has a singularity at infinity of rank $k + 1$, the coefficients $f(z)$ and $g(z)$ can be expanded in series

$$f(z) = z^k \sum_{s=0}^{\infty} \frac{f_s}{z^s}; \quad g(z) = z^{2k} \sum_{s=0}^{\infty} \frac{g_s}{z^s}$$

which converge for large $|z|$, at least one of f_0, g_0, and g_1 being nonzero.

Provided that $f_0^2 \neq 4g_0$, formal series solutions of the form

$$e^{\xi_j(z)} \sum_{s=0}^{\infty} \frac{a_{s,j}}{z^s} \qquad (j = 1, 2)$$

*For numerical purposes, however, the second series should be retained when $\frac{3}{2}\pi \leqslant \arg z \leqslant 2\pi$.

**Proofs of results in this subsection are given in Olver and Stenger 1965.

can be constructed. Here

$$\xi_j(z) = z^{k+1} \sum_{s=0}^{k} \frac{(-)^{j-1}\phi_s - \frac{1}{2} f_s}{(k+1-s)\, z^s} + \{(-)^{j-1}\phi_{k+1} - \tfrac{1}{2} f_{k+1} - \tfrac{1}{2} k\} \ln z$$

the coefficients ϕ_s being defined by the expansion

$$\{\tfrac{1}{4} f^2(z) + \tfrac{1}{2} f'(z) - g(z)\}^{1/2} = z^k \sum_{s=0}^{\infty} \frac{\phi_s}{z^s}$$

The other coefficients $a_{s,j}$ may be calculated recursively by substituting in the differential equation and equating coefficients, the value of $a_{0,j}$ being arbitrary.

Let the z-plane be divided into $2k+2$ sectors of equal angle, as indicated in Figure 12.7-1 in the case $k = 3$. In any closed sector lying properly within* the union

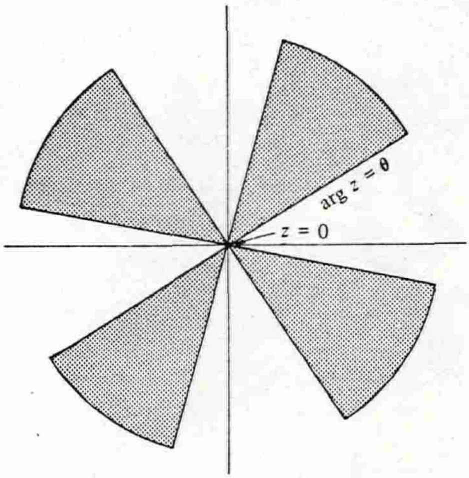

Fig. 12.7-1 z-plane, $k = 3$: $\theta = (\frac{1}{2}\pi - \arg \phi_0)/(k+1)$.

of a shaded sector and the two adjacent unshaded sectors there is a solution of the differential equation having the formal series, with $j = 1$, as its asymptotic expansion. This solution is recessive in the shaded sector and dominant in the abutting sectors. Similarly in any closed sector within the union of an unshaded sector and the two adjacent shaded sectors there is a solution having the formal series, with $j = 2$, as its asymptotic expansion, and this solution is recessive in the unshaded sector and dominant in the abutting sectors.

Again, the exceptional case $f_0^2 = 4g_0$ is amenable to an appropriate preliminary transformation of independent and dependent variables.

*Except for the common vertex.

12.8 DIFFERENTIAL EQUATIONS WITH A PARAMETER

12.8.1 Classification and Preliminary Transformations

In this section we discuss asymptotic solutions of differential equations of the form

$$\frac{d^2w}{dz^2} = \{u^2 f(z) + g(z)\}\, w$$

in which u is a large real or complex parameter, and z ranges over a real interval or complex domain **D**, say. Equations of this type occur frequently in mathematical physics. The form of the asymptotic solutions depends on the nature of the *transition points* in **D**, that is, points at which $f(z)$ or $g(z)$ is singular, or $f(z)$ vanishes; compare 12.6.1.

In the case in which **D** is free from transition points–which we shall call henceforth *Case I*–it was shown in section 12.6 that the LG functions

$$f^{-1/4}(z) \exp\left\{\pm u \int f^{1/2}(z)\, dz\right\}$$

furnish asymptotic solutions with uniform relative error $O(u^{-1})$ as $|u| \to \infty$. In 12.8.2 we construct asymptotic expansions in descending powers of u, the initial terms of which are the LG approximations.

Later subsections treat cases in which **D** contains a single transition point z_0, say. If, at z_0, $f(z)$ has a pole of order $m \geqslant 2$ and $g(z)$ is either analytic or has a pole of order less than $\frac{1}{2} m + 1$, then the LG approximation or the expansions of 12.8.2 may be used. In *Case II*, treated in 12.8.4, z_0 is a simple zero of $f(z)$ and an analytic point of $g(z)$; in *Case III*, treated in 12.8.6, z_0 is a simple pole of $f(z)$ and $(z - z_0)^2 g(z)$ is analytic there.

Basically the same approach is made in all cases. First, the Liouville transformation (12.6.1) is applied. This introduces new variables W and ξ, related by

$$W = \dot{z}^{-1/2} w$$

the dot denoting differentiation with respect to ξ. Then

$$\frac{d^2W}{d\xi^2} = \{u^2 \dot{z}^2 f(z) + \psi(\xi)\}\, W$$

where

$$\psi(\xi) = \dot{z}^2 g(z) + \dot{z}^{1/2} \frac{d^2}{d\xi^2}(\dot{z}^{-1/2})$$

The transformation is now prescribed in such a way that: (i) ξ and z are analytic functions of each other at the transition point (if any); (ii) the approximating differential equation obtained by neglect of $\psi(\xi)$, or part of $\psi(\xi)$, has solutions which are functions of a single variable. The actual prescriptions are as follows:

Case I:

$$\dot{z}^2 f(z) = 1, \quad \text{giving} \quad \xi = \int f^{1/2}(z)\,dz$$

Case II:

$$\dot{z}^2 f(z) = \xi, \quad \text{giving} \quad \frac{2}{3}\xi^{3/2} = \int_{z_0}^{z} f^{1/2}(t)\,dt$$

Case III:

$$\dot{z}^2 f(z) = \frac{1}{\xi}, \quad \text{giving} \quad 2\xi^{1/2} = \int_{z_0}^{z} f^{1/2}(t)\,dt$$

The transformed differential equation becomes

$$\frac{d^2 W}{d\xi^2} = \{u^2 \xi^m + \psi(\xi)\}\, W$$

with $m = 0$ (Case I), $m = 1$ (Case II), or $m = -1$ (Case III).

In Cases I and II approximate solutions of the new equation are obtained by neglecting $\psi(\xi)$. In Case I this is the LG approximating procedure used in section 12.6. In Case II the approximants are Airy functions. In Case III the basic approximating equation is

$$\frac{d^2 W}{d\xi^2} = \left(\frac{u^2}{\xi} + \frac{\rho}{\xi^2}\right) W$$

where ρ is the value of $\xi^2 \psi(\xi)$ at $\xi = 0$. The solutions are expressible in terms of modified Bessel functions of order $\pm\sqrt{1 + 4\rho}$ and argument $2u\sqrt{\xi}$.

12.8.2 Case I: No Transition Points

The standard form of differential equation is given by

$$\frac{d^2 W}{d\xi^2} = \{u^2 + \psi(\xi)\}\, W$$

The variable ξ ranges over a bounded or unbounded complex domain Δ, being the map of the original z-domain **D**. The function $\psi(\xi)$ is holomorphic in Δ.

A formal series solution can be constructed in the form

$$W = e^{u\xi} \sum_{s=0}^{\infty} \frac{A_s(\xi)}{u^s}$$

This gives

$$\frac{dW}{d\xi} = ue^{u\xi} \sum_{s=0}^{\infty} \frac{A_s(\xi) + A'_{s-1}(\xi)}{u^s}; \quad \frac{d^2W}{d\xi^2} = u^2 e^{u\xi} \sum_{s=0}^{\infty} \frac{A_s(\xi) + 2A'_{s-1}(\xi) + A''_{s-2}(\xi)}{u^s}$$

Satisfaction of the given differential equation requires

$$2A'_s(\xi) = -A''_{s-1}(\xi) + \psi(\xi)A_{s-1}(\xi) \qquad (s = 0, 1, \ldots)$$

Thus $A_0(\xi)$ = constant, which we take to be unity without loss of generality, and higher coefficients are found recursively by

$$A_{s+1}(\xi) = -\frac{1}{2}A'_s(\xi) + \frac{1}{2}\int \psi(\xi)A_s(\xi)\,d\xi \qquad (s = 0, 1, \ldots)$$

the constants of integration being arbitrary. Each coefficient $A_s(\xi)$ is holomorphic in Δ.

A second formal solution is obtainable by replacing u by $-u$ throughout. In general both formal series diverge. However, corresponding to any positive integer n there exist solutions $W_{n,j}(u, \xi)$, $j = 1, 2$, which are holomorphic in Δ, depend on arbitrary reference points α_j, and are given by

$$W_{n,1}(u, \xi) = e^{u\xi} \sum_{s=0}^{n-1} \frac{A_s(\xi)}{u^s} + \epsilon_{n,1}(u, \xi)$$

$$W_{n,2}(u, \xi) = e^{-u\xi} \sum_{s=0}^{n-1} (-)^s \frac{A_s(\xi)}{u^s} + \epsilon_{n,2}(u, \xi)$$

where the error terms are bounded by

$$|\epsilon_{n,j}(u, \xi)|, \quad \left|\frac{\partial \epsilon_{n,j}(u, \xi)}{u\partial\xi}\right| \leqslant 2\left|e^{(-)^{j-1}u\xi}\right| \exp\left\{\frac{2\mathcal{V}_{\alpha_j,\xi}(A_1)}{|u|}\right\} \frac{\mathcal{V}_{\alpha_j,\xi}(A_n)}{|u|^n}$$

These bounds apply at each point ξ of Δ which can be linked to α_j by a path $\mathcal{Q}_j$ lying in Δ and such that $\mathrm{Re}(uv)$ is nondecreasing ($j = 1$) or nonincreasing ($j = 2$) as v passes along $\mathcal{Q}_j$ from α_j to ξ. The variations in the error bounds must be evaluated along $\mathcal{Q}_j$. The reference points α_j can be at infinity, subject to convergence of the variations.

In the context of the present result, the condition on the path $\mathcal{Q}_j$ is called the *monotonicity condition*, and admissible paths are said to be *($u\xi$)-progressive*; compare 12.6.6. Again, points of Δ excluded by the monotonicity condition are called shadow zones; the zones depend on j, $\arg u$, and the choice of α_j.

Example: Modified Bessel functions of large order—The functions $z^{1/2}I_\nu(\nu z)$ and $z^{1/2}K_\nu(\nu z)$ satisfy

$$\frac{d^2w}{dz^2} = \left\{\nu^2\,\frac{1+z^2}{z^2} - \frac{1}{4z^2}\right\} w$$

The preceding theory will now be applied to derive uniform asymptotic expansions for large positive real values of ν.

The appropriate Liouville transformation is given by

$$\xi = \int \frac{(1+z^2)^{1/2}}{z}\,dz; \quad w = \left(\frac{z^2}{1+z^2}\right)^{1/4} W$$

Then

$$\frac{d^2W}{d\xi^2} = \{\nu^2 + \psi(\xi)\}\, W$$

where

$$\psi(\xi) = z^2(1 - \tfrac{1}{4}z^2)(1+z^2)^{-3}$$

Integration yields

$$\xi = (1+z^2)^{1/2} + \ln z - \ln\{1 + (1+z^2)^{1/2}\}$$

it being convenient to take the arbitrary constant of integration to be zero. The mapping between the planes of z and ξ is indicated in Figs. 12.8-1 and 12.8-2. We take **D** to be the sector $|\arg z| < \pi/2$. Its map Δ comprises the union of the sector $|\arg \xi| < \pi/2$ and the strip $|\mathrm{Im}\,\xi| < \pi/2$.

From the above results it follows that the transformed differential equation has solutions

$$W_{n,1}(\nu,\xi) = e^{\nu\xi}\left\{\sum_{s=0}^{n-1} \frac{A_s}{\nu^s} + \eta_{n,1}(\nu,\xi)\right\}$$

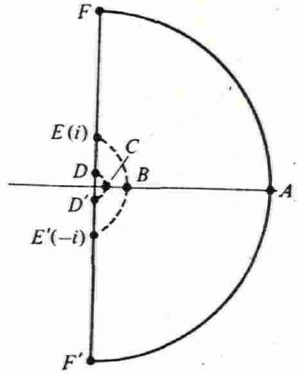

Fig. 12.8-1 z-plane: domain **D**.

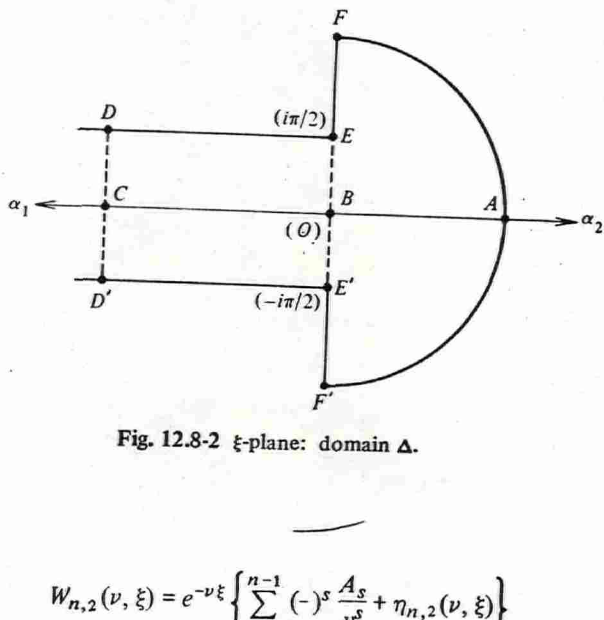

Fig. 12.8-2 ξ-plane: domain $\boldsymbol{\Delta}$.

$$W_{n,2}(\nu,\xi) = e^{-\nu\xi}\left\{\sum_{s=0}^{n-1}(-)^s\frac{A_s}{\nu^s} + \eta_{n,2}(\nu,\xi)\right\}$$

With the reference points α_1 and α_2 taken to be $-\infty$ and $+\infty$ respectively, the present error terms are bounded by

$$|\eta_{n,1}(\nu,\xi)| \leqslant 2\exp\left\{\frac{2\mathcal{V}_{-\infty,\xi}(A_1)}{\nu}\right\}\frac{\mathcal{V}_{-\infty,\xi}(A_n)}{\nu^n}$$

$$|\eta_{n,2}(\nu,\xi)| \leqslant 2\exp\left\{\frac{2\mathcal{V}_{\xi,\infty}(A_1)}{\nu}\right\}\frac{\mathcal{V}_{\xi,\infty}(A_n)}{\nu^n}$$

The condition on the variational paths is that Re ξ is monotonic; from Figs. 12.8-1 and 12.8-2 it is clear that the whole of Δ is admissible both for $j = 1$ and $j = 2$. In other words, shadow zones are absent in the present case.*

On reverting to the original variables we find that the recurrence relation for the coefficients $A_s(\xi)$ becomes

$$A_{s+1} = -\frac{1}{2}\frac{z}{(1+z^2)^{1/2}}\frac{dA_s}{dz} + \frac{1}{8}\int \frac{z(4-z^2)}{(1+z^2)^{5/2}} A_s\,dz$$

From this equation it ~~can be deduced~~ that A_s is a polynomial in $p \equiv (1+z^2)^{-1/2}$ of degree $3s$. If the integration constants are chosen in such a way that A_s vanishes at $z = \infty$ when $s \geqslant 1$, then we obtain

$$A_0 = 1;\quad A_1 = (3p - 5p^3)/24;\quad A_2 = (81p^2 - 462p^4 + 385p^6)/1152$$

To identify $I_\nu(\nu z)$ and $K_\nu(\nu z)$ in terms of the solutions just constructed, we note that as $z \to 0$, that is, $\xi \to -\infty$, both $I_\nu(\nu z)$ and $(1+z^2)^{-1/4}W_{n,1}(\nu,\xi)$ are recessive; hence their ratio is independent of z. Similarly, by letting $z \to +\infty$, that is, $\xi \to +\infty$, we see that the ratio of $K_\nu(\nu z)$ and $(1+z^2)^{-1/4}W_{n,2}(\nu,\xi)$ is independent of z. In both cases the ratio may be found by considering asymptotic forms as $z \to +\infty$, ν being fixed. The final expansions are

$$I_\nu(\nu z) = \frac{1}{1+\eta_{n,1}(\nu,\infty)}\frac{e^{\nu\xi}}{(2\pi\nu)^{1/2}(1+z^2)^{1/4}}\left\{\sum_{s=0}^{n-1}\frac{A_s}{\nu^s} + \eta_{n,1}(\nu,\xi)\right\}$$

$$K_\nu(\nu z) = \left(\frac{\pi}{2\nu}\right)^{1/2}\frac{e^{-\nu\xi}}{(1+z^2)^{1/4}}\left\{\sum_{s=0}^{n-1}(-)^s\frac{A_s}{\nu^s} + \eta_{n,2}(\nu,\xi)\right\}$$

valid when $\nu > 0$, $|\arg z| < \pi/2$, and n is any positive integer.

If z is restricted to the sector $|\arg z| \leqslant (\pi/2) - \delta\ (< \pi/2)$, then the error bounds show that

$$I_\nu(\nu z) \sim \frac{e^{\nu\xi}}{(2\pi\nu)^{1/2}(1+z^2)^{1/4}}\sum_{s=0}^{\infty}\frac{A_s}{\nu^s}$$

$$K_\nu(\nu z) \sim \left(\frac{\pi}{2\nu}\right)^{1/2}\frac{e^{-\nu\xi}}{(1+z^2)^{1/4}}\sum_{s=0}^{\infty}(-)^s\frac{A_s}{\nu^s}$$

as $\nu \to \infty$, uniformly with respect to z.

*This would not be so if ν were complex.

12.8.3 Auxiliary Functions for the Airy Functions

For negative values of x, the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are of oscillatory character, the period and amplitude of the oscillation diminishing as $x \to -\infty$. On the other hand, for increasing positive x both functions are positive and monotonic, $\mathrm{Ai}(x)$ tending rapidly to zero and $\mathrm{Bi}(x)$ to infinity. To make a combined assessment of the magnitudes of both functions applicable to both negative and positive arguments, we introduce a *weight function* $E(x)$, *modulus function* $M(x)$, and *phase function* $\theta(x)$.

The weight function is defined by

$$E(x) = \sqrt{\mathrm{Bi}(x)/\mathrm{Ai}(x)} \quad (c \leqslant x < \infty); \qquad E(x) = 1 \quad (-\infty < x \leqslant c)$$

Here $c = -0.36605\ldots$ is the negative root of the equation

$$\mathrm{Ai}(x) = \mathrm{Bi}(x)$$

having smallest absolute value. The function $E(x)$ is continuous and nondecreasing.

The modulus and phase functions are defined by

$$E(x)\,\mathrm{Ai}(x) = M(x)\sin\theta(x); \quad E^{-1}(x)\,\mathrm{Bi}(x) = M(x)\cos\theta(x)$$

where $E^{-1}(x) = 1/E(x)$. In consequence

$$M(x) = \sqrt{2\,\mathrm{Ai}(x)\,\mathrm{Bi}(x)}; \quad \theta(x) = \tfrac{1}{4}\pi \quad (x \geqslant c)$$

or

$$M(x) = \sqrt{\mathrm{Ai}^2(x) + \mathrm{Bi}^2(x)}; \quad \theta(x) = \tan^{-1}\{\mathrm{Ai}(x)/\mathrm{Bi}(x)\} \quad (x \leqslant c)$$

the branch of the inverse tangent being $\pi/4$ at $x = c$, and continuous elsewhere. The modulus is a slowly changing function with the property

$$M(x) \sim \pi^{-1/2}|x|^{-1/4} \qquad (x \to \pm\infty)$$

The phase is a nonincreasing function.

The following constant is required in the next subsection:

$$\lambda = \sup_{(-\infty,\infty)} \{\pi|x|^{1/2}M^2(x)\} = 1.04\ldots$$

12.8.4 Case II: Simple Turning Point

The standard form of differential equation for Case II is

$$\frac{d^2W}{d\xi^2} = \{u^2\xi + \psi(\xi)\}\,W$$

For simplicity, it is supposed here that the variables are real, u being positive, and also that $\psi(\xi)$ is infinitely differentiable in a finite or infinite interval (α, β) which contains the turning point $\xi = 0$.

The basic approximating equation is

$$\frac{d^2 W}{d\xi^2} = u^2 \xi W$$

solutions of which are $\mathrm{Ai}(u^{2/3}\xi)$ and $\mathrm{Bi}(u^{2/3}\xi)$. A formal series solution of the given differential equation can be constructed in the form

$$W = \mathrm{Ai}(u^{2/3}\xi) \sum_{s=0}^{\infty} \frac{A_s(\xi)}{u^{2s}} + \frac{\mathrm{Ai}'(u^{2/3}\xi)}{u^{4/3}} \sum_{s=0}^{\infty} \frac{B_s(\xi)}{u^{2s}}$$

with $A_0(\xi) = 1$. Differentiating and equating coefficients, we find that

$$B_s(\xi) = \frac{1}{2\xi^{1/2}} \int_0^{\xi} \{\psi(v) A_s(v) - A_s''(v)\} \frac{dv}{v^{1/2}} \qquad (\xi > 0)$$

$$B_s(\xi) = \frac{1}{2(-\xi)^{1/2}} \int_{\xi}^{0} \{\psi(v) A_s(v) - A_s''(v)\} \frac{dv}{(-v)^{1/2}} \qquad (\xi < 0)$$

and

$$A_{s+1}(\xi) = -\frac{1}{2} B_s'(\xi) + \frac{1}{2} \int \psi(\xi) B_s(\xi)\, d\xi$$

These equations determine recursively sequences of functions which are infinitely differentiable throughout (α, β), *including* $\xi = 0$.

Again, the formal series diverges in general, but corresponding to each nonnegative integer n there exists an infinitely differentiable solution $W_{2n+1,1}(u, \xi)$, such that

$$W_{2n+1,1}(u, \xi) = \mathrm{Ai}(u^{2/3}\xi) \sum_{s=0}^{n} \frac{A_s(\xi)}{u^{2s}} + \frac{\mathrm{Ai}'(u^{2/3}\xi)}{u^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,1}(u, \xi)$$

with

$$|\epsilon_{2n+1,1}(u, \xi)| \leqslant 2 \frac{M(u^{2/3}\xi)}{E(u^{2/3}\xi)} \exp\left\{\frac{2\lambda \mathcal{V}_{\xi,\beta}(|\xi|^{1/2} B_0)}{u}\right\} \frac{\mathcal{V}_{\xi,\beta}(|\xi|^{1/2} B_n)}{u^{2n+1}}$$

E, M, and λ being defined as in 12.8.3. For positive or negative ξ, the bound for $\epsilon_{2n+1,1}(u, \xi)$ is essentially $\mathrm{Ai}(u^{2/3}\xi)O(u^{-2n-1})$, except near the zeros of $\mathrm{Ai}(u^{2/3}\xi)$.

A second solution is furnished by

$$W_{2n+1,2}(u, \xi) = \mathrm{Bi}(u^{2/3}\xi) \sum_{s=0}^{n} \frac{A_s(\xi)}{u^{2s}} + \frac{\mathrm{Bi}'(u^{2/3}\xi)}{u^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,2}(u, \xi)$$

with

$$|\epsilon_{2n+1,2}(u, \xi)| \leqslant 2E(u^{2/3}\xi)\, M(u^{2/3}\xi) \exp\left\{\frac{2\lambda \mathcal{V}_{\alpha,\xi}(|\xi|^{1/2}B_0)}{u}\right\} \frac{\mathcal{V}_{\alpha,\xi}(|\xi|^{1/2}B_n)}{u^{2n+1}}$$

Analogous results are also available for complex variables; see Olver 1974a, Chapter 11.

12.8.5 Auxiliary Functions for Bessel Functions

For nonnegative real values of ν and positive values of x, a weight function $E_\nu(x)$ is defined by

$$E_\nu(x) = \{-Y_\nu(x)/J_\nu(x)\}^{1/2} \quad (0 < x \leqslant X_\nu); \quad E_\nu(x) = 1 \quad (X_\nu \leqslant x < \infty)$$

where $x = X_\nu$ is the smallest root of the equation

$$J_\nu(x) + Y_\nu(x) = 0$$

$E_\nu(x)$ is a continuous, positive, nonincreasing function of x.

Corresponding modulus and phase functions are defined by

$$J_\nu(x) = E_\nu^{-1}(x)\, M_\nu(x) \cos\theta_\nu(x); \quad Y_\nu(x) = E_\nu(x)\, M_\nu(x) \sin\theta_\nu(x)$$

thus

$$M_\nu(x) = \{2|Y_\nu(x)|J_\nu(x)\}^{1/2}; \quad \theta_\nu(x) = -\tfrac{1}{4}\pi \quad (0 < x \leqslant X_\nu)$$

$$M_\nu(x) = \{J_\nu^2(x) + Y_\nu^2(x)\}^{1/2}; \quad \theta_\nu(x) = \tan^{-1}\{Y_\nu(x)/J_\nu(x)\} \quad (x \geqslant X_\nu)$$

the branch of the inverse tangent being chosen to make $\theta_\nu(x)$ continuous everywhere.

12.8.6 Case III: Simple Pole

The final form of differential equation to be considered in the present section is

$$\frac{d^2W}{d\xi^2} = \left\{\frac{u^2}{4\xi} + \frac{\nu^2 - 1}{4\xi^2} + \frac{\psi(\xi)}{\xi}\right\} W$$

in which u is again a large positive parameter, ν is a nonnegative real constant, and $\psi(\xi)$ is infinitely differentiable in a finite or infinite interval (α, β) containing $\xi = 0$.

For $\xi \in (0, \beta)$ a formal series solution is supplied by

$$W = \xi^{1/2} I_\nu(u\xi^{1/2}) \sum_{s=0}^{\infty} \frac{A_s(\xi)}{u^{2s}} + \frac{\xi}{u} I_{\nu+1}(u\xi^{1/2}) \sum_{s=0}^{\infty} \frac{B_s(\xi)}{u^{2s}}$$

where I_ν is the modified Bessel function, and the coefficients are determined recursively by $A_0(\xi) = 1$

$$B_s(\xi) = -A_s'(\xi) + \frac{1}{\xi^{1/2}} \int_0^{\xi} \left\{\psi(v)\, A_s(v) - \left(\nu + \frac{1}{2}\right) A_s'(v)\right\} \frac{dv}{v^{1/2}}$$

and

$$A_{s+1}(\xi) = \nu B_s(\xi) - \xi B_s'(\xi) + \int \psi(\xi)\, B_s(\xi)\, d\xi$$

Each coefficient tends to a finite limit as $\xi \to 0$.

Corresponding to any nonnegative integer n, there exists a solution $W_{2n+1,1}(u, \xi)$ of the given differential equation of the form

$$W_{2n+1,1}(u, \xi) = \xi^{1/2} I_\nu(u\xi^{1/2}) \sum_{s=0}^{n} \frac{A_s(\xi)}{u^{2s}} + \frac{\xi}{u} I_{\nu+1}(u\xi^{1/2}) \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,1}(u, \xi)$$

with

$$\left|\epsilon_{2n+1,1}(u, \xi)\right| \leqslant \lambda_1(\nu)\, \xi^{1/2} I_\nu(u\xi^{1/2}) \exp\left\{\frac{\lambda_1(\nu)}{u} \mathcal{V}_{0,\xi}(\xi^{1/2} B_0)\right\} \frac{\mathcal{V}_{0,\xi}(\xi^{1/2} B_n)}{u^{2n+1}}$$

Here $\lambda_1(\nu)$ is the (finite) constant defined by

$$\lambda_1(\nu) = \sup_{x \in (0,\infty)} \{2x I_\nu(x) K_\nu(x)\}$$

For each n an independent second solution is given by

$$W_{2n+1,2}(u, \xi) = \xi^{1/2} K_\nu(u\xi^{1/2}) \sum_{s=0}^{n} \frac{A_s(\xi)}{u^{2s}} - \frac{\xi}{u} K_{\nu+1}(u\xi^{1/2}) \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,2}(u, \xi)$$

where K_ν is the second modified Bessel function, and

$$|\epsilon_{2n+1,2}(u,\xi)| \leqslant \lambda_1(\nu)\,\xi^{1/2}K_\nu(u\xi^{1/2})\exp\left\{\frac{\lambda_1(\nu)}{u}\mathcal{V}_{\xi,\beta}(\xi^{1/2}B_0)\right\}\frac{\mathcal{V}_{\xi,\beta}(\xi^{1/2}B_n)}{u^{2n+1}}$$

For negative ξ the solutions $W_{2n+1,1}(u,\xi)$ and $W_{2n+1,2}(u,\xi)$ are no longer appropriate since they have branch-points at $\xi = 0$ and become complex when continued to negative values. The coefficients $A_s(\xi)$ and $B_s(\xi)$ are free from singularity at $\xi = 0$, however, and may be continued to negative ξ; thus

$$B_s(\xi) = -A_s'(\xi) + \frac{1}{|\xi|^{1/2}}\int_\xi^0 \left\{\psi(v)\,A_s(v) - \left(\nu + \frac{1}{2}\right)A_s'(v)\right\}\frac{dv}{|v|^{1/2}}$$

and $A_{s+1}(\xi)$ is related to $B_s(\xi)$ by the same formula as for positive ξ.

Solutions of the given differential equation for $\xi \in (\alpha, 0)$ are given by

$$W_{2n+1,3}(u,\xi) = |\xi|^{1/2}J_\nu(u|\xi|^{1/2})\sum_{s=0}^{n}\frac{A_s(\xi)}{u^{2s}} - \frac{|\xi|}{u}J_{\nu+1}(u|\xi|^{1/2})\sum_{s=0}^{n-1}\frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,3}(u,\xi)$$

$$W_{2n+1,4}(u,\xi) = |\xi|^{1/2}Y_\nu(u|\xi|^{1/2})\sum_{s=0}^{n}\frac{A_s(\xi)}{u^{2s}} - \frac{|\xi|}{u}Y_{\nu+1}(u|\xi|^{1/2})\sum_{s=0}^{n-1}\frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,4}(u,\xi)$$

where n is again an arbitrary nonnegative integer, and

$$|\epsilon_{2n+1,3}(u,\xi)| \leqslant \lambda_3(\nu)|\xi|^{1/2}\frac{M_\nu(u|\xi|^{1/2})}{E_\nu(u|\xi|^{1/2})}\exp\left\{\frac{\lambda_2(\nu)}{u}\mathcal{V}_{\xi,0}(|\xi|^{1/2}B_0)\right\}\frac{\mathcal{V}_{\xi,0}(|\xi|^{1/2}B_n)}{u^{2n+1}}$$

$$|\epsilon_{2n+1,4}(u,\xi)| \leqslant \lambda_4(\nu)|\xi|^{1/2}E_\nu(u|\xi|^{1/2})\,M_\nu(u|\xi|^{1/2}) \times \exp\left\{\frac{\lambda_2(\nu)}{u}\mathcal{V}_{\alpha,\xi}(|\xi|^{1/2}B_0)\right\}\frac{\mathcal{V}_{\alpha,\xi}(|\xi|^{1/2}B_n)}{u^{2n+1}}$$

Here E_ν and M_ν are defined as in 12.8.5, and $\lambda_2(\nu)$, $\lambda_3(\nu)$, $\lambda_4(\nu)$ denote the suprema of the functions

$$\pi x M_\nu^2(x),\quad \pi x|J_\nu(x)|E_\nu(x)\,M_\nu(x),\quad \pi x|Y_\nu(x)|M_\nu(x)/E_\nu(x)$$

respectively, for $x \in (0, \infty)$. Each is finite.

Again, analogous results are available for complex u and ξ.

12.9 ESTIMATION OF REMAINDER TERMS

12.9.1 Numerical Use of Asymptotic Approximations

When a realistic analytical bound for the error term in a given expansion is unavailable, it is unsafe to infer the size of the error term simply by inspecting the rate of numerical decrease of the terms in the series. Even in the case of a convergent power-series expansion this cannot be done: the tail has to be majorized analytically —for example, by a geometric progression—before final accuracy can be guaranteed. For a divergent asymptotic expansion the situation is much worse. First, it is impossible to majorize the tail. Secondly, the series represents an infinite class of functions, and the error term depends on which particular member of the class we have in mind.

In cases where the asymptotic variable x, say, is real and positive and the distinguished point is at infinity, the wanted function should be computed by an independent (preferably non-asymptotic) method at the smallest value of x it is intended to apply the asymptotic approximation. If the results are in agreement to S significant figures, then it is likely (but not *certain*) that the approximation will be accurate to at least S significant figures for all greater values of x.

For a complex variable z, both $|z|$ and $\arg z$ have to be considered in appraising accuracy. Suppose that an asymptotic approximation is valid as $z \to \infty$ in any closed sector within $\theta_1 < \arg z < \theta_2$, but not within a larger sector. Then the accuracy of the approximation deteriorates severely as the rays $\arg z = \theta_1, \theta_2$ are approached. In consequence, numerical work should be confined to a sector $\theta_1' \leqslant \arg z \leqslant \theta_2'$ lying well within $\theta_1 < \arg z < \theta_2$, and independent evaluations made at $\arg z = \theta_1'$ and θ_2' for the smallest value of $|z|$ intended to be used.

When the regions of validity in the complex plane are not sectors error appraisal is more complicated. Basically, however, the guiding principle is to keep a safe distance from the true boundaries of the region of validity.

12.9.2 Converging Factors

Let $f(z)$ be a function of z having the asymptotic expansion

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

as $z \to \infty$ in a sector **S**: $\theta_1 \leqslant \arg z \leqslant \theta_2$. As a special case **S** can degenerate into the positive or negative real axis. Suppose that successive terms in the series diminish in absolute value until the $(n+1)^{\text{th}}$ term is reached, thereafter they increase. Clearly $n = n(z)$, where $n(z)$ is a discontinuous function of $|z|$ which is independent of $\arg z$. Generally the expansion yields its greatest accuracy when truncated at $n(z)$ terms. We write

$$R_n(z) = f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s}$$

and call $R_{n(z)}(z)$ the *optimum remainder term*, whether or not it is actually the least.

Now define

$$C(z) = R_{n(z)}(z)/\{a_{n(z)}z^{-n(z)}\}$$

so that

$$f(z) = \sum_{s=0}^{n(z)-1} \frac{a_s}{z^s} + C(z)\,\frac{a_{n(z)}}{z^{n(z)}}$$

If we have a way of assessing $C(z)$ when $|z|$ is large, then the magnitude of the optimum remainder term can be estimated. In some cases it is actually possible to construct an asymptotic expansion for $C(z)$ in descending powers of z or $n(z)$. In these fortunate circumstances $C(z)$ can be calculated to several significant figures, considerably increasing the attainable accuracy in the computed value of $f(z)$. For this reason $C(z)$ is called a *converging factor*.

Example: The exponential integral—From (7.1-2) we have

$$E_1(z) = e^{-z}\int_0^\infty \frac{e^{-zt}}{1+t}\,dt \qquad (|\arg z| < \tfrac{1}{2}\pi)$$

Since

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-)^{n-1}t^{n-1} + \frac{(-)^n t^n}{1+t}$$

for any nonnegative integer n, it follows that

$$E_1(z) = e^{-z}\left\{\sum_{s=0}^{n-1} u_s(z) + R_n(z)\right\}$$

where

$$u_s(z) = (-)^s s!\, z^{-s-1}$$

and

$$R_n(z) = (-)^n \int_0^\infty \frac{e^{-zt}t^n}{1+t}\,dt$$

The series $\sum u_s(z)$ is the asymptotic expansion of $e^z E_1(z)$ for large $|z|$, and the term of smallest absolute value is $u_{[|z|]}(z)$, unless $|z|$ is an integer in which event there are two equally small terms $u_{|z|-1}(z)$ and $u_{|z|}(z)$.

We write $\theta = \arg z$ and seek an asymptotic approximation to

$$R_n\{(n+\zeta)e^{i\theta}\} \equiv (-)^n \int_0^\infty \frac{\exp\{-(n+\zeta)e^{i\theta}t\}\,t^n}{1+t}\,dt$$

for large n and fixed values of ζ and θ. The saddle-point of the integrand is given by

$$\frac{d}{dt}\{t\exp(-e^{i\theta}t)\} = 0$$

that is, by $t = e^{-i\theta}$. The path of integration is made to pass through this point by rotation through an angle $-\theta$. Then setting $t = \tau e^{-i\theta}$, we obtain

$$R_n\{(n+\zeta)e^{i\theta}\} = (-)^n e^{-i(n+1)\theta} \int_0^\infty \frac{e^{-(n+\zeta)\tau}\tau^n}{1+\tau e^{-i\theta}}\,d\tau$$

And although this result has been derived on the assumption that $|\theta| < \pi/2$, it is easily extended to $|\theta| < \pi$ by further rotation of the path and appeal to analytic continuation. The desired asymptotic expansion is then found by Laplace's method (12.2.6) to be

$$R_n\{(n+\zeta)e^{i\theta}\} \sim (-)^n(1-\alpha)\,e^{-n-\zeta-i(n+1)\theta}\left(\frac{2\pi}{n}\right)^{1/2}$$

$$\times\left\{1 + \frac{\zeta^2 - 2\zeta + 2\alpha\zeta + \frac{1}{6} - 2\alpha + 2\alpha^2}{2n} + \cdots\right\}$$

as $n \to \infty$, uniformly with respect to ζ in any compact set and $\theta \in [-\pi+\delta, \pi-\delta]$. Here $\alpha \equiv 1/(1+e^{i\theta})$, and δ is an arbitrary positive constant less than π.

To derive the corresponding expansion for the converging factor

$$C_n(z) \equiv R_n(z)/u_n(z)$$

we make use of Stirling's series (7.2-11) for $n!$. In this way we obtain

$$C_n\{(n+\zeta)e^{i\theta}\} \sim (1-\alpha)\left\{1 + \frac{\alpha(\zeta - 1 + \alpha)}{n} + \cdots\right\}$$

again as $n \to \infty$, uniformly with respect to bounded ζ and $\theta \in [-\pi+\delta, \pi-\delta]$. To apply this result for an assigned value of z, we take $\theta = \arg z$, $n = [|z|]$, and $\zeta = |z| -$

$[|z|]$, so that $(n + \zeta) e^{i\theta} = z$. Truncating the expansion for $C_n\{(n + \zeta)e^{i\theta}\}$ at its first term, for example, we conclude that if the original expansion for $E_1(z)$ is truncated at its $[|z|]^{\text{th}}$ term, then the remainder term is approximately equal to the first neglected term multiplied by $1 - \alpha$, that is, $1/(1 + e^{-i\theta})$. For positive real z this becomes $\frac{1}{2}$.

As a numerical illustration, take $z = 5$. Then

$$u_0(5) = 0.2;\quad u_1(5) = -0.04;\quad u_2(5) = 0.016;\quad u_3(5) = -0.0096;\quad u_4(5) = 0.00768$$

whence

$$u_0(5) + u_1(5) + \cdots + u_4(5) = 0.17408$$

compared with the correct value $e^5 E_1(5) = 0.170422176 \ldots$. From the asymptotic expansion for the converging factor, we calculate

$$C_5(5) \sim \tfrac{1}{2}\left(1 - \tfrac{1}{4} \cdot \tfrac{1}{5} + \cdots\right) = 0.475$$

on neglecting terms beyond the second. Since $u_5(5) = -0.00768$, the estimate $C_5(5)\, u_5(5)$ for the remainder term is $-0.00365 \ldots$. Impressively, this equals the discrepancy between the partial sum $u_0(5) + u_1(5) + \cdots + u_4(5)$ and $e^5 E_1(5)$, to within one unit of the fifth decimal place.

12.9.3 Euler's Transformation

Another way of increasing the accuracy obtainable from an asymptotic expansion is to transform it into a new series in which the initial terms decrease at a faster rate. Then it is often the case that the optimum remainder term is smaller for the new series than for the original series. It might even happen that the new series converges.

The most frequently used transformation is due to Euler, and is as follows. Let $a_0, a_1, a_2, \ldots$ be a given sequence and $b_0, b_1, b_2, \ldots$ a derived sequence, defined by

$$b_s = k^s [\Delta^s (a_j k^{-j})]_{j=0}$$

Here k is an arbitrary number and Δ the forward difference operator: $\Delta v_j = v_{j+1} - v_j$, $\Delta^2 v_j = \Delta v_{j+1} - \Delta v_j$, and so on. Suppose that $f(z)$ is an analytic function of the complex variable z such that

$$f(z) \sim \sum_{s=0}^{\infty} \frac{a_s}{z^{s+1}}$$

as $z \to \infty$ in a given sector **S**. Then

$$f(z) \sim \sum_{s=0}^{\infty} \frac{b_s}{(z - k)^{s+1}} \qquad (z \to \infty \text{ in } \mathbf{S})$$

With $k = 1$ and $z = -1$, the transformation reduces to

$$\sum_{s=0}^{\infty} (-)^s a_s \sim \sum_{s=0}^{\infty} (-)^s \frac{\Delta^s a_0}{2^{s+1}}$$

Example: Let us consider again the asymptotic expansion of the function $e^5 E_1(5)$ of 12.9.2. Application of Euler's transformation is delayed until the smallest term is reached; thus $a_s = (-)^s u_{s+5}(5)$. Relevant forward differences are given in units of the fifth decimal place in the accompanying table.

s	a_s	Δa_s	$\Delta^2 a_s$	$\Delta^3 a_s$	$\Delta^4 a_s$	$\Delta^5 a_s$	$\Delta^6 a_s$
0	-0.00768	- 154	- 214	- 192	- 280	- 434	-804
1	-0.00922	- 368	- 406	- 472	- 714	-1238	
2	-0.01290	- 774	- 878	-1186	-1952		
3	-0.02064	-1652	-2064	-3138			
4	-0.03716	-3716	-5202				
5	-0.07432	-8918					
6	-0.16350						

The first few terms of the transformed series are

$$\sum_{s=0}^{\infty} (-)^s \frac{\Delta^s a_0}{2^{s+1}} = 10^{-5} \left(-384 + 38 - 27 + 12 - 9 + 7 - 6 + \cdots\right)$$

Truncation at the sixth term gives -0.00363, then addition to $u_0(5) + u_1(5) + \cdots + u_4(5)$ yields 0.17045, agreeing with the value of $e^5 E_1(5)$ to within 0.00003; compare 12.9.2. Even closer agreement is attainable by working with more terms and more decimal places, and applying a second Euler transformation.

From the analytical standpoint the numerical procedure is equivalent to expanding the remainder term

$$R_5(z) = e^z E_1(z) - \sum_{s=0}^{4} u_s(z)$$

as an asymptotic series in descending powers of $z + 5$, and truncating the new series at the optimum stage.

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